Asymptotic analysis at any order of Helmholtz's problem in a corner with a thin layer: an algebraic approach

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Abstract: We consider the Helmholtz equation in an angular sector partially covered by a homogeneous layer of small thickness, denoted ε. We propose in this work an asymptotic expansion of the solution with respect to ε at any order. This is done using matched asymptotic expansion, which consists here in introducing different asymptotic expansions of the solution in three subdomains: the vicinity of the corner, the layer and the rest of the domain. These expansions are linked through matching conditions. The presence of the corner makes these matching conditions delicate to derive because the fields have singular behaviors. Our approach is to reformulate these matching conditions purely algebraically by writing all asymptotic expansions as formal series. By using algebraic calculus we reduce the matching conditions to scalar relations linking the singular behaviors of the fields. These relations have a convolutive structure and involve some coefficients that can be computed analytically. Our asymptotic expansion is justified rigorously with error estimates.

Keywords: asymptotic analysis, Helmholtz's equation, matched asymptotic expansions, corner singularities, algebraic formal series.

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Introduction

Problems that involve thin layers appear in many areas, from composite materials engineering [35] to biology [12], including elasticity [8, 20], fluid mechanics [31, 1, 24] and electrochemistry [37]. Applications are especially numerous in electromagnetism, let us mention the studies of thin dielectric layers [22, 26, 34], ferromagnetic films [5, 21] and the skin effect [15]. All these situations are numerically challenging because they require finely meshing the thin structure, which is very costly when its thickness is very small compared to the wavelength and the size of the objects. In this work we propose to overcome this difficulty by using an asymptotic expansion of the solution, such that each term of the expansion is cheaper to compute than the solution itself.

Infinite planar layers and smooth curved layers were studied during the 90s in [9, 19]. Their method is to stretch the layer in its transverse direction into a standard layer of thickness 1, and look for a Taylor-type asymptotic expansion as a sum of integer powers of the original thickness, denoted ε. The terms of this expansion can be computed by induction. Those results were later extended to heterogeneous and periodic layers in [3, 2, 4] and more recently in [16, 10].

Here we want to handle more realistic situations where the coating has angles or covers only partially the obstacle. We consider a two-dimensional model where the domain is the union of an infinite angular sector and the coating, potentially with a perturbation at the corner of size proportional to ε. This was studied for Poisson's problems in [11, 6, 7], providing an asymptotic expansion at any order and approximate models. These works show the presence of non-integer powers of ε and integer powers of $\ln \varepsilon$ in the asymptotic expansion, that are linked to the corner singularities of the solution. That asymptotic expansion at any order was generalized to periodic layer in [17], still for Poisson's problems. In comparison, Helmholtz's problems not only present the same difficulties, but they also lead to more complex singularities, which prompted us to introduce new and more efficient algebraic calculus tools in order to

obtain an expansion at any order. Let us mention also that [33] proposes an asymptotic expansion of Helmholtz's problem up to order 2 in presence of periodic layers.

We can identify in these works two methods of analysis: multiscale asymptotic expansions and matched asymptotic expansions (see [38, 28, 29] and [36, 23, 18] respectively for a general presentation). They both involve two types of fields: "far fields" depending on the macroscopic scale described by (x, y) and "near fields" depending on the microscopic scale described by $(\frac{x}{\varepsilon}, \frac{y}{\varepsilon})$. In multiscale expansions, far and near fields are defined in the whole domain and the near fields tend to 0 towards infinity so that they describe a boundary layer effect in the "near zone" (the vicinity of the corner or the layer, depending on the situation). In contrast, matched asymptotic expansions involve near fields only in the near zone and far fields only in the "far zone" (the rest of the domain), and the near and far fields have to coincide in an intermediate zone.

In this paper, we chose the method of matched expansions. In addition, we propose a new algebraic approach to derive the matching conditions, that are especially intricate in our problem. It avoids specific cumbersome calculations, replacing them with abstract generalizable ones. We believe that this approach gives a better understanding of the structure of the asymptotic expansion at any order. It reveals a convolutive structure and it provides explicit expressions to compute exactly and very cheaply the constants that appear in the obtained matching formulas.

We consider the Helmholtz equation with absorption because it brings obvious well-posedness and stability of the problem uniformly in ε, which allows us to focus on asymptotic expansion techniques. The case without absorption requires to design a specific radiation condition, that will be the object of a future paper. Moreover, we apply a Dirichlet condition on the boundary. The extension to Neumann is not obvious and will be presented in a forthcoming article.

This paper is organized as follows. In Section 1, we define the problem, state the main result and introduce the method based on matched asymptotic expansion. The matching condition around the corner are derived using an algebraic approach in Section 2. It is the most original part of the article. In Section 3, we introduce appropriate frameworks which allow to define uniquely the terms of the asymptotic expansion. Error estimates are performed in Section 4, proving the main result of the paper.

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1 Setting of the problem and the method

1.1 Definition of the problem and main result

To describe the domain, let us introduce $\Theta \in (0, 2\pi)$, $\Omega := \{(r\cos\theta, r\sin\theta) \mid r \in \mathbb{R}_+^*, \theta \in (0, \Theta)\}$, $\Lambda := \mathbb{R}_+^* \times (-1, 0)$, $\Gamma = \mathbb{R}_+^* \times \{0\}$, $\Sigma_\Omega = \{(r\cos\Theta, r\sin\Theta) \mid r \in \mathbb{R}_+^*\}$ and $\Sigma_\Lambda := \mathbb{R}_+^* \times \{-1\}$. All these sets are shown in Figure 1. Then let $\Omega_1 \subset \mathbb{R}^2$ be an open set that coincides with $\Omega \cup \Gamma \cup \Lambda$ outside of the disc $B(0, R_c)$ for some $R_c \in \mathbb{R}_+^*$. In addition, let $\mu, \rho \in L^\infty(\Omega_1)$ be two functions greater than a positive constant (ellipticity assumption), and equal to μ_0 and ρ_0 in $\Omega \setminus B(0, R_c)$ and to μ_1 and ρ_1 in $\Lambda \setminus B(0, R_c)$. See Figures 2 and 3 for different configurations.

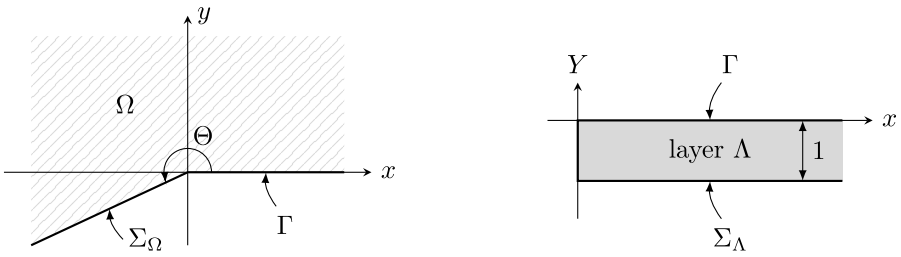

Figure 1: The domains Ω (on the left) and Λ (on the right)

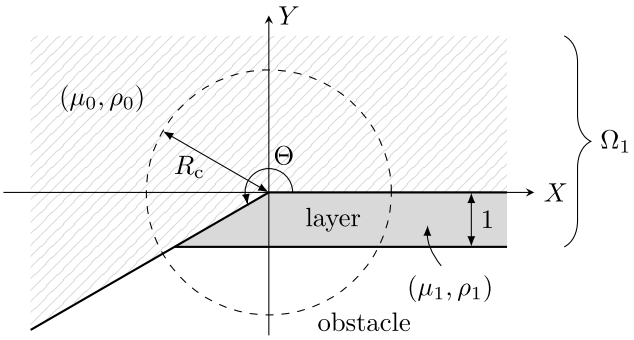

Figure 2: The domain Ω_1 with a configuration example of $\Omega_1 \cap B(0, R_c)$

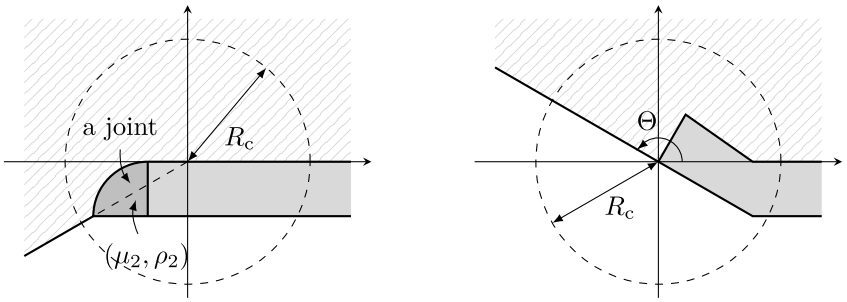

Figure 3: Other configuration examples of $\Omega_1 \cap B(0, R_c)$ for different values of Θ

Let $\varepsilon > 0$. The physical domain is given by $\Omega_\varepsilon := \{(x,y) \in \mathbb{R}^2 \mid (\frac{x}{\varepsilon}, \frac{y}{\varepsilon}) \in \Omega_1\}$. We introduce the scaled coefficients $\mu_\varepsilon : (x,y) \in \Omega_\varepsilon \mapsto \mu(\frac{x}{\varepsilon}, \frac{y}{\varepsilon})$ and $\rho_\varepsilon : (x,y) \in \Omega_\varepsilon \mapsto \rho(\frac{x}{\varepsilon}, \frac{y}{\varepsilon})$, and the scaled variables $X := \frac{x}{\varepsilon}$ and $Y := \frac{y}{\varepsilon}$. Let $\omega \in \mathbb{C} \setminus \mathbb{R}$ and $f_\mathrm{s} \in H^{-1}(\Omega)$ a source term s.t. $\mathrm{dist}(\mathrm{supp}(f_\mathrm{s}), \Gamma) > 0$. We denote u_ε the unique solution in $H^1_0(\Omega_\varepsilon)$ of

$$\mathrm{div}(\mu_\varepsilon \nabla u_\varepsilon) + \omega^2 \rho_\varepsilon u_\varepsilon = f_\mathrm{s} \qquad \text{in } \Omega_\varepsilon \tag{1.1}$$

$\Im(\omega) \neq 0$ is a technical assumption that makes this problem well-posed (it suffices to use the Lax-Milgram theorem) with a stability constant independent of ε :

$$\exists C > 0,\ \forall f_\mathrm{s} \in H^{-1}(\Omega),\ \forall \varepsilon > 0, \quad \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant C \|f_\mathrm{s}\|_{H^{-1}(\Omega)} \tag{1.2}$$

The case $\Im(\omega) = 0$ is an open question and will be the object of a future work.

The main result of this paper is given in the following theorem, proven in Section 4, page 36.

Theorem 1.1: asymptotic expansion of u_ε
Let $\mathbb{P} := \mathbb{N} + \frac{\pi}{\Theta}\mathbb{N}$. There exist $(n_p) \in \mathbb{N}^\mathbb{P}$ and a family $(u_{p,\ell})_{p \in \mathbb{P}, \ell \in [\![0, n_p]\!]}$ of elements of $H^1_{\mathrm{loc}}(\Omega)$ that can be build recursively w.r.t. p (see Theorem 3.17 for the construction) such that

$$\forall P \in \mathbb{R}_+, \forall \delta > 0, \qquad \left\| u_\varepsilon - \sum_{p \in \mathbb{P} \cap [0,P]} \sum_{\ell=0}^{n_p} \varepsilon^p \ln^\ell \varepsilon\, u_{p,\ell} \right\|_{H^1(\Omega \setminus B(0,\delta))} = o(\varepsilon^P) \qquad \text{when } \varepsilon \to 0.$$

The presence of integer powers of ε is entirely classical in asymptotic analysis. Integer powers of $\varepsilon^{\pi/\Theta}$ and $\ln \varepsilon$ can be found in other asymptotic expansions involving corners, see [11, 6, 7]. Theorem 1.1 can be extended to the case where Σ_Ω is covered by another layer (see Remark 2.7 for a useful point).

Notations: We denote (x,y) the cartesian coordinates, (r,θ) the polar coordinates with $\theta \in [0, 2\pi)$, $B(0,r)$ the disc of $\mathbb{R}^2$ of radius r centered at $(0,0)$, $k_i := \omega\sqrt{\rho_i/\mu_i}$ for any $i \in \{0,1\}$ and $\alpha := e^{-\mathrm{i}\Theta}$.

1.2 The matched asymptotic expansion method

To take into account the different behaviors of the solution in the layer, near the corner and far from the corner and the layer, we divide Ω_ε in three zones, illustrated in Figure 4. In each zone we postulate an asymptotic expansion in powers of ε and $\ln \varepsilon$, called "ansatz".

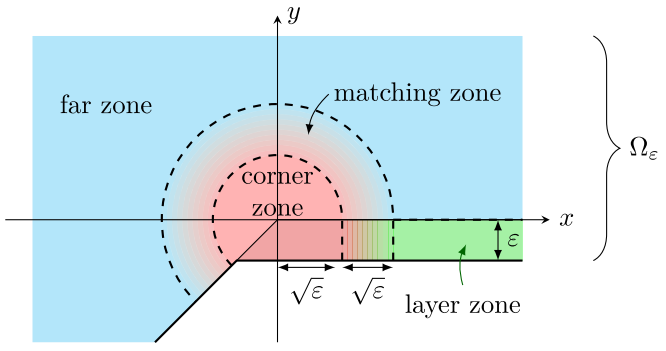

Figure 4: Zones of the matched asymptotic expansion

Let us assume that for any $(p,\ell) \in \mathbb{P} \times \mathbb{N}$ there exist three functions independent of ε – namely $u_{p,\ell}$ defined on Ω called "far field", $U_{p,\ell}$ defined on Λ called "layer field" and $S_{p,\ell}$ defined on Ω_1 called "corner field" – such that u_ε is formally written as:

- $u_\varepsilon(x,y) = \sum_{p \in \mathbb{P}} \sum_{\ell \in \mathbb{N}} \varepsilon^p \ln^\ell \varepsilon\, u_{p,\ell}(x,y)$ when $r := \|(x,y)\| \geqslant \sqrt{\varepsilon}$ and $(x,y) \notin \mathbb{R}_+ \times (-\varepsilon, 0)$ (far zone),

- $u_\varepsilon(x,y) = \sum_{p\in\mathbb{P}}\sum_{\ell\in\mathbb{N}} \varepsilon^p \ln^\ell\varepsilon\, U_{p,\ell}\Big(x, \frac{y}{\varepsilon}\Big)$ when $x \geqslant \sqrt{\varepsilon}$ and $y \in (-\varepsilon, 0)$ (layer zone),
- $u_\varepsilon(x,y) = \sum_{p\in\mathbb{P}}\sum_{\ell\in\mathbb{N}} \varepsilon^p \ln^\ell\varepsilon\, S_{p,\ell}\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)$ when $r \leqslant 2\sqrt{\varepsilon}$ (corner zone).

Remark: We will see in Proposition 3.14 that: $\forall p \in \mathbb{P}, \exists n_p \in \mathbb{N}, \forall \ell > n_p,\ u_{p,\ell} = 0$ and $U_{p,\ell} = 0$ and $S_{p,\ell} = 0$.

Injecting the above sums in the Helmholtz equation, using that $\partial_x^2\big[\varphi(\frac{x}{\varepsilon})\big] = \varepsilon^{-2}[\partial_X^2\varphi](X = \frac{x}{\varepsilon})$ and $\partial_y^2\big[\varphi(\frac{y}{\varepsilon})\big] = \varepsilon^{-2}[\partial_Y^2\varphi](Y = \frac{y}{\varepsilon})$ for any function φ, and formally identifying the powers of ε and $\ln\varepsilon$, one can easily derive the following volume equations and edge conditions for the various fields.

$$\begin{cases} \mu_0\Delta u_{p,\ell} + \omega^2\rho_0 u_{p,\ell} = f\,\delta_{p,0}\,\delta_{\ell,0} & \text{in } \Omega \\ u_{p,\ell} = U_{p,\ell} & \text{on } \Gamma \\ u_{p,\ell} = 0 & \text{on } \Sigma_\Omega \end{cases} \tag{1.3}$$

$$\begin{cases} \mu_1\partial_Y^2 U_{p,\ell} = -(\mu_1\partial_x^2 + \omega^2\rho_1)U_{p-2,\ell} & \text{in } \Lambda \\ \mu_1\partial_Y U_{p,\ell} = \mu_0\,\partial_y u_{p-1,\ell} & \text{on } \Gamma \\ U_{p,\ell} = 0 & \text{on } \Sigma_\Lambda \end{cases} \tag{1.4}$$

$$\begin{cases} \operatorname{div}(\mu\nabla S_{p,\ell}) = -\omega^2\rho\, S_{p-2,\ell} & \text{in } \Omega_1 \\ S_{p,\ell} = 0 & \text{on } \partial\Omega_1 \end{cases} \tag{1.5}$$

where we denote by convention $u_{p,\ell} = 0$, $U_{p,\ell} = 0$ and $S_{p,\ell} = 0$ for any $(p,\ell) \in \mathbb{R}\setminus\mathbb{P}\times\mathbb{N}$, and $\delta_{i,j} := 1$ if $i = j$ and 0 if not.

Remarks:

- The condition $\mu_1\partial_Y U_{p,\ell} = \mu_0\partial_y u_{p-1,\ell}$ is included in the problem satisfied by $U_{p,\ell}$ whereas $u_{p,\ell} = U_{p,\ell}$ is included in the problem satisfied by $u_{p,\ell}$ so that the construction is inductive: $u_{p-1,\ell}$ allows to build $U_{p,\ell}$, which allows to build $u_{p,\ell}$.
- The problem satisfied by $U_{p,\ell}$ depends only on Y, the variable x playing the role of a parameter.

(1.3)–(1.5) would be sufficient to uniquely define the fields, if they were in their natural variational spaces (e.g. $H^1(\Omega)$ for $u_{p,\ell}$). But we need to take into account a matching condition: the far and corner fields must coincide in the intersection of the far and corner zones, and similarly for the layer and corner fields. These intersections form the matching zone (see Figure 4). Given that $\varepsilon \to 0$ and $\frac{\sqrt{\varepsilon}}{\varepsilon} \to \infty$, this zone tends to $(0,0)$ w.r.t. the far and layer fields, but it tends to infinity w.r.t. the corner fields. Thus, the matching condition links the asymptotic behavior of far and layer fields at the corner to the one of corner fields at infinity:

$$\begin{cases} \sum \varepsilon^p \ln^\ell\varepsilon\, u_{p,\ell}(x,y) \approx \sum \varepsilon^p \ln^\ell\varepsilon\, S_{p,\ell}(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}) & \text{in } \Omega \text{ when } r \to 0 \text{ and } \frac{r}{\varepsilon} \to \infty \\ \sum \varepsilon^p \ln^\ell\varepsilon\, U_{p,\ell}(x, \frac{y}{\varepsilon}) \approx \sum \varepsilon^p \ln^\ell\varepsilon\, S_{p,\ell}(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}) & \text{in } \Lambda \text{ when } x \to 0 \text{ and } \frac{x}{\varepsilon} \to \infty \end{cases} \tag{1.6}$$

We will see that the far fields $u_{p,\ell}$ have an asymptotic expansion at the corner which is roughly a sum of powers of r, some of which are positive (like in a Taylor expansion e.g.). The matching conditions imply that these positive powers of r have to appear in the asymptotic expansions of the corner fields at infinity. We call them singularities for the corner fields. Conversely, the asymptotic expansion of the corner fields at infinity contain negative powers of r corresponding to the decay of the variational part, and these powers must be found in the far fields, which corresponds to singularities at the corner. Thus the fields cannot be searched in their natural variational spaces.

2 Matching conditions

This section establishes the matching condition linking corner fields to far and layer fields. This is by far the most difficult relation to derive, while all the others have been easily stated in (1.3)–(1.5). In this section we assume that the various fields exist and that they satisfy (1.3)–(1.5) and we give a necessary

and sufficient condition for the matching assumptions (1.6) to be satisfied. Our approach is based on an algebraic formulation of the problem, that reveals the structure of the matching relations by a rigorous algebraic calculus.

To perform the matching of the corner fields with the far and the layer fields at the same time, we merge the latter two into a single field denoted $\mathbf{u}_{p,\ell}$ and called "far-and-layer field". It is defined on a new domain Π, defined as follows:

- If $\Theta \leqslant \frac{3\pi}{2}$, then $\Pi := \Omega \sqcup \Gamma \sqcup \Lambda$ (disjoint union) and it is an open of $\mathbb{R}^2$.
- If $\Theta > \frac{3\pi}{2}$, then Ω and Λ intersect as subsets of $\mathbb{R}^2$, so the previous definition is not valid anymore (see Figure 5). Thus, we define Π as the disjoint gluing of Ω and Λ on Γ (which is a flat Riemannian manifold).

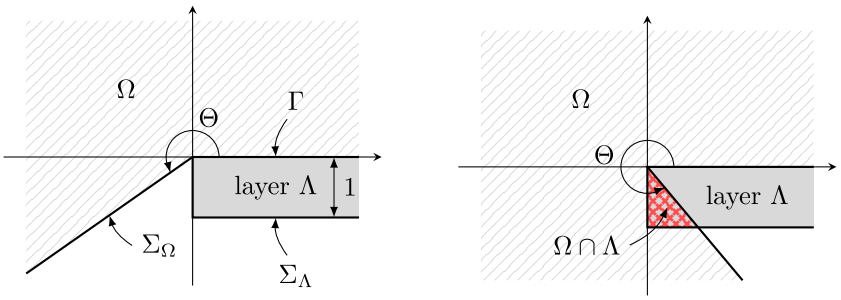

Figure 5: The domain Π is equal to $\Omega \sqcup \Gamma \sqcup \Lambda$ when $\Theta \leqslant \frac{3\pi}{2}$ (open subset of $\mathbb{R}^2$) and it is a flat Riemannian manifold when $\Theta > \frac{3\pi}{2}$ ($\Omega \cap \Lambda \neq \varnothing$).

For all $(p,\ell) \in \mathbb{P} \times \mathbb{N}$, we define $\mathbf{u}_{p,\ell} := \begin{cases} u_{p,\ell} & \text{in } \Omega \\ U_{p,\ell} & \text{in } \Lambda \end{cases}$ and the generalized radial variable $\mathbf{r} := \begin{cases} r & \text{in } \Omega \\ x & \text{in } \Lambda \end{cases}$

A straightforward reformulation of (1.3) and (1.4) gives that for any $(p,\ell) \in \mathbb{P} \times \mathbb{N}$:

$$\begin{cases} \mu_0 \Delta \mathbf{u}_{p,\ell} + \omega^2 \rho_0 \mathbf{u}_{p,\ell} = f\, \delta_{p,0}\, \delta_{\ell,0} & \text{in } \Omega \\ \mu_1 \partial_Y^2 \mathbf{u}_{p,\ell} = -(\mu_1 \partial_x^2 + \omega^2 \rho_1) \mathbf{u}_{p-2,\ell} & \text{in } \Lambda \\ \mu_1 \partial_Y \mathbf{u}_{p,\ell|Y=0^-} = \mu_0\, \partial_y \mathbf{u}_{p-1,\ell|y=0^+} & \text{on } \Gamma \\ \mathbf{u}_{p,\ell|y=0^+} - \mathbf{u}_{p,\ell|Y=0^-} = 0 & \text{on } \Gamma \\ \mathbf{u}_{p,\ell} = 0 & \text{on } \Sigma_\Omega \cup \Sigma_\Lambda \end{cases} \tag{2.1}$$

where by convention $\mathbf{u}_{p,\ell} = 0$ when $(p,\ell) \in \mathbb{R} \setminus \mathbb{P} \times \mathbb{N}$.

Let us give some starting point ideas to dive into this section. The matching assumption under study links the asymptotic behaviors of $\sum_{p,\ell} \varepsilon^p \ln^\ell \varepsilon\, \mathbf{u}_{p,\ell}$ when $\mathbf{r} \to 0$ and $\sum_{p,\ell} \varepsilon^p \ln^\ell \varepsilon\, S_{p,\ell}$ when $\mathbf{r} \to \infty$. So we can begin with a look at the asymptotic of $\mathbf{u}_{p,\ell}$ when $\mathbf{r} \to 0$, especially on Ω because it is the most interesting part. First, by (2.1), $u_{0,0}$ satisfies $\mu_0 \Delta u_{0,0} + \omega^2 \rho_0 u_{0,0} = 0$ in the vicinity of the corner in Ω, with homogeneous Dirichlet condition on $\partial\Omega$. So using separation of variables, it is easy to show that:

$$u_{0,0}(r,\theta) \underset{r \ll 1}{=} \sum_{d \in \frac{\pi}{\Theta}\mathbb{N}^*} \sigma_d(\mathbf{u}_{0,0})\, J_d(k_0 r)\, \sin(d\theta) = \sum_{d \in \frac{\pi}{\Theta}\mathbb{N}^*} \sigma_d(\mathbf{u}_{0,0}) \left(\sum_{n=0}^{\infty} a_{d,n} r^{d+2n} \right) \sin(d\theta) \tag{2.2}$$

with $\sigma_d(\mathbf{u}_{0,0})$ and $a_{d,n}$ some coefficients in $\mathbb{C}$, J_d Bessel functions of the first kind and $k_0 := \omega\sqrt{\rho_0/\mu_0}$. Then, one can show that (2.1) implies that $u_{1,0}$ satisfies the Helmholtz equation but with condition $\mu_1 u_{1,0} = \mu_0\, \partial_y u_{0,0|y=0^+}$ on Γ. Using (2.2), one can show that there exist some functions $f_{q,i}$ and coefficients $\sigma_d(\mathbf{u}_{1,0})$ s.t.:

$$u_{1,0} \underset{r \ll 1}{=} \sum_{d \in \mathbb{P}} r^{d-1}(f_{d,0}(\theta) + \ln(r)\, f_{d,1}(\theta)) + \sum_{d \in \frac{\pi}{\Theta}\mathbb{N}^*} \sigma_d(\mathbf{u}_{1,0})\, J_d(k_0 r)\, \sin(d\theta)$$

where the first sum is a particular solution of Helmholtz's equation that has trace $\frac{\mu_0}{\mu_1}\, \partial_y u_{0|y=0^+}$ on Γ, and the second one is a homogeneous solution. More generally, $u_{p,\ell}$ has a similar decomposition with

potentially lower powers of r (the smallest power being $-p$) and higher powers of $\ln r$.

$u_{1,0}(\Omega)$ is not in H^1 because it behaves as $\frac{1}{r}$ when $r \to 0$. More generally $u_{p,\ell}$ is not in $H^1(\Omega)$. It can be decomposed as the sum of a H^1 function and a function that is singular at $(0,0)$. Proposition 3.6 will show how to find solutions in such spaces and it reveals that they depend on some coefficients $\sigma_d(\mathbf{u}_{p,\ell})$ for $d < 0$ (corresponding to the singular part) while the coefficients for $d > 0$ (corresponding to the variational part) are uniquely fixed. So the $\sigma_d(\mathbf{u}_{p,\ell})$ for $d < 0$ are degrees of freedom in the construction of far-and-layer fields, and they will be fixed by the matching conditions. Likewise, the corner fields have similar decompositions when $\mathbf{r} \to \infty$, and singular coefficients $\sigma_d(S_{p,\ell})$ for $d > 0$ (but here singular when $\mathbf{r} \to \infty$) that will be fixed by the matching conditions.

In Section 2.1 we introduce algebraic formal series later used to perform efficient calculations on the asymptotic expansions. In Section 2.2 we define spaces of explicit functions called $\mathcal{A}(\dots)$ that contain all possible singularities at 0 (for the far-and-layer fields) or at ∞ (for the corner fields). In the previous paragraph, it corresponds to the functions $r^q \ln^i r\, f_{q,i}(\theta)$. In Section 2.3 we give tools to compute these singularities. In Section 2.4, we write the asymptotic expansions w.r.t. $\mathbf{r}$ of the various fields. Finally, in Section 2.5, we re-express the matching conditions (1.6) with equations that can be used to build the fields.

2.1 Algebraic preliminaries

To handle infinite series that may not converge, e.g. "$\sum_{p,\ell} \varepsilon^p \ln^\ell \varepsilon\, u_{p,\ell}$", we use the algebraic notion of *formal series* introduced in this section. Let E be a vector space and $(E_i)_{i\in I}$ be a family of vector subspaces of E. To begin, let us remind that $\sum_{i\in I} E_i$ designates the vector subspace of E made of *finite* sums of elements of the E_i. If this sum is direct, we denote it $\bigoplus_{i\in I} E_i$. From now on, we assume that the sum is direct. In order to deal with *infinite* sums we introduce the following definition.

Notation 2.1: Let us denote

$$\forall (\varphi_i) \in \prod_{i\in I} E_i, \quad \sum_{i\in I} \varphi_i := (\varphi_i) \qquad \text{and} \qquad \sum_{i\in I} E_i := \prod_{i\in I} E_i.$$

Note the boldness of the symbol $\sum$. $\sum_{i\in I} \varphi_i$ is not a real sum that can be computed, but just a notation called "formal series". Its support is defined as $\{i \in I \mid \varphi_i \neq 0\}$. In additional for any $J \subset I$ that contains this support, we also denote $\sum_{j\in J} \varphi_j := (\varphi_i)_{i\in I}$.

There is a canonical injection $\bigoplus E_i \to \sum E_i$, that maps any sum $\sum_{i\in I} \varphi_i$ with finite support (and $\forall i,\ \varphi_i \in E_i$) to the formal series $\sum_{i\in I} \varphi_i$. So we can consider in practice that $\bigoplus E_i$ is included in $\sum E_i$.

We will use Notation 2.1 with $I = \mathbb{R}$ and $E_d = \mathcal{A}_d(\dots)$ a space of functions that behave like $\mathbf{r}^d$ defined in Section 2.2. In Section 2.3 we build some operators in the spaces $\mathcal{A}$ that have a translation action on the index d. We say that they have a "degree" (cf. Definition 2.2 and Figure 6). That allows us to naturally extend them to the formal series of the spaces $\mathcal{A}$ via the construction below.

Definition 2.2: operators with a degree
Let F be another vector space and $(F_d)_{d\in\mathbb{R}}$ be a family of subspaces of F s.t. the sum $\sum F_d$ is direct. Let $f : \bigoplus E_d \to \bigoplus F_d$ be a linear map and let $d_0 \in \mathbb{R}$.
We say that f has degree d_0 iff: $\forall d \in \mathbb{R}$, $\forall \varphi \in E_d$, $f(\varphi) \in F_{d+d_0}$. In this case we denote $\deg f := d_0$ and we extend f from $\sum E_d$ to $\sum F_d$ by setting $f\big(\sum_{d\in\mathbb{R}} \varphi_d\big) := \sum_{d\in\mathbb{R}} f(\varphi_d)$ for any (φ_d).

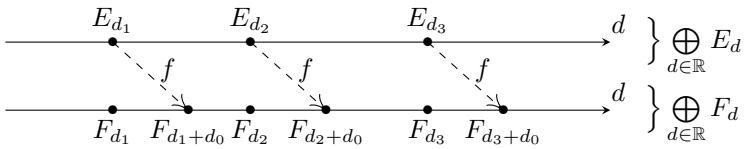

Figure 6: Schematic illustration of an operator that has degree d_0 (here $d_0 > 0$)

Definition 2.3: Let $\sum_{d\in\mathbb{R}} \varphi_d$ be an element of $\sum E_d$ with a support bounded from below and $d_{\inf} := \inf \operatorname{supp}(\varphi_d)$. For any linear map $f : \bigoplus E_d \to \bigoplus E_d$ that has a positive degree, we define

$$\sum_{n=0}^{\infty} f^n\Big(\sum_{d\in\mathbb{R}} \varphi_d\Big) := \sum_{d\in\mathbb{R}} \sum_{n\in\mathbb{N}} f^n(\varphi_{d-n\deg f}) \tag{2.3}$$

(where f^n is the n-th iterated composition of f). See Figure 7. More generally, for any finite set $\mathcal{F}$ of linear maps $\bigoplus E_d \to \bigoplus E_d$ that have positive degrees, we set

$$\sum_{n=0}^{\infty} \sum_{(f_1,\dots,f_n)\in\mathcal{F}^n} f_1\circ\cdots\circ f_n\Big(\sum_{d\in\mathbb{R}} \varphi_d\Big) := \sum_{d\in\mathbb{R}} \sum_{n\in\mathbb{N}} \sum_{(f_1,\dots,f_n)\in\mathcal{F}^n} f_1\circ\cdots\circ f_n(\varphi_{d-\Sigma_{1\leqslant i\leqslant n}\deg f_i}). \tag{2.4}$$

We also denote it as $\langle\mathcal{F}\rangle\big(\sum\varphi_d\big)$, or $\langle\tilde f_1,\dots,\tilde f_k\rangle\big(\sum\varphi_d\big)$ if $\mathcal{F} = \{\tilde f_1,\dots,\tilde f_k\}$. This is well-defined because, for any d, the sums over n in the right-hand sides of (2.3)–(2.4) have a finite number of non-zero terms and belong to E_d.
If $\operatorname{supp}(\varphi_d)$ is bounded from above and the elements of $\mathcal{F}$ have a negative degree, we can do the same definition.

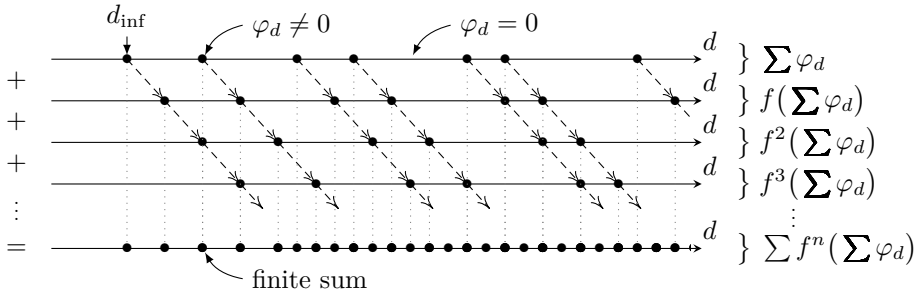

Figure 7: Schematic illustration of $\sum_{n=0}^{\infty} f^n$ for f a linear map that has a positive degree

In Section 2.5, we introduce formal series with powers of ε in order to express the matching condition. They are defined similarly to Notation 2.1: for any set $P \subset \mathbb{R}$ and any family $(\tilde E_p)_{p\in P}$ of vector spaces, we denote

$$\forall(\varphi_p) \in \prod_{p\in P} \tilde E_p, \quad \sum_{p\in P} \varepsilon^p\varphi_p := (\varphi_p) \qquad \text{and} \qquad \sum_{p\in P} \varepsilon^p \tilde E := \prod_{p\in P} \tilde E_p. \tag{2.5}$$

Again this is only a notation and here ε is not a real number but an algebraic indeterminate. This is similar to Notation 2.1 and we will later choose to use either notation depending on the physical meaning of the formal series.

For instance, the case $\tilde E = \mathbb{C}$ and $P = \mathbb{N}$ gives the classical set of formal power series, usually denoted $\mathbb{C}[[\varepsilon]]$ (see [30, 32]). The Taylor approximations at 0 of any smooth function f can be represented by $\sum_{p=0}^{\infty} \varepsilon^p \frac{f^{(p)}(0)}{p!} \in \mathbb{C}[[\varepsilon]]$. Truncations of this series give approximations at a given order. We will use similar representations for the asymptotic expansion of u_ε.

Let us take $P := \mathbb{R}$, $\tilde E := \sum_{d\in\mathbb{R}} E_d$ and $\tilde E_p := \tilde E$ for any p. Let $p_0 \in \mathbb{R}$ and $f : \bigoplus E_d \to \bigoplus E_d$ a linear

map that has a degree d_0. For any $(\varphi_p) \in \tilde{E}^{\mathbb{R}}$, we define:

$$(\varepsilon^{p_0} f)\Big(\sum_{p\in\mathbb{R}} \varepsilon^p \varphi_p\Big) := \sum_{p\in\mathbb{R}} \varepsilon^p f(\varphi_{p-p_0}) \tag{2.6}$$

These kind of linear maps $\sum_p \varepsilon^p \tilde{E} \to \sum_p \varepsilon^p \tilde{E}$ are the one said to have a degree. We denote $\deg(\varepsilon^{p_0} f) := (p_0, d_0)$.

Definition 2.4: Let $\sum_{(p,d)\in\mathbb{R}^2} \varepsilon^p \varphi_{p,d} \in \sum_{p\in\mathbb{R}} \varepsilon^p \tilde{E}$. Let $\mathcal{G}$ be a finite set of linear maps $\sum \varepsilon^p \tilde{E} \to \sum \varepsilon^p \tilde{E}$ that have degrees. There is a finite set $\mathcal{F}$ of linear maps $\bigoplus E_d \to \bigoplus E_d$ and $(p_f)_f \in \mathbb{R}^{\mathcal{F}}$ s.t. $\mathcal{G} = \{\varepsilon^{p_f} f \mid f \in \mathcal{F}\}$. We assume that there is $v \in \mathbb{R}^2$ s.t. $\{\langle (p,d), v\rangle \mid (p,d) \in \mathbb{R}^2, \varphi_{p,d} \neq 0\}$ is bounded from below and $\forall g \in \mathcal{G}$, $\langle \deg g, v\rangle > 0$. We denote:

$$\sum_{n=0}^{\infty} \sum_{(g_1,\dots,g_n)\in\mathcal{G}^n} g_1 \circ \cdots \circ g_n \Big(\sum_{(p,d)\in\mathbb{R}^2} \varepsilon^p \varphi_{p,d}\Big) := \sum_{(p,d)\in\mathbb{R}^2} \varepsilon^p \sum_{n\in\mathbb{N}} \sum_{(f_1,\dots,f_n)\in\mathcal{F}^n} f_1 \circ \cdots \circ f_n(\varphi_{p-\sum_i p_{f_i}, d-\sum_i \deg f_i}).$$

which is well-defined in $\sum \varepsilon^p \tilde{E}$. We also denote it as $\langle \mathcal{G}\rangle\big(\sum_{p,d} \varepsilon^p \varphi_{p,d}\big)$.

Definition 2.5: Let "$\ln \varepsilon$" be here an algebraic indeterminate independent from the indeterminate ε. We denote $E[\ln \varepsilon]$ the set of polynomials with coefficients in E. More precisely it is the set of elements of $E^{\mathbb{N}}$ with finite support and, for any $(\varphi_\ell) \in E[\ln \varepsilon]$, we denote $\sum_{\ell\in\mathbb{N}} \ln^\ell \varepsilon\, \varphi_\ell := (\varphi_\ell)$.

2.2 Definition of the spaces $\mathcal{A}$

In [14, p.10], Costabel and Dauge build a similar asymptotic expansion for the Poisson equation in the half plane with mixed boundary condition: Neumann in a part of the boundary and Robin $u + \varepsilon \partial_n u = 0$ in another. They quickly mention that their singularities can be written as $\Re[(-z)^q P(\ln(-z))]$ with $z = x + \mathrm{i}y$, $q \in \mathbb{R}$ and P a real polynomial. To define the spaces $\mathcal{A}$, we adapted this idea to take into account the layer, the angle Θ and the Helmholtz equation. These simple expressions give both powerful algebraic tools for the theory and fast precise algorithms for the numerical resolution (see Section 2.3).

Definition 2.6: the spaces $\mathcal{A}$
Let $\alpha := e^{-\mathrm{i}\Theta}$. We define in Ω the complex variable $z := x + \mathrm{i}y = re^{\mathrm{i}\theta}$. For all $q \in \mathbb{R}$, we take the following conventions: $(\alpha z)^q := r^q e^{q\mathrm{i}(\theta-\Theta)}$, $\overline{\alpha z}^q := r^q e^{-q\mathrm{i}(\theta-\Theta)}$ and $\log(\alpha z) := \ln r + \mathrm{i}(\theta - \Theta)$. Let $d \in \mathbb{R}$. We denote:

- $\mathcal{A}_d(\Omega)$ the vector subspace of $\mathcal{C}^0(\Omega, \mathbb{C})$ generated by the functions $z \mapsto \Im[(\alpha z)^q\, \overline{\alpha z}^k P(\log(\alpha z))]$ with $q \in \mathbb{R}$, $k \in \mathbb{N}$, $q + k = d$ and $P \in \mathbb{R}[T]$,
- $\mathcal{A}_d(\Lambda) := \{(x, Y) \mapsto x^d\, Q(\ln x, Y) \mid Q \in \mathbb{C}[T, Y] \text{ and } Q(T, -1) = 0\}$,
- $\mathcal{A}_d(\Gamma) := \{x \mapsto x^d\, Q(\ln x) \mid Q \in \mathbb{C}[T]\}$,
- $\mathcal{A}_d(\Pi) := \{\varphi \in \mathcal{C}^0(\Pi, \mathbb{C}) \mid \varphi_{|\Omega} \in \mathcal{A}_d(\Omega) \text{ and } \varphi_{|\Lambda} \in \mathcal{A}_d(\Lambda)\}$,
- and for any $D \in \{\Pi, \Omega, \Lambda, \Gamma\}$, $\mathcal{A}(D) := \sum_{d\in\mathbb{R}} \mathcal{A}_d(D)$ (cf. the introduction of Section 2.1).

Note that elements of $\mathcal{A}(\Omega)$ and $\mathcal{A}(\Pi)$ vanish on Σ_Ω, and elements of $\mathcal{A}(\Lambda)$ and $\mathcal{A}(\Pi)$ vanish on Σ_Λ. In addition, elements of $\mathcal{A}(\Omega)$ are naturally functions depending on the polar coordinates. For instance:

- $\Im[(\alpha z)^q\, \overline{\alpha z}^k] = r^{q+k} \sin\big((q-k)(\theta - \Theta)\big)$
- $\Im[(\alpha z)^q\, \overline{\alpha z}^k \log(\alpha z)] = r^{q+k} \big[\ln(r)\, \cos\big((q-k)(\theta-\Theta)\big) - (\theta - \Theta) \sin\big((q-k)(\theta-\Theta)\big)\big]$

Note also that in this definition we used the variables x, y, which are relevant for far fields, but all the tools developed in this section can also be used for corner fields, replacing (x, y) by (X, Y).

Remark 2.7: Definition 2.6 can be extended to the case where Σ_Ω is covered by another layer, by defining $\mathcal{A}_d(\Omega)$ as the vector space generated by the functions $z \mapsto \Im[(\alpha z)^q\,\overline{\alpha z}^k P(\log(\alpha z))]$ and $z \mapsto \Im[z^q \bar{z}^k P(\log z)]$ with $q \in \mathbb{R}$, $k \in \mathbb{N}$, $q+k=d$ and $P \in \mathbb{R}[T]$.

In order to build particular solutions of PDEs in $\mathcal{A}$, we will need the three following lemmas. The proof of the first one can be found in Appendix A.

Lemma 2.8: For any $D \in \{\Pi, \Omega, \Gamma, \Lambda\}$, we have the following decomposition: $\mathcal{A}(D) = \bigoplus_{d\in\mathbb{R}} \mathcal{A}_d(D)$. Furthermore, for any $d \in \mathbb{R}$, $\mathcal{A}_d(\Omega)$ can itself be decomposed as follows:

$$\mathcal{A}_d(\Omega) = \bigoplus_{\substack{(q,k)\in\mathbb{R}\times\mathbb{N}\\ q+k=d}} \left\{ z \mapsto \lambda\,\Im[(\alpha z)^q\,\overline{\alpha z}^k P(\log(\alpha z))] \;\middle|\; \lambda \in \mathbb{C}, P \in \mathbb{R}[T] \text{ and } \mathcal{P}(q,k,P) \right\} \tag{2.7}$$

where $\mathcal{P}$ is the property defined by $\mathcal{P}(q,k,P) := (q \notin \mathbb{N} \text{ or } q > k \text{ or } P(0)=0)$.

Remark: The condition $\mathcal{P}$ is a way to exclude the functions $z \mapsto \Im[(\alpha z)^q \overline{\alpha z}^k]$ with $q \in \mathbb{N}$ and $q < k$, which are already present in the direct sum as they are equal to $z \mapsto -\Im[(\alpha z)^k \overline{\alpha z}^q]$.

Let φ be a function of $\mathcal{A}(\Omega)$ of the form $\Im[(\alpha z)^q\,\overline{\alpha z}^k P(\log(\alpha z))]$. Note that on Γ, φ is equal to $x^{q+k}\Im[\alpha^{q-k}P(\ln x - \mathrm{i}\Theta)]$. Let us define $\Im[\alpha^{q-k}P(T-\mathrm{i}\Theta)] := \sum_{i=1}^{\deg P} \Im(a_i)T^i$ in $\mathbb{C}[T]$, where $\sum_{i=1}^{\deg P} a_i T^i := \alpha^{q-k}P(T-\mathrm{i}\Theta)$. Then $\varphi_{|\Gamma}(x) = x^{q+k}Q(\ln x)$ for some $Q \in \mathbb{R}[X]$, which implies $\varphi_{|\Gamma} \in \mathcal{A}(\Gamma)$. Conversely, for future constructions, it will be important to solve the equation:

$$\text{given } Q \in \mathbb{R}[T],\ \text{find } P \in \mathbb{R}[T]\ \text{ s.t. }\ \Im[\alpha^d P(T-\mathrm{i}\Theta)] = Q(T). \tag{2.8}$$

Lemma 2.9: Let $\alpha := e^{-\mathrm{i}\Theta}$, $d \in \mathbb{R}$ and $Q \in \mathbb{R}[T]$.

1. If $d \in \mathbb{R} \setminus \frac{\pi}{\Theta}\mathbb{Z}$, then there is a unique solution $P \in \mathbb{R}[T]$ of (2.8). Moreover $\deg P = \deg Q$. We denote the solution $L_d(Q)$.
2. If $d \in \frac{\pi}{\Theta}\mathbb{Z}$, then the set of solutions of (2.8) is of the form $\{P_0 + c \mid c \in \mathbb{R}\}$ with $P_0 \in \mathbb{R}[T]$ and $\deg(P_0) = \deg(Q)+1$. We denote $L_d(Q)$ the unique solution that vanishes at 0.

In both cases, L_d is a linear map from $\mathbb{R}[T]$ into itself.

Proof: There are two cases whether the coefficient of degre m of $\Im[\alpha^d(T-\mathrm{i}\Theta)^m]$ vanishes or not.

1. If $d \in \mathbb{R} \setminus \frac{\pi}{\Theta}\mathbb{Z}$, then $\alpha^d \in \mathbb{C}\setminus\mathbb{R}$, so: $\forall m \in \mathbb{N}$, $\deg \Im[\alpha^d(T-\mathrm{i}\Theta)^m] = m$. Therefore, $(\Im[\alpha^d(T-\mathrm{i}\Theta)^m])_{m\in\mathbb{N}}$ is a basis of $\mathbb{R}[T]$. So writing Q in this basis gives a unique solution of (2.8).
2. If $d \in \frac{\pi}{\Theta}\mathbb{Z}$, since $\alpha^d \in \mathbb{R}$, we have: $\forall m \in \mathbb{N}$, $\deg \Im[\alpha^d(T-\mathrm{i}\Theta)^m] = m-1$. So in this case $(\Im[\alpha^d(T-\mathrm{i}\Theta)^m])_{m\in\mathbb{N}^*}$ is a basis of $\mathbb{R}[T]$. Thus, $F : P \mapsto \Im[\alpha^d P(T-\mathrm{i}\Theta)]$ is surjective and its kernel is the set of constant polynomials. Its restriction to $E := \{P \in \mathbb{R}[T] \mid P(0)=0\}$ is therefore an isomorphism and we set $L_d := (F_{|E})^{-1}$. Finally, for any $Q \in \mathbb{R}[T]$ we have $F^{-1}(\{Q\}) = \{L_d(Q)\} + \operatorname{Ker} F$. □

For any $q \in \mathbb{R}$, the maps $Q \in \mathbb{C}[T] \mapsto \Im[x^q Q(\ln x)] \in \mathcal{A}(\Gamma)$ and $Q \in \mathbb{C}[T,Y] \mapsto \Im[x^q Q(\ln x, Y)] \in \mathcal{A}(\Lambda)$ are clearly injective. In the following lemma we investigate the injectivity of $P \in \mathbb{R}[T] \mapsto \Im[(\alpha z)^q\,\overline{\alpha z}^k P(\log(\alpha z))] \in \mathcal{A}(\Omega)$.

Lemma 2.10: Let $(q,k) \in \mathbb{R}\times\mathbb{N}$. The map $P \mapsto \Im[(\alpha z)^q\,\overline{\alpha z}^k P(\log(\alpha z))]$ is injective from the set of real polynomials P for which $\mathcal{P}(q,k,P)$ is true into $\mathcal{A}(\Omega)$.

Proof: For any $\theta \in (0,\Theta)$ and $r \in \mathbb{R}_+^*$, we have $\varphi(re^{\mathrm{i}\theta}) = r^{q+k}\Im[e^{\mathrm{i}(q-k)(\theta-\Theta)}P(\ln r + \mathrm{i}(\theta-\Theta))] = r^{q+k}\Im[\alpha'^{q-k}P(\ln r - \mathrm{i}\Theta')]$ with $\Theta' := \Theta - \theta$ and $\alpha' := e^{-\mathrm{i}\Theta'}$.

- If $q \neq k$, we can choose θ so that $q-k \in \mathbb{R} \setminus \frac{\pi}{\Theta'}\mathbb{Z}$. So Lemma 2.9 applied to (Θ', α') instead of (Θ, α) implies that P is unique.
- Otherwise, $q-k=0 \in \frac{\pi}{\Theta'}\mathbb{Z}$ for any θ. So according to 2.9, P is unique up to a constant a priori. But the property $\mathcal{P}$ implies that $P(0)=0$, so this constant is fixed. □

2.3 Tools for solving the Poisson and Helmholtz equations in the spaces $\mathcal{A}$

In this section, we show how to solve canonical problems set in Π in the spaces $\mathcal{A}$. More precisely, let $(\psi_\Omega, \psi_\Lambda, \psi_\Gamma) \in \mathcal{A}(\Omega) \times \mathcal{A}(\Lambda) \times \in \mathcal{A}(\Gamma)$, we look for the solutions $\varphi \in \mathcal{A}(\Pi)$ of: problems of the form

$$\left\{ \begin{array}{rl} \Delta\varphi = \psi_\Omega & \text{in } \Omega \\ \partial_Y^2 \varphi = \psi_\Lambda & \text{in } \Lambda \\ \partial_Y \varphi_{|Y=0^-} = \psi_\Gamma & \text{on } \Gamma \end{array} \right. \tag{2.9}$$

Note that by definition of $\mathcal{A}(\Pi)$, φ also satisfies $\varphi_{|y=0^+} - \varphi_{|Y=0^-}$ on Γ and $\varphi_{|\Sigma_\Omega \cup \Sigma_\Lambda} = 0$. Solving this system will enable us to build in Section 2.4 the asymptotic expansion of $\mathbf{u}_{p,\ell}$ and $S_{p,\ell}$. Indeed, note for instance that this system is identical to (2.1) except for the first line. We first describe the homogeneous solutions of (2.9), then build explicitly some particular solutions. Since functions of $\mathcal{A}$ are uniquely determined by some polynomials (see Lemma 2.10), we are able to code an *exact, very fast and memory-thrifty* solver of (2.9). This is one of the key advantages of the $\mathcal{A}$ framework.

Definition 2.11: For any $d \in \frac{\pi}{\Theta}\mathbb{Z}^*$, we define on Ω the function $\phi_d^\Omega := (-1)^{d\Theta/\pi} r^d \sin(d\theta) = r^d \sin\big(d(\theta-\Theta)\big) = \Im[(\alpha z)^d] \in \mathcal{A}_d(\Omega)$ and $\phi_d \in \mathcal{A}_d(\Pi)$ its extension by 0 in Λ.

These functions play an important role in the sequel because they solve the homogeneous Laplace equation in Ω, resp. Π.

Proposition 2.12: the Laplace problem in $\mathcal{A}$

1. $\mathrm{Span}(\{\phi_d^\Omega \mid d \in \frac{\pi}{\Theta}\mathbb{Z}^*\})$ is the set of solutions in $\mathcal{A}(\Omega)$ of

$$\left\{ \begin{array}{rl} \Delta\varphi = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \Gamma \cup \Sigma_\Omega \end{array} \right. \tag{2.10}$$

2. $\mathrm{Span}(\{\phi_d \mid d \in \frac{\pi}{\Theta}\mathbb{Z}^*\})$ is the set of solutions in $\mathcal{A}(\Pi)$ of

$$\left\{ \begin{array}{rl} \Delta\varphi = 0 & \text{in } \Omega \\ \partial_Y^2 \varphi = 0 & \text{in } \Lambda \\ \partial_Y \varphi_{|Y=0^-} = 0 & \text{on } \Gamma \end{array} \right. \tag{2.11}$$

Proof:

1. ϕ_d^Ω is clearly solution of (2.10) for any $d \in \frac{\pi}{\Theta}\mathbb{Z}^*$. Conversely, let φ be a solution. Let us denote

$$\forall d \in \tfrac{\pi}{\Theta}\mathbb{N}^*,\ \forall r \in \mathbb{R}_+^*, \quad c_d(r) := \frac{2}{\Theta}\int_0^\Theta \varphi(r,\theta)\,\sin(d\theta)\,\mathrm{d}\theta.$$

Using separation of variables and $\varphi_{|\partial\Omega} = 0$, it is easy to show that $\varphi = \sum_{d \in \frac{\pi}{\Theta}\mathbb{N}^*} c_d(r)\sin(d\theta)$ with convergence in $H^2(\Omega \cap \{r_1 < r < r_2\})$ for any $0 < r_1 < r_2 < \infty$. Since $\Delta\varphi = 0$, we have $(r\frac{\mathrm{d}}{\mathrm{d}r})^2 c_d = d^2 c_d$ for any d. Hence we get: $\forall d \in \frac{\pi}{\Theta}\mathbb{N}^*, \exists a_d, a_{-d} \in \mathbb{C}, \forall r \in \mathbb{R}_+^*,\ c_d(r) = a_d r^d + a_{-d} r^{-d}$. Moreover, by definition of $\mathcal{A}(\Omega)$, there is $q \in \mathbb{R}_+^*$ s.t. $\varphi \in \sum_{d\in[-q,q]} \mathcal{A}_d(\Omega)$. So

$$\forall d \in \tfrac{\pi}{\Theta}\mathbb{Z}^* \cap (q,\infty), \quad a_d = \lim_{r\to\infty} r^{-d} c_d(r) = \lim_{r\to\infty} \frac{2}{\Theta}\int_0^\Theta \underbrace{r^{-d}\varphi(r,\theta)}_{\longrightarrow 0}\,\sin(d\theta)\,\mathrm{d}\theta = 0.$$

Similarly, looking at $r \to 0$ one gets: $\forall d \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap (-\infty, -q),\ a_d = 0$. So $\varphi = \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap [-q,q]} a_d \,\mathrm{sgn}(d)\,\phi_d^\Omega$ (where $\mathrm{sgn}(d) := d/|d|$), which is a finite sum. Therefore, φ is in the desired span.

2. Any solution of the system vanishes in Λ, so point 2 easily follows from point 1. □

Let us now define the following linear forms σ_d which satisfy $\sigma_d(\phi_q) = \delta_{d,q}$ for any $d, q \in \frac{\pi}{\Theta}\mathbb{Z}^*$ and which enable us to "project" any element of $\mathcal{A}(\Pi)$ on $\mathrm{Span}(\{\phi_d \mid d \in \frac{\pi}{\Theta}\mathbb{Z}^*\})$. These linear forms appear later as key singularity coefficients in the matching condition.

Definition 2.13: linear forms σ_d
Let $d \in \frac{\pi}{\Theta}\mathbb{Z}^*$. For any $(q,k,P) \in \mathbb{R}\times\mathbb{N}\times\mathbb{R}[T]$ s.t. $\mathcal{P}(q,k,P)$ is true and $\varphi : z \mapsto \Im[(\alpha z)^q\,\overline{\alpha z}^k P(\log(\alpha z))]$, let us define:

$$\begin{cases} \sigma_d(\varphi) = 0 & \text{if } q \neq d \text{ or } k \neq 0 \\ \sigma_d\big(\Im[(\alpha z)^d P(\log(\alpha z))]\big) = P(0) & \text{otherwise} \end{cases}$$

It is well-defined by Lemma 2.10. By Lemma 2.8, σ_d can be extended into a linear form $\mathcal{A}(\Omega) \to \mathbb{C}$. Finally, for any $\varphi \in \mathcal{A}(\Pi)$, $\sigma_d(\varphi) := \sigma_d(\varphi_{|\Omega})$.

Let us now build particular solutions of (2.9). By linearity, it suffices to build particular solutions of three sub-problems. According to Definition 2.2, for any $D_1, D_2 \in \{\Pi, \Omega, \Gamma, \Lambda\}$, we say that a linear map $F : \mathcal{A}(D_1) \to \mathcal{A}(D_2)$ has degree $d \in \mathbb{R}$ iff: $\forall q \in \mathbb{R}$, $\forall \varphi \in \mathcal{A}_q(D_1)$, $F(\varphi) \in \mathcal{A}_{q+d}(D_2)$.

Proposition 2.14: particular solutions of (2.9)
Let us denote $\mathcal{A}_\perp(\Pi) := \{\varphi \in \mathcal{A}(\Pi) \mid \forall d \in \frac{\pi}{\Theta}\mathbb{Z}^*,\ \sigma_d(\varphi) = 0\}$, which is a supplementary of $\mathrm{Span}(\{\varphi_d \mid d \in \frac{\pi}{\Theta}\mathbb{Z}^*\})$ in $\mathcal{A}(\Pi)$.

1. For any $\psi_\Omega \in \mathcal{A}(\Omega)$ there exists a unique solution $\varphi_\Delta \in \mathcal{A}_\perp(\Pi)$ of

$$\begin{cases} \Delta\varphi = \psi_\Omega & \text{in } \Omega \\ \partial_Y^2\varphi = 0 & \text{in } \Lambda \\ \partial_Y\varphi_{|Y=0^-} = 0 & \text{on } \Gamma \end{cases} \tag{2.12}$$

   The associated map $R_\Delta : \psi_\Omega \in \mathcal{A}(\Omega) \mapsto \varphi_\Delta \in \mathcal{A}_\perp(\Pi)$ is linear and has degree 2.

2. For any $\psi_\Lambda \in \mathcal{A}(\Lambda)$ there exists a unique solution $\varphi_{\partial_Y^2} \in \mathcal{A}_\perp(\Pi)$ of

$$\begin{cases} \Delta\varphi = 0 & \text{in } \Omega \\ \partial_Y^2\varphi = \psi_\Lambda & \text{in } \Lambda \\ \partial_Y\varphi_{|Y=0^-} = 0 & \text{on } \Gamma \end{cases} \tag{2.13}$$

   The associated map $R_{\partial_Y^2} : \psi_\Lambda \in \mathcal{A}(\Lambda) \mapsto \varphi_{\partial_Y^2} \in \mathcal{A}_\perp(\Pi)$ is linear and has degree 0.

3. For any $\psi_\Gamma \in \mathcal{A}(\Gamma)$ there exists a unique solution $\varphi_\mathrm{N} \in \mathcal{A}_\perp(\Pi)$ of

$$\begin{cases} \Delta\varphi = 0 & \text{in } \Omega \\ \partial_Y^2\varphi = 0 & \text{in } \Lambda \\ \partial_Y\varphi_{|Y=0^-} = \psi_\Gamma & \text{on } \Gamma \end{cases} \tag{2.14}$$

   The associated map $R_\mathrm{N} : \psi_\Gamma \in \mathcal{A}(\Gamma) \mapsto \varphi_\mathrm{N} \in \mathcal{A}_\perp(\Pi)$ is linear and has degree 0.

Using Propositions 2.12 and 2.14, it is then easy to see that the set of solutions of (2.9) is $R_\Delta(\varphi_\Omega) + R_{\partial_Y^2}(\varphi_\Lambda) + R_\mathrm{N}(\varphi_\Gamma) + \mathrm{Span}(\{\phi_d \mid d \in \frac{\pi}{\Theta}\mathbb{Z}^*\})$. Moreover, the functions φ_Δ, $\varphi_{\partial_Y^2}$ and φ_N in Proposition 2.14 have explicit expressions (see the proof below), which allows to compute them easily in practice.

Proof: Proposition 2.12 gives the uniqueness of the solutions φ_Δ, $\varphi_{\partial_Y^2}$ and φ_N, so only their existence remains to prove. This is done by a construction. For any $D \in \{\Pi, \Omega, \Lambda, \Gamma\}$, let $\mathcal{A}(D,\mathbb{R}) := \mathcal{A}(D) \cap \mathcal{C}^0(D,\mathbb{R})$. Since $\mathcal{A}(D) = \mathcal{A}(D,\mathbb{R}) \oplus \mathrm{i}\mathcal{A}(D,\mathbb{R})$, it suffices to build the solutions when $(\psi_\Omega, \psi_\Lambda, \psi_\Gamma) \in \mathcal{A}(\Omega,\mathbb{R}) \times \mathcal{A}(\Lambda,\mathbb{R}) \times \mathcal{A}(D,\Gamma)$, and then extend it to any source term by complexification.

1. According to Lemma 2.8, it suffices to build φ_Δ when $\psi_\Omega = \Im[(\alpha z)^q\,\overline{\alpha z}^k P_\psi(\log(\alpha z))]$ with $(q,k,P_\psi) \in \mathbb{R}\times\mathbb{N}\times\mathbb{R}[T]$ s.t. $\mathcal{P}(q,k,P_\psi)$ is true. First, $\varphi_{\Delta|\Lambda} = 0$ because φ_Δ satisfies:

$$\begin{cases} \partial_Y^2\varphi_\Delta = 0 & \text{in } \Lambda \\ \partial_Y\varphi_{\Delta|Y=0^-} = 0 & \text{on } \Gamma \\ \varphi_\Delta = 0 & \text{on } \Sigma_\Lambda \end{cases}$$

Given that $\Delta = 4\partial_z\partial_{\bar z}$, we have for any $\varphi_1 : z \mapsto \Im[(\alpha z)^{q_1}\,\overline{\alpha z}^{k_1}P_1(\log(\alpha z))]$ that

$$\Delta\varphi_1 = 4\,\Im[(\alpha z)^{q_1-1}\,k_{1\overline{\alpha z}^{k_1-1}}(q_1P_1 + P_1')(\log(\alpha z))].$$

So taking $q_1 := q+1$, $k_1 := k+1$ and $P_1 \in \mathbb{R}[T]$ a solution of $4k_1(q_1P_1 + P_1') = P_\psi$, we have $\Delta\varphi_1 = \psi_\Omega$ in Ω.

- If $q_1 = 0$, P_1 is unique up to a constant, Moreover we can write

$$\varphi_1 = \Im\big[\overline{\alpha z}^{k_1}\big(P_1(\log(\alpha z)) - P_1(0)\big)\big] - P_1(0)\,\Im[(\alpha z)^{k_1}]$$

  where each term satisfy the property $\mathcal{P}$. So for any $d \in \frac{\pi}{\Theta}\mathbb{Z}^*$ different from k_1, we have $\sigma_d(\varphi_1) = 0$, while $\sigma_{k_1}(\varphi_1) = -P_1(0)$ if $k_1 \in \mathbb{N}\cap\frac{\pi}{\Theta}\mathbb{Z}^*$. Taking $P_1(0) := 0$ thus gives: $\forall d \in \frac{\pi}{\Theta}\mathbb{Z}^*,\ \sigma_d(\varphi_1) = 0$.
- If $q_1 \neq 0$, there is a unique solution P_1. Given that $q_1 \neq 0$ and $k_1 \neq 0$, Definition 2.13 implies that: $\forall d \in \frac{\pi}{\Theta}\mathbb{Z}^*,\ \sigma_d(\varphi_1) = 0$.

However, we cannot set $\varphi_{\Delta|\Omega} = \varphi_1$, because φ_1 does not vanish on Γ. Let us then introduce $\varphi_2 : z \mapsto \Im[(\alpha z)^{q+k+2}P_2(\log(\alpha z))]$ with $P_2(T) := L_{q+k+2}\big(\Im[\alpha^{q-k}P_1(T - \mathrm{i}\Theta)]\big)$ that satisfies by Lemma 2.9:

$$\begin{cases} \Delta\varphi_2 = 0 & \text{in } \Omega \\ \varphi_2 = x^{q+k+2}\Im[\alpha^{q+k+2}P_2(\ln x - \mathrm{i}\Theta)] = x^{q+k+2}\Im[\alpha^{q-k}P_1(\ln x - \mathrm{i}\Theta)] = \varphi_1 & \text{on } \Gamma \end{cases}$$

In addition Lemma 2.9 implies that $q+k+2 \in \frac{\pi}{\Theta}\mathbb{Z}^* \Rightarrow P_2(0) = 0$, so: $\forall d \in \frac{\pi}{\Theta}\mathbb{Z}^*,\ \sigma_d(\varphi_2) = 0$. Finally we set $\varphi_{\Delta|\Omega} := \varphi_1 - \varphi_2$, which is in $\mathcal{A}_\perp(\Pi)$.

2. Similarly, it suffices to build $\varphi_{\partial_Y^2}$ for $\psi_\Lambda = x^q\,Q_\psi(\ln x, Y)$. We look for $\varphi_{\partial_Y^2}$ of the form $x^q\,Q_\varphi(\ln x, Y)$ in Λ with $Q_\varphi \in \mathbb{R}[T,Y]$. Then necessarily we have

$$\begin{cases} \partial_Y^2 Q_\varphi = Q_\psi \\ \partial_Y Q_\varphi(\cdot,0) = 0 \\ Q_\varphi(\cdot,-1) = 0 \end{cases}$$

This uniquely defines Q_φ. Taking $P_\varphi := L_q(Q_\varphi(\cdot,0))$ and $\varphi_{\partial_Y^2|\Omega} := \Im[(\alpha z)^q P_\varphi(\log(\alpha z))]$ then implies that $\Delta\varphi_{\partial_Y^2} = 0$ in Ω, φ continuous and $\varphi_{\partial_Y^2} \in \mathcal{A}_\perp(\Pi)$.

3. Again it suffices to consider $\psi_\Gamma = x^q\,P_\psi(\ln x)$ with $P_\psi \in \mathbb{R}[T]$. We take φ_{N} of the same form as in point 2. Then $\partial_Y\varphi_{\mathrm{N}|Y=0^-} = x^q\,\partial_Y Q_\varphi(\ln x, 0)$ so it suffices to take the polynomial solutions of:

$$\begin{cases} \partial_Y^2 Q_\varphi = 0 \\ \partial_Y Q_\varphi(\cdot,0) = P_\psi \\ Q_\varphi(\cdot,-1) = 0 \\ P_\varphi = L_q(Q_\varphi(\cdot,0)) \end{cases}$$

Finally the linearity of Problems 2.12–2.14 and the uniqueness of φ_Δ, $\varphi_{\partial_Y^2}$ and φ_{N} imply that R_Δ, $R_{\partial_Y^2}$ and R_{N} are linear maps. □

Moreover, we will need analogous operators in $\mathcal{A}(\Omega)$ to build the far fields in Theorem 3.6. The proof is entirely similar to the one of Proposition 2.14, so we omit it. Again, these operators have explicit expressions.

Proposition 2.15: Let us denote $\mathcal{A}_\perp(\Omega) = \{\varphi \in \mathcal{A}(\Omega) \mid \forall d \in \frac{\pi}{\Theta}\mathbb{Z}^*,\ \sigma_d(\varphi) = 0\}$, which is a supplementary of $\mathrm{Span}(\{\phi_d^\Omega \mid d \in \frac{\pi}{\Theta}\mathbb{Z}^*\})$ in $\mathcal{A}(\Omega)$.

1. For any $\psi_\Omega \in \mathcal{A}(\Gamma)$ there exists a unique solution $\varphi_\Delta \in \mathcal{A}_\perp(\Omega)$ of

$$\begin{cases} \Delta\varphi = \psi_\Omega & \text{in } \Omega \\ \varphi = 0 & \text{on } \Gamma \end{cases} \tag{2.15}$$

   The associated map $R_\Delta^\Omega : \psi_\Gamma \in \mathcal{A}(\Omega) \mapsto \varphi_\Delta \in \mathcal{A}_\perp(\Omega)$ is linear and has degree 2.

2. For any $\psi_\Gamma \in \mathcal{A}(\Gamma)$ there exists a unique solution $\varphi_\mathrm{D} \in \mathcal{A}_\perp(\Omega)$ of

$$\begin{cases} \Delta\varphi = 0 & \text{in } \Omega \\ \varphi = \psi & \text{on } \Gamma \end{cases} \tag{2.16}$$

   The associated map $R_\mathrm{D}^\Omega : \psi_\Gamma \in \mathcal{A}(\Gamma) \mapsto \varphi_\mathrm{D} \in \mathcal{A}_\perp(\Omega)$ is linear and has degree 0.

To end this section, let us show that the spaces $\mathcal{A}$ are stable under some differential operators.

Lemma 2.16:

1. $\partial^2_{x|\Lambda} : \varphi \mapsto \partial_x^2\varphi_{|\Lambda}$ maps $\mathcal{A}(\Pi)$ to $\mathcal{A}(\Lambda)$ and has degree -2 (see Definition 2.2).
2. $\partial_{y|\Gamma,y=0^+} : \varphi \mapsto \partial_y\varphi_{|\Gamma,y=0^+}$ maps $\mathcal{A}(\Pi)$ to $\mathcal{A}(\Gamma)$ and has degree -1.
3. $\Delta : \varphi \mapsto \Delta\varphi_{|\Omega}$ maps $\mathcal{A}(\Omega)$ (and $\mathcal{A}(\Pi)$) to $\mathcal{A}(\Omega)$ and has degree -2.

Note that, when used on corner fields (which depend on (X, Y)), the first two operators will rather be denoted $\partial^2_{X|\Lambda}$ and $\partial_{Y|\Gamma,Y=0^+}$.

Proof: It suffices to verify it when φ has the form $\Im[(\alpha z)^q\,\overline{\alpha z}^k P(\log(\alpha z))]$ in Ω and $x^{q+k}\,Q(\ln x, Y)$ in Λ, with $q \in \mathbb{R}$, $k \in \mathbb{N}$, $P[T]$ and $Q \in \mathbb{R}[T, Y]$.

1. We have $\partial_x^2\varphi(x, Y) = x^{q+k-2}((q+k)(q+k-1) + 2(q+k)\partial_T + \partial_T^2)\,Q(\ln x, Y)$ which is in $\mathcal{A}(\Lambda)$.
2. Let $\psi : z \mapsto (\alpha z)^q\,\overline{\alpha z}^k P(\log(\alpha z))$. From $\partial_z\psi = \frac{1}{2}(\partial_x\psi - \mathrm{i}\partial_y\psi)$ and $\partial_{\bar z}\psi = \frac{1}{2}(\partial_x\psi + \mathrm{i}\partial_y\psi)$ it follows that $\partial_y\varphi = \Im[\partial_y\psi] = \Re[\partial_z\psi - \partial_{\bar z}\psi]$. Therefore,

$$\begin{aligned} \partial_y\varphi_{|\Gamma,y=0^+} &= \Re[\alpha(\alpha x)^{q-1}\,\overline{\alpha x}^k(qP + P')(\log(\alpha x)) - (\alpha x)^q\,\overline{\alpha}\,k\,\overline{\alpha x}^{k-1}P(\log(\alpha x))] \\ &= x^{q+k-1}\Re[\alpha^{q-k}((q-k)P + P')(\ln x - \mathrm{i}\Theta)] \quad \in \mathcal{A}(\Gamma) \end{aligned}$$

3. Since $\Delta = 4\partial_z\partial_{\bar z}$, we have $\Delta\varphi = 4\,\Im[(\alpha z)^{q-1}\,k\overline{\alpha z}^{k-1(qP_1} + P_1')(\log(\alpha z))]$ which is in $\mathcal{A}(\Omega)$. □

2.4 Asymptotic behaviors w.r.t. r of solutions of model problems

In this section we give tools that will be used to compute the asymptotic behaviors of $\mathbf{u}_{p,\ell}$ and $S_{p,\ell}$ resp. when $\mathbf{r} \to 0$ and $\mathbf{r} \to \infty$ using the spaces $\mathcal{A}$. To do so, we will use series of elements of $\mathcal{A}$, which is made rigorous by the following definition.

Definition 2.17: the spaces $\mathcal{A}^\pm$
Let $D \in \{\Pi, \Omega, \Gamma, \Lambda\}$. We denote $\mathcal{A}^+(D)$, resp. $\mathcal{A}^-(D)$, the set of elements of $\sum_{d\in\mathbb{R}} \mathcal{A}_d(D)$ whose support is included in the image of a sequence that tends to ∞, resp. $-\infty$. We write their elements as formal series according to Definition 2.1.

Remarks:

- The asymptotic of $\mathbf{u}_{p,\ell}$ when $\mathbf{r} \to 0$ involve increasing powers of r so it will be expressed in $\mathcal{A}^+(D)$. Similarly, $S_{p,\ell}$ when $\mathbf{r} \to \infty$ involves decreasing powers of r, so it will be expressed in $\mathcal{A}^-(D)$.

- As seen in Section 2.1, $\mathcal{A}(D)$ is included in $\mathcal{A}^+(D)$ and $\mathcal{A}^-(D)$. But elements of $\mathcal{A}^\pm(D)$ are not in general $D \to \mathbb{R}$ functions, as the formal series may diverge pointwise.

Using Definitions 2.2, we can extend $\partial^2_{x|\Lambda}$, $\partial_{y|\Gamma,y=0^+}$, $R_{\partial^2_Y}$, $R_{\rm N}$ and R_Δ to the spaces $\mathcal{A}^\pm$. We also use the notation $\langle . \rangle$ introduced in Definition 2.3. E.g. $\langle -k_0^2\, R_\Delta \rangle = \sum_{n=0}^\infty (-k_0^2\, R_\Delta)^n$. Moreover, we extend σ_d to $\mathcal{A}^\pm(\Pi)$ for any $d \in \frac{\pi}{\Theta}\mathbb{Z}^*$, by setting $\sigma_d(\sum_{d'} \varphi_{d'}) := \sigma_d(\varphi_d)$ for any $\sum_{d'} \varphi_{d'} \in \mathcal{A}^\pm(\Pi)$.

Lemma 2.18: Let $g^0 \in \mathcal{A}^+(\Lambda)$ and $h^0 \in \mathcal{A}^+(\Gamma)$. The solutions in $\mathcal{A}^+(\Pi)$ of

$$\left\{ \begin{array}{rl} \mu_0 \Delta \mathbf{u}^0 + \omega^2 \rho_0 \mathbf{u}^0 = 0 & \text{in } \Omega \\ \mu_1 \partial_Y^2 \mathbf{u}^0 = g & \text{in } \Lambda \\ \mu_1 \partial_{Y|Y=0^-} \mathbf{u}^0 = h & \text{on } \Gamma \end{array} \right. \tag{2.17}$$

are the formal series of the following form (where $\sigma_d(\mathbf{u}^0)$ vanishes when d is small enough)

$$\mathbf{u}^0 = \left\langle -k_0^2\, R_\Delta \right\rangle \left(\frac{1}{\mu_1} R_{\partial^2_Y}(g^0) + \frac{1}{\mu_1} R_{\rm N}(h^0) + \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^*} \sigma_d(\mathbf{u}^0)\, \phi_d \right) \tag{2.18}$$

Proof: Let $\mathbf{v}^0 := (\mathrm{id} + k_0^2 R_\Delta)\mathbf{u}^0$. Since $\langle -k_0^2 R_\Delta \rangle = \sum_{n=0}^\infty (-k_0^2 R_\Delta)^n$, $\mathrm{id} + k_0^2 R_\Delta$ is the inverse of $\langle -k_0^2 R_\Delta \rangle$, so $\mathbf{u}^0 = \langle -k_0^2 R_\Delta \rangle \mathbf{v}^0$. Moreover, Proposition 2.14 states that $\Delta \circ R_\Delta = \mathrm{id}$, $\partial^2_{Y|\Lambda} \circ R_\Delta = 0$ and $\partial_{Y|\Gamma,Y=0^-} \circ R_\Delta = 0$, which imply resp. $(\mu_0 \Delta + \omega^2 \rho_0) \circ \langle -k_0^2 R_\Delta \rangle = \mu_0 \Delta \circ (\mathrm{id} + k_0^2 R_\Delta) \circ \langle -k_0^2 R_\Delta \rangle = \mu_0 \Delta$, $\partial^2_{Y|\Lambda} \circ \langle -k_0^2 R_\Delta \rangle = \partial^2_{Y|\Lambda}$ and $\partial_{Y|\Gamma,Y=0^-} \circ \langle -k_0^2 R_\Delta \rangle = \partial^2_{Y|\Gamma,Y=0^-}$. Therefore, (2.17) is equivalent to

$$\left\{ \begin{array}{rl} \mu_0 \Delta \mathbf{v}^0 = 0 & \text{in } \Omega \\ \mu_1 \partial_Y^2 \mathbf{v}^0 = g^0 & \text{in } \Lambda \\ \mu_1 \partial_Y \mathbf{v}^0_{|Y=0^-} = h^0 & \text{on } \Gamma \end{array} \right. \tag{2.19}$$

Then Propositions 2.12 and 2.14 imply that the solutions of (2.19) in $\mathcal{A}^+(\Pi)$ are the $\frac{1}{\mu_1} R_{\partial^2_Y}(g^0) + \frac{1}{\mu_1} R_{\rm N}(h^0) + \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^*} c_d\, \phi_d$ where c_d vanishes for small enough d. Finally we have $c_d = \sigma_d(\mathbf{v}^0) = \sigma_d(\mathbf{u}^0)$ for any $d \in \frac{\pi}{\Theta}\mathbb{Z}^*$ because $\sigma_d \circ R_\Delta$, $\sigma_d \circ R_{\partial^2_Y}$ and $\sigma_d \circ R_{\rm N}$ vanish by Proposition 2.14. □

Lemma 2.19: Let $F^\infty_\Omega \in \mathcal{A}^-(\Omega)$ and $F^\infty_\Lambda \in \mathcal{A}^-(\Lambda)$. The solutions in $\mathcal{A}^-(\Pi)$ of

$$\left\{ \begin{array}{rl} \mu_0 \Delta S^\infty = F^\infty_\Omega & \text{in } \Omega \\ \mu_1 \Delta S^\infty = F^\infty_\Lambda & \text{in } \Lambda \\ \mu_0 \partial_{Y|Y=0^+} S^\infty - \mu_1 \partial_{Y|Y=0^-} S^\infty = 0 & \text{on } \Gamma \end{array} \right. \tag{2.20}$$

are the formal series of the following form (where $\sigma_d(S^\infty)$ vanishes when d is big enough)

$$S^\infty = \left\langle -R_{\partial^2_Y} \circ \partial^2_{X|\Lambda},\ \frac{\mu_0}{\mu_1} R_{\rm N} \circ \partial_{Y|\Gamma,Y=0^+} \right\rangle \left(\frac{1}{\mu_0} R_\Delta(F^\infty_\Omega) + \frac{1}{\mu_1} R_{\partial^2_Y}(F^\infty_\Lambda) + \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^*} \sigma_d(S^\infty)\, \phi_d \right) \tag{2.21}$$

Proof: This is similar to Lemma 2.18. Let $R_1 := -R_{\partial^2_Y} \circ \partial^2_{X|\Lambda}$, $R_2 := \frac{\mu_0}{\mu_1} R_{\rm N} \circ \partial_{Y|\Gamma,Y=0^+}$ and $\mathbf{v}^\infty := (\mathrm{id} - R_1 - R_2) S^\infty$. By Definition 2.3 we have

$$\langle R_1, R_2 \rangle = \sum_{n=0}^\infty \sum_{(i_1,\dots,i_n) \in \{1,2\}^n} R_{i_1} \circ \cdots \circ R_{i_n} = \sum_{n=0}^\infty (R_1 + R_2)^n = (\mathrm{id} - R_1 - R_2)^{-1}.$$

So $S^\infty = \langle R_1, R_2 \rangle \mathbf{v}^\infty$. Moreover, Proposition 2.14 implies that $\Delta \circ R_1 = \Delta \circ R_2 = 0$, $\partial^2_{Y|\Lambda} \circ R_1 = -\partial^2_{X|\Lambda}$, $\partial^2_{Y|\Lambda} \circ R_2 = 0$, $\partial_{Y|\Gamma,Y=0^-} \circ R_1 = 0$ and $\partial_{Y|\Gamma,Y=0^-} \circ R_2 = \frac{\mu_0}{\mu_1} \partial_{Y|\Gamma,Y=0^+}$. We deduce that $\Delta \circ \langle R_1, R_2 \rangle = \Delta$,

$$(\partial^2_{Y|\Lambda} + \partial^2_{X|\Lambda}) \circ \langle R_1, R_2 \rangle = \partial^2_{Y|\Lambda} \circ (\mathrm{id} - R_1) \circ \langle R_1, R_2 \rangle = \partial^2_{Y|\Lambda} \circ (\mathrm{id} - R_1 - R_2) \circ \langle R_1, R_2 \rangle = \partial^2_{Y|\Lambda}$$

and similarly $(\mu_0\partial_{Y|\Gamma,Y=0^+} - \mu_1\partial_{Y|\Gamma,Y=0^-})\circ\langle R_1,R_2\rangle = \mu_1\partial_{Y|\Gamma,Y=0^-}$. Therefore, (2.20) is equivalent to

$$\begin{cases} \mu_0\Delta\mathbf{v}^\infty = F_\Omega^\infty & \text{in } \Omega \\ \mu_1\partial_Y^2\mathbf{v}^\infty = F_\Lambda^\infty & \text{in } \Lambda \\ \mu_1\partial_Y\mathbf{v}^\infty_{|Y=0^-} = 0 & \text{on } \Gamma \end{cases} \tag{2.22}$$

Then Propositions 2.12 and 2.14 imply that the solutions of (2.22) in $\mathcal{A}^+(\Pi)$ are the $\frac{1}{\mu_0}R_\Delta(F_\Omega^\infty) + \frac{1}{\mu_1}R_{\partial_Y^2}(F_\Lambda^\infty) + \sum_{d\in\frac{\pi}{\Theta}\mathbb{Z}^*} c_d\,\phi_d$ where $c_d = 0$ for big enough d. Finally, $c_d = \sigma_d(S^\infty)$ as for Lemma 2.18. □

Definition 2.20: Let $d\in\mathbb{R}$ and $a\in\{0,\infty\}$. We define o_∂, a kind of differentiable small o, as follows.

- For any $\varphi:\Omega\to\mathbb{R}$, we say that $\varphi = o_\partial(r^d)$ when $r\to a$ if φ is $\mathcal{C}^\infty$ in a vicinity of $r=a$ and $\forall(j,k)\in\mathbb{N}^2$, $\partial_r^j\partial_\theta^k\varphi = o(r^{d-j})$ uniformly w.r.t. θ when $r\to a$.
- For any $\varphi:\Lambda\to\mathbb{R}$, we say that $\varphi = o_\partial(x^d)$ when $x\to a$ if φ is $\mathcal{C}^\infty$ in a vicinity of $x=a$ and $\forall(j,k)\in\mathbb{N}^2$, $\partial_r^j\partial_Y^k\varphi = o(x^{d-j})$ uniformly w.r.t. Y when $x\to a$.
- For any $\varphi:\Gamma\to\mathbb{R}$, we say that $\varphi = o_\partial(x^d)$ when $x\to a$ if φ is $\mathcal{C}^\infty$ in a vicinity of $x=a$ and $\forall j\in\mathbb{N}$, $\partial_r^j\varphi = o(x^{d-j})$ when $x\to a$.
- For any $\varphi:\Pi\to\mathbb{R}$, we say that $\varphi = o_\partial(\mathbf{r}^d)$ when $\mathbf{r}\to a$ if $\varphi_{|\Omega} = o_\partial(r^d)$ and $\varphi_{|\Lambda} = o_\partial(x^d)$.

Definition 2.21: Let $d\in\mathbb{R}$, $D\in\{\Pi,\Omega,\Gamma,\Lambda\}$ and $\varphi = \sum_{q\in\mathbb{R}}\varphi_q$ in $\mathcal{A}^+(D)$ or $\mathcal{A}^-(D)$ with $\varphi_q\in\mathcal{A}_q(D)$ for all $q\in\mathbb{R}$. We denote $T_{\leqslant d}(\varphi) := \sum_{q\in\mathbb{R},q\leqslant d}\varphi_q$ and $T_{\geqslant d}(\varphi) := \sum_{q\in\mathbb{R},q\geqslant d}\varphi_q$ the truncations of φ below and above d.

Using Lemmas 2.18–2.19 and Kondratiev's theory (involving weighted Sobolev spaces, Laplace's transform and the residue theorem), we proved the following theorems, giving asymptotic behaviors for solutions of model problems of the type of far-and-layer fields and corner fields. The proofs can be found in Appendix B.

Theorem 2.22: Let $\mathbf{u}\in H^1_{\text{loc}}(\Pi)$, $f\in\mathcal{D}'(\Omega)$, $g\in L^2_{\text{loc}}(\Lambda)$ and $h\in L^2_{\text{loc}}(\Gamma)$ be such that

$$\begin{cases} \mu_0\Delta\mathbf{u} + \omega^2\rho_0\mathbf{u} = f & \text{in } \Omega \\ \mu_1\partial_Y^2\mathbf{u} = g & \text{in } \Lambda \\ \mu_1\partial_{Y|Y=0^-}\mathbf{u} = h & \text{on } \Gamma \\ \mathbf{u}_{|y=0^+} - \mathbf{u}_{|Y=0^-} = 0 & \text{on } \Gamma \\ \mathbf{u} = 0 & \text{on } \Sigma_\Omega\cup\Sigma_\Lambda \end{cases} \tag{2.23}$$

We assume that:

- f vanishes in the vicinity of the corner $(0,0)$,
- there exists $g^0\in\mathcal{A}^+(\Lambda)$ s.t.: $\forall d\in\mathbb{R}$, $g(x,Y) = T_{\leqslant d}(g^0)(x,Y) + o_\partial(x^d)$ when $x\to 0$,
- there exists $h^0\in\mathcal{A}^+(\Gamma)$ s.t.: $\forall d\in\mathbb{R}$, $h(x) = T_{\leqslant d}(h^0)(x) + o_\partial(x^d)$ when $x\to 0$,
- there is $\eta>0$, $u^{\text{v}}\in H^1(\Omega\cap B(0,\eta))$ and $\varphi\in\mathcal{A}(\Omega)$ s.t. $\mathbf{u}_{|\Omega\cap B(0,\eta)} = u^{\text{v}} + \varphi$.

Then there is $\mathbf{u}^0\in\mathcal{A}^+(\Pi)$ that has the form (2.18) s.t.: $\forall d\in\mathbb{R}$, $\mathbf{u} = T_{\leqslant d}(\mathbf{u}^0) + o_\partial(\mathbf{r}^d)$ when $\mathbf{r}\to 0$.

Theorem 2.23: Let $S\in H^1_{\text{loc}}(\Omega_1)$ and $F\in L^2_{\text{loc}}(\overline{\Omega_1})$ (i.e. L^2 on any bounded subset of Ω_1) such that

$$\begin{cases} \operatorname{div}(\mu\nabla S) = F & \text{in } \Omega_1 \\ S = 0 & \text{on } \partial\Omega_1 \end{cases} \tag{2.24}$$

We assume that:

- there exists $F_\Omega^\infty\in\mathcal{A}^-(\Omega)$ such that $\forall d\in\mathbb{R}$, $F_{|\Omega} = T_{\geqslant d}(F_\Omega^\infty) + o_\partial(r^d)$ when $r\to\infty$,
- there exists $F_\Lambda^\infty\in\mathcal{A}^-(\Lambda)$ such that $\forall d\in\mathbb{R}$, $F_{|\Lambda} = T_{\geqslant d}(F_\Lambda^\infty) + o_\partial(x^d)$ when $x\to\infty$,
- S belongs to $V + \chi_\infty\mathcal{A}(\Pi)$ (the space in which the corner fields will be build in Section 3.2).

Then there is $S^\infty \in \mathcal{A}^-(\Pi)$ that has the form (2.21) s.t.: $\forall d \in \mathbb{R},\ S = T_{\geqslant d}(S^\infty) + o_\partial(\mathbf{r}^d)$ when $\mathbf{r} \to \infty$.

Note that, since $g^0 \in \mathcal{A}^+(\Lambda)$, we have $T_{\leqslant d}(g^0) \in \mathcal{A}(\Lambda)$, so the formula $g = T_{\leqslant d}(g^0) + o_\partial(x^d)$ makes sense. The same applies to the truncations of h^0, $\mathbf{u}^0$, F^∞_Ω and F^∞_Λ and S^∞.

A consequence of Theorems 2.22–2.23 is Proposition 3.13 that states that for any (p,ℓ) there is $\mathbf{u}^0_{p,\ell} \in \mathcal{A}^+(\Pi)$ and $S^\infty_{p,\ell} \in \mathcal{A}^-(\Pi)$ s.t.:

$$\forall d \in \mathbb{R}, \quad \begin{cases} \mathbf{u}_{p,\ell} = T_{\leqslant d}(\mathbf{u}^0_{p,\ell}) + o_\partial(\mathbf{r}^d) & \text{when } \mathbf{r} \to 0 \\ S_{p,\ell} = T_{\geqslant d}(S^\infty_{p,\ell}) + o_\partial(\mathbf{r}^d) & \text{when } \mathbf{r} \to \infty \end{cases}$$

In the rest of this section, we will assume that such formal series exist. In addition, given the equations satisfied by $\mathbf{u}_{p,\ell}$ and $S_{p,\ell}$ (see (2.1) and (1.5)), (2.18) and (2.21) rewrite here as

$$\mathbf{u}^0_{p,\ell} = \big\langle -k_0^2\, R_\Delta \big\rangle \Big(- R_{\partial_Y^2} \circ (\partial^2_{x|\Lambda} + k_1^2)(\mathbf{u}^0_{p-2,\ell}) + \frac{\mu_0}{\mu_1} R_{\mathrm{N}} \circ \partial_{y|\Gamma, y=0^+}(\mathbf{u}^0_{p-1,\ell}) + \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^*} \sigma_d(\mathbf{u}^0_{p,\ell})\, \phi_d \Big) \tag{2.25}$$

$$S^\infty_{p,\ell} = \Big\langle -R_{\partial_Y^2} \circ \partial^2_{X|\Lambda}, \frac{\mu_0}{\mu_1} R_{\mathrm{N}} \circ \partial_{Y|\Gamma, Y=0^+} \Big\rangle \Big(- k_0^2 R_\Delta(S^\infty_{p-2,\ell|\Omega}) - k_1^2 R_{\partial_Y^2}(S^\infty_{p-2,\ell|\Lambda}) + \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^*} \sigma_d(S^\infty_{p,\ell})\, \phi_d \Big)$$

where by convention $\mathbf{u}^0_{p,\ell}$ and $S^\infty_{p,\ell}$ vanish when $p \in \mathbb{R} \setminus \mathbb{P}$. Therefore $(\mathbf{u}^0_{p,\ell})_{p,\ell}$ and $(S^\infty_{p,\ell})_{p,\ell}$ are uniquely defined by $(\sigma_d(\mathbf{u}^0_{p,\ell}))_{d,p,\ell}$ and $(\sigma_d(S^\infty_{p,\ell}))_{d,p,\ell}$. When $d < 0$ (resp. $d > 0$), ϕ_d is non-variational for the far-and-layer fields (resp. corner fields), and Theorems 3.6 and 3.11 show that $\sigma_d(\mathbf{u}^0_{p,\ell})$ (resp. $\sigma_d(S^\infty_{p,\ell})$) can be fixed arbitrarily. The rest of this section is devoted to finding how to fix them in order to satisfy the matching conditions (1.6). On the contrary, when $d > 0$ (resp. $d < 0$), ϕ_d is variational for the far-and-layer fields (resp. corner fields) and the values of $\sigma_d(\mathbf{u}^0_{p,\ell})$ (resp. $\sigma_d(S^\infty_{p,\ell})$) are uniquely defined once $(\sigma_{d'}(\mathbf{u}^0_{p,\ell}))_{d'<0}$ (resp. $(\sigma_{d'}(S^\infty_{p,\ell}))_{d'>0}$) has been fixed. Ways to compute these values numerically will be investigated in a future work.

2.5 Specifying of the matching conditions

In this section we express $\sum_{p,\ell} \varepsilon^p \ln^\ell\varepsilon \cdot \mathbf{u}^0_{p,\ell}$ and $\sum_{p,\ell} \varepsilon^p \ln^\ell\varepsilon \cdot S^\infty_{p,\ell}$ in function of the $\sigma_d(\mathbf{u}^0_{p,\ell})$ and $\sigma_d(S^\infty_{p,\ell})$, we then rewrite rigorously the matching conditions (1.6), and we finally show that they are equivalent to a set of equations on the coefficients $\sigma_d(.)$. Here $\sum_{p,\ell} \varepsilon^p \ln^\ell\varepsilon \cdot \mathbf{u}^0_{p,\ell}$ and $\sum_{p,\ell} \varepsilon^p \ln^\ell\varepsilon \cdot S^\infty_{p,\ell}$ are formal series that belong resp. to the spaces $\mathcal{A}^+_\varepsilon(\Pi)$ and $\mathcal{A}^-_\varepsilon(\Pi)$ defined below.

In this section, "ε" and "$\ln\varepsilon$" denote two algebraic indeterminates independent of each other (so they are not numbers).

Definition 2.24: We denote

$$\mathcal{A}^+_\varepsilon(\Pi) := \sum_{p\in\mathbb{P}} \varepsilon^p \sum_{d\in\mathbb{P}-p} \mathcal{A}_d(\Pi)[\ln\varepsilon] \qquad \text{and} \qquad \mathcal{A}^-_\varepsilon(\Pi) := \sum_{p\in\mathbb{P}} \varepsilon^p \sum_{d\in p-\mathbb{P}} \mathcal{A}_d(\Pi)[\ln\varepsilon].$$

According to Section 2.1, we write their elements as formal series like

$$\sum_{p\in\mathbb{P}} \sum_{d\in\mathbb{P}-p} \sum_{\ell\in\mathbb{N}} \varepsilon^p \ln^\ell\varepsilon \cdot \varphi_{p,d,\ell}, \qquad \text{resp.} \qquad \sum_{p\in\mathbb{P}} \sum_{d\in p-\mathbb{P}} \sum_{\ell\in\mathbb{N}} \varepsilon^p \ln^\ell\varepsilon \cdot \varphi_{p,d,\ell}.$$

Ansatz 2.25: We assume that

$$\sum_{p\in\mathbb{P},\ell\in\mathbb{N}} \varepsilon^p \ln^\ell\varepsilon \cdot \mathbf{u}^0_{p,\ell} \in \mathcal{A}^+_\varepsilon(\Pi) \qquad \text{and} \qquad \sum_{p\in\mathbb{P},\ell\in\mathbb{N}} \varepsilon^p \ln^\ell\varepsilon \cdot S^\infty_{p,\ell} \in \mathcal{A}^-_\varepsilon(\Pi).$$

Compared to the ansatz of Section 1.2, it adds that, for any $(p,\ell) \in \mathbb{P}\times\mathbb{N}$, $\mathbf{u}_{p,\ell}$ has components only in the $\mathcal{A}_d(\Pi)$ s.t. $d \in \mathbb{P}-p$ and $S_{p,\ell}$ has only in the $\mathcal{A}_d(\Pi)$ s.t. $d \in p-\mathbb{P}$. This is necessary for the matching

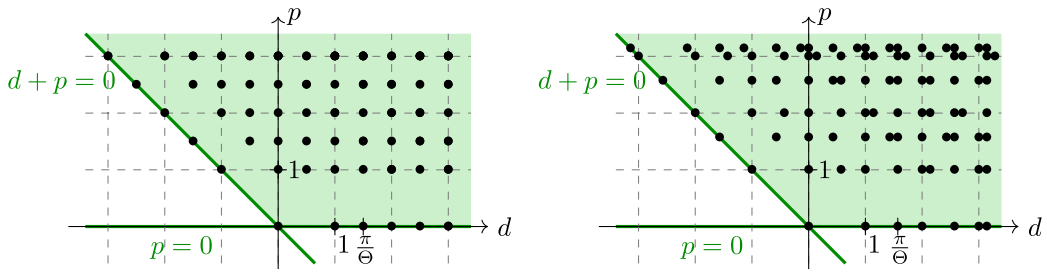

Figure 8: Points of $\{(d,p) \mid p \in \mathbb{P} \text{ and } d \in \mathbb{P} - p\}$ for $\Theta = \frac{3\pi}{2}$ (on the left) and $\Theta = 2$ (on the right)

because e.g. we will see that for any $\varphi \in \varepsilon^p \mathcal{A}_d(\Pi)[\ln \varepsilon] \subset \mathcal{A}_\varepsilon^-(\Pi)$ that is a term of $\sum \varepsilon^p \ln^\ell \varepsilon \, S^\infty_{p,\ell}$, $\varphi(\frac{\cdot}{\varepsilon})$ must appear in $\sum \varepsilon^p \ln^\ell \varepsilon \, \mathbf{u}^0_{p,\ell}$ and $\varphi(\frac{\cdot}{\varepsilon}) \in \varepsilon^{p-d}\mathcal{A}_d(\Pi)[\ln \varepsilon]$, so $p - d \in \mathbb{P}$.

Let us denote:
$$\mathcal{R}_\varepsilon^+ := \left\{-k_0^2\, R_\Delta,\ -\varepsilon^2 R_{\partial_Y^2} \circ \partial^2_{x|\Lambda},\ -\varepsilon^2 k_1^2 R_{\partial_Y^2},\ \varepsilon \tfrac{\mu_0}{\mu_1} R_{\mathrm{N}} \circ \partial_{y|\Gamma, y=0^+}\right\}$$
$$\mathcal{R}_\varepsilon^- := \left\{-\varepsilon^2 k_0^2\, R_\Delta,\ -R_{\partial_Y^2} \circ \partial^2_{X|\Lambda},\ -\varepsilon^2 k_1^2 R_{\partial_Y^2},\ \tfrac{\mu_0}{\mu_1} R_{\mathrm{N}} \circ \partial_{Y|\Gamma, Y=0^+}\right\}$$

where the above operators are defined as in (2.6). For any $R \in \mathcal{R}_\varepsilon^+ \cup \mathcal{R}_\varepsilon^-$, we denote $(\deg_\varepsilon R, \deg_{\mathcal{A}} R) := \deg R$. See Figure 9. Thanks to Definition 2.4, we can consider $\langle \mathcal{R}_\varepsilon^\pm \rangle$ which is well-defined on $\mathcal{A}_\varepsilon^\pm(\Pi)$ (it suffices to take $v := (\pm 1, 2)$ in Definition 2.4). Moreover one can check that it maps $\mathcal{A}_\varepsilon^\pm(\Pi)$ into itself.

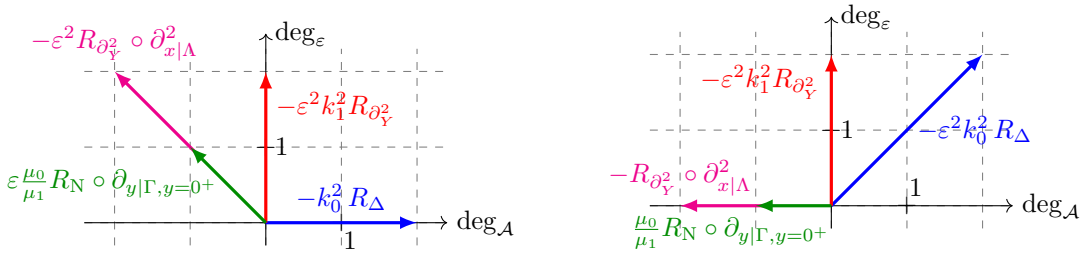

Figure 9: Degrees of the elements of $\mathcal{R}_\varepsilon^+$ (on the left) and $\mathcal{R}_\varepsilon^-$ (on the right)

Proposition 2.26: We have the following equalities in $\mathcal{A}_\varepsilon^+(\Pi)$ and $\mathcal{A}_\varepsilon^-(\Pi)$ respectively:

$$\sum_{p\in\mathbb{P},\ell\in\mathbb{N}} \varepsilon^p \ln^\ell \varepsilon \cdot \mathbf{u}^0_{p,\ell} = \langle \mathcal{R}_\varepsilon^+ \rangle \left(\sum_{p\in\mathbb{P},\ell\in\mathbb{N}} \varepsilon^p \ln^\ell \varepsilon \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap (\mathbb{P}-p)} \sigma_d(\mathbf{u}^0_{p,\ell}) \phi_d \right) \tag{2.26}$$

$$\sum_{p\in\mathbb{P},\ell\in\mathbb{N}} \varepsilon^p \ln^\ell \varepsilon \cdot S^\infty_{p,\ell} = \langle \mathcal{R}_\varepsilon^- \rangle \left(\sum_{p\in\mathbb{P},\ell\in\mathbb{N}} \varepsilon^p \ln^\ell \varepsilon \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap (p-\mathbb{P})} \sigma_d(S^\infty_{p,\ell}) \phi_d \right) \tag{2.27}$$

Proof: We only prove (2.26), as (2.27) is similar. We could do it by inductively composing (2.25), but we chose instead a proof similar to Lemma 2.19 to avoid heavy calculations. Let $(p,\ell) \in \mathbb{P} \times \mathbb{N}$. Given the equations satisfied by $\mathbf{u}_{p,\ell}$ in (2.1), and the fact that $\mathbf{u}_{p,\ell} = T_{\leqslant d}(\mathbf{u}^0_{p,\ell}) + o_\partial(\mathbf{r}^d)$ for any $d \in \mathbb{R}$, we have

$$\begin{cases} \mu_0 \Delta \mathbf{u}^0_{p,\ell} + \omega^2 \rho_0 \mathbf{u}^0_{p,\ell} = 0 & \text{in } \Omega \\ \mu_1 \partial_Y^2 \mathbf{u}^0_{p,\ell} = -(\mu_1 \partial_x^2 + \omega^2 \rho_1) \mathbf{u}^0_{p-2,\ell} & \text{in } \Lambda \\ \mu_1 \partial_{Y|Y=0^-} \mathbf{u}^0_{p,\ell} = \mu_0\, \partial_{y|y=0^+} \mathbf{u}^0_{p-1,\ell} & \text{on } \Gamma \end{cases} \tag{2.28}$$

Let $\mathbf{u}^0_\varepsilon := \sum_{p,\ell} \varepsilon^p \ln^\ell \varepsilon \, \mathbf{u}^0_{p,\ell}$. Summing over (p,ℓ) (2.28) times $\varepsilon^p \ln^\ell \varepsilon$ yields

$$\begin{cases} \mu_0 \Delta \mathbf{u}^0_\varepsilon + \omega^2 \rho_0 \mathbf{u}^0_\varepsilon = 0 & \text{in } \Omega \\ \mu_1 \partial_Y^2 \mathbf{u}^0_\varepsilon + \varepsilon^2 (\mu_1 \partial_x^2 + \omega^2 \rho_1) \mathbf{u}^0_\varepsilon = 0 & \text{in } \Lambda \\ \mu_1 \partial_{Y|Y=0^-} \mathbf{u}^0_\varepsilon - \varepsilon\, \mu_0\, \partial_{y|y=0^+} \mathbf{u}^0_\varepsilon = 0 & \text{on } \Gamma \end{cases}$$

Let $\mathbf{v}_\varepsilon^0 := (\mathrm{id} - \sum_{R\in\mathcal{R}_\varepsilon^+} R)\mathbf{u}_\varepsilon^0$. It is easy to see that $\mathrm{id} - \sum_{R\in\mathcal{R}_\varepsilon^+} R$ maps $\mathcal{A}_\varepsilon^+(\Pi)$ into itself, so $\mathbf{v}_\varepsilon^0 \in \mathcal{A}_\varepsilon^+(\Pi)$. By Definition 2.4, we have

$$\langle \mathcal{R}_\varepsilon^+ \rangle := \sum_{n=0}^{\infty} \sum_{(R_1,\dots,R_n)\in(\mathcal{R}_\varepsilon^+)^n} R_1 \circ \cdots \circ R_n = \sum_{n=0}^{\infty} \Big(\sum_{R\in\mathcal{R}_\varepsilon^+} R \Big)^n = \Big(\mathrm{id} - \sum_{R\in\mathcal{R}_\varepsilon^+} R \Big)^{-1}.$$

So $\mathbf{u}_\varepsilon^0 = \langle \mathcal{R}_\varepsilon^+ \rangle \mathbf{v}_\varepsilon^0$. Moreover, Proposition 2.14 implies $\forall R \in \mathcal{R}_\varepsilon^+ \setminus \{-k_0^2 R_\Delta\},\ \Delta \circ R = 0$. Hence

$$(\mu_0\Delta + \omega^2\rho_0) \circ \langle \mathcal{R}_\varepsilon^+ \rangle = \mu_0\Delta \circ (\mathrm{id} + k_0^2 R_\Delta) \circ \langle \mathcal{R}_\varepsilon^+ \rangle = \mu_0\Delta \circ \Big(\mathrm{id} - \sum_{R\in\mathcal{R}_\varepsilon^+} R \Big) \circ \langle \mathcal{R}_\varepsilon^+ \rangle = \mu_0\Delta$$

Similarly, one can check that $\big[\mu_1\partial_{Y|\Lambda}^2 + \varepsilon^2(\mu_1\partial_{x|\Lambda}^2 + \omega^2\rho_1)\big] \circ \langle \mathcal{R}_\varepsilon^+ \rangle = \mu_1\partial_{Y|\Lambda}^2$ and $(\mu_1\partial_{Y|\Gamma,Y=0^-} - \varepsilon\,\mu_0\,\partial_{y|\Gamma,y=0^+}) \circ \langle \mathcal{R}_\varepsilon^+ \rangle = \mu_1\partial_{Y|\Gamma,Y=0^-}$. Therefore $\mathbf{v}_\varepsilon^0$ satisfies

$$\begin{cases} \mu_0\Delta\mathbf{v}_\varepsilon^0 = 0 & \text{in } \Omega \\ \mu_1\partial_Y^2\mathbf{v}_\varepsilon^0 = 0 & \text{in } \Lambda \\ \mu_1\partial_{Y|Y=0^-}\mathbf{v}_\varepsilon^0 = 0 & \text{on } \Gamma \end{cases}$$

Then Proposition 2.14 implies that there are numbers $c_{p,\ell,d}$ s.t. $\mathbf{v}_\varepsilon^0 = \sum_{p,\ell,d} \varepsilon^p \ln^\ell\varepsilon\, c_{p,\ell,d}\, \phi_d$. Finally we have $c_{p,\ell,d} = \sigma_d(\mathbf{u}_{p,\ell}^0)$ for any (p,ℓ,d) because Proposition 2.14 gives that $\forall R \in \mathcal{R}_\varepsilon^+,\ \sigma_d \circ R = 0$. □

Now, we want to define $\mathcal{H}_\varepsilon : \mathcal{A}_\varepsilon^+(\Pi) \to \mathcal{A}_\varepsilon^-(\Pi)$ as the scaling operator

$$\forall \varphi \in \mathcal{A}_\varepsilon^+(\Pi), \qquad \mathcal{H}_\varepsilon(\varphi) = \begin{cases} (X,Y) \mapsto \varphi(\varepsilon X, \varepsilon Y) & \text{in } \Omega \\ (X,Y) \mapsto \varphi(\varepsilon X, Y) & \text{in } \Lambda \end{cases} \tag{2.29}$$

However, ε is an indeterminate. Let us first define $\mathcal{H}_\varepsilon$ for any $\varphi \in \mathcal{A}_d(\Pi)$, $d \in \mathbb{R}$. If $\varphi_{|\Omega}$ has the form $\Im[(\alpha z)^q\, \overline{\alpha z}^k P(\log(\alpha z))]$ with $(q,k,P) \in \mathbb{R}\times\mathbb{N}\times\mathbb{R}[T]$ s.t. $q+k=d$ and $\mathcal{P}(q,k,P)$ is true, we set:

$$\mathcal{H}_\varepsilon(\varphi)_{|\Omega} := \sum_{j=0}^{\deg P} \varepsilon^d \frac{\ln^j\varepsilon}{j!}\, \Im[(\alpha z)^q\, \overline{\alpha z}^k P^{(j)}(\log(\alpha z))] \in \varepsilon^d \mathcal{A}_d(\Omega)[\ln\varepsilon]. \tag{2.30}$$

And if $\varphi_{|\Lambda}$ has the form $x^d\, Q(\ln(x), Y)$ with $Q \in \mathbb{C}[T]$, we set:

$$\mathcal{H}_\varepsilon(\varphi)_{|\Lambda} := \sum_{j=0}^{\deg Q} \varepsilon^d \frac{\ln^j\varepsilon}{j!}\, x^d\, \partial_T^j Q(\ln x, Y) \in \varepsilon^d \mathcal{A}_d(\Lambda)[\ln\varepsilon]. \tag{2.31}$$

Thanks to Lemmas 2.8 and 2.10, it defines well $\mathcal{H}_\varepsilon$ from $\mathcal{A}_d(\Pi)$ to $\varepsilon^d\,\mathcal{A}_d(\Pi)[\ln\varepsilon]$. Then we extend $\mathcal{H}_\varepsilon$ to $\mathcal{A}_\varepsilon^+(\Pi)$ by setting $\mathcal{H}_\varepsilon(\sum_{p,d,\ell} \varepsilon^p \ln^\ell\varepsilon\, \varphi_{p,d,\ell}) := \sum_{p,d} \sum_\ell \varepsilon^p \ln^\ell\varepsilon\, \mathcal{H}_\varepsilon(\varphi_{p,d,\ell})$ for any $(\varphi_{p,d,\ell})$ s.t. $\varphi_{p,d,\ell} \in \mathcal{A}_d(\Pi)$ for any (p,d,ℓ). One can check that is in $\mathcal{A}_\varepsilon^-(\Pi)$.

Remarks:

- In practice, we will only use the informal definition of (2.29), but everything we will do can be checked using (2.30) and (2.31).
- $\mathcal{H}_\varepsilon$ is invertible and $\mathcal{H}_\varepsilon^{-1}$ is roughly the scaling: $\mathcal{H}_\varepsilon^{-1}(\varphi) = \begin{cases} (x,y) \mapsto \varphi(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}) & \text{in } \Omega \\ (x,Y) \mapsto \varphi(\frac{x}{\varepsilon}, Y) & \text{in } \Lambda \end{cases}$
- For the first time, powers of $\ln\varepsilon$ naturally appear because of the power of $\ln r$. This explains why the presence of these powers in the ansatz is necessary from the beginning.

Definition 2.27:
We rigorously rewrite the matching condition as: $\displaystyle\sum_{p\in\mathbb{P},\ell\in\mathbb{N}} \varepsilon^p \ln^\ell\varepsilon \cdot \mathbf{u}_{p,\ell}^0 = \mathcal{H}_\varepsilon^{-1}\Big(\sum_{p\in\mathbb{P},\ell\in\mathbb{N}} \varepsilon^p \ln^\ell\varepsilon \cdot S_{p,\ell}^\infty \Big)$.

When composing Definition 2.27 and Proposition 2.26, $\mathcal{H}_\varepsilon^{-1} \circ \langle \mathcal{R}_\varepsilon^- \rangle$ appears. Since $\mathcal{R}_\varepsilon^+$ and $\mathcal{R}_\varepsilon^-$ only differ on powers of ε due to the scaling, one could expect that $\mathcal{H}_\varepsilon^{-1} \circ \langle \mathcal{R}_\varepsilon^- \rangle \circ \mathcal{H}_\varepsilon$ is equal to $\langle \mathcal{R}_\varepsilon^+ \rangle$. However, R_Δ

(resp. $R_{\partial^2_Y}$, $R_{\rm N}$) picks the particular solution of Equation 2.12 (resp. 2.13, 2.14) whose image by the σ_d vanish. This means that they are the solutions in the kernels of the following projectors of $\mathcal{A}^\pm_\varepsilon(\Pi)$:

$$\pi^\pm_\sigma\left(\sum_{p\in\mathbb{P}}\sum_{d\in\pm(\mathbb{P}-p)}\sum_{\ell\in\mathbb{N}}\varepsilon^p\ln^\ell\varepsilon\cdot\varphi_{p,d,\ell}\right):=\sum_{p\in\mathbb{P}}\sum_{d\in\frac{\pi}{\Theta}\mathbb{Z}^*\cap\pm(\mathbb{P}-p)}\sum_{\ell\in\mathbb{N}}\varepsilon^p\ln^\ell\varepsilon\cdot\sigma_d(\varphi_{p,d,\ell})\,\phi_d \tag{2.32}$$

for any $(\varphi_{p,d,\ell})$ s.t. $\varphi_{p,d,\ell}\in\mathcal{A}_d(\Pi)$ for any (p,d,ℓ). Lemma 2.28 below implies that $\mathcal{H}^{\pm1}_\varepsilon$ does not map $\operatorname{Ker}\pi^\pm_\sigma$ to $\operatorname{Ker}\pi^\mp_\sigma$. Therefore $\mathcal{H}^{-1}_\varepsilon\circ R_\Delta\circ\mathcal{H}_\varepsilon$ selects other solutions than R_Δ, and likewise for $R_{\partial^2_Y}$ and $R_{\rm N}$. We will see that it implies that $\mathcal{H}^{-1}_\varepsilon\circ\langle\mathcal{R}^-_\varepsilon\rangle\circ\mathcal{H}_\varepsilon$ it is equal to $\langle\mathcal{R}^+_\varepsilon\rangle$ times a correction operator given in Theorem 2.29.

To precise the action of $\mathcal{H}_\varepsilon$ on $\pi^\pm_\sigma$, we introduce π^+_ε and π^-_ε, which map any $\varphi\in\mathcal{A}_d(\Pi)$ of the form $\Im[(\alpha z)^q\,\overline{\alpha z}^k P(\log(\alpha z))]$ in Ω (with $(q,k,P)\in\mathbb{R}\times\mathbb{N}\times\mathbb{R}[T]$, $q+k=d$ and $\mathcal{P}(q,k,P)$) to:

$$\pi^\pm_\varepsilon(\varphi):=\begin{cases}P(\pm\ln\varepsilon)\,\phi_d & \text{if } d\in\frac{\pi}{\Theta}\mathbb{Z}^* \text{ and } k=0\\ 0 & \text{otherwise}\end{cases} \tag{2.33}$$

By Lemmas 2.8 and 2.10, it defines a linear map $\mathcal{A}_d(\Pi)\to\mathcal{A}_d(\Pi)[\ln\varepsilon]$. Then we extend $\pi^\pm_\varepsilon$ into an endomorphism of $\mathcal{A}^\pm_\varepsilon(\Pi)$ by setting $\pi^\pm_\varepsilon(\sum_{p,d,\ell}\varepsilon^p\ln^\ell\varepsilon\,\varphi_{p,d,\ell}):=\sum_{p,d}\sum_\ell\varepsilon^p\ln^\ell\varepsilon\,\pi^\pm_\varepsilon(\varphi_{p,d,\ell})$.

Lemma 2.28: We have $\mathcal{H}^{\mp1}_\varepsilon\circ\pi^\mp_\sigma\circ\mathcal{H}^{\pm1}_\varepsilon=\pi^\pm_\varepsilon$.

Proof: Since the involved operators only depend on the part on Ω, and by Lemma 2.8, it suffices to prove that $\mathcal{H}^{\mp1}_\varepsilon\circ\pi^\mp_\sigma\circ\mathcal{H}^{\pm1}_\varepsilon(\varphi)=\pi^\pm_\varepsilon(\varphi)$ for any φ of the form $\Im[(\alpha z)^q\,\overline{\alpha z}^k P(\log(\alpha z))]$ in Ω (with $(q,k,P)\in\mathbb{R}\times\mathbb{N}\times\mathbb{R}[T]$ and $\mathcal{P}(q,k,P)$). If $k>0$ or $q\notin\frac{\pi}{\Theta}\mathbb{Z}^*$, then both images vanish, so we can assume that $k=0$ and $q\in\frac{\pi}{\Theta}\mathbb{Z}^*$. Then both images vanish in Λ, and we have in Ω:

$$\begin{aligned}\mathcal{H}^{\pm1}_\varepsilon(\varphi):z\mapsto\varepsilon^{\pm q}\Im[(\alpha z)^q P(\log(\alpha z)\pm\ln\varepsilon)]\quad&\text{(with abuse of notation)}\\ \pi^\mp_\sigma\circ\mathcal{H}^{\pm1}_\varepsilon(\varphi)=\varepsilon^q P(\ln\varepsilon)\,\phi_q&\\ \mathcal{H}^{\mp1}_\varepsilon\circ\pi^\mp_\sigma\circ\mathcal{H}^{\pm1}_\varepsilon(\varphi)=P(\pm\ln\varepsilon)\,\phi_q=\pi^\pm_\varepsilon(\varphi)&\end{aligned}$$

□

The next theorem is the key ingredient to match the far-and-layer fields with the corner fields because it links $\langle\mathcal{R}^+_\varepsilon\rangle$ to $\langle\mathcal{R}^-_\varepsilon\rangle$.

Theorem 2.29: We have $\mathcal{H}^{\mp1}_\varepsilon\circ\langle\mathcal{R}^\mp_\varepsilon\rangle\circ\mathcal{H}^{\pm1}_\varepsilon=\langle\mathcal{R}^\pm_\varepsilon\rangle\circ(\mathrm{id}-\langle\pi^\pm_\varepsilon\mid\mathcal{R}^\pm_\varepsilon\rangle)$ where we denote:

$$\begin{aligned}\langle\pi^\pm_\varepsilon\mid\mathcal{R}^\pm_\varepsilon\rangle&:=\pi^\pm_\varepsilon\circ\left(\sum_{R\in\mathcal{R}^\pm_\varepsilon}R\right)\circ\left\langle\left\{(\mathrm{id}-\pi^\pm_\varepsilon)\circ R\;\middle|\;R\in\mathcal{R}^\pm_\varepsilon\right\}\right\rangle\\ &=\sum_{n=1}^\infty\sum_{(R_1,\dots,R_n)\in(\mathcal{R}^\pm_\varepsilon)^n}\pi^\pm_\varepsilon\circ R_1\circ(\mathrm{id}-\pi^\pm_\varepsilon)\circ R_2\circ\cdots\circ(\mathrm{id}-\pi^\pm_\varepsilon)\circ R_n\end{aligned}$$

Proof: We only prove that $\mathcal{H}^{-1}_\varepsilon\circ\langle\mathcal{R}^-_\varepsilon\rangle\circ\mathcal{H}_\varepsilon=\langle\mathcal{R}^+_\varepsilon\rangle\circ(\mathrm{id}-\langle\pi^+_\varepsilon\mid\mathcal{R}^+_\varepsilon\rangle)$, but the other formula works the same. We have $\mathcal{H}^{-1}_\varepsilon\circ\langle\mathcal{R}^-_\varepsilon\rangle\circ\mathcal{H}_\varepsilon=\langle\{\mathcal{H}^{-1}_\varepsilon\circ R\circ\mathcal{H}_\varepsilon\mid R\in\mathcal{R}^-_\varepsilon\}\rangle$, so we need to calculate the $\mathcal{H}^{-1}_\varepsilon\circ R\circ\mathcal{H}_\varepsilon$ for each $R\in\mathcal{R}^-_\varepsilon$. We claim that:

$$\begin{aligned}\mathcal{H}^{-1}_\varepsilon\circ\quad(-\varepsilon^2k_0^2\,R_\Delta)\quad\circ\mathcal{H}_\varepsilon&=\mathcal{H}^{-1}_\varepsilon\circ(\mathrm{id}-\pi^-_\sigma)\circ\mathcal{H}_\varepsilon\circ\quad(-k_0^2\,R_\Delta)\\ \mathcal{H}^{-1}_\varepsilon\circ\quad(-R_{\partial^2_Y}\circ\partial^2_{x|\Lambda})\quad\circ\mathcal{H}_\varepsilon&=\mathcal{H}^{-1}_\varepsilon\circ(\mathrm{id}-\pi^-_\sigma)\circ\mathcal{H}_\varepsilon\circ\quad(-\varepsilon^2R_{\partial^2_Y}\circ\partial^2_{x|\Lambda})\\ \mathcal{H}^{-1}_\varepsilon\circ\quad(-\varepsilon^2k_1^2R_{\partial^2_Y})\quad\circ\mathcal{H}_\varepsilon&=\mathcal{H}^{-1}_\varepsilon\circ(\mathrm{id}-\pi^-_\sigma)\circ\mathcal{H}_\varepsilon\circ\quad(-\varepsilon^2k_1^2R_{\partial^2_Y})\\ \mathcal{H}^{-1}_\varepsilon\circ\underbrace{(\tfrac{\mu_0}{\mu_1}R_{\rm N}\circ\partial_{y|\Gamma,y=0^+})}_{\in\mathcal{R}^-_\varepsilon}\circ\mathcal{H}_\varepsilon&=\mathcal{H}^{-1}_\varepsilon\circ(\mathrm{id}-\pi^-_\sigma)\circ\mathcal{H}_\varepsilon\circ\underbrace{(\varepsilon\tfrac{\mu_0}{\mu_1}R_{\rm N}\circ\partial_{y|\Gamma,y=0^+})}_{\in\mathcal{R}^+_\varepsilon}\end{aligned}$$

Let us show the first line (the others are similar).

Proposition 2.14 implies that for any $\psi \in \mathcal{A}_\varepsilon^-(\Pi)$, $R_\Delta(\psi)$ is a solution of:

$$\begin{cases} \Delta\varphi = \psi & \text{in } \Omega \\ \partial_Y^2 \varphi = 0 & \text{in } \Lambda \\ \partial_Y \varphi_{|Y=0^-} = 0 & \text{on } \Gamma \end{cases} \tag{2.34}$$

and $\pi_\sigma^- \circ R_\Delta = 0$. Moreover, by Proposition 2.12, the solution of (2.34) is unique modulo $\operatorname{Im} \pi_\sigma^-$. But one can check that $\varepsilon^{-2}\mathcal{H}_\varepsilon \circ R_\Delta \circ \mathcal{H}_\varepsilon^{-1}(\psi)$ is also solution of (2.34). So the image of $\varepsilon^{-2}\mathcal{H}_\varepsilon \circ R_\Delta \circ \mathcal{H}_\varepsilon^{-1} - R_\Delta$ is included in $\operatorname{Im} \pi_\sigma^-$. Hence $(\mathrm{id} - \pi_\sigma^-) \circ (\varepsilon^{-2}\mathcal{H}_\varepsilon \circ R_\Delta \circ \mathcal{H}_\varepsilon^{-1} - R_\Delta) = 0$. Therefore:

$$R_\Delta = (\mathrm{id} - \pi_\sigma^-) \circ R_\Delta = (\mathrm{id} - \pi_\sigma^-) \circ \varepsilon^{-2}\mathcal{H}_\varepsilon \circ R_\Delta \circ \mathcal{H}_\varepsilon^{-1}$$

which implies $\mathcal{H}_\varepsilon^{-1} \circ R_\Delta \circ \mathcal{H}_\varepsilon = \mathcal{H}_\varepsilon^{-1} \circ (\mathrm{id} - \pi_\sigma^-) \circ \mathcal{H}_\varepsilon \circ \varepsilon^{-2} R_\Delta$.

Hence:
$$\begin{aligned} \mathcal{H}_\varepsilon^{-1} \circ \langle \mathcal{R}_\varepsilon^- \rangle \circ \mathcal{H}_\varepsilon &= \langle \{ \mathcal{H}_\varepsilon^{-1} \circ R \circ \mathcal{H}_\varepsilon \mid R \in \mathcal{R}_\varepsilon^- \} \rangle \\ &= \langle \{ \mathcal{H}_\varepsilon^{-1} \circ (\mathrm{id} - \pi_\sigma^-) \circ \mathcal{H}_\varepsilon \circ R \mid R \in \mathcal{R}_\varepsilon^+ \} \rangle \\ &= \langle \{ (\mathrm{id} - \pi_\varepsilon^+) \circ R \mid R \in \mathcal{R}_\varepsilon^+ \} \rangle \\ &= \sum_{n=0}^{\infty} \sum_{(R_1,\dots,R_n) \in (\mathcal{R}_\varepsilon^+)^n} (\mathrm{id} - \pi_\varepsilon^+) \circ R_1 \circ \cdots \circ (\mathrm{id} - \pi_\varepsilon^+) \circ R_n \\ &= \sum_{n=0}^{\infty} \sum_{\substack{(R_1,\dots,R_n) \in (\mathcal{R}_\varepsilon^+)^n \\ (i_1,\dots,i_n) \in \{0,1\}^n}} (-\pi_\varepsilon^+)^{i_1} \circ R_1 \circ \cdots \circ (-\pi_\varepsilon^+)^{i_n} \circ R_n \end{aligned}$$

The terms s.t. $(i_1, \dots, i_n) = (0, \dots, 0)$ sum up to $\langle \mathcal{R}_\varepsilon^+ \rangle$, and for the rest we split the sums depending on $\ell := \min\{k \mid i_k = 1\}$:

$$\mathcal{H}_\varepsilon^{-1} \circ \langle \mathcal{R}_\varepsilon^- \rangle \circ \mathcal{H}_\varepsilon = \langle \mathcal{R}_\varepsilon^+ \rangle + \sum_{n=0}^{\infty} \sum_{\ell=1}^{n} \sum_{\substack{(R_1,\dots,R_n) \in (\mathcal{R}_\varepsilon^+)^n \\ i_1 = \cdots = i_{\ell-1} = 0,\, i_\ell = 1 \\ (i_{\ell+1},\dots,i_n) \in \{0,1\}^{n-\ell}}} (-\pi_\varepsilon^+)^{i_1} \circ R_1 \circ \cdots \circ (-\pi_\varepsilon^+)^{i_n} \circ R_n$$

Denoting $m = n - \ell + 1$, $(\tilde{R}_1, \dots, \tilde{R}_m) := (R_\ell, \dots, R_n)$ and $(j_2, \dots, j_m) := (i_{\ell+1}, \dots, i_n)$ yields

$$\begin{aligned} \mathcal{H}_\varepsilon^{-1} \circ \langle \mathcal{R}_\varepsilon^- \rangle \circ \mathcal{H}_\varepsilon &= \langle \mathcal{R}_\varepsilon^+ \rangle + \sum_{\ell,m \in \mathbb{N}^*} \sum_{(R_1,\dots,R_{\ell-1}) \in (\mathcal{R}_\varepsilon^+)^{\ell-1}} \sum_{\substack{(\tilde{R}_1,\dots,\tilde{R}_m) \in (\mathcal{R}_\varepsilon^+)^m \\ (j_2,\dots,j_m) \in \{0,1\}^{m-1}}} R_1 \circ \cdots \circ R_{\ell-1} \circ (-\pi_\varepsilon^+) \circ \tilde{R}_1 \\ &\qquad \circ (-\pi_\varepsilon^+)^{j_2} \circ \tilde{R}_2 \circ \cdots \circ (-\pi_\varepsilon^+)^{j_m} \circ \tilde{R}_m \\ &= \langle \mathcal{R}_\varepsilon^+ \rangle + \Bigg(\sum_{\substack{\ell \in \mathbb{N}^* \\ (R_k) \in (\mathcal{R}_\varepsilon^+)^{\ell-1}}} R_1 \circ \cdots \circ R_{\ell-1} \Bigg) \Bigg(\sum_{\substack{m \in \mathbb{N}^*,\, (\tilde{R}_k) \in (\mathcal{R}_\varepsilon^+)^m \\ (j_k) \in \{0,1\}^{m-1}}} (-\pi_\varepsilon^+) \circ \tilde{R}_1 \circ (-\pi_\varepsilon^+)^{j_2} \circ \tilde{R}_2 \circ \cdots \circ (-\pi_\varepsilon^+)^{j_m} \circ \tilde{R}_m \Bigg) \\ &= \langle \mathcal{R}_\varepsilon^+ \rangle - \Bigg(\sum_{\ell' \in \mathbb{N},\, (R_k) \in (\mathcal{R}_\varepsilon^+)^{\ell'}} R_1 \circ \cdots \circ R_{\ell'} \Bigg) \Bigg(\sum_{m \in \mathbb{N}^*,\, (\tilde{R}_k) \in (\mathcal{R}_\varepsilon^+)^m} \pi_\varepsilon^+ \circ \tilde{R}_1 \circ (\mathrm{id} - \pi_\varepsilon^+) \circ \tilde{R}_2 \circ \cdots \circ (\mathrm{id} - \pi_\varepsilon^+) \circ \tilde{R}_m \Bigg) \\ &= \langle \mathcal{R}_\varepsilon^+ \rangle - \langle \mathcal{R}_\varepsilon^+ \rangle \circ \langle \pi_\varepsilon^+ \mid \mathcal{R}_\varepsilon^+ \rangle \end{aligned}$$

□

Corollary 2.30: The matching condition of Definition 2.27 is equivalent to each of these equalities:

$$\sum_{p \in \mathbb{P}, \ell \in \mathbb{N}} \varepsilon^p \ln^\ell \varepsilon \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap (\mathbb{P} - p)} \sigma_d(\mathbf{u}_{p,\ell}) \phi_d = (\mathrm{id} - \langle \pi_\varepsilon^+ \mid \mathcal{R}_\varepsilon^+ \rangle) \circ \mathcal{H}_\varepsilon^{-1} \Bigg(\sum_{p \in \mathbb{P}, \ell \in \mathbb{N}} \varepsilon^p \ln^\ell \varepsilon \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap (p - \mathbb{P})} \sigma_d(S_{p,\ell}^\infty) \phi_d \Bigg)$$

$$\sum_{p \in \mathbb{P}, \ell \in \mathbb{N}} \varepsilon^p \ln^\ell \varepsilon \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap (p - \mathbb{P})} \sigma_d(S_{p,\ell}) \phi_d = (\mathrm{id} - \langle \pi_\varepsilon^- \mid \mathcal{R}_\varepsilon^- \rangle) \circ \mathcal{H}_\varepsilon \Bigg(\sum_{p \in \mathbb{P}, \ell \in \mathbb{N}} \varepsilon^p \ln^\ell \varepsilon \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap (p - \mathbb{P})} \sigma_d(\mathbf{u}_{p,\ell}^0) \phi_d \Bigg)$$

Proof: Let us prove only the first condition. Combining Proposition 2.26 with Theorem 2.29 shows that the matching condition is equivalent to:

$$\langle \mathcal{R}_\varepsilon^+ \rangle \Bigg(\sum_{p,\ell} \varepsilon^p \ln^\ell \varepsilon \sum_d \sigma_d(\mathbf{u}_{p,\ell}^0) \phi_d \Bigg) = \langle \mathcal{R}_\varepsilon^+ \rangle \circ (\mathrm{id} - \langle \pi_\varepsilon^+ \mid \mathcal{R}_\varepsilon^+ \rangle) \circ \mathcal{H}_\varepsilon^{-1} \Bigg(\sum_{p,\ell} \varepsilon^p \ln^\ell \varepsilon \sum_d \sigma_d(S_{p,\ell}^\infty) \phi_d \Bigg) \tag{2.35}$$

It remains to simplify both $\langle \mathcal{R}_\varepsilon^+ \rangle$. Proposition 2.14 implies: $\forall R \in \mathcal{R}_\varepsilon^+,\ \pi_\sigma^+ \circ R = 0$. Since $\langle \mathcal{R}_\varepsilon^+ \rangle$ is a sum of the identity and non-trivial products of elements of $\mathcal{R}_\varepsilon^+$, we have $\pi_\sigma^+ \circ \langle \mathcal{R}_\varepsilon^+ \rangle = \pi_\sigma^+ \circ \mathrm{id} = \pi_\sigma^+$. So applying π_σ^+ to (2.35) replaces the factors $\langle \mathcal{R}_\varepsilon^+ \rangle$ by π_σ^+. Now, $\operatorname{Im} \pi_\sigma^+$ is stable by $\mathrm{id} - \langle \pi_\varepsilon^+ \mid \mathcal{R}_\varepsilon^+ \rangle$ and the projector π_σ^+ is the identity on it, so we can finally simplify π_σ^+ on both sides of (2.35). □

For any (p,ℓ), let us define $\tau_{p,\ell} : \mathcal{A}_\varepsilon^\pm(\Pi) \to \mathcal{A}^\pm(\Pi)$ by: $\forall \varphi = \sum_{p',\ell'} \varepsilon^{p'} \ln^{\ell'} \varepsilon\, \varphi_{p',\ell'} \in \mathcal{A}_\varepsilon^\pm(\Pi),\ \tau_{p,\ell}(\varphi) := \varphi_{p,\ell}$. It allows us to define the matching coefficients, that are for any $(d,d',p,\ell) \in (\frac{\pi}{\Theta}\mathbb{Z}^*)^2 \times \mathbb{P} \times \mathbb{N}$ the following complex numbers:

$$c^{\mathbf{u} \leftarrow S}_{d,d',p,\ell} := \sigma_d \circ \tau_{p+d',\ell} \circ (\mathrm{id} - \langle \pi_\varepsilon^+ \mid \mathcal{R}_\varepsilon^+ \rangle)(\phi_{d'}) \quad \text{and} \quad c^{S \leftarrow \mathbf{u}}_{d,d',p,\ell} := \sigma_d \circ \tau_{p-d',\ell} \circ (\mathrm{id} - \langle \pi_\varepsilon^- \mid \mathcal{R}_\varepsilon^- \rangle)(\phi_{d'})$$

In addition, for any $a, b \in \mathbb{R}$, we denote $[\![a,b]\!]_{\mathbb{P},\mathbb{N}} := \{c \in \mathbb{R} \mid c - a \in \mathbb{P} \text{ and } b - c \in \mathbb{N}\}$. It is a finite subset of $[a,b]$.

Theorem 2.31 gives equations to concretely build the fields $\mathbf{u}_{p,\ell}$ and $S_{p,\ell}$ so that they match around the corner. It fixes their non-variational, which are determined by $\sigma_d(\mathbf{u}^0_{p,\ell})$ when $d < 0$ and $\sigma_d(S^\infty_{p,\ell})$ when $d > 0$ (see Theorems 3.6 and 3.11). It also provides inductive formulas, depending on the fields with smaller p. Moreover these formulas have a convolutive structure w.r.t. p and ℓ.

Theorem 2.31: The matching condition of Definition 2.27 is equivalent to the following set of equations:

$$\begin{cases} \forall (d,p,\ell) \in (-\frac{\pi}{\Theta}\mathbb{N}^*) \times \mathbb{P} \times \mathbb{N},\quad \sigma_d(\mathbf{u}^0_{p,\ell}) = \displaystyle\sum_{p' \in [\![0,p+d]\!]_{\mathbb{P},\mathbb{N}}} \sum_{\substack{d' \in \frac{\pi}{\Theta}\mathbb{Z}^* \\ p'-d' \in [\![0,p]\!]_{\mathbb{P},\mathbb{N}}}} \sum_{\ell'=0}^{\ell} c^{\mathbf{u} \leftarrow S}_{d,d',p-p',\ell-\ell'} \cdot \sigma_{d'}(S^\infty_{p',\ell'}) & \text{(2.36a)} \\ \forall (d,p,\ell) \in \frac{\pi}{\Theta}\mathbb{N}^* \times \mathbb{P} \times \mathbb{N},\quad \sigma_d(S^\infty_{p,\ell}) = \displaystyle\sum_{p' \in [\![0,p-d]\!]_{\mathbb{P},\mathbb{N}}} \sum_{\substack{d' \in \frac{\pi}{\Theta}\mathbb{Z}^* \\ p'+d' \in [\![0,p]\!]_{\mathbb{P},\mathbb{N}}}} \sum_{\ell'=0}^{\ell} c^{S \leftarrow \mathbf{u}}_{d,d',p-p',\ell-\ell'} \cdot \sigma_{d'}(\mathbf{u}^0_{p',\ell'}) & \text{(2.36b)} \end{cases}$$

Proof: Let $\mathtt{V}^{\mathtt{u}}$ be the set of families of complex numbers $(\mathtt{u}_{p,d,\ell})_{p \in \mathbb{P}, d \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap (\mathbb{P}-p), \ell \in \mathbb{N}}$ s.t. for any (p,d) only a finite number of the $\mathtt{u}_{p,d,\ell}$ are non-zero. We define $\mathtt{V}^{\mathtt{S}}$ similarly by replacing "$d \in \mathbb{P}-p$" with "$d \in p-\mathbb{P}$". Let $\sigma(\mathbf{u}) := (\sigma_d(\mathbf{u}^0_{p,\ell}))_{p,d,\ell} \in \mathtt{V}^{\mathtt{u}}$ and $\sigma(S) := (\sigma_d(S^\infty_{p,\ell}))_{p,d,\ell} \in \mathtt{V}^{\mathtt{S}}$. We define the following linear maps:

$$\mathtt{P} : \mathtt{V}^{\mathtt{S}} \to \mathtt{V}^{\mathtt{u}},\ (\mathtt{S}_{p,d,\ell})_{p,d,\ell} \mapsto \left(\sum_{p' \in [\![0,p+d]\!]_{\mathbb{P},\mathbb{N}}} \sum_{\substack{d' \in \frac{\pi}{\Theta}\mathbb{Z}^* \\ p'-d' \in [\![0,p]\!]_{\mathbb{P},\mathbb{N}}}} \sum_{\ell'=0}^{\ell} c^{\mathbf{u} \leftarrow S}_{d,d',p-p',\ell-\ell'} \cdot \mathtt{S}_{p',d',\ell'} \right)_{p,d,\ell}$$

$$\mathtt{Q} : \mathtt{V}^{\mathtt{u}} \to \mathtt{V}^{\mathtt{S}},\ (\mathtt{u}_{p,d,\ell})_{p,d,\ell} \mapsto \left(\sum_{p' \in [\![0,p-d]\!]_{\mathbb{P},\mathbb{N}}} \sum_{\substack{d' \in \frac{\pi}{\Theta}\mathbb{Z}^* \\ p'+d' \in [\![0,p]\!]_{\mathbb{P},\mathbb{N}}}} \sum_{\ell'=0}^{\ell} c^{S \leftarrow \mathbf{u}}_{d,d',p-p',\ell-\ell'} \cdot \mathtt{u}_{p',d',\ell'} \right)_{p,d,\ell}$$

These sums have a finite number of terms, so they are well-defined.

Step 1: Let $(d,p,\ell) \in \frac{\pi}{\Theta}\mathbb{Z}^* \times \mathbb{P} \times \mathbb{N}$. Applying $\sigma_d \circ \tau_{p,\ell}$ to the first equation of Corollary 2.30 gives:

$$\begin{aligned} \sigma_d(\mathbf{u}^0_{p,\ell}) &= \sum_{p' \in \mathbb{P}, \ell' \in \mathbb{N}} \sum_{d' \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap (\mathbb{P}-p')} \sigma_d \circ \tau_{p-p',\ell-\ell'} \circ (\mathrm{id} - \langle \pi_\varepsilon^+ \mid \mathcal{R}_\varepsilon^+ \rangle) \circ \mathcal{H}_\varepsilon^{-1}(\phi_{d'}) \cdot \sigma_{d'}(S^\infty_{p',\ell'}) \\ &= \sum_{p' \in \mathbb{P}, \ell' \in \mathbb{N}} \sum_{d' \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap (\mathbb{P}-p')} \sigma_d \circ \tau_{p-p'+d',\ell-\ell'} \circ (\mathrm{id} - \langle \pi_\varepsilon^+ \mid \mathcal{R}_\varepsilon^+ \rangle)(\phi_{d'}) \cdot \sigma_{d'}(S^\infty_{p',\ell'}) \\ &= \sum_{p' \in \mathbb{P}} \sum_{d' \in \frac{\pi}{\Theta}\mathbb{Z}^*, p'+d' \in \mathbb{P}} \sum_{\ell' \in \mathbb{N}} c^{\mathbf{u} \leftarrow S}_{d,d',p-p',\ell-\ell'} \cdot \sigma_{d'}(S^\infty_{p',\ell'}). \end{aligned}$$

Moreover, for any $R \in \mathcal{R}_\varepsilon^+$, we have $\deg_\varepsilon R \in \mathbb{N}$ and $\deg_\varepsilon R + \deg_{\mathcal{A}} R \in \mathbb{N}$ (see Figure 9). So for $c^{\mathbf{u} \leftarrow S}_{d,d',p-p',\ell-\ell'}$ to be non-zero, we need $p - p' + d' \in \mathbb{N}$ and $(p - p' + d') + d - d' \in \mathbb{N}$. In addition, $\ell - \ell' \in \mathbb{N}$. Therefore, the matching condition in equivalent to $\sigma(\mathbf{u}) = \mathtt{P}\,\sigma(S)$. Similarly, it is equivalent

to $\sigma(S) = \mathtt{Q}\,\sigma(\mathbf{u})$.

Step 2: We define the subspaces $\mathtt{V}^{\mathtt{u}}_{\pm} = \{\mathtt{u} \in \mathtt{V}^{\mathtt{u}} \mid \forall (p,d,\ell),\ \mathtt{u}_{p,d,\ell} \neq 0 \Rightarrow \pm d > 0\}$ and likewise $\mathtt{V}^{\mathtt{S}}_{\pm}$. So $\mathtt{V}^{\mathtt{u}} = \mathtt{V}^{\mathtt{u}}_{+} \oplus \mathtt{V}^{\mathtt{u}}_{-}$ and similarly for $\mathtt{V}^{\mathtt{S}}$. Let $\sigma(\mathbf{u})_{\pm}$ be the components of $\sigma(\mathbf{u})$ on $\mathtt{V}^{\mathtt{u}}_{\pm}$, and similarly for $\sigma(S)_{\pm}$. We decompose $\mathtt{P}$ and $\mathtt{Q}$ on those subspaces, which gives in block matrix notation:

$$\mathtt{P} = \underbrace{\left(\begin{array}{c}\mathtt{P}_+\\\hline \mathtt{P}_-\end{array}\right)}_{\mathtt{V}^{\mathtt{S}}} \begin{matrix}\}\,\mathtt{V}^{\mathtt{u}}_+\\ \}\,\mathtt{V}^{\mathtt{u}}_-\end{matrix} \qquad \text{and} \qquad \mathtt{Q} = \left(\begin{array}{c}\mathtt{Q}_+\\\hline \mathtt{Q}_-\end{array}\right) = \underbrace{\left(\begin{array}{c|c}\mathtt{Q}^+_+ & \mathtt{Q}^-_+\\\hline \mathtt{Q}^+_- & \mathtt{Q}^-_-\end{array}\right)}_{\mathtt{V}^{\mathtt{u}}_+ \quad \mathtt{V}^{\mathtt{u}}_-} \begin{matrix}\}\,\mathtt{V}^{\mathtt{S}}_+\\ \}\,\mathtt{V}^{\mathtt{S}}_-\end{matrix}$$

The present theorem rewrites as:

$$\left\{\begin{array}{l}\sigma(\mathbf{u})_- = \mathtt{P}_-\,\sigma(S)\\ \sigma(S)_+ = \mathtt{Q}_+\,\sigma(\mathbf{u})\end{array}\right. \tag{2.37}$$

By Step 1, this is clearly a necessary condition for the matching. It remains to prove that it is sufficient. Let us show that (2.37) implies one of the two conditions of Step 1, e.g. $\sigma(\mathbf{u}) = \mathtt{P}\,\sigma(S)$.
Note that by Step 1 we have for any $\mathtt{u} \in \mathtt{V}^{\mathtt{u}}$ and $\mathtt{S} \in \mathtt{V}^{\mathtt{S}}$: $\mathtt{u} = \mathtt{PS} \iff \mathtt{S} = \mathtt{Qu}$. So $\mathtt{P}$ and $\mathtt{Q}$ are inverses of each other. Hence $\mathtt{Q}(\sigma(\mathbf{u}) - \mathtt{P}\,\sigma(S)) = \mathtt{Q}\,\sigma(\mathbf{u}) - \sigma(S)$. Projecting this onto $\mathtt{V}^{\mathtt{S}}_+$ we deduce:

$$\mathtt{Q}^+_+(\sigma(\mathbf{u})_+ - \mathtt{P}_+\,\sigma(S)) + \mathtt{Q}^-_+\underbrace{(\sigma(\mathbf{u})_- - \mathtt{P}_-\,\sigma(S))}_{=0} = \underbrace{\mathtt{Q}_+\,\sigma(\mathbf{u}) - \sigma(S)_+}_{=0}.$$

Let us show that $\mathtt{Q}^+_+$ is injective.

> Let $\mathtt{u} \in \mathtt{V}^{\mathtt{u}}_+ \setminus \{0\}$ and let us show that $\mathtt{Q}^+_+\mathtt{u} \neq 0$. Let (p,d,ℓ) be the smallest triplet for lexicographic order such that $\mathtt{u}_{p,d,\ell} \neq 0$. The term of $\mathtt{Q}^+_+\mathtt{u}$ of index $(p+d,d,\ell)$ is :
>
> $$\sum_{p' \in [\![0,p]\!]_{\mathbb{Z},\mathbb{N}}} \sum_{\substack{d' \in \frac{\pi}{\Theta}\mathbb{N}^* \\ p'+d' \in [\![0,p+d]\!]_{\mathbb{Z},\mathbb{N}}}} \sum_{\ell'=0}^{\ell} c^{S \leftarrow \mathbf{u}}_{d,d',p+d-p',\ell-\ell'} \cdot \mathtt{u}_{p',d',\ell'} \;=\; c^{S \leftarrow \mathbf{u}}_{d,d,d,0} \cdot \mathtt{u}_{p,d,\ell}.$$
>
> We claim that it is non-zero because $c^{S \leftarrow \mathbf{u}}_{d,d,d,0} = 1$. Indeed, in the sum
>
> $$\langle \pi^+_\varepsilon \mid \mathcal{R}^+_\varepsilon \rangle := \sum_{n=1}^{\infty} \sum_{(R_1,\dots,R_n) \in (\mathcal{R}^+_\varepsilon)^n} \pi^+_\varepsilon \circ R_1 \circ (\mathrm{id} - \pi^+_\varepsilon) \circ R_2 \circ \cdots \circ (\mathrm{id} - \pi^+_\varepsilon) \circ R_n$$
>
> there is no term of degree $(0,0)$ w.r.t. $\mathcal{A}$ and ε. So the component of $\langle \pi^+_\varepsilon \mid \mathcal{R}^+_\varepsilon \rangle(\phi_d)$ in $\varepsilon^0 \mathcal{A}_d(\Pi)[\ln \varepsilon]$ is zero. Thus $c^{S \leftarrow \mathbf{u}}_{d,d,d,0} = \sigma_d \circ \tau_{0,0} \circ (\mathrm{id} - \langle \pi^+_\varepsilon \mid \mathcal{R}^+_\varepsilon \rangle)(\phi_d) = \sigma_d(\phi_d) = 1$. So $\mathtt{Q}^+_+\mathtt{u} \neq 0$.

We have proven that $\sigma(\mathbf{u})_+ = \mathtt{P}_+\,\sigma(S)$. Given that $\sigma(\mathbf{u})_- = \mathtt{P}_-\,\sigma(S)$, we deduce $\sigma(\mathbf{u}) = \mathtt{P}\,\sigma(S)$. □

Remarks 2.32:

- Thanks to the tools of Section 2.3, we can compute exactly and very quickly the coefficients $c^{\mathbf{u} \leftarrow S}_{d,d',p,\ell}$ and $c^{S \leftarrow \mathbf{u}}_{d,d',p,\ell}$. Moreover, these coefficients depend only on Θ, ω and $(\mu_0,\mu_1,\rho_0,\rho_1)$, but not on Ω_1 nor precisely on the functions μ and ρ.
- In the sums of Theorem 2.31, the indexes d and d' satisfy $d - d' \in \mathbb{Z} \cap \frac{\pi}{\Theta}\mathbb{Z}$. Indeed, on the one hand $d, d' \in \frac{\pi}{\Theta}\mathbb{Z}^*$. On the other we have in (2.36a) that $p' \in [\![0, p+d]\!]_{\mathbb{Z},\mathbb{N}}$ and $p' - d' \in [\![0,p]\!]_{\mathbb{Z},\mathbb{N}}$, so $d - d' = (p+d-p') - (p-(p'-d')) \in \mathbb{N} - \mathbb{N} \subset \mathbb{Z}$ (and likewise in (2.36b)). Note that the set $\mathbb{Z} \cap \frac{\pi}{\Theta}\mathbb{Z}$ can be very small. If $\Theta \in \pi\mathbb{Q}$, then $\mathbb{Z} \cap \frac{\pi}{\Theta}\mathbb{Z} = b\mathbb{Z}$ where $\Theta = \pi\frac{a}{b}$ with $(a,b) \in \mathbb{N} \times \mathbb{N}^*$ and $\gcd(a,b) = 1$. Otherwise, $\mathbb{Z} \cap \frac{\pi}{\Theta}\mathbb{Z} = \{0\}$.

3 Construction of the asymptotic expansion

Equations 1.3–1.5 and Theorem 2.31 give the equations that the fields $u_{p,\ell}$, $U_{p,\ell}$ and $S_{p,\ell}$ must satisfy. In this section, we will build these fields according to those conditions. First of all, let us express the layer fields with the far fields, so that only two types of fields remain to build. Let $(\mathcal{U}_n) \in \mathbb{R}[Y]^{\mathbb{N}}$ be the unique sequence of polynomials s.t. for any $n \in \mathbb{N}^*$:

$$\left\{\begin{array}{r}\mathcal{U}_0'' = 0\\ \mathcal{U}_0'(0) = \frac{\mu_0}{\mu_1}\\ \mathcal{U}_0(-1) = 0\end{array}\right. \qquad \text{and} \qquad \left\{\begin{array}{r}\mathcal{U}_n'' = -\mathcal{U}_{n-1}\\ \mathcal{U}_n'(0) = 0\\ \mathcal{U}_n(-1) = 0\end{array}\right. \tag{3.1}$$

Lemma 3.1: expression of the layer fields
Let us assume that the fields $u_{p,\ell}$ and $U_{p,\ell}$ are regular enough (we will check later that they are). (1.4) implies for any $(p,\ell) \in \mathbb{P} \times \mathbb{N}$ and $(x,Y) \in \Lambda$:

$$U_{p,\ell}(x,Y) = \sum_{n=0}^{\infty} (\partial_x^2 + k_1^2)^n \partial_y u_{p-1-2n,\ell}(x,0) \cdot \mathcal{U}_n(Y). \tag{3.2}$$

where this sum has a finite number of non-zero terms by the convention: $\forall p \in \mathbb{R} \setminus \mathbb{P}, \forall \ell \in \mathbb{N},\ u_{p,\ell} := 0$.

Proof: There exists an increasing sequence (p_m) s.t. $\mathbb{P} = \{p_m \mid m \in \mathbb{N}\}$. So we can prove the result by induction on $p \in \mathbb{P}$. For $p = 0$, (1.4) states:

$$\begin{cases} \partial_Y^2 U_{0,\ell} = 0 & \text{in } \Lambda \\ \partial_Y U_{0,\ell} = 0 & \text{on } \Gamma \\ U_{0,\ell} = 0 & \text{on } \Sigma_\Lambda \end{cases}$$

so $U_{0,\ell} = 0$ for any ℓ. It is coherent (3.2) (which is a sum of zeros in this case).
Next, for the inductive step, we assume that (3.2) holds for ranks smaller that p. (1.4) gives:

$$\begin{cases} \partial_Y^2 U_{p,\ell} = -(\partial_x^2 + k_1^2) U_{p-2,\ell} = -\sum_{n=0}^{\infty} (\partial_x^2 + k_1^2)^{n+1} \partial_y u_{(p-2)-1-2n,\ell}(x,0) \cdot \mathcal{U}_n(Y) \\ \qquad\qquad = -\sum_{n=1}^{\infty} (\partial_x^2 + k_1^2)^{n} \partial_y u_{p-1-2n,\ell}(x,0) \cdot \mathcal{U}_{n-1}(Y) & \text{in } \Lambda \\ \partial_Y U_{p,\ell} = \frac{\mu_0}{\mu_1} \partial_y u_{p-1,\ell} & \text{on } \Gamma \\ U_{p,\ell} = 0 & \text{on } \Sigma_\Lambda \end{cases}$$

It is easy to see that $\sum_{n=0}^{\infty} (\partial_x^2 + k_1^2)^n \partial_y u_{p-1-2n,\ell}(x,0) \cdot \mathcal{U}_n(Y)$ is the only solution of this. □

We saw in Section 2 that far and corner fields possess singularities when $r \to 0$, resp. $\mathbf{r} \to \infty$. So the usual variational frameworks are not sufficient to build these fields and we need to design new frameworks. It is done in Sections 3.1 and 3.2. In both sections we start by introducing the natural space H_{var} in which an ad hoc variational problem is well-posed. Then we define a bigger space $H := H_{\text{var}} + \chi \mathcal{A}(D) = \{u + \chi\varphi \mid u \in H_{\text{var}}, \varphi \in \mathcal{A}(D)\}$ that contains the singularities, where χ is a $\mathcal{C}^\infty$ truncation function in the vicinity of 0 (for $u_{p,\ell}$) or infinity (for $S_{p,\ell}$), and $D \in \{\Omega, \Pi\}$. Next we determine the elements of $H_{\text{var}} \cap \chi\mathcal{A}(D)$, which allows us to define on H the linear forms σ_d associated to the singularities. Finally we show that some model problems are well-posed in H.

Before we start, the following lemma is a tool to estimate the behavior at 0 and ∞ of functions of $\mathcal{A}$.

Lemma 3.2: Let $a < b$ in $\mathbb{R}$, $n \in \mathbb{N}^*$, $(d_i, \ell_i)_{i \in [\![1,n]\!]}$ be n distinct elements of $\mathbb{R} \times \mathbb{N}$, $(f_i)_{i \in [\![1,n]\!]} \in (\mathcal{C}([a,b], \mathbb{C}) \setminus \{0\})^n$ and:

$$\varphi : (r,\theta) \in \mathbb{R}_+^* \times [a,b] \ \mapsto \ \sum_{i=1}^{n} r^{d_i} \ln^{\ell_i} r \cdot f_i(\theta).$$

Then there is an interval $I \subset [a,b]$ with non-empty interior, $c \in \mathbb{R}_+^*$ and $r_1, r_2 \in \mathbb{R}$ such that:

$$\forall r \in (0, r_1), \forall \theta \in I, \quad |\varphi(r,\theta)| > c\, r^{\min_j d_j} \quad \text{and} \quad \forall r \in (r_2, \infty), \forall \theta \in I, \quad |\varphi(r,\theta)| > c\, r^{\max_j d_j}.$$

Proof: Let $j \in [\![1,n]\!]$ be s.t. (d_j, ℓ_j) is maximal for the lexicographic order. Let $I \subset [a,b]$ be a non trivial interval on which $|f_j|$ is greater than a positive constant. Since $\sum_{i \neq j} r^{d_i} \ln^{\ell_i} r \cdot f_i(\theta) = o(r^{d_j} \ln^{\ell_j} r)$, we have when $r \to \infty$ and $\theta \in I$: $|\varphi(r,\theta)| \gtrsim r^{d_j} \ln^{\ell_j} r \gtrsim r^{d_j}$. And we can similarly treat the vicinity of 0. □

We can apply Lemma 3.2 to any $\varphi \in \mathcal{A}_d(\Omega)$ with $(a,b) := (0,\Theta)$, or to any $\varphi \in \mathcal{A}_d(\Lambda)$ with $(a,b) := (-1,0()$ (replacing the variables (r,θ) by (x,Y)). We can also apply it to $\partial_r\varphi$ and $\partial_\theta\varphi$ when $\varphi \in \mathcal{A}_d(\Omega)$ and to $\partial_x\varphi$ and $\partial_Y\varphi$ when $\varphi \in \mathcal{A}_d(\Lambda)$.

Definition: For any $D \in \{\Pi, \Omega, \Gamma, \Lambda\}$ and $\varphi \in \mathcal{A}(D)$, we denote $\deg_{\min} \varphi := \sup\{d \in \mathbb{R} \mid \varphi \in \sum_{q \geqslant d} \mathcal{A}_q(D)\}$ and $\deg_{\max} \varphi := \inf\{d \in \mathbb{R} \mid \varphi \in \sum_{q \leqslant d} \mathcal{A}_q(D)\}$.

3.1 Existence and uniqueness for far fields-like problems

We denote $H_{00}^{1/2}(\Gamma)$ the set of functions of $H^{1/2}(\Gamma)$ whose extension by 0 to $\partial\Omega = \Gamma \cup \{(0,0)\} \cup \Sigma_\Omega$ is in $H^{1/2}(\partial\Omega)$. Using Lax-Milgram theorem, it is easy to prove the following lemma.

Lemma 3.3: the Helmholtz problem in $H^1(\Omega)$
Let $f \in (H^1(\Omega))'$ and $g \in H_{00}^{1/2}(\Gamma)$. The following system has a unique solution in $H^1(\Omega)$.

$$\begin{cases} \mu_0 \Delta u + \omega^2 \rho_0 u = f & \text{in } \Omega \\ u = g & \text{on } \Gamma \\ u = 0 & \text{on } \Sigma_\Omega \end{cases}$$

Let χ_0 be a radial function of $\mathcal{C}^\infty(\mathbb{R}^2)$ equal to 1 in the vicinity of 0 and to 0 in the vicinity of infinity. The appropriate space to build the far fields is $H^1(\Omega) + \chi_0 \mathcal{A}(\Omega)$. One can check that it does not depend on the choice of χ_0.

Lemma 3.4: $H^1(\Omega) \cap \chi_0 \mathcal{A}(\Omega) = \chi_0 \sum_{d>0} \mathcal{A}_d(\Omega)$.

Proof: The inclusion $\supset$ is easy to check, so we focus on $\subset$. Let $\varphi \in \mathcal{A}(\Omega) \setminus \{0\}$ be s.t. $\chi_0 \varphi \in H^1(\Omega)$. There is $d \in \mathbb{R}$, $\varphi_1 \in \mathcal{A}_d(\Omega) \setminus \{0\}$ and $\varphi_2 \in \sum_{q>d} \mathcal{A}_q(\Omega)$ s.t. $\varphi = \varphi_1 + \varphi_2$. If $\partial_\theta \varphi_1$ were null everywhere, then would so too φ_1 because $\varphi_{1|\Sigma_\Omega} = 0$. But we assumed the contrary, so $\partial_\theta \varphi_1 \neq 0$. Thus Lemma 3.2 implies that there is a non-trivial interval $I \subset [0, \Theta]$ s.t. $|\partial_\theta \varphi_1(r,\theta)| \gtrsim r^d$ when $r \to 0$ and $\theta \in I$. Finally $r^{-1} \partial_\theta(\chi_0 \varphi) \in L^2(\Omega)$ implies that $d > 0$. □

Définition 3.5: For any $u \in H^1(\Omega) + \chi_0 \mathcal{A}(\Omega)$ and $d \in -\frac{\pi}{\Theta}\mathbb{N}^*$, we denote $\sigma_d(u) := \sigma_d(\varphi)$ where $\varphi \in \mathcal{A}(\Omega)$ is s.t. $u - \chi_0 \varphi \in H^1(\Omega)$ (see Definition 2.13 for $\sigma_d(\varphi)$). It does not depend on the choice of φ thanks to Lemma 3.4.

Theorem 3.6: existence and uniqueness for a far fields-like model problem
Let $f \in (H^1(\Omega))'$, $g \in H_{00}^{1/2}(\Gamma) + \chi_0 \mathcal{A}(\Gamma)$ and $(s_d) \in \mathbb{C}^{-\frac{\pi}{\Theta}\mathbb{N}^*}$ with finite support. The following system has a unique solution in $H^1(\Omega) + \chi_0 \mathcal{A}(\Omega)$.

$$\begin{cases} \mu_0 \Delta u + \omega^2 \rho_0 u = f & \text{in } \Omega \\ u = g & \text{on } \Gamma \\ u = 0 & \text{on } \Sigma_\Omega \\ \sigma_d(u) = s_d & \forall d \in -\frac{\pi}{\Theta}\mathbb{N}^* \end{cases}$$

Proof: Let us show the existence first, and then the uniqueness.
Existence: Let $\tilde{g} \in H_{00}^{1/2}(\Gamma)$ and $\varphi \in \mathcal{A}(\Gamma)$ be s.t. $g = \tilde{g} + \chi_0 \varphi$. We look for the solution in the form $u = \tilde{u} + \chi_0 \psi$ with $\tilde{u} \in H^1(\Omega)$ and $\psi \in \mathcal{A}(\Omega)$. Let

$$\psi^+ := \sum_{n=0}^{\infty} (-k_0^2 R_\Delta^\Omega)^n \left(R_{\mathrm{D}}^\Omega(\varphi) + \sum_{d \in -\frac{\pi}{\Theta}\mathbb{N}^*} s_d \, \phi_d^\Omega \right) \in \mathcal{A}^+(\Omega)$$

(where $\phi_d^\Omega := \phi_{d|\Omega}$) and $\psi := T_{\leqslant 2}(\psi^+)$. Using Proposition 2.15 one can check by calculus the first system below (see also Lemma 2.18 for a similar result). Then, the second system below derives from $\psi \in \mathcal{A}(\Omega)$,

$\psi = \psi^+ - (\mathrm{id} - T_{\leqslant 2})(\psi^+)$ and $\deg \Delta = -2$ (by Lemma 2.16).

$$\begin{cases} (\mu_0\Delta + \omega^2\rho_0)\psi^+ = 0 & \text{in } \Omega \\ \psi^+ = \varphi & \text{on } \Gamma \\ \psi^+ = 0 & \text{on } \Sigma_\Omega \\ \sigma_d(\psi^+) = s_d & \forall d \in -\frac{\pi}{\Theta}\mathbb{N}^* \end{cases} \quad \text{so} \quad \begin{cases} (\mu_0\Delta + \omega^2\rho_0)\psi \in \mathcal{A}(\Omega) \cap \sum\limits_{d>0} \mathcal{A}_d(\Omega) = \sum\limits_{d>0} \mathcal{A}_d(\Omega) \\ \psi_{|\Gamma} - \varphi \in \mathcal{A}(\Gamma) \cap \sum\limits_{d>2} \mathcal{A}_d(\Gamma) = \sum\limits_{d>2} \mathcal{A}_d(\Gamma) \\ \psi_{|\Sigma_\Omega} = 0 \\ \sigma_q(\psi) = s_d \quad \forall d \in -\frac{\pi}{\Theta}\mathbb{N}^* \end{cases}$$

Therefore $(\mu_0\Delta + \omega^2\rho_0)(\chi_0\psi) \in L^2(\Omega)$ and $\chi_0\,(\psi_{|\Gamma} - \varphi) \in H^{1/2}_{00}(\Gamma)$. Finally, thanks to Lemma 3.3, we can take $\tilde{u}$ as the unique solution in $H^1(\Omega)$ of:

$$\begin{cases} \mu_0\Delta\tilde{u} + \omega^2\rho_0\tilde{u} = f - (\mu_0\Delta + \omega^2\rho_0)(\chi_0\psi) & \text{in } \Omega \\ \tilde{u} = \tilde{g} - \chi_0\,(\psi - \varphi) & \text{on } \Gamma \\ \tilde{u} = 0 & \text{on } \Sigma_\Omega \end{cases}$$

Uniqueness: Let u be a homogeneous solution. There is $\tilde{u} \in H^1(\Omega)$ and $\psi \in \mathcal{A}(\Omega)$ such that $u = \tilde{u} + \chi_0\psi$. Let $\varphi := \Delta\psi + k_0^2\psi$. By Lemma 2.16, $\varphi \in \mathcal{A}(\Omega)$. Let us show that $\deg_{\min}\varphi > -2$.

> We denote $d := \deg_{\min}\varphi$ and $\widetilde{\varphi} := (\Delta + k_0^2)(\chi_0\psi)$. By Lemma 3.2, there is a non-trivial interval $I \subset [0,\Theta]$ s.t. when $r \to 0$ and $\theta \in I$: $\widetilde{\varphi}(r,\theta) = \varphi(r,\theta) \gtrsim r^d$. Besides, $\widetilde{\varphi} = -(\Delta + k_0^2)\tilde{u} \in (H^1_0(\Omega))'$. Let us test it with $\zeta_q : (r,\theta) \mapsto \widetilde{\varphi}(r,\theta)\cdot r^{q-2d-2}\left(1-\chi_0(2^{1/q}r)\right)\chi(\theta)$ where $q > 0$ and $\chi \in \mathcal{D}(0,\Theta)\setminus\{0\}$ is everywhere non negative. Since $\zeta_q \in \mathcal{D}(\Omega)$:
>
> $$\langle\widetilde{\varphi},\zeta_q\rangle := \int_0^\infty \int_0^\Theta \widetilde{\varphi}\cdot\zeta_q \cdot r\mathrm{d}\theta\,\mathrm{d}r \lesssim \|\zeta_q\|_{H^1(\Omega)} \qquad \forall q > 0$$
>
> Let us assume by contradiction that $d \leqslant -2$. It is not difficult to check that when $q \to 0^+$:
>
> $$\langle\widetilde{\varphi},\zeta_q\rangle \gtrsim \int_{c_1 2^{-\frac{1}{q}}}^{c_2} r^d\, r^{q-d-2}\, r\mathrm{d}r \gtrsim \frac{1}{q} \quad \text{and} \quad \|\zeta_q\|_{H^1} \lesssim \|\nabla\zeta_q\|_{L^2} \lesssim \left(\int_{c_3 2^{-\frac{1}{q}}}^{c_4} r^{2(q-d-3)}\, r\mathrm{d}r\right)^{\frac{1}{2}} \lesssim \frac{1}{\sqrt{q}}$$
>
> with some constants c_i and Poincaré's inequality. But it contradicts $\langle\widetilde{\varphi},\zeta_q\rangle \lesssim \|\zeta_q\|_{H^1}$. So $d > -2$.

Now let us show that $\deg_{\min}\psi > 0$.

> Let us assume the contrary. Then there is $d \leqslant 0$, $\psi_1 \in \mathcal{A}_d(\Omega)\setminus\{0\}$ and $\psi_2 \in \sum_{q>d}\mathcal{A}_q(\Omega)$ s.t. $\psi = \psi_1 + \psi_2$. Lemma 2.16 states that $\deg\Delta = -2$, so $\Delta\psi_1 \in \mathcal{A}_{d-2}(\Omega)$ and $\Delta\psi_2 \in \sum_{q>d-2}\mathcal{A}_q(\Omega)$. In addition, $\varphi \in \sum_{q>-2}\mathcal{A}_q(\Omega)$ and $\mathcal{A}(\Omega) = \bigoplus_{q\in\mathbb{R}}\mathcal{A}_q(\Omega)$ according to Lemma 2.8. So identifying the coordinate of $\Delta\psi + k_0^2\psi = \varphi$ in $\mathcal{A}_{d-2}(\Omega)$ gives $\Delta\psi_1 = 0$.
> Moreover, $\psi_{|\Gamma} = -\tilde{u}_{|\Gamma} \in H^{1/2}(\Gamma)$, so $\psi_{|\Gamma} \in \sum_{q>0}\mathcal{A}_q(\Gamma)$. Hence:
>
> $$\begin{cases} \Delta\psi_1 = 0 & \text{in } \Omega \\ \psi_1 = 0 & \text{on } \Sigma_\Omega \cup \Gamma \end{cases}$$
>
> By Proposition 2.12, it implies $d \in -\frac{\pi}{\Theta}\mathbb{N}^*$ and $\psi_1 = \sigma_d(\psi_1)\,\phi_d^\Omega$. However, $\sigma_d(\psi_1) = \sigma_d(\psi) = \sigma_d(u) = 0$, which contradicts $\psi_1 \neq 0$.

Finally, Lemma 3.4 implies $\chi_0\psi \in H^1(\Omega)$, so $u \in H^1(\Omega)$. This means that $u = 0$ by Lemma 3.3. □

3.2 Existence and uniqueness for corner fields-like problems

Definition 3.7: the variational space V
We define $V := \{v \in H^1_{\mathrm{loc}}(\Omega_1) \mid \nabla v \in L^2(\Omega_1) \text{ and } v_{|\partial\Omega_1} = 0\}$ and the norm $\|v\|_V := \|\nabla v\|_{L^2(\Omega_1)}$.
Here "$v_{|\partial\Omega_1} = 0$" means that $\chi_0(\frac{\cdot}{R})\,v \in H^1_0(\Omega_1)$ for any $R > 0$, where $\chi_0(\frac{\cdot}{R}) : (X,Y) \mapsto \chi_0\big(\frac{X}{R},\frac{Y}{R}\big)$.

Lemma 3.7: Any $v \in V$ satisfies $\|v\|_{H^1(\Lambda\cap\{X>R_c\})} \lesssim \|v\|_V$ and $\left\|\frac{v}{1+r}\right\|_{L^2(\Omega_1)} \lesssim \|v\|_V$.

Proof: We denote $A := \Lambda \cap \{X > R_c\}$ and $B := B(0, R_c)$, and we recall that $R_c > 0$ is s.t. $\Omega_1 \setminus B = \Pi \setminus B$. Poincaré's inequality gives for a.e. $X > R_c$: $\|v(X,\cdot)\|_{L^2(-1,0)} \lesssim \|\partial_Y v(X,\cdot)\|_{L^2(-1,0)}$. Integrating w.r.t. X the square of this then gives: $\|v\|_{L^2(A)} \lesssim \|\partial_Y v\|_{L^2(A)} \leqslant \|v\|_V$. So $\|v\|_{H^1(A)} \lesssim \|v\|_V$.
Next, a generalized Poincaré inequality on circular slices of $\Omega \setminus B$ gives for a.e. $r > R_c$: $\|v(r,\cdot)\|_{L^2(0,\pi)} \lesssim \|\partial_\theta v(r,\cdot)\|_{L^2(0,\pi)} + |v(r,\theta=0)|$. Then integrating w.r.t. r the square of this times $\frac{1}{1+r}$ gives:

$$\int_{\Omega\setminus B} \frac{v^2}{(1+r)r}\, r\mathrm{d}\theta\, \mathrm{d}r \lesssim \int_{\Omega\setminus B} \frac{(\partial_\theta v)^2}{(1+r)r}\, r\mathrm{d}\theta\, \mathrm{d}r + \int_{\Gamma\setminus B} \frac{v^2}{1+r}\mathrm{d}r.$$

Therefore $\|\frac{v}{1+r}\|^2_{L^2(\Omega\setminus B)} \lesssim \|\nabla v\|^2_{L^2(\Omega\setminus B)} + \|v\|^2_{L^2(\Gamma\setminus B)}$. But we know that $\|v\|_{L^2(\Gamma\setminus B)} \lesssim \|v\|_{H^1(A)} \lesssim \|v\|_V$, so $\|\frac{v}{1+r}\|_{L^2(\Pi\setminus B)} \lesssim \|v\|_V$. Finally Poincaré's inequality in B also gives $\|\frac{v}{1+r}\|_{L^2(\Omega_1\cap B)} \lesssim \|v\|_V$. □

Lemma 3.8: the Poisson problem in Ω_1
Let $F : \Omega_1 \to \mathbb{C}$ be s.t. $(1+r)F \in L^2(\Omega_1)$, and $g \in L^2(\Gamma \cap \{X > R_c\})$. The following system is well-posed in V.

$$\begin{cases} \mathrm{div}(\mu\nabla S) = F & \text{in } \Omega_1 \setminus (\Gamma \cap \{X > R_c\}) \\ S = 0 & \text{on } \partial\Omega_1 \\ S_{|Y=0^+} - S_{|Y=0^-} = 0 & \text{on } \Gamma \cap \{X > R_c\} \\ \mu_0 \partial_Y S_{|Y=0^+} - \mu_1 \partial_y S_{|Y=0^-} = g & \text{on } \Gamma \cap \{X > R_c\} \end{cases}$$

Proof: The variational formulation of this problem is:

$$\forall v \in V, \qquad \int_{\Omega_1} \mu\nabla S \cdot \nabla v = -\int_{\Omega_1} Fv + \int_{\Gamma\cap\{X>R_c\}} gv.$$

The left-hand side is coercive by definition of V. Lemma 3.7 ensures that the right-hand side is continuous. Moreover it is easy to see that V is complete. So we can conclude using the Lax-Milgram theorem. □

Let χ_∞ be a function of $\mathcal{C}^\infty(\mathbb{R}^2)$ equal to 0 on $B(0,R_c)$ and 1 in a vicinity of infinity. The proper space to build the corner fields is $V + \chi_\infty \mathcal{A}(\Pi)$. One can check that it does not depend on the choice of χ_∞.

Lemma 3.9: $V \cap \chi_\infty \mathcal{A}(\Pi) = \left\{ \chi_\infty \varphi \;\middle|\; \varphi \in \mathcal{A}(\Pi),\ \deg_{\max}(\varphi_{|\Omega}) < 0 \text{ and } \deg_{\max}(\varphi_{|\Lambda}) < -\frac{1}{2} \right\}$.

Proof: The inclusion $\supset$ is easy to check, so we focus on $\subset$. Let $\varphi \in \mathcal{A}(\Pi)$ be s.t. $\chi_\infty\varphi \in V$. The same method as Lemma 3.4 shows that $\deg_{\max} \varphi_{|\Omega} < 0$. And using that $\partial_Y(\chi_\infty\varphi) \in L^2(\Lambda)$ we likewise get $\deg_{\max}(\varphi_{|\Lambda}) < -\frac{1}{2}$. □

Definition 3.10: For any $S \in V + \chi_\infty\mathcal{A}(\Pi)$ and $d \in \frac{\pi}{\Theta}\mathbb{N}^*$, we denote $\sigma_d(S) := \sigma_d(\varphi)$ where $\varphi \in \mathcal{A}(\Pi)$ is s.t. $S - \chi_\infty\varphi \in V$. It does not depend on the choice of φ thanks to Lemma 3.9.

Theorem 3.11: existence and uniqueness for a corner fields-like model problem
Let $f : \Omega_1 \to \mathbb{C}$ be s.t. $(1+r)F \in L^2(\Omega_1)$, $\varphi : \Pi \to \mathbb{C}$ be s.t. $\varphi_{|\Omega} \in \mathcal{A}(\Omega)$ and $\varphi_{|\Lambda} \in \mathcal{A}(\Lambda)$, and $(s_d) \in \mathbb{C}^{\frac{\pi}{\Theta}\mathbb{N}^*}$ with finite support. The following system has a unique solution in $V + \chi_\infty\mathcal{A}(\Pi)$.

$$\begin{cases} \mathrm{div}(\mu\nabla S) = F + \chi_\infty\varphi & \text{in } \Omega_1 \\ S = 0 & \text{on } \partial\Omega_1 \\ \sigma_d(S) = s_d & \forall d \in \frac{\pi}{\Theta}\mathbb{N}^* \end{cases}$$

Remark: φ is not just an element of $\mathcal{A}(\Pi)$ because it may be discontinuous on Γ.

Proof: This proof is similar to Theorem 3.6. Let us show the existence first, and then the uniqueness.
Existence: We look for the solution in the form $S = \tilde{S} + \chi_\infty \psi$ with $\tilde{S} \in V$ and $\psi \in \mathcal{A}(\Pi)$. More precisely we denote (using notation $\langle . \rangle$ from Definition 2.3)

$$\varphi^- := \left\langle -R_{\partial_Y^2} \circ \partial^2_{X|\Lambda},\ \frac{\mu_0}{\mu_1} R_{\mathrm{N}} \circ \partial_{Y|\Gamma, Y=0^+} \right\rangle \left(\frac{1}{\mu_0} R_\Delta(\varphi_{|\Omega}) + \frac{1}{\mu_1} R_{\partial_Y^2}(\varphi_{|\Lambda}) + \sum_{d \in \frac{\pi}{\Theta}\mathbb{N}^*} s_d\, \phi_d \right) \in \mathcal{A}^-(\Pi)$$

and $\psi := T_{\geqslant -2}(\varphi^-)$. By Lemma 2.19, we have the first system below. Then, similarly to the proof of Theorem 3.6, one can check that it implies the second system below.

$$\left\{ \begin{array}{rl} \mu_0 \Delta \varphi^- = \varphi & \text{in } \Omega \\ \mu_1 \Delta \varphi^- = \varphi & \text{in } \Lambda \\ [\varphi^-]_\Gamma = 0 & \text{on } \Gamma \\ [\mu \partial_Y \varphi^-]_\Gamma = 0 & \text{on } \Gamma \\ \varphi^- = 0 & \text{on } \Sigma_\Omega \cup \Sigma_\Lambda \\ \sigma_d(\varphi^-) = s_d & \forall d \in \frac{\pi}{\Theta}\mathbb{N}^* \end{array} \right. \quad \text{so} \quad \left\{ \begin{array}{rl} (\mu_0 \Delta \psi - \varphi)_{|\Omega} \in \mathcal{A}(\Omega) \cap \sum\limits_{d<-4} \mathcal{A}_d(\Omega) = \sum\limits_{d<-4} \mathcal{A}_d(\Omega) \\ (\mu_1 \Delta \psi - \varphi)_{|\Lambda} \in \mathcal{A}(\Lambda) \cap \sum\limits_{d<-2} \mathcal{A}_d(\Lambda) = \sum\limits_{d<-2} \mathcal{A}_d(\Lambda) \\ [\psi]_\Gamma = 0 \\ [\mu \partial_Y \psi]_\Gamma \in \mathcal{A}(\Gamma) \cap \sum\limits_{d<-2} \mathcal{A}_d(\Gamma) = \sum\limits_{d<-2} \mathcal{A}_d(\Gamma) \\ \psi_{|\Sigma_\Omega \cup \Sigma_\Lambda} = 0 \\ \sigma_q(\psi) = s_d \quad \forall d \in \frac{\pi}{\Theta}\mathbb{N}^* \end{array} \right.$$

where $[\ldots]_\Gamma$ stands for the jump on Γ. Therefore $(1+r)\left[\operatorname{div}\big(\mu \nabla(\chi_\infty \psi)\big) - \chi_\infty \varphi\right] \in L^2(\Omega_1)$ and $[\mu \partial_Y(\chi_\infty \psi)]_\Gamma \in L^2(\Gamma \cap \{X > R_{\mathrm{c}}\})$. Finally, thanks to Lemma 3.8, we take $\tilde{S}$ as the unique solution in V of:

$$\left\{ \begin{array}{ll} \operatorname{div}(\mu \nabla \tilde{S}) = F + \chi_\infty \varphi - \operatorname{div}\big(\mu \nabla(\chi_\infty \psi)\big) & \text{in } \Omega_1 \setminus (\Gamma \cap \{X > R_{\mathrm{c}}\}) \\ \tilde{S} = 0 & \text{on } \partial\Omega_1 \\ [\mu \partial_Y \tilde{S}]_\Gamma = -[\mu \partial_Y(\chi_\infty \psi)]_\Gamma & \text{on } \Gamma \cap \{X > R_{\mathrm{c}}\} \end{array} \right.$$

Uniqueness: Let S be a homogeneous solution and let us show that $S = 0$. There is $\tilde{S} \in V$ and $\psi \in \mathcal{A}(\Omega)$ s.t. $S = \tilde{S} + \chi_\infty \psi$. The proof decomposes into the following steps, all proven by contradiction.

1. $\deg_{\max}(\Delta \psi_{|\Omega}) < -2$: Like in step 1 of the uniqueness proof of Theorem 3.6, we test the inequality:

$$\int_\Omega \Delta(\chi_\infty \psi) \cdot \zeta = -\int_\Omega \Delta \tilde{S} \cdot \zeta = \int_\Omega \nabla \tilde{S} \cdot \nabla \zeta \lesssim \|\zeta\|_V \qquad \forall \zeta \in \mathcal{D}(\Omega)$$

   with $\zeta_q : (r, \theta) \mapsto \Delta(\chi_\infty \psi)(r, \theta) \cdot r^{-q-2d-2}\, \chi_0(2^{-1/q} r)\, \chi(\theta)$ when $q \to 0^+$, where $d := \deg_{\max}(\Delta \psi_{|\Omega})$ and $\chi \in \mathcal{D}(0, \Theta) \setminus \{0\}$ is everywhere non negative.

2. $\deg_{\max}(\Delta \psi_{|\Lambda}) < -\frac{1}{2}$: This step works like the previous one.

3. $\deg_{\max}[\mu \partial_Y \psi]_\Gamma < -\frac{1}{2}$: Let $d := \deg_{\max}[\mu \partial_Y \psi]_\Gamma$ and $\varphi := [\mu \partial_Y(\chi_\infty \psi)]_\Gamma$. We have $\nabla \tilde{S} \in L^2(\Omega_1)$ and, by steps 1 and 2, $\operatorname{div}(\mu \nabla \tilde{S}) = -\operatorname{div}\big(\mu \nabla(\chi_\infty \psi)\big) \in L^2(\Omega_1)$. Thus $\varphi = -[\mu \partial_Y \tilde{S}]_\Gamma \in H^{-1/2}(\Gamma)$. Like previously, we test it with $\zeta_q : x \mapsto \varphi(x) \cdot x^{-q-2d-1}\, \chi_0(2^{-1/q} x)$ when $q \to 0^+$. If $d \geqslant -\frac{1}{2}$, we get $\langle \varphi, \zeta_q \rangle_\Gamma \gtrsim \frac{1}{q}$ and $\|\zeta_q\|_{H^{1/2}(\Gamma)} \lesssim \|\zeta_q\|_{H^1(\Gamma)} \lesssim \frac{1}{\sqrt{q}}$, which contradicts $\langle \varphi, \zeta_q \rangle_\Gamma \lesssim \|\zeta_q\|_{H^{1/2}(\Gamma)}$.

4. $\deg_{\max}(\psi_{|\Lambda}) < -\frac{1}{2}$: Let $d := \deg_{\max}(\psi_{|\Lambda})$. There is $\psi_1 \in \mathcal{A}_d(\Lambda) \setminus \{0\}$ and $\psi_2 \in \sum_{q<d} \mathcal{A}_q(\Lambda)$ s.t. $\psi_{|\Lambda} = \psi_1 + \psi_2$. By Lemma 2.16, $\deg(\partial^2_{x|\Lambda}) = -2$, $\deg(\partial_{Y|\Gamma, Y=0^+}) = -1$ and $\deg(\partial^2_{Y|\Lambda}) = \deg(\partial_{Y|\Gamma, Y=0^-}) = 0$. In addition, Proposition 2.8 states that $\mathcal{A}(D) = \bigoplus_{q \in \mathbb{R}} \mathcal{A}_q(D)$ for any $D \in \{\Lambda, \Gamma\}$. So by taking the coordinates of $\Delta \psi_{|\Lambda}$ in $\mathcal{A}_d(\Lambda)$ and of $[\mu \partial_Y \psi]_\Gamma$ in $\mathcal{A}_d(\Gamma)$, we get if $d \geqslant -\frac{1}{2}$:

$$\left\{ \begin{array}{rl} \partial_Y^2 \psi_1 = 0 & \text{in } \Lambda \\ \partial_{Y|\Gamma, Y=0^-} \psi_1 = 0 & \text{on } \Gamma \end{array} \right.$$

   But since $\psi_{|\Sigma_\Lambda} = 0$, it implies $\psi_1 = 0$, which is contradictory.

5. $\deg_{\max}(\psi_{|\Omega}) < 0$: We denote $d := \deg_{\max}(\psi_{|\Omega})$. There is $\psi_1 \in \mathcal{A}_d(\Omega) \setminus \{0\}$ and $\psi_2 \in \sum_{q<d} \mathcal{A}_q(\Omega)$ s.t. $\psi_{|\Omega} = \psi_1 + \psi_2$. Since $\deg(\Delta_{|\Omega}) = -2$, taking the coordinate of $\Delta \psi_{|\Omega}$ in $\mathcal{A}_{d-2}(\Omega)$ and of $\psi_{|\Gamma}$ in $\mathcal{A}_d(\Lambda)$ gives if $d \geqslant 0$:

$$\left\{ \begin{array}{rl} \Delta \psi_1 = 0 & \text{in } \Omega \\ \psi_1 = 0 & \text{on } \Gamma \end{array} \right.$$

   By Proposition 2.12, it implies $d \in \frac{\pi}{\Theta}\mathbb{N}^*$ and $\psi_1 = \sigma_d(\psi_1) \phi_q$. But $\sigma_d(\psi_1) = \sigma_d(\psi) = \sigma_d(S) = 0$.

Finally, Lemma 3.9 implies $\chi_\infty \psi \in V$, so $S \in V$. So the uniqueness in Lemma 3.8 implies $S = 0$. □

3.3 Construction of the fields

Definition 3.12: For any $(p,\ell) \in (\mathbb{R}\setminus\mathbb{P})\times\mathbb{N}$ we denote by convention $u_{p,\ell}=0$, $U_{p,\ell}=0$ and $S_{p,\ell}=0$. We define by induction on $p\in\mathbb{P}$ that for any $\ell\in\mathbb{N}$:

- $U_{p,\ell} : (x,Y)\in\Lambda \mapsto \sum_{n=0}^{\infty} (\partial_x^2+k_1^2)^n \partial_y u_{p-1-2n,\ell}(x,0)\cdot \mathcal{U}_n(Y)$. (3.3)
- $u_{p,\ell}$ is the unique solution in $H^1(\Omega)+\chi_0\mathcal{A}(\Omega)$ of :

$$\begin{cases} \mu_0\Delta u_{p,\ell} + \omega^2\rho_0 u_{p,\ell} = f_{\mathrm{s}}\,\delta_{p,0}\,\delta_{\ell,0} & \text{in } \Omega \\ u_{p,\ell} = 0 & \text{on } \Sigma_\Omega \\ u_{p,\ell} = U_{p,\ell} & \text{on } \Gamma \\ \sigma_d(u_{p,\ell}) = \displaystyle\sum_{p'\in[\![0,p+d]\!]_{\mathbb{P},\mathbb{N}}} \sum_{\substack{d'\in\frac{\pi}{\Theta}\mathbb{Z}^* \\ p'-d'\in[\![0,p]\!]_{\mathbb{P},\mathbb{N}}}} \sum_{\ell'=0}^{\ell} c^{\mathbf{u}\leftarrow S}_{d,d',p-p',\ell-\ell'}\cdot\sigma_{d'}(S^\infty_{p',\ell'}) & \forall d\in -\frac{\pi}{\Theta}\mathbb{N}^* \end{cases} \tag{3.4}$$

- $S_{p,\ell}$ is the unique solution in $V+\chi_\infty\mathcal{A}(\Pi)$ of :

$$\begin{cases} \mathrm{div}(\mu\nabla S_{p,\ell}) = -\omega^2\rho\, S_{p-2,\ell} & \text{in } \Omega_1 \\ S_{p,\ell} = 0 & \text{on } \partial\Omega_1 \\ \sigma_d(S_{p,\ell}) = \displaystyle\sum_{p'\in[\![0,p-d]\!]_{\mathbb{P},\mathbb{N}}} \sum_{\substack{d'\in\frac{\pi}{\Theta}\mathbb{Z}^* \\ p'+d'\in[\![0,p]\!]_{\mathbb{P},\mathbb{N}}}} \sum_{\ell'=0}^{\ell} c^{S\leftarrow\mathbf{u}}_{d,d',p-p',\ell-\ell'}\cdot\sigma_{d'}(\mathbf{u}^0_{p',\ell'}) & \forall d\in\frac{\pi}{\Theta}\mathbb{N}^* \end{cases} \tag{3.5}$$

We will show that these fields are well-defined at the same time as the following proposition.

Proposition 3.13: For any $(p,\ell)\in\mathbb{P}\times\mathbb{N}$ there exist $\mathbf{u}^0_{p,\ell}\in\mathcal{A}^+(\Pi)$ and $S^\infty_{p,\ell}\in\mathcal{A}^-(\Pi)$ s.t., for any $d\in\mathbb{R}$, $\mathbf{u}_{p,\ell} = T_{\leqslant d}(\mathbf{u}^0_{p,\ell}) + o_\partial(\mathbf{r}^d)$ when $\mathbf{r}\to 0$ and $S_{p,\ell} = T_{\geqslant d}(S^\infty_{p,\ell}) + o_\partial(\mathbf{r}^d)$ when $\mathbf{r}\to\infty$.

Justification of Definition 3.12 and Proposition 3.13: Let us show by induction on p that for any ℓ:

(H1) $u_{p,\ell}$, $U_{p,\ell}$ and $S_{p,\ell}$ exist and are unique,

(H2) Proposition 3.13 is true at rank (p,ℓ),

(H3) $\forall m\in\mathbb{N}$, $\partial_x^m\partial_y u_{p,\ell|\Gamma}\in H^1_0(\Gamma)+\chi_0\mathcal{A}(\Gamma)$ (additional property that will be useful during the proof).

Since all fields are null for $p<0$, the initial case is trivial and only the inductive step remains to prove. Let $p\in\mathbb{P}$. Let us assume (H1)–(H3) at any rank $p'<p$ and prove it at rank p.

1. Existence and uniqueness of the fields:
   - $U_{p,\ell}$: By (H3), the $\partial_y u_{p',\ell|\Gamma}$ with $p'<p$ are all in $H^m_{\mathrm{loc}}(\Gamma)$ for any $m\in\mathbb{N}$. So the functions $(\partial_x^2+k_1^2)^n\partial_y u_{p-1-2n,\ell|\Gamma}$ are continuous and Formula (3.3) is well-defined at any point of Λ.
   - $u_{p,\ell}$: By (H2), $S^\infty_{p',\ell'}$ exists for any $p'<p$ and ℓ', so the coefficients $\sigma_{d'}(S^\infty_{p',\ell'})$ are well-defined. Thus, (3.4) defines well $\sigma_d(u_{p,\ell})$. And we have: $\sigma_d(u_{p,\ell})\neq 0 \Rightarrow [\![0,p+d]\!]_{\mathbb{P},\mathbb{N}}\neq\varnothing \Rightarrow d\geqslant -p$, so $(\sigma_d(u_{p,\ell}))_{d\in-\frac{\pi}{\Theta}\mathbb{N}^*}$ has finite support. Next, we apply Theorem 3.6. Its hypotheses are satisfied, because (H3) implies:
   $$U_{p,\ell|\Gamma} = \sum_{n=0}^{\infty}(\partial_x^2+k_1^2)^n\partial_y u_{p-1-2n,\ell}(x,0)\cdot\mathcal{U}_n(0) \;\in\; H^1_0(\Gamma)+\chi_0\mathcal{A}(\Gamma) \;\subset\; H^{1/2}_{00}(\Gamma)+\chi_0\mathcal{A}(\Gamma)$$
   - $S_{p,\ell}$: Similarly, (H2) implies that $(\sigma_d(S_{p,\ell}))_{d\in\frac{\pi}{\Theta}\mathbb{N}^*}$ is well defined and has finite support. Next, we apply Theorem 2.23. We need to check its hypotheses, i.e. there is $f:\Omega_1\to\mathbb{C}$ and $\varphi:\Pi\to\mathbb{C}$ s.t. $\omega^2\rho\, S_{p-2,\ell} = F+\chi_\infty\varphi$, $(1+r)F\in L^2(\Omega_1)$, $\varphi_{|\Omega}\in\mathcal{A}(\Omega)$ and $\varphi_{|\Lambda}\in\mathcal{A}(\Lambda)$. By (H2), there is $\psi\in\mathcal{A}(\Pi)$ s.t. $S_{p-2,\ell}=\psi+o_\partial(\mathbf{r}^{-4})$. So it suffices to set $\varphi := \omega^2\rho\psi$ and $F := \omega^2\rho\, S_{p-2,\ell} - \chi_\infty\varphi$.
2. Asymptotic expansions:

- $\underline{\mathbf{u}_{p,\ell}}$: We use Theorem 2.22. To do so, we must check that there is $g_\Gamma \in \mathcal{A}^+(\Gamma)$ and $g_\Lambda \in \mathcal{A}^+(\Lambda)$ s.t.:
$$\forall d \in \mathbb{R}, \qquad \begin{cases} \partial_Y^2 U_{p,\ell} = T_{\leqslant d}(g_\Lambda) + o_\partial(x^d) & \text{in } \Lambda \\ \partial_Y U_{p,\ell|Y=0^-} = T_{\leqslant d}(g_\Gamma) + o_\partial(x^d) & \text{on } \Gamma \end{cases}$$
Given the definition of $U_{p,\ell}$ (3.3) , it suffices to show that, for any $p' < p$ there is $h \in \mathcal{A}^+(\Gamma)$ s.t.: $\forall d \in \mathbb{R}$, $\partial_n u_{p',\ell|\Gamma} = T_{\leqslant d}(h) + o_\partial(x^d)$. But it derives from (H2).
- $\underline{S_{p,\ell}}$: Similarly, we use Theorem 2.23 thanks to (H2).

3. Let $m \in \mathbb{N}$. The asymptotic expansion of $\mathbf{u}_{p,\ell}$ implies: $\exists h \in \mathcal{A}(\Gamma)$, $\partial_x^m \partial_y u_{p,\ell|\Gamma} = h + o_\partial(x^1)$. So there is $x_0 \in \mathbb{R}_+^*$ s.t. $\partial_x^m \partial_y u_{p,\ell|\Gamma} - h$ is H^1 on $\Gamma \cap \{x < x_0\} = (0, x_0) \times \{0\}$ and it vanishes at 0. To prove (H3) at rank p, it remains to show that $\partial_x^m \partial_y u_{p,\ell} \in H^1((\frac{x_0}{2}, \infty) \times \{0\})$. To do so, it suffices to get $u \in H^{m+3}((\frac{x_0}{2}, \infty) \times (0, \frac{\delta}{2}))$ with $\delta := \operatorname{dist}(\operatorname{supp}(f), \Gamma)$. But it follows from classical elliptic regularity because on one side $\mu_0 \Delta u_{p,\ell} + \omega^2 \rho_0 u_{p,\ell} = 0$ on $(\frac{x_0}{4}, \infty) \times (0, \delta)$, and on the other $u_{p,\ell|\Gamma} = U_{p,\ell|\Gamma} \in H^{m+3}((\frac{x_0}{4}, \infty) \times \{0\})$ by (3.3) and (H3). □

Proposition 3.14: $\forall p \in \mathbb{P}, \exists n_p \in \mathbb{N}, \forall \ell > n_p, \quad (u_{p,\ell} = 0 \text{ and } U_{p,\ell} = 0 \text{ and } S_{p,\ell} = 0)$.

Proof: For any $(d, d', p) \in (\frac{\pi}{\Theta}\mathbb{Z}^*)^2 \times \mathbb{P}$, one has $(\mathrm{id} - \langle \pi_\varepsilon^+ \mid \mathcal{R}_\varepsilon^+ \rangle)(\phi_{d'}) \in \mathcal{A}_\varepsilon^+(\Pi)$, so by definition of $\mathcal{A}_\varepsilon^+(\Pi)$ there is n (depending of d, d', p) s.t. for any $\ell > n$: $c^{\mathbf{u} \leftarrow S}_{d,d',p,\ell} := \sigma_d \circ \tau_{p+d',\ell} \circ (\mathrm{id} - \langle \pi_\varepsilon^+ \mid \mathcal{R}_\varepsilon^+ \rangle)(\phi_{d'}) = 0$. The same is true for the coefficients $c^{S \leftarrow \mathbf{u}}_{d,d',p,\ell}$. Finally the result follows by induction from Definition 3.12. □

Proposition 3.15: Ansatz 2.25 and the matching condition of Definition 2.27 is satisfied.

Proof: First let us note that for any (p, ℓ)
$$\sigma_d(u_{p,\ell}) = \sigma_d(\mathbf{u}^0_{p,\ell}) \text{ when } d \in -\tfrac{\pi}{\Theta}\mathbb{N}^* \quad \text{and} \quad \sigma_d(S_{p,\ell}) = \sigma_d(S^\infty_{p,\ell}) \text{ when } d \in \tfrac{\pi}{\Theta}\mathbb{N}^*. \tag{3.6}$$
where $\sigma_d(u_{p,\ell})$, resp. $\sigma_d(S_{p,\ell})$, is set by Definition 3.5, resp. 3.10, whereas $\sigma_d(\mathbf{u}^0_{p,\ell})$ and $\sigma_d(S^\infty_{p,\ell})$ rest on the definition of σ_d on $\mathcal{A}^\pm(\Pi)$ at page 15. Indeed we have $u_{p,\ell} - T_{\leqslant 1}(\mathbf{u}^0_{p,\ell})_{|\Omega} = o_\partial(r^1)$, so $\chi_0 \cdot (u_{p,\ell} - T_{\leqslant 1}(\mathbf{u}^0_{p,\ell})_{|\Omega}) \in H^1(\Omega)$ which implies by Definition 3.5 that $\sigma_d(u_{p,\ell}) = \sigma_d\big(T_{\leqslant 1}(\mathbf{u}^0_{p,\ell})_{|\Omega}\big) = \sigma_d(\mathbf{u}^0_{p,\ell})$. Likewise for $S_{p,\ell}$.
Now, given Proposition 3.14, to prove Ansatz 2.25 it suffices to check that, for any (p, ℓ), $\mathbf{u}^0_{p,\ell} \in \sum_{d \in \mathbb{P}-p} \mathcal{A}_d(\Pi)$ and $S^\infty_{p,\ell} \in \sum_{d \in p-\mathbb{P}} \mathcal{A}_d(\Pi)$. Let us show it for $\mathbf{u}^0_{p,\ell}$ only, using induction.

> Since all fields vanish for $p < 0$, only the inductive step is non-trivial. (2.25) states that
> $$\mathbf{u}^0_{p,\ell} = \sum_{n=0}^{\infty} (-k_0^2 R_\Delta)^n \Bigg(\underbrace{R_{\partial_Y^2} \circ (\partial^2_{x|\Lambda} + k_1^2)(\mathbf{u}^0_{p-2,\ell}) + \tfrac{\mu_0}{\mu_1} R_{\mathrm{N}} \circ \partial_{y|\Gamma, y=0^+}(\mathbf{u}^0_{p-1,\ell})}_{:=A} + \sum_{d \in \frac{\pi}{\Theta}\mathbb{Z}^*} \sigma_d(\mathbf{u}^0_{p,\ell})\, \phi_d \Bigg).$$
> Since $\deg R_\Delta \in \mathbb{N}$, it suffices to show that the big brackets belong to $\sum_{d \in \mathbb{P}-p} \mathcal{A}_d(\Pi)$. This is true for A using the induction hypothesis and $\deg R_{\partial_Y^2} = 0$, $\deg(\partial^2_{x|\Lambda}) = -2$ and $\deg(R_{\mathrm{N}} \circ \partial_{y|\Gamma, y=0^+}) = -1$ (see Proposition 2.14 and Lemma 2.16). It remains to show that $\forall d \in \frac{\pi}{\Theta}\mathbb{Z}^*$, $\sigma_d(\mathbf{u}^0_{p,\ell}) \neq 0 \Rightarrow d \in \mathbb{P} - p$. This last assertion holds because on one hand $\frac{\pi}{\Theta}\mathbb{N}^* \subset \mathbb{P} - p$, and on the other (3.6) and (3.4) imply: $\forall d \in -\frac{\pi}{\Theta}\mathbb{N}^*$, $\sigma_d(\mathbf{u}^0_{p,\ell}) \neq 0 \Leftrightarrow \sigma_d(u_{p,\ell}) \neq 0 \Rightarrow [\![0, p+d]\!]_{\mathbb{P},\mathbb{N}} \neq \varnothing \Rightarrow d \in \mathbb{P} - p$.

Finally (3.4)–(3.5) and (3.6) show that the matching relations of Theorem 2.31 are satisfied, and we can apply Theorem 2.31 thanks to Ansatz 2.25. Thus the matching condition of Definition 2.27 is satisfied. □

3.4 Practical way to build the far fields

This section shows how to build directly the far fields without computing the layer and corner fields. Thanks to the explicit expression of the layer fields in (3.3), the layer is replaced by boundary conditions on Γ, while the corner fields are replaced by corner conditions depending on corner profiles.

Definition 3.16: corner profiles
Let $d \in \frac{\pi}{\Theta}\mathbb{N}^*$. We denote $(\mathcal{S}_{d,n})_{n\in\mathbb{N}}$ the unique sequence of $V + \chi_\infty \mathcal{A}(\Pi)$ s.t. for any $n \in \mathbb{N}^*$:

$$\begin{cases} \operatorname{div}(\mu\nabla\mathcal{S}_{d,0}) = 0 & \text{in } \Omega_1 \\ \mathcal{S}_{d,0} = 0 & \text{on } \partial\Omega_1 \\ \sigma_q(\mathcal{S}_{d,0}) = \delta_{d,q} & \forall q \in \frac{\pi}{\Theta}\mathbb{N}^* \end{cases} \quad \text{and} \quad \begin{cases} \operatorname{div}(\mu\nabla\mathcal{S}_{d,n}) = -\omega^2\rho\,\mathcal{S}_{d,n-1} & \text{in } \Omega_1 \\ \mathcal{S}_{d,n} = 0 & \text{on } \partial\Omega_1 \\ \sigma_q(\mathcal{S}_{d,n}) = 0 & \forall q \in \frac{\pi}{\Theta}\mathbb{N}^* \end{cases}$$

And for any (d,n) we denote $\mathcal{S}^\infty_{d,n}$ the element of $\mathcal{A}^-(\Pi)$ s.t. $\forall d \in \mathbb{R}$, $\mathcal{S}_{d,n} = T_{\geqslant d}(\mathcal{S}^\infty_{d,n}) + o_\partial(\mathbf{r}^d)$.
These objects are well-defined thanks to Theorems 3.11 and 2.23. The proof is the same as for Definition 3.12 and Proposition 3.13.

Using (3.5) and the uniqueness in Theorem 3.11, one can easily show by induction on p:

$$\forall (p,\ell) \in \mathbb{P}\times\mathbb{N}, \quad S_{p,\ell} = \sum_{n=0}^{\infty} \sum_{d\in\frac{\pi}{\Theta}\mathbb{N}^*\cap(p-2n-\mathbb{P})} \sigma_d(S_{p-2n,\ell})\,\mathcal{S}_{d,n} \tag{3.7}$$

Thus, the same holds replacing $S_{p,\ell}$ and $\mathcal{S}_{d,n}$ by resp. $S^\infty_{p,\ell}$ and $\mathcal{S}^\infty_{d,n}$.
For any $(d,d',p,\ell) \in (-\frac{\pi}{\Theta}\mathbb{N}^*)\times\frac{\pi}{\Theta}\mathbb{Z}^*\times\mathbb{P}\times\mathbb{N}$, we introduce the corner coefficient $c^{\mathbf{u}\leftarrow\mathbf{u}}_{d,d',p,\ell}$, we have

$$c^{\mathbf{u}\leftarrow\mathbf{u}}_{d,d',p,\ell} := \sum_{\substack{(p_1,p_2)\in(\mathbb{N}-d)\times(\mathbb{N}+d')\\ n\in\mathbb{N},\,p_1+p_2+2n=p}} \sum_{\substack{d_1\in\frac{\pi}{\Theta}\mathbb{Z}^*\\ p_1+d_1\in\mathbb{N}}} \sum_{\substack{d_2\in\frac{\pi}{\Theta}\mathbb{N}^*\\ p_2-d_2\in\mathbb{N}}} \sum_{\substack{(\ell_1,\ell_2)\in\mathbb{N}^2\\ \ell_1+\ell_2=\ell}} c^{\mathbf{u}\leftarrow S}_{d,d_1,p_1,\ell_1}\cdot\sigma_{d_1}(\mathcal{S}^\infty_{d_2,n})\cdot c^{S\leftarrow\mathbf{u}}_{d_2,d',p_2,\ell_2} \tag{3.8a}$$

$$= \sum_{\substack{(p_1,p_2)\in(\mathbb{N}-d)\times(\mathbb{N}+d')\\ n\in\mathbb{N},\,p_1+p_2+2n=p}} \sum_{\substack{d_1\in-\frac{\pi}{\Theta}\mathbb{N}^*\\ p_1+d_1\in\mathbb{N}}} \sum_{\substack{d_2\in\frac{\pi}{\Theta}\mathbb{N}^*\\ p_2-d_2\in\mathbb{N}}} \sum_{\substack{(\ell_1,\ell_2)\in\mathbb{N}^2\\ \ell_1+\ell_2=\ell}} c^{\mathbf{u}\leftarrow S}_{d,d_1,p_1,\ell_1}\cdot\sigma_{d_1}(\mathcal{S}^\infty_{d_2,n})\cdot c^{S\leftarrow\mathbf{u}}_{d_2,d',p_2,\ell_2} \tag{3.8b}$$

$$+ \sum_{\substack{(p_1,p_2)\in(\mathbb{N}-d)\times(\mathbb{N}+d')\\ p_1+p_2=p}} \sum_{\substack{d_1\in\frac{\pi}{\Theta}\mathbb{N}^*\\ p_1+d_1\in\mathbb{N},\,p_2-d_1\in\mathbb{N}}} \sum_{\substack{(\ell_1,\ell_2)\in\mathbb{N}^2\\ \ell_1+\ell_2=\ell}} c^{\mathbf{u}\leftarrow S}_{d,d_1,p_1,\ell_1}\cdot c^{S\leftarrow\mathbf{u}}_{d_1,d',p_2,\ell_2} \tag{3.8c}$$

(the two given formulas are equal because, when $d_1 > 0$, $\sigma_{d_1}(\mathcal{S}^\infty_{d_2,n}) = \sigma_{d_1}(\mathcal{S}_{d_2,n}) = \delta_{d_1,d_2}\,\delta_{n,0}$). Like in Remark 2.32, we have $d - d_1, d_2 - d' \in \mathbb{Z}\cap\frac{\pi}{\Theta}\mathbb{Z}$ in (3.8a). Thus, if $d - d' \notin \mathbb{Z}\cap\frac{\pi}{\Theta}\mathbb{Z}$, line (3.8c) vanishes. Moreover, for any (p,ℓ), we denote $u^0_{p,\ell} := \mathbf{u}^0_{p,\ell|\Omega} \in \mathcal{A}^+(\Omega)$, which satisfies: $\forall d \in \mathbb{R}$, $u_{p,\ell} \underset{r\to0}{=} T_{\leqslant d}(u^0_{p,\ell}) + o_\partial(r^d)$.

Theorem 3.17: direct construction of the far fields
Let $(T_n)_{n\in\mathbb{N}}$ be the sequence of Taylor coefficients of the tangent: $\forall t \in (-\frac{\pi}{2},\frac{\pi}{2})$, $\tan t = \sum_{n=0}^\infty T_n t^{2n+1}$.
$(u_{p,\ell})_{p\in\mathbb{P},\ell\in\mathbb{N}}$ is the unique family of $H^1(\Omega) + \chi_0\mathcal{A}(\Omega)$ s.t. for any $(p,\ell) \in \mathbb{P}\times\mathbb{N}$:

$$\begin{cases} \mu_0\Delta u_{p,\ell} + \omega^2\rho_0 u_{p,\ell} = f_{\mathrm{s}}\,\delta_{p,0}\,\delta_{\ell,0} & \text{in } \Omega \\ u_{p,\ell} = 0 & \text{on } \Sigma_\Omega \\ u_{p,\ell} = \displaystyle\sum_{n=0}^\infty \frac{\mu_0}{\mu_1} T_n\cdot(\partial_x^2 + k_1^2)^n\partial_y u_{p-1-2n,\ell} & \text{on } \Gamma \\ \sigma_d(u_{p,\ell}) = \displaystyle\sum_{\substack{p'\in\mathbb{P}\\ p-p'\in\mathbb{P}+\frac{2\pi}{\Theta}}} \sum_{\substack{d'\in\frac{\pi}{\Theta}\mathbb{Z}^*\\ p'+d'\in[\![0,p+d]\!]_{\mathbb{P},\mathbb{N}}}} \sum_{\ell'=0}^{\ell} c^{\mathbf{u}\leftarrow\mathbf{u}}_{d,d',p-p',\ell-\ell'}\cdot\sigma_{d'}(u^0_{p',\ell'}) & \forall d \in -\frac{\pi}{\Theta}\mathbb{N}^* \end{cases}$$

Proof: Uniqueness follows from the uniqueness in Theorem 3.6. So it suffices to prove that the far fields satisfy the equations above.

Boundary condition: By (3.3) and (3.4), we have

$$u_{p,\ell}(x,0) = U_{p,\ell}(x,0) = \sum_{n=0}^\infty (\partial_x^2 + k_1^2)^n\partial_y u_{p-1-2n,\ell}(x,0)\cdot\mathcal{U}_n(0).$$

So we need to calculate $\mathcal{U}_n(0)$. Let $\mathrm{U} : (Y,t) \mapsto \sum_{n=0}^{\infty} \mathcal{U}_n(Y)\, t^{2n+1}$. The definition of $(\mathcal{U}_n)$ (reminded in (3.9) below) formally implies a differential equation on U given in (3.10).

$$\forall n \in \mathbb{N},\ \begin{cases} \mathcal{U}_n'' = -\mathcal{U}_{n-1} \\ \mathcal{U}_n'(0) = \frac{\mu_0}{\mu_1}\delta_{n,0} \\ \mathcal{U}_n(-1) = 0 \end{cases} \tag{3.9}$$

$$\begin{cases} \partial_Y^2 \mathrm{U} = -t^2 \mathrm{U} \\ \partial_Y \mathrm{U}_{|Y=0} = \mathcal{U}_0'(0)\cdot t = \frac{\mu_0}{\mu_1} t \\ \mathrm{U}_{|Y=-1} = 0 \end{cases} \tag{3.10}$$

Thus $\mathrm{U}(Y,t) = \frac{\mu_0}{\mu_1}\frac{\sin(t(Y+1))}{\cos(t)} := \varphi(Y,t)$. This is formal, as we do not know whether the series U converges. However, there is a sequence of polynomial functions (Φ_n) s.t., for any $(Y,t) \in (-1,0)\times(-\frac{\pi}{2},\frac{\pi}{2})$, $\varphi(Y,t) = \sum_{n=0}^{\infty} \Phi_n(Y)\, t^{2n+1}$ (because φ is odd w.r.t. t). Since φ satisfies (3.10), (Φ_n) is a solution of (3.9). But this solution is unique, so $(\Phi_n) = (\mathcal{U}_n)$ and $\mathrm{U} = \varphi$. Hence $\forall t \in (-\frac{\pi}{2},\frac{\pi}{2})$, $\sum_{n=0}^{\infty} \mathcal{U}_n(0)\, t^{2n+1} = \frac{\mu_0}{\mu_1}\tan(t)$. That is to say $\forall n \in \mathbb{N}$, $\mathcal{U}_n(0) = \frac{\mu_0}{\mu_1} T_n$.

Corner condition: For any $(d,p,\ell) \in \frac{\pi}{\Theta}\mathbb{N}^* \times \mathbb{R} \times \mathbb{N}$, let $\widetilde{\mathcal{S}}^{\infty}_{d,p,\ell} := \mathcal{S}^{\infty}_{d,p/2}$ if $\frac{p}{2} \in \mathbb{N}$ and $\ell = 0$ and $\widetilde{\mathcal{S}}^{\infty}_{d,p,\ell} := 0$ otherwise. For any family $(x_{p,\ell})$, we denote $x_{\bullet,\bullet} := (x_{p,\ell})$ the family itself. We also denote $*$ the convolution product w.r.t. (p,ℓ). Let $d \in -\frac{\pi}{\Theta}\mathbb{N}^*$. We have

$$\begin{aligned}
\sigma_d(u_{\bullet,\bullet}) &= \sum_{d_1 \in \frac{\pi}{\Theta}\mathbb{Z}^*} c^{\mathbf{u}\leftarrow S}_{d,d_1,\bullet,\bullet} * \sigma_{d_1}(S^{\infty}_{\bullet,\bullet}) && \text{by (3.4)} \\
&= \sum_{d_1 \in \frac{\pi}{\Theta}\mathbb{Z}^*} c^{\mathbf{u}\leftarrow S}_{d,d_1,\bullet,\bullet} * \sum_{d_2 \in \frac{\pi}{\Theta}\mathbb{N}^*} \sigma_{d_1}(\widetilde{\mathcal{S}}^{\infty}_{d_2,\bullet,\bullet}) * \sigma_{d_2}(S_{\bullet,\bullet}) && \text{by (3.7)} \\
&= \sum_{d_1 \in \frac{\pi}{\Theta}\mathbb{Z}^*} c^{\mathbf{u}\leftarrow S}_{d,d_1,\bullet,\bullet} * \sum_{d_2 \in \frac{\pi}{\Theta}\mathbb{N}^*} \sigma_{d_1}(\widetilde{\mathcal{S}}^{\infty}_{d_2,\bullet,\bullet}) * \sum_{d' \in \frac{\pi}{\Theta}\mathbb{Z}^*} c^{S\leftarrow\mathbf{u}}_{d_2,d',\bullet,\bullet} * \sigma_{d'}(\mathbf{u}^0_{\bullet,\bullet}) && \text{by (3.5)} \\
&= \sum_{d' \in \frac{\pi}{\Theta}\mathbb{Z}^*} \left(\sum_{d_1 \in \frac{\pi}{\Theta}\mathbb{Z}^*} \sum_{d_2 \in \frac{\pi}{\Theta}\mathbb{N}^*} c^{\mathbf{u}\leftarrow S}_{d,d_1,\bullet,\bullet} * \sigma_{d_1}(\widetilde{\mathcal{S}}_{d_2,\bullet,\bullet}) * c^{S\leftarrow\mathbf{u}}_{d_2,d',\bullet,\bullet} \right) * \sigma_{d'}(u^0_{\bullet,\bullet}) \\
&= \sum_{d' \in \frac{\pi}{\Theta}\mathbb{Z}^*} \tilde{c}^{\mathbf{u}\leftarrow\mathbf{u}}_{d,d',\bullet,\bullet} * \sigma_{d'}(u^0_{\bullet,\bullet})
\end{aligned}$$

with $\tilde{c}^{\mathbf{u}\leftarrow\mathbf{u}}_{d,d',p,\ell} := \sum_{\substack{(p_1,p_2),n\in\mathbb{N} \\ p_1+p_2+2n=p}} \sum_{d_1 \in \frac{\pi}{\Theta}\mathbb{Z}^*} \sum_{d_2 \in \frac{\pi}{\Theta}\mathbb{N}^*} \sum_{\substack{(\ell_1,\ell_2)\in\mathbb{N}^2 \\ \ell_1+\ell_2=\ell}} c^{\mathbf{u}\leftarrow S}_{d,d_1,p_1,\ell_1} \cdot \sigma_{d_1}(\mathcal{S}^{\infty}_{d_2,n}) \cdot c^{S\leftarrow\mathbf{u}}_{d_2,d',p_2,\ell_2}$.

Moreover, for any $R \in \mathcal{R}^{\pm}_{\varepsilon}$, we have $\deg_{\varepsilon} R \in \mathbb{N}$ and $\deg_{\varepsilon} R \pm \deg_{\mathcal{A}} R \in \mathbb{N}$ (see Figure 9). So by definition of the coefficients $c^{\mathbf{u}\leftarrow S}_{\dots}$ and $c^{S\leftarrow\mathbf{u}}_{\dots}$:

- $c^{\mathbf{u}\leftarrow S}_{d,d_1,p_1,\ell_1} \neq 0 \Rightarrow p_1 + d \in \mathbb{N}$ and $p_1 + d_1 \in \mathbb{N}$,
- $c^{S\leftarrow\mathbf{u}}_{d_2,d',p_2,\ell_2} \neq 0 \Rightarrow p_2 - d_2 \in \mathbb{N}$ and $p_2 - d' \in \mathbb{N}$.

This implies that $\tilde{c}^{\mathbf{u}\leftarrow\mathbf{u}}_{d,d',p,\ell} = c^{\mathbf{u}\leftarrow\mathbf{u}}_{d,d',p,\ell}$ for any (d,d',p,ℓ).
Furthermore, those conditions on (p_1,p_2,d_1,d_2) imply that, if $c^{\mathbf{u}\leftarrow\mathbf{u}}_{d,d',p,\ell} \neq 0$, then:

- $p = (p_1+d) + 2n + (p_2-d_2) - d + d_2 \in \mathbb{N} + \frac{\pi}{\Theta}\mathbb{N}^* + \frac{\pi}{\Theta}\mathbb{N}^* \subset \mathbb{P} + \frac{2\pi}{\Theta}$
- and $p + d - d' = (p_1+d) + 2n + (p_2-d') \in \mathbb{N}$.

This and the property $(\sigma_{d'}(u_{p',\ell'}) \neq 0 \Rightarrow p' \in \mathbb{P}$ and $p' + d' \in \mathbb{P})$ explain the sum indexes of the formula given for $\sigma_d(u_{p,\ell})$ in Theorem 3.17. □

Example: Using Theorem 3.17, one can check that $u_{\pi/\Theta,\ell}$ vanishes for any ℓ. In addition $\frac{2\pi}{\Theta} > 1$, so the first non-zero far fields are $u_{0,0}$ and $u_{1,0}$ and they satisfy

$$\begin{cases} \mu_0\Delta u_{0,0} + \omega^2\rho_0 u_{0,0} = f_{\mathrm{s}} & \text{in } \Omega \\ u_{0,0} = 0 & \text{on } \Sigma_\Omega \\ u_{0,0} = 0 & \text{on } \Gamma \\ \sigma_d(u_{p,\ell}) = 0 & \forall d \in -\frac{\pi}{\Theta}\mathbb{N}^* \end{cases} \quad \text{and} \quad \begin{cases} \mu_0\Delta u_{1,0} + \omega^2\rho_0 u_{1,0} = 0 & \text{in } \Omega \\ u_{1,0} = 0 & \text{on } \Sigma_\Omega \\ u_{1,0} = \frac{\mu_0}{\mu_1}\partial_y u_{0,0} & \text{on } \Gamma \\ \sigma_d(u_{1,0}) = 0 & \forall d \in -\frac{\pi}{\Theta}\mathbb{N}^* \end{cases}$$

4 Error estimates

Let $\chi \in \mathcal{C}^\infty(\mathbb{R}^2)$ be equal to 1 on $B(0,1)$ and 0 outside $B(0,2)$, and, for any $\eta > 0$, $\chi_\eta : x \mapsto \chi(\frac{x}{\eta})$. We denote $\Lambda_\varepsilon := (-\varepsilon, 1) \times \mathbb{R}_+^*$ and $\Pi_\varepsilon := \Omega \sqcup \Gamma \sqcup \Lambda_\varepsilon$ (defined similarly as Π at page 6). We define on Π_ε the following variant of the far-and-layer fields $\mathbf{u}_{p,\ell}$

$$\mathbf{u}^\varepsilon_{p,\ell}(x,y) := \begin{cases} u_{p,\ell}(x,y) & \text{in } \Omega \\ U_{p,\ell}(x,\frac{y}{\varepsilon}) & \text{in } \Lambda_\varepsilon \end{cases}$$

We also denote, for any $p \in \mathbb{P}$, $n_p := \max\{\ell \in \mathbb{N} \mid u_{p,\ell} \neq 0 \text{ or } U_{p,\ell} \neq 0 \text{ or } S_{p,\ell} \neq 0\}$.

Let us define the approximate global field at order $P \in \mathbb{R}_+$ as follows for any ε small enough

$$\forall (x,y) \in \Omega_\varepsilon, \quad u_{\varepsilon,P}(x,y) := (1-\chi_\eta(x,y)) \sum_{\substack{p \in \mathbb{P} \cap [0,P] \\ \ell \in [\![0,n_p]\!]}} \varepsilon^p \ln^\ell \varepsilon \, \mathbf{u}^\varepsilon_{p,\ell}(x,y) \; + \; \chi_\eta(x,y) \sum_{\substack{p \in \mathbb{P} \cap [0,P] \\ \ell \in [\![0,n_p]\!]}} \varepsilon^p \ln^\ell \varepsilon \, S_{p,\ell}(\tfrac{x}{\varepsilon}, \tfrac{y}{\varepsilon})$$

where $\eta := \sqrt{\varepsilon}$. Note that $(1-\chi_\eta(x,y))\mathbf{u}^\varepsilon_{p,\ell}(x,y)$ is well-defined on Ω_ε when $\eta > \varepsilon R_c$ since $\Pi_\varepsilon \setminus B(0,\varepsilon R_c) = \Omega_\varepsilon \setminus B(0,\varepsilon R_c)$. We will see as a consequence of Theorem 4.2 that $u_{\varepsilon,P} \in H^1(\Omega_\varepsilon)$.

The matching zone is $\Omega_\varepsilon \cap C_\eta$ where C_η is the annulus $C_\eta := B(0,2\eta) \setminus B(0,\eta)$. Letting $\eta = \sqrt{\varepsilon}$ makes the matching zone tend to 0 w.r.t. the far fields (because $\eta \to 0$ when $\varepsilon \to 0$) and to infinity w.r.t. the corner fields (because $\frac{\eta}{\varepsilon} \to \infty$). Thanks to the matching assumption, we can state the following first error estimate concercing the error in the matching zone. We use the symbol $\lesssim$ for majorations valid up to a constant independent of ε.

Lemma 4.1: Let $P \in \mathbb{R}_+$. For ε small enough, we have

$$\Big\| \sum_{p \in \mathbb{P} \cap [0,P]} \sum_{\ell \in \mathbb{N}} \varepsilon^p \ln^\ell \varepsilon \left(\mathbf{u}^\varepsilon_{p,\ell}(x,y) - S_{p,\ell}(\tfrac{x}{\varepsilon}, \tfrac{y}{\varepsilon}) \right) \Big\|_{H^1(\Omega_\varepsilon \cap C_\eta)} \lesssim \varepsilon^{\frac{P}{2}-1}.$$

Proof: We will compare $\mathbf{u}^\varepsilon_{p,\ell}$ and $S_{p,\ell}$ in C_η to their asymptotic expansions at 0, resp ∞. Let us denote:

- $\mathbf{u}_{\varepsilon,P} := \sum_{p \in \mathbb{P} \cap [0,P]} \sum_{\ell \in \mathbb{N}} \varepsilon^p \ln^\ell \varepsilon \, \mathbf{u}^\varepsilon_{p,\ell}$ in Π_ε
- $S_{\varepsilon,P} := \sum_{p \in \mathbb{P} \cap [0,P]} \sum_{\ell \in \mathbb{N}} \varepsilon^p \ln^\ell \varepsilon \, S_{p,\ell}(\frac{x}{\varepsilon}, \frac{y}{\varepsilon})$ in Ω_ε
- $\mathbf{u}^{\mathcal{A}}_{\varepsilon,P} := \sum_{p \in \mathbb{P} \cap [0,P]} \sum_{\ell \in \mathbb{N}} \varepsilon^p \ln^\ell \varepsilon \cdot \begin{cases} T_{\leqslant P-p}(\mathbf{u}^0_{p,\ell})(x,y) & \text{in } \Omega \\ T_{\leqslant P-p}(\mathbf{u}^0_{p,\ell})(x,\frac{y}{\varepsilon}) & \text{in } \Lambda_\varepsilon \end{cases}$
- $S^{\mathcal{A}}_{\varepsilon,P} := \sum_{p \in \mathbb{P} \cap [0,P]} \sum_{\ell \in \mathbb{N}} \varepsilon^p \ln^\ell \varepsilon \cdot T_{\geqslant p-P}(S^\infty_{p,\ell})(\frac{x}{\varepsilon}, \frac{y}{\varepsilon})$ in Π_ε.

(see Definition 2.21 for $T_\bullet$). We split the estimate into three parts that we will majorize separately:

$$\|\mathbf{u}_{\varepsilon,P} - S_{\varepsilon,P}\|_{H^1(\Omega_\varepsilon \cap C_\eta)} \leqslant \|\mathbf{u}_{\varepsilon,P} - \mathbf{u}^{\mathcal{A}}_{\varepsilon,P}\|_{H^1(\Omega_\varepsilon \cap C_\eta)} + \|\mathbf{u}^{\mathcal{A}}_{\varepsilon,P} - S^{\mathcal{A}}_{\varepsilon,P}\|_{H^1(\Omega_\varepsilon \cap C_\eta)} + \|S^{\mathcal{A}}_{\varepsilon,P} - S_{\varepsilon,P}\|_{H^1(\Omega_\varepsilon \cap C_\eta)}.$$

In addition, we will split some of the norms $\|\ldots\|_{H^1(\Omega_\varepsilon \cap C_\eta)}$ into $\|\ldots\|_{H^1(\Omega \cap C_\eta)} + \|\ldots\|_{H^1(\Lambda_\varepsilon \cap C_\eta)}$.

1. $\|\mathbf{u}_{\varepsilon,P} - \mathbf{u}^{\mathcal{A}}_{\varepsilon,P}\|_{H^1(\Omega \cap C_\eta)}$: By Proposition 3.13, for any (p,ℓ) we have $u_{p,\ell} - T_{\leqslant P-p}(\mathbf{u}^0_{p,\ell}) = o_\partial(r^{P-p})$ in Ω when $r \to 0$. By Definition 2.20 of o_∂, it implies $u_{p,\ell} - T_{\leqslant P-p}(\mathbf{u}^0_{p,\ell}) = O(r^{P-p}) = O(r^{P-p-1})$ and $\nabla\big[u_{p,\ell} - T_{\leqslant P-p}(\mathbf{u}^0_{p,\ell})\big] = O(r^{P-p-1})$ uniformly in θ. Thus:

$$\begin{aligned} \|\mathbf{u}_{\varepsilon,P} - \mathbf{u}^{\mathcal{A}}_{\varepsilon,P}\|_{H^1(\Omega \cap C_\eta)} &\leqslant \sum_{p \in \mathbb{P} \cap [0,P]} \sum_{\ell=0}^{n_p} \varepsilon^p \ln^\ell \varepsilon \cdot \|u_{p,\ell} - T_{\leqslant P-p}(\mathbf{u}^0_{p,\ell})\|_{H^1(\Omega \cap C_\eta)} \\ &\lesssim \sum_{p \in \mathbb{P} \cap [0,P]} \sum_{\ell=0}^{n_p} \varepsilon^p \ln^\ell \varepsilon \cdot \left(\int_{B(0,2\eta)} r^{2(P-p-1)} \, \mathrm{d}r \, r\mathrm{d}\theta \right)^{1/2} \\ &\lesssim \sum_{p \in \mathbb{P} \cap [0,P]} \sum_{\ell=0}^{n_p} \varepsilon^p \ln^\ell \varepsilon \cdot \eta^{P-p} \\ &\lesssim \eta^{P-2} \qquad \text{because } \varepsilon^p \lesssim \eta^p \text{ and } \ln^\ell \varepsilon \lesssim \eta^{-2} \end{aligned}$$

2. $\|\mathbf{u}_{\varepsilon,P} - \mathbf{u}^{\mathcal{A}}_{\varepsilon,P}\|_{H^1(\Lambda_\varepsilon \cap C_\eta)}$: Similarly, $U_{p,\ell} - T_{\leqslant P-p}(\mathbf{u}^0_{p,\ell}) = o_\partial(x^{P-p})$ in Λ when $x \to \infty$. Thus

- $U_{p,\ell} - T_{\leqslant P-p}(\mathbf{u}^0_{p,\ell}) = O(x^{P-p})$ so $U_{p,\ell}(x, \frac{y}{\varepsilon}) - T_{\leqslant P-p}(\mathbf{u}^0_{p,\ell})(x, \frac{y}{\varepsilon}) = O(x^{P-p})$
- $\partial_x\big[U_{p,\ell} - T_{\leqslant P-p}(\mathbf{u}^0_{p,\ell})\big] = O(x^{P-p-1})$ so $\partial_x\big[U_{p,\ell}(x, \frac{y}{\varepsilon}) - T_{\leqslant P-p}(\mathbf{u}^0_{p,\ell})(x, \frac{y}{\varepsilon})\big] = O(\varepsilon^{-1}x^{P-p-1})$
- $\partial_Y\big[U_{p,\ell} - T_{\leqslant P-p}(\mathbf{u}^0_{p,\ell})\big] = O(x^{P-p})$ so $\partial_Y\big[U_{p,\ell}(x, \frac{y}{\varepsilon}) - T_{\leqslant P-p}(\mathbf{u}^0_{p,\ell})(x, \frac{y}{\varepsilon})\big] = O(\varepsilon^{-1}x^{P-p})$

which are all $O(\varepsilon^{-1}x^{P-p-1})$ (uniformly in Y). Hence:

$$\begin{aligned}
\|\mathbf{u}_{\varepsilon,P} - \mathbf{u}^{\mathcal{A}}_{\varepsilon,P}\|_{H^1(\Lambda_\varepsilon \cap C_\eta)} &\lesssim \sum_{p\in\mathbb{P}\cap[0,P]} \sum_{\ell=0}^{n_p} \varepsilon^p \ln^\ell\varepsilon \cdot \varepsilon^{-1} \left(\int_{[0,2\eta]\times[-\varepsilon,0]} x^{2(P-p-1)}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} \\
&\lesssim \sum_{p\in\mathbb{P}\cap[0,P]} \sum_{\ell=0}^{n_p} \varepsilon^p \ln^\ell\varepsilon \cdot \varepsilon^{-1} \cdot \varepsilon^{\frac12}\eta^{P-p-\frac12} \\
&\lesssim \eta^{P-2} \qquad \text{because } \varepsilon = \eta^2 \text{ and } \ln^\ell\varepsilon \lesssim \eta^{-\frac12}
\end{aligned}$$

3. $\|S_{\varepsilon,P} - S^{\mathcal{A}}_{\varepsilon,P}\|_{H^1(\Omega\cap C_\eta)}$: Similarly $S_{p,\ell} - T_{\geqslant p-P}(S^\infty_{p,\ell}) = o_\partial(r^{p-P})$ in Ω when $r \to \infty$, so $S_{p,\ell} - T_{\geqslant p-P}(S^\infty_{p,\ell}) = O(r^{p-P})$ and $\nabla_{(X,Y)}\big[S_{p,\ell} - T_{\geqslant p-P}(S^\infty_{p,\ell})\big] = O(r^{p-P-1}) = O(r^{p-P})$ uniformly in θ. Since $\nabla_{(x,y)} = \varepsilon^{-1}\nabla_{(X,Y)}$, we deduce:

$$\begin{aligned}
\|S_{\varepsilon,P} - S^{\mathcal{A}}_{\varepsilon,P}\|_{H^1(\Omega\cap C_\eta)} &\leqslant \sum_{p\in\mathbb{P}\cap[0,P]} \sum_{\ell=0}^{n_p} \varepsilon^p \ln^\ell\varepsilon \cdot \Big\|\big[S_{p,\ell} - T_{\geqslant p-P}(S^\infty_{p,\ell})\big](\tfrac{x}{\varepsilon}, \tfrac{y}{\varepsilon})\Big\|_{H^1(\Omega\cap C_\eta)} \\
&\lesssim \sum_{p\in\mathbb{P}\cap[0,P]} \sum_{\ell=0}^{n_p} \varepsilon^p \ln^\ell\varepsilon \cdot \varepsilon^{-1} \left(\int_{\mathbb{R}^2\setminus B(0,\eta)} \Big(\frac{r}{\varepsilon}\Big)^{2(p-P)}\,\mathrm{d}r\,r\mathrm{d}\theta\right)^{1/2} \\
&\lesssim \sum_{p\in\mathbb{P}\cap[0,P]} \sum_{\ell=0}^{n_p} \varepsilon^p \ln^\ell\varepsilon \cdot \varepsilon^{P-p-1} \cdot \eta^{p-P+1} \\
&\lesssim \eta^{P-2} \qquad \text{because } \varepsilon = \eta^2 \text{ and } \ln^\ell\varepsilon \lesssim \eta^{-1}
\end{aligned}$$

4. $\|S_{\varepsilon,P} - S^{\mathcal{A}}_{\varepsilon,P}\|_{H^1(\Lambda_\varepsilon\cap C_\eta)}$: $S_{p,\ell} - T_{\geqslant p-P}(S^\infty_{p,\ell}) = o_\partial(X^{p-P})$ in Λ when $x \to 0$, so:

- $S_{p,\ell} - T_{\geqslant p-P}(\mathbf{u}^0_{p,\ell}) = O(X^{p-P})$
- $\partial_X\big[S_{p,\ell} - T_{\geqslant p-P}(S^\infty_{p,\ell})\big] = O(X^{p-P-1}) = O(X^{p-P})$
- $\partial_Y\big[S_{p,\ell} - T_{\geqslant p-P}(S^\infty_{p,\ell})\big] = O(X^{p-P})$

uniformy in Y. Hence:

$$\begin{aligned}
\|S_{\varepsilon,P} - S^{\mathcal{A}}_{\varepsilon,P}\|_{H^1(\Lambda_\varepsilon\cap C_\eta)} &\leqslant \sum_{p\in\mathbb{P}\cap[0,P]} \sum_{\ell=0}^{n_p} \varepsilon^p \ln^\ell\varepsilon \Big\|\big[S_{p,\ell} - T_{\geqslant p-P}(S^\infty_{p,\ell})\big](\tfrac{x}{\varepsilon}, \tfrac{y}{\varepsilon})\Big\|_{H^1(\Lambda_\varepsilon\cap C_\eta)} \\
&\lesssim \sum_{p\in\mathbb{P}\cap[0,P]} \sum_{\ell=0}^{n_p} \varepsilon^p \ln^\ell\varepsilon \cdot \varepsilon^{-1} \left(\int_{[\eta,\infty[\times[-\varepsilon,0]} \Big(\frac{x}{\varepsilon}\Big)^{2(p-P)}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} \\
&\lesssim \sum_{p\in\mathbb{P}\cap[0,P]} \sum_{\ell=0}^{n_p} \varepsilon^p \ln^\ell\varepsilon \cdot \varepsilon^{P-p-1} \cdot \varepsilon^{\frac12}\eta^{p-P+\frac12} \\
&\lesssim \eta^{P-2} \qquad \text{because } \varepsilon = \eta^2 \text{ and } \ln^\ell\varepsilon \lesssim \eta^{-\frac32}
\end{aligned}$$

5. $\|\mathbf{u}^{\mathcal{A}}_{\varepsilon,P} - S^{\mathcal{A}}_{\varepsilon,P}\|_{H^1(\Omega_\varepsilon\cap C_\eta)}$: Let us show that this norm vanishes. It suffices to prove that in $\mathcal{A}^+_\varepsilon(\Pi)$:

$$\sum_{p\in\mathbb{P}\cap[0,P]} \sum_{\ell\in\mathbb{N}} \epsilon^p \ln^\ell\epsilon \cdot T_{\leqslant P-p}(\mathbf{u}^0_{p,\ell}) = \mathcal{H}_\varepsilon^{-1}\Big(\sum_{p\in\mathbb{P}\cap[0,P]} \sum_{\ell\in\mathbb{N}} \epsilon^p \ln^\ell\epsilon \cdot T_{\geqslant p-P}(S^\infty_{p,\ell})\Big) \tag{4.1}$$

where $\mathcal{H}_\varepsilon$ is defined in (2.29)–(2.31), and ϵ and $\ln\epsilon$ denote the algebraic indeterminates of $\mathcal{A}^+_\varepsilon(\Pi)$ (we denote them differently from Sections 2.1 and 2.5 to avoid confusion with the real number ε).

For any (p,d) let $\tilde{\mathbf{u}}_{p,d}$ be the coordinate of $\sum_{p,\ell}\epsilon^p\ln^\ell\epsilon\,\mathbf{u}^0_{p,\ell}$ in $\mathcal{A}_d(\Pi)[\ln\epsilon]$ and $\tilde{S}_{p,d}$ be the one of $\sum_{p,\ell}\epsilon^p\ln^\ell\epsilon\,S^\infty_{p,\ell}$. Moreover, for any d and $\varphi\in\mathcal{A}_d(\Pi)[\ln\epsilon]$, let $\widetilde{\mathcal{H}}_\varepsilon^{-1}(\varphi):=\varepsilon^d\mathcal{H}_\varepsilon^{-1}(\varphi)\in\mathcal{A}_d(\Pi)[\ln\epsilon]$. Then:

$$\begin{aligned}
(4.1)&\iff\sum_{p\leqslant P}\sum_{d\leqslant P-p}\epsilon^p\tilde{\mathbf{u}}_{p,d}=\mathcal{H}_\varepsilon^{-1}\Big(\sum_{p\leqslant P}\sum_{d\geqslant p-P}\epsilon^p\tilde{S}_{p,d}\Big)\\
&\iff\sum_{p\leqslant P}\sum_{p+d\leqslant P}\epsilon^p\tilde{\mathbf{u}}_{p,d}=\sum_{p\leqslant P}\sum_{p-d\leqslant P}\epsilon^{p-d}\widetilde{\mathcal{H}}_\varepsilon^{-1}(\tilde{S}_{p,d})\\
&\iff\sum_{p\leqslant P}\sum_{p+d\leqslant P}\epsilon^p\tilde{\mathbf{u}}_{p,d}=\sum_{p+d\leqslant P}\sum_{p\leqslant P}\epsilon^p\widetilde{\mathcal{H}}_\varepsilon^{-1}(\tilde{S}_{p+d,d})\\
&\iff\forall(p,d)\text{ s.t. }p\leqqslant P\text{ and }p+d\leqslant P,\quad\tilde{\mathbf{u}}_{p,d}=\widetilde{\mathcal{H}}_\varepsilon^{-1}(\tilde{S}_{p+d,d})\qquad(4.2)
\end{aligned}$$

because two formal series coincide iff their coordinates coincide one by one.
But Proposition 3.15 shows the matching condition $\sum_{p,d}\epsilon^p\tilde{\mathbf{u}}_{p,d}=\mathcal{H}_\varepsilon^{-1}\Big(\sum_{p,d}\epsilon^p\tilde{S}_{p,d}\Big)=\sum_{p,d}\epsilon^p\widetilde{\mathcal{H}}_\varepsilon^{-1}(\tilde{S}_{p+d,d})$, which is equivalent to: $\forall(p,d),\ \tilde{\mathbf{u}}_{p,d}=\widetilde{\mathcal{H}}_\varepsilon^{-1}(\tilde{S}_{p+d,d})$. Thus we get (4.2), and then (4.1). □

Theorem 4.2: global error estimate
For any $P\in\mathbb{R}_+$ we have $\big\|u_\varepsilon-u_{\varepsilon,P}\big\|_{H^1(\Omega_\varepsilon)}=o(\varepsilon^{\frac{P}{2}-2})$.

Proof: Let $r_{\varepsilon,P}:=u_{\varepsilon,P}-u_\varepsilon$. It satisfies, for some functions f_ε and g_ε,

$$\begin{cases}
\operatorname{div}(\mu_\varepsilon\nabla r_{\varepsilon,P})+\omega^2\rho_\varepsilon r_{\varepsilon,P}=f_\varepsilon & \text{in }\Omega_\varepsilon\\
r_{\varepsilon,P}=0 & \text{on }\partial\Omega_\varepsilon\\
r_{\varepsilon,P|y=0^+}-r_{\varepsilon,P|y=0^-}=0 & \text{on }\Gamma\cap\{x>\eta\}\\
\mu_0\partial_y r_{\varepsilon,P|y=0^+}-\mu_1\partial_y r_{\varepsilon,P|y=0^-}=g_\varepsilon & \text{on }\Gamma\cap\{x>\eta\}
\end{cases}$$

As for (1.2), this problem is well-posed with a stability constant independent of ε:

$$\|r_{\varepsilon,P}\|_{H^1(\Omega_\varepsilon)}\lesssim\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)}+\|g_\varepsilon\|_{L^2(\Gamma\cap\{x>\eta\})}.$$

To get our error estimate, it suffices to show that $\|f_\varepsilon\|_{L^2}+\|g_\varepsilon\|_{L^2}\lesssim\varepsilon^{\frac{P}{2}-2}=\eta^{P-4}$.

Estimate of $\|f_\varepsilon\|_{L^2}$: We denote:

- $\mathbf{u}_{\varepsilon,P}:=\sum_{p\in\mathbb{P}\cap[0,P]}\sum_{\ell\in\mathbb{N}}\varepsilon^p\ln^\ell\varepsilon\,\mathbf{u}^\varepsilon_{p,\ell}$ in Π_ε
- $S_{\varepsilon,P}:=\sum_{p\in\mathbb{P}\cap[0,P]}\sum_{\ell\in\mathbb{N}}\varepsilon^p\ln^\ell\varepsilon\,S_{p,\ell}(\frac{x}{\varepsilon},\frac{y}{\varepsilon})$ in Ω_ε
- and $\mathcal{D}_\varepsilon:u\mapsto\operatorname{div}(\mu_\varepsilon\nabla u)+\omega^2\rho_\varepsilon u$ the differential operator of Helmholtz's equation.

Since $\mathcal{D}_\varepsilon u_\varepsilon=f$ by definition of u_ε, we have

$$\begin{aligned}
f_\varepsilon&=\mathcal{D}_\varepsilon r_{\varepsilon,P}\\
&=\mathcal{D}_\varepsilon\big((1-\chi_\eta)\mathbf{u}_{\varepsilon,P}+\chi_\eta S_{\varepsilon,P}-u_\varepsilon\big)\\
&=\big((1-\chi_\eta)\mathcal{D}_\varepsilon\mathbf{u}_{\varepsilon,P}-f\big)+\chi_\eta\mathcal{D}_\varepsilon S_{\varepsilon,P}+[\mathcal{D}_\varepsilon,\chi_\eta](S_{\varepsilon,P}-\mathbf{u}_{\varepsilon,P})
\end{aligned}$$

where $[.,.]$ is the commutator. Let us estimate these terms one by one.

1. For ε small enough, $1-\chi_\eta$ is equal to 0 in $B(0,\varepsilon R_c)$ and 1 in $\operatorname{supp}(f)$, so $(1-\chi_\eta)\mathcal{D}_\varepsilon\mathbf{u}_{\varepsilon,P|\Omega}=(1-\chi_\eta)(\mu_0\Delta+\omega^2\rho_0)\mathbf{u}_{\varepsilon,P|\Omega}=(1-\chi_\eta)f=f$.
Moreover, using that $\mu_1\partial_Y^2U_{p,\ell}=-(\mu_1\partial_x^2+\omega^2\rho_1)U_{p-2,\ell}$ for any (p,ℓ), we get in $\Lambda_\varepsilon\setminus B(0,\varepsilon R_c)$:

$$\begin{aligned}
\mathcal{D}_\varepsilon\mathbf{u}_{\varepsilon,P}&=(\mu_1\Delta+\omega^2\rho_1)\mathbf{u}_{\varepsilon,P}\\
&=\sum_{p\in\mathbb{P}\cap(P-2,P]}\sum_{\ell\in\mathbb{N}}\varepsilon^p\ln^\ell\varepsilon\,(\mu_1\partial_x^2+\omega^2\rho_1)U_{p,\ell}\\
&=\sum_{p\in\mathbb{P}\cap(P-2,P]}\sum_{\ell\in\mathbb{N}}\varepsilon^{p+\frac12}\ln^\ell\varepsilon\sum_{n\in\mathbb{N}}\mu_1\,(\partial_x^2+k_1^2)^{n+1}\partial_y u_{p-1-2n,\ell}(x,0)\cdot\mathcal{U}_n(\tfrac{y}{\varepsilon})\qquad\text{by (3.3)}
\end{aligned}$$

Since $\mathcal{U}_n \in L^\infty(-1,0)$ for any n (see (3.1)), we deduce:

$$\|(1-\chi_\eta)\mathcal{D}_\varepsilon \mathbf{u}_{\varepsilon,P}\|_{L^2(\Lambda_\varepsilon)} \lesssim \sum_{p\in\mathbb{P}\cap(P-2,P]} \sum_{\ell=0}^{n_p} \sum_{n=0}^{\lfloor (p-1)/2\rfloor} \varepsilon^{p+\frac12} \ln^\ell\varepsilon \cdot \big\|(\partial_x^2+k_1^2)^{n+1}\partial_y u_{p-1-2n,\ell}\big\|_{L^2(\Gamma\cap\{x>\eta\})}.$$

Let us show that, for any p,ℓ,m, $\|\partial_x^m\partial_y u_{p,\ell|\Gamma}\|_{L^2(\Gamma\cap\{x>\eta\})} \lesssim \eta^{-p-m-1}$ (4.3).

- The proof of Proposition 3.13 shows that $\partial_x^m\partial_y u_{p,\ell} \in H_0^1(\Gamma) + \chi_0\mathcal{A}(\Gamma)$ for any p,ℓ,m. So $\|\partial_x^m\partial_y u_{p,\ell|\Gamma}\|_{L^2(\Gamma\cap\{x>1\})} < \infty$.
- By Proposition 3.13, $\partial_y u_{p,\ell|\Gamma} = \partial_y T_{\leqslant -p-\frac12}(\mathbf{u}^0_{p,\ell}) + o_\partial(x^{-p-\frac32})$ when $x\to 0$, and Ansatz 2.25 is satisfied, thus $T_{\leqslant -p-\frac12}(\mathbf{u}^0_{p,\ell}) = 0$. So: $\forall m\in\mathbb{N}$, $\partial_x^m\partial_y u_{p,\ell|\Gamma} = O(x^{-p-\frac32-m})$ when $x\to 0$. It implies $\|\partial_x^m\partial_y u_{p,\ell|\Gamma}\|_{L^2(\Gamma\cap\{\eta<x<1\})} \lesssim \eta^{-p-m-1}$.

Hence:

$$\begin{aligned}\|(1-\chi_\eta)\mathcal{D}_\varepsilon \mathbf{u}_{\varepsilon,P}\|_{L^2(\Lambda_\varepsilon)} &\lesssim \sum_{p\in\mathbb{P}\cap(P-2,P]} \sum_{\ell=0}^{n_p} \varepsilon^{p+\frac12}\ln^\ell\varepsilon \sum_{n=0}^{\lfloor (p-1)/2\rfloor} \eta^{-(p-1-2n)-2(n+1)-1}\\ &\lesssim \sum_{p\in\mathbb{P}\cap(P-2,P]} \sum_{\ell=0}^{n_p} \varepsilon^{p+\frac12}\ln^\ell\varepsilon\cdot\eta^{-p-2}\\ &\lesssim \eta^{P-4} \qquad \text{because } \ln^\ell\varepsilon \lesssim \eta^{-1}\end{aligned}$$

So $\|(1-\chi_\eta)\mathcal{D}_\varepsilon \mathbf{u}_{\varepsilon,P} - f\|_{L^2(\Omega_\varepsilon)} \lesssim \eta^{P-4}$.

2. For the second term of f_ε, (3.5) implies that $\mathcal{D}_\varepsilon S_{\varepsilon,P} = \sum_{p\in\mathbb{P}\cap(P-2,P]}\sum_{\ell\in\mathbb{N}} \varepsilon^p\ln^\ell\varepsilon\,\omega^2\rho S_{p,\ell}$. But by Proposition 3.13 and Ansatz 2.25: $\forall(p,\ell)$, $S_{p,\ell} = O(r^p)$. Thus, using that ρ is bounded and $\|S_{p,\ell}(\frac{x}{\varepsilon},\frac{y}{\varepsilon})\|_{L^2(\Omega_\varepsilon\cap B(0,2\eta))} = \varepsilon\,\|S_{p,\ell}\|_{L^2(\Omega_1\cap B(0,2\eta/\varepsilon))}$, we get

$$\begin{aligned}\|\chi_\eta\mathcal{D}_\varepsilon S_{\varepsilon,P}\|_{L^2(\Omega_\varepsilon)} &\lesssim \sum_{p\in\mathbb{P}\cap(P-2,P]} \sum_{\ell=0}^{n_p} \varepsilon^p\ln^\ell\varepsilon\cdot\varepsilon\,\|S_{p,\ell}\|_{L^2(\Omega_1\cap B(0,2\eta/\varepsilon))}\\ &\lesssim \sum_{p\in\mathbb{P}\cap(P-2,P]} \sum_{\ell=0}^{n_p} \varepsilon^{p+1}\ln^\ell\varepsilon\cdot\Big(\frac{\eta}{\varepsilon}\Big)^{p+1}\\ &\lesssim \eta^{P-4} \qquad \text{because } \ln^\ell\varepsilon\lesssim\eta^{-3}\end{aligned}$$

3. Using that $\|\Delta\chi_\eta\|_{L^2} + \|\nabla\chi_\eta\|_{L^2} = O(\eta^{-2})$, the last term of f_ε satisfies

$$\begin{aligned}\|[\mathcal{D}_\varepsilon,\chi_\eta](S_{\varepsilon,P}-\mathbf{u}_{\varepsilon,P})\|_{L^2(\Omega_\varepsilon)} &\lesssim \|\Delta\chi_\eta\cdot(S_{\varepsilon,P}-\mathbf{u}_{\varepsilon,P}) + 2\nabla\chi_\eta\cdot\nabla(S_{\varepsilon,P}-\mathbf{u}_{\varepsilon,P})\|_{L^2(\Omega_\varepsilon)}\\ &\lesssim \eta^{-2}\|S_{\varepsilon,P}-\mathbf{u}_{\varepsilon,P}\|_{H^1(\Omega_\varepsilon\cap B(0,2\eta)\setminus B(0,\eta))}\\ &\lesssim \eta^{P-4} \qquad \text{by Lemma 4.1.}\end{aligned}$$

Estimate of $\|g_\varepsilon\|_{L^2}$: Without loss of generality, we assume $\partial_y\chi_{\eta|\Gamma} = 0$. (3.5) implies that $\mu_0\partial_Y S_{p,\ell|Y=0^+} = \mu_1\partial_Y S_{p,\ell|Y=0^-}$ on $\Gamma\cap\{X>R_c\}$ for any (p,ℓ). Thus we have on $\Gamma\cap\{x>\eta\}$:

$$\begin{aligned}g_\varepsilon &= \mu_0\partial_y r_{\varepsilon,P|y=0^+} - \mu_1\partial_y r_{\varepsilon,P|y=0^-}\\ &= \chi_\eta\cdot\big(\mu_0\partial_y\mathbf{u}_{\varepsilon,P|y=0^+} - \mu_1\partial_y\mathbf{u}_{\varepsilon,P|y=0^-}\big)\\ &= \chi_\eta \sum_{p\in\mathbb{P}\cap(P-1,P]}\sum_{\ell\in\mathbb{N}} \varepsilon^p\ln^\ell\varepsilon\cdot\mu_0\partial_y u_{p,\ell|y=0^+}\end{aligned}$$

By (4.3), we have that $\|\partial_y u_{p,\ell|\Gamma}\|_{L^2(\Gamma\cap\{x>\eta\})} \lesssim \eta^{-p-1}$. So $\|g_\varepsilon\|_{L^2(\Gamma\cap\{x>\eta\})} \lesssim \eta^{P-4}$ since $\ln^\ell\varepsilon\lesssim\eta^{-2}$. □

Remark: Theorem 4.2 can be improved to $\|u_\varepsilon - u_{\varepsilon,P}\|_{H^1(\Omega_\varepsilon)} = o(\varepsilon^{P/2})$. Indeed with the same notations as the above proof, one can show that, for any $(p,\ell)\in\mathbb{P}\times\mathbb{N}$, $\|(1-\chi_\eta)\mathbf{u}^\varepsilon_{p,\ell}\|_{H^1(\Omega_\varepsilon)} \lesssim \eta^{-p}$ and $\|\chi_\eta S_{p,\ell}\|_{H^1(\Omega_\varepsilon)} \lesssim \eta^{-p}$. Therefore:

$$\|r_{\varepsilon,P}\|_{H^1(\Omega_\varepsilon)} \lesssim \|r_{\varepsilon,P+4}\|_{H^1(\Omega_\varepsilon)} + \sum_{p\in\mathbb{P}\cap(P,P+4]}\sum_{\ell\in\mathbb{N}} \varepsilon^p\ln^\ell\varepsilon\cdot\big(\|(1-\chi_\eta)\mathbf{u}^\varepsilon_{p,\ell}\|_{H^1(\Omega_\varepsilon)} + \|\chi_\eta S_{p,\ell}\|_{H^1(\Omega_\varepsilon)}\big) = o(\varepsilon^{\frac{P}{2}}).$$

Proof of Theorem 1.1: It follows from Theorem 4.2 applied at order $2P+4$ and from the fact that, for small enough ε, $\sum_{p\in\mathbb{P}\cap[0,2P]}\sum_{\ell\in\mathbb{N}}\varepsilon^p\ln^\ell\varepsilon\,u_{p,\ell}$ and $u_{\varepsilon,2P}$ coincide in $\Omega\setminus B(0,\delta)$:

$$\left\|u_\varepsilon-\sum_{p\in\mathbb{P}\cap[0,P]}\sum_{\ell\in\mathbb{N}}\varepsilon^p\ln^\ell\varepsilon\,u_{p,\ell}\right\|_{H^1(\Omega\setminus B(0,\delta))}\lesssim\|u_\varepsilon-u_{\varepsilon,2P+4}\|_{H^1(\Omega_\varepsilon)}+\sum_{p\in\mathbb{P}\cap(P,2P+4]}\sum_{\ell\in\mathbb{N}}\varepsilon^p\ln^\ell\varepsilon\,\|u_{p,\ell}\|_{H^1(\Omega\setminus B(0,\delta))}$$
$$=o(\varepsilon^P)+o(\varepsilon^P).$$

□

Remark: An alternative way of stating the error estimate is:

$$\left\|u_\varepsilon-\sum_{p\in\mathbb{P}\cap[0,P)}\sum_{\ell\in\mathbb{N}}\varepsilon^p\ln^\ell\varepsilon\,u_{p,\ell}\right\|_{H^1(\Omega\setminus B(0,\delta))}=O(\varepsilon^P\ln^n\varepsilon)\qquad\text{with } n:=\max\{\ell\in\mathbb{N}\mid u_{P,\ell}\neq 0\}.$$

A Appendix: proof of Lemma 2.8

By Definition 2.6, we already know that the formulas to prove are true replacing $\bigoplus$ by $\sum$. So it suffices to show that those sums are direct.

1. Let us show that $\mathcal{A}(\Omega)=\bigoplus_{d\in\mathbb{R}}\mathcal{A}_d(\Omega)$. Let $n\in\mathbb{N}$ and, for any $j\in[\![1,n]\!]$, $d_j\in\mathbb{R}$ and $\varphi_j\in\mathcal{A}_{d_j}(\Omega)$. We assume that $d_1<d_2<\cdots<d_n$ and $\sum_{j=1}^n\varphi_j=0$. Let us show that $\forall j\in[\![1,n]\!],\ \varphi_j=0$. Let $\theta\in(0,\Theta)$. By definition of $\mathcal{A}_{d_j}(\Omega)$, for any $j\in[\![1,n]\!]$, there is $P_{\theta,j}\in\mathbb{C}[T]$ s.t.: $\forall r\in\mathbb{R}_+^*,\ \varphi_j(r,\theta)=r^{d_j}P_{\theta,j}(\ln r)$. Let us assume by contradiction that: $\exists j\in[\![1,n]\!],\ P_{\theta,j}\neq 0$. Let $j_0:=\max\{j\in[\![1,n]\!]\mid P_{\theta,j}\neq 0\}$. Then $0=\sum_{i=1}^n\lambda_i\varphi_i(r,\theta)\sim r^{d_{j_0}}P_{\theta,j_0}(\ln r)$ when $r\to\infty$, so $P_{\theta,j_0}=0$. This is contradictory, so: $\forall\theta\in(0,\Theta),\forall j\in[\![1,n]\!],\forall r\in\mathbb{R}_+^*,\ \varphi_j(r,\theta)=0$.
2. By the same method, one can show that $\mathcal{A}(D)=\bigoplus_{d\in\mathbb{R}}\mathcal{A}_d(D)$ for any $D\in\{\Pi,\Gamma,\Lambda\}$.
3. Now we will prove (2.7). Let $d\in\mathbb{R}$ and $I:=\{(q,k)\in\mathbb{R}\times\mathbb{N}\mid q+k=d\}$. Separating real and imaginary parts, it is enough to show that:

$$\mathcal{A}_d(\Omega)\cap\mathcal{C}^0(\Omega,\mathbb{R})=\bigoplus_{(q,k)\in I}\left\{z\mapsto\Im[(\alpha z)^q\,\overline{\alpha z}^k P(\log(\alpha z))]\;\middle|\;P\in\mathbb{R}[T]\text{ and }\mathcal{P}(q,k,P)\right\}.$$

Let, for any $(q,k)\in I$, $P_{q,k}\in\mathbb{R}[T]$ and $\varphi_{q,k}:z\mapsto\Im[(\alpha z)^q\,\overline{\alpha z}^k P_{q,k}(\log(\alpha z))]$. We assume that $\mathcal{P}(q,k,P_{q,k})$ holds for any $(q,k)\in I$, that the $P_{q,k}$ are all null except for a finite number, and that $\sum_{(q,k)\in I}\varphi_{q,k}=0$. Let us show by induction on $m:=\max_{(q,k)\in I}\deg P_{q,k}$ that: $\forall(q,k)\in I,\ P_{q,k}=0$ (which implies in turn: $\forall(q,k),\ \varphi_{q,k}=0$). We initialize at $m=-\infty$, i.e. $(\forall(q,k),\ P_{q,k}=0)$, which is trivial. Thus, only the inductive step ($m\in\mathbb{N}$) remains to prove.
To do so, we first note that in Ω:

$$0=r^{-d}\sum_{(q,k)\in I}\varphi_{q,k}(r,\theta)=\sum_{(q,k)\in I}\Im[e^{\mathrm{i}(q-k)(\theta-\Theta)}P_{q,k}(\ln r+\mathrm{i}(\theta-\Theta))]\tag{A.1}$$

By applying $r\partial_r$, we deduce: $0=\sum_{(q,k)\in I}\Im[e^{\mathrm{i}(q-k)(\theta-\Theta)}P'_{q,k}(\ln r+\mathrm{i}(\theta-\Theta))]$.
For any $(q,k)\in I$, let us define (note the switch of indexes at line 2):

$$Q_{q,k}:=\begin{cases}P'_{q,k} & \text{if } q\notin\mathbb{N}\\ P'_{q,k}-P'_{q,k}(0) & \text{if } q\in\mathbb{N}\text{ and }q\leqslant k\\ P'_{q,k}-P'_{k,q}(0) & \text{if } q\in\mathbb{N}\text{ and }q>k\end{cases}$$

Then $(\forall(q,k)\in I,\ \mathcal{P}(q,k,Q_{q,k}))$. Since $\Im[(\alpha z)^q\,\overline{\alpha z}^k]=-\Im[(\alpha z)^k\,\overline{\alpha z}^q]$ (and it is null when $q=k$), the previous equality rewrites as

$$0=\sum_{(q,k)\in I}\Im[(\alpha z)^q\,\overline{\alpha z}^k Q_{q,k}(\log(\alpha z))].$$

Then, by induction hypothesis: $\forall(q,k)\in I,\ Q_{q,k}=0$. This means for $P_{q,k}$ that:

- If $q \notin \mathbb{N}$, then $P'_{q,k} = 0$.
- If $q \in \mathbb{N}$ and $q \neq k$, then $P''_{q,k} = 0$ and $P'_{q,k}(0) - P'_{k,q}(0) = 0$ (A.2).
- If $q \in \mathbb{N}$ and $q = k$, then $P''_{q,k} = 0$.

So (A.1) reduces to :

$$\begin{aligned}
0 &= \sum_{(q,k)\in I} \Im[e^{\mathrm{i}(q-k)(\theta-\Theta)} P_{q,k}(0)] + \mathbf{1}_{q\in\mathbb{N}} \cdot \Im[e^{\mathrm{i}(q-k)(\theta-\Theta)} P'_{q,k}(0)(\ln r + \mathrm{i}(\theta-\Theta))] \\
&= \sum_{(q,k)\in I} \mathbf{1}_{(q\notin\mathbb{N} \text{ or } q>k)} \cdot \Im[e^{\mathrm{i}(q-k)(\theta-\Theta)} P_{q,k}(0)] && \text{by } \mathcal{P} \\
&\quad + \mathbf{1}_{(q\in\mathbb{N} \text{ and } q=k)} \cdot \Im[e^{\mathrm{i}(q-k)(\theta-\Theta)} P'_{q,k}(0)(\ln r + \mathrm{i}(\theta-\Theta))] \\
&\quad + \mathbf{1}_{(q\in\mathbb{N} \text{ and } q>k)} \cdot \Im[e^{\mathrm{i}(q-k)(\theta-\Theta)} P'_{q,k}(0)\, 2\mathrm{i}(\theta-\Theta)] && \text{by (A.2)} \\
&= \sum_{(q,k)\in I} \mathbf{1}_{(q\notin\mathbb{N} \text{ or } q>k)} \cdot P_{q,k}(0) \cdot \sin((q-k)(\theta-\Theta)) \\
&\quad + \mathbf{1}_{(q\in\mathbb{N} \text{ and } q=k)} \cdot P'_{q,k}(0) \cdot (\theta-\Theta) \\
&\quad + \mathbf{1}_{(q\in\mathbb{N} \text{ and } q>k)} \cdot 2P'_{q,k}(0) \cdot \sin\big((q-k)(\theta-\Theta) + \tfrac{\pi}{2}\big) \cdot (\theta-\Theta)
\end{aligned}$$

The functions of θ present here are linearly independent. So the coefficients $P_{q,k}(0)$ and $P'_{q,k}(0)$ are all zero. This concludes the proof. □

B Appendix: asymptotic behaviors w.r.t. r

In this section, we prove Theorems 2.22 and 2.23. We will use Sections 3.1 and 3.2 which are after Theorems 2.22 and 2.23 in the paper, but are independent of them.

B.1 Proof of Theorem 2.23: asymptotic behavior for corner fields-like problems

The proof relies on the Kondratiev theory, usually used to analyse singularities of solutions of elliptic equations, see [25, 27, 11, 13, 26]. We use it in a way that gives an expansion in $\mathcal{A}^+(\Pi)$.

First we introduce the variables (t,θ), defined as $(\ln r, \theta)$ in Ω and $(\ln x, Y)$ in Λ. The pair (t,θ) lies in $\overline{\overline{\Pi}} := \mathbb{R} \times (-1, \Theta)$. Moreover, we denote $\overline{\overline{\Omega}}$, $\overline{\overline{\Lambda}}$, $\overline{\overline{\Sigma_\Omega}}$, $\overline{\overline{\Gamma}}$ and $\overline{\overline{\Sigma_\Lambda}}$ the images of Ω, Λ, Σ_Ω, Γ and Σ_Λ by the change of variable $(x,y) \rightsquigarrow (t,\theta)$, see Figure 10. The notation $\overline{\overline{\dots}}$ is intended to remind the strip shape of $\overline{\overline{\Pi}}$. Finally, for any $u : \Pi \to \mathbb{C}$, we denote

$$\bar{\bar{u}} : (t,\theta) \in \overline{\overline{\Pi}} \mapsto \begin{cases} u(r = e^t, \theta) & \text{in } \overline{\overline{\Omega}} \\ u(x = e^t, Y = \theta) & \text{in } \overline{\overline{\Lambda}} \end{cases}$$

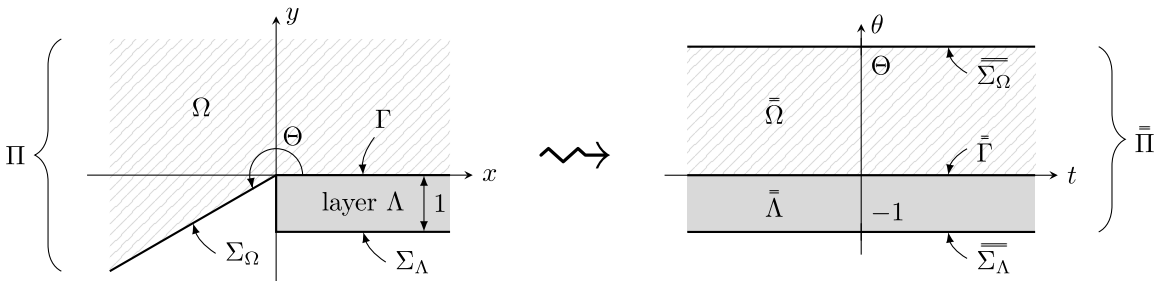

Figure 10: The change of variables $(x,y) \rightsquigarrow (t,\theta)$ and the associated domains.

Definition B.1: Kondratiev spaces
Let H be a Hilbert space and $(s,\beta)\in\mathbb{R}^2$. We define $K^s_\beta(\mathbb{R},H) := \{t\mapsto e^{\beta t}u(t) \mid u\in H^s(\mathbb{R},H)\}$, equipped with the norm $\|u\|_{K^s_\beta(\mathbb{R},H)} := \|t\mapsto e^{-\beta t}u(t)\|_{H^s(\mathbb{R},H)}$. We also denote $K^\infty_\beta(\mathbb{R},H) := \bigcap_{s\in\mathbb{R}} K^s_\beta(\mathbb{R},H)$, and $K^s_\beta(\bar{\bar{\Gamma}}) := K^s_\beta(\mathbb{R},\mathbb{C})$ (identifying the line $\bar{\bar{\Gamma}}$ with $\mathbb{R}$).

Remarks:

- If $s=m\in\mathbb{N}$, then $K^m_\beta(\mathbb{R},H) = \{u\in H^m_{\text{loc}}(\mathbb{R},H) \mid \forall k\in[\![0,m]\!],\ e^{-\beta t}\partial_t^k u(t)\in L^2(\mathbb{R},H)\}$.
- Note that if $u\in K^s_\beta(\mathbb{R},H)$ and u is zero in a vicinity of $-\infty$, then: $\forall\beta'\geqslant\beta,\ u\in K^s_{\beta'}(\mathbb{R},H)$.
- Kondratiev's spaces are linked to o_∂ (see Definition 2.20) in the following way. Let $\chi\in\mathcal{C}^\infty(\mathbb{R})$ be equal to 0 in a vicinity of $-\infty$ and 1 in a vicinity of $+\infty$. Then for any $u:\Omega\to\mathbb{C}$:

$$\begin{cases} \forall\beta\in\mathbb{R}, & \chi\bar{u}\in\bigcap\limits_{m\in\mathbb{N}} K^\infty_\beta(\mathbb{R},H^m(0,\Theta)) \ \Rightarrow\ \forall d>\beta,\ u\underset{r\to\infty}{=} o_\partial(r^d) \\ \forall d\in\mathbb{R}, & u\underset{r\to\infty}{=} o_\partial(r^d) \ \Rightarrow\ \exists t_0\in\mathbb{R},\ \forall\beta>d,\ \ \chi(\cdot+t_0)\,\bar{u}\in\bigcap\limits_{m\in\mathbb{N}} K^\infty_\beta(\mathbb{R},H^m(0,\Theta)) \end{cases}$$

  And there are similar implications in Λ and Γ.

Let H be a Hilbert space, $\beta\in\mathbb{R}$ and $\varphi\in K^0_\beta(\mathbb{R},H)$. For any $\lambda\in\mathbb{C}$ s.t. $\Re(\lambda)=\beta$ we define the (bilateral) Laplace transform of φ at λ as

$$\widehat{\varphi}(\lambda) := \int_{-\infty}^{+\infty} e^{-\lambda t}\varphi(t)\,\mathrm{d}t = \mathcal{F}\big[t\mapsto e^{-\beta t}\varphi(t)\big](\Im(\lambda)) \tag{B.1}$$

where $\mathcal{F}$ denotes the Fourier transform. By properties of $\mathcal{F}$, we have $\widehat{\varphi}\in L^2(\{\lambda\in\mathbb{C}\mid\Re(\lambda)=\beta\})$. If φ depends on t and θ, $\widehat{\varphi}$ implies that we see φ as a function from $\mathbb{R}$ to a space of functions of θ.

To introduce the method, let us use the Laplace transform on $\bar{\bar{S}}$. Let χ_∞ be the truncation function introduced in Section 3.2 and $s := \overline{\overline{\chi_\infty S}}$. Since $S\in V+\chi_\infty\mathcal{A}(\Pi)$, there is $\beta\in\mathbb{R}$ s.t. $s\in K^0_\beta(\mathbb{R},L^2(-1,\Theta))$, so $\hat{s}(\lambda,\theta)$ is well-defined for any $\lambda\in\mathbb{C}$ s.t. $\Re(\lambda)>\beta$. In addition we have for any $f\in L^2(\Omega)$, $g\in L^2(\Lambda)$, $h\in H^{-1/2}(\Gamma)$ and $u\in H^1_{\text{loc}}(\Pi)$:

$$\begin{cases} -\mu_0\Delta u = f & \text{in } \Omega \\ -\mu_1\Delta u = g & \text{in } \Lambda \\ u_{|y=0^+} - u_{|y=0^-} = 0 & \text{on } \Gamma \\ \mu_0\partial_y u_{|y=0^+} - \mu_1\partial_y u_{|y=0^-} = h & \text{on } \Gamma \\ u = 0 & \text{on } \Sigma_\Omega\cup\Sigma_\Lambda \end{cases} \iff \begin{cases} -e^{-2t}\mu_0\Delta\bar{\bar{u}} = \bar{\bar{f}} & \text{in } \bar{\bar{\Omega}} \\ -\mu_1\left(e^{-2t}(\partial_t^2-\partial_t)+\partial_\theta^2\right)\bar{\bar{u}} = \bar{\bar{g}} & \text{in } \bar{\bar{\Lambda}} \\ \bar{\bar{u}}_{|\theta=0^+} - \bar{\bar{u}}_{|\theta=0^-} = 0 & \text{on } \bar{\bar{\Gamma}} \\ e^{-t}\mu_0\partial_\theta\bar{\bar{u}}_{|\theta=0^+} - \mu_1\partial_\theta\bar{\bar{u}}_{|\theta=0^-} = \bar{\bar{h}} & \text{on } \bar{\bar{\Gamma}} \\ \bar{\bar{u}} = 0 & \text{on } \overline{\overline{\Sigma_\Omega}}\cup\overline{\overline{\Sigma_\Lambda}} \end{cases} \tag{B.2}$$

Taking $u := \chi_\infty S$ in (B.2) and applying the Laplace transform yield that $\hat{s}$ satisfies:

$$\begin{cases} (\partial_\theta^2+\lambda^2)\hat{s}(\lambda,\theta) = f & \text{if } \theta\in(0,\Theta) \\ \partial_\theta^2\hat{s}(\lambda,\theta) = g - \left[(\lambda+2)^2-(\lambda+2)\right]\hat{s}(\lambda+2,\theta) & \text{if } \theta\in(-1,0) \\ \hat{s}(\lambda,0^+) - \hat{s}(\lambda,0^-) = 0 & \\ \mu_1\partial_\theta\hat{s}(\lambda,0^-) = \mu_0\partial_\theta\hat{s}(\lambda+1,0^+) & \\ \hat{s}(\lambda,\Theta) = \hat{s}(\lambda,-1) = 0 & \end{cases} \tag{B.3}$$

for some functions functions f and g depending on F and χ_∞. Solving this system w.r.t. θ allows us to extend $\hat{s}$ w.r.t. λ further to the left in the complex plane, except at the λ for which (B.3) is ill-posed. These λ are poles of $\hat{s}$ and they will be used in Proposition B.3 to identify terms of the asymptotic expansion of S.

For any $m\in\mathbb{N}^*$, let us define the Hilbert spaces $\mathcal{H}^m := \{u\in H^1_0(-1,\Theta)\mid u_{|(0,\Theta)}\in H^m(0,\Theta)$ and $u_{|(-1,0)}\in H^m(-1,0)\}$ with the norm $\|u\|^2_{\mathcal{H}^m} := \|u\|^2_{H^m(-1,0)} + \|u\|^2_{H^m(0,\Theta)}$, and $\mathcal{H}^m_\times := H^m(0,\Theta)\times H^m(-1,0)\times\mathbb{C}$. We need to solve problems of the following form with $u\in\mathcal{H}^{m+2}$ and $(f,g,a)\in\mathcal{H}^m_\times$, $m\in\mathbb{N}$:

$$\begin{cases} u''+\lambda^2 u = f & \text{on } (0,\Theta) \\ u'' = g & \text{on } (-1,0) \\ u(0^+) - u(0^-) = 0 & \\ u'(0^-) = a & \\ u(\Theta) = u(-1) = 0 & \end{cases} \tag{B.4}$$

Denoting $A(\lambda) : u \mapsto \big((u''+\lambda^2 u)_{|(0,\Theta)},\, u''_{|(-1,0)},\, u'(0^-)\big)$, (B.4) is equivalent to $A(\lambda)(u) = (f,g,a)$. Note that for any $m \in \mathbb{N}$, $A(\lambda) \in \mathcal{L}(\mathcal{H}^{m+2}, \mathcal{H}^m_\times)$ where $\mathcal{L}(\dots)$ denotes the space of continuous linear maps between two normed vector spaces.

Lemma B.2: Let $m \in \mathbb{N}$.

1. $A(\lambda) : \mathcal{H}^{m+2} \to \mathcal{H}^m_\times$ is invertible iff $\lambda \in \mathbb{C} \setminus \frac{\pi}{\Theta}\mathbb{Z}^*$.
2. $\lambda \mapsto A(\lambda)^{-1}$ is meromorphic from $\mathbb{C}$ to $\mathcal{L}(\mathcal{H}^m_\times, \mathcal{H}^{m+2})$. Its poles are in $\frac{\pi}{\Theta}\mathbb{Z}^*$ and are simple. Moreover, for any $q \in \frac{\pi}{\Theta}\mathbb{Z}^*$ and $v \in \mathcal{H}^m_\times$, the residue $(\mathrm{Res}_{\lambda=q} A(\lambda)^{-1})(v)$ is proportional to the function $\theta \in [-1,\Theta] \mapsto \mathbf{1}_{[0,\Theta]}(\theta)\cdot\sin(q\theta)$.
3. Let $\beta_1 < \beta_2$ be some reals. There is $C > 0$ depending only on (m,β_1,β_2) s.t., for any $\lambda \in \mathbb{C}$ satisfying $\beta_1 < \Re(\lambda) < \beta_2$ and $|\Im(\lambda)| > 1$, we have: $\|A(\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}^m_\times, \mathcal{H}^{m+2})} \leqslant C\,|\Im(\lambda)|^{m+2}$ (B.5).

Proof:

1. Let $\lambda \in \mathbb{C}$. An easy calculation gives that any element of $\mathrm{Ker}\, A(\lambda)$ must be proportional to $\theta \in [-1,\Theta] \mapsto \mathbf{1}_{[0,\Theta]}(\theta)\cdot\sin(\lambda\theta)$. This function belongs to $\mathcal{H}^m \setminus \{0\}$ iff $\lambda \in \frac{\pi}{\Theta}\mathbb{Z}^*$. Therefore $\mathrm{Ker}\, A(\lambda) \neq 0 \Longleftrightarrow \lambda \in \frac{\pi}{\Theta}\mathbb{Z}^*$.
   Moreover, it is easy to see that, for any $(f,g,a) \in \mathcal{H}^m_\times$, (B.4) with $\lambda := 0$ has a unique solution in $\mathcal{H}^{m+2}$. Thus $A(0) : \mathcal{H}^{m+2} \to \mathcal{H}^m_\times$ is invertible. Now, for any $\lambda \in \mathbb{C}$, we have $A(\lambda) = A(0) + \lambda^2 B$ with $B : u \mapsto (u_{|(0,\Theta)}, 0, 0)$. B is a compact operator from $\mathcal{H}^{m+2}$ to $\mathcal{H}^m_\times$, so the Fredholm alternative holds for $A(\lambda)$. Therefore $A(\lambda) : \mathcal{H}^{m+2} \to \mathcal{H}^m_\times$ is invertible iff $\lambda \in \mathbb{C} \setminus \frac{\pi}{\Theta}\mathbb{Z}^*$.
2. Since $\lambda \mapsto A(\lambda)$ is holomorphic on $\mathbb{C} \setminus \frac{\pi}{\Theta}\mathbb{Z}^*$, so too is $\lambda \mapsto A(\lambda)^{-1}$. Let us describe its behavior near the points $q \in \frac{\pi}{\Theta}\mathbb{Z}^*$ using [25, Theorem 5.1.1, p.147]. It depends on the "Jordan chains" of $A(q)$, which are the sequences $(u_0,\dots,u_n) \in (\mathcal{H}^m)^{n+1}$, $n \in \mathbb{N}$, s.t.
$$\forall k \in [\![0,n]\!], \qquad \sum_{j=0}^{k} \frac{1}{j!}\frac{\mathrm{d}^j A}{\mathrm{d}\lambda^j}\Big|_{\lambda=q}(u_{k-j}) = 0 \tag{B.6}$$
   Let us compute these chains. Taking $k := 0$ in (B.6) gives $A(q)(u_0) = 0$, so u_0 is proportional to $\theta \mapsto \mathbf{1}_{[0,\Theta]}(\theta)\cdot\sin(q\theta)$ by step 1. In addition, if $n \geqslant 1$, taking $k := 1$ gives $A(q)(u_1) + \frac{\mathrm{d}A}{\mathrm{d}\lambda}(q)(u_0) = 0$. This implies on one hand $u_1'' + q^2 u_1 + 2qu_0 = 0$ in $(0,\Theta)$. On the other we get $u''_{1|(-1,0)} = 0$ and $u_1'(0^-) = 0$, so $u_{1|(-1,0)} = 0$, which gives $u_1(0) = u_1(\Theta) = 0$. Therefore
$$0 \neq \int_0^\Theta 2q|u_0|^2 = \int_0^\Theta -(u_1''+q^2u_1)\cdot\overline{u_0} = \int_0^\Theta -u_1\cdot(\underbrace{\overline{u_0''+q^2u_0}}_{=0}) = 0.$$
   This is absurd, so we must have $n = 0$ for any Jordan chain of $A(q)$. Therefore, [25, Theorem 5.1.1, p.147] states that $A(\lambda)^{-1}$ has a simple pole at q, and $\mathrm{Im}(\mathrm{Res}_{\lambda=q} A(\lambda)^{-1}) = \mathbb{C}u_0$.
3. Let $(f,g,a) \in \mathcal{H}^m_\times$ and $u := A(\lambda)^{-1}(f,g,a) \in \mathcal{H}^{m+2}$. We will write $\lesssim$ for inequalities valid up to a constant that depends on (m,β_1,β_2) but not on (λ,f,g,a). By Poincaré's inequality and integration by parts, we have
$$\|u\|^2_{H^1(-1,0)} \lesssim \int_{-1}^0 |u'|^2 = -\int_0^{-1} g\bar{u} + a\,\bar{u}(0) \lesssim (\|g\|_{L^2(-1,0)} + |a|)\cdot\|u\|_{H^1(-1,0)}$$
   thus $\|u\|_{H^1(-1,0)} \lesssim \|g\|_{L^2(-1,0)} + |a|$. Since $u''_{|(-1,0)} = g$, we deduce $\|u\|_{H^{m+2}(-1,0)} \lesssim \|g\|_{H^m} + |a|$. Then, let $u_1 : \theta \in [0,\Theta] \mapsto u(0)\big(1-\frac{\theta}{\Theta}\big)$, $v := u - u_1$ and $f_\lambda := f - \lambda^2 u_1$. We have $v(0) = v(\Theta) = 0$ and $v'' + \lambda^2 v = f_\lambda$, so
$$\int_0^\Theta (-|v'|^2 + \lambda^2|v|^2) = \int_0^\Theta f_\lambda\bar{v}$$
   Dividing by λ and taking the absolute value of the imaginary part, we get: $\frac{|\Im(\lambda)|}{|\lambda|^2}\|v'\|^2_{L^2} + |\Im(\lambda)|\|v\|^2_{L^2} \leqslant \frac{1}{|\lambda|}\|f_\lambda\|_{L^2}\|v\|_{L^2}$. Now we asume $|\Im(\lambda)| > 1$ and $\beta_1 < \Re(\lambda) < \beta_2$, so $|\lambda| \lesssim |\Im(\lambda)|$. Thus $\|v'\|^2_{L^2} + |\lambda|^2\|v\|^2_{L^2} \leqslant \|f_\lambda\|_{L^2}\|v\|_{L^2}$, which yields $\|v\|_{L^2} \lesssim |\lambda|^{-2}\|f_\lambda\|_{L^2}$ and then $\|v'\|_{L^2} \lesssim |\lambda|^{-1}\|f_\lambda\|_{L^2}$. Now from $v'' = f_\lambda - \lambda^2 v$, one can easily derive by induction on m that $\|v\|_{H^{m+2}} \lesssim |\lambda|^m\|f_\lambda\|_{H^m}$. This implies $\|u\|_{H^{m+2}(0,\Theta)} \lesssim |\lambda|^m\|f\|_{H^m} + |\lambda|^{m+2}|u(0)|$, with $|u(0)| \lesssim \|u\|_{H^{m+2}(-1,0)}$.
   So finally $\|u\|_{\mathcal{H}^{m+2}} \lesssim |\Im(\lambda)|^{m+2}\|(f,g,a)\|_{\mathcal{H}^m_\times}$. □

Proposition B.3 is the base step of the asymptotic expansion. It will be applied to s the rest of the at a given order, and the function s_{r} below will be the rest at the next order. Iterating this process provides an asymptotic expansion of S at any order. Since Theorem 2.23 implies infinite regularity on S, we work in the spaces K^∞_β and $\mathcal{H}^m$ for any m.

Proposition B.3: Let $d \in \mathbb{R}$, $m \in \mathbb{N} \setminus \{0,1\}$, $s \in \bigcap_{\beta>d} K^\infty_\beta(\mathbb{R}, \mathcal{H}^m)$, $f \in \bigcap_{\beta>d-1} K^\infty_\beta(\mathbb{R}, H^{m-2}(0,\Theta))$, $g \in \bigcap_{\beta>d-1} K^\infty_\beta(\mathbb{R}, H^{m-2}(-1,0))$, $h \in \bigcap_{\beta>d-1} K^\infty_\beta(\bar{\Gamma})$ and $d_{\min} := \min(\frac{\pi}{\Theta}\mathbb{Z}^* \cap (d-1, d])$. We assume:

$$\begin{cases} \mu_0 \Delta s = f & \text{in } \bar{\bar{\Omega}} \\ \mu_1 \left(e^{-2t}(\partial_t^2 - \partial_t) + \partial_\theta^2\right) s = g & \text{in } \bar{\bar{\Lambda}} \\ s_{|\theta=0^+} - s_{|\theta=0^-} = 0 & \text{on } \bar{\bar{\Gamma}} \\ e^{-t}\mu_0 \partial_\theta s_{|\theta=0^+} - \mu_1 \partial_\theta s_{|\theta=0^-} = h & \text{on } \bar{\bar{\Gamma}} \\ s = 0 & \text{on } \overline{\overline{\Sigma_\Omega}} \cup \overline{\overline{\Sigma_\Lambda}} \end{cases}$$

Then there is $s_{\mathrm{r}} \in \bigcap_{d-1<\beta<d_{\min}} K^\infty_\beta(\mathbb{R}, \mathcal{H}^m)$ and $(c_q) \in \mathbb{C}^{\frac{\pi}{\Theta}\mathbb{Z}^* \cap (d-1,d]}$ s.t. $s = \sum_{q \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap (d-1,d]} c_q \, \bar{\bar{\phi_q}} + s_{\mathrm{r}}$.

Proof: For any $\beta \in \mathbb{R}$, let $\mathbb{C}_{\Re>\beta} := \{\lambda \in \mathbb{C} \mid \Re(\lambda) > \beta\}$. The Laplace transform of s is well-defined and holomorphic from $\mathbb{C}_{\Re>d}$ to $\mathcal{H}^m$. We will show that $\hat{s}$ has a meromorphic extension on $\mathbb{C}_{\Re>d-1}$, with poles belonging in $\mathfrak{P} := \frac{\pi}{\Theta}\mathbb{Z}^* \cap (d-1, d]$, then we will apply the residue theorem on a rectangle surrounding these poles. The functions ϕ_q will appear in the residues.

Step 1 : $\hat{s}$ satisfies in $\mathbb{C}_{\Re>d}$:

$$\begin{cases} \mu_0(\partial_\theta^2 + \lambda^2)\hat{s}(\lambda,\theta) = \hat{f}(\lambda,\theta) & \text{if } \theta \in (0,\Theta) \\ \mu_1 \partial_\theta^2 \hat{s}(\lambda,\theta) = \hat{g}(\lambda,\theta) - \mu_1\left[(\lambda+2)^2 - (\lambda+2)\right]\hat{s}(\lambda+2,\theta) := \tilde{g}(\lambda,\theta) & \text{if } \theta \in (-1,0) \\ \hat{s}(\lambda,0^+) - \hat{s}(\lambda,0^-) = 0 \\ \mu_1 \partial_\theta \hat{s}(\lambda,0^-) = \hat{h}(\lambda) + \mu_0 \partial_\theta \hat{s}(\lambda+1,0^+) := \tilde{h}(\lambda) \\ \hat{s}(\lambda,\Theta) = \hat{s}(\lambda,-1) = 0 \end{cases}$$

For any $\lambda \in \mathbb{C}_{\Re>d-1} \setminus \frac{\pi}{\Theta}\mathbb{Z}^*$, let $\tilde{s}(\lambda) := A(\lambda)^{-1}\left(\frac{1}{\mu_0}\hat{f}(\lambda), \frac{1}{\mu_1}\tilde{g}(\lambda), \frac{1}{\mu_1}\tilde{h}(\lambda)\right)$. Then $\tilde{s}$ is also solution of the above system, so $\tilde{s}$ and $\hat{s}$ coincide on $\mathbb{C}_{\Re>d} \setminus \frac{\pi}{\Theta}\mathbb{Z}^*$ by Lemma B.2. Hence $\tilde{s}$ is an extension of $\hat{s}$ on $\mathbb{C}_{\Re>d-1} \setminus \mathfrak{P}$, that we will still denote $\hat{s}$. Lemma B.2 implies that this extension is meromorphic with simple poles and: $\forall q \in \mathfrak{P}$, $\mathrm{Res}_q \, \hat{s} = (\mathrm{Res}_{\lambda=q} \, A(\lambda)^{-1})\left(\frac{1}{\mu_0}\hat{f}(q), \frac{1}{\mu_1}\tilde{g}(q), \frac{1}{\mu_1}\tilde{h}(q)\right)$.

Step 2 : Let $t \in \mathbb{R}$, $b \in (d-1, d_{\min})$ and $k \in \mathbb{R}_+^*$. The residue theorem applied to $\lambda \mapsto e^{\lambda t}\hat{s}(\lambda)$ on the rectangle $[b, d+1] \times [-k, k] \subset \mathbb{C}$ gives the following equality in $\mathcal{H}^m$.

$$\begin{aligned} 2\mathrm{i}\pi \sum_{q \in \mathfrak{P}} \mathrm{Res}_q(\lambda \mapsto e^{\lambda t}\hat{s}(\lambda)) \quad = \quad & \int_{-k}^{k} e^{(d+1+\mathrm{i}\gamma)t}\hat{s}(d+1+\mathrm{i}\gamma)\,\mathrm{d}\gamma - \int_{-k}^{k} e^{(b+\mathrm{i}\gamma)t}\hat{s}(b+\mathrm{i}\gamma)\,\mathrm{d}\gamma \\ & + \int_{b}^{d+1} e^{(\beta-\mathrm{i}k)t}\hat{s}(\beta-\mathrm{i}k)\,\mathrm{d}\beta - \int_{b}^{d+1} e^{(\beta+\mathrm{i}k)t}\hat{s}(\beta+\mathrm{i}k)\,\mathrm{d}\beta \end{aligned}$$

Note that when $k \to \infty$ the first integral tends to the inverse Laplace transform of s (up to a constant). We will show that the last two integrals tend to 0, by proving the following:

$$\forall n \in \mathbb{N}, \qquad \sup_{\beta \in [b, d+1]} \int_{\mathbb{R}\setminus[-1,1]} \gamma^n \|\hat{s}(\beta+\mathrm{i}\gamma)\|_{\mathcal{H}^m}\,\mathrm{d}\gamma < \infty \tag{B.7}$$

Let $n \in \mathbb{N}$ and $\beta \in [b, d+1]$. (B.5) and the expressions of $\tilde{g}$ and $\tilde{h}$ yield

$$\begin{aligned} \int_{\mathbb{R}\setminus[-1,1]} \gamma^n \|\hat{s}(\beta+\mathrm{i}\gamma)\|_{\mathcal{H}^m}\mathrm{d}\gamma &\leqslant \int_{\mathbb{R}\setminus[-1,1]} \gamma^n \|A(\lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}_\times^{m-2}, \mathcal{H}^m)} \left\| \left(\tfrac{1}{\mu_0}\hat{f}(\beta+\mathrm{i}\gamma), \tfrac{1}{\mu_1}\tilde{g}(\beta+\mathrm{i}\gamma), \tfrac{1}{\mu_1}\tilde{h}(\beta+\mathrm{i}\gamma)\right)\right\|_{\mathcal{H}_\times^{m-2}} \mathrm{d}\gamma \\ &\lesssim \int_{\mathbb{R}\setminus[-1,1]} \gamma^{n+m+2} \Big(\|\hat{f}(\beta+\mathrm{i}\gamma)\|_{H^{m-2}} + \|\hat{g}(\beta+\mathrm{i}\gamma)\|_{H^{m-2}} + |\hat{h}(\beta+\mathrm{i}\gamma)| \\ &\qquad + \|\hat{s}(\beta+\mathrm{i}\gamma+1)\|_{\mathcal{H}^m} + \gamma^2 \|\hat{s}(\beta+\mathrm{i}\gamma+2)\|_{\mathcal{H}^m} \Big)\,\mathrm{d}\gamma \end{aligned}$$

Let us treat the term with f, given that the others are similar.

$$\begin{aligned}\int_{\mathbb{R}\setminus[-1,1]}\gamma^{n+m+2}\|\hat f(\beta+\mathrm{i}\gamma)\|_{H^{m-2}}\,\mathrm{d}\gamma &\underset{\text{C.S.}}{\leqslant} \|\gamma^{-1}\|_{L^2(\mathbb{R}\setminus[-1,1])}\cdot\Big\|\gamma^{n+m+3}\|\hat f(\beta+\mathrm{i}\gamma)\|_{H^{m-2}}\Big\|_{L^2(\mathbb{R}\setminus[-1,1])}\\ &\lesssim \|e^{-\beta t}\partial_t^{n+m+3}f\|_{L^2(\mathbb{R},H^{m-2}(0,\Theta))}\\ &\lesssim \max_{\beta'\in\{b,d+1\}}\|e^{-\beta' t}\partial_t^{n+m+3}f\|_{L^2(\mathbb{R},H^{m-2}(0,\Theta))}\\ &= \max_{\beta'\in\{b,d+1\}}\|f\|_{K^{n+m+3}_{\beta'}(\mathbb{R},H^{m-2}(0,\Theta))}\end{aligned}$$

thanks to an interpolation between b and $d+1$. We have majorized by a finite constant independent of β, so the supremum on β is finite. Thus (B.7) is proven.

Therefore: $\displaystyle\int_1^\infty\Big\|\int_b^{d+1}e^{(\beta\pm\mathrm{i}k)t}\hat s(\beta\pm\mathrm{i}k)\,\mathrm{d}\beta\Big\|_{\mathcal{H}^m}\mathrm{d}k\lesssim\int_1^\infty\int_b^{d+1}\|\hat s(\beta\pm\mathrm{i}k)\|_{\mathcal{H}^m}\,\mathrm{d}\beta\,\mathrm{d}k<+\infty.$
So by taking arbitrarily large k, we can make the terms $\int_b^{d+1}e^{(\beta\pm\mathrm{i}\gamma)t}\hat s(\beta\pm\mathrm{i}k)\,\mathrm{d}\beta$ tend to 0 in $\mathcal{H}^m$. Hence:

$$\sum_{q\in\mathfrak{P}}\mathrm{Res}_q(\lambda\mapsto e^{\lambda t}\hat s(\lambda))=\frac{1}{2\mathrm{i}\pi}\int_{\mathbb{R}}e^{(d+1+\mathrm{i}\gamma)t}\hat s(d+1+\mathrm{i}\gamma)\,\mathrm{d}\gamma-\frac{1}{2\mathrm{i}\pi}\int_{\mathbb{R}}e^{(b+\mathrm{i}\gamma)t}\hat s(b+\mathrm{i}\gamma)\,\mathrm{d}\gamma.$$

Now, since $\hat s$ has only simple poles, the terms of the sum are equal to

$$e^{qt}\,\mathrm{Res}_q\,\hat s=e^{qt}(\mathrm{Res}_{\lambda=q}\,A(\lambda)^{-1})\big(\tfrac{1}{\mu_0}\hat f(q),\tfrac{1}{\mu_1}\tilde g(q),\tfrac{1}{\mu_1}\tilde h(q)\big)=e^{qt}c_q\mathbf{1}_{[0,\Theta]}(\theta)\sin(d\theta)=c_q\,\overline{\overline{\phi_q}}(t,\theta)$$

for some constant $c_q\in\mathbb{C}$ by point 2 of Lemma B.2. The first intergral is equal to $s(t)$ by inverse Laplace transform (because $s\in K^\infty_{d+1}(\mathbb{R},\mathcal{H}^m)$). We define $s_{\mathrm{r}}(t)$ to be equal to the last integral. Thus we get the desired formula, and (B.7) shows that $s_{\mathrm{r}}\in K^\infty_b(\mathbb{R},\mathcal{H}^m)$. □

Let S be the function set in Theorem 2.23 and $(\sigma_d(S))_{d\in\frac{\pi}{\Theta}\mathbb{N}^*}$ the coefficients of Definition 3.10, which vanish for d big enough. With the $\sigma_d(S)$, the non-variational part of S (denoted $S^{\mathcal{A}}$ below) can be explicitly computed, so it remains to get an asymptotic expansion of its variational part (denoted S^V below). In order to apply Proposition B.3, we need that $\overline{\overline{\chi_\infty S^V}}$ belongs to some space K^∞_β (since $\Omega_1\setminus B(0,R_{\mathrm{c}})=\Pi\setminus B(0,R_{\mathrm{c}})$, $\chi_\infty S^V$ can be seen as a function defined on Π, which allows us to consider $\overline{\overline{\chi_\infty S^V}}$). We will use again the notations $\langle.\rangle$ and $T_{\geqslant d}$ of Definitions 2.3 and 2.21.

Proposition B.4: Let $d\in\mathbb{R}_-$,

$$S^{\mathcal{A}}:=T_{\geqslant d}\left[\left\langle -R_{\partial_Y^2}\circ\partial^2_{X|\Lambda},\,\frac{\mu_0}{\mu_1}R_{\mathrm{N}}\circ\partial_{Y|\Gamma,Y=0^+}\right\rangle\left(\frac{1}{\mu_0}R_\Delta(F^\infty_\Omega)+\frac{1}{\mu_1}R_{\partial_Y^2}(F^\infty_\Lambda)+\sum_{d\in\frac{\pi}{\Theta}\mathbb{N}^*}\sigma_d(S)\,\phi_d\right)\right]$$

in $\mathcal{A}(\Pi)$ and $S^V:=S-\chi_\infty S^{\mathcal{A}}$. There is $d\in\mathbb{R}_-$ s.t. $S^V\in V$ and: $\forall m\in\mathbb{N}$, $\overline{\overline{\chi_\infty S^V}}\in K^\infty_{1/2}(\mathbb{R},\mathcal{H}^m)$.

Proof: We consider any $d\in\mathbb{R}_-$ and we will fix it later. The proof has three steps: showing that $S^V\in V$, showing that S^V is regular w.r.t t, and deducing that it is regular w.r.t. θ. For any $r_1,r_2\in\mathbb{R}$ s.t. $R_{\mathrm{c}}<r_1<r_2$, we will denote $\chi\in\mathcal{C}^\infty(\mathbb{R}^2)$ a radial function equal to 0 on $B(0,r_1)$ and 1 on $\mathbb{R}^2\setminus B(0,r_2)$ (r_1,r_2 are implicit in this notation).

Step 1: Let $\varphi:\Pi\to\mathbb{C}$ be equal to $T_{\leqslant d}(F^\infty_D)$ on D for any $D\in\{\Omega,\Lambda\}$, and $F_{\mathrm{v}}=F-\chi_\infty\varphi$. If $d<-2$ then $(1+r)F_{\mathrm{v}}\in L^2(\Omega_1)$. So the same construction than in Theorem 3.11 shows that for d small enough, the problem

$$\begin{cases}\mathrm{div}(\mu\nabla S^V)=f_1:=F_{\mathrm{v}}+\chi_\infty\varphi-\mathrm{div}\big(\mu\nabla(\chi_\infty\cdot S^{\mathcal{A}})\big) & \text{in }\Omega_1\setminus(\Gamma\cap\{X>R_{\mathrm{c}}\})\\ [\mu\partial_Y S^V]_\Gamma=g_1:=-[\mu\partial_Y(\chi_\infty S^{\mathcal{A}})]_\Gamma & \text{on }\Gamma\cap\{X>R_{\mathrm{c}}\}\\ S^V=0 & \text{on }\partial\Omega_1\end{cases}\tag{B.8}$$

has a unique solution in V and that $S=S^V+\chi_\infty S^{\mathcal{A}}$. In addition, it also gives that $(\mu\Delta S^{\mathcal{A}}-\varphi)_{|D}\in\sum_{d'<d}\mathcal{A}_{d'}(D)$ for any $D\in\{\Omega,\Lambda\}$, and $[\mu\partial_Y S^{\mathcal{A}}]_\Gamma\in\sum_{d'<d}\mathcal{A}_{d'}(\Gamma)$. And we have $F_{\mathrm{v}}=o_\partial(\mathbf{r}^d)$ by

hypothesis, so for d small enough and up to increasing R_c (without loss of generality): $\forall m \in \mathbb{N}$,

$$(\overline{\overline{\chi f_1}})_{|\bar{\bar{\Omega}}} \in K^\infty_{-2}(\mathbb{R}, H^m(0,\Theta)) \quad \text{and} \quad (\overline{\overline{\chi f_1}})_{|\bar{\bar{\Lambda}}} \in K^\infty_{-3/2}(\mathbb{R}, H^m(-1,0)) \quad \text{and} \quad \overline{\overline{\chi g_1}} \in K^\infty_{-1/2}(\bar{\bar{\Gamma}}) \quad \text{(B.9)}$$

Step 2: For any $\beta \in \mathbb{R}$, we denote $K^0_\beta(\bar{\bar{\Omega}}) := K^0_\beta(\mathbb{R}, L^2(0,\Theta)) = \{(t,\theta) \mapsto e^{\beta t} u(t,\theta) \mid u \in L^2(\bar{\bar{\Omega}})\}$ and $K^0_\beta(\bar{\bar{\Lambda}}) := K^0_\beta(\mathbb{R}, L^2(-1,0))$. And we define

$$\bar{\bar{V}} := \{w \in H^1_{\text{loc}}(\bar{\bar{\Pi}}) \mid \nabla w_{|\bar{\bar{\Omega}}} \in L^2(\bar{\bar{\Omega}}),\ \partial_t w_{|\bar{\bar{\Lambda}}} \in K^0_{1/2}(\bar{\bar{\Lambda}}),\ \partial_\theta w_{|\bar{\bar{\Lambda}}} \in K^0_{-1/2}(\bar{\bar{\Lambda}}) \text{ and } w_{|\overline{\overline{\Sigma_\Omega} \cup \Sigma_\Lambda}} = 0\}.$$

Note that if Ω_1 were equal to Π, $\bar{\bar{V}}$ would simply be $\{\bar{\bar{u}} \mid u \in V\}$. So it is the natural variational space for Poisson's problem transferred into $\bar{\bar{\Pi}}$. Looking at χS^V as a function defined on Π, we will show by induction on n that:

$$\forall n \in \mathbb{N},\ \forall R_c < r_1 < r_2, \qquad \partial_t^n \overline{\overline{\chi S^V}} \in \bar{\bar{V}}.$$

Since $S^V \in V$, the initial case is trivial, so only the inductive step remains to prove. Let us assume the property at rank n and show it at rank $n+1$. We will use the method of finite differences. Letting $\mu := \mu_0$ on Ω and $\mu := \mu_1$ on Λ, (B.8) implies:

$$\begin{cases} \mu\Delta(\chi S^V) = f_2 := \chi f_1 - 2\mu\nabla\chi\cdot\nabla S^V - \mu\Delta\chi\cdot S^V & \text{in } D,\ \forall D \in \{\Omega,\Lambda\} \\ [\mu\partial_Y(\chi S^V)]_\Gamma = g_2 := \chi g_1 & \text{on } \Gamma \\ \chi S^V = 0 & \text{on } \Sigma_\Omega \cup \Sigma_\Lambda \end{cases}$$

By induction hypothesis applied to $(r_1', r_2') := (\frac{R_c+r_1}{2}, r_1)$, $\partial_t^k \nabla S^V$ and $\partial_t^k S^V$ are L^2 on $\{r_1 < r < r_2\}$ for any $k \in [\![0,n]\!]$. So (B.9) implies $\overline{\overline{f_2}}_{|\bar{\bar{\Omega}}} \in K^n_{-2}(\mathbb{R}, L^2(0,\Theta))$ and $\overline{\overline{f_2}}_{|\bar{\bar{\Lambda}}} \in K^n_{-3/2}(\mathbb{R}, L^2(-1,0))$. Similarly $\overline{\overline{g_2}} \in K^\infty_{-1/2}(\bar{\bar{\Gamma}})$. Next, we apply the change variables $(x,y) \rightsquigarrow (t,\theta)$ using (B.2), and then ∂_t^n. We get that $s := \partial_t^n(\overline{\overline{\chi S^V}})$ satisfies:

$$\begin{cases} -e^{-2t}\mu_0\Delta s = f_\Omega := e^{-2t}\partial_t^n(e^{2t}\overline{\overline{f_2}}_{|\Omega}) & \text{in } \bar{\bar{\Omega}} \\ -\mu_1\,(e^{-2t}(\partial_t^2 - \partial_t) + \partial_\theta^2)s = f_\Lambda := \partial_t^n \overline{\overline{f_2}}_{|\Lambda} + \sum_{k=0}^{n-1}(-2)^{n-k}e^{-2t}\mu_1(\partial_t^2-\partial_t)\partial_t^k(\overline{\overline{\chi S^V}}) & \text{in } \bar{\bar{\Lambda}} \\ s_{|\theta=0^+} - s_{|\theta=0^-} = 0 & \text{on } \bar{\bar{\Gamma}} \\ e^{-t}\mu_0\partial_\theta s_{|\theta=0^+} - \mu_1\partial_\theta s_{|\theta=0^-} = g := \partial_t^n\overline{\overline{g_2}} & \text{on } \bar{\bar{\Gamma}} \\ s = 0 & \text{on } \overline{\overline{\Sigma_\Omega}} \cup \overline{\overline{\Sigma_\Lambda}} \end{cases} \quad \text{(B.10)}$$

with $f_\Omega \in K^0_{-2}(\Omega)$, $f_\Lambda \in K^0_{-3/2}(\bar{\bar{\Lambda}})$ and $g \in K^\infty_{-1/2}(\bar{\bar{\Gamma}})$ by induction hypothesis. The variational formulation of (B.10) is: $\forall \varphi \in \bar{\bar{V}}$,

$$\int_{\bar{\bar{\Omega}}} \mu_0\nabla s\cdot\nabla\varphi + \int_{\bar{\bar{\Lambda}}} \mu_1\,(e^{-t}\,\partial_t s\,\partial_t\varphi + e^t\,\partial_\theta s\,\partial_\theta\varphi) = \int_{\bar{\bar{\Omega}}} e^{2t} f_\Omega\varphi + \int_{\bar{\bar{\Lambda}}} e^t f_\Lambda\varphi + \int_{\bar{\bar{\Gamma}}} e^t g\varphi.$$

Let us denote $D_\eta\varphi(t,\theta) := \frac{\varphi(t+\eta,\theta)-\varphi(t,\theta)}{\eta}$ for any $\eta \in \mathbb{R}^*$ and any function φ. Taking $\varphi := D_{-\eta}D_\eta\bar{s}$ and discretely integrating by parts $D_{-\eta}$, we have:

$$\int_{\bar{\bar{\Omega}}} \mu_0|D_\eta\nabla s|^2 + \int_{\bar{\bar{\Lambda}}} \mu_1\,(D_\eta(e^{-t}\partial_t s)\cdot D_\eta\partial_t\bar{s} + e^t\,|D_\eta\partial_\theta s|^2)$$
$$= \int_{\bar{\bar{\Omega}}} e^{2t} f_\Omega\cdot D_{-\eta}D_\eta\bar{s} + \int_{\bar{\bar{\Lambda}}} e^t f_\Lambda\cdot D_{-\eta}D_\eta\bar{s} + \int_{\bar{\bar{\Gamma}}} D_{-\eta}D_\eta(e^t g)\cdot\bar{s}$$

But for any functions φ,ψ, we have $D_\eta(\varphi\psi) = \varphi\cdot D_\eta\psi + D_\eta\varphi\cdot\psi$. Therefore:

- $D_\eta(e^{-t}\partial_t s)\cdot D_\eta\partial_t\bar{s} = e^{-t}|D_\eta\partial_t s|^2 + \frac{e^{-\eta}-1}{\eta}e^{-t}\partial_t s\cdot D_\eta\partial_t\bar{s}$
- $D_{-\eta}D_\eta(e^t g) = e^t D_{-\eta}D_\eta g + 2\frac{e^\eta-1}{\eta}e^t D_\eta g + \left(\frac{e^\eta-1}{\eta}\right)^2 e^t g$

Let us assume that η is small enough so that $\frac{|e^\eta-1|}{\eta} < 2$ and $\frac{|e^{-\eta}-1|}{\eta} < 2$. Then:

$$\|D_\eta\nabla s\|^2_{L^2(\bar{\bar{\Omega}})} + \|D_\eta\partial_t s\|^2_{K^0_{1/2}(\bar{\bar{\Lambda}})} + \|D_\eta\partial_\theta s\|^2_{K^0_{-1/2}(\bar{\bar{\Lambda}})} + \int_{\bar{\bar{\Lambda}}} \mu_1\frac{e^{-\eta}-1}{\eta}e^{-t}\partial_t s\cdot D_\eta\partial_t\bar{s}$$
$$\lesssim \|f_\Omega\|_{K^0_{-2}(\bar{\bar{\Omega}})}\|D_{-\eta}D_\eta s\|_{L^2(\bar{\bar{\Omega}})} + \|f_\Lambda\|_{K^0_{-3/2}(\bar{\bar{\Lambda}})}\|D_{-\eta}D_\eta s\|_{K^0_{1/2}(\bar{\bar{\Lambda}})} + \|g\|_{K^2_{-1/2}(\bar{\bar{\Gamma}})}\|s\|_{K^0_{-1/2}(\bar{\bar{\Gamma}})}$$

Then, moving the intergral to the right-hand side and majorizing some $\|D_\eta \cdot\|$ by $\|\partial_t \cdot\|$ or $\|\nabla \cdot\|$, we get:

$$\begin{aligned}\|D_\eta \nabla s\|^2_{L^2(\bar{\bar{\Omega}})} + \|D_\eta \partial_t s\|^2_{K^0_{1/2}(\bar{\bar{\Lambda}})} + \|D_\eta \partial_\theta s\|^2_{K^0_{-1/2}(\bar{\bar{\Lambda}})} &\lesssim \|f_\Omega\|_{K^0_{-2}(\bar{\bar{\Omega}})}\|D_\eta \nabla s\|_{L^2(\bar{\bar{\Omega}})} \\ &+ \|f_\Lambda\|_{K^0_{-3/2}(\bar{\bar{\Lambda}})}\|D_\eta \partial_t s\|_{K^0_{1/2}(\bar{\bar{\Lambda}})} + \|g\|_{K^2_{-1/2}(\bar{\bar{\Gamma}})}\|s\|_{K^0_{-1/2}(\bar{\bar{\Gamma}})} + \|\partial_t s\|_{K^0_{1/2}(\bar{\bar{\Lambda}})}\|D_\eta \partial_t s\|_{K^0_{1/2}(\bar{\bar{\Lambda}})}\end{aligned}$$

But, since $s_{|\overline{\overline{\Sigma_\Lambda}}} = 0$, a Poincaré-type inequality gives $\|s\|^2_{K^0_{-1/2}(\bar{\bar{\Gamma}})} \lesssim \|\partial_\theta s\|^2_{K^0_{-1/2}(\bar{\bar{\Lambda}})}$. Finally, using Young's inequality $ab \lesssim \frac{1}{\delta}a + \delta b$ on norm products with δ small enough and moving the $\|D_\eta \partial_t s\|^2_{K^0_{1/2}(\bar{\bar{\Lambda}})}$ and $\|D_\eta \partial_\theta s\|^2_{K^0_{-1/2}(\bar{\bar{\Lambda}})}$ from the right-hand side to the left-hand one, we get:

$$\begin{aligned}\|\nabla \partial_t s\|^2_{L^2(\bar{\bar{\Omega}})} + \|\partial_t^2 s\|^2_{K^0_{1/2}(\bar{\bar{\Lambda}})} + \|\partial_\theta \partial_t s\|^2_{K^0_{-1/2}(\bar{\bar{\Lambda}})} &\\ \leqslant \limsup_{\eta \to 0} \|D_\eta \nabla s\|^2_{L^2(\bar{\bar{\Omega}})} + \|D_\eta \partial_t s\|^2_{K^0_{1/2}(\bar{\bar{\Lambda}})} + \|D_\eta \partial_\theta s\|^2_{K^0_{-1/2}(\bar{\bar{\Lambda}})} &\\ \lesssim \|f_\Omega\|^2_{K^0_{-2}(\bar{\bar{\Omega}})} + \|f_\Lambda\|^2_{K^0_{-3/2}(\bar{\bar{\Lambda}})} + \|g\|^2_{K^1_{-1/2}(\bar{\bar{\Gamma}})} + \|\partial_\theta s\|^2_{K^0_{-1/2}(\bar{\bar{\Lambda}})} + \|\partial_t s\|^2_{K^0_{1/2}(\bar{\bar{\Lambda}})} &\\ < \infty &\end{aligned}$$

By definition of $\bar{\bar{V}}$, it implies that $\partial_t s \in \bar{\bar{V}}$ and completes the induction.

Step 3: Let $K^0_{1/2}(\bar{\bar{\Pi}}) := K^0_{1/2}(\mathbb{R}, L^2(-1, \Theta))$. For now we have proven that, for any $(n, i) \in (\mathbb{N} \times \{0, 1\}) \setminus (\{(0, 0)\})$ and $R_c < r_1 < r_2$, $\partial_t^n \partial_\theta^i \overline{\overline{\chi S^V}} \in K^0_{1/2}(\bar{\bar{\Pi}})$. It generalizes to the case $(n, i) = (0, 0)$ thanks to a Poincaré inequality: $\|\overline{\overline{\chi S^V}}\|_{K^0_{1/2}(\bar{\bar{\Pi}})} \lesssim \|\partial_\theta \overline{\overline{\chi S^V}}\|_{K^0_{1/2}(\bar{\bar{\Pi}})} \lesssim \|\partial_\theta \overline{\overline{\chi S^V}}\|_{L^2(\bar{\bar{\Omega}})} + \|\partial_\theta \overline{\overline{\chi S^V}}\|_{K^0_{-1/2}(\bar{\bar{\Lambda}})} < \infty$.
To treat higher-order θ-derivatives, we start from the equality $\mu\Delta(\chi S^V) = \chi f_1 - 2\mu \nabla\chi \cdot \nabla S^V - \mu \Delta\chi \cdot S^V$ in $\Omega \cup \Lambda$ proven in to step 2. Applying the change of variables $(x, y) \rightsquigarrow (t, \theta)$ gives by (B.2):

$$\begin{cases} -e^{-2t}\mu_0 \Delta \overline{\overline{\chi S^V}} = \overline{\overline{\chi f_1}} - 2\mu_0 \overline{\overline{\nabla\chi \cdot \nabla S^V}} - \mu_0 \overline{\overline{\Delta\chi \cdot S^V}} & \text{in } \bar{\bar{\Omega}} \\ -\mu_1 \left(e^{-2t}(\partial_t^2 - \partial_t) + \partial_\theta^2\right)(\overline{\overline{\chi S^V}}) = \overline{\overline{\chi f_1}} - 2\mu_1 \overline{\overline{\nabla\chi \cdot \nabla S^V}} - \mu_1 \overline{\overline{\Delta\chi \cdot S^V}} & \text{in } \bar{\bar{\Lambda}} \end{cases} \tag{B.11}$$

Moreover, (B.9) implies that, for any $m \in \mathbb{N}$, $\overline{\overline{\chi f_1}}_{|\bar{\bar{\Omega}}} \in K^\infty_{-3/2}(\mathbb{R}, H^m(0, \Theta))$ and $\overline{\overline{\chi f_1}}_{|\bar{\bar{\Lambda}}} \in K^\infty_{1/2}(\mathbb{R}, H^m(-1, 0))$. So deriving (B.11) w.r.t. t and θ enough times gives by induction: $\forall (n, i) \in \mathbb{N}^2, \forall R_c < r_1 < r_2, \forall D \in \{\bar{\bar{\Omega}}, \bar{\bar{\Lambda}}\}$, $\partial_t^n \partial_\theta^i \overline{\overline{\chi S^V}}_{|D} \in K^0_{1/2}(D)$. Finally, applying this to $\chi := \chi_\infty$ completes the proof. □

Proof of Theorem 2.23: For any $q \in \frac{\pi}{\Theta}\mathbb{N}^*$, let $\sigma_q(S^\infty) := \sigma_q(S)$. We denote, for any $d \in \mathbb{R}$, $T_{>d} := \mathrm{id} - T_{\leqslant d}$ and

$$S^\infty_d := \left\langle -R_{\partial^2_Y} \circ \partial^2_{x|\Lambda},\ \frac{\mu_0}{\mu_1} R_N \circ \partial_{y|\Gamma, y=0^+} \right\rangle \left(\frac{1}{\mu_0} R_\Delta(F^\infty_\Omega) + \frac{1}{\mu_1} R_{\partial^2_Y}(F^\infty_\Lambda) + \sum_{q \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap (d, \infty)} \sigma_q(S^\infty)\, \phi_q \right)$$

and, for any $n \in \mathbb{N}$, $d_n := \frac{1}{2} - n$ and $S_{r,n} := \chi_\infty S - \chi_\infty T_{>d_n}(S^\infty_{d_n})$. Let $m \in \mathbb{N}$. We will show by induction on n that there are coefficients $(\sigma_q(S^\infty))_{q \in -\frac{\pi}{\Theta}\mathbb{N}^*}$ s.t.:

$$\forall n \in \mathbb{N},\ \forall \beta > d_n, \qquad \overline{\overline{S_{r,n}}} \in K^\infty_\beta(\mathbb{R}, \mathcal{H}^m).$$

For now we only know $\sigma_q(S^\infty)$ for $q \geqslant \frac{\pi}{\Theta} > d_0$, and $S_{r,n}$ involves $\sigma_q(S^\infty)$ only when $q > d_n$, so the n-th inductive step involves fixing $\sigma_q(S^\infty)$ for all $q \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap (d_n, d_{n-1}]$.

Initial case: By Proposition B.4, there is $d \in \mathbb{R}_-$ and S^V s.t. $S = S^V + \chi_\infty T_{\geqslant d}(S^\infty_{d_0})$ and $\overline{\overline{\chi_\infty S^V}} \in K^\infty_{1/2}(\mathbb{R}, \mathcal{H}^m)$. Hence

$$\begin{aligned} S_{r,0} &= \chi_\infty S - \chi_\infty T_{>d_0}(S^\infty_{d_0}) \\ &= \chi_\infty \left(S^V + \chi_\infty T_{\geqslant d}(S^\infty_{d_0})\right) - \chi_\infty T_{>d_0}(S^\infty_{d_0}) \\ &= \chi_\infty S^V + \chi_\infty \cdot (\chi_\infty - 1) \cdot T_{\geqslant d}(S^\infty_{d_0}) + \chi_\infty \cdot \left(T_{\geqslant d}(S^\infty_{d_0}) - T_{>0}(S^\infty_{d_0})\right) \end{aligned}$$

So $\overline{\overline{S_{r,0}}} \in K^\infty_{1/2}(\mathbb{R}, \mathcal{H}^m)$. But $\overline{\overline{S_{r,0}}}$ is null in a vicinity of $-\infty$, so: $\forall \beta > d_0 = \frac{1}{2}$, $\overline{\overline{S_{r,0}}} \in K^\infty_\beta(\mathbb{R}, \mathcal{H}^m)$.

Inductive step: We assume the property at rank n and will show it at rank $n+1$. Let $\tilde{S}_{\mathrm{r},n+1} := \chi_\infty S - \chi_\infty T_{>d_{n+1}}(S^\infty_{d_n})$ and $\mathfrak{D} := \frac{\pi}{\Theta}\mathbb{Z}^* \cap (d_{n+1}, d_n]$. $\tilde{S}_{\mathrm{r},n+1}$ is a variant of $S_{\mathrm{r},n+1}$ that does not involve $\sigma_q(S^\infty)$ for $q \in \mathfrak{D}$. We will apply Proposition B.3 to $d := d_n$ and $s := \overline{\overline{\tilde{S}_{\mathrm{r},n+1}}}$. To do so, we must check that:

$$\begin{cases} \mu_0 \Delta \overline{\overline{\tilde{S}_{\mathrm{r},n+1|\bar{\Omega}}}} \in \bigcap_{\beta > d_{n+1}} K^\infty_\beta(\mathbb{R}, H^{m-2}(0,\Theta)) \\ \mu_1 \left(e^{-2t}(\partial_t^2 - \partial_t) + \partial_\theta^2\right) \overline{\overline{\tilde{S}_{\mathrm{r},n+1|\bar{\Lambda}}}} \in \bigcap_{\beta > d_{n+1}} K^\infty_\beta(\mathbb{R}, H^{m-2}(-1,0)) \\ (e^{-t}\mu_0 \partial_{\theta|\theta=0^+} - \mu_1 \partial_{\theta|\theta=0^-}) \overline{\overline{\tilde{S}_{\mathrm{r},n+1}}} \in \bigcap_{\beta > d_{n+1}} K^\infty_\beta(\bar{\Gamma}) \end{cases}$$

Let us show only the first line, the others being similar.

> Since $\overline{\overline{\Delta \tilde{S}_{\mathrm{r},n+1}}} = e^{-2t} \Delta \overline{\overline{\tilde{S}_{\mathrm{r},n+1}}}$, it suffices to have that: $\forall \beta > d_{n+1} - 2$, $\overline{\overline{\Delta \tilde{S}_{\mathrm{r},n+1}}} \in K^\infty_\beta(\mathbb{R}, H^{m-2}(0,\Theta))$. And this is true because we have in Ω in a vicinity of $r \to \infty$:
>
> $$\begin{aligned} \mu_0 \Delta \tilde{S}_{\mathrm{r},n+1} &= F - \mu_0 \Delta \left[T_{>d_{n+1}}(S^\infty_{d_n})\right] && \text{by definition of } \tilde{S}_{\mathrm{r},n+1}, \text{ and } \mu_0 \Delta S = F \text{ in } \Omega \\ &= T_{\geqslant d_{n+1}-2}(F^\infty_\Omega) + o_\partial(r^{d_{n+1}-2}) && \text{by hypothesis on } F \\ &\quad - \mu_0 T_{>d_{n+1}-2}(\Delta S^\infty_{d_n}) && \text{by } \deg \Delta = -2 \text{ (Lemma 2.16)} \\ &= (T_{\leqslant d_{n+1}-2} - T_{<d_{n+1}-2})(F^\infty_\Omega) + o_\partial(r^{d_{n+1}-2}) && \text{by } \mu_0 \Delta S^\infty_{d_n} = F^\infty_\Omega \text{ (Lemma 2.19)} \end{aligned}$$

Therefore, Proposition B.3 states that there are coefficients $(c_q)_{q\in\mathfrak{D}}$, s_{r} and $d_{\min} := \min(\mathfrak{D})$ s.t.: $\forall \beta \in (d_{n+1}, d_{\min})$, $s_{\mathrm{r}} \in K^\infty_\beta(\mathbb{R}, \mathcal{H}^m)$ and $\overline{\overline{\tilde{S}_{\mathrm{r},n+1}}} = \sum_{q\in\mathfrak{D}} c_q \overline{\phi_q} + s_{\mathrm{r}}$. Let $\sigma_q(S^\infty) := -c_q$ for any $q \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap (d_{n+1}, d_n]$. Then:

$$\overline{\overline{S_{\mathrm{r},n+1}}} = \overline{\overline{\tilde{S}_{\mathrm{r},n+1}}} + \sum_{q\in\mathfrak{D}} \sigma_q(S^\infty) \overline{\overline{\chi_\infty \phi_q}} = s_{\mathrm{r}} + \sum_{q\in\mathfrak{D}} c_q \overline{\overline{(1-\chi_\infty)\phi_q}} \in \bigcap_{\beta \in (d_{n+1}, d_{\min})} K^\infty_\beta(\mathbb{R}, \mathcal{H}^m)$$

Moreover, $\overline{\overline{S_{\mathrm{r},n+1}}}$ is null in a vicinity of $-\infty$, so: $\forall \beta > d_{n+1}$, $\overline{\overline{S_{\mathrm{r},n+1}}} \in K^\infty_\beta(\mathbb{R}, \mathcal{H}^m)$. This concludes the induction.

To complete the proof of the theorem, we must show that: $\forall d \in \mathbb{R}$, $S = T_{\geqslant d}(S^\infty) + o_\partial(\mathbf{r}^d)$ when $\mathbf{r} \to \infty$. We will do it in Ω, but it works the same in Λ. Let $d \in \mathbb{R}$ and $n \in \mathbb{N}$ s.t. $d_n < d$. We have in Ω:

$$s := \chi_\infty S - \chi_\infty T_{\geqslant d}(S^\infty) = S_{\mathrm{r},n} + \chi_\infty \cdot T_{<d} \circ T_{>d_n}(S^\infty) \quad \text{with} \quad T_{<d} \circ T_{>d_n}(S^\infty) \in \sum_{d'<d} \mathcal{A}_{d'}(\Omega).$$

So there is $d' < d$ s.t.: $\forall m \in \mathbb{N}$, $\bar{s} \in K^\infty_{d'}(\mathbb{R}, H^m(0,\Theta))$. Thus for any $(i,j) \in \mathbb{N}^2$:

$$\overline{\overline{r^{-d'}(r\partial_r)^i \partial_\theta^j s}} = e^{-d't} \partial_t^i \partial_\theta^j \bar{s} \in H^2(\mathbb{R}, H^2(0,\Theta)) \subset L^\infty(\bar{\Omega}).$$

So $r^{-d'+i}\partial_r^i \partial_\theta^j s$ is also bounded. By definition of o_∂ (given in 2.20), this concludes the proof. □

B.2 Proof of Theorem 2.22: asymptotic behavior for far-and-layer fields-like problems

This proof is very similar to Section B.1. The main difference is that we look at the asymptotic expansion when $\mathbf{r} \to 0$ (i.e. $t \to -\infty$) instead of $\mathbf{r} \to \infty$ (i.e. $t \to +\infty$). So this time we use a truncation function $\chi_f \in \mathcal{C}^\infty(\Pi)$ that is equal to 1 in a vicinity of $\mathbf{r} = 0$ and to 0 in a vicinity of infinity. In addition, we can assume that $\chi_f f = 0$. Moreover the Laplace transform of $\overline{\chi_f \mathbf{u}}$ is first defined in a left half-plane of the complex plan (instead of a right one), and then extended to the right.

Applying the change of variables $(x,y) \rightsquigarrow (t,\theta)$ on the equations satisfied by $\chi_f \mathbf{u}$ (that one can easily deduce from (2.23)) and then the Laplace transform yields that $s := \overline{\overline{\chi_f \mathbf{u}}}$ satisfies:

$$\begin{cases} \mu_0(\partial_\theta^2 + \lambda^2)\hat{s}(\lambda,\theta) = \tilde{f} - \omega^2 \rho_0 \hat{s}(\lambda - 2, \theta) & \text{if } \theta \in (0,\Theta) \\ \partial_\theta^2 \hat{s}(\lambda,\theta) = \tilde{g} & \text{if } \theta \in (-1,0) \\ \hat{s}(\lambda, 0^+) - \hat{s}(\lambda, 0^-) = 0 \\ \partial_\theta \hat{s}(\lambda, 0^-) = \tilde{h} \\ \hat{s}(\lambda,\Theta) = \hat{s}(\lambda,-1) = 0 \end{cases} \tag{B.12}$$

for some functions $\tilde f, \tilde g, \tilde h$ depending on g, h, χ_f. This system has the form of (B.4), so Lemma B.2 gives the tools to solve it and to extend $\hat s$ to the right (by steps of 2 here). This is stated in Proposition B.5, whose proof is very similar to Proposition B.3.

Proposition B.5: Let $d \in \mathbb{R}$, $m \in \mathbb{N} \setminus \{0,1\}$, $s \in \bigcap_{\beta<d} K^\infty_\beta(\mathbb{R}, \mathcal{H}^m)$, $\tilde f \in \bigcap_{\beta<d+2} K^\infty_\beta(\mathbb{R}, H^{m-2}(0,\Theta))$, $\tilde g \in \bigcap_{\beta<d+2} K^\infty_\beta(\mathbb{R}, H^{m-2}(-1,0))$, $\tilde h \in \bigcap_{\beta<d+2} K^\infty_\beta(\bar\Gamma)$ and $d_{\max} = \max(\frac{\pi}{\Theta}\mathbb{Z}^* \cap [d, d+2))$. We assume:

$$\begin{cases} \mu_0 \Delta s + e^{2t}\omega^2\rho_0 s = \tilde f & \text{in } \bar{\bar\Omega} \\ \partial_\theta^2 s = \tilde g & \text{in } \bar{\bar\Lambda} \\ s_{|\theta=0^+} - s_{|\theta=0^-} = 0 & \text{on } \bar{\bar\Gamma} \\ \mu_1 \partial_\theta s_{|\theta=0^-} = \tilde h & \text{on } \bar{\bar\Gamma} \\ s = 0 & \text{on } \overline{\overline{\Sigma_\Omega}} \cup \overline{\overline{\Sigma_\Lambda}} \end{cases}$$

Then there is $s_{\mathrm{r}} \in \bigcap_{d_{\max}<\beta<d+2} K^\infty_\beta(\mathbb{R}, \mathcal{H}^m)$ and $(c_q) \in \mathbb{C}^{\frac{\pi}{\Theta}\mathbb{Z}^* \cap [d,d+2)}$ s.t. $s = \sum_{q \in \frac{\pi}{\Theta}\mathbb{Z}^* \cap [d,d+2)} c_q \, \overline{\phi_q} + s_{\mathrm{r}}$.

Assumption B.6: Replacing $\mathbf{u}$ by $\chi_\infty \mathbf{u}$, we assume without loss of generality that $\mathbf{u}$ has a compact support.

For any $q \in -\frac{\pi}{\Theta}\mathbb{N}^*$, let $\sigma_q(\mathbf{u}) := \sigma_q(\mathbf{u}_{|\Omega})$, where $\sigma_q(\mathbf{u}_{|\Omega})$ is set in Definition 3.5. This quantity vanishes when q is small enough. Proposition B.7 is the analogue of Proposition B.4.

Proposition B.7: Let $d \in \mathbb{R}_+$,

$$\mathbf{u}^{\mathcal{A}} := T_{\leqslant d}\left[\left\langle -k_0^2\, R_\Delta \right\rangle \left(\frac{1}{\mu_1} R_{\partial_Y^2}(g^0) + \frac{1}{\mu_1} R_{\mathrm{N}}(h^0) + \sum_{q \in -\frac{\pi}{\Theta}\mathbb{N}^*} \sigma_q(\mathbf{u})\, \phi_q \right)\right]$$

in $\mathcal{A}(\Pi)$ and $\mathbf{u}^{\mathrm{v}} := \mathbf{u} - \chi_f \mathbf{u}^{\mathcal{A}}$. There is $d \in \mathbb{R}_+$ s.t. $\mathbf{u}^{\mathrm{v}}_{|\Omega} \in H^1(\Omega)$ and: $\forall m \in \mathbb{N}$, $\overline{\overline{\chi_f \mathbf{u}^{\mathrm{v}}}} \in K^\infty_{-1}(\mathbb{R}, \mathcal{H}^m)$.

Proof: We consider any $d \in \mathbb{R}_+$ and we will fix it later. The proof has five steps: writing the equations satisfied by $\mathbf{u}^{\mathrm{v}}$, and then showing that $\mathbf{u}^{\mathrm{v}}$ is regular in the layer, that it is H^1 in Ω, that it is regular in Ω w.r.t t, and that it is regular in Ω w.r.t. θ. For any $r_1, r_2 \in \mathbb{R}$ s.t. $r_1 < r_2 < r_f$, we denote $\chi \in \mathcal{C}^\infty(\mathbb{R}^2)$ a radial function equal to 1 on $B(0,r_1)$ and 0 on $\mathbb{R}^2 \setminus B(0,r_2)$ (r_1, r_2 are implicit in this notation).

Step 1: Similarly to the proofs of theorems 3.6 and 3.11 and Proposition B.4, one can show that:

$$\begin{cases} (\mu_0\Delta + \omega^2\rho_0)\mathbf{u}^{\mathrm{v}} = f_1 := f - (\mu_0\Delta + \omega^2\rho_0)(\chi_f \mathbf{u}^{\mathcal{A}}) & \text{in } \Omega \\ \mu_1 \partial_Y^2 \mathbf{u}^{\mathrm{v}} = g_1 := g - \mu_1 \partial_Y^2 (\chi_f \mathbf{u}^{\mathcal{A}}) & \text{in } \Lambda \\ \mathbf{u}^{\mathrm{v}}_{|y=0^+} - \mathbf{u}^{\mathrm{v}}_{|y=0^-} = 0 & \text{on } \Gamma \\ \mu_1 \partial_Y \mathbf{u}^{\mathrm{v}}_{|Y=0^-} = h_1 := h - \mu_1 \partial_Y (\chi_f \mathbf{u}^{\mathcal{A}}) & \text{on } \Gamma \\ \mathbf{u}^{\mathrm{v}} = 0 & \text{on } \Sigma_\Omega \cup \Sigma_\Lambda \end{cases} \quad \text{with} \quad \begin{cases} (\mu_0\Delta + \omega^2\rho_0)\mathbf{u}^{\mathcal{A}} \in \sum_{d'>d} \mathcal{A}_{d'}(\Omega) \\ \partial_Y^2 \mathbf{u}^{\mathcal{A}}_{|\Lambda} - T_{\leqslant d}(g^0) \in \sum_{d'>d} \mathcal{A}_{d'}(\Lambda) \\ \partial_Y \mathbf{u}^{\mathcal{A}}_{|Y=0^-} - T_{\leqslant d}(h^0) \in \sum_{d'>d} \mathcal{A}_{d'}(\Gamma) \end{cases}$$

Moreover, given that $\chi f = 0$ for any $r_1 < r_2 < r_f$, one can check that for d big enough and up to decreasing r_f (without loss of generality), we have $\overline{\overline{\chi f_1}} \in K^\infty_0(\mathbb{R}, H^m(0,\Theta))$, $\overline{\overline{\chi g_1}} \in K^\infty_0(\mathbb{R}, H^m(-1,0))$ and $\overline{\chi h_1} \in K^\infty_0(\bar\Gamma)$ for any $m \in \mathbb{N}$.

Step 2: Let us show that: $\forall r_1 < r_2 < r_f, \forall m \in \mathbb{N}$, $\overline{\overline{\chi \mathbf{u}^{\mathrm{v}}_{|\Lambda}}} \in K^\infty_0(\mathbb{R}, H^m(-1,0))$. By assumption, for any $x > 0$, $\mathbf{u}^{\mathrm{v}}$ in H^1 in a vicinity of $\{x\} \times [-1,0] \subset \Lambda$, so $\mathbf{u}^{\mathrm{v}}_{|\{x\}\times[-1,0]}$ is well-defined in $L^2(\{x\} \times [-1,0])$. Moreover, g_1 and h_1 are $o_\partial(x^d)$, so they are differentiable on $(0, r_f) \times [-1,0]$ (up to decreasing r_f). Hence, $Y \in (-1,0) \mapsto \mathbf{u}^{\mathrm{v}}(x,Y)$ is $\mathcal{C}^2$ and the ODE it satisfies (see step 1) is explicitly solvable:

$$\forall (x,Y) \in (0,r_f) \times [-1,0], \qquad \mathbf{u}^{\mathrm{v}}(x,Y) = \int_{-1}^{0} \left((Y-Y')^+ - Y - 1 \right) g_1(x,Y')\, \mathrm{d}Y' + h_1(x) \cdot (Y+1).$$

Since $\overline{\overline{\chi g_1}} \in K_0^\infty(\mathbb{R}, H^m(-1,0))$ for any m and $\overline{\overline{\chi h_1}} \in K_0^\infty(\bar{\bar{\Gamma}})$, we deduce $\overline{\overline{\chi \mathbf{u}^{\mathrm{v}}_{|\Lambda}}} \in K_0^\infty(\mathbb{R}, H^m(-1,0))$.

Step 3: Let us show that $\mathbf{u}^{\mathrm{v}}_{|\Omega} \in H^1(\Omega)$. First, step 2 implies $\overline{\overline{\chi \mathbf{u}^{\mathrm{v}}_{|\Gamma}}} \in K_0^\infty(\bar{\bar{\Gamma}})$ so $\chi \mathbf{u}^{\mathrm{v}}_{|\Gamma} \in H^{1/2}(\Gamma)$. Since $\mathbf{u} \in H^1_{\mathrm{loc}}$ and $\mathbf{u}$ has a compact support by assumption B.6, we get $\mathbf{u}^{\mathrm{v}}_{|\Gamma} \in H^{1/2}(\Gamma)$. So the following system has a solution in $H^1(\Omega)$ by Lemma 3.3:

$$\begin{cases} \mu_0 \Delta u^{\mathrm{v}} + \omega^2 \rho_0 u^{\mathrm{v}} = f_1 & \text{in } \Omega \\ u^{\mathrm{v}}_{|y=0^+} = \mathbf{u}^{\mathrm{v}}_{|y=0^-} & \text{on } \Gamma \\ u^{\mathrm{v}} = 0 & \text{on } \Sigma_\Omega \end{cases}$$

But $\mathbf{u}^{\mathrm{v}}$ also satisfies these equations, so $u^{\mathrm{v}} + \chi_f \mathbf{u}^{\mathcal{A}}_{|\Omega}$ satisfies the same problem as $\mathbf{u}_{|\Omega}$, and they both belong to $H^1(\Omega) + \chi_0 \mathcal{A}(\Omega)$. Then by uniqueness in Theorem 3.6, $u^{\mathrm{v}} + \chi_f \mathbf{u}^{\mathcal{A}}_{|\Omega} = \mathbf{u}_{|\Omega}$, i.e. $u^{\mathrm{v}} = \mathbf{u}^{\mathrm{v}}_{|\Omega}$. Thus, $\mathbf{u}^{\mathrm{v}}_{|\Omega} \in H^1(\Omega)$.

Step 4: Let $v : \Omega \to \mathbb{C}$ defined by $v := \mathbf{u}^{\mathrm{v}}_{|\Omega} - v_{\mathrm{lift}}$ with $v_{\mathrm{lift}}(r,\theta) := \big(1 - \frac{\theta}{\Theta}\big)\mathbf{u}^{\mathrm{v}}_{|\Gamma}(r)$ in polar coordinates. Let $K^0_\beta(\bar{\Omega}) := K^0_\beta(\mathbb{R}, L^2(0,\Theta))$ for any $\beta \in \mathbb{R}$. We will show by induction on n that:

$$\forall n \in \mathbb{N}^*,\ \forall r_1 < r_2 < r_f, \qquad \partial_t^n \overline{\overline{\chi} v} \in K^0_{-1}(\bar{\bar{\Omega}}) \quad \text{and} \quad \nabla \partial_t^n \overline{\overline{\chi} v} \in L^2(\bar{\bar{\Omega}}).$$

For the initial case, we have on the one hand $\chi \mathbf{u}^{\mathrm{v}}_{|\Omega} \in H^1(\Omega)$ so $\overline{\overline{\chi \mathbf{u}^{\mathrm{v}}_{|\Omega}}} \in K^0_{-1}(\bar{\bar{\Omega}})$ and $\nabla \overline{\overline{\chi \mathbf{u}^{\mathrm{v}}_{|\Omega}}} \in L^2(\bar{\bar{\Omega}})$, and on the other $\overline{\overline{\chi \mathbf{u}^{\mathrm{v}}_{|\Gamma}}} \in K^1_0(\bar{\bar{\Gamma}})$ so $\overline{\overline{\chi v_{\mathrm{lift}}}} \in K^0_{-1}(\bar{\bar{\Omega}})$ and $\nabla \overline{\overline{\chi v_{\mathrm{lift}}}} \in L^2(\bar{\bar{\Omega}})$. Thus the initial case is proven and only the inductive step remains to prove. We assume the property at rank n and we will show it at rank $n+1$, using the method of finite differences. Step 1 implies that:

$$\begin{cases} (\mu_0 \Delta + \omega^2 \rho_0)(\chi v) = f_2 := \chi f_1 - \chi\, \mu_0 \Delta v_{\mathrm{lift}} - 2\mu_0 \nabla \chi \cdot \nabla \mathbf{u}^{\mathrm{v}} - \mu_0 \Delta \chi \cdot \mathbf{u}^{\mathrm{v}} & \text{in } \Omega \\ \chi v = 0 & \text{on } \Sigma_\Omega \cup \Gamma \end{cases}$$

In addition, we have $\overline{\overline{f_2}} \in K^n_{-2}(\mathbb{R}, L^2(0,\Theta))$ since:

- by step 1, $\overline{\overline{\chi f_1}} \in K^\infty_0(\mathbb{R}, L^2(0,\Theta))$, so $\overline{\overline{\chi f_1}} \in K^\infty_{-2}(\mathbb{R}, L^2(0,\Theta))$,
- by step 2, $\overline{\overline{\chi \mathbf{u}^{\mathrm{v}}_{|\Gamma}}} \in K^\infty_0(\bar{\bar{\Gamma}})$, so $\overline{\overline{\chi v_{\mathrm{lift}}}} \in K^\infty_0(\mathbb{R}, H^2(0,\Theta))$, and applying it at $(r_1', r_2') := (r_2, \frac{r_2 + r_f}{2})$ gives $\overline{\overline{\chi \Delta v_{\mathrm{lift}}}} \in K^\infty_{-2}(\mathbb{R}, L^2(0,\Theta))$,
- and by induction hypothesis applied to $(r_1', r_2') := (r_2, \frac{r_2+r_f}{2})$, $\partial_t^k \nabla v$ and $\partial_t^k v$ are L^2 on $\{r_1 < r < r_2\}$ for any $k \in [\![0, n]\!]$.

Let $s := \partial_t^n(\overline{\overline{\chi} v})$. By changing variables $(x,y) \rightsquigarrow (t,\theta)$ in the previous system and applying ∂_t^n, we get:

$$\begin{cases} e^{-2t} \mu_0 \Delta s + \omega^2 \rho_0 s = f_3 := e^{-2t} \partial_t^n (e^{2t} \overline{\overline{f_2}}) - \sum_{k=0}^{n-1} \omega^2 \rho_0 2^{n-k} \partial_t^k (\overline{\overline{\chi} v}) & \text{in } \bar{\bar{\Omega}} \\ s = 0 & \text{on } \overline{\overline{\Sigma_\Omega}} \cup \bar{\bar{\Gamma}} \end{cases}$$

with $f_3 \in K^0_{-2}(\bar{\bar{\Omega}})$ by induction hypothesis. And the variational formulation of this is:

$$\forall \varphi \in K^0_{-1}(\bar{\bar{\Omega}}),\ \nabla \varphi \in L^2(\bar{\bar{\Omega}}) \text{ and } \varphi_{|\overline{\overline{\Sigma_\Omega}} \cup \bar{\bar{\Gamma}}} = 0 \ \Rightarrow \quad \int_{\bar{\bar{\Omega}}} (\mu_0 \nabla s \cdot \nabla \varphi + \omega^2 \rho_0 e^{2t} s \varphi) = \int_{\bar{\bar{\Omega}}} e^{2t} f_3 \varphi.$$

Let us denote $D_\eta \varphi(t,\theta) := \frac{\varphi(t+\eta,\theta) - \varphi(t,\theta)}{\eta}$ for any $\eta \in \mathbb{R}^*$ and any function φ. Taking $\varphi := D_{-\eta} D_\eta \bar{s}$ and discretely integrating by parts $D_{-\eta}$ gives:

$$\int_{\bar{\bar{\Omega}}} (\mu_0 |D_\eta \nabla s|^2 - \omega^2 \rho_0 e^{2t} |s|^2) = \int_{\bar{\bar{\Omega}}} e^{2t} f_3 \cdot D_{-\eta} D_\eta \bar{s}$$

Then by coercivity (since $\mathrm{Im}(\omega) \neq 0$), we have for any $\delta > 0$:

$$\|D_\eta \nabla s\|^2_{L^2(\bar{\bar{\Omega}})} + \|D_\eta s\|^2_{K^0_{-1}(\bar{\bar{\Omega}})} \lesssim \|f_3\|_{K^0_{-2}(\bar{\bar{\Omega}})} \|D_{-\eta} D_\eta s\|_{K^0_{-1}(\bar{\bar{\Omega}})} \lesssim \tfrac{1}{\delta} \|f_3\|^2_{K^0_{-2}(\bar{\bar{\Omega}})} + \delta \|D_\eta \partial_t s\|^2_{L^2(\bar{\bar{\Omega}})}$$

Taking δ small enough and moving $\|D_\eta \partial_t s\|^2_{L^2(\bar{\bar{\Omega}})}$ from the right-hand side to the left-hand one, we get:

$$\|\partial_t \nabla s\|^2_{L^2(\bar{\bar{\Omega}})} + \|\partial_t s\|^2_{K^0_{-1}(\bar{\bar{\Omega}})} \leqslant \limsup_{\eta \to 0} \|D_\eta \nabla s\|^2_{L^2(\bar{\bar{\Omega}})} + \|D_\eta s\|^2_{K^0_{-1}(\bar{\bar{\Omega}})} \lesssim \|f_3\|^2_{K^0_{-2}(\bar{\bar{\Omega}})} < \infty.$$

This completes the induction.

Step 5: We have shown that for any $(n,i) \in \mathbb{N} \times \{0,1\}$ and $r_1 < r_2 < r_f$, $\partial_t^n \partial_\theta^i \overline{\overline{\chi} v} \in K^0_{-1}(\bar{\bar{\Omega}})$. Since $\overline{\overline{\chi v_{\text{lift}}}} \in K^\infty_{-1}(\mathbb{R}, H^1(0,\Theta))$, we also have $\partial_t^n \partial_\theta^i \overline{\overline{\chi \mathbf{u}^{\text{v}}_{|\Omega}}} \in K^0_{-1}(\bar{\bar{\Omega}})$ for these (n,i).
To treat higher-order θ-derivatives, we begin with the equality $(\mu_0\Delta + \omega^2\rho_0)(\chi \mathbf{u}^{\text{v}}) = \chi f_1 - 2\mu_0 \nabla\chi \cdot \nabla \mathbf{u}^{\text{v}} - \mu_0 \Delta\chi \cdot \mathbf{u}^{\text{v}}$ in Ω, which follows from step 1. Applying the change of variables $(x,y) \rightsquigarrow (t,\theta)$, we get:

$$(e^{-2t}\mu_0\Delta + \omega^2\rho_0)(\chi \mathbf{u}^{\text{v}}) = \overline{\overline{\chi f_1}} - 2\mu_0 \overline{\overline{\nabla\chi \cdot \nabla \mathbf{u}^{\text{v}}}} - \mu_0 \overline{\overline{\Delta\chi \cdot \mathbf{u}^{\text{v}}}}. \tag{B.13}$$

Now $\overline{\overline{\chi f_1}} \in K^\infty_0(\mathbb{R}, H^m(0,\Theta))$ for any $m \in \mathbb{N}$ by step 1. So deriving (B.13) w.r.t. t and θ enough times gives by induction: $\forall (n,i) \in \mathbb{N}^2, \forall r_1 < r_2 < r_f$, $\partial_t^n \partial_\theta^i \overline{\overline{\chi \mathbf{u}^{\text{v}}_{|\Omega}}} \in K^0_{-1}(\bar{\bar{\Omega}})$. Combining it with step 2, we get $\overline{\overline{\chi \mathbf{u}^{\text{v}}}} \in K^\infty_{-1}(\mathbb{R}, \mathcal{H}^m)$. Finally, we apply it to $\chi := \chi_\infty$ to conclude. □

Finally, Propositions B.5 and B.7 give all the ingredients to prove Theorem 2.22. The proof is very similar to the one of Theorem 2.23 on page 44, so we do not go into details again.

References

[1] Yves Achdou, O. Pironneau, and F. Valentin. "Effective boundary conditions for laminar flows over periodic rough boundaries". In: *J. Comput. Phys.* 147.1 (1998), pp. 187–218. ISSN: 0021-9991. DOI: 10.1006/jcph.1998.6088. URL: https://doi.org/10.1006/jcph.1998.6088.

[2] H. Ammari and S. He. "Effective impedance boundary conditions for an inhomogeneous thin layer on a curved metallic surface". In: *IEEE Transactions on Antennas and Propagation* 46.5 (1998), pp. 710–715. DOI: 10.1109/8.668915.

[3] H. Ammari and S. He. "Gerneralized Effective Impedance Boundary Conditions for an Imhomogeneous Thin Layer in Electromagnetic Scattering". In: *Journal of Electromagnetic Waves and Applications* 11.9 (1997), pp. 1197–1212. DOI: 10.1163/156939397X01106. eprint: https://doi.org/10.1163/156939397X01106. URL: https://doi.org/10.1163/156939397X01106.

[4] H. Ammari and C. Latiri-Grouz. "Approximate boundary conditions for thin periodic coatings". In: *Mathematical and numerical aspects of wave propagation (Golden, CO, 1998)*. SIAM, Philadelphia, PA, 1998, pp. 297–301.

[5] Habib Ammari, Laurence Halpern, and Kamel Hamdache. "Asymptotic behaviour of thin ferromagnetic films". In: *Asymptotic Analysis* 24 (Dec. 2000).

[6] Alexis Auvray and Grégory Vial. "Asymptotic expansions and effective boundary conditions: a short review for smooth and nonsmooth geometries with thin layers". In: *ESAIM: ProcS* 61 (2018), pp. 38–54. DOI: 10.1051/proc/201861038. URL: https://doi.org/10.1051/proc/201861038.

[7] Alexis Auvray and Grégory Vial. "Improved impedance conditions for a thin layer problem in a nonsmooth domain". In: *Mathematical Methods in the Applied Sciences* 42 (Mar. 2019). DOI: 10.1002/mma.5434.

[8] Michel Bellieud, Giuseppe Geymonat, and Françoise Krasucki. "Asymptotic analysis of a linear isotropic elastic composite reinforced by a thin layer of periodically distributed isotropic parallel stiff fibres". In: *J. Elasticity* 122.1 (2016), pp. 43–74. ISSN: 0374-3535. DOI: 10.1007/s10659-015-9532-7. URL: https://doi.org/10.1007/s10659-015-9532-7.

[9] A. Bendali and K. Lemrabet. "The Effect of a Thin Coating on the Scattering of a Time-Harmonic Wave for the Helmholtz Equation". In: *SIAM Journal on Applied Mathematics* 56.6 (1996), pp. 1664–1693. DOI: 10.1137/S0036139995281822. eprint: https://doi.org/10.1137/S0036139995281822. URL: https://doi.org/10.1137/S0036139995281822.

[10] Amandine Boucart. "Diffraction électromagnétique par une couche mince de nanoparticules réparties aléatoirement : développement asymptotique, conditions effectives et simulations." PhD thesis. Institut Polytechnique de Paris, Apr. 2023. URL: https://theses.hal.science/tel-04400789.

[11] Gabriel Caloz et al. "Asymptotic expansion of the solution of an interface problem in a polygonal domain with thin layer". In: *Asymptotic Analysis* 50.1-2 (2006), pp. 121–173.

[12] Fabien Caubet et al. "New transmission condition accounting for diffusion anisotropy in thin layers applied to diffusion MRI". en. In: *ESAIM: Mathematical Modelling and Numerical Analysis* 51.4 (2017), pp. 1279–1301. DOI: 10.1051/m2an/2016060. URL: http://www.numdam.org/articles/10.1051/m2an/2016060/.

[13] Lucas Chesnel. “Étude de quelques problèmes de transmission avec changement de signe. Application aux métamatériaux”. 2012EPXX0044. PhD thesis. 2012, 1 vol. (316 p.) URL: http://www.theses.fr/2012EPXX0044/document.

[14] Martin Costabel and Monique Dauge. “A singularly perturbed mixed boundary value problem”. In: *Communications in Partial Differential Equations* 21 (Jan. 1996), pp. 1919–1949. DOI: 10.1080/03605309608821249.

[15] D. De Zutter and L. Knockaert. “Skin effect modeling based on a differential surface admittance operator”. In: *IEEE Transactions on Microwave Theory and Techniques* 53.8 (2005), pp. 2526–2538. DOI: 10.1109/TMTT.2005.852766.

[16] Bérangère Delourme, Houssem Haddar, and Patrick Joly. “Approximate models for wave propagation across thin periodic interfaces”. In: *Journal de Mathématiques Pures et Appliquées* 98.1 (2012), pp. 28–71. ISSN: 0021-7824. DOI: https://doi.org/10.1016/j.matpur.2012.01.003. URL: https://www.sciencedirect.com/science/article/pii/S0021782412000128.

[17] Bérangère Delourme, Kersten Schmidt, and Adrien Semin. “On the homogenization of thin perforated walls of finite length”. In: *Asymptotic Analysis* 97 (2016), pp. 211–264.

[18] Wiktor Eckhaus. “Fundamental Concepts of Matching”. In: *SIAM Review* 36.3 (1994), pp. 431–439. DOI: 10.1137/1036099. eprint: https://doi.org/10.1137/1036099. URL: https://doi.org/10.1137/1036099.

[19] Björn Engquist and Jean-Claude Nédélec. “Effective boundary conditions for electromagnetic scattering in thin layers”. Rapport interne 278, CMAP.

[20] Gero Friesecke, Richard D. James, and Stefan Müller. “The Föppl-von Kármán plate theory as a low energy Γ-limit of nonlinear elasticity”. In: *C. R. Math. Acad. Sci. Paris* 335.2 (2002), pp. 201–206. ISSN: 1631-073X. DOI: 10.1016/S1631-073X(02)02388-9. URL: https://doi.org/10.1016/S1631-073X(02)02388-9.

[21] H. Haddar and P. Joly. “Effective boundary conditions for thin ferromagnetic coatings. Asymptotic analysis of the 1D model”. In: *Asymptot. Anal.* 27.2 (2001), pp. 127–160. ISSN: 0921-7134.

[22] Houssem Haddar and Zixian Jiang. “Axisymmetric eddy current inspection of highly conducting thin layers via asymptotic models”. In: *Inverse Problems* 31.11 (2015), pp. 115005, 25. ISSN: 0266-5611. DOI: 10.1088/0266-5611/31/11/115005. URL: https://doi.org/10.1088/0266-5611/31/11/115005.

[23] A. M. Il’in. *Matching of asymptotic expansions of solutions of boundary value problems*. Vol. 102. Translations of Mathematical Monographs. Translated from the Russian by V. Minachin [V. V. Minakhin]. American Mathematical Society, Providence, RI, 1992, pp. x+281. ISBN: 0-8218-4561-6. DOI: 10.1090/mmono/102. URL: https://doi.org/10.1090/mmono/102.

[24] Willi Jäger and Andro Mikelić. “On the roughness-induced effective boundary conditions for an incompressible viscous flow”. In: *J. Differential Equations* 170.1 (2001), pp. 96–122. ISSN: 0022-0396. DOI: 10.1006/jdeq.2000.3814. URL: https://doi.org/10.1006/jdeq.2000.3814.

[25] V. A. Kozlov, V. G. Maz’ya, and J. Rossmann. *Elliptic boundary value problems in domains with point singularities*. Vol. 52. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997, pp. x+414. ISBN: 0-8218-0754-4. DOI: 10.1090/surv/052. URL: https://doi.org/10.1090/surv/052.

[26] Abdelkader Makhlouf. “Justification et amélioration de modèles d’antennes patch par la méthode des développements asymptotiques raccordés”. 2008ISAT0041. PhD thesis. 2008, 1 vol. (246 p.) URL: http://www.theses.fr/2008ISAT0041/document.

[27] Vladimir Maz’ya, Serguei Nazarov, and Boris Plamenevskij. *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains. Vol. I*. Vol. 111. Operator Theory: Advances and Applications. Translated from the German by Georg Heinig and Christian Posthoff. Birkhäuser Verlag, Basel, 2000, pp. xxiv+435. ISBN: 3-7643-6397-5.

[28] Serguei Nazarov. “The Vishik–Lyusternik method for elliptic boundary value problems in regions with conic points. I. Problem in a cone”. In: *Siberian Math. J.* 22 (1981), pp. 594–611. DOI: https://doi.org/10.1007/BF00967765.

[29] Serguei Nazarov. “The Vishik–Lyusternik method for elliptic boundary value problems in regions with conic points. II. Problem in a bounded domain”. In: *Siberian Math. J.* 22 (1981), pp. 753–769. DOI: https://doi.org/10.1007/BF00968072.

[30] Ivan Niven. “Formal power series”. In: *Amer. Math. Monthly* 76 (1969), pp. 871–889. ISSN: 0002-9890. DOI: 10.2307/2317940. URL: https://doi.org/10.2307/2317940.

[31] G. L. Richard, C. Ruyer-Quil, and J. P. Vila. “A three-equation model for thin films down an inclined plane”. In: *J. Fluid Mech.* 804 (2016), pp. 162–200. ISSN: 0022-1120. DOI: 10.1017/jfm.2016.530. URL: https://doi.org/10.1017/jfm.2016.530.

[32] Benjamin Sambale. “An invitation to formal power series”. In: *Jahresber. Dtsch. Math.-Ver.* 125.1 (2023), pp. 3–69. ISSN: 0012-0456. DOI: 10.1365/s13291-022-00256-6. URL: https://doi.org/10.1365/s13291-022-00256-6.

[33] Adrien Semin, Bérangère Delourme, and Kersten Schmidt. "On the homogenization of the Helmholtz problem with thin perforated walls of finite length". In: *ESAIM: M2AN* 52.1 (2018), pp. 29–67. DOI: 10.1051/m2an/2017030. URL: https://doi.org/10.1051/m2an/2017030.

[34] Thomas B. A. Senior and John Leonidas Volakis. *Approximate boundary conditions in electromagnetics / T. B. A. Senior and J. L. Volakis.* eng. IEE electromagnetic waves series ; 41. London: Institution of Electrical Engineers, 1995. ISBN: 1-84919-169-7.

[35] Seyyed Fakhrabadi, Mir Masoud and Khani, Navid. "Investigation of interphase effects on mechanical behaviors of carbon nanocone-based composites". In: *Mechanics & Industry* 15.4 (2014), pp. 287–292. DOI: 10.1051/meca/2014038. URL: https://doi.org/10.1051/meca/2014038.

[36] Milton Van Dyke. *Perturbation methods in fluid mechanics*. Applied Mathematics and Mechanics, Vol. 8. Academic Press, New York-London, 1964, pp. x+229.

[37] Pallavi Verma, Pascal Maire, and Petr Novák. "A review of the features and analyses of the solid electrolyte interphase in Li-ion batteries". In: *Electrochimica Acta* 55.22 (2010), pp. 6332–6341. ISSN: 0013-4686. DOI: https://doi.org/10.1016/j.electacta.2010.05.072. URL: https://www.sciencedirect.com/science/article/pii/S0013468610007747.

[38] M. I. Višik and L. A. Lyusternik. "Regular degeneration and boundary layer for linear differential equations with small parameter". In: *Uspehi Mat. Nauk (N.S.)* 12.5(77) (1957), pp. 3–122.