



HAL
open science

Stable minimal hypersurface in R^6

Laurent Mazet

► **To cite this version:**

| Laurent Mazet. Stable minimal hypersurface in R^6 . 2024. hal-04583971v1

HAL Id: hal-04583971

<https://hal.science/hal-04583971v1>

Preprint submitted on 22 May 2024 (v1), last revised 23 May 2024 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

STABLE MINIMAL HYPERSURFACES IN \mathbb{R}^6

LAURENT MAZET

ABSTRACT. Following the strategy developed by Chodosh, Li, Minter and Stryker, and using the volume estimate of Antonelli and Xu, we prove that, in \mathbb{R}^6 , a complete, two-sided, stable minimal hypersurfaces is flat.

1. INTRODUCTION

A minimal hypersurface M^n of \mathbb{R}^{n+1} is a critical point of the n -volume functional. It is characterized by its vanishing mean curvature. If a unit normal vectorfield ν is defined along M and φ is a function with compact support on M , one can consider a deformation of M with initial speed $\varphi\nu$. The computation of the second derivative of the n -volume along this deformation at initial time gives

$$\int_M |\nabla\varphi|^2 - |A_M|^2\varphi^2$$

where A_M is the second fundamental form of M . So asking that this quantity is non negative for any φ means that M is a minimum at order 2 of the n -volume. Such a minimal hypersurface is called stable.

The stable Bernstein problem asks whether a complete stable minimal hypersurface is a flat affine hyperplane. We give a positive answer in the case $n = 5$.

Theorem 1.1. *Let $M^5 \looparrowright \mathbb{R}^6$ be an immersed, complete, connected, two-sided, stable minimal hypersurface. Then M is a Euclidean hyperplane.*

A particular class of stable minimal hypersurface is given by minimal graphs over \mathbb{R}^n . In [4], Bernstein proved that a minimal graph over \mathbb{R}^2 has to be a plane. In the sixties, the same question for higher dimensions was studied in a series of papers by Fleming [17], De Giorgi [14], Almgren [1] and Simons [26]. They proved that minimal graphs over \mathbb{R}^n are planes if $n \leq 7$. For $n \geq 8$, Bombieri, De Giorgi and Giusti [5] were able to construct counter-examples and gave also in \mathbb{R}^8 an example of a stable minimal hypersurfaces that is not a hyperplane.

Concerning the stable Bernstein problem, the question was solved positively in \mathbb{R}^3 by Do Carmo and Peng [15], Fischer-Colbrie and Schoen [16] and Porogelov [22] in the early eighties. In higher dimension, Schoen, Simon and Yau [23, 24] were able to settle the stable Bernstein in \mathbb{R}^{n+1} , $n \leq 6$, under a Euclidean volume growth assumption (see also the recent work of Bellettini [3]).

Recently Chodosh and Li [9] were able to answer positively the stable Bernstein problem in \mathbb{R}^4 . Later two alternative proofs came out: one by Catino, Mastrolia and Roncoroni [8] and one by Chodosh and Li [10]. Actually in [10], Chodosh and Li develop a second strategy to prove the result. Then, in a joint work Minter and Stryker [12], they were able to apply this strategy in the case of \mathbb{R}^5 to solve the stable Bernstein problem in this dimension as well.

As in [13, 12], it is well known that Theorem 1.1 comes with corollaries like curvature estimates for stable minimal immersions in 6-dimensional manifolds and characterization of finite Morse index minimal hypersurfaces in \mathbb{R}^6 . For example, we have

Corollary 1.1. *Let (X^6, g) be a complete Riemannian manifold whose sectional curvature satisfies $|sec_g| \leq K$. Then any compact, two-sided, stable minimal immersion $M^5 \looparrowright X$ satisfies*

$$|A_M|(q) \min(1, d_M(q, \partial M)) \leq C(K)$$

for $q \in M$.

The basic idea to prove Theorem 1.1 is to obtain a Euclidean growth estimate for the volume of M and then apply the work of Schoen, Simon and Yau. The strategy of Chodosh and Li is a way towards this estimate. We refer to [10, 12] for a good presentation of their ideas. Let us give some elements. Let M be a stable minimal hypersurface in \mathbb{R}^{n+1} with induced metric g . Inspired by the work of Gulliver and Lawson [18], they consider the conformal metric $\tilde{g} = r^{-2}g$ where r is the Euclidean distance to 0 in \mathbb{R}^{n+1} . If M was a hyperplane passing through the origin $(M \setminus \{0\}, \tilde{g})$ would be isometric to the Euclidean product $\mathbb{S}^{n-1} \times \mathbb{R}$. In the general case, the idea of Chodosh and Li is that the stability assumption implies that the geometry of $(M \setminus \{0\}, \tilde{g})$ should look like $\mathbb{S}^{n-1} \times \mathbb{R}$. In [12], the authors consider the bi-Ricci curvature which is a certain combination of sectional curvatures. It was introduced by Shen and Ye in [25], already to study minimal surfaces (see precise definition in Section 2). Notice that on $\mathbb{S}^{n-1} \times \mathbb{R}$, the bi-Ricci curvature is lower bounded by $n-2$. In [12], the authors prove that the stability of M implies a positive spectral lower bound for the bi-Ricci curvature of $(M \setminus \{0\}, \tilde{g})$. More precisely they prove that, on $(M \setminus \{0\}, \tilde{g})$, the operator $-a\tilde{\Delta} + (\widetilde{\text{BRic}}_- - 1)$ is non-negative where $\widetilde{\text{BRic}}_-$ is the punctual minimum of the bi-Ricci curvature of \tilde{g} . This should be understood as a weak form of $\widetilde{\text{BRic}} \geq 1$.

The second step of the strategy consists in the construction of a μ -bubble in $(M \setminus \{0\}, \tilde{g})$ with a spectral lower bound for its Ricci curvature. In some sense, they identify in any sufficiently large part of $(M \setminus \{0\}, \tilde{g})$ a hypersurface that play the role of $\mathbb{S}^{n-1} \times \{t\}$ in $\mathbb{S}^{n-1} \times \mathbb{R}$.

The last step in [12] is to obtain a Bishop-Gromov volume estimate for the μ -bubble under the spectral lower bound on the Ricci curvature. In their paper, the proof of this volume estimate was specific to dimension 3. Recently, Antonelli and Xu [2] have proved such a Bishop-Gromov estimate in any dimension.

Once the \tilde{g} -volume of the μ -bubble is controlled, this gives an estimate of its volume in the original metric g and then control the growth of the volume of M tanks to an isoperimetric inequality due to Michael and Simon [20] and Brendle [6].

In the present paper, we also follow the above strategy of [12]. Here we consider a weighted bi-Ricci curvature BRic_α where the parameter α does not give the same weight to all sectional curvature in the combination (a similar idea appear in the recent article by Hong and Yan [19]). We prove a spectral lower bound for the weighted bi-Ricci curvature: the operator $-a\tilde{\Delta} + (\widetilde{\text{BRic}}_{\alpha-} - \delta)$ is non-negative where $a, \delta \in \mathbb{R}$. At that step, a, α are two parameters that should be chosen such that $\delta > 0$.

By imposing some new constraints on a and α , we are then able to construct the μ -bubble with a spectral lower bound on the Ricci curvature. At the last step, we apply the Bishop-Gromov estimate of Antonelli and Xu [2]. In order to do so, this imposes some new

constraints on the parameters a and α . Nevertheless, the choice $a = \frac{11}{10}$ and $\alpha = \frac{40}{43}$ fits all the constraints. The end of the proof then follows the line of [12].

Organization. In Section 2, we fix some notations that we use all along the paper. Section 3 is devoted to the proof the spectral lower bound for $\widetilde{\text{BRic}}_\alpha$ for the Gulliver-Lawson metric on a stable minimal hypersurface. In Section 4, we construct the μ -bubble with a spectral lower Ricci bound. We end the proof of Theorem 1.1 in Section 5. Along the paper, we specify the value of n , a and α only when it is necessary, we hope this allows to understand where the constraints come from.

Acknowledgments. The author was partially supported by the ANR-19-CE40-0014 grant. Part of this work was carried out during a stay at the Instituto de Matemáticas de la Universidad de Granada (IMAG), the author would to thank its members for their hospitality.

2. PRELIMINARIES

Let (M^n, g) be a Riemannian manifold and $(e_i)_{1 \leq i \leq n}$ be an orthonormal basis of $T_p M$. For $\alpha \in \mathbb{R}$, we recall or define

- the Ricci curvature $\text{Ric}(e_1, e_1) = \sum_{i=2}^n R(e_1, e_i, e_i, e_1)$,
- the punctual minimum of the Ricci curvature $\lambda(p) = \min_{v \in T_p M, |v|=1} \text{Ric}(e, e)$,
- the weighted bi-Ricci or α -bi-Ricci curvature

$$\text{BRic}_\alpha(e_1, e_2) = \sum_{i=2}^n R(e_1, e_i, e_i, e_1) + \alpha \sum_{j=3}^n R(e_2, e_j, e_j, e_2)$$

- the minimum of the α -bi-Ricci curvature $\Lambda_\alpha(p) = \min_{(e,f) \text{ orthonormal in } T_p M} \text{BRic}_\alpha(e, f)$

Notice that for $\alpha = 1$, BRic_1 is the classical bi-Ricci curvature as defined in [25].

If $\Sigma \looparrowright M$ is a hypersurface with unit normal ν . We use the following conventions:

- the second fundamental form of Σ is $A_\Sigma(X, Y) = (\nabla_X \nu, Y) = -(\nabla_X Y, \nu)$ and
- the mean curvature of Σ is $H = \text{tr } A_\Sigma$.

If Ω is a subset of M , we denote by $\mathcal{N}_\rho(\Omega)$ the ρ -tubular neighborhood of Ω : the set of points at distance less than ρ from Ω .

We finish by a simple remark that we use in Section 3.

Remark 1. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a positive definite symmetric matrix and $B \in \mathbb{R}^n$. Then the function $f : X \in \mathbb{R}^n \mapsto X^\top A X + B^\top X \in \mathbb{R}$ is lower bounded and its minimum is given by $-\frac{1}{4} B^\top A^{-1} B$.

3. SPECTRAL LOWER BOUND FOR THE WEIGHTED BI-RICCI CURVATURE

Let $F : M^n \looparrowright \mathbb{R}^{n+1}$ be a complete two-sided minimal hypersurface and g its induced metric. We consider the Gulliver-Lawson conformal metric $\tilde{g} = r^{-2}g$ where r is the Euclidean distance function to 0. Notice that if $F(p) = 0$, \tilde{g} is not defined. So we consider $N = M \setminus F^{-1}(0)$. As it was observed by Gulliver and Lawson [18], the metric (N, \tilde{g}) is complete.

The first step of the proof of Theorem 1.1 consists in proving that the stability assumption can be translated in a spectral lower bound for the α -bi-Ricci curvature of the metric \tilde{g} . Actually we have the following result.

Theorem 3.1. *Let $M^n \looparrowright \mathbb{R}^{n+1}$ be a two-sided stable minimal hypersurface. Suppose $n = 5$, then, for $a = \frac{11}{10}$, $\alpha = \frac{40}{43}$ and $\delta = \frac{3}{10}$, there is a smooth function V such that*

$$V \geq \delta - \tilde{\Lambda}_\alpha$$

and

$$(1) \quad \int_N |\nabla \varphi|_{\tilde{g}}^2 dv_{\tilde{g}} \geq \int_N \frac{1}{a} V \varphi^2 dv_{\tilde{g}}$$

for any $\varphi \in C_c^1(N)$.

3.1. Recalling some computations. We first recall some computations and results of [12].

We denote by ν the unit normal to M and by $|dr|$ the norm of the differential of r along M with respect to the g -metric.

Let $(e_i)_{1 \leq i \leq n}$ be an orthonormal basis for the metric g then, for the conformal metric \tilde{g} , an orthonormal basis is given by $\tilde{e}_i = re_i$. The sectional curvatures of g and \tilde{g} are related by

$$(2) \quad \tilde{R}_{ijji} = r^2 R_{ijji} + 2 - |dr|^2 - dr(e_i)^2 - dr(e_j)^2 - (p, \nu)(A_{ii} + A_{jj})$$

(see [12, Proposition 3.5]).

The second result that we want to recall is the writing of the stability inequality in the conformal metric \tilde{g} . We have

$$(3) \quad \int_N |\nabla \varphi|^2 dv_{\tilde{g}} \geq \int_N \left(r^2 |A|^2 - \frac{n(n-2)}{2} + \frac{n^2-4}{4} |dr|^2 \right) \varphi^2 dv_{\tilde{g}}$$

for any $\varphi \in C_c^1(N)$ (see [12, Proposition 3.10]).

3.2. Estimating the curvature terms. In this subsection, we want to relate the curvature term in the stability inequality (3) to the α -bi-Ricci curvature. We use the notations of the preceding subsection.

Proposition 3.1.

$$\begin{aligned} \widetilde{\text{BRic}}_\alpha(\tilde{e}_1, \tilde{e}_2) &= r^2 \text{BRic}_\alpha(e_1, e_2) + 2(n-1 + \alpha(n-2)) - (n + \alpha(n-1))|dr|^2 \\ &\quad - ((n-2 - \alpha)dr(e_1)^2 + \alpha(n-3)dr(e_2)^2) \\ &\quad - (p, \nu)((n-2 - \alpha)A_{11} + \alpha(n-3)A_{22}) \end{aligned}$$

Proof. Summing (2) and using $\text{tr } A = 0$, we have

$$\begin{aligned} \widetilde{\text{BRic}}_\alpha(\tilde{e}_1, \tilde{e}_2) &= \sum_{i=2}^n \tilde{R}_{1ii1} + \alpha \sum_{j=3}^n \tilde{R}_{2jj2} \\ &= r^2 \text{BRic}_\alpha(e_1, e_2) + 2(n-1) - (n-1)|dr|^2 - (n-1)dr(e_1)^2 \\ &\quad - (|dr|^2 - dr(e_1)^2) - (p, \nu)((n-1)A_{11} - A_{11}) + 2\alpha(n-2) \\ &\quad - \alpha(n-2)|dr|^2 - \alpha(n-2)dr(e_2)^2 - \alpha(|dr|^2 - dr(e_1)^2 - dr(e_2)^2) \\ &\quad - \alpha(p, \nu)((n-2)A_{22} - A_{11} - A_{22}) \\ &= r^2 \text{BRic}_\alpha(e_1, e_2) + 2(n-1 + \alpha(n-2)) - (n + \alpha(n-1))|dr|^2 \\ &\quad - ((n-2 - \alpha)dr(e_1)^2 + \alpha(n-3)dr(e_2)^2) \\ &\quad - (p, \nu)((n-2 - \alpha)A_{11} + \alpha(n-3)A_{22}) \end{aligned}$$

□

Proposition 3.2.

$$\text{BRic}_\alpha(e_1, e_2) = - \sum_{i=1}^n A_{1i}^2 - \alpha \sum_{j=2}^n A_{2j}^2 - \alpha A_{11} A_{22}$$

Proof. Applying Gauss formula we have

$$\begin{aligned} \text{BRic}_\alpha(e_1, e_2) &= \sum_{i=2}^n (A_{11} A_{ii} - A_{1i}^2) + \alpha \sum_{j=3}^n (A_{22} A_{jj} - A_{2j}^2) \\ &= - \sum_{i=1}^n A_{1i}^2 + \alpha (-A_{22} (A_{11} + A_{22}) - \sum_{j=3}^n A_{2j}^2) \\ &= - \sum_{i=1}^n A_{1i}^2 - \alpha \sum_{j=2}^n A_{2j}^2 - \alpha A_{11} A_{22} \end{aligned}$$

□

Using the above computation, we obtain the following estimate of the curvature term. This estimate introduces some constraints on α and a second parameter a .

Proposition 3.3. *Let $a, \alpha > 0$ such that $a > \frac{1}{2}$, $2a \geq \alpha$ and*

$$W = (a - \frac{1}{2}) (a - \frac{n-2}{2n} (1 + 2\alpha)) - \frac{n-2}{4n} (1 - \alpha)^2 > 0$$

Let us define

$$f = \frac{(n-2)^2}{8W} \left((a - \frac{1}{2}) \frac{n-2}{n} (1 + \alpha \frac{n-4}{n-2})^2 + (1 - \alpha)^2 (a + \frac{n-2}{2n} - \frac{2}{n} \alpha) \right)$$

Then

$$ar^2|A|^2 + f(1 - |dr|^2) \geq -r^2 \text{BRic}_\alpha(e_1, e_2) + (p, \nu) ((n-2-\alpha)A_{11} + \alpha(n-3)A_{22})$$

Proof. By Proposition 3.2, the right-hand side of the expected inequality satisfies to

$$\begin{aligned} &-r^2 \text{BRic}_\alpha(e_1, e_2) + (p, \nu) ((n-2-\alpha)A_{11} + \alpha(n-3)A_{22}) \\ &= r^2 \left(\sum_{i=1}^n A_{1i}^2 + \alpha \sum_{j=2}^n A_{2j}^2 + \alpha A_{11} A_{22} \right. \\ &\quad \left. + (\frac{p}{r^2}, \nu) ((n-2-\alpha)A_{11} + \alpha(n-3)A_{22}) \right) \\ (4) \quad &= r^2 \left(A_{11}^2 + \alpha A_{22}^2 + \alpha A_{11} A_{22} + \sum_{i=2}^n A_{1i}^2 + \alpha \sum_{j=3}^n A_{2j}^2 \right. \\ &\quad \left. + (\frac{p}{r^2}, \nu) ((n-2-\alpha)A_{11} + \alpha(n-3)A_{22}) \right) \end{aligned}$$

The vector $A_\Delta = (A_{11}, \dots, A_{nn})$ belongs to the sub-space $F_n = \{X \in \mathbb{R}^n \mid x_1 + \dots + x_n = 0\}$. We write a decomposition in an orthonormal basis of F_n as

$$\begin{pmatrix} A_{11} \\ \vdots \\ A_{nn} \end{pmatrix} = \sum_{i=1}^{n-3} \begin{pmatrix} 0 \\ 0 \\ E_i \end{pmatrix} x_i + \frac{1}{\sqrt{2n(n-2)}} \begin{pmatrix} n-2 \\ n-2 \\ -2 \\ \vdots \\ -2 \end{pmatrix} z_1 + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} z_2$$

where $(E_i)_{1 \leq i \leq n-3}$ is an orthonormal basis of F_{n-2} . So we have

$$\begin{aligned} & A_{11}^2 + \alpha A_{22}^2 + \alpha A_{11} A_{22} + \left(\frac{p}{r^2}, \nu\right) ((n-2-\alpha)A_{11} + \alpha(n-3)A_{22}) \\ &= \left(\frac{\sqrt{n-2}}{\sqrt{2n}} z_1 + \frac{z_2}{\sqrt{2}}\right)^2 + \alpha \left(\frac{\sqrt{n-2}}{\sqrt{2n}} z_1 - \frac{z_2}{\sqrt{2}}\right)^2 + \alpha \left(\frac{n-2}{2n} z_1^2 - \frac{1}{2} z_2^2\right) \\ (5) \quad &+ \left(\frac{p}{r^2}, \nu\right) \left(\frac{\sqrt{n-2}}{\sqrt{2n}} (n-2 + \alpha(n-4)) z_1 + \frac{n-2}{\sqrt{2}} (1-\alpha) z_2\right) \\ &= \frac{n-2}{2n} (1+2\alpha) z_1^2 + \sqrt{\frac{n-2}{n}} (1-\alpha) z_1 z_2 + \frac{1}{2} z_2^2 \\ &+ \left(\frac{p}{r^2}, \nu\right) \frac{n-2}{\sqrt{2}} \left(\sqrt{\frac{n-2}{n}} \left(1 + \alpha \frac{n-4}{n-2}\right) z_1 + (1-\alpha) z_2\right) \end{aligned}$$

For $a > 0$, we are interested in the minimum (if it exists) of

$$\begin{aligned} (6) \quad & a(z_1^2 + z_2^2) - \frac{n-2}{2n} (1+2\alpha) z_1^2 - \sqrt{\frac{n-2}{n}} (1-\alpha) z_1 z_2 - \frac{1}{2} z_2^2 \\ & - \left(\frac{p}{r^2}, \nu\right) \frac{n-2}{\sqrt{2}} \left(\sqrt{\frac{n-2}{n}} \left(1 + \alpha \frac{n-4}{n-2}\right) z_1 + (1-\alpha) z_2\right) \end{aligned}$$

The matrix of the quadratic part of the above expression is

$$\begin{pmatrix} a - \frac{n-2}{2n} (1+2\alpha) & -\sqrt{\frac{n-2}{4n}} (1-\alpha) \\ -\sqrt{\frac{n-2}{4n}} (1-\alpha) & a - \frac{1}{2} \end{pmatrix}$$

This matrix is positive definite if $a > \frac{1}{2}$ and its determinant is positive:

$$W = \left(a - \frac{1}{2}\right) \left(a - \frac{n-2}{2n} (1+2\alpha)\right) - \frac{n-2}{4n} (1-\alpha)^2 > 0$$

If it's the case, by Remark 1 with vector $B = -(\frac{p}{r^2}, \nu)^{\frac{n-2}{\sqrt{2}}}(\sqrt{\frac{n-2}{n}}(1 + \alpha\frac{n-4}{n-2}), (1 - \alpha))$, the quantity in (6) is lower bounded by

$$\begin{aligned} & -(\frac{p}{r^2}, \nu)^2 \frac{(n-2)^2}{8W} \left((a - \frac{1}{2})\frac{n-2}{n} (1 + \alpha\frac{n-4}{n-2})^2 + \frac{n-2}{n} (1 - \alpha)^2 (1 + \alpha\frac{n-4}{n-2}) \right. \\ & \quad \left. + (a - \frac{n-2}{2n} (1 + 2\alpha))(1 - \alpha)^2 \right) \\ &= -(\frac{p}{r^2}, \nu)^2 \frac{(n-2)^2}{8W} \left((a - \frac{1}{2})\frac{n-2}{n} (1 + \alpha\frac{n-4}{n-2})^2 + (1 - \alpha)^2 (a + \frac{n-2}{2n} - \frac{2}{n}\alpha) \right) \\ &= -(\frac{p}{r^2}, \nu)^2 f \end{aligned}$$

Since $(\frac{p}{r}, \nu)^2 = (1 - |dr|^2)$, we have then proved that

$$\begin{aligned} a(z_1^2 + z_2^2) + \frac{f}{r^2}(1 - |dr|^2) &\geq \frac{n-2}{2n}(1 + 2\alpha)z_1^2 + \sqrt{\frac{n-2}{n}}(1 - \alpha)z_1z_2 + \frac{1}{2}z_2^2 \\ &\quad + (\frac{p}{r^2}, \nu) \frac{n-2}{\sqrt{2}} \left(\sqrt{\frac{n-2}{n}}(1 + \alpha\frac{n-4}{n-2})z_1 + (1 - \alpha)z_2 \right) \end{aligned}$$

Combining this with (4) and (5), if $2a \geq \alpha$, we have

$$\begin{aligned} a|A|^2 + \frac{f}{r^2}(1 - |dr|^2) &\geq a(|A_\Delta|^2 + \sum_{i \neq j} A_{ij}^2) + \frac{f}{r^2}(1 - |dr|^2) \\ &\geq A_{11}^2 + \alpha A_{22}^2 + \alpha A_{11}A_{22} + (\frac{p}{r^2}, \nu)((n-2 - \alpha)A_{11} + \alpha(n-3)A_{22}) \\ &\quad + a \sum_{i \neq j} A_{ij}^2 \\ &\geq A_{11}^2 + \alpha A_{22}^2 + \alpha A_{11}A_{22} + \sum_{i=2}^n A_{1i}^2 + \alpha \sum_{j=3}^n A_{2j}^2 \\ &\quad + (\frac{p}{r^2}, \nu)((n-2 - \alpha)A_{11} + \alpha(n-3)A_{22}) \\ &\geq -\text{BRic}_\alpha(e_1, e_2) + \frac{(p, \nu)}{r^2}((n-2 - \alpha)A_{11} + \alpha(n-3)A_{22}) \end{aligned}$$

This is the expected estimate □

3.3. Proof of Theorem 3.1. Let us assume that the basis is chosen such that $\tilde{\Lambda}_\alpha = \text{BRic}_\alpha(\tilde{e}_1, \tilde{e}_2)$. From (3), we are looking for a lower bound for $r^2|A|^2 - \frac{n(n-2)}{2} + \frac{n^2-4}{4}|dr|^2$. Under the assumptions of Proposition 3.3, $\alpha \leq 1$ (such that $n-2 - \alpha \geq \alpha(n-3)$) and using

Proposition 3.1, we have

$$\begin{aligned}
& a\left(r^2|A|^2 - \frac{n(n-2)}{2} + \frac{n^2-4}{4}|dr|^2\right) \\
& \geq -r^2 \text{BRic}_\alpha(e_1, e_2) + (p, \nu)((n-2-\alpha)A_{11} + \alpha(n-3)A_{22}) \\
& \quad - f(1-|dr|^2) - a\frac{n(n-2)}{2} + a\frac{n^2-4}{4}|dr|^2 \\
& \geq -\widetilde{\text{BRic}}_\alpha(\tilde{e}_1, \tilde{e}_2) + 2(n-1+\alpha(n-2)) - (n+\alpha(n-1))|dr|^2 \\
& \quad - ((n-2-\alpha)dr(e_1)^2 + \alpha(n-3)dr(e_2)^2) \\
& \quad - f(1-|dr|^2) - a\frac{n(n-2)}{2} + a\frac{n^2-4}{4}|dr|^2 \\
& \geq C(|dr|^2) - \tilde{\Lambda}_\alpha
\end{aligned}$$

where

$$C(t) = 2(n-1+\alpha(n-2)) - (2n-2+\alpha(n-2))t - f(1-t) - a\frac{n(n-2)}{2} + a\frac{n^2-4}{4}t$$

C is an affine function and $0 \leq |dr|^2 \leq 1$, so $C(|dr|^2) \geq \min(C(0), C(1))$. We have

$$\begin{aligned}
C(1) &= 2(n-1+\alpha(n-2)) - (2n-2+\alpha(n-2)) - a\frac{n(n-2)}{2} + a\frac{n^2-4}{4} \\
&= \alpha(n-2) - a\frac{(n-2)^2}{4} = (n-2)\left(\alpha - a\frac{n-2}{4}\right)
\end{aligned}$$

and

$$C(0) = 2(n-1+\alpha(n-2)) - f - a\frac{n(n-2)}{2}$$

If we consider $a = \frac{11}{10}$ and $\alpha = \frac{40}{43}$, we have $a > \frac{1}{2}$, $2a \geq \alpha$, $\alpha \leq 1$ and $W = \frac{26697}{184900} > 0$. So the above computations apply. We have

$$C(0) = \frac{731975}{1530628} \simeq 0.47 \quad \text{and} \quad C(1) = \frac{543}{1720} \simeq 0.31$$

So for these values of a and α , and with $\delta = \frac{3}{10} \leq \min(C(0), C(1))$, we have

$$V = a\left(r^2|A|^2 - \frac{n(n-2)}{2} + \frac{n^2-4}{4}|dr|^2\right) \geq \delta - \tilde{\Lambda}_\alpha$$

By (3), the spectral estimate (1) is true. Theorem 3.1 is proved.

4. THE μ -BUBBLE CONSTRUCTION

In this section, we produce a warped μ -bubble with a spectral Ricci curvature lower bound. So we start with a complete non-compact Riemannian manifold (N^n, \bar{g}) with a spectral lower bound on the α -bi-Ricci curvature: there is a smooth function \bar{V} on N such that

$$\bar{V} \geq \delta - \bar{\Lambda}_\alpha$$

and

$$(7) \quad \int_N |\bar{\nabla} \varphi|_{\bar{g}}^2 dv_{\bar{g}} \geq \int_N \frac{1}{a} \bar{V} \varphi^2 dv_{\bar{g}}$$

for any $\varphi \in C_c^1(N)$

Theorem 4.1. *Assume (N, \bar{g}) as above with $n = 5$, $a = \frac{11}{10}$, $\alpha = \frac{40}{43}$ and $\delta = \frac{3}{10}$. Let Ω_+ be a domain in N . Then there is a domain Ω_* with*

- $\Omega_+ \subset \Omega_* \subset \overline{\mathcal{N}}_{100\pi}(\Omega_+)$ and
- there is a smooth function V on $\Sigma = \partial\Omega_*$ such that

$$V \geq \frac{\delta}{2} - \alpha\lambda^\Sigma$$

and

$$(8) \quad \frac{4}{4-a} \int_\Sigma |\nabla\varphi|^2 dv_g \geq \int_\Sigma V\varphi^2 dv_g$$

for any $\varphi \in C^1(\Sigma)$ where g is the induced metric on Σ .

4.1. Construction of the μ -bubble. Because of the spectral control (7) on N , we know (see [16]) that there is a positive function w on N such that

$$(9) \quad -a\overline{\Delta}w = \overline{V}w \geq (\delta - \overline{\Lambda}_\alpha)w$$

Let us recall quickly the construction of the μ -bubble. Let Ω_- be a domain in N such that $\Omega_+ \subset\subset \Omega_- \subset \overline{\mathcal{N}}_{100\pi}(\Omega_+)$. Let $h : \Omega_- \setminus \Omega_+ \rightarrow \mathbb{R}$ be a smooth function such that $\lim_{p \rightarrow \partial\Omega_+} h(p) = +\infty$ and $\lim_{p \rightarrow \partial\Omega_-} h(p) = -\infty$. Let $\underline{\Omega}$ be a domain with $\Omega_+ \subset\subset \underline{\Omega} \subset\subset \Omega_-$.

For any sets of finite perimeter Ω with $\Omega_+ \subset\subset \Omega \subset\subset \Omega_-$, we consider the quantity

$$\mathcal{A}(\Omega) = \int_{\partial^*\Omega} w^a - \int_U (\chi_\Omega - \chi_{\underline{\Omega}})hw^a$$

where $\partial^*\Omega$ is the reduced boundary of Ω . By similar argument to the ones in [11, 28], there is a set of finite perimeter Ω_* ($\Omega_+ \subset\subset \Omega_* \subset\subset \Omega_-$) which minimize the functional \mathcal{A} . Moreover its reduced boundary $\partial^*\Omega_* = \Sigma$ is smooth (see for example [21, 27]).

4.2. Spectral Ricci-curvature bound of the μ -bubble. We denote by $k = n - 1$ the dimension of Σ and by η the outgoing unit normal to Σ .

As in [12, Proposition 4.2], if φ is a function on Σ , writing the first variation of \mathcal{A} for a variation $\{\Omega_t\}$ of Ω_* generated by $\varphi\eta$ gives

$$0 = \frac{d}{dt}\mathcal{A}(\Omega_t)|_{t=0} = \int_\Sigma (Hw^a + aw^{a-1}dw(\eta) - hw^a)\varphi = \int_\Sigma (H + aw^{-1}dw(\eta) - h)w^a\varphi$$

Since this is true for any φ ,

$$(10) \quad H = h - ad \ln w(\eta)$$

Computing the second derivative of $\mathcal{A}(\Omega_t)$, we obtain

$$0 \leq \frac{d^2}{dt^2}\mathcal{A}(\Omega_t)|_{t=0} = \int_\Sigma w^a \left(-\varphi\Delta\varphi - (|B|^2 + \overline{\text{Ric}}(\eta, \eta))\varphi^2 - aw^{-2}dw(\eta)^2\varphi^2 \right. \\ \left. + aw^{-1}\overline{\nabla}^2 w(\eta, \eta)\varphi^2 - aw^{-1}(\overline{\nabla}w, \nabla\varphi)\varphi - dh(\eta)\varphi^2 \right)$$

where B is the second fundamental form of Σ . So

$$0 \leq \int_\Sigma -\text{div}(w^a\varphi\nabla\varphi) + w^a \left(|\nabla\varphi|^2 - (|B|^2 + \overline{\text{Ric}}(\eta, \eta))\varphi^2 - aw^{-2}dw(\eta)^2\varphi^2 \right. \\ \left. + aw^{-1}\overline{\nabla}^2 w(\eta, \eta)\varphi^2 - dh(\eta)\varphi^2 \right)$$

Using $\overline{\nabla}^2 w(\eta, \eta) = \overline{\Delta}w - \Delta w - Hdw(\eta)$, we obtain

$$(11) \quad 0 \leq \int_{\Sigma} w^a \left(|\nabla\varphi|^2 - (|B|^2 + \overline{\text{Ric}}(\eta, \eta))\varphi^2 - aw^{-2}dw(\eta)^2\varphi^2 \right. \\ \left. + aw^{-1}(\overline{\Delta}w - \Delta w - Hdw(\eta))\varphi^2 - dh(\eta)\varphi^2 \right)$$

For $\varphi = w^{-a/2}\psi$, we have $\nabla\varphi = w^{-a/2}\nabla\psi - \frac{a}{2}w^{-a/2-1}\psi\nabla w$. So we can write

$$\begin{aligned} \int_{\Sigma} w^a (|\nabla\varphi|^2 - aw^{-1}\Delta w\varphi^2) &= \int_{\Sigma} |\nabla\psi|^2 - aw^{-1}\psi(\nabla w, \nabla\psi) + \frac{a^2}{4}\psi^2w^{-2}|\nabla w|^2 - a\psi^2w^{-1}\Delta w \\ &= \int_{\Sigma} |\nabla\psi|^2 - a \operatorname{div}(\psi^2w^{-1}\nabla w) + aw^{-1}\psi(\nabla w, \nabla\psi) \\ &\quad - \left(a - \frac{a^2}{4}\right)\psi^2w^{-2}|\nabla w|^2 \\ &= \int_{\Sigma} |\nabla\psi|^2 + aw^{-1}\psi(\nabla w, \nabla\psi) - \left(a - \frac{a^2}{4}\right)\psi^2w^{-2}|\nabla w|^2 \end{aligned}$$

Using that $w^{-1}\psi(\nabla w, \nabla\psi) \leq \varepsilon|\nabla\psi|^2 + \frac{1}{4\varepsilon}\psi^2w^{-2}|\nabla w|^2$ with $\varepsilon = \frac{1}{4-a}$, we get

$$\int_{\Sigma} w^a (|\nabla\varphi|^2 - aw^{-1}\Delta w\varphi^2) \leq \frac{4}{4-a} \int_{\Sigma} |\nabla\psi|^2$$

From (11) and (9), we then obtain

$$(12) \quad \begin{aligned} \frac{4}{4-a} \int_{\Sigma} |\nabla\psi|^2 &\geq \int_{\Sigma} \left(|B|^2 + \overline{\text{Ric}}(\eta, \eta) + aw^{-2}dw(\eta)^2 - aw^{-1}\overline{\Delta}w + aHd\ln(\eta) \right) \psi^2 \\ &\quad + adh(\eta)\psi^2 \\ &\geq \int_{\Sigma} \left(|B|^2 + \overline{\text{Ric}}(\eta, \eta) + \delta - \overline{\Lambda}_{\alpha} + aw^{-2}dw(\eta)^2 + aHd\ln(\eta) \right) \psi^2 \\ &\quad + adh(\eta)\psi^2 \end{aligned}$$

Let (e_1, \dots, e_k) be an orthonormal basis of Σ . Using Gauss equation, we have

$$\begin{aligned} \alpha \operatorname{Ric}^{\Sigma}(e_1, e_1) &= \alpha \sum_{j=2}^k R_{1jj1}^{\Sigma} = \alpha \sum_{j=2}^k (\overline{R}_{1jj1} + B_{11}B_{jj} - B_{1j}^2) \\ &= \overline{\text{BRic}}_{\alpha}(\eta, e_1) - \overline{\text{Ric}}(\eta, \eta) + \alpha \sum_{j=2}^k (B_{11}B_{jj} - B_{1j}^2) \end{aligned}$$

So assuming that e_1 is such that $\operatorname{Ric}^{\Sigma}(e_1, e_1) = \lambda^{\Sigma}$, we have

$$\overline{\text{Ric}}(\eta, \eta) - \overline{\Lambda}_{\alpha} \geq \overline{\text{Ric}}(\eta, \eta) - \overline{\text{BRic}}_{\alpha}(\eta, e_1) = -\alpha\lambda^{\Sigma} + \alpha \sum_{j=2}^k (B_{11}B_{jj} - B_{1j}^2)$$

So the above inequality and using $\text{tr } B = H$ in (12), we then get the inequality

$$\begin{aligned} \frac{4}{4-a} \int_{\Sigma} |\nabla \psi|^2 &\geq \int_{\Sigma} \psi^2 \left(\delta - \alpha \lambda^{\Sigma} + |B|^2 + \alpha \sum_{j=2}^k (B_{11} B_{jj} - B_{1j}^2) + a(d \ln w(\eta))^2 \right. \\ &\quad \left. + aH d \ln w(\eta) + adh(\eta) \right) \\ &\geq \int_{\Sigma} \psi^2 \left(\delta - \alpha \lambda^{\Sigma} + |B|^2 + \alpha H B_{11} - \alpha \sum_{j=1}^k B_{1j}^2 + a(d \ln w(\eta))^2 \right. \\ &\quad \left. + aH d \ln w(\eta) + adh(\eta) \right) \end{aligned}$$

Using (10), we have

$$\begin{aligned} K &:= |B|^2 + \alpha H B_{11} - \alpha \sum_{j=1}^k B_{1j}^2 + a(d \ln w(\eta))^2 + aH d \ln w(\eta) = \\ &\quad |B|^2 + \alpha H B_{11} - \alpha \sum_{j=1}^k B_{1j}^2 + \frac{1}{a}(H-h)^2 + H(h-H) \end{aligned}$$

Let us denote by Φ the traceless part of B and let Φ_{Δ} denote the vector $(\Phi_{11}, \dots, \Phi_{kk}) \in F_k$. Thus, for $\alpha \leq 2$, we have

$$K \geq \frac{1}{k} H^2 + |\Phi_{\Delta}|^2 + \frac{\alpha}{k} H^2 + \alpha H \Phi_{11} - \alpha \left(\frac{1}{k} H + \Phi_{11} \right)^2 + \frac{1}{a} (H-h)^2 + H(h-H)$$

We can write a decomposition of Φ_{Δ} in an orthonormal basis of F_k

$$\Phi_{\Delta} = \sum_{i=1}^{k-2} \begin{pmatrix} 0 \\ E_i \end{pmatrix} x_i + \frac{1}{\sqrt{k(k-1)}} \begin{pmatrix} k-1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} z$$

where $(E_i)_{1 \leq i \leq k-2}$ is an orthonormal basis of F_{k-1} . We then have

$$\begin{aligned} K &\geq \frac{1}{k} H^2 + z^2 + \frac{\alpha}{k} H^2 + \alpha H \sqrt{\frac{k-1}{k}} z - \alpha \left(\frac{1}{k} H + \sqrt{\frac{k-1}{k}} z \right)^2 + \frac{1}{a} (H-h)^2 + H(h-H) \\ &\geq \left(\frac{1}{k} + \frac{\alpha}{k} - \frac{\alpha}{k^2} + \frac{1}{a} - 1 \right) H^2 + \left(1 - \alpha \frac{k-1}{k} \right) z^2 + \frac{1}{a} h^2 + \alpha \sqrt{\frac{k-1}{k}} \left(1 - \frac{2}{k} \right) H z + \left(1 - \frac{2}{a} \right) H h \end{aligned}$$

The above expression is a quadratic form in (H, z, h) associated to the matrix

$$G = \begin{pmatrix} \frac{1}{k} + \frac{\alpha}{k} - \frac{\alpha}{k^2} + \frac{1}{a} - 1 & \frac{\alpha}{2} \sqrt{\frac{k-1}{k}} \left(1 - \frac{2}{k} \right) & \frac{1}{2} - \frac{1}{a} \\ \frac{\alpha}{2} \sqrt{\frac{k-1}{k}} \left(1 - \frac{2}{k} \right) & 1 - \alpha \frac{k-1}{k} & 0 \\ \frac{1}{2} - \frac{1}{a} & 0 & \frac{1}{a} \end{pmatrix}$$

Notice that this matrix is positive definite if $1 - \alpha \frac{k-1}{k} > 0$ and $\det(G) > 0$. Actually for $k = 4$, $a = \frac{11}{10}$, $\alpha = \frac{40}{43}$, we have $1 - \alpha \frac{k-1}{k} = \frac{13}{43} > 0$ and

$$\det \left(G - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{22} \end{pmatrix} \right) = \frac{2599}{1789832} > 0$$

So $K \geq \frac{1}{22}h^2$. Finally, for our values of the parameters, we have

$$(13) \quad \frac{4}{4-a} \int_{\Sigma} |\nabla \psi|^2 \geq \int_{\Sigma} \psi^2 \left(\frac{\delta}{2} - \alpha \lambda^{\Sigma} \right) + \psi^2 \left(\frac{\delta}{2} + \frac{1}{22}h^2 + adh(\eta) \right)$$

4.3. End of the proof. We need to choose the domain Ω_- and the function h . Let $\Psi : N \setminus \Omega_+ \rightarrow \mathbb{R}_+$ be a smoothing of the distance function $d_{\tilde{g}}(\cdot, \partial\Omega_+)$ such that $\frac{1}{2}d_{\tilde{g}}(\cdot, \partial\Omega_+) \leq \Psi \leq 2d_{\tilde{g}}(\cdot, \partial\Omega_+)$ and $|\bar{\nabla} \Psi|_{\tilde{g}} \leq 2$. Let $\varepsilon > 0$ small such that $(1+\varepsilon)11\pi\sqrt{\frac{88}{15}}$ is a regular value of Ψ . Let us define $\Omega_- = \Omega_+ \cup \{\Phi \leq (1+\varepsilon)11\pi\sqrt{\frac{88}{15}}\}$. On Ω_- , $d_{\tilde{g}}(\cdot, \partial\Omega_+) \leq 2(1+\varepsilon)11\pi\sqrt{\frac{88}{15}} \leq 100\pi$, so $\Omega_- \subset \bar{\mathcal{N}}_{100\pi}(\Omega_+)$.

On $\{0 < \Psi < (1+\varepsilon)11\pi\sqrt{\frac{88}{15}}\}$, we consider the function h defined by $h = k \circ \frac{\Psi}{1+\varepsilon}$ where

$$k(t) = -\sqrt{\frac{33}{10}} \tan\left(\frac{1}{11}\sqrt{\frac{15}{88}}t - \frac{\pi}{2}\right)$$

for $t \in (0, 11\pi\sqrt{\frac{88}{15}})$. We have $\lim_{p \rightarrow \partial\Omega_{\pm}} h(p) = \pm\infty$. Notice that k solves $-k' = \frac{3}{44} + \frac{5}{242}k^2$ so

$$|adh(\eta)| = a|k' \left(\frac{\Psi(p)}{1+\varepsilon} \right)| \frac{|\Psi'(p)|}{1+\varepsilon} \leq \frac{2a}{1+\varepsilon} \left(\frac{3}{44} + \frac{5}{242}h^2 \right) \leq \frac{3}{20} + \frac{1}{22}h^2 = \frac{\delta}{2} + \frac{1}{22}h^2$$

Hence, the above construction applies and, for our choices of parameters, (13) becomes

$$\frac{4}{4-a} \int_{\Sigma} |\nabla \psi|^2 \geq \int_{\Sigma} \psi^2 \left(\frac{\delta}{2} - \alpha \lambda^{\Sigma} \right)$$

This ends the proof of Theorem 4.1.

5. STABLE BERNSTEIN PROBLEM

In this section we prove Theorem 1.1. This is a consequence of the following volume growth estimate.

Proposition 5.1. *Let $F : M^5 \looparrowright \mathbb{R}^6$ be a complete, immersed, two-sided, simply-connected stable minimal hypersurface. Let \mathcal{B}_{ρ} denote the geodesic ball of radius $\rho > 0$ centered at some point p_0 in M (for the induced metric g). Then*

$$\text{Vol}(\mathcal{B}_{\rho}) \leq \text{Vol}(\mathbb{B}^5) \left(\frac{800}{43} \right)^{5/2} (2 \exp(100\pi))^5 \rho^5$$

Proof. First, up to a translation, we may assume that $F(p_0) = 0$. Let Ω_+ be a smooth compact domain in M such that $\mathcal{B}_{\rho} \subset \Omega_+ \subset \mathcal{B}_{2\rho}$ and such that $0 \notin F(\partial\Omega_+)$. We consider the Gulliver-Lawson conformal metric $\tilde{g} = r^{-2}g$. By Theorem 3.1 and Theorem 4.1, there is Ω_* a domain in M such that $\Omega_+ \subset \Omega_* \subset \tilde{\mathcal{N}}_{100\pi}(\Omega_+)$ and $\partial\Omega_*$ satisfies the spectral Ricci lower bound (8) for the metric induced by \tilde{g} .

By [7, Theorem 1], M has one end. We consider Ω_{**} the connected component of Ω_* that contains \mathcal{B}_ρ . We assume M is simply connected so the unbounded component of $M \setminus \Omega_{**}$ has only boundary component Σ_0 . Let Ω' be the bounded component of $M \setminus \Sigma_0$. We have $\mathcal{B}_\rho \subset \Omega'$ and $\partial\Omega' \subset \tilde{\mathcal{N}}_{100\pi}(\mathcal{B}_{2\rho})$.

On $\partial\mathcal{B}_{2\rho}$, the Euclidean distance function r is bounded by 2ρ . So, by [10, Lemma 6.2], on $\tilde{\mathcal{N}}_{100\pi}(\mathcal{B}_{2\rho})$, the Euclidean distance function r is bounded by $2\rho \exp(100\pi)$.

Now, because of the spectral Ricci lower bound (8) and since $\frac{4}{(4-a)\alpha} = \frac{43}{29} < \frac{3}{2} = \frac{k-1}{k-2}$, we can apply the volume estimate of Antonelli and Xu [2, Theorem 1] for the metric \tilde{g} and obtain

$$\text{Vol}_{\tilde{g}}(\Sigma_0) \leq \left(\frac{\delta}{6\alpha}\right)^{-2} \text{Vol}(\mathbb{S}^4) = \left(\frac{800}{43}\right)^2 \text{Vol}(\mathbb{S}^4)$$

So scaling back to the Euclidean metric

$$\text{Vol}(\Sigma_0) \leq \left(\frac{800}{43}\right)^2 \text{Vol}(\mathbb{S}^4) (2 \exp(100\pi))^4 \rho^4$$

Finally we can apply the isoperimetric inequality for minimal hypersurfaces in \mathbb{R}^{n+1} [6, 20] to obtain

$$\text{Vol}_g(\mathcal{B}_\rho) \leq \text{Vol}_g(\Omega') \leq \text{Vol}(\mathbb{B}^5) \left(\frac{800}{43}\right)^{5/2} (2 \exp(100\pi))^5 \rho^5$$

□

Proof of Theorem 1.1. Let $M \looparrowright \mathbb{R}^6$ be an immersed, connected, complete, two-sided, stable minimal hypersurface. The stability assumption lifts to the universal cover, so we can assume M to be simply connected. By Proposition 5.1, M has Euclidean volume growth. So by [23] (see also [3]), we obtain that M is a flat hyperplane. □

REFERENCES

- [1] F. J. Almgren, Jr. Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem. *Ann. of Math. (2)*, 84:277–292, 1966.
- [2] Gioacchino Antonelli and Kai Xu. New spectral Bishop-Gromov and Bonnet-Myers theorems and application to isoperimetry. preprint, arXiv:2405.08918.
- [3] Costante Bellettini. Extensions of Schoen–Simon–Yau and Schoen–Simon theorems via iteration à la De Giorgi. preprint, arXiv:2310.01340.
- [4] Serge Bernstein. Über ein geometrisches Theorem und seine Anwendung auf die partiellen Differentialgleichungen vom elliptischen Typus. *Math. Z.*, 26:551–558, 1927.
- [5] E. Bombieri, E. De Giorgi, and E. Giusti. Minimal cones and the Bernstein problem. *Invent. Math.*, 7:243–268, 1969.
- [6] Simon Brendle. The isoperimetric inequality for a minimal submanifold in Euclidean space. *J. Amer. Math. Soc.*, 34:595–603, 2021.
- [7] Huai-Dong Cao, Ying Shen, and Shunhui Zhu. The structure of stable minimal hypersurfaces in \mathbf{R}^{n+1} . *Math. Res. Lett.*, 4:637–644, 1997.
- [8] Giovanni Catino, Paolo Mastrolia, and Alberto Roncoroni. Two rigidity results for stable minimal hypersurfaces. *Geom. Funct. Anal.*, 34:1–18, 2024.
- [9] Otis Chodosh and Chao Li. Stable minimal hypersurfaces in \mathbf{R}^4 . preprint, arXiv:2108.11462.
- [10] Otis Chodosh and Chao Li. Stable anisotropic minimal hypersurfaces in \mathbf{R}^4 . *Forum Math. Pi*, 11:Paper No. e3, 22, 2023.
- [11] Otis Chodosh and Chao Li. Generalized soap bubbles and the topology of manifolds with positive scalar curvature. *Ann. of Math. (2)*, 199:707–740, 2024.
- [12] Otis Chodosh, Chao Li, Paul Minter, and Douglas Stryker. Stable minimal hypersurfaces in \mathbf{R}^5 . preprint, arXiv:2401.01492.

- [13] Otis Chodosh, Chao Li, and Douglas Stryker. Complete stable minimal hypersurfaces in positively curved 4-manifolds. preprint, arXiv:2202.07708.
- [14] Ennio De Giorgi. Una estensione del teorema di Bernstein. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 19:79–85, 1965.
- [15] M. do Carmo and C. K. Peng. Stable complete minimal surfaces in \mathbf{R}^3 are planes. *Bull. Amer. Math. Soc. (N.S.)*, 1:903–906, 1979.
- [16] Doris Fischer-Colbrie and Richard Schoen. The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. *Comm. Pure Appl. Math.*, 33:199–211, 1980.
- [17] Wendell H. Fleming. On the oriented Plateau problem. *Rend. Circ. Mat. Palermo (2)*, 11:69–90, 1962.
- [18] Robert Gulliver and H. Blaine Lawson, Jr. The structure of stable minimal hypersurfaces near a singularity. In *Geometric measure theory and the calculus of variations (Arcata, Calif., 1984)*, volume 44 of *Proc. Sympos. Pure Math.*, pages 213–237. Amer. Math. Soc., Providence, RI, 1986.
- [19] Han Hong and Zeitan Yan. Rigidity and nonexistence of CMC hypersurfaces in 5-manifolds. preprint, arXiv:2405.06867.
- [20] J. H. Michael and L. M. Simon. Sobolev and mean-value inequalities on generalized submanifolds of R^n . *Comm. Pure Appl. Math.*, 26:361–379, 1973.
- [21] Frank Morgan. Regularity of isoperimetric hypersurfaces in riemannian manifolds. *Trans. Amer. Math. Soc.*, 355:5041–5052, 2003.
- [22] A. V. Pogorelov. On the stability of minimal surfaces. *Dokl. Akad. Nauk SSSR*, 260:293–295, 1981.
- [23] R. Schoen, L. Simon, and S. T. Yau. Curvature estimates for minimal hypersurfaces. *Acta Math.*, 134:275–288, 1975.
- [24] Richard Schoen and Leon Simon. Regularity of stable minimal hypersurfaces. *Comm. Pure Appl. Math.*, 34:741–797, 1981.
- [25] Ying Shen and Rugang Ye. On the geometry and topology of manifolds of positive bi-ricci curvature. preprint, arXiv:9708014.
- [26] James Simons. Minimal varieties in riemannian manifolds. *Ann. of Math. (2)*, 88:62–105, 1968.
- [27] Italo Tamanini. Regularity results for almost minimal oriented hypersurfaces in R^N . *Quaderni del Dipartimento di Matematica dell' Università di Lecce*, 1, 1984.
- [28] Jintian Zhu. Width estimate and doubly warped product. *Trans. Amer. Math. Soc.*, 374:1497–1511, 2021.

INSTITUT DENIS POISSON, CNRS UMR 7013, UNIVERSITÉ DE TOURS, UNIVERSITÉ D'ORLÉANS, PARC DE GRANDMONT, 37200 TOURS, FRANCE

Email address: laurent.mazet@univ-tours.fr