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THE L_p -DUAL SPACE OF A SEMISIMPLE LIE GROUP

BACHIR BEKKA

ABSTRACT. Let G be a semisimple Lie group. We describe the irreducible representations of G by linear isometries on L_p -spaces for $p \in (1, +\infty)$ with $p \neq 2$. More precisely, we show that, for every such representation π , there exists a parabolic subgroup Q of G such that π is equivalent to the natural representation of G on $L_p(G/Q)$ twisted by a unitary character of Q . When G is of real rank one, we give a complete classification of the possible irreducible representations of G on an L_p -space for $p \neq 2$, up to equivalence.

1. INTRODUCTION

Let G be a locally compact topological group and E a Banach space. Denote by $\mathcal{B}(E)$ the algebra of bounded operators on E , equipped with the strong operator topology, that is, the weakest topology for which the map $\mathcal{B}(E) \rightarrow E, T \mapsto Tv$ is continuous for every $v \in E$. An *isometric representation* (for short, a *representation*) of G on E is a continuous group homomorphism $\pi : G \rightarrow \text{Iso}(E)$, where $\text{Iso}(E)$ is the subgroup of $GL(E)$ consisting of the linear surjective isometries on E .

In the context of semisimple Lie groups, there is strong evidence (see [HC53], [God52], [War72], [Fel65]) that the right notions of irreducibility and equivalence of Banach representations are defined as follows. Recall that to a representation $\pi : G \rightarrow \text{Iso}(E)$ of G there is associated an algebra homomorphism $C_c(G) \rightarrow \mathcal{B}(E)$, denoted by π again, defined by

$$\pi(f) = \int_G f(g)\pi(g)d\mu_G(g) \quad \text{for all } f \in C_c(G),$$

where $C_c(G)$ is the convolution algebra of continuous functions on G with compact support and μ_G is a left Haar measure on G .

Definition 1. (i) A representation $\pi : G \rightarrow \text{Iso}(E)$ is *completely irreducible* if the algebra $\pi(C_c(G))$ is dense in $\mathcal{B}(E)$ for the strong operator topology.

(ii) Two representations $\pi_1 : G \rightarrow \text{Iso}(E_1)$ and $\pi_2 : G \rightarrow \text{Iso}(E_2)$ on Banach spaces E_1 and E_2 are *Naimark equivalent* (for short, *equivalent*) if, for $i \in \{1, 2\}$, there exists a dense subspace V_i of E_i which are $\pi_i(C_c(G))$ -invariant and a closed injective linear map $T : V_1 \rightarrow V_2$ such that

$$\pi_2(f)Tv = T\pi_1(f)v \quad \text{for all } v \in V_1, f \in C_c(G).$$

(iii) Let \mathcal{E} be a class of Banach spaces. The \mathcal{E} -*dual* of G , denoted by $\widehat{G}_{\mathcal{E}}$, is the set of equivalence classes of completely irreducible representations of G on some Banach space $E \in \mathcal{E}$;

When E is a Hilbert space, a representation $G \rightarrow \text{Iso}(E)$ is traditionally called a unitary representation; a unitary representation π is completely irreducible if and only if π is irreducible (that is, $\{0\}$ and E are the only G -invariant closed subspaces

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of E). Moreover, unitary representations π_1 and π_2 are Naimark equivalent if and only if π_1 and π_2 are unitarily equivalent (see [War72, 4.3]); in particular, if \mathcal{E} denotes the class of Hilbert spaces, then $\widehat{G}_{\mathcal{E}}$ coincides with the usual unitary dual \widehat{G} of G .

Let G a noncompact connected semisimple linear Lie group. For such a group, we will be concerned with \widehat{G}_{L_p} where L_p is the class of L_p -spaces for $p \in [1, +\infty[$. By an L_p -space, we mean the usual space $L_p(X, \mathcal{B}, \mu)$ of equivalence classes (modulo null sets) of measurable p -integrable functions $f : X \rightarrow \mathbf{C}$, where μ is a positive σ -finite measure defined on a standard Borel space (X, \mathcal{B}) . The unitary dual space $\widehat{G} = \widehat{G}_{L_2}$ is of course a classical much studied object (see for instance [Kna01], [War72]). We will deal here with \widehat{G}_{L_p} for $p \neq 2$.

Let θ be a Cartan involution on the Lie algebra \mathfrak{g} of G and \mathfrak{k} (respectively \mathfrak{p}) the eigenspace for the eigenvalue 1 (respectively -1) of θ . Let \mathfrak{a} be a maximal commutative subspace of \mathfrak{p} and let $\Sigma \subset \mathfrak{a}^*$ be the corresponding root system, where $\mathfrak{a}^* = \text{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{R})$ is the real dual space of \mathfrak{a} . Let Σ^+ be the set of positive roots for an ordering of Σ and \mathfrak{n} the sum of the root spaces \mathfrak{g}_{α} for $\alpha \in \Sigma^+$. Let K , A , and N be the subgroups of G with Lie algebras \mathfrak{k} , \mathfrak{a} and \mathfrak{n} , respectively. Then K is a maximal compact, A an abelian and N a nilpotent subgroup; moreover, we have an *Iwasawa decomposition* $G = KAN$.

Let M be the centralizer of A in K . The group MAN is a minimal parabolic subgroup of G . Recall (see [Kna01, Section V.5] or [War72, Chap. I, 1.2]) that a *parabolic subgroup* of G is a closed subgroup containing a conjugate of MAN . Every such group is conjugate to a *standard parabolic subgroup* $Q = Q_F$, which is parametrized by a subset F of the set Δ of simple positive roots; more precisely, let \mathfrak{a}_F be the intersection of all $\ker \alpha$ for $\alpha \in F$ and \mathfrak{n}_F the sum of the root spaces \mathfrak{g}_{α} for $\alpha \in \Sigma^+ \setminus \text{span}(F)$. We have a *Langlands decomposition*

$$Q = M_Q A_Q N_Q,$$

where M_Q is reductive and contains M , $A_Q = \exp(\mathfrak{a}_F)$ and $N_Q = \exp(\mathfrak{n}_F)$. Moreover, M_Q and A_Q commute, and both normalize N_Q . In particular, the standard minimal parabolic subgroup is given by $Q_{\emptyset} = MAN$.

Fix a parabolic subgroup $Q = Q_F$ of G and a real number $p \in [1, +\infty[$. We are going to define the L_p analog of the (unitary) principal series representations associated to Q .

Let $K_Q := K \cap M_Q$ and choose a Borel fundamental domain $\Omega_Q \subset K$ for the coset space K/K_Q . Then Ω_Q is also a fundamental domain for G/Q , since $G = KQ$ and $K \cap Q = K_Q$. Every $g \in G$ has a unique decomposition

$$g = \kappa(g)\mu(g)\exp(H(g))n(g) \quad \text{for } \kappa(g) \in \Omega_Q, \mu(g) \in M_Q, H(g) \in \mathfrak{a}_F, n(g) \in N_Q.$$

Let μ_Q be the unique quasi-invariant Borel probability measure on G/Q which is K -invariant. We identify G/Q with Ω_Q and transfer μ_Q and the natural G -action on G/Q to Ω_Q . Let $\rho_{Q,p} \in \mathfrak{a}^*$ be defined by

$$\rho_{Q,p} := \frac{1}{p} \sum_{\alpha \in \Sigma^+ \setminus \text{span}(F)} \dim(\mathfrak{g}_{\alpha})\alpha.$$

Observe that $\rho_{MAN,2}$ is the familiar half-sum of positive roots.

Fix a real linear form $\lambda \in \mathfrak{a}^*$ and a unitary character

$$\chi : M_Q \rightarrow \mathbf{S}^1 = \mathbf{U}(1)$$

of the reductive group M_Q . We define a representation $\pi(Q, \chi, \lambda, p)$ of G on

$$L_p(G/Q, \mu_Q) = L_p(\Omega_Q, \mu_Q)$$

by

$$\pi(Q, \chi, \lambda, p)(g)f(x) = \overline{\chi(\mu(g^{-1}x))} e^{-(i\lambda + \rho_{Q,p})H(g^{-1}x)} f(\kappa(g^{-1}x))$$

for $f \in L_p(\Omega_Q, \mu_F)$, $x \in \Omega_Q$, $g \in G$. It is easily verified (see Section 2) that $\pi(Q, \chi, \lambda, p)$ is indeed an isometric representation of G on $L_p(G/Q, \mu_Q)$.

We denote by $\widehat{M_Q^{\text{Ab}}}$ the group of unitary characters of M_Q . Here is our main result.

Theorem A. *Let G be a noncompact connected semisimple linear Lie group, and let $p \in (1, +\infty)$ with $p \neq 2$. Let π be a completely irreducible representation of G on an L_p -space. Then there exists a parabolic subgroup Q of G , a real linear form $\lambda \in \mathfrak{a}^*$, and a unitary character $\chi \in \widehat{M_Q^{\text{Ab}}}$ such that π is equivalent to the representation $\pi(Q, \chi, \lambda, p)$ on $L_p(G/Q, \mu_Q)$ described above.*

In order to study the irreducibility of a representation $\pi(Q, \chi, \lambda, p)$ as in the previous theorem, we consider its associated Harish-Chandra (\mathfrak{g}, K) -module. For a unitary representation σ of M_Q and for $\nu \in \mathfrak{a}_{\mathbf{C}}^* = \text{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C})$, following [Wal88, 5.2], we denote by $I_{Q, \sigma, \nu}$ the associated infinitesimal nonunitary principal series of G , that is, the (\mathfrak{g}, K) -module underlying the induced representation

$$\text{Ind}_Q^G(\sigma \otimes \nu \otimes 1_{N_Q}) = \text{Ind}_{M_Q A_Q N_Q}^G(\sigma \otimes \nu \otimes 1_{N_Q}).$$

Proposition 2. (i) *Let $\pi(Q, \chi, \lambda, p)$ be a representation of G on $L_p(G/Q, \mu_Q)$ as in Theorem A. Then $\pi(Q, \chi, \lambda, p)$ is completely irreducible if and only if the principal series representation $I_{Q, \chi, \nu}$ is (algebraically) irreducible, where*

$$\nu = i\lambda + \delta_p \rho_{Q,2}$$

for $\delta_p = \frac{2}{p} - 1 \in (-1, 1)$.

(ii) *For $i = 1, 2$, let $\pi(Q_i, \chi_i, \lambda_i, p_i)$ be representations of G on $L_{p_i}(G/Q_i, \mu_{Q_i})$ as in Theorem A. Then $\pi(Q_1, \chi_1, \lambda_1, p_1)$ and $\pi(Q_2, \chi_2, \lambda_2, p_2)$ are equivalent if and only if the corresponding principal series representations I_{Q_1, χ_1, ν_1} and I_{Q_2, χ_2, ν_2} as in (i) are (algebraically) equivalent.*

Principal series representations $I_{Q, \sigma, \nu}$ are of course central objects in the representation theory of semisimple Lie groups; their structure has been much studied, especially in the case where Q is minimal parabolic, and it has been shown that “most” of them are irreducible (see [GN57], [Bru56], [PRRV67], [Kos69], [Wal71], [LW73]).

Using known irreducibility and equivalence results from the literature, we can settle the case where G is a simple Lie group with real rank one. So, G belongs to one of the series $\text{SO}_0(n, 1)$, $\text{SU}(n, 1)$, $\text{Sp}(n, 1)$ or $\text{Sp}(n, 1)$ for $n \geq 2$ or G is the exceptional Lie group $F_{4(-20)}$. We set $P := MAN$.

Theorem B. *Let $p \in (1, +\infty)$ with $p \neq 2$.*

- (i) *Let $G = \text{SO}_0(n, 1)$ for $n \geq 2$. Then $\widehat{M^{\text{Ab}}} = 1_M$ for $n \neq 3$ and $\widehat{M^{\text{Ab}}} \cong \mathbf{Z}$ for $n = 3$. The representation $\pi(P, \chi, \lambda, p)$ is irreducible for every $\lambda \in \mathfrak{a}^*$ and every $\chi \in \widehat{M^{\text{Ab}}}$.*
- (ii) *Let $G = \text{SU}(n, 1)$ for $n \geq 2$. Then $\widehat{M^{\text{Ab}}} \cong \mathbf{Z}$.*
 - (ii1) *The representation $\pi(P, \chi, \lambda, p)$ is irreducible for every $\lambda \in \mathfrak{a}^* \setminus \{0\}$ and every $\chi \in \widehat{M^{\text{Ab}}}$.*
 - (ii2) *After an appropriate identification of $\widehat{M^{\text{Ab}}}$ with \mathbf{Z} , the representation $\pi(P, m, 0, p) = \pi(P, \chi, 0, p)$ for $m \in \mathbf{Z}$ is not irreducible if and only if*

p belongs to the finite set

$$\left\{ \frac{2n}{k} : k \in \{1, \dots, 2n-1\}, k \equiv m \pmod{2}, k \neq m, k \neq 2n+1 \right\}.$$

- (iii) Let $G = \mathrm{Sp}(n, 1)$ for $n \geq 2$. Then $\widehat{M^{\mathrm{Ab}}} = 1_M$.
 - (iii1) The representation $\pi(P, 1_M, \lambda, p)$ is irreducible for every $\lambda \in \mathfrak{a}^* \setminus \{0\}$
 - (iii2) The representation $\pi(P, 1_M, 0, p)$ is not irreducible if and only if $p = \frac{2n+1}{2n}$ or $p = 2n+1$.
- (iv) Let $G = \mathrm{F}_{4(-20)}$. Then $\widehat{M^{\mathrm{Ab}}} = 1_M$.
 - (iv1) The representation $\pi(P, 1_M, \lambda, p)$ is irreducible for every $\lambda \in \mathfrak{a}^* \setminus \{0\}$
 - (iv2) The representation $\pi(P, 1_M, 0, p)$ is not irreducible if and only if p belongs to the set $\left\{ \frac{11}{10}, \frac{11}{9}, \frac{11}{8}, \frac{11}{3}, \frac{11}{2}, 11 \right\}$.

Corollary C. Let G be one of the groups as in Theorem B.

- (i) Let $p \in (1, +\infty)$ with $p \neq 2$. Every completely irreducible representation of G on an L_p -space is equivalent either to the trivial representation 1_G or to one of the irreducible representations of the form $\pi(P, \chi, \lambda, p)$ from Theorem B.
- (ii) Two irreducible representations $\pi(P, \chi, \lambda, p)$ and $\pi(P, \chi', \lambda', p')$ from Theorem B with $(\chi', \lambda', p') \neq (\chi, \lambda, p)$ are equivalent if and only if

$$(\chi', \lambda', p') = (\bar{\chi}, \lambda, p) \quad \text{or} \quad (\chi', \lambda', p') = (\bar{\chi}, -\lambda, q),$$
 where q is the conjugate exponent of p .

The paper is organized as follows. The proofs of Theorem A, Proposition 3, and Theorem B are given in Sections 2, 3, and 4 respectively. As an example, the case of the group $G = \mathrm{SL}_2(\mathbf{R})$, which is a twofold cover of $\mathrm{SO}_0(2, 1)$, is treated in Section 5.

2. PROOF OF THEOREM A

Let G be a noncompact connected semisimple linear Lie group, Let (X, \mathcal{B}, μ) be a standard Borel space equipped with a σ -finite measure μ on \mathcal{B} . Let

$$\pi : G \rightarrow \mathrm{Iso}(L_p(X, \mu))$$

be a completely irreducible representation of G by linear isometries on $L_p(X, \mu)$ for $p \in (1, +\infty)$ with $p \neq 2$. Assume that π is not the trivial representation 1_G of G .

Observe that $L_p(X, \mu)$ is isometrically isomorphic to $L_p(X, \mu')$ for every σ -finite measure μ' on X which is equivalent to μ ; indeed, if $\varphi = \frac{d\mu'}{d\mu}$ denotes the Radon-Nikodym derivative of μ' with respect to μ , then

$$f \mapsto f\varphi^{1/p}$$

is a bijective linear isometry from $L_p(X, \mu')$ to $L_p(X, \mu)$. So, upon choosing a probability measure in the measure class of μ , we can and will assume in the sequel that μ is a probability measure.

• *First step.* We claim that there exists a measure-class preserving measurable action

$$G \times X \rightarrow X, \quad (g, x) \mapsto gx$$

of G on X and a measurable map $c : G \times X \rightarrow \mathbf{S}^1$ such that, for $f \in L_p(X, \mu)$, $g \in G$, and $x \in X$ we have

$$(1) \quad \pi(g)f(x) = c(g^{-1}, x) \left(\frac{dg_*^{-1}(\mu)}{d\mu}(x) \right)^{1/p} f(g^{-1}x),$$

where \mathbf{S}^1 is the set of complex numbers of modulus 1 and $g_*(\mu)$ is the image of μ under the map $x \mapsto gx$; moreover c satisfies the cocycle relation

$$(2) \quad c(g_1g_2, x) = c(g_1, g_2x)c(g_2, x) \quad \text{for all } g_1, g_2 \in G, \text{ for almost all } x \in X.$$

Indeed, let $g \in G$. By the Banach-Lamperti theorem (see [Ban93], [Lam58]), there exists a measure-class preserving measurable map $\varphi(g) : X \rightarrow X$ and a measurable map $c(g) : X \rightarrow \mathbf{S}^1$ such that

$$\pi(g)f(x) = c(g)(x) \left(\frac{d\varphi(g)_*(\mu)}{d\mu}(x) \right)^{1/p} f(\varphi(g)(x))$$

for $f \in L_p(X, \mu)$, $g \in G$, and $x \in X$. Since $\pi(g_1g_2) = \pi(g_1)\pi(g_2)$ for all $g_1, g_2 \in G$ and since $g \mapsto \pi(g)f$ is measurable for every $f \in L_p(X, \mu)$, it is readily checked that $(g, x) \mapsto \varphi(g^{-1})x$ is a measure-class preserving G -action on X and that

$$c : (g, x) \rightarrow c(g^{-1})x$$

is a measurable map satisfying Equation (2).

Recall that another measurable cocycle $c' : G \times X \rightarrow \mathbf{S}^1$ is said to be cohomologous to c if there exists a measurable map $b : X \rightarrow \mathbf{S}^1$ such that

$$c'(g, x) = b(gx)c(g, x)b(x)^{-1} \quad \text{for all } g \in G, \text{ for almost all } x \in X.$$

Let π' be the representation of G on $L_p(X, \mu)$ defined by the same formula (1), with c' instead of c . Then π' is equivalent to π ; indeed, the map $U : L_p(X, \mu) \rightarrow L_p(X, \mu)$, defined by

$$Uf(x) = b(x)f(x) \quad \text{for all } f \in L_p(X, \mu), x \in X,$$

is a bijective isometry which intertwines π' and π .

• *Second step.* We claim that the measure μ is continuous (that is, μ has no atoms).

Indeed, let A be the (at most countable) set of atoms of μ . The decomposition of μ into its atomic and continuous parts is $\mu = \mu_a + \mu_c$, where $\mu_a = \mu|_A$ and $\mu_c = \mu - \mu_a$. We can write

$$L_p(X, \mu) = L_p(X, \mu_c) \oplus \ell_p(A)$$

and it follows from the Banach-Lamperti theorem that $L_p(X, \mu_c)$ and $\ell_p(A)$ are $\pi(G)$ -invariant and so define subrepresentations of π . Since π is irreducible, it follows that either μ is purely atomic or μ is continuous.

Assume by contradiction that μ is purely atomic, that is, $L_p(X, \mu) = \ell_p(A)$. Let $G \curvearrowright A$ be the action of G on A and $c : G \times A \rightarrow \mathbf{S}^1$ the cocycle as above. Since π is continuous, the map

$$G \rightarrow \ell_p(A), \quad g \mapsto c(g, x)\delta_{g \cdot x}$$

is continuous for every $x \in A$. As G is connected, this implies that G fixes pointwise A . By irreducibility of π , it follows that A is a singleton $\{x_0\}$. Equation (2) shows that then that $g \mapsto c(g, x_0)$ is homomorphism from G to \mathbf{S}^1 . Since G is semisimple, G has a trivial abelianization and hence $c(g, x_0) = 1$ for all $g \in G$. So, $\pi = 1_G$; this is a contradiction and the claim is proved.

As is well-known (see [Zim84, Theorem 2.1.19]), the measurable G -system (X, μ, G) admits a continuous compact model, that is, there exists a compact metric space Y on which G acts continuously, a probability measure ν on the Borel subsets of Y and a G -equivariant Borel isomorphism $\Phi : Y \rightarrow X$ with $\Phi_*(\nu) = \mu$.

Let $c' : G \times Y \rightarrow \mathbf{S}^1$ be defined by $c'(g, y) = c(g, \Phi(y))$. The representation π' , defined on $L_p(Y, \nu)$ by the formula

$$\pi'(g)f(y) = c'(g^{-1}, y) \left(\frac{dg_*^{-1}(\nu)}{d\nu}(y) \right)^{1/p} f(g^{-1}y),$$

is equivalent to π . So, without loss of generality, we may assume that X is a compact metric space on which G acts continuously and that μ is a quasi-invariant probability measure on the Borel subsets of X .

Next, let K be a maximal compact subgroup of G and denote by dk the normalized Haar measure on K . Then $\mu' = \int_K k_*(\mu)dk$, defined by

$$\mu'(A) = \int_K \mu(kA)dk$$

for all Borel subsets A of X , is a probability measure on X which is K -invariant; moreover, μ' is equivalent to μ . So, we can and will assume in the sequel that μ is K -invariant.

• *Third step.* We claim that there are countably many K -orbits $\mathcal{O}_1, \mathcal{O}_2, \dots$, in X such that

$$\mu \left(\bigcup_{i \geq 1} \mathcal{O}_i \right) = 1.$$

Indeed, let $Y := X/K$ be the space of K -orbits, equipped with the quotient topology structure given by the quotient map $r : X \rightarrow Y$ and with the probability measure $\nu := r_*(\mu)$.

Since K is compact, Y is a Hausdorff space. So, Y is a second countable compact space and is therefore a standard Borel space. Let $\nu = \nu_a + \nu_c$ be the decomposition of ν into its atomic and continuous parts. The claim will be proved if we show that $\nu_c = 0$.

By a well-known disintegration lemma (see e.g. Lemma 11.1 [Mac52] or Theorem 5.14 in [EW11]), there exists a map

$$\theta : Y \rightarrow \text{Prob}(X), y \mapsto \theta_y,$$

where $\text{Prob}(X)$ is the set of Borel probability measures on X , with the following properties:

- (a) for every $y \in Y$, we have $\theta_y(r^{-1}(y)) = 1$;
- (b) for every $f \in L_1(X, \mu)$, the map $y \mapsto \int_X f(x)d\theta_y(x)$ is measurable and we have

$$\int_X f d\mu = \int_Y \left(\int_X f(x)d\theta_y(x) \right) d\nu(y).$$

Moreover, if a second map $\theta' : Y \rightarrow \text{Prob}(X)$ satisfies (a) and (b), then $\theta_y = \theta'_y$ for ν -almost every $y \in Y$.

Since μ is K -invariant, it follows from the uniqueness of θ that θ_y is the unique K -invariant probability measure on the K -orbit $y \subset X$, for ν -almost every y .

For $f \in C(X)$ and $y \in Y$, let $f|_y \in C(y)$ denote the restriction of f to y . The map $f \mapsto (y \mapsto f|_y)$ extends to an isometric isomorphism

$$S : L_p(X, \mu) \rightarrow \int_Y^\oplus L_p(y, \theta_y) d\nu(y),$$

where $\int_Y^\oplus L_p(y, \theta_y) d\nu(y)$ is the L_p -direct integral of the family $(L_p(y, \theta_y))_{y \in Y}$ in the sense of [dJR17]. Let π_y denote the representation of K on $L_p(y, \theta_y)$ given by

$$\pi_y(k)f(x) = c(k^{-1}, x)f(k^{-1}x) \quad \text{for all } f \in L_p(y, \theta_y), k \in K, x \in y.$$

The direct integral $\int_Y^\oplus \pi_y d\nu(y)$ of the π_y 's is a representation of K defined on $\int_Y^\oplus L_p(y, \theta_y) d\nu(y)$ and the map S intertwines $\pi|_K$ with $\int_Y^\oplus \pi_y d\nu(y)$; for this, see [dJR17, Theorem 4.9] (the proof given there extends to representations twisted by a cocycle as in our situation).

For $\delta \in \widehat{K}$ (recall that \widehat{K} is the unitary dual of K), set

$$\chi_\delta = d(\delta) \text{ch}_\delta,$$

where $\text{ch}_\delta = \text{Tr} \circ \delta$ is the usual character of δ , and define

$$P_\pi(\delta) := \pi(\overline{\chi_\delta}) = \int_K \overline{\chi_\delta}(k) \pi(k) dk.$$

Then $P_\pi(\delta)$ is a continuous projection of $L_p(X, \mu)$ onto the isotypical K -submodule of $L_p(X, \mu)$ of type δ . Similarly, for every $y \in Y$,

$$P_{\pi_y}(\delta) := \pi_y(\overline{\chi_\delta}) = \int_K \overline{\chi_\delta}(k) \pi_y(k) dk$$

is a continuous projection of $L_p(y, \theta_y)$ onto the isotypical K -submodule of $L_p(y, \theta_y)$ of type δ .

The field $y \rightarrow P_{\pi_y}(\delta)$ defines a decomposable operator $\int_Y^\oplus P_{\pi_y}(\delta) d\nu(y)$ on $\int_Y^\oplus L_p(y, \theta_y) d\nu(y)$ and, since S intertwines $\pi|_K$ and $\int_Y^\oplus \pi_y d\nu(y)$, we have

$$(3) \quad SP_\pi(\delta)S^{-1} = \int_Y^\oplus P_{\pi_y}(\delta) d\nu(y)$$

Let Ω be a Borel subset of Y . Denote by V_Ω the range of the multiplication operator on $\int_Y^\oplus L_p(y, \theta_y) d\nu(y)$ by $\mathbf{1}_\Omega$, that is,

$$\begin{aligned} V_\Omega &= \left\{ f = (f_y)_{y \in Y} \in \int_Y^\oplus L_p(y, \theta_y) d\nu(y) : \mathbf{1}_\Omega f = f \right\} \\ &= \left\{ f = (f_y)_{y \in Y} \in \int_Y^\oplus L_p(y, \theta_y) d\nu(y) : f_y = 0 \text{ for } \nu\text{-almost every } y \notin \Omega \right\}. \end{aligned}$$

Observe that

$$(4) \quad \int_Y^\oplus P_{\pi_y}(\delta) d\nu(y) \text{ leaves invariant } V_\Omega.$$

Assume, by contradiction, that the continuous part ν_c of ν is non-zero; so, denoting by Y_c the complement in Y of set of atoms of ν , we have $\nu(Y_c) > 0$ and the measure $\nu|_{Y_c}$ is continuous.

For $\delta \in \widehat{K}$, the subset

$$D_\delta := \{y \in Y : P_{\pi_y}(\delta) = 0\}$$

of Y is measurable. Indeed, let $(f_n)_n$ be a dense sequence in $C(X)$ for the uniform convergence; then, by Urysohn lemma, $(f_n|_y)_n$ is dense in $C(y)$ and hence in $L_p(y, \theta_y)$ for every $y \in Y$, and so

$$D_\delta = \bigcap_{n \in \mathbf{N}} \{y \in Y : P_{\pi_y}(\delta)(f_n|_y) = 0\}.$$

Since $y \mapsto P_{\pi_y}(\delta)(f_n|_y)$ belongs to $\int_Y^\oplus L_p(y, \theta_y) d\nu(y)$ and is therefore a measurable field, D_δ is measurable.

Observe that \widehat{K} is countable, since K is a separable compact group. So, the set

$$D := \bigcap_{\delta \in \widehat{K}} D_\delta$$

is measurable. We claim that

$$(5) \quad \nu(D) = 0.$$

Indeed, assume, by contradiction that $\nu(D) > 0$. Then $V_D \neq \{0\}$. However, for every $(f_y)_{y \in Y} \in V_D$, we have

$$P_{\pi_y}(\delta)(f_y) = 0 \quad \text{for all } \delta \in \widehat{K}$$

and hence $f_y = 0$ for every $y \in D$. This contradiction shows that (5) holds.

It follows from (5) that there exists $\delta \in \widehat{K}$ such that $\nu(Y_c \cap N_\delta) < \nu(Y_c)$. So, for

$$\Omega := Y_c \setminus N_\delta$$

we have $\nu(\Omega) > 0$. Since $\nu|_{Y_c}$ is continuous, we can therefore find a partition

$$\Omega = \coprod_{n \in \mathbf{N}} \Omega_n$$

of Ω in countably many Borel subsets Ω_n with $\nu(\Omega_n) > 0$ for all $n \in \mathbf{N}$. We have

$$\left(\int_Y^{\oplus} P_{\pi_y}(\delta) d\nu(y) \right) V_{\Omega_n} \neq \{0\} \quad \text{for all } n \in \mathbf{N}.$$

since $P_{\pi_y}(\delta) \neq 0$ for every $y \in \Omega$. Hence, for every $n \in \mathbf{N}$, we can find

$$f_n \in \left(\int_Y^{\oplus} P_{\pi_y}(\delta) d\nu(y) \right) V_{\Omega_n} \quad \text{with } f_n \neq 0.$$

Since $\Omega_n \cap \Omega_m = \emptyset$ for $n \neq m$, it follows from (4) that the family $(f_n)_{n \in \mathbf{N}}$ is linearly independent. Therefore, on the one hand, the family $(S^{-1}f_n)_{n \in \mathbf{N}}$ of elements in $L_p(X, \mu)$ is linearly independent. On the other hand, it follows from (3) that

$$S^{-1}f_n \in P_\pi(\delta)(S^{-1}(V_{\Omega_n})) \subset P_\pi(\delta)(S^{-1}(V_\Omega)) \quad \text{for all } n \in \mathbf{N}.$$

This implies that the subspace $P_\pi(\delta)(S^{-1}(V_\Omega))$ of $L_p(X, \mu)$ is infinite dimensional. So, the projection $P_\pi(\delta)$ does not have finite-dimensional range. This is not possible since π is completely irreducible (see [God52, Theorem 2] or [HC53, Lemma 33]). This is a contradiction shows that ν is an atomic measure.

• *Fourth step.* We claim that there exists a point $x_0 \in X$ such that $\mu(Gx_0) = 1$. Indeed, it follows from the third step that there exists a point $x_0 \in X$ such that $\mu(Kx_0) > 0$ and therefore $\mu(Gx_0) > 0$. If we show that the G -action of (X, μ) is ergodic, then it will follow that $\mu(Gx_0) = 1$.

Assume, by contradiction, that there exists a partition $X = X_1 \coprod X_2$ into two G -invariant Borel subsets X_i with $\mu(X_i) > 0$ for $i = 1, 2$. Then

$$L_p(X, \mu) = V_{X_1} \oplus V_{X_2},$$

where

$$V_{X_i} = \{f \in L_p(X, \mu) : \mathbf{1}_{X_i} f = f\} \cong L_p(X_i, \mu|_{X_i});$$

moreover, Formula (1) shows that V_{X_i} is $\pi(G)$ -invariant. Since $V_{X_i} \neq \{0\}$ for $i = 1, 2$, this contradicts the fact that π is irreducible and the claim is proved.

The stabilizer of $x_0 \in X$ as above is a closed subgroup H . Hence, upon disregarding a set of measure zero, we may and will identify X , as *Borel G -space*, with the space G/H , equipped with its quotient Borel structure.

Next, we draw some consequences for the cocycle $c : G \times X \rightarrow \mathbf{S}^1$ over the transitive G -space X . First, upon passing to a cohomologous cocycle, we can assume that c is a strict cocycle, that is, Equation (2) holds for all $g_1, g_2 \in G$ and $x \in X$. Moreover, choose a Borel section $s : G/H \rightarrow G$ for the projection $G \rightarrow G/H$ with $s(H) = e$. Observe that $gs(x)s(gx)^{-1}$ belongs to H , for every

$g \in G$ and $x \in X$. Then there exists a continuous homomorphism $\chi : H \rightarrow \mathbf{S}^1$ such that c is cohomologous to the cocycle $c_\chi : G \times X \rightarrow \mathbf{S}^1$ given by

$$c_\chi(g, x) = \chi(gs(x)s(gx)^{-1}) \quad \text{for all } g \in G, x \in X;$$

for all this, see Theorem 5.27 in [Var85].

We may and will in the sequel assume that the cocycle c coincides with c_χ . Also, as is well-known (see e.g. [Mac52, Theorem 1.1]), all G -quasi-invariant σ -finite Borel measures on G/H are mutually equivalent; so, we may assume that the G -quasi-invariant μ on G/H is one of the standard G -invariant Borel measures on G/H constructed in [Mac52, §.1].

- *Fifth step.* We claim that K acts transitively on the space G/H .

Indeed, on the one hand, since G is a connected Lie group, $X = G/H$ is a C^∞ -smooth connected manifold; moreover, the measure μ on X is locally Lebesgue measure, that is, μ is given by a smooth density times Lebesgue measure in any local coordinates on X ; indeed, this follows from the fact that the maps $x \mapsto gx$ are diffeomorphisms of X for every $g \in G$.

On the other hand, it follows from the third step that there exists a K -orbit \mathcal{O} in X with $\mu(\mathcal{O}) > 0$. As is well-known, \mathcal{O} is a submanifold of X , diffeomorphic to K/L where L is the stabilizer in K of a point in K (see e.g. [Bou72, III, §1, 7.]). In view of what we said about μ , this is possible only if $\dim \mathcal{O} = \dim X$, that is, if \mathcal{O} is open in X . Since X is connected, it follows that $\mathcal{O} = X$.

- *Sixth step.* We claim that H is contained as a finite index subgroup in a parabolic subgroup of G , as defined in the Introduction. Indeed, it follows from the fifth step and Lemma 3 below, that there exists a parabolic subgroup Q which contains H and which has the following properties; let $Q = M_Q A_Q N_Q$ be the Langlands decomposition of Q . Write $Q^0 = C S A_Q N_Q$ for the connected identity component Q^0 of Q , where C is the maximal compact factor of M_Q^0 and S is the product of all the non-compact simple factors of M_Q^0 . Then

$$H^0 = C' S A_Q N_Q,$$

where C' is a connected closed subgroup of C .

We claim H has finite index in Q . Assume, by contradiction, that H has infinite index in Q . Observe that Q is a direct product $Q = ZQ^0$ for a finite subgroup Z of its center (see [Kna96, Theorem 7.53]). So, H is of the form $H = T S A_Q N_Q$ for a subgroup T of infinite index in the compact group $L := ZC$.

Let χ be the unitary character of H associated to c . Observe that χ is trivial on the semisimple Lie group S . Moreover, we have $[\mathfrak{a}_F, \mathfrak{n}_F] = \mathfrak{n}_F$ for the Lie algebras \mathfrak{a}_F and \mathfrak{n}_F of A_Q and N_Q , where F is the set of simple roots associated to Q as in the Introduction; it follows that the commutator subgroup of $A_Q N_Q$ coincides with N_Q and hence that χ is also trivial on N_Q .

Choose a Borel fundamental domain $\Omega_0 \subset G$ for G/Q and a fundamental domain $\Omega_1 \subset L$ for L/T . Then $\Omega_0 \Omega_1$ is a fundamental domain for G/H and every $g \in G$ has unique decompositions

$$g = \omega_0(g)q(g) = \omega_0(g)\omega_1(g)h(g) \quad \text{for } \omega_i(g) \in \Omega_i, q(g) \in Q, h(g) \in H.$$

Let ν_1 be the L -invariant probability measure on $\Omega_1 \cong L/T$. Identify $L_p(G/H, \mu)$ with $L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$ for a G -quasi-invariant probability measure ν_0 on $\Omega_0 \cong G/Q$. For $g \in G$ and $F \in L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$, we have

$$\pi(g)F(x) = \chi(h(g^{-1}x))F(\omega_0(g^{-1}x)\omega_1(g^{-1}x)) \quad \text{for all } x \in \Omega_0 \times \Omega_1.$$

Define a representation π_Q of Q on $L_p(\Omega_1, \nu_1)$ by

$$\pi_Q(q)f(x_1) = \chi(h(q^{-1}x_1))f(\omega_1(q^{-1}x_1))$$

for $f \in L_p(\Omega_1, \nu_1)$, $x_1 \in \Omega_1$, $q \in Q$.

Since $\Omega_1 \cong Q/H$ is infinite, $L_p(\Omega_1, \nu_1)$ is infinite dimensional; therefore, the restriction $\pi_Q|_L$ of π_Q to the compact group L is not irreducible (see [War72, 4.2.2.4]). Hence, there exists a closed $\pi_Q(L)$ -invariant subspace V of $L_p(\Omega_1, \nu_1)$ with $V \neq \{0\}$ and $V \neq L_p(\Omega_1, \nu_1)$. Observe that $\pi_Q(q)$ is a multiple of $\chi(q)$ for $q \in A_Q$ and $\pi_Q(q)$ is the identity for $q \in SN_Q$. Since $Q = LSA_QN_Q$, it follows that V is Q -invariant.

For $F \in L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$ and $x_0 \in \Omega_0$, let $F_{x_0} \in L_p(\Omega_1, \nu_1)$ be defined by

$$F_{x_0}(x_1) = F(x_0x_1).$$

Let \tilde{V} be subspace of $L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$ of all $F \in L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$ such that $F_{x_0} \in V$ for ν_0 -almost every $x_0 \in \Omega_0$. Then \tilde{V} is a proper non trivial closed subspace of $L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$.

We claim that \tilde{V} is $\pi(G)$ -invariant. Indeed, let $F \in \tilde{V}$, $g \in G$ and $F' = \pi(g)F$; for $x_0 \in \Omega_0$, $x_1 \in \Omega_1$, set $x = x_0x_1$ and let $q = q(x_0^{-1}g)$ be the Q -component of $x_0^{-1}g$. We have $\omega_0(g^{-1}x) = \omega_0(g^{-1}x_0)$ and

$$\begin{aligned} F'_{x_0}(x_1) &= F'(x_0x_1) = \chi(h(g^{-1}x))F(\omega_0(g^{-1}x_0)\omega_1(g^{-1}x)) \\ &= \chi(h((x_0^{-1}g)^{-1}x_1))F_{\omega_0(g^{-1}x_0)}(\omega_1((x_0^{-1}g)^{-1}x_1)) \\ &= \chi(h(q^{-1}x_1))F_{\omega_0(g^{-1}x_0)}(\omega_1(q^{-1}x_1)) \\ &= (\pi_Q(q)F_{\omega_0(g^{-1}x_0)})(x_1) \end{aligned}$$

Since $F_{\omega_0(g^{-1}x_0)} \in V$ for almost every x_0 and since V is Q -invariant, it follows that $F' \in \tilde{V}$ and the claim is proved.

Now, π is irreducible. So, we have obtained a contradiction which shows that H has finite index in Q .

- *Seventh step.* We claim that $H = Q$. Indeed, H contains Q^0 , since H has finite index in Q . As mentioned in the sixth step, Q is a direct product $Q = ZQ^0$ for a finite abelian subgroup Z . Hence, we have $Q = Z'H$ for a subgroup Z' of Z . We have to show that Z' is trivial.

Assume, by contradiction, that $Z' \neq \{e\}$. Then there exists a non trivial character δ of Z' . We proceed as in the sixth step. Let $\Omega_0 \subset G$ a fundamental domain for G/Q . We identify $L_p(G/H, \mu)$ with $L_p(\Omega_0 \times Z', \nu_0 \otimes \nu_1)$ for a convenient G -quasi-invariant probability measure ν_0 on $\Omega_0 \cong G/Q$ and a probability measure ν_1 on Z' .

For $F \in L_p(\Omega_0 \times Z', \nu_0 \otimes \nu_1)$ and $x_0 \in \Omega_0$, let $F_{x_0} \in L_p(\Omega_1, \nu_1)$ be defined by $F_{x_0}(x_1) = F(x_0x_1)$. Let \tilde{V}_δ be subspace of $L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$ of all F such that $F_{x_0}(zx_1) = \delta(z)F_{x_0}(x_1)$ for all $z \in Z'$ and ν_0 -almost every $x_0 \in \Omega_0$. Then \tilde{V}_δ is a proper non trivial closed subspace of $L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$. Moreover, one checks that \tilde{V}_δ is $\pi(G)$ -invariant; this contradicts the irreducibility of π .

- *Eighth step.* There exists a real linear form $\lambda \in \mathfrak{a}^*$ and a unitary character χ of M_Q such that π is equivalent to the representation $\pi(Q, \chi, \lambda, p)$ on $L_p(G/Q, \mu_Q)$ described in Theorem A.

Indeed, by the seventh step, we have $H = Q = M_Q A_Q N_Q$. The unitary character of Q associated to the cocycle c is trivial on N_Q and M_Q centralizes A_Q ; hence, this character is of the form $\chi \otimes \chi'$ for unitary characters χ of M_Q and χ' of A_Q . Let $F \subset \Sigma$ be the set of simple positive roots associated to Q . There exists $\lambda \in \mathfrak{a}^*$ such that

$$\chi'(\exp X) = e^{-i\lambda(X)} \quad \text{for all } X \in \mathfrak{a}_F = \log(A_Q).$$

Let μ be the unique quasi-invariant Borel probability measure on G/Q which is K -invariant. For $g \in G$, decompose g according to $G = KM_QA_QN_Q$ as

$$g = \kappa(g)\mu(g)\exp(H(g))n(g) \quad \text{for } \kappa(g) \in \Omega_Q, \mu(g) \in M_Q, H(g) \in \mathfrak{a}_F, n(g) \in N_Q.$$

The Radon-Nikodym derivative of $g_*\mu$ with respect to μ is given by (see e.g. [Kna96, Proposition 8.44])

$$\frac{dg_*(\mu)}{d\mu}(x) = e^{-\rho_F(H(g))}$$

for

$$\rho_F = \sum_{\alpha \in \Sigma^+ \setminus \text{span}(\mathbb{F})} \dim(\mathfrak{g}_\alpha)\alpha.$$

This finishes the proof of Theorem A.

The following lemma, which was used in the proof above, is a consequence of the description of cocompact subgroups of semisimple Lie groups from [Wit90] (see also [GW72]).

Lemma 3. *Let G be a noncompact connected semisimple linear Lie group with maximal compact subgroup K . Let H be a closed subgroup of G with the property that the natural action of K on G/H is transitive. Upon replacing H by one of its conjugate, there exists a standard parabolic subgroup Q which contains H and which has the following properties; denote by $Q = M_QA_QN_Q$ the Langlands decomposition of Q ; write $Q^0 = CSA_QN_Q$ for the identity component Q^0 of Q , where C is the maximal compact factor of M_Q^0 and S is the product of all the non-compact simple factors of M_Q^0 . Then there exists a connected closed subgroup C' of C such that*

$$H^0 = C'SA_QN_Q.$$

Proof. Since K acts transitively on G/H , we have $G = KH$. In particular, H is a cocompact subgroup of G . By [Wit90, Theorem 1.2] and upon conjugating H , there exists a standard parabolic subgroup Q of G with the following properties:

- $Q = M_QA_QN_Q$ contains H ;
- there exist a connected, normal subgroup X of S and a connected closed subgroup Y of CA_Q such that $H^0 = YXN_Q$, where S and C are as in the statement of the Lemma.

Since H/H^0 is at most countable, we have

$$G = \bigcup_{n \geq 1} KH_0g_n$$

for some sequence $(g_n)_{n \geq 1}$ of elements in G . As K is compact and H_0 is closed, KH_0g_n is a closed subset of G for every n . Hence, by Baire's category theorem, KH_0 has a non empty interior. By homogeneity, it follows that KH_0 is open in G . Since G is connected, we have therefore $G = KH_0$, that is,

$$(6) \quad G = KYXN_Q.$$

Recall that the multiplication map $K \times A \times N \rightarrow G$ from the Iwasawa decomposition $G = KAN$ is a diffeomorphism. The reductive Lie group M_Q has also an Iwasawa decomposition

$$M_Q = K_1A_1N_1,$$

for $K_1 := K \cap M_Q$; we have $A = A_1A_Q$ as a direct product and $N = N_1N_Q$ as semi-direct product (see [Kna96, Proposition 7.82]). In particular, we have $C \subset K_1$. Let $\Omega \subset K$ be a fundamental domain for K/K_1 . So,

$$(7) \quad \text{the product map } \Omega \times K_1 \times A_1 \times A_Q \times N_1 \times N_Q \rightarrow G \quad \text{is a bijection.}$$

The Iwasawa decomposition of the semisimple Lie group $X \subset S$ is $X = K_2 A_2 N_2$, where $K_2 := K_1 \cap X$ and where A_2 and N_2 are subgroups of A_1 and N_2 . By (6), we have therefore

$$G = KYXN_Q = KK_2YA_2N_2N_Q.$$

Since $Y \subset CA_Q$, it follows from (7) that

$$\dim A_2 = \dim A_1, \quad \dim N_2 = \dim N_1, \quad \text{and} \quad A_Q \subset Y.$$

So, we have $S = X$ and $Y = C'A_Q$ for a connected closed subgroup C' of C . \square

3. PROOF OF PROPOSITION 2

Let $Q = M_Q A_Q N_Q$ be a parabolic subgroup of G ; for a real linear form $\lambda \in \mathfrak{a}^*$ and a unitary character $\chi : M_Q \rightarrow \mathbf{S}^1$, consider the representation $\pi(Q, \lambda, \chi, p)$ of G on $L_p(G/Q, \mu)$ as in the Introduction.

We may realize $\pi(Q, \lambda, \chi, p)$ in the so-called compact picture as follows. Let $C(K, \chi)$ be the vector space of continuous functions $F : K \rightarrow \mathbf{C}$ such that

$$F(xk) = \overline{\chi(k)}F(x) \quad \text{for all } k \in K \cap M_Q, x \in K;$$

equip $C(K, \chi)$ with the norm

$$\|F\|_p = \left(\int_{K/(K \cap M_Q)} |F(x)|^p d\mu(x) \right)^{1/p},$$

where μ is the unique K -invariant probability measure on $K/(K \cap M_Q)$. Let $L_p(K, \chi)$ be the completion of $(C(K, \chi), \|\cdot\|_p)$. Let $\rho_{Q,p} \in \mathfrak{a}^*$ be defined as in the Introduction. Then $\pi(Q, \lambda, \chi, p)$ acts on $L_p(K, \chi)$ through

$$\pi(Q, \chi, \lambda, p)(g)F(x) = \overline{\chi(\mu(g^{-1}x))} e^{-(i\lambda + \rho_{Q,p})H(g^{-1}x)} F(\kappa(g^{-1}x))$$

for $F \in L_p(K, \chi), x \in K$ and $g \in G$ with decomposition

$$g = \kappa(g)\mu(g) \exp(H(g))n(g), \quad \kappa(g) \in K, \mu(g) \in M_Q, H(g) \in \log(A_Q), n(g) \in N_Q$$

Observe that, for $g \in K$, the operator $\pi(Q, \chi, \lambda, p)(g)$ is simply left translation on $L_p(K, \mu)$ by g^{-1} .

Set $E := L_p(K, \chi)$. Recall that the subspace E_K of K -finite vectors in E is the space of functions $F \in E$ such that

$$\pi(Q, \chi, \lambda, p)(K)F = \{\pi(Q, \chi, \lambda, p)(g)F : g \in K\}$$

spans a finite-dimensional subspace of E . The space E^∞ of C^∞ -vectors in E consists of smooth functions $F \in C(K, \chi)$. As a K -module, E^∞ is the representation of K differentiably induced by $\chi|_{K \cap M_Q}$ (in the sense of [War72, 5.3.1]). Therefore, by Frobenius reciprocity (which remains valid in this context), the multiplicity of $\delta \in \widehat{K}$ in E^∞ is equal to the multiplicity of χ in $\delta|_{K \cap M_Q}$ and is therefore finite, since δ is finite dimensional. Hence, the multiplicity of δ in $E_K \cap E^\infty$ is finite. Since $E_K \cap E^\infty$ is dense in E (see e.g. [War72, Theorem 4.4.5.16]), it follows that the multiplicity of δ in E is finite. In other words, $\pi(Q, \chi, \lambda, p)$ is an *admissible* representation of G (as defined [Kna01, p.207] and [Wal71, 3.3.5]) or is a *K-finite* representation (as defined in [War72, 4.5.1]).

As is well-known, E_K is a module over the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbf{C}})$ of the complexification of the Lie algebra \mathfrak{g} of G . Set

$$(8) \quad \nu := i\lambda + \rho_{Q,p} - \rho_{Q,2} = i\lambda + \left(\frac{2}{p} - 1 \right) \rho_{Q,2};$$

then E_K coincides with the (\mathfrak{g}, K) -module associated to the (nonunitary) induced representation $\text{Ind}_Q^G(\chi \otimes \nu \otimes 1_{N_Q})$. This (\mathfrak{g}, K) -module consists of the functions $f : G \rightarrow \mathbf{C}$ which are C^∞ and have the following properties:

- $f(gman) = \overline{\chi(m)} e^{-(i\lambda + \nu + \rho_{Q,2})(\log a)} f(g)$ for all $g \in G$ and $man \in Q$;
- the linear span of $K \cdot f$ is finite dimensional space, where $K \cdot f$ is the space of translates $g \mapsto f(k^{-1}g)$ for $k \in K$.

(for this, see [Wal71, 5.2.1]). So, the (\mathfrak{g}, K) module associated to $\pi(Q, \chi, \lambda, p)$ is the principal series representation denoted $I_{Q, \chi, \nu}$ in [Wal71] for ν as in (8).

Now, since $\pi(Q, \chi, \lambda, p)$ is admissible, $\pi(Q, \chi, \lambda, p)$ is completely irreducible if and only if its associated (\mathfrak{g}, K) -module $I_{Q, \chi, \nu}$ is algebraically irreducible. Moreover, two such representations $\pi(Q_1, \chi_1, \lambda_1, p_1)$ and $\pi(Q_2, \chi_2, \lambda_2, p_2)$ are Naimark equivalent if and only if the (\mathfrak{g}, K) -modules I_{Q_1, χ_1, ν_1} and I_{Q_2, χ_2, ν_2} are algebraically equivalent (for all this, see Theorems 4.5.5.4 and 4.5.5.2 in [War72]).

4. PROOF OF THEOREM B

We assume from now on that G is a simple Lie group with real rank one. In this case, we have $\dim \mathfrak{a} = 1$.

Choose $\alpha \in \Sigma^+$ and $H \in \mathfrak{a}$ such that $\frac{1}{2}\alpha \notin \Sigma$ and $\alpha(H) = 1$. We identify $\mathfrak{a}_{\mathbf{C}}^*$ with \mathbf{C} by means of the map $\lambda \mapsto \lambda(H)$.

There is, up to conjugation, only one parabolic subgroup, namely $P := MAN$. For $p \in (1 + \infty)$, we will write ρ_p instead of $\rho_{P,p}$. In view of Proposition 2, we have to study, given $\lambda \in \mathbf{R}$ and $\chi \in \widehat{M^{\text{Ab}}}$, the irreducibility of the (non unitary) principal series representation $I_{P, \chi, \nu_{\lambda, p}}$ for

$$\nu_{\lambda, p} = i\lambda + \left(\frac{2}{p} - 1\right) \rho_2$$

and to decide when two such representations are equivalent. We will apply several times the following result of Kostant about the case where $\nu = 1_M$. Set

$$t_\alpha := \begin{cases} (\dim \mathfrak{g}_\alpha)/2 & \text{if } 2\alpha \notin \Sigma \\ (\dim \mathfrak{g}_\alpha)/2 + 1 & \text{if } 2\alpha \in \Sigma, \end{cases}$$

where $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is the usual root space corresponding to α ; set also

$$n_\alpha := \begin{cases} 1 & \text{if } 2\alpha \notin \Sigma \\ 2 & \text{if } 2\alpha \in \Sigma. \end{cases}$$

Recall that

$$\rho_2 = (\dim \mathfrak{g}_\alpha)/2 + \dim \mathfrak{g}_{2\alpha}.$$

For $\nu \in \mathbf{C}$, we have (see Theorem 2 in [Kos69] and Theorem 2.9.8 in [Kos75]):

$$(9) \quad I_{P, 1_M, \nu} \text{ is not irreducible} \iff \nu \notin (-t_\alpha, t_\alpha) \text{ and } \nu + \rho_2 \in n_\alpha \mathbf{Z}.$$

We will treat separately the different simple groups G which may occur; for the data we will use concerning these groups, we refer to [Hel78, Chap. X].

• Let $G = \text{SO}_0(n, 1)$ for $n \geq 2$. Here, $K \cong \text{SO}(n)$ and $M \cong \text{SO}(n-1)$. Moreover, we have $\dim \mathfrak{g}_\alpha = n-1$ and $2\alpha \notin \Sigma$. So,

$$t_\alpha = \frac{n-1}{2}, \quad n_\alpha = 1, \quad \text{and} \quad \rho_2 = \frac{n-1}{2};$$

hence, for $\lambda \in \mathbf{R}$, we have

$$\nu_{\lambda, p} = i\lambda + \left(\frac{2}{p} - 1\right) \frac{n-1}{2}.$$

Since $\left| \frac{2}{p} - 1 \right| < 1$, it follows from (9) that

$I_{P,1M,\nu_{\lambda,p}}$ is irreducible for all $\lambda \in \mathbf{R}$ and $p \in (1, +\infty)$ with $p \neq 2$.

The case where $n \neq 3$ is settled since the abelianization of M is then trivial and so $\widehat{M^{\text{Ab}}} = 1_M$.

Assume now that $n = 3$. Observe that in this case $G \cong \text{SL}_2(\mathbf{C})$ and $M \cong \text{U}(1)$. We use here the results from [Thi73, §. 13, B]: identifying $\widehat{M^{\text{Ab}}}$ with $\frac{1}{2}\mathbf{Z}$, we have, for $m \in \widehat{M^{\text{Ab}}}$ and $\nu \in \mathbf{C}$,

$$I_{P,m,\nu} \text{ is irreducible} \iff (\nu + l + 1)(\nu - l - 1) \neq 0$$

for all $l \in \{|m| + k : k = 0, 1, 2, \dots\}$. Since $(n - 1)/2 = 1$, this condition is clearly satisfied for $\nu_{\lambda,p}$ and so we obtain that $I_{P,\chi,\nu_{\lambda,p}}$ is irreducible for all $\lambda \in \mathbf{R}$, $\chi \in \widehat{M^{\text{Ab}}}$, and $p \in (1, +\infty)$ with $p \neq 2$.

• Let $G = \text{SU}(n, 1)$ for $n \geq 2$. Here, $K \cong \text{SU}(n)$ and $M \cong \text{U}(n - 1)$. Moreover, we have $\dim \mathfrak{g}_\alpha = 2(n - 1)$ and $\dim \mathfrak{g}_{2\alpha} = 1$. So,

$$t_\alpha = n, \quad n_\alpha = 2, \quad \text{and} \quad \rho_2 = n;$$

hence, for $\lambda \in \mathbf{R}$, we have

$$\nu_{\lambda,p} = i\lambda + \left(\frac{2}{p} - 1 \right) n.$$

Since $\left| \frac{2}{p} - 1 \right| < 1$, it follows from (9) that

$I_{P,1M,\nu_{\lambda,p}}$ is irreducible for all $\lambda \in \mathbf{R}$ and $p \in (1, +\infty)$ with $p \neq 2$.

In [Kra76, Proposition 1] precise necessary and sufficient conditions are given on $\sigma \in \widehat{M}$ and $\nu \in \mathbf{C}$ for $I_{P,\sigma,\nu}$ to be irreducible. In the special case of $\sigma \in \widehat{M^{\text{Ab}}}$, this criterion reads as follows. Identify $\widehat{M^{\text{Ab}}}$ with $\frac{1}{n+1}\mathbf{Z}$ as in [Kra76] and for $\sigma \in \widehat{M^{\text{Ab}}}$ and $\nu \in \mathbf{C}$, define $(s_1, \dots, s_{n+1}) \in \mathbf{C}^{n+1}$ by

$$s_i = \begin{cases} \frac{1}{2}(\nu - (n - 1)\sigma) & \text{if } i = 1 \\ \sigma + \frac{n}{2} - i + 1 & \text{if } 2 \leq i \leq n \\ -\frac{1}{2}(\nu + (n - 1)\sigma) & \text{if } i = n + 1. \end{cases}$$

Then $I_{P,m,\nu}$ is not irreducible if and only if either $s_1 - s_i \in \mathbf{Z} \setminus \{0\}$ for all $2 \leq i \leq n$ or $s_{n+1} - s_i \in \mathbf{Z} \setminus \{0\}$ for all $2 \leq i \leq n$.

It follows immediately that $I_{P,\sigma,\nu_{\lambda,p}}$ is irreducible if $\lambda \neq 0$. So, we may assume that $\lambda = 0$.

Writing $\sigma = \frac{m}{n+1}$ for $m \in \mathbf{Z}$, the criterion above implies that $I_{P,\sigma,\nu_{0,p}}$ is not irreducible if and only if either

$$\frac{1}{2} \left(\nu_{p,0} - (n - 1) \frac{m}{n+1} \right) - \frac{m}{n+1} + \frac{n}{2} \in \mathbf{Z} \setminus \{0\}$$

or

$$-\frac{1}{2} \left(\nu_{p,0} + (n - 1) \frac{m}{n+1} \right) - \frac{m}{n+1} + \frac{n}{2} \in \mathbf{Z} \setminus \{0\}$$

that is, if and only if

$$\text{either } \nu_{p,0} - (m + n) \in 2\mathbf{Z} \setminus \{0\} \quad \text{or} \quad \nu_{p,0} - (m + n) \in 2\mathbf{Z} \setminus \{0\}$$

As $\nu_{\lambda,0} = \left(\frac{2}{p} - 1\right)n$, we see that $I_{P,\sigma,\nu_0,p}$ is not irreducible if only if

$$(10) \quad \text{either } 2\left(\frac{1}{p} - 1\right)n - m \in 2\mathbf{Z} \setminus \{0\} \quad \text{or} \quad \frac{2n}{p} - m \in 2\mathbf{Z} \setminus \{0\}.$$

It is clear that (10) is equivalent to

$$p \in \left\{ \frac{2n}{k} : k \in \{1, \dots, 2n-1\}, k \equiv m \pmod{2}, k \neq m, k \neq 2n+m \right\}.$$

• Let $G = \text{Sp}(n, 1)$ for $n \geq 2$. Here, $K \cong \text{Sp}(n) \times \text{Sp}(1)$ and $M \cong \text{Sp}(n-1) \times \text{Sp}(1)$. Moreover, we have $\dim \mathfrak{g}_\alpha = 4(n-1)$ and $\dim \mathfrak{g}_{2\alpha} = 3$. So,

$$t_\alpha = 2n-1, \quad n_\alpha = 2, \quad \text{and} \quad \rho_2 = 2n+1;$$

hence, for $\lambda \in \mathbf{R}$, we have

$$\nu_{\lambda,p} = i\lambda + \left(\frac{2}{p} - 1\right)(2n+1).$$

Observe that the abelianization of M is trivial and so $\widehat{M^{\text{Ab}}} = \{1_M\}$. Moreover, we have

$$\left| \frac{2}{p} - 1 \right| (2n+1) \geq 2n-1 \iff p \in (1, \frac{2n+1}{2n}] \cup [2n+1, +\infty).$$

and

$$\left(\frac{2}{p} - 1\right)(2n+1) + 2n-1 \in 2\mathbf{Z} \iff p \in \left\{ \frac{2n+1}{k} : k \in \mathbf{N}^* \right\}.$$

It follows from (9) that

- $I_{P,1_M,\nu_{\lambda,p}}$ is irreducible for every $\lambda \in \mathfrak{a}^* \setminus \{0\}$
- $I_{P,1_M,\nu_0,p}$ is not irreducible if and only if $p = \frac{2n+1}{2n}$ or $p = 2n+1$.

• Let $G = \text{F}_{4(-20)}$. Here, $K \cong \text{Spin}(9)$ and $M \cong \text{Spin}(7)$. Moreover, we have $\dim \mathfrak{g}_\alpha = 8$ and $\dim \mathfrak{g}_{2\alpha} = 7$. So,

$$t_\alpha = 5, \quad n_\alpha = 2, \quad \text{and} \quad \rho_2 = 11;$$

hence, for $\lambda \in \mathbf{R}$, we have

$$\nu_{\lambda,p} = i\lambda + 11\left(\frac{2}{p} - 1\right).$$

The abelianization of M is trivial and so $\widehat{M^{\text{Ab}}} = \{1_M\}$. Moreover, we have

$$11\left| \frac{2}{p} - 1 \right| \geq 5 \iff p \in (1, \frac{11}{8}] \cup [\frac{11}{3}, +\infty).$$

and

$$11\left(\frac{2}{p} - 1\right) + 5 \in 2\mathbf{Z} \iff p \in \left\{ \frac{11}{k} : k \in \mathbf{N}^* \right\}.$$

It follows from (9) that

- $I_{P,1_M,\nu_{\lambda,p}}$ is irreducible for every $\lambda \in \mathfrak{a}^* \setminus \{0\}$
- $I_{P,1_M,\nu_0,p}$ is not irreducible if and only if $p \in \left\{ \frac{11}{10}, \frac{11}{9}, \frac{11}{8}, \frac{11}{3}, \frac{11}{2}, 11 \right\}$.

In all the cases above, if two representations $I_{P,\chi,\nu}$ and $I_{P,\chi',\nu'}$ are irreducible, then they are equivalent if and only if

$$(\chi', \nu) = (\bar{\chi}, \nu) \quad \text{or} \quad (\chi', \nu') = (\bar{\chi}, -\nu).$$

Now,

$$-\nu_{\lambda,p} = -(i\lambda + \left(\frac{2}{p} - 1\right)\rho_2) = (-i\lambda + \left(\frac{2}{q} - 1\right)\rho_2) = \nu_{\lambda,q}.$$

Therefore, $I_{P,\chi',\nu_{\lambda',p'}}$ and $I_{P,\chi,\nu_{\lambda,p}}$ are equivalent if and only if

$$(\chi', \lambda', p') = (\bar{\chi}, \lambda, p) \quad \text{or} \quad (\chi', \lambda', p') = (\bar{\chi}, -\lambda, q).$$

5. AN EXAMPLE: $SL_2(\mathbf{R})$

Let $G = SL_2(\mathbf{R})$, with maximal compact subgroup $K = SO(2)$. The standard minimal parabolic subgroup is

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbf{R}, a \neq 0, b \in \mathbf{R} \right\}.$$

We have $P = MAN$ for

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbf{R}, a > 0 \right\},$$

$$M = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

and

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbf{R} \right\}.$$

Identifying $\mathfrak{a}_{\mathbf{C}}^*$ with \mathbf{C} as above, we have

$$\rho_{P,p} = \frac{1}{p}$$

for every $p \in (1, \infty)$. We identify G/P as G -space with the real projective line $\mathbf{P}(\mathbf{R})$, with G acting by Möbius transformations on this latter space. The Lebesgue measure μ is the unique K -invariant probability Borel measure on $\mathbf{P}(\mathbf{R})$. The measurable space G/P can further be identified with $\mathbf{R} = \mathbf{P}(\mathbf{R}) \setminus \{\infty\}$.

The Radon-Nikodym derivative for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting on \mathbf{R} is given by

$$\frac{dg_*(\mu)}{d\mu}(x) = \frac{1}{(cx + d)^2}.$$

Let ε denote the non trivial character of M and let $\lambda \in \mathbf{R}$. The representation $\pi(P, \chi, \lambda, p)$ of G by isometries on $L_p(\mathbf{R}, \mu)$ is defined by

$$\pi(P, 1_M, \lambda, p)(g)f(x) = |cx + d|^{-i\lambda - \frac{2}{p}} f\left(\frac{ax + b}{cx + d}\right) \quad \text{if} \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$\pi(P, \varepsilon, \lambda, p)(g)f(x) = \text{sgn}(cx + d)|cx + d|^{-i\lambda - \frac{2}{p}} f\left(\frac{ax + b}{cx + d}\right) \quad \text{if} \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The Harish-Chandra (\mathfrak{g}, K) -module underlying $\pi(P, \varepsilon, \lambda, p)$ is $I_{P,\chi,\nu}$ for

$$\nu = i\lambda + \delta_p \rho_{P,2} = i\lambda + \frac{2}{p} - 1$$

Assume that $p \neq 2$. It is well-known that $I_{P,\chi,\nu}$ is irreducible (see e.g. Proposition 1.3.3 in [Vog81]); observe that the irreducibility of $I_{P,\varepsilon,\nu}$ depends on the fact that

$\frac{2}{p} - 1$ is not an integer). Moreover, the modules $I_{P,\chi,\nu}$ are pairwise non equivalent for fixed p .

In summary, the L_p -dual space of $G = \mathrm{SL}_2(\mathbf{R})$ for $p \in (1, +\infty)$ with $p \neq 2$ is

$$\widehat{G}_{L_p} = \{\pi(P, 1_M, \lambda, p) : \lambda \in \mathbf{R}\} \cup \{\pi(P, \varepsilon, \lambda, p) : \lambda \in \mathbf{R}\} \cup \{1_G\}.$$

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