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A Power Tower Control: A New Sliding Mode Control

Malek GHANES and Jean-Pierre BARBOT

Abstract—A control based power tower function at order 2 is proposed in this paper. This leads to a new sliding mode control, which allows employing backstepping technique that combines both guaranteed and finite time convergence. The proposed control is applied to a double integrator subject to perturbation d . Both guaranteed and finite convergence are ensured by the controller when d is considered constant and bounded, without knowing its upper bound. For the case, when d is variable and bounded with its upper bound known, only a finite time convergence is obtained. Simulation results are given to show the well founded of the proposed novel control.

Index Terms—Power tower, Backstepping, Guaranteed-finite time convergence, sliding mode.

I. INTRODUCTION

There are a large number of iterative techniques to built control or differentiator. These include: High order sliding mode [7], [16], [25], homogeneous [1], [19], [20], [22] backstepping [6], [14], [23], singular perturbation [13], [24], [26], high gain [2], [10], [11], ... In this article we study the possibility of constructing a finite-time control using the power tower function [12], [18] truncated to second order. Our original motivation was, similar to the case of the variable exponent “Homogeneous” differentiator, to propose a continuous variable exponent “Homogeneous” control. In the case of the differentiator the variation law of the homogeneity exponent is a function of the measurement noise [8] or of time to ensure guaranteed and finite convergence [9]. For the control, in order to ensure both guaranteed and finite time convergence with a continuous law exponential variation, theoretical obstructions prevented us from finding the control (see [27] for a discontinuous law) without a specific continuous power function. Indeed, for our best knowledge, a control combining guaranteed and finite time without singular problems has not yet been considered in the literature. Moreover, this kind of control has not been formally associated with the backstepping approach. To solve the problem of ensuring at the same time, guaranteed and finite time convergence, the addition of at least two different controllers are proposed in the literature (see [3], [17], [20], [21],...). In this paper we propose to fix this control problem by using only one controller. For this purpose, a new control based on the power tower function is introduced. This function, beyond its specific properties at the limits or on the fractal topology [18] obtained the property of its

derivative, allowed us to render possible the use of backstepping techniques for convergence with both guaranteed¹ and finite time. By doing so, two parameters, power exponent and linear gain, are needed for the control tuning in absence of perturbation. In the presence of the latter, when it is variable, bounded with its upper bound known, one exponent of the power tower function is set to zero. When the perturbation is constant and bounded, an integral action is necessary.

The remaining of the paper is organized as follows. Section II presents the main result, which render possible the use of backstepping technique that ensures a guaranteed-finite convergence by using only one controller. In Section III, the performances of the proposed control based power tower function, in absence and presence of perturbation, are put forward. A conclusion is given in section IV with some future works.

II. MAIN RESULT

Our control exploits the **power tower function** of order 2 allowing us to propose a new guaranteed-finite time backstepping based control. Before presenting this new idea, we need to introduce the following lemma:

Lemma 1: The time derivative of the function $\lceil a \rceil^{|a|^\alpha}$ with a a function of time at least \mathbb{C}^1 and α a strictly positive constant is for $a \neq 0$:

$$\begin{aligned} \frac{d\lceil a \rceil^{|a|^\alpha}}{dt} &= |a|^{|a|^\alpha} |a|^{\alpha-1} (1 + \alpha \ln(|a|)) \dot{a} \\ &= |a|^{|a|^\alpha + \alpha - 1} (1 + \alpha \ln(|a|)) \dot{a} \end{aligned} \quad (1)$$

with $\dot{a} = \frac{da}{dt}$.

Proof: As

$$\lceil a \rceil^{|a|^\alpha} = |a|^{|a|^\alpha} \operatorname{sgn}(a),$$

we first time derive

$$|a|^{|a|^\alpha}.$$

In order to use the derivative of the composition of functions, we set

$$|a|^{|a|^\alpha} = e^{\ln(|a|)|a|^\alpha}$$

and we obtain

$$\frac{d|a|^{|a|^\alpha}}{dt} = e^{\ln(|a|)|a|^\alpha} \frac{d \ln(|a|) |a|^\alpha}{da} \dot{a}$$

or again

¹The convergence is ensured to reach a vicinity of the equilibrium point in a guaranteed time whatever the initial conditions are.

M. GHANES and J.-P. BARBOT are with Nantes Université-École Centrale de Nantes-LS2N, UMR 6004 CNRS, Nantes, France malek.ghanes@ec-nantes.fr

J.-P. BARBOT is also with QUARTZ EA 7393, ENSEA, Cergy-Pontoise, France barbot@ensea.fr

$$\frac{d|a|^{\alpha}}{dt} = e^{\ln(|a|)|a|^{\alpha}} (|a|^{\alpha-1} + \ln(|a|) \alpha |a|^{\alpha-1}) \text{sgn}(a) \dot{a}$$

which gives, multiplying both parts by $\text{sgn}(a)$

$$\frac{d|a|^{\alpha}}{dt} = e^{\ln(|a|)|a|^{\alpha}} (|a|^{\alpha-1} + \ln(|a|) \alpha |a|^{\alpha-1}) \dot{a}.$$

This ends the proof. \square

Remark 1: If $\alpha > 1$ then we have

$$\text{Lim}_{a \rightarrow 0} |a|^{\alpha+\alpha-1} (1 + \alpha \ln(|a|)) = 0 \quad (2)$$

which will be a guarantee of a bounded control law and therefore feasible in the vicinity of $a = 0$. This also ensures that for $\alpha > 1$ and \dot{a} bounded the equation (1) is also defined at $a = 0$ and is equal to zero.

Remark 2: An other property for $\alpha > 0$ of the proposed truncated power tower function is

$$\text{Lim}_{a \rightarrow 0} |a|^{\alpha} \text{sgn}(a) = \text{Lim}_{a \rightarrow 0} \text{sgn}(a). \quad (3)$$

Consequently the closed loop behavior obtained with such function refers to a sliding mode behavior.

Now, let us consider the following system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + d \end{aligned} \quad (4)$$

where u is the control input and d the disturbance.

Remark 3: To design a backstepping type control law in guaranteed-finite time we will refer to the lemma 1 and not to the derivative of $|a|^{\theta}$ which has an unbounded limit for $1 > \theta > 0$ and $a \rightarrow 0^+$.

Theorem 1: If $d = 0$, the following power tower control law:

$$\begin{aligned} u &= -K_2 [z_2]^{|z_2|^{\gamma}} - \left(x_1 K_1 |x_1|^{|x_1|^{\beta} + \beta - 1} \right. \\ &\quad \left. + (1 + \beta \ln(|x_1|)) (-K_1 [x_1]^{|x_1|^{\beta}} + z_2) \right) \end{aligned} \quad (5)$$

where $\beta > 1$, $\gamma > 0$, $K_1 > 0$, $K_2 > 0$ and

$$z_2 = x_2 - x_2^* \quad (6)$$

with $x_2^* = -K_1 [x_1]^{|x_1|^{\beta}}$, ensures a guaranteed-finite time convergence of (4) to $x_1 = x_2 = 0$.

Proof: Based on the well known backstepping method [15], the proof is given in two steps.

First step. The first step consists on stabilizing x_1 by a fictive control x_2^* . For that, we define this control as a power tower control, that is

$$x_2^* = -K_1 [x_1]^{|x_1|^{\beta}} \quad (7)$$

with $\beta > 1$. Setting

$$V_1 = \frac{x_1^2}{2}. \quad (8)$$

The time-derivative of (8) reads

$$\dot{V}_1 = -K_1 |x_1|^{|x_1|^{\beta}} + x_1 z_2 \quad (9)$$

If $x_2^* = x_2$ (i.e., $z_2 = 0$), then the guaranteed-finite time convergence of x_1 to zero is achieved when $d = 0$. However, at this step, (6) is not converged to zero, that is why the following second step is important to design the real control.

Second step. Let first compute the time-derivative of (6). For that, we use the result of lemma 1 (see equation (1)). Then we obtain

$$\dot{z}_2 = u + K_1 |x_1|^{|x_1|^{\beta} + \beta - 1} (1 + \beta \ln(|x_1|)) \dot{x}_1. \quad (10)$$

Now considering the Lyapunov function

$$V_2 = V_1 + \frac{z_2^2}{2}. \quad (11)$$

The time-derivative of (11) is:

$$\begin{aligned} \dot{V}_2 &= -K_1 |x_1|^{|x_1|^{\beta}} + x_1 z_2 \\ &\quad + z_2 \left(u + K_1 |x_1|^{|x_1|^{\beta} + \beta - 1} \right. \\ &\quad \left. (1 + \beta \ln(|x_1|)) (-K_1 [x_1]^{|x_1|^{\beta}} + z_2) \right). \end{aligned} \quad (12)$$

By setting u as proposed in (5) (including (7)), (12) becomes

$$\dot{V}_2 = -K_1 |x_1|^{|x_1|^{\beta}} - K_2 |z_2|^{|z_2|^{\gamma}} \quad (13)$$

From (13) the guaranteed-finite time convergence follows and this end the proof. \square

From Theorem 1, we can set our first corollary:

Corollary 1: If d is bounded and its bound is know i.e. $|d| < D_{max}$, the following control law:

$$\begin{aligned} u &= -K_2 \text{sgn}(z_2) - \left(x_1 + K_1 |x_1|^{|x_1|^{\beta} + \beta - 1} \right. \\ &\quad \left. (1 + \beta \ln(|x_1|)) (-K_1 [x_1]^{|x_1|^{\beta}} + z_2) \right) \end{aligned} \quad (14)$$

with $K_2 > D_{max}$, $\beta > 1$, $K_1 > 0$, and $z_2 = x_2 + K_1 [x_1]^{|x_1|^{\beta}}$ ensures a finite time convergence of (4) to $x_1 = x_2 = 0$.

Proof: By using the same Lyapunov function V_2 defined in (11), we obtain:

$$\dot{V}_2 = -K_1 |x_1|^{|x_1|^{\beta}} - z_2 (-K_2 \text{sgn}(z_2) + d) \quad (15)$$

and as $K > D_{max}$, we have a convergence in finite time but note in guaranteed time. \square

Corollary 2: If d is constant, bounded and its bound is unknown, the following control law:

$$\begin{aligned} u &= w - K_2 [z_2]^{|z_2|^{\gamma}} - \left(x_1 + K_1 |x_1|^{|x_1|^{\beta} + \beta - 1} \right. \\ &\quad \left. (1 + \beta \ln(|x_1|)) (-K_1 [x_1]^{|x_1|^{\beta}} + z_2) \right) \\ w &= -z_2 \end{aligned} \quad (16)$$

with $\beta > 1$, $\gamma > 0$ and $z_2 = x_2 + K_1|x_1|^{\beta}$ ensures a guaranteed-finite time convergence of (4) to $x_1 = x_2 = 0$.

Proof: By considering a new Lyapunov function as follows

$$V_3 = V_2 + \frac{(d-w)^2}{2}, \quad (17)$$

where V_2 is defined in (11), we get:

$$\dot{V}_3 = -K_1|x_1|^{1+|x_1|^\beta} - K_2|z_2|^{1+|z_2|^\gamma}. \quad (18)$$

From LaSalle theorem, we deduce that the system (4) controlled by the input (16) converges in guaranteed-finite time to the invariant set IS :

$$IS = \{x_1 = 0, x_2 = 0 \quad \text{and} \quad w \in \mathbb{R}\} \quad (19)$$

This ends the proof. \square

We end this section with a remark that highlights the usefulness of the power tower function in the case of terminal sliding mode.

Remark 4: Let us consider the system (4), taking the following terminal sliding surface [28] based on a power tower function

$$s = x_2 + k|x_1|^{\beta} \quad (20)$$

instead of the one proposed in [28]

$$s = x_2 + k|x_1|^{q/p},$$

with $p > q > 0$. The time derivative of s defined in (20) is

$$\dot{s} = u + d + k|x_1|^{\beta-1}x_2(1 + \beta \ln(|x_1|)). \quad (21)$$

From (21), we can deduce following power tower terminal control law

$$u = -[s]^{|\beta|^\gamma} - k|x_1|^{\beta-1}x_2(1 + \beta \ln(|x_1|)), \quad (22)$$

which has no singularity at $x_1 = 0$ for $\beta > 1$, because $\lim_{|x_1| \rightarrow 0} |x_1|^{\beta-1} \ln(|x_1|) = 0$. The choice of $\beta > 1$ bypasses the singularity in a different way that the one proposed in [4], [5] (i.e., define an equivalent sliding mode surface $s = x_1 + k|x_2|^{p/q}$, $1 < p/q < 2$).

III. SIMULATIONS

To test the validity of the proposed power tower control, 4 simulations are presented. They are conducted using Matlab software, with a solver based on explicit Euler type where the sampling time is fixed to $50\mu s$. System (4) is considered with the following initial conditions: $x_1(0) = 1$, $x_2(0) = -1.5$. The control “gain” is very high when the initial conditions are far from zero, which requires very small sample step. As our sample step is limited to $10^{-6} sec$ and our solver is an explicit Euler scheme, we have taken initial conditions not too far from zero. When the perturbation d is different from zero, controllers (16) and (14) are used, where $w(t=0) = 0$ in (16) and $K = 10$ in (14).

A. Results with $d = 0$ and with sign function

In this part, the controller (5) is applied to (4). The control gains are selected as follows $\beta = 2$, $\gamma = 1.5$, $K_1 = 1$ and $K_2 = 20$. The obtained results are depicted in Fig. 1. We can notice the very good performance of the proposed controller. The state x_1 converges to zero in a guaranteed-finite time. The same conclusion is stated for x_2 , which converges to zero in fixed-finite time. As z_2 is function of x_1 and x_2 , its convergence to zero is also achieved. For the behavior of the control (5), we can observe a chattering phenomena in z_2 when x_1 converges to zero at $t = 1.2s$. This behavior is natural and can be explained by (3). Moreover, at this time, when $x_1 = 0$, a peak on u is observed, it comes from the fact that in the control law, to avoid any problems with $\ln(x_1)$ in the vicinity of $x_1 = 0$, we cancel the product $|x_1|^{\beta-1} \ln(|x_1|)$ around $x_1 = 0$. To overcome the chattering behavior, we propose to replace the discontinuous function of this control by a continuous one when z_2 and x_1 reaches zero. This introduces the next subsection.

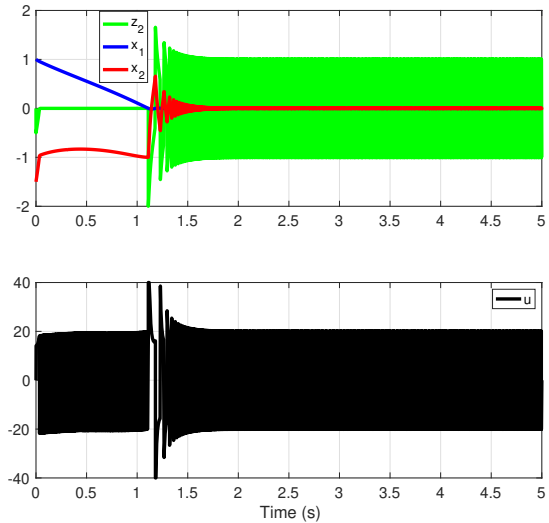


Fig. 1: z_2 , x_1 , x_2 (top) and u (bottom)

B. Results with $d = 0$ and with tanh function

In this part, the sign function used in (5) is replaced by the tanh function, where the gain of this function is fixed to 50 to be more close to the behaviour of the sign function. The obtained results are depicted in Fig. 2. As expected the same results about the guaranteed-finite time convergences of the states x_1 , x_2 and z_2 to zero are obtained. However, we can notice that the chattering disappeared in the control u , and the pic is reduced thanks to the tanh function.

C. Results with $d \neq 0$ constant and with tanh function

In this part, we test the performances of the controller defined in (16) when it is applied to system (4) in presence of a constant bounded perturbation d . The control gains as selected as follows: $\beta = 2$, $\gamma = 1.5$, $K_1 = 1$ and $K_2 = 20$.

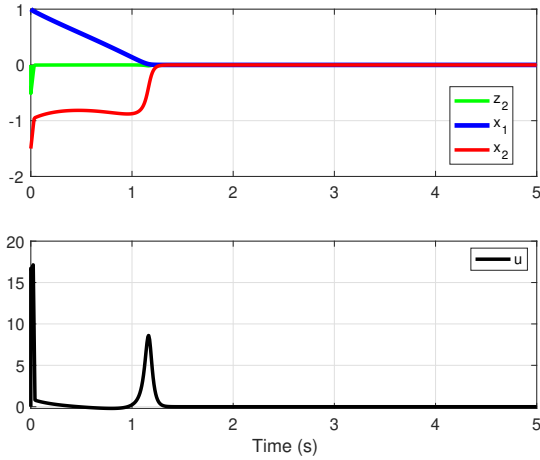


Fig. 2: z_2 , x_1 , x_2 (top) and u (bottom)

We fixed $d = 10$ (any another value can be chosen). The initial condition of the integrator in (16) is chosen equal to zero ($w_0 = 0$). The simulation results are shown in (3). In Fig. 3 we can show that the control (16) performs well in the sense that the perturbation d is exactly canceled ($u = -d$ in steady state) thanks to the integral term w in (16). The latter is replaced by the \tanh function with high gain (50) to avoid chattering phenomenon. Even if the sign function is approximated by the \tanh function for avoiding the chattering, the convergence of the states x_1 and x_2 seems to be in guaranteed-finite time.

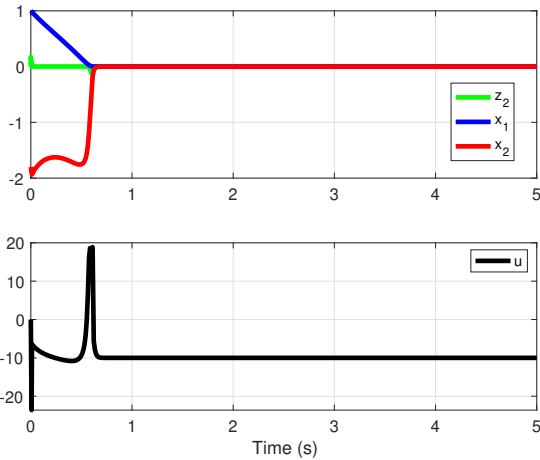


Fig. 3: z_2 , x_1 , x_2 (top) and u (bottom)

D. Results with d variable and with \tanh function

In this part, we test the performances of the controller defined in (14) when it is applied to system (4) in presence of a variable bounded perturbation d where its upper bound is known. For that we took d as a sinus function: $d = \sin(t)$ and $\beta = 2$ and $\gamma = 0$. The control gains are fixed $\beta = 2$, $\gamma = 0$, $K_1 = 1$ and $K_2 = 20$. Then, $K_2 > D_{max}$, where $D_{max} = 1$ is the upper bound of the sinus function.

The obtained results, plotted in Fig. (4), show very good performances of the control (14), the perturbation d is exactly canceled ($u = -d$) thanks to the sign function in (16), which is replaced by \tanh function (gain= 50) to avoid chattering phenomenon. In this case the convergence of the states x_1 and x_2 are in finite time and not in guaranteed time for the sign function and only asymptotic for \tanh function. It can be noticed also that z_2 has not converged exactly to zero. This is due to the fact the gain of the \tanh function is not chosen so big to avoid control pics.

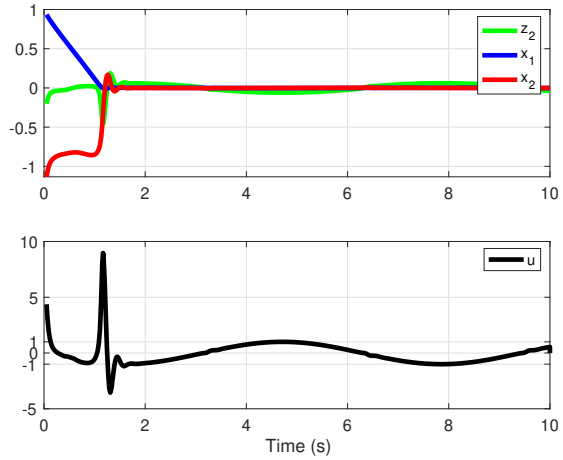


Fig. 4: z_2 , x_1 , x_2 (top) and u (bottom)

IV. CONCLUSION

In this paper, we propose a new control law on a power tower function truncated to order 2. This function makes it possible to use the backstepping technique in order to propose convergence in guaranteed and finite time. In a future work, the convergence finite times including the fixed one will be derived and the non-matching perturbations problem will be studied. The fixed-time will be computed in function of K_1 and K_2 which multiply respectively the term $|x_1|^{|\beta|}$ and the term $|z_2|^{|\gamma|}$ in the control design. Moreover, the approach will be extended to higher dimensional systems. In a second step, the behavior of such law in an observer-based control scheme or with respect to noisy measurements or actuator saturation or again under sampling must be investigated.

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