THE L_p -DUAL SPACE OF A SEMISIMPLE LIE GROUP

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Abstract. Let G be a semisimple Lie group. We describe the irreducible representations of G by linear isometries on L_p-spaces for $p \in (1, +\infty)$ with $p \neq 2$. More precisely, we show that, for every such representation π, there exists a parabolic subgroup Q of G such that π is equivalent to the natural representation of G on $L_p(G/Q)$ twisted by a unitary character of Q. When G is of real rank one, we give a complete classification of the possible irreducible representations of G on an L_p-space for $p \neq 2$, up to equivalence.

1. Introduction

Let G be a locally compact topological group and E a Banach space. Denote by $\mathcal{B}(E)$ the algebra of bounded operators on E, equipped with the strong operator topology, that is, the weakest topology for which the map $\mathcal{B}(E) \to E, T \mapsto Tv$ is continuous for every $v \in V$. An *isometric representation* (for short, a *representation*) of G on E is a continuous group homomorphism $\pi : G \to \mathrm{Iso}(E)$, where $\mathrm{Iso}(E)$ is the subgroup of $GL(E)$ consisting of the linear surjective isometries on E.

In the context of semisimple Lie groups, there is strong evidence (see [HC53], [God52], [War72], [Fel65]) that the right notions of irreducibility and equivalence of Banach representations are defined as follows. Recall that to a representation $\pi : G \to \mathrm{Iso}(E)$ of G there is associated an algebra homomorphism $C_c(G) \to \mathcal{B}(E)$, denoted by π again, defined by

$$\pi(f) = \int_G f(g)\pi(g)d\mu_G(g) \qquad \text{for all} \quad f \in C_c(G),$$

where $C_c(G)$ is the convolution algebra of continuous functions on G with compact support and μ_G is a left Haar measure on G.

Definition 1. (i) A representation $\pi : G \to \mathrm{Iso}(E)$ is *completely irreducible* if the algebra $\pi(C_c(G))$ is dense in $\mathcal{B}(E)$ for the strong operator topology.
(ii) Two representations $\pi_1 : G \to \mathrm{Iso}(E_1)$ and $\pi_2 : G \to \mathrm{Iso}(E_2)$ on Banach spaces E_1 and E_2 are *Naimark equivalent* (for short, *equivalent*) if, for $i \in \{1, 2\}$, there there exists a dense subspace V_i of E_i which are $\pi_i(C_c(G)$-invariant and a closed injective linear map $T : V_1 \to V_2$ such that

$$\pi_2(f)Tv = T\pi_1(f)v \qquad \text{for all} \quad v \in E_1 \;\; f \in C_c(G).$$

(iii) Let $\mathcal{E}$ be a class of Banach spaces. The *$\mathcal{E}$-dual* of G, denoted by $\widehat{G}_{\mathcal{E}}$, is the set of equivalence classes of completely irreducible representations of G on some Banach space $E \in \mathcal{E}$;

When E is a Hilbert space, a representation $G \to \mathrm{Iso}(E)$ is traditionally called a unitary representation; a unitary representation π is completely irreducible if and only if π is irreducible (that is, $\{0\}$ and E are the only G-invariant closed subspaces

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of E). Moreover, unitary representations π_1 and π_2 are Naimark equivalent if and only if π_1 and π_2 are unitarily equivalent (see [War72, 4.3]); in particular, if $\mathcal{E}$ denotes the class of Hilbert spaces, then $\widehat{G}_{\mathcal{E}}$ coincides with the usual unitary dual $\widehat{G}$ of G.

Let G a noncompact connected semisimple linear Lie group. For such a group, we will be concerned with $\widehat{G}_{L_p}$ where L_p is the class of L_p-spaces for $p \in [1, +\infty[$. By an L_p-space, we mean the usual space $L_p(X, \mathcal{B}, \mu)$ of equivalence classes (modulo null sets) of measurable p-integrable functions $f : X \to \mathbf{C}$, , where μ is a positive σ-finite measure defined on a standard Borel space $(X, \mathcal{B})$. The unitary dual space $\widehat{G} = \widehat{G}_{L_2}$ is of course a classical much studied object (see for instance [Kna01], [War72]). We will deal here with $\widehat{G}_{L_p}$ for $p \neq 2$.

Let θ be a Cartan involution on the Lie algebra $\mathfrak{g}$ of G and $\mathfrak{k}$ (respectively $\mathfrak{p}$) the eigenspace for the eigenvalue 1 (respectively -1) of θ. Let $\mathfrak{a}$ be a maximal commutative subspace of $\mathfrak{p}$ and let $\Sigma \subset \mathfrak{a}^*$ be the corresponding root system, where $\mathfrak{a}^* = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{R})$ is the real dual space of $\mathfrak{a}$. Let Σ^+ be the set of positive roots for an ordering of Σ and $\mathfrak{n}$ the sum of the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Sigma^+$. Let K, A, and N be the subgroups of G with Lie algebras $\mathfrak{k}, \mathfrak{a}$ and $\mathfrak{n}$, respectively. Then K is a maximal compact, A an abelian and N a nilpotent subgroup; moreover, we have an *Iwasawa decomposition* $G = KAN$.

Let M be the centralizer of A in K. The group MAN is a minimal parabolic subgroup of G. Recall (see [Kna01, Section V.5] or [War72, Chap. I, 1.2]) that a *parabolic subgroup* of G is a closed subgroup containing a conjugate of MAN. Every such group is conjugate to a *standard parabolic subgroup* $Q = Q_F$, which is parametrized by a subset F of the set Δ of simple positive roots; more precisely, let $\mathfrak{a}_F$ be the intersection of all $\ker \alpha$ for $\alpha \in F$ and $\mathfrak{n}_F$ the sum of the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Sigma^+ \setminus \mathrm{span(F)}$. We have a *Langlands decomposition*

$$Q = M_Q A_Q N_Q,$$

where M_Q is reductive and contains M, $A_Q = \exp(\mathfrak{a}_F)$ and $N_Q = \exp(\mathfrak{n}_F)$. Moreover, M_Q and A_Q commute, and both normalize N_Q. In particular, the standard minimal parabolic parabolic subgroup is given by $Q_\emptyset = MAN$.

Fix a parabolic subgroup $Q = Q_F$ of G and a real number $p \in [1, +\infty[$. We are going to define the L_p analog of the (unitary) principal series representations associated to Q.

Let $K_Q := K \cap M_Q$ and choose a Borel fundamental domain $\Omega_Q \subset K$ for the coset space K/K_Q. Then Ω_Q is also a fundamental domain for G/Q, since $G = KQ$ and $K \cap Q = K_Q$. Every $g \in G$ has a unique decomposition

$$g = \kappa(g)\mu(g)\exp(H(g))n(g) \qquad \text{for} \quad \kappa(g) \in \Omega_Q, \mu(g) \in M_Q, H(g) \in \mathfrak{a}_F, n(g) \in N_Q.$$

Let μ_Q be the unique quasi-invariant Borel probability measure on G/Q which is K-invariant. We identify G/Q with Ω_Q and transfer μ_Q and the natural G-action on G/Q to Ω_Q. Let $\rho_{Q,p} \in \mathfrak{a}^*$ be defined by

$$\rho_{Q,p} := \frac{1}{p} \sum_{\alpha \in \Sigma^+ \setminus \mathrm{span(F)}} \dim(\mathfrak{g}_\alpha)\alpha.$$

Observe that $\rho_{MAN,2}$ is the familiar half-sum of positive roots.

Fix a real linear form $\lambda \in \mathfrak{a}^*$ and a unitary character

$$\chi : M_Q \to \mathbf{S}^1 = \mathbf{U}(1)$$

of the reductive group M_Q. We define a representation $\pi(Q, \chi, \lambda, p)$ of G on

$$L_p(G/Q, \mu_Q) = L_p(\Omega_Q, \mu_Q)$$

by

$$\pi(Q,\chi,\lambda,p)(g)f(x)=\overline{\chi(\mu(g^{-1}x))}e^{-(i\lambda+\rho_{Q,p})H(g^{-1}x)}f(\kappa(g^{-1}x))$$

for $f\in L_p(\Omega_Q,\mu_F), x\in\Omega_Q, g\in G$. It is easily verified (see Section 2) that $\pi(Q,\chi,\lambda,p)$ is indeed an isometric representation of G on $L_p(G/Q,\mu_Q)$.

We denote by $\widehat{M_Q^{\mathrm{Ab}}}$ the group of unitary characters of M_Q. Here is our main result.

Theorem A. *Let G be a noncompact connected semisimple linear Lie group, and let $p\in(1,+\infty)$ with $p\neq 2$. Let π be a completely irreducible representation of G on an L_p-space. Then there exists a parabolic subgroup Q of G, a real linear form $\lambda\in\mathfrak{a}^*$, and a unitary character $\chi\in\widehat{M_Q^{\mathrm{Ab}}}$ such that π is equivalent to the representation $\pi(Q,\chi,\lambda,p)$ on $L_p(G/Q,\mu_Q)$ described above.*

In order to study the irreducibility of a representation $\pi(Q,\chi,\lambda,p)$ as in the previous theorem, we consider its associated Harish-Chandra $(\mathfrak{g},K)$-module. For a unitary representation σ of M_Q and for $\nu\in\mathfrak{a}_{\mathbf{C}}^*=\mathrm{Hom}_{\mathbf{R}}(\mathfrak{a},\mathbf{C})$, following [Wal88, 5.2], we denote by $I_{Q,\sigma,\nu}$ the associated infinitesimal nonunitary principal series of G, that is, the $(\mathfrak{g},K)$-module underlying the induced representation

$$\mathrm{Ind}_Q^G(\sigma\otimes\nu\otimes 1_{N_Q})=\mathrm{Ind}_{M_QA_QN_Q}^G(\sigma\otimes\nu\otimes 1_{N_Q}).$$

Proposition 2. (i) *Let $\pi(Q,\chi,\lambda,p)$ be a representation of G on $L_p(G/Q,\mu_Q)$ as in Theorem A. Then $\pi(Q,\chi,\lambda,p)$ is completely irreducible if and only if the principal series representation $I_{Q,\chi,\nu}$ is (algebraically) irreducible, where*

$$\nu=i\lambda+\delta_p\rho_{Q,2}$$

for $\delta_p=\dfrac{2}{p}-1\in(-1,1)$.

(ii) *For $i=1,2$, let $\pi(Q_i,\chi_i,\lambda_i,p_i)$ be representations of G on $L_{p_i}(G/Q_i,\mu_{Q_i})$ as in Theorem A. Then $\pi(Q_1,\chi_1,\lambda_1,p_1)$ and $\pi(Q_2,\chi_2,\lambda_2,p_2)$ are equivalent if and only if the corresponding principal series representations I_{Q_1,χ_1,ν_1} and I_{Q_2,χ_2,ν_2} as in (i) are (algebraically) equivalent.*

Principal series representations $I_{Q,\sigma,\nu}$ are of course central objects in the representation theory of semisimple Lie groups; their structure has been much studied, especially in the case where Q is minimal parabolic, and it has been shown that "most" of them are irreducible (see [GN57], [Bru56], [PRRV67], [Kos69], [Wal71], [LW73]).

Using known irreducibility and equivalence results from the literature, we can settle the case where G is a simple Lie group with real rank one. So, G belongs to one of the series $\mathrm{SO}_0(n,1),\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ or $\mathrm{Sp}(n,1)$ for $n\geq 2$ or G is the exceptional Lie group $F_{4(-20)}$. We set $P:=MAN$.

Theorem B. *Let $p\in(1,+\infty)$ with $p\neq 2$.*

(i) *Let $G=\mathrm{SO}_0(n,1)$ for $n\geq 2$. Then $\widehat{M^{\mathrm{Ab}}}=1_M$ for $n\neq 3$ and $\widehat{M^{\mathrm{Ab}}}\cong\mathbf{Z}$ for $n=3$. The representation $\pi(P,\chi,\lambda,p)$ is irreducible for every $\lambda\in\mathfrak{a}^*$ and every $\chi\in\widehat{M^{\mathrm{Ab}}}$.*

(ii) *Let $G=\mathrm{SU}(n,1)$ for $n\geq 2$. Then $\widehat{M^{\mathrm{Ab}}}\cong\mathbf{Z}$.*

(ii1) *The representation $\pi(P,\chi,\lambda,p)$ is irreducible for every $\lambda\in\mathfrak{a}^*\setminus\{0\}$ and every $\chi\in\widehat{M^{\mathrm{Ab}}}$.*

(ii2) *After an appropriate identification of $\widehat{M^{\mathrm{Ab}}}$ with $\mathbf{Z}$, the representation $\pi(P,m,0,p)=\pi(P,\chi,0,p)$ for $m\in\mathbf{Z}$ is* not *irreducible if and only if*

p belongs to the finite set

$$\left\{\frac{2n}{k} : k \in \{1,\dots,2n-1\}, k \equiv m \pmod 2, k \neq m, k \neq 2n+m\right\}.$$

(iii) *Let $G = \mathrm{Sp}(n,1)$ for $n \geq 2$. Then $\widehat{M^{\mathrm{Ab}}} = 1_M$.*
  (iii1) *The representation $\pi(P, 1_M, \lambda, p)$ is irreducible for every $\lambda \in \mathfrak{a}^* \setminus \{0\}$*
  (iii2) *The representation $\pi(P, 1_M, 0, p)$ is* not *irreducible if and only if $p = \dfrac{2n+1}{2n}$ or $p = 2n+1$.*

(iv) *Let $G = \mathrm{F}_{4(-20)}$. Then $\widehat{M^{\mathrm{Ab}}} = 1_M$.*
  (iv1) *The representation $\pi(P, 1_M, \lambda, p)$ is irreducible for every $\lambda \in \mathfrak{a}^* \setminus \{0\}$*
  (iv2) *The representation $\pi(P, 1_M, 0, p)$ is* not *irreducible if and only if p belongs to the set $\left\{\frac{11}{10}, \frac{11}{9}, \frac{11}{8}, \frac{11}{3}, \frac{11}{2}, 11\right\}$.*

Corollary C. *Let G be one of the groups as in Theorem B.*

(i) *Let $p \in (1, +\infty)$ with $p \neq 2$. Every completely irreducible representation of G on an L_p-space is equivalent either to the trivial representation 1_G or to one of the irreducible representations of the form $\pi(P, \chi, \lambda, p)$ from Theorem B.*

(ii) *Two irreducible representations $\pi(P, \chi, \lambda, p)$ and $\pi(P, \chi', \lambda', p')$ from Theorem B with $(\chi', \lambda', p') \neq (\chi, \lambda, p)$ are equivalent if and only if*

$$(\chi', \lambda', p') = (\overline{\chi}, \lambda, p) \quad \text{or} \quad (\chi', \lambda', p') = (\overline{\chi}, -\lambda, q),$$

where q is the conjugate exponent of p..

The paper is organized as follows. The proofs of Theorem A, Proposition 3, and Theorem B are is given in Sections 2, 3, and 4 respectively. As an example, the case of the group $G = SL_2(\mathbf{R})$, which is a twofold cover of $SO_0(2,1)$, is treated in Section 5.

2. Proof of Theorem A

Let G be a noncompact connected semisimple linear Lie group, Let $(X, \mathcal{B}, \mu)$ be a standard Borel space equipped with a σ-finite measure μ on $\mathcal{B}$. Let

$$\pi : G \to \mathrm{Iso}(L_p(X, \mu))$$

be a completely irreducible representation of G by linear isometries on $L_p(X, \mu)$ for $p \in (1, +\infty)$ with $p \neq 2$. Assume that π is not the trivial representation 1_G of G.

Observe that $L_p(X, \mu)$ is isometrically isomorphic to $L_p(X, \mu')$ for every σ-finite measure μ' on X which is equivalent to μ; indeed, if $\varphi = \frac{d\mu'}{d\mu}$ denotes the Radon-Nikodym derivative of μ' with respect to μ, then

$$f \mapsto f\varphi^{1/p}$$

is a bijective linear isometry from $L_p(X, \mu')$ to $L_p(X, \mu)$. So, upon choosing a probability measure in the measure class of μ, we can and will assume in the sequel that μ is a probability measure.

• *First step.* We claim that there exists a measure-class preserving measurable action

$$G \times X \to X, \qquad (g, x) \mapsto gx$$

of G on X and a measurable map $c : G \times X \to \mathbf{S}^1$ such that, for $f \in L_p(X, \mu), g \in G$, and $x \in X$ we have

$$\pi(g)f(x) = c(g^{-1}, x)\left(\frac{dg_*^{-1}(\mu)}{d\mu}(x)\right)^{1/p} f(g^{-1}x), \tag{1}$$

where $\mathbf{S}^1$ is the set of complex numbers of modulus 1 and $g_*(\mu)$ is the image of μ under the map $x \mapsto gx$; moreover c satisfies the cocycle relation

$$c(g_1g_2, x) = c(g_1, g_2x)c(g_2, x) \qquad \text{for all} \quad g_1, g_2 \in G, \text{ for almost all } x \in X. \tag{2}$$

Indeed, let $g \in G$. By the Banach-Lamperti theorem (see [Ban93], [Lam58]), there exists a measure-class preserving measurable map $\varphi(g) : X \to X$ and a measurable map $c(g) : X \to \mathbf{S}^1$ such that

$$\pi(g)f(x) = c(g)(x)\left(\frac{d\varphi(g)_*(\mu)}{d\mu}(x)\right)^{1/p} f(\varphi(g)(x))$$

for $f \in L_p(X, \mu), g \in G$, and $x \in X$. Since $\pi(g_1g_2) = \pi(g_1)\pi(g_2)$ for all $g_1, g_2 \in G$ and since $g \mapsto \pi(g)f$ is measurable for every $f \in L_p(X, \mu)$, it is readily checked that $(g, x) \mapsto \varphi(g^{-1})x$ is a measure-class preserving G-action on X and that

$$c : (g, x) \to c(g^{-1})x$$

is a measurable map satisfying Equation (2).

Recall that another measurable cocycle $c' : G \times X \to \mathbf{S^1}$ is said to be cohomologous to c if there exists a measurable map $b : X \to \mathbf{S^1}$ such that

$$c'(g, x) = b(gx)c(g, x)b(x)^{-1} \qquad \text{for all} \quad g \in G, \text{ for almost all } x \in X.$$

Let π' be the representation of G on $L_p(X, \mu)$ defined by the same formula (1), with c' instead of c. Then π' is equivalent to π; indeed, the map $U : L_p(X, \mu) \to L_p(X, \mu)$, defined by

$$Uf(x) = b(x)f(x) \qquad \text{for all} \quad f \in L_p(X, \mu), x \in X,$$

is a bijective isometry which intertwines π' and π.

• *Second step.* We claim that the measure μ is continuous (that is, μ has no atoms).

Indeed, let A be the (at most countable) set of atoms of μ. The decomposition of μ into its atomic and continuous parts is $\mu = \mu_a + \mu_c$, where $\mu_a = \mu|_A$ and $\mu_c = \mu - \mu_a$. We can write

$$L_p(X, \mu) = L_p(X, \mu_c) \oplus \ell_p(A)$$

and it follows from the Banach-Lamperti theorem that $L_p(X, \mu_c)$ are $\ell_p(A)$ are $\pi(G)$-invariant and so define subrepresentations of π. Since π is irreducible, it follows that either μ is purely atomic or μ is continuous.

Assume by contradiction that μ is purely atomic, that is, $L_p(X, \mu) = \ell_p(A)$. Let $G \curvearrowright A$ be the action of G on A and $c : G \times A \to \mathbf{S}^1$ the cocycle as above. Since π is continuous, the map

$$G \to \ell_p(A), \qquad g \mapsto c(g, x)\delta_{g\cdot x}$$

is continuous for every $x \in A$. As G is connected, this implies that G fixes pointwise A. By irreducibility of π, it follows that A is a singleton $\{x_0\}$. Equation (2) shows that then that $g \mapsto c(g, x_0)$ is homomorphism from G to $\mathbf{S}^1$. Since G is semisimple, G has a trivial abelianization and hence $c(g, x_0) = 1$ for all $g \in G$. So, $\pi = 1_G$; this is a contradiction and the claim is proved.

As is well-known (see [Zim84, Theorem 2.1.19]), the measurable G-system (X, μ, G) admits a continuous compact model, that is, there exists a compact metric space Y on which G acts continuously, a probability measure ν on the Borel subsets of Y and a G-equivariant Borel isomorphism $\Phi : Y \to X$ with $\Phi_*(\nu) = \mu$.

Let $c' : G \times Y \to \mathbf{S}^1$ be defined by $c'(g, y) = c(g, \Phi(y))$. The representation π', defined on $L_p(Y, \nu)$ by the formula

$$\pi'(g)f(y) = c'(g^{-1}, y)\left(\frac{dg_*^{-1}(\nu)}{d\nu}(y)\right)^{1/p} f(g^{-1}y),$$

is equivalent to π. So, without loss of generality, we may assume that X is a compact metric space on which G acts continuously and that μ is a quasi-invariant probability measure on the Borel subsets of X.

Next, let K be a maximal compact subgroup of G and denote by dk the normalized Haar measure on K. Then $\mu' = \int_K k_*(\mu)dk$, defined by

$$\mu'(A) = \int_K \mu(kA)dk$$

for all Borel subsets A of X, is a probability measure on X which is K-invariant; moreover, μ' is equivalent to μ. So, we can and will assume in the sequel that μ is K-invariant.

• *Third step.* We claim that there are countably many K-orbits $\mathcal{O}_1, \mathcal{O}_2, \cdots$, in X such that

$$\mu\left(\bigcup_{i\geq 1} \mathcal{O}_i\right) = 1.$$

Indeed, let $Y := X/K$ be the space of K-orbits, equipped with the quotient topology structure given by the quotient map $r : X \to Y$ and with the probability measure $\nu := r_*(\mu)$.

Since K is compact, Y is a Hausdorff space. So, Y is a second countable compact space and is therefore a standard Borel space. Let $\nu = \nu_a + \nu_c$ be the decomposition of ν into its atomic and continuous parts. The claim will be proved if we show that $\nu_c = 0$.

By a well-known disintegration lemma (see e.g. Lemma 11.1 [Mac52] or Theorem 5.14 in [EW11]), there exists a map

$$\theta : Y \to \mathrm{Prob}(X), y \mapsto \theta_y,$$

where $\mathrm{Prob}(X)$ is the set of Borel probability measures on X, with the following properties:

(a) for every $y \in Y$, we have $\theta_y(r^{-1}(y)) = 1$;
(b) for every $f \in L_1(X, \mu)$, the map $y \mapsto \int_X f(x)d\theta_y(x)$ is measurable and we have

$$\int_X f d\mu = \int_Y \left(\int_X f(x)d\theta_y(x)\right) d\nu(y).$$

Moreover, if a second map $\theta' : Y \to \mathrm{Prob}(X)$ satisfies (a) and (b), then $\theta_y = \theta'_y$ for ν-almost every $y \in Y$.

Since μ is K-invariant, it follows from the uniqueness of θ that θ_y is the unique K-invariant probability measure on the K-orbit $y \subset X$, for ν-almost every y.

For $f \in C(X)$ and $y \in Y$, let $f|_y \in C(y)$ denote the restriction of f to y. The map $f \mapsto (y \mapsto f_y)$ extends to an isometric isomorphism

$$S : L_p(X, \mu) \to \int_Y^{\oplus} L_p(y, \theta_y)d\nu(y),$$

where $\int_Y^{\oplus} L_p(y, \theta_y)d\nu(y)$ is the L_p-direct integral of the family $(L_p(y, \theta_y))_{y\in Y}$ in the sense of [dJR17]. Let π_y denote the representation of K on $L_p(y, \theta_y)$ given by

$$\pi_y(k)f(x) = c(k^{-1}, x)f(k^{-1}x)) \qquad \text{for all} \quad f \in L_p(y, \theta_y), k \in K, x \in y.$$

The direct integral $\int_Y^\oplus \pi_y d\nu(y)$ of the π_y's is a representation of K defined on $\int_Y^\oplus L_p(y,\theta_y)d\nu(y)$ and the map S intertwines $\pi|_K$ with $\int_Y^\oplus \pi_y d\nu(y)$; for this, see [dJR17, Theorem 4.9] (the proof given there extends to representations twisted by a cocycle as in our situation).

For $\delta \in \widehat{K}$ (recall that $\widehat{K}$ is the unitary dual of K), set

$$\chi_\delta = d(\delta)\mathrm{ch}_\delta,$$

where $\mathrm{ch}_\delta = \mathrm{Tr}\circ\delta$ is the usual character of δ, and define

$$P_\pi(\delta) := \pi(\overline{\chi_\delta}) = \int_K \overline{\chi_\delta}(k)\pi(k)dk.$$

Then $P_\pi(\delta)$ is a continuous projection of $L_p(X,\mu)$ onto the isotypical K-submodule of $L_p(X,\mu)$ of type δ. Similarly, for every $y \in Y$,

$$P_{\pi_y}(\delta) := \pi_y(\overline{\chi_\delta}) = \int_K \overline{\chi_\delta}(k)\pi_y(k)dk$$

is a continuous projection of $L_p(y,\theta_y)$ onto the isotypical K-submodule of $L_p(y,\theta_y)$ of type δ.

The field $y \to P_{\pi_y}(\delta)$ defines a decomposable operator $\int_Y^\oplus P_{\pi_y}(\delta)d\nu(y)$ on $\int_Y^\oplus L_p(y,\theta_y)d\nu(y)$ and, since S intertwines $\pi|_K$ and $\int_Y^\oplus \pi_y d\nu(y)$, we have

$$SP_\pi(\delta)S^{-1} = \int_Y^\oplus P_{\pi_y}(\delta)d\nu(y) \tag{3}$$

Let Ω be a Borel subset of Y. Denote by V_Ω the range of the multiplication operator on $\int_Y^\oplus L_p(y,\theta_y)d\nu(y)$ by $\mathbf{1}_\Omega$, that is,

$$\begin{aligned} V_\Omega &= \left\{ f = (f_y)_{y\in Y} \in \int_Y^\oplus L_p(y,\theta_y)d\nu(y) : \mathbf{1}_\Omega f = f \right\} \\ &= \left\{ f = (f_y)_{y\in Y} \in \int_Y^\oplus L_p(y,\theta_y)d\nu(y) : f_y = 0 \quad \text{for} \quad \nu\text{-almost every} \quad y \notin \Omega \right\}. \end{aligned}$$

Observe that

$$\int_Y^\oplus P_{\pi_y}(\delta)d\nu(y) \quad \text{leaves invariant} \quad V_\Omega. \tag{4}$$

Assume, by contradiction, that the continuous part ν_c of ν is non-zero; so, denoting by Y_c the complement in Y of set of atoms of ν, we have $\nu(Y_c) > 0$ and the measure $\nu|_{Y_c}$ is continuous.

For $\delta \in \widehat{K}$, the subset

$$D_\delta := \{y \in Y : P_{\pi_y}(\delta) = 0\}$$

of Y is measurable. Indeed, let $(f_n)_n$ be a dense sequence in $C(X)$ for the uniform convergence; then, by Urysohn lemma, $(f_n|_y)_n$ is dense in $C(y)$ and hence in $L_p(y,\theta_y)$ for every $y \in Y$, and so

$$D_\delta = \bigcap_{n\in\mathbf{N}} \{y \in Y : P_{\pi_y}(\delta)(f_n|_y) = 0\}.$$

Since $y \mapsto P_{\pi_y}(\delta)(f_n|_y)$ belongs to $\int_Y^\oplus L_p(y,\theta_y)d\nu(y)$ and is therefore a measurable field, D_δ is measurable.

Observe that $\widehat{K}$ is countable, since K is a separable compact group. So, the set

$$D := \bigcap_{\delta\in\widehat{K}} D_\delta$$

is measurable. We claim that

$$\nu(D) = 0. \tag{5}$$

Indeed, assume, by contradiction that $\nu(D) > 0$. Then $V_D \neq \{0\}$. However, for every $(f_y)_{y\in Y} \in V_D$, we have

$$P_{\pi_y}(\delta)(f_y) = 0 \qquad \text{for all} \quad \delta \in \widehat{K}$$

and hence $f_y = 0$ for every $y \in D$. This contradiction shows that (5) holds.

It follows from (5) that there exists $\delta \in \widehat{K}$ such that $\nu(Y_c \cap N_\delta) < \nu(Y_c)$. So, for

$$\Omega := Y_c \setminus N_\delta$$

we have $\nu(\Omega) > 0$. Since $\nu|_{Y_c}$ is continuous, we can therefore find a partition

$$\Omega = \coprod_{n\in\mathbf{N}} \Omega_n$$

of Ω in countably many Borel subsets Ω_n with $\nu(\Omega_n) > 0$ for all $n \in \mathbf{N}$. We have

$$\left(\int_Y^{\oplus} P_{\pi_y}(\delta) d\nu(y)\right) V_{\Omega_n} \neq \{0\} \qquad \text{for all} \quad n \in \mathbf{N}.$$

since $P_{\pi_y}(\delta) \neq 0$ for every $y \in \Omega$. Hence, for every $n \in \mathbf{N}$, we can find

$$f_n \in \left(\int_Y^{\oplus} P_{\pi_y}(\delta) d\nu(y)\right) V_{\Omega_n} \qquad \text{with} \qquad f_n \neq 0.$$

Since $\Omega_n \cap \Omega_m = \emptyset$ for $n \neq m$, it follows from (4) that the family $(f_n)_{n\in\mathbf{N}}$ is linearly independent. Therefore, on the one hand, the family $(S^{-1}f_n)_{n\in\mathbf{N}}$ of elements in $L_p(X,\mu)$ is linearly independent. On the other hand, it follows from (3) that

$$S^{-1}f_n \in P_\pi(\delta)(S^{-1}(V_{\Omega_n})) \subset P_\pi(\delta)(S^{-1}(V_\Omega)) \qquad \text{for all} \quad n \in \mathbf{N}.$$

This implies that the subspace $P_\pi(\delta)(S^{-1}(V_\Omega))$ of $L_p(X,\mu)$ is infinite dimensional. So, the projection $P_\pi(\delta)$ does not have finite-dimensional range. This is not possible since π is completely irreducible (see [God52, Theorem 2] or [HC53, Lemma 33]). This is a contradiction shows that ν is an atomic measure.

• *Fourth step.* We claim that there exists a point $x_0 \in X$ such that $\mu(Gx_0) = 1$. Indeed, it follows from the third step that there exists a point $x_0 \in X$ such that $\mu(Kx_0) > 0$ and therefore $\mu(Gx_0) > 0$. If we show that the G-action of (X,μ) is ergodic, then it will follow that $\mu(Gx_0) = 1$.

Assume, by contradiction, that there exists a partition $X = X_1 \coprod X_2$ into two G-invariant Borel subsets X_i with $\mu(X_i) > 0$ for $i = 1, 2$. Then

$$L_p(X,\mu) = V_{X_1} \oplus V_{X_2},$$

where

$$V_{X_i} = \{f \in L_p(X,\mu) : \mathbf{1}_{X_i} f = f\} \cong L_p(X_i, \mu|_{X_i});$$

moreover, Formula (1) shows that V_{X_i} is $\pi(G)$-invariant. Since $V_{X_i} \neq \{0\}$ for $i = 1, 2$, this contradicts the fact that π is irreducible and the claim is proved.

The stabilizer of $x_0 \in X$ as above is a closed subgroup H. Hence, upon disregarding a set of measure zero, we may and will identify X, as *Borel G-space,* with the space G/H, equipped with its quotient Borel structure.

Next, we draw some consequences for the cocycle $c : G \times X \to \mathbf{S^1}$ over the transitive G-space X. First, upon passing to a cohomologous cocycle, we can assume that c is a strict cocycle, that is, Equation (2) holds for all $g_1, g_2 \in G$ and $x \in X$. Moreover, choose a Borel section $s : G/H \to G$ for the projection $G \to G/H$ with $s(H) = e$. Observe that $gs(x)s(gx)^{-1}$ belongs to H, for every

$g \in G$ and $x \in X$. Then there exists a continuous homomorphism $\chi : H \to \mathbf{S^1}$ such that c is cohomologous to the cocycle $c_\chi : G \times X \to \mathbf{S^1}$ given by

$$c_\chi(g, x) = \chi(gs(x)s(gx)^{-1}) \qquad \text{for all} \quad g \in G, x \in X;$$

for all this, see Theorem 5.27 in [Var85].

We may and will in the sequel assume that the cocycle c coincides with c_χ. Also, as is well-known (see e.g. [Mac52, Theorem 1.1]), all G-quasi-invariant σ-finite Borel measures on G/H are mutually equivalent; so, we may assume that the G-quasi-invariant μ on G/H is one of the standard G-invariant Borel measures on G/H constructed in [Mac52, §.1].

• *Fifth step.* We claim that K acts transitively on the space G/H.

Indeed, on the one hand, since G is a connected Lie group, $X = G/H$ is a C^∞-smooth connected manifold; moreover, the measure μ on X is locally Lebesgue measure, that is, μ is given by a smooth density times Lebesgue measure in any local coordinates on X; indeed, this follows from the fact that the maps $x \mapsto gx$ are diffeomorphisms of X for every $g \in G$.

On the other hand, it follows from the third step that there exists a K-orbit $\mathcal{O}$ in X with $\mu(\mathcal{O}) > 0$. As is well-known, $\mathcal{O}$ is a submanifold of X, diffeomorphic to K/L where L is the stabilizer in K of a point in K (see e.g. [Bou72, III, §1, 7.]). In view of what we said about μ, this is possible only if $\dim \mathcal{O} = \dim X$, that is, if $\mathcal{O}$ is open in X. Since X is connected, it follows that $\mathcal{O} = X$.

• *Sixth step.* We claim that H is contained as a finite index subgroup in a parabolic subgroup of G, as defined in the Introduction. Indeed, it follows from the fifth step and Lemma 3 below, that there exists a parabolic subgroup Q which contains H and which has the following properties; let $Q = M_Q A_Q N_Q$ be the Langlands decomposition of Q. Write $Q^0 = CSA_Q N_Q$ for the connected identity component Q^0 of Q, where C is the maximal compact factor of M_Q^0 and S is the product of all the non-compact simple factors of M_Q^0. Then

$$H^0 = C'SA_Q N_Q,$$

where C' is a connected closed subgroup of C.

We claim H has finite index in Q. Assume, by contradiction, that H has infinite index in Q. Observe that Q is a direct product $Q = ZQ^0$ for a finite subgroup Z of its center (see [Kna96, Theorem 7.53]). So, H is of the form $H = TSA_Q N_Q$ for a subgroup T of infinite index in the compact group $L := ZC$.

Let χ be the unitary character of H associated to c. Observe that χ is trivial on the semisimple Lie group S. Moreover, we have $[\mathfrak{a}_F, \mathfrak{n}_F] = \mathfrak{n}_F$ for the Lie algebras $\mathfrak{a}_F$ and $\mathfrak{n}_F$ of A_Q and N_Q, where F is the set of simple roots associated to Q as in the Introduction; it follows that the commutator subgroup of $A_Q N_Q$ coincides with N_Q and hence that χ is also trivial on N_Q.

Choose a Borel fundamental domain $\Omega_0 \subset G$ for G/Q and a fundamental domain $\Omega_1 \subset L$ for L/T. Then $\Omega_0\Omega_1$ is a fundamental domain for G/H and every $g \in G$ has unique decompositions

$$g = \omega_0(g)q(g) = \omega_0(g)\omega_1(g)h(g) \qquad \text{for} \quad \omega_i(g) \in \Omega_i, q(g) \in Q, h(g) \in H.$$

Let ν_1 be the L-invariant probability measure on $\Omega_1 \cong L/T$. Identify $L_p(G/H, \mu)$ with $L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$ for a G-quasi-invariant probability measure ν_0 on $\Omega_0 \cong G/Q$. For $g \in G$ and $F \in L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$, we have

$$\pi(g)F(x) = \chi(h(g^{-1}x))F(\omega_0(g^{-1}x)\omega_1(g^{-1}x)) \qquad \text{for all} \quad x \in \Omega_0 \times \Omega_1.$$

Define a representation π_Q of Q on $L_p(\Omega_1, \nu_1)$ by

$$\pi_Q(q)f(x_1) = \chi(h(q^{-1}x_1))f(\omega_1(q^{-1}x_1))$$

for $f \in L_p(\Omega_1, \nu_1), x_1 \in \Omega_1, q \in Q$.

Since $\Omega_1 \cong Q/H$ is infinite, $L_p(\Omega_1, \nu_1)$ is infinite dimensional; therefore, the restriction $\pi_Q|_L$ of π_Q to the compact group L is not irreducible (see [War72, 4.2.2.4]). Hence, there exists a closed $\pi_Q(L)$-invariant subspace V of $L_p(\Omega_1, \nu_1)$ with $V \neq \{0\}$ and $V \neq L_p(\Omega_1, \nu_1)$. Observe that $\pi_Q(q)$ is a multiple of $\chi(q)$ for $q \in A_Q$ and $\pi_Q(q)$ is the identity for $q \in SN_Q$. Since $Q = LSA_QN_Q$, it follows that V is Q-invariant.

For $F \in L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$ and $x_0 \in \Omega_0$, let $F_{x_0} \in L_p(\Omega_1, \nu_1)$ be defined by

$$F_{x_0}(x_1) = F(x_0 x_1).$$

Let $\widetilde{V}$ be subspace of $L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$ of all $F \in L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$ such that $F_{x_0} \in V$ for ν_0-almost every $x_0 \in \Omega_0$. Then $\widetilde{V}$ is a proper non trivial closed subspace of $L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$.

We claim that $\widetilde{V}$ is $\pi(G)$-invariant. Indeed, let $F \in \widetilde{V}, g \in G$ and $F' = \pi(g)F$; for $x_0 \in \Omega_0, x_1 \in \Omega_1$, set $x = x_0 x_1$ and let $q = q(x_0^{-1} g)$ be the Q-component of $x_0^{-1} g$. We have $\omega_0(g^{-1}x) = \omega_0(g^{-1}x_0)$ and

$$\begin{aligned} F'_{x_0}(x_1) &= F'(x_0 x_1) = \chi(h(g^{-1}x))F(\omega_0(g^{-1}x_0)\omega_1(g^{-1}x)) \\ &= \chi(h(((x_0^{-1}g)^{-1}x_1))F_{\omega_0(g^{-1}x_0)}(\omega_1((x_0^{-1}g)^{-1}x_1)) \\ &= \chi(h(q^{-1}x_1))F_{\omega_0(g^{-1}x_0)}(\omega_1(q^{-1}x_1))) \\ &= (\pi_Q(q)F_{\omega_0(g^{-1}x_0)})(x_1) \end{aligned}$$

Since $F_{\omega_0(g^{-1}x_0)} \in V$ for almost every x_0 and since V is Q-invariant, it follows that $F' \in \widetilde{V}$ and the claim is proved.

Now, π is irreducible. So, we have obtained a contradiction which shows that H has finite index in Q.

• *Seventh step.* We claim that $H = Q$. Indeed, H contains Q^0, since H has finite index in Q. As mentioned in the sixth step, Q is a direct product $Q = ZQ^0$ for a finite abelian subgroup Z. Hence, we have $Q = Z'H$ for a subgroup Z' of Z. We have to show that Z' is trivial.

Assume, by contradiction, that $Z' \neq \{e\}$. Then there exists a non trivial character δ of Z'. We proceed as in the sixth step. Let $\Omega_0 \subset G$ a fundamental domain for G/Q. We identify $L_p(G/H, \mu)$ with $L_p(\Omega_0 \times Z', \nu_0 \otimes \nu_1)$ for a convenient G-quasi-invariant probability measure ν_0 on $\Omega_0 \cong G/Q$ and a probability measure ν_1 on Z'.

For $F \in L_p(\Omega_0 \times Z', \nu_0 \otimes \nu_1)$ and $x_0 \in \Omega_0$, let $F_{x_0} \in L_p(\Omega_1, \nu_1)$ be defined by $F_{x_0}(x_1) = F(x_0 x_1)$. Let $\widetilde{V}_\delta$ be subspace of $L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$ of all F such that $F_{x_0}(zx_1) = \delta(z)F_{x_0}(x_1)$ for all $z \in Z'$ and ν_0-almost every $x_0 \in \Omega_0$. Then $\widetilde{V}_\delta$ is a proper non trivial closed subspace of $L_p(\Omega_0 \times \Omega_1, \nu_0 \otimes \nu_1)$. Moreover, one checks that $\widetilde{V}_\delta$ is $\pi(G)$-invariant; this contradicts the irreducibility of π.

• *Eighth step.* There exists a a real linear form $\lambda \in \mathfrak{a}^*$ and a unitary character χ of M_Q such that π is equivalent to the representation $\pi(Q, \chi, \lambda, p)$ on $L_p(G/Q, \mu_Q)$ described in Theorem A.

Indeed, by the seventh step, we have $H = Q = M_Q A_Q N_Q$. The unitary character of Q associated to the cocycle c is trivial on N_Q and M_Q centralizes A_Q; hence, this character is of the form $\chi \otimes \chi'$ for unitary characters χ of M_Q and χ' of A_Q. Let $F \subset \Sigma$ be the set of simple positive roots associated to Q There exists $\lambda \in \mathfrak{a}^*$ such that

$$\chi'(\exp X) = e^{-i\lambda(X)} \qquad \text{for all} \quad X \in \mathfrak{a}_F = \log(A_Q).$$

Let μ be the unique quasi-invariant Borel probability measure on G/Q which is K-invariant. For $g \in G$, decompose g according to $G = KM_QA_QN_Q$ as

$$g = \kappa(g)\mu(g)\exp(H(g))n(g) \quad \text{for} \quad \kappa(g) \in \Omega_Q, \mu(g) \in M_Q, H(g) \in \mathfrak{a}_F, n(g) \in N_Q.$$

The Radon-Nikodym derivative of $g_*\mu$ with respect to μ is given by (see e.g. [Kna96, Proposition 8.44])

$$\frac{dg_*(\mu)}{d\mu}(x) = e^{-\rho_F(H(g))}$$

for

$$\rho_F = \sum_{\alpha \in \Sigma^+ \setminus \mathrm{span(F)}} \dim(\mathfrak{g}_\alpha)\alpha.$$

This finishes the proof of Theorem A.

The following lemma, which was used in the proof above, is a consequence of the description of cocompact subgroups of semisimple Lie groups from [Wit90] (see also [GW72]).

Lemma 3. *Let G be a noncompact connected semisimple linear Lie group with maximal compact subgroup K. Let H be a closed subgroup of G with the property that the natural action of K on G/H is transitive. Upon replacing H by one of its conjugate, there exists a standard parabolic subgroup Q which contains H and which has the following properties; denote by $Q = M_QA_QN_Q$ the Langlands decomposition of Q; write $Q^0 = CSA_QN_Q$ for the identity component Q^0 of Q, where C is the maximal compact factor of M_Q^0 and S is the product of all the non-compact simple factors of M_Q^0. Then there exists a connected closed subgroup C' of C such that*

$$H^0 = C'SA_QN_Q.$$

Proof. Since K acts transitively on G/H, we have $G = KH$. In particular, H is a cocompact subgroup of G. By [Wit90, Theorem 1.2] and upon conjugating H, there exists a standard parabolic subgroup Q of G with the following properties:

- $Q = M_QA_QN_Q$ contains H;
- there exist a connected, normal subgroup X of S and a connected closed subgroup Y of CA_Q such that $H^0 = YXN_Q$, where S and C are as in the statement of the Lemma.

Since H/H^0 is at most countable, we have

$$G = \bigcup_{n \geq 1} KH_0g_n$$

for some sequence $(g_n)_{n\geq 1}$ of elements in G. As K is compact and H_0 is closed, KH_0g_n is a closed subset of G for every n. Hence, by Baire's category theorem, KH_0 has a non empty interior. By homogeneity, it follows that KH_0 is open in G. Since G is connected, we have therefore $G = KH_0$, that is,

$$G = KYXN_Q. \tag{6}$$

Recall that the multiplication map $K \times A \times N \to G$ from the Iwasawa decomposition $G = KAN$ is a diffeomorphism. The reductive Lie group M_Q has also an Iwasawa decomposition

$$M_Q = K_1A_1N_1,$$

for $K_1 := K \cap M_Q$; we have $A = A_1A_Q$ as a direct product and $N = N_1N_Q$ as semi-direct product (see [Kna96, Proposition7.82]). In particular, we have $C \subset K_1$. Let $\Omega \subset K$ be a fundamental domain for K/K_1. So,

$$\text{(7) the product map} \qquad \Omega \times K_1 \times A_1 \times A_Q \times N_1 \times N_Q \to G \qquad \text{is a bijection.}$$

The Iwasawa decomposition of the semisimple Lie group $X \subset S$ is $X = K_2 A_2 N_2$, where $K_2 := K_1 \cap X$ and where A_2 and N_2 are subgroups of A_1 and N_2. By (6), we have therefore

$$G = KYXN_Q = KK_2YA_2N_2N_Q.$$

Since $Y \subset CA_Q$, it follows from (7) that

$$\dim A_2 = \dim A_1, \quad \dim N_2 = \dim N_1, \quad \text{and} \quad A_Q \subset Y.$$

So, we have $S = X$ and $Y = C'A_Q$ for a connected closed subgroup C' of C.

□

3. Proof of Proposition 2

Let $Q = M_Q A_Q N_Q$ be a parabolic subgroup of G; for a real linear form $\lambda \in \mathfrak{a}^*$ and a unitary character $\chi : M_Q \to \mathbf{S}^1$, consider the representation $\pi(Q, \lambda, \chi, p)$ of G on $L_p(G/Q, \mu)$ as in the Introduction.

We may realize $\pi(Q, \lambda, \chi, p)$ in the so-called compact picture as follows. Let $C(K, \chi)$ be the vector space of continuous functions $F : K \to \mathbf{C}$ such that

$$F(xk) = \overline{\chi(k)}F(x) \qquad \text{for all} \quad k \in K \cap M_Q,\ x \in K;$$

equip $C(K, \chi)$ with the norm

$$\|F\|_p = \left(\int_{K/(K\cap M_Q)} |F(x)|^p d\mu(x)) \right)^{1/p},$$

where μ is the unique K-invariant probability measure on $K/(K \cap M_Q)$. Let $L_p(K, \chi)$ be the completion of $(C(K, \chi), \|\cdot\|_p)$. Let $\rho_{Q,p} \in \mathfrak{a}^*$ be defined as in the Introduction. Then $\pi(Q, \lambda, \chi, p)$ acts on $L_p(K, \chi)$ through

$$\pi(Q, \chi, \lambda, p)(g)F(x) = \overline{\chi(\mu(g^{-1}x))} e^{-(i\lambda + \rho_{Q,p})H(g^{-1}x)} F(\kappa(g^{-1}x))$$

for $F \in L_p(K, \chi), x \in K$ and $g \in G$ with decomposition

$$g = \kappa(g)\mu(g)\exp(H(g))n(g), \qquad \kappa(g) \in K, \mu(g) \in M_Q, H(g) \in \log(A_Q), n(g) \in N_Q$$

Observe that, for $g \in K$, the operator $\pi(Q, \chi, \lambda, p)(g)$ is simply left translation on $L_p(K, \mu)$ by g^{-1}.

Set $E := L_p(K, \chi)$. Recall that the subspace E_K of K-finite vectors in E is the space of functions $F \in E$ such that

$$\pi(Q, \chi, \lambda, p)(K)F = \{\pi(Q, \chi, \lambda, p)(g)F : g \in K\}$$

spans a finite-dimensional subspace of E. The space E^∞ of C^∞-vectors in E consists of smooth functions $F \in C(K, \chi)$. As a K-module, E^∞ is the representation of K differentiably induced by $\chi|_{K\cap M_Q}$ (in the sense of [War72, 5.3.1]). Therefore, by Frobenius reciprocity (which remains valid in this context), the multiplicity of $\delta \in \widehat{K}$ in E^∞ is equal to the multiplicity of χ in $\delta|_{K\cap M_Q}$ and is therefore finite, since δ is finite dimensional. Hence, the multiplicity of δ in $E_K \cap E^\infty$ is finite. Since $E_K \cap E^\infty$ is dense in E (see e.g. [War72, Theorem 4.4.5.16]), it follows that the multiplicity of δ in E is finite. In other words, $\pi(Q, \chi, \lambda, p)$ is an *admissible* representation of G (as defined [Kna01, p.207] and [Wal71, 3.3.5]) or is a K-*finite* representation (as defined in [War72, 4.5.1]).

As is well-known, E_K is a module over the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\mathbf{C})$ of the complexification of the Lie algebra $\mathfrak{g}$ of G. Set

$$\nu := i\lambda + \rho_{Q,p} - \rho_{Q,2} = i\lambda + \left(\frac{2}{p} - 1\right)\rho_{Q,2}; \tag{8}$$

then E_K coincides with the $(\mathfrak{g}, K)$-module associated to the (nonunitary) induced representation $\mathrm{Ind}_Q^G(\chi \otimes \nu \otimes 1_{N_Q})$. This $(\mathfrak{g}, K)$-module consists of the functions $f : G \to \mathbf{C}$ which are C^∞ and have the following properties:

- $f(gman) = \overline{\chi(m)} e^{-(i\lambda+\nu+\rho_{Q,2})(\log a)}\ f(g)$ for all $g \in G$ and $man \in Q$;
- the linear span of $K \cdot f$ is finite dimensional space, where $K \cdot f$ is the space of translates $g \mapsto f(k^{-1}g)$ for $k \in K$.

(for this, see [Wal71, 5.2.1]). So, the $(\mathfrak{g}, K)$ module associated to $\pi(Q, \chi, \lambda, p)$ is the principal series representation denoted $I_{Q,\chi,\nu}$ in [Wal71] for ν as in (8).

Now, since $\pi(Q, \chi, \lambda, p)$ is admissible, $\pi(Q, \chi, \lambda, p)$ is completely irreducible if and only if its associated $(\mathfrak{g}, K)$-module $I_{Q,\chi,\nu}$ is algebraically irreducible. Moreover, two such representations $\pi(Q_1, \chi_1, \lambda_1, p_1)$ and $\pi(Q_1, \chi_1, \lambda_1, p_1)$ are Naimark equivalent if and only if the $(\mathfrak{g}, K)$-modules I_{Q_1,χ_1,ν_1} and I_{Q_2,χ_2,ν_2} are algebraically equivalent (for all this, see Theorems 4.5.5.4 and 4.5.5.2 in [War72]).

4. Proof of Theorem B

We assume from now on that G is a simple Lie group with real rank one. In this case, we have $\dim \mathfrak{a} = 1$.

Choose $\alpha \in \Sigma^+$ and $H \in \mathfrak{a}$ such that $\frac{1}{2}\alpha \notin \Sigma$ and $\alpha(H) = 1$. We identify $\mathfrak{a}_{\mathbf{C}}^*$ with $\mathbf{C}$ by means of the map $\lambda \mapsto \lambda(H)$.

There is, up to conjugation, only one parabolic subgroup, namely $P := MAN$. For $p \in (1 + \infty)$, we will write ρ_p instead of $\rho_{P,p}$. In view of Proposition 2, we have to study, given $\lambda \in \mathbf{R}$ and $\chi \in \widehat{M^{\mathrm{Ab}}}$, the irreducibility of the (non unitary) principal series representation $I_{P,\chi,\nu_{\lambda,p}}$ for

$$\nu_{\lambda,p} = i\lambda + \left(\frac{2}{p} - 1\right)\rho_2$$

and to decide when two such representations are equivalent. We will apply several times the following result of Kostant about the case where $\nu = 1_M$. Set

$$t_\alpha := \begin{cases} (\dim \mathfrak{g}_\alpha)/2 & \text{if} \quad 2\alpha \notin \Sigma \\ (\dim \mathfrak{g}_\alpha)/2 + 1 & \text{if} \quad 2\alpha \in \Sigma, \end{cases}$$

where $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is the usual root space corresponding to α; set also

$$n_\alpha := \begin{cases} 1 & \text{if} \quad 2\alpha \notin \Sigma \\ 2 & \text{if} \quad 2\alpha \in \Sigma. \end{cases}$$

Recall that

$$\rho_2 = (\dim \mathfrak{g}_\alpha)/2 + \dim \mathfrak{g}_{2\alpha}.$$

For $\nu \in \mathbf{C}$, we have (see Theorem 2 in [Kos69] and Theorem 2.9.8 in [Kos75]):

$$(9) \qquad I_{P,1_M,\nu} \text{ is } \textit{not} \text{ irreducible } \iff \nu \notin (-t_\alpha, t_\alpha) \text{ and } \nu + \rho_2 \in n_\alpha \mathbf{Z}.$$

We will treat separately the different simple groups G which may occur; for the data we will use concerning these groups, we refer to [Hel78, Chap. X].

• Let $G = \mathrm{SO}_0(n, 1)$ for $n \geq 2$. Here, $K \cong SO(n)$ and $M \cong SO(n-1)$. Moreover, we have $\dim \mathfrak{g}_\alpha = n - 1$ and $2\alpha \notin \Sigma$. So,

$$t_\alpha = \frac{n-1}{2}, \qquad n_\alpha = 1, \qquad \text{and} \qquad \rho_2 = \frac{n-1}{2};$$

hence, for $\lambda \in \mathbf{R}$, we have

$$\nu_{\lambda,p} = i\lambda + \left(\frac{2}{p} - 1\right)\frac{n-1}{2}.$$

Since $\left|\frac{2}{p}-1\right|<1$, it follows from (9) that

$$I_{P,1_M,\nu_{\lambda,p}} \quad \text{is irreducible for all } \lambda\in\mathbf{R} \text{ and } p\in(1,+\infty) \text{ with } p\neq 2.$$

The case where $n\neq 3$ is settled since the abelianization of M is then trivial and so $\widehat{M^{\mathrm{Ab}}}=1_M$.

Assume now that $n=3$. Observe that in this case $G\cong \mathrm{SL}_2(\mathbf{C})$ and $M\cong \mathrm{U}(1)$. We use here the results from [Thi73, §. 13, B]: identifying $\widehat{M^{\mathrm{Ab}}}$ with $\frac{1}{2}\mathbf{Z}$, we have, for $m\in\widehat{M^{\mathrm{Ab}}}$ and $\nu\in\mathbf{C}$,

$$I_{P,m,\nu} \text{ is irreducible } \iff (\nu+l+1)(\nu-l-1)\neq 0$$

for all $l\in\{|m|+k : k=0,1,2,\cdots\}$. Since $(n-1)/2=1$, this condition is clearly satisfied for $\nu_{\lambda,p}$ and so we obtain that $I_{P,\chi,\nu_{\lambda,p}}$ is irreducible for all $\lambda\in\mathbf{R}$, $\chi\in\widehat{M^{\mathrm{Ab}}}$, and $p\in(1,+\infty)$ with $p\neq 2$.

• Let $G=\mathrm{SU}(n,1)$ for $n\geq 2$. Here, $K\cong\mathrm{SU}(n)$ and $M\cong\mathrm{U}(n-1)$. Moreover, we have $\dim\mathfrak{g}_\alpha=2(n-1)$ and $\dim\mathfrak{g}_{2\alpha}=1$. So,

$$t_\alpha=n, \qquad n_\alpha=2, \qquad \text{and} \qquad \rho_2=n;$$

hence, for $\lambda\in\mathbf{R}$, we have

$$\nu_{\lambda,p}=i\lambda+\left(\frac{2}{p}-1\right)n.$$

Since $\left|\frac{2}{p}-1\right|<1$, it follows from (9) that

$$I_{P,1_M,\nu_{\lambda,p}} \quad \text{is irreducible for all } \lambda\in\mathbf{R} \text{ and } p\in(1,+\infty) \text{ with } p\neq 2.$$

In [Kra76, Proposition 1] precise necessary and sufficient conditions are given on $\sigma\in\widehat{M}$ and $\nu\in\mathbf{C}$ for $I_{P,\sigma,\nu}$ to be irreducible. In the special case of $\sigma\in\widehat{M^{\mathrm{Ab}}}$, this criterion reads as follows. Identify $\widehat{M^{\mathrm{Ab}}}$ with $\frac{1}{n+1}\mathbf{Z}$ as in [Kra76] and for $\sigma\in\widehat{M^{\mathrm{Ab}}}$ and $\nu\in\mathbf{C}$, define $(s_1,\dots,s_{n+1})\in\mathbf{C}^{n+1}$ by

$$s_i=\begin{cases}\frac{1}{2}(\nu-(n-1)\sigma) & \text{if} \quad i=1\\ \sigma+\frac{n}{2}-i+1 & \text{if} \quad 2\leq i\leq n\\ -\frac{1}{2}(\nu+(n-1)\sigma) & \text{if} \quad i=n+1.\end{cases}$$

Then $I_{P,m,\nu}$ is not irreducible if and only if either $s_1-s_i\in\mathbf{Z}\setminus\{0\}$ for all $2\leq i\leq n$ or $s_{n+1}-s_i\in\mathbf{Z}\setminus\{0\}$ for all $2\leq i\leq n$.

It follows immediately that $I_{P,\sigma,\nu_{\lambda,p}}$ is irreducible if $\lambda\neq 0$. So, we may assume that $\lambda=0$.

Writing $\sigma=\dfrac{m}{n+1}$ for $m\in\mathbf{Z}$, the criterion above implies that $I_{P,\sigma,\nu_{0,p}}$ is not irreducible if and only if either

$$\frac{1}{2}\left(\nu_{p,0}-(n-1)\frac{m}{n+1}\right)-\frac{m}{n+1}+\frac{n}{2}\in\mathbf{Z}\setminus\{0\}$$

or

$$-\frac{1}{2}\left(\nu_{p,0}+(n-1)\frac{m}{n+1}\right)-\frac{m}{n+1}+\frac{n}{2}\in\mathbf{Z}\setminus\{0\}$$

that is, if and only if

$$\text{either} \quad \nu_{p,0}-(m+n)\in 2\mathbf{Z}\setminus\{0\} \quad \text{or} \quad \nu_{p,0}-(m+n)\in 2\mathbf{Z}\setminus\{0\}$$

As $\nu_{\lambda,0} = \left(\frac{2}{p} - 1\right) n$, we see that $I_{P,\sigma,\nu_{0,p}}$ is not irreducible if only if

$$\text{either} \quad 2\left(\frac{1}{p} - 1\right) n - m \in 2\mathbf{Z} \setminus \{0\} \quad \text{or} \quad \frac{2n}{p} - m \in 2\mathbf{Z} \setminus \{0\}. \tag{10}$$

It is clear that (10) is equivalent to

$$p \in \left\{ \frac{2n}{k} : k \in \{1, \dots, 2n-1\}, k \equiv m \pmod 2, k \neq m, k \neq 2n+m \right\}.$$

• Let $G = \mathrm{Sp}(n,1)$ for $n \geq 2$. Here, $K \cong \mathrm{Sp}(n) \times \mathrm{Sp}(1)$ and $M \cong \mathrm{Sp}(n-1) \times \mathrm{Sp}(1)$. Moreover, we have $\dim \mathfrak{g}_\alpha = 4(n-1)$ and $\dim \mathfrak{g}_{2\alpha} = 3$. So,

$$t_\alpha = 2n-1, \qquad n_\alpha = 2, \qquad \text{and} \qquad \rho_2 = 2n+1;$$

hence, for $\lambda \in \mathbf{R}$, we have

$$\nu_{\lambda,p} = i\lambda + \left(\frac{2}{p} - 1\right)(2n+1).$$

Observe that the abelianization of M is trivial and so $\widehat{M^{\mathrm{Ab}}} = \{1_M\}$. Moreover, we have

$$\left|\frac{2}{p} - 1\right| (2n+1) \geq 2n-1 \iff p \in (1, \frac{2n+1}{2n}] \cup [2n+1, +\infty).$$

and

$$\left(\frac{2}{p} - 1\right)(2n+1) + 2n - 1 \in 2\mathbf{Z} \iff p \in \left\{ \frac{2n+1}{k} : k \in \mathbf{N}^* \right\}.$$

It follows from (9) that

- $I_{P,1_M,\nu_{\lambda,p}}$ is irreducible for every $\lambda \in \mathfrak{a}^* \setminus \{0\}$
- $I_{P,1_M,\nu_{0,p}}$ is not irreducible if and only if $p = \dfrac{2n+1}{2n}$ or $p = 2n+1$.

• Let $G = \mathrm{F}_{4(-20)}$. Here, $K \cong \mathrm{Spin}(9)$ and $M \cong \mathrm{Spin}(7)$. Moreover, we have $\dim \mathfrak{g}_\alpha = 8$ and $\dim \mathfrak{g}_{2\alpha} = 7$. So,

$$t_\alpha = 5, \qquad n_\alpha = 2, \qquad \text{and} \qquad \rho_2 = 11;$$

hence, for $\lambda \in \mathbf{R}$, we have

$$\nu_{\lambda,p} = i\lambda + 11\left(\frac{2}{p} - 1\right).$$

The abelianization of M is trivial and so $\widehat{M^{\mathrm{Ab}}} = \{1_M\}$. Moreover, we have

$$11\left|\frac{2}{p} - 1\right| \geq 5 \iff p \in (1, \frac{11}{8}] \cup [\frac{11}{3}, +\infty).$$

and

$$11\left(\frac{2}{p} - 1\right) + 5 \in 2\mathbf{Z} \iff p \in \left\{ \frac{11}{k} : k \in \mathbf{N}^* \right\}.$$

It follows from (9) that

- $I_{P,1_M,\nu_{\lambda,p}}$ is irreducible for every $\lambda \in \mathfrak{a}^* \setminus \{0\}$
- $I_{P,1_M,\nu_{0,p}}$ is not irreducible if and only if $p \in \left\{ \frac{11}{10}, \frac{11}{9}, \frac{11}{8}, \frac{11}{3}, \frac{11}{2}, 11 \right\}$.

In all the cases above, if two representations $I_{P,\chi,\nu}$ and $I_{P,\chi',\nu'}$ are irreducible, then they are equivalent if and only if

$$(\chi',\nu)=(\overline{\chi},\nu) \quad \text{or} \quad (\chi',\nu')=(\overline{\chi},-\nu).$$

Now,

$$-\nu_{\lambda,p}=-(i\lambda+\left(\frac{2}{p}-1\right)\rho_2=(-i\lambda+\left(\frac{2}{q}-1\right)\rho_2=\nu_{\lambda,q}.$$

Therefore, $I_{P,\chi',\nu_{\lambda',p'}}$ and $I_{P,\chi,\nu_{\lambda,p}}$ are equivalent if and only if

$$(\chi',\lambda',p')=(\overline{\chi},\lambda,p) \quad \text{or} \quad (\chi',\lambda',p')=(\overline{\chi},-\lambda,q).$$

5. An example: $SL_2(\mathbf{R})$

Let $G=\mathrm{SL}_2(\mathbf{R})$, with maximal compact subgroup $K=SO(2)$. The standard minimal parabolic subgroup is

$$P=\left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} : a\in\mathbf{R}, a\neq 0, b\in\mathbf{R}\right\}.$$

We have $P=MAN$ for

$$A=\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} : a\in\mathbf{R}, a>0\right\},$$

$$M=\left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\right\},$$

and

$$N=\left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} : b\in\mathbf{R}\right\}.$$

Identifying $\mathfrak{a}_{\mathbf{C}}^*$ with $\mathbf{C}$ as above, we have

$$\rho_{P,p}=\frac{1}{p}$$

for every $p\in(1+\infty)$. We identify G/P as G-space with the real projective line $\mathbf{P}(\mathbf{R})$, with G acting by Möbius transformations on this latter space. The Lebesgue measure μ is the unique K-invariant probability Borel measure on $\mathbf{P}(\mathbf{R})$. The measurable space G/P can further be identified with $\mathbf{R}=\mathbf{P}(\mathbf{R})\setminus\{\infty\}$.

The Radon-Nikodym derivative for $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ acting on $\mathbf{R}$ is given by

$$\frac{dg_*(\mu)}{d\mu}(x)=\frac{1}{(cx+d)^2}.$$

Let ε denote the non trivial character of M and let $\lambda\in\mathbf{R}$. The representation $\pi(P,\chi,\lambda,p)$ of G by isometries on $L_p(\mathbf{R},\mu)$ is defined by

$$\pi(P,1_M,\lambda,p)(g)f(x)=|cx+d|^{-i\lambda-\frac{2}{p}}f\left(\frac{ax+b}{cx+d}\right) \quad \text{if} \quad g^{-1}=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$$

and

$$\pi(P,\varepsilon,\lambda,p)(g)f(x)=\mathrm{sgn}(cx+d)|cx+d|^{-i\lambda-\frac{2}{p}}f\left(\frac{ax+b}{cx+d}\right) \quad \text{if} \quad g^{-1}=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$$

The Harish-Chandra $(\mathfrak{g},K)$-module underlying $\pi(P,\varepsilon,\lambda,p)$ is $I_{P,\chi,\nu}$ for

$$\nu=i\lambda+\delta_p\rho_{P,2}=i\lambda+\frac{2}{p}-1$$

Assume that $p\neq 2$. It is well-known that $I_{P,\chi,\nu}$ is irreducible (see e.g. Proposition 1.3.3 in [Vog81]; observe that the irreducibility of $I_{P,\varepsilon,\nu}$ depends on the fact that

$\frac{2}{p} - 1$ is not an integer). Moreover, the modules $I_{P,\chi,\nu}$ are pairwise non equivalent for fixed p.

In summary, the L_p-dual space of $G = \mathrm{SL}_2(\mathbf{R})$ for $p \in (1, +\infty)$ with $p \neq 2$ is

$$\widehat{G}_{L_p} = \{\pi(P, 1_M, \lambda, p) : \lambda \in \mathbf{R}\} \cup \{\pi(P, \varepsilon, \lambda, p) : \lambda \in \mathbf{R}\} \cup \{1_G\}.$$

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