Sharp analysis on the joint distribution of the number of descents and inverse descents in a random permutation

Luis Fredes, Bernard Bercu, Michel Bonnefont, Adrien Richou

SHARP ANALYSIS ON THE JOINT DISTRIBUTION OF THE NUMBER OF DESCENTS AND INVERSE DESCENTS IN A RANDOM PERMUTATION

BERNARD BERCU, MICHEL BONNEFONT, LUIS FREDES, AND ADRIEN RICHOU

Abstract. Chatteerjee and Diaconis have recently shown the asymptotic normality for the joint distribution of the number of descents and inverse descents in a random permutation. A noteworthy point of their results is that the asymptotic variance of the normal distribution is diagonal, which means that the number of descents and inverse descents are asymptotically uncorrelated. The goal of this paper is to go further in this analysis by proving a large deviation principle for the joint distribution. We shall show that the rate function of the joint distribution is the sum of the rate functions of the marginal distributions, which also means that the number of descents and inverse descents are asymptotically independent at the large deviation level. However, we are going to prove that they are finely dependent at the sharp large deviation level.

1. Introduction

Let $\mathcal{S}_n$ be the symmetric group of permutations on the set of integers $\{1, \ldots, n\}$ where $n \geq 1$. A permutation $\pi_n \in \mathcal{S}_n$ has a descent at position $k \in \{1, \ldots, n-1\}$ if $\pi_n(k) > \pi_n(k+1)$. Let $D_n = D_n(\pi_n)$ be the random variable counting the number of descents of a permutation π_n chosen uniformly at random from $\mathcal{S}_n$ and denote by $D'_n = D_n(\pi_n^{-1})$ the number of descents of the inverse π_n^{-1} of π_n. We clearly have for all $n \geq 2$,

$$D_n = \sum_{k=1}^{n-1} \mathrm{I}_{\{\pi_n(k) > \pi_n(k+1)\}} \qquad \text{and} \qquad D'_n = \sum_{k=1}^{n-1} \mathrm{I}_{\{\pi_n^{-1}(k) > \pi_n^{-1}(k+1)\}}. \tag{1.1}$$

A wide range of literature is available on the asymptotic behavior of the marginal distribution (D_n). It is well-known [12], [13], [19] that

$$\lim_{n \to \infty} \frac{D_n}{n} = \frac{1}{2} \qquad \text{a.s.} \tag{1.2}$$

Université de Bordeaux, Institut de Mathématiques de Bordeaux, UMR CNRS 5251, 351 Cours de la Libération, 33405 Talence cedex, France.

E-mail addresses: bernard.bercu@math.u-bordeaux.fr, michel.bonnefont@math.u-bordeaux.fr, luis.fredes@math.u-bordeaux.fr, adrien.richou@math.u-bordeaux.fr.

Date: May 21, 2024.

2020 *Mathematics Subject Classification.* 60F10, 05A05.

Key words and phrases. Large deviations, random permutations.

The corresponding author is Michel Bonnefont, michel.bonnefont@math.u-bordeaux.fr.

and

$$\sqrt{n}\left(\frac{D_n}{n}-\frac{1}{2}\right)\xrightarrow[n\to+\infty]{\mathcal{L}}\mathcal{N}\Big(0,\frac{1}{12}\Big). \tag{1.3}$$

Moreover, Bryc, Minda and Sethuraman [4] have shown that (D_n/n) satisfies a large deviation principle (LDP) with good rate function given, for all $x \in \mathbb{R}$, by

$$I(x)=\sup_{t\in\mathbb{R}}\big\{xt-L(t)\big\} \tag{1.4}$$

where

$$L(t)=\log\left(\frac{\exp(t)-1}{t}\right). \tag{1.5}$$

Surprisingly, to the best of our knowledge, very few result are available on the asymptotic behavior of the joint distribution (D_n, D'_n), except the recent contribution of Chatteerjee and Diaconis [7]. By using a normal approximation method due to Chatterjee [6], it is shown in [7] that

$$\sqrt{n}\left(\frac{D_n}{n}-\frac{1}{2},\frac{D'_n}{n}-\frac{1}{2}\right)\xrightarrow[n\to+\infty]{\mathcal{L}}\mathcal{N}\big(0,\Gamma\big), \tag{1.6}$$

where Γ stands for the diagonal matrix

$$\Gamma=\frac{1}{12}\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

It clearly means that D_n and D'_n are asymptotically uncorrelated. One can observe that the almost sure convergence of the couple $(D_n/n, D'_n/n)$ to $(1/2, 1/2)$ can be obtained from Remark 3.2 in [3]. As suggested in Section 3 of [7], it is possible to go deeper into the analysis of the joint distribution (D_n, D'_n) by proving that the couple $(D_n/n, D'_n/n)$ satisfies an LDP with good rate function I defined, for all $x, y \in \mathbb{R}$, by

$$I(x,y)=I(x)+I(y). \tag{1.7}$$

It also implies that, at the large deviation level, D_n and D'_n are asymptotically independent. Moreever, it was recently proven in [3] that the sequence (D_n) satisfies a sharp large deviation principle (SLDP). We are going to improve this result by showing that the couple (D_n, D'_n) satisfies a SLDP which will highlight the very fine dependence between D_n and D'_n at the sharp large deviation level.

The paper is organized as follows. Section 2 deals with an explicit construction of the probability space on which we will be working. Section 3 is devoted to our martingale approach which allows us to find again a direct proof of the asymptotic normality (1.6) and to propose new standard results for the joint distribution (D_n, D'_n) such as a functional central limit theorem. The main results of the paper are given in Section 4 where we establish the large deviation properties of the couple (D_n, D'_n). A short conclusion is given in Section 5. All technical proofs are postponed to Sections 6 to 8.

2. The probability space

Before investigating the joint distribution of (D_n, D'_n), it is necessary to provide an explicit construction of the probability space on which we will be working. For that purpose, we are going to study how a permutation π_{n+1} of size $n+1$ adds descents when it is created from a permutation π_n of size n. We will do this by means of a graphical representation as explained below. We start by presenting a permutation π_n as the set of all points $GR(\pi_n)$ inside $[0,1]^2$ where

$$\pi_n(i) = j \implies \left(\frac{i}{n+1}, \frac{j}{n+1}\right) \in GR(\pi_n).$$

One can observe that the image of this transformation has the property of having exactly one point in each line and column of type $i/(n+1)$ for $i \in \{1, \dots, n\}$ and to send all the points inside $[0,1]^2$. We call $\mathcal{GR}_n$ the set of all points with this property. It is easy to see that the sets $\mathcal{S}_n$ and $\mathcal{GR}_n$ are in bijection since the points in each element of $\mathcal{GR}_n$ defines the map of a bijective function. We define the cell $\mathcal{C}_n(i,j)$ to be the region inside $[0,1]^2$ defined, for all $(i,j) \in \{1, \dots, n+1\}^2$, by

$$\mathcal{C}_n(i,j) = \left]\frac{i-1}{n+1}, \frac{i}{n+1}\right[\times \left]\frac{j-1}{n+1}, \frac{j}{n+1}\right[.$$

For a point $u \in [0,1]^2$, we construct π_{n+1} as a function of u and π_n as described in Figure 1 below,

$$\pi_{n+1}(i) = \begin{cases} \pi_n(i) & \text{if } u \in \mathcal{C}_n(k,\ell),\ i < k \text{ and } \pi_n(i) < \ell \\ \pi_n(i)+1 & \text{if } u \in \mathcal{C}_n(k,\ell),\ i < k \text{ and } \pi_n(i) \geq \ell \\ \pi_n(i-1) & \text{if } u \in \mathcal{C}_n(k,\ell),\ i > k \text{ and } \pi_n(i) < \ell \\ \pi_n(i-1)+1 & \text{if } u \in \mathcal{C}_n(k,\ell),\ i > k \text{ and } \pi_n(i) \geq \ell \\ \ell & \text{if } u \in \mathcal{C}_n(k,\ell),\ i = k \end{cases}$$

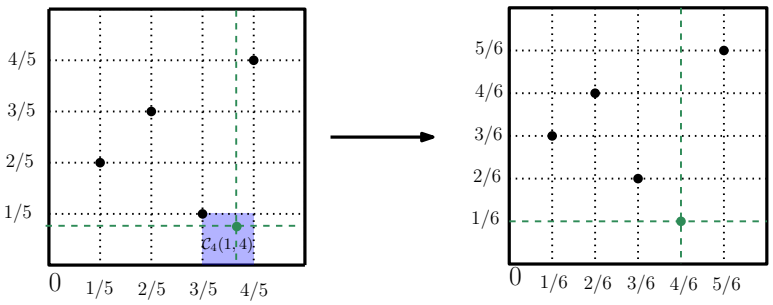


FIGURE 1. Graphical representation of $\pi_4 = (2314)$ to the left. We colored in light blue the cell $\mathcal{C}_4(1,4)$. We represent to the right-side the resulting permutation $\pi_5 = (34215)$ when $u \in \mathcal{C}_4(1,4)$. We present the point u and vertical and horizontal lines crossing it in green in order to visually see the transformation.

It is clear that our construction of π_{n+1} only depends on π_n and on the choice of the indexes (k, ℓ) of the cell where the point u lands. One can think of this transformation as the result of adding, in the graphic representation of π_n, new horizontal and vertical lines crossing the point u and after rescaling these lines in order to keep the relative order of points with spacing $1/(n+2)$, we obtain the graphical representation of π_{n+1}.

Up to now, π_{n+1} is built in a deterministic way. In what follows, what will play the role of u is a random variable U_{n+1} uniformly chosen in $[0,1]^2$ and independent of the permutation π_n. Since the boundary lines of cells have Lebesgue measure 0, we suppose without loss of generality that U_{n+1} does not belong to the boundary of any cell. It is simple to prove that if π_n is chosen uniformly at random from $\mathcal{S}_n$, then π_{n+1} is a uniform permutation over $\mathcal{S}_{n+1}$. From this construction, we can built the sequence (D_n, D'_n) on a unique probability space only depending on the whole sequence (U_n).

First of all, we start at the origin $(D_1, D'_1) = (0, 0)$. Then, for all $n \geq 1$, we denote by ΔD_{n+1} and $\Delta D'_{n+1}$ the increments $\Delta D_{n+1} = D_{n+1} - D_n$ and $\Delta D'_{n+1} = D'_{n+1} - D'_n$. Thanks to our representation, we can study the number of cells $\mathcal{C}_n(k, \ell)$ that generate prescribed increments $\Delta D_{n+1} = a$ and $\Delta D'_{n+1} = b$, $a, b \in \{0, 1\}$ for the permutation π_n and its inverse π_n^{-1} when $u \in \mathcal{C}_n(k, \ell)$. For $a, b \in \{0, 1\}$, let

$$N_{n+1}(a, b, \pi_n) = \mathrm{Card}\big\{(k, \ell) \in \{1, \ldots, n+1\}^2 / \Delta D_{n+1} = a, \Delta D'_{n+1} = b, u \in \mathcal{C}_n(k, \ell)\big\}.$$

Our first result shows that $N_{n+1}(a, b, \pi_n)$ depends only on D_n and D'_n as follows.

Theorem 2.1. *For every permutation π_{n+1} chosen uniformly at random from $\mathcal{S}_{n+1}$, we have*

$$N_{n+1}(a, b, \pi_n) = \begin{cases} (n - D_n)(n - D'_n) + n & \text{if} \quad (a, b) = (1, 1), \\ (n - D_n)(D'_n + 1) - n & \text{if} \quad (a, b) = (1, 0), \\ (D_n + 1)(n - D'_n) - n & \text{if} \quad (a, b) = (0, 1), \\ (D_n + 1)(D'_n + 1) + n & \text{if} \quad (a, b) = (0, 0). \end{cases}$$

Proof. The proof is given in Section 7. □

3. Our martingale approach

Let V_n be the two-dimensional random vector $V_n = (D_n, D'_n)$. Denote by $\mathcal{F}_n$ the σ-algebra $\mathcal{F}_n = \sigma(D_1, \ldots, D_n, D'_1, \ldots, D'_n)$. It follows from Theorem 2.1 that for all $n \geq 1$,

$$V_{n+1} = V_n + \xi_{n+1} \tag{3.1}$$

where

$$\mathbb{P}(\xi_{n+1} = (a,b)|\mathcal{F}_n) = \begin{cases} \dfrac{(n-D_n)(n-D'_n)+n}{(n+1)^2} & \text{if} \quad (a,b) = (1,1), \\ \dfrac{(n-D_n)(D'_n+1)-n}{(n+1)^2} & \text{if} \quad (a,b) = (1,0), \\ \dfrac{(D_n+1)(n-D'_n)-n}{(n+1)^2} & \text{if} \quad (a,b) = (0,1), \\ \dfrac{(D_n+1)(D'_n+1)+n}{(n+1)^2} & \text{if} \quad (a,b) = (0,0). \end{cases}$$

It implies that the conditional distribution of the marginals of ξ_{n+1} given $\mathcal{F}_n$ are the correlated Bernoulli $\mathcal{B}(p_n)$ and $\mathcal{B}(p'_n)$ distributions where

$$p_n = \frac{n-D_n}{n+1} \qquad \text{and} \qquad p'_n = \frac{n-D'_n}{n+1}. \tag{3.2}$$

Moreover, we also have

$$\mathbb{E}[\xi_{n+1}\xi_{n+1}^T|\mathcal{F}_n] = \begin{pmatrix} p_n & p_np'_n + r_n \\ p_np'_n + r_n & p'_n \end{pmatrix}$$

where

$$r_n = \frac{n}{(n+1)^2}.$$

Consequently, we obtain from (3.1) that

$$\mathbb{E}[V_{n+1}|\mathcal{F}_n] = \mathbb{E}[V_n + \xi_{n+1}|\mathcal{F}_n] = \begin{pmatrix} D_n + p_n \\ D'_n + p'_n \end{pmatrix}. \tag{3.3}$$

Let (M_n) be the sequence defined for all $n \geq 1$ by

$$M_n = n\left(V_n - \frac{(n-1)}{2}v\right) \tag{3.4}$$

where $v^T = (1,1)$. It follows from (3.3) that (M_n) is a locally square integrable martingale. One can easily see that its predictable quadratic variation reduces to

$$\begin{aligned} \langle M \rangle_n &= \sum_{k=1}^{n-1} \mathbb{E}[(M_{k+1} - M_k)(M_{k+1} - M_k)^T|\mathcal{F}_k], \\ &= \sum_{k=1}^{n-1} \begin{pmatrix} (k-D_k)(D_k+1) & k \\ k & (k-D'_k)(D'_k+1) \end{pmatrix}. \end{aligned} \tag{3.5}$$

Our martingale decomposition (3.4) allows us to propose a straightforward proof of the asymptotic normality (1.6) previously established via a more sophisticated approach by Chatteerjee and Diaconis [7]. We can also establish new asymptotic results such as a functional central limit theorem and a law of iterated logarithm as follows.

Proposition 3.1. *Let $D([0,\infty[)$ be the Skorokhod space of right-continuous functions with left-hand limits. We have the distributional convergence in $D([0,+\infty[)$,*

$$\left(\sqrt{n}\Big(\frac{D_{\lfloor nt\rfloor}}{\lfloor nt\rfloor}-\frac{1}{2},\frac{D'_{\lfloor nt\rfloor}}{\lfloor nt\rfloor}-\frac{1}{2}\Big),\ t\geq 0\right)\Longrightarrow (W_t,\ t\geq 0) \tag{3.6}$$

where (W_t) is a continuous two-dimensional centered Gaussian process starting from the origin with covariance matrix given, for $0<s\leq t$, by

$$\mathbb{E}[W_tW_s^T]=\frac{s}{12t^2}\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

In particular, we find again the asymptotic normality (1.6).

Proposition 3.2. *We have the quadratic strong law*

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\begin{pmatrix}\frac{D_k}{k}-\frac{1}{2}\\ \frac{D'_k}{k}-\frac{1}{2}\end{pmatrix}\begin{pmatrix}\frac{D_k}{k}-\frac{1}{2}\\ \frac{D'_k}{k}-\frac{1}{2}\end{pmatrix}^T=\frac{1}{12}\begin{pmatrix}1&0\\0&1\end{pmatrix}\qquad a.s. \tag{3.7}$$

In particular,

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\Big(\frac{D_k}{k}-\frac{1}{2}\Big)^2+\Big(\frac{D'_k}{k}-\frac{1}{2}\Big)^2=\frac{1}{6}\qquad a.s. \tag{3.8}$$

Moreover, we also have the law of iterated logarithm

$$\limsup_{n\to\infty}\frac{n}{2\log\log n}\left(\Big(\frac{D_n}{n}-\frac{1}{2}\Big)^2+\Big(\frac{D'_n}{n}-\frac{1}{2}\Big)^2\right)=\frac{1}{6}\qquad a.s. \tag{3.9}$$

Proof. The proofs are postponed to Section 8 □

4. Main results

Our first result is the LDP for the couple $(D_n/n, D'_n/n)$ which extends the LDP for (D_n/n) previously established by Bryc, Minda and Sethuraman [4].

Theorem 4.1. *The couple $(D_n/n, D'_n/n)$ satisfies an LDP with good rate function I given by* (1.7).

Our second result is the SLDP for the sequence (D_n, D'_n) which nicely improves the SLDP recently established for (D_n) in Theorem 3.1 of [3]. For any positive real number x and $n\in\mathbb{N}^*$, denote $\{x_n\}=nx-\lceil(n-1)x\rceil$. Let us remark that $\{x_n\}\in[x-1,x]$ for any $n\in\mathbb{N}^*$. Instead of dividing (D_n,D'_n) by n, it is more natural to divide them by $n-1$ for the following reason. Let A_n be the number of ascents of $\pi_n\in\mathcal{S}_n$. Then, it is clear that $D_n+A_n=n-1$. Moreover, D_n and A_n share the same distribution. Consequently, $D_n/(n-1)$ is symmetric with respect to $1/2$. The same applies for D'_n.

Theorem 4.2. *For any x and y in $]1/2,1[$, we have*

$$\mathbb{P}\Big(\frac{D_n}{n-1}\geq x,\frac{D'_n}{n-1}\geq y\Big)=\frac{\exp(-nI(x,y)+\varphi_n(x,y))}{2\pi n\sigma_x t_x\sigma_y t_y}\big[1+o(1)\big] \tag{4.1}$$

where the value t_x is the unique solution of $L'(t_x) = x$, $\sigma_x^2 = L''(t_x)$ and

$$\varphi_n(x,y) = \{x_n\}t_x + \{y_n\}t_y + \frac{1}{2}t_x t_y. \tag{4.2}$$

In the same spirit, we have for any x and y in $]1/2,1[$,

$$\mathbb{P}\Big(\frac{D_n}{n-1} \le 1-x, \frac{D'_n}{n-1} \le 1-y\Big) = \frac{\exp(-nI(x,y)+\varphi_n(x,y))}{2\pi n\sigma_x t_x \sigma_y t_y}\big[1+o(1)\big]. \tag{4.3}$$

Moreover, we also have for any x and y in $]1/2,1[$,

$$\mathbb{P}\Big(\frac{D_n}{n-1} \le 1-x, \frac{D'_n}{n-1} \ge y\Big) = \frac{\exp(-nI(x,y)+\phi_n(x,y))}{2\pi n\sigma_x t_x \sigma_y t_y}\big[1+o(1)\big], \tag{4.4}$$

and

$$\mathbb{P}\Big(\frac{D_n}{n-1} \ge x, \frac{D'_n}{n-1} \le 1-y\Big) = \frac{\exp(-nI(x,y)+\phi_n(x,y))}{2\pi n\sigma_x t_x \sigma_y t_y}\big[1+o(1)\big] \tag{4.5}$$

where

$$\phi_n(x,y) = \{x_n\}t_x + \{y_n\}t_y - \frac{1}{2}t_x t_y. \tag{4.6}$$

By using a direct modification in the proof of Theorem 3.1 in [3], we obtain that for any x in $]1/2,1[$,

$$\mathbb{P}\Big(\frac{D_n}{n-1} \ge x\Big) = \mathbb{P}\Big(\frac{D'_n}{n-1} \ge x\Big) = \frac{\exp(-nI(x)+\{x_n\}t_x)}{\sigma_x t_x\sqrt{2\pi n}}\big[1+o(1)\big].$$

Moreover, we have by symmetry that

$$\mathbb{P}\Big(\frac{D_n}{n-1} \ge x\Big) = \mathbb{P}\Big(\frac{D_n}{n-1} \le 1-x\Big).$$

Therefore, an immediately consequence of Theorem 4.2 is as follows.

Corollary 4.1. *For all $x, y \in]1/2,1[$, we have*

$$\mathbb{P}\Big(\frac{D_n}{n-1} \ge x, \frac{D'_n}{n-1} \ge y\Big) = \mathbb{P}\Big(\frac{D_n}{n-1} \ge x\Big)\mathbb{P}\Big(\frac{D'_n}{n-1} \ge y\Big)\exp\left(\frac{t_x t_y}{2}\right)\big[1+o(1)\big],$$

$$\mathbb{P}\Big(\frac{D_n}{n-1} \le 1-x, \frac{D'_n}{n-1} \le 1-y\Big) = \mathbb{P}\Big(\frac{D_n}{n-1} \le 1-x\Big)\mathbb{P}\Big(\frac{D'_n}{n-1} \le 1-y\Big)\exp\left(\frac{t_x t_y}{2}\right)\big[1+o(1)\big],$$

$$\mathbb{P}\Big(\frac{D_n}{n-1} \le 1-x, \frac{D'_n}{n-1} \ge y\Big) = \mathbb{P}\Big(\frac{D_n}{n-1} \le 1-x\Big)\mathbb{P}\Big(\frac{D'_n}{n-1} \ge y\Big)\exp\left(-\frac{t_x t_y}{2}\right)\big[1+o(1)\big],$$

$$\mathbb{P}\Big(\frac{D_n}{n-1} \ge x, \frac{D'_n}{n-1} \le 1-y\Big) = \mathbb{P}\Big(\frac{D_n}{n-1} \ge x\Big)\mathbb{P}\Big(\frac{D'_n}{n-1} \le 1-y\Big)\exp\left(-\frac{t_x t_y}{2}\right)\big[1+o(1)\big].$$

Remark 4.1. *Corollary 4.1 highlights the very fine dependence between D_n and D'_n at the sharp large deviation level.*

Chatteerjee and Diaconis found challenging to study the asymptotic behavior of

$$T_n = D_n + D'_n.$$

They proved in Theorem 1.1 of [7] that

$$\sqrt{n}\left(\frac{T_n}{n}-1\right)\xrightarrow[n\to+\infty]{\mathcal{L}}\mathcal{N}\Big(0,\frac{1}{6}\Big). \tag{4.7}$$

We immediately deduce from Theorem 4.1 together with the contraction principle [8] that (T_n/n) satisfies an LDP as follows.

Corollary 4.2. *The sequence (T_n/n) satisfies an LDP with good rate function J defined, for all $y \in \mathbb{R}$, by*

$$J(y) = \inf\Big\{I(x) + I(y-x),\ x \in \mathbb{R}\Big\}. \tag{4.8}$$

5. Conclusion

Our strategy of proof may be extended to some of other well-known statistics on random permutations. More precisely, if we are interested in the major index [16], a central limit theorem for the joint distribution of the major index and the major index of the inverse has been obtained in [1, 23]. However, the LDP is available only for the marginal distribution [17]. The LDP for the joint distribution, or even for the quadruplet distribution adding the couple (D_n, D'_n) is still an open problem and may be tackled using some old results concerning their Laplace transform [14, 21], see also the link with inversions in [11]. These questions are left for future work.

6. Proofs of the large deviation results

6.1. **Some preliminary results.** Denote by m_n the Laplace transform of the couple (D_n, D'_n) defined, for all $t, s \in \mathbb{R}$, by

$$m_n(t,s) = \mathbb{E}\big[\exp(tD_n + sD'_n)\big].$$

One can notice that $m_n(t,s)$ is always finite for all $t, s \in \mathbb{R}$ since D_n and D'_n are bounded by n. We shall make use of the unsigned Stirling numbers of the first kind, which are defined algebraically as the coefficients of the rising factorial via the identity given for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(x)^{(n)} = x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} x^k. \tag{6.1}$$

Lemma 6.1. *For all $t, s \in \mathbb{R}^*$, we have*

$$m_n(t,s) = \left(\frac{e^t-1}{t}\right)^n\left(\frac{e^s-1}{s}\right)^n\left(\frac{1-e^{-t}}{t}\right)\left(\frac{1-e^{-s}}{s}\right)S_n^1(t,s) \tag{6.2}$$

where for $a \in \{0,1\}$,

$$S_n^a(t,s) = \sum_{k=0}^{n-1}\begin{bmatrix} n \\ n-k \end{bmatrix}\left(\frac{(n-k)!}{n!}\right)^2 (st)^k\big(1+ar_{n-k}(t)\big)\big(1+ar_{n-k}(s)\big)$$

and

$$r_n(t) = \sum_{\ell\in\mathbb{Z}^*}\left(1+\frac{2i\pi\ell}{t}\right)^{-(n+1)}.$$

Let us remark that the notation S_n^0 is not useful for the moment but it will be used in the proof of Theorem 4.2.

Proof. It follows from Section 7 in [5], see also Theorem 2 in [20] or more recently Section 3 in [7], that for all $p, q \in (-1, 1)$,

$$\mathbb{E}\big[p^{D_n} q^{D'_n}\big] = \frac{(1-p)^{n+1}(1-q)^{n+1}}{pqn!} \sum_{k,\ell \geq 0} \binom{k\ell + n - 1}{n} p^k q^\ell. \tag{6.3}$$

Let us start by assuming that $t < 0$ and $s < 0$. By using (6.3) with $p = e^t < 1$ and $q = e^s < 1$, we obtain that

$$\begin{aligned} m_n(t,s) &= \frac{(1-e^t)^{n+1}(1-e^s)^{n+1}}{e^t e^s n!} \sum_{k,\ell \geq 0} \binom{k\ell + n - 1}{n} e^{tk} e^{s\ell}, \\ &= \frac{(e^t-1)^{n+1}(e^s-1)^{n+1}}{e^t e^s (n!)^2} \sum_{k,\ell \geq 0} (k\ell)^{(n)} e^{tk} e^{s\ell} \end{aligned}$$

where the rising factorial $(kl)^{(n)}$ was previously defined in (6.1). Hence, we obtain from (6.1) that for all $n \geq 1$,

$$m_n(t,s) = \frac{(e^t-1)^{n+1}(e^s-1)^{n+1}}{e^t e^s (n!)^2} \sum_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \sum_{k \geq 0} k^j e^{tk} \sum_{\ell \geq 0} l^j e^{s\ell}.$$

It follows from the series representation of the polylogarithm function given e.g. by formula (13.1) in [15, 24] that for all $n \geq 1$,

$$\begin{aligned} m_n(t,s) &= \frac{(e^t-1)^{n+1}(e^s-1)^{n+1}}{e^t e^s (n!)^2} \sum_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \frac{(j!)^2}{(ts)^{j+1}} \sum_{k \in \mathbb{Z}} \frac{1}{\big(1 + \frac{2i\pi k}{t}\big)^{j+1}} \sum_{\ell \in \mathbb{Z}} \frac{1}{\big(1 + \frac{2i\pi \ell}{s}\big)^{j+1}} \\ &= \frac{(e^t-1)^{n+1}(e^s-1)^{n+1}}{e^t e^s (st)^{n+1}} \sum_{j=0}^{n-1} \begin{bmatrix} n \\ n-j \end{bmatrix} \left(\frac{(n-j)!}{n!}\right)^2 (st)^j R_{n-j}(t,s) \end{aligned}$$

where

$$R_{n-j}(t,s) = \big(1 + r_{n-j}(t)\big)\big(1 + r_{n-j}(s)\big),$$

which is exactly what we wanted to prove. From now on, we can remark that m_n is clearly an holomorphic function with respect to t, resp. s, on $\mathbb{C}$. Moreover, the right hand-side of (6.2) is also an analytical function with respect to t, resp. s, at least on $\mathbb{C} \setminus 2i\pi\mathbb{Z}$, which means that (6.2) holds true on $(\mathbb{R}^*)^2$. □

Remark 6.1. *One can observe that for all $t \in \mathbb{R}^*$,*

$$m_n(t,0) = m_n(0,t) = \left(\frac{e^t - 1}{t}\right)^n \left(\frac{1 - e^{-t}}{t}\right) \sum_{\ell \in \mathbb{Z}} \left(1 + \frac{2i\pi\ell}{t}\right)^{-(n+1)}.$$

This formula can be used to simplify several computations done in [3].

We have seen in the proof of Lemma 6.1 that for all $n \geq 1$, r_n is an analytical function on $\mathbb{C} \setminus (2i\pi\mathbb{Z})$. Then, S_n is defined on $\mathbb{C}^2 \setminus (2i\pi\mathbb{Z})^2$, which implies that m_n is defined on $\mathbb{C}^2$.

The next lemma recalls some useful known properties of unsigned Stirling numbers.

Lemma 6.2. *We have for all $k \leq n$,*

$$\begin{bmatrix} n \\ n-k \end{bmatrix} \left(\frac{(n-k)!}{n!} \right)^2 \leq \frac{1}{k!}.$$

Moreover,

$$\lim_{n \to +\infty} \begin{bmatrix} n \\ n-k \end{bmatrix} \left(\frac{(n-k)!}{n!} \right)^2 = \frac{1}{2^k k!}.$$

Proof. For the first property, we use Definition 26.8.3 in [18] of the unsigned Stirling numbers of first kind. The inequality follows by bounding the factors inside the sum by the maximum element $(n-1)!/(n-k-1)!$ and by noticing that there are $\binom{n-1}{k}$ elements in the sum. The second property follows from formula 26.8.43 in [18] and the Stirling approximation. □

The last lemma gives an upper-bound for the functions r_n.

Lemma 6.3. *For all $n \geq 1$ and for all $t \in \mathbb{R}^+$, there exists a constants C_t such that for any $v \in [-\pi, \pi]$,*

$$|r_n(t+iv)| \leq C_t \left| \frac{t+iv}{t+i(2\pi - |v|)} \right|^{n+1} \leq C_t.$$

Proof. We have for any $v \in [-\pi, \pi]$,

$$|r_n(t+iv)| \left| \frac{t+i(2\pi-|v|)}{t+iv} \right|^{n+1} \leq \sum_{\ell \in \mathbb{Z}^*} \left| \frac{t^2 + (2\pi - |v|)^2}{t^2 + (v+2\pi\ell)^2} \right|^{\frac{n+1}{2}}.$$

Hereafter, assuming that $v \in [-\pi, 0]$, we have

$$\begin{aligned}
\sum_{\ell \in \mathbb{Z}^*} \left| \frac{t^2 + (2\pi - |v|)^2}{t^2 + (v+2\pi\ell)^2} \right|^{\frac{n+1}{2}} &= 1 + \sum_{\ell \geq 2} \left| \frac{t^2 + (2\pi - |v|)^2}{t^2 + (v+2\pi\ell)^2} \right|^{\frac{n+1}{2}} + \sum_{\ell \leq -1} \left| \frac{t^2 + (2\pi - |v|)^2}{t^2 + (v+2\pi\ell)^2} \right|^{\frac{n+1}{2}} \\
&\leq 1 + \sum_{\ell \geq 2} \left| \frac{t^2 + \pi^2}{t^2 + 4\pi^2(\ell-1)^2} \right|^{\frac{n+1}{2}} + \sum_{\ell \leq -1} \left| \frac{t^2 + \pi^2}{t^2 + 4\pi^2\ell^2} \right|^{\frac{n+1}{2}} \\
&= 1 + 2 \sum_{\ell \in \mathbb{N}^*} \left| \frac{t^2 + \pi^2}{t^2 + 4\pi^2\ell^2} \right|^{\frac{n+1}{2}} < +\infty,
\end{aligned}$$

since we have assumed that $n+1 \geq 2$. By symmetry the previous upper-bound stays true for $v \in [0, \pi]$, which gives us the announced result. □

6.2. **Proof of Theorem 4.1.** For all $t, s \in \mathbb{R}$, denote by L_n the normalized cumulant generating function

$$L_n(t,s) = \frac{1}{n} \log m_n(t,s).$$

It follows from Lemma 6.1 that for all $t, s \in \mathbb{R}$,

$$\lim_{n\to\infty} L_n(t,s) = L(t,s) = \log\left(\Big(\frac{\exp(t)-1}{t}\Big)\Big(\frac{\exp(s)-1}{s}\Big)\right) = L(t) + L(s) \tag{6.4}$$

where

$$L(t) = \log\left(\frac{\exp(t)-1}{t}\right).$$

The function L is finite and differentiable on all the real line. Then, we deduce from the Gärtner-Ellis theorem, see e.g. Theorem 2.3.6 in [8], that the couple $(D_n/n, D'_n/n)$ satisfies an LDP with good rate function I given by (1.7). □

6.3. **Proof of Theorem 4.2.** Our goal is to establish, for all x and y in $]1/2, 1[$, a sharp asymptotic expansion of the probability

$$\mathbb{P}\Big(\frac{D_n}{n-1} \geq x, \frac{D'_n}{n-1} \geq y\Big) = \sum_{k=\lceil (n-1)x\rceil}^{n-1} \sum_{k'=\lceil (n-1)y\rceil}^{n-1} \mathbb{P}(D_n = k, D'_n = k'). \tag{6.5}$$

For all $t, v, s, w \in \mathbb{R}$, we have

$$m_n(t+iv, s+iw) = \sum_{k=0}^{n-1}\sum_{k'=0}^{n-1} e^{(t+iv)k+(s+iw)k'}\mathbb{P}(D_n = k, D'_n = k').$$

Therefore, for all $0 \leq k, k' \leq n-1$ and in fact for all $k, k' \in \mathbb{Z}$, we obtain that

$$\mathbb{P}(D_n = k, D'_n = k') = e^{-tk-sk'}\frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} m_n(t+iv, s+iw)e^{-ikv-ik'w}dvdw.$$

As D_n and D'_n are smaller than n, we clearly have $|m_n(t+iv, s+iw)| \leq e^{|t|n+|s|n}$. Hence, it follows from (6.5) together with Fubini's theorem that for all $t, s > 0$,

$$\mathbb{P}\Big(\frac{D_n}{n-1} \geq x, \frac{D'_n}{n-1} \geq y\Big) = \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} m_n(t+iv, s+iw)\Sigma_n(t+iv, s+iw)dvdw \tag{6.6}$$

where

$$\begin{aligned}\Sigma_n(t+iv, s+iw) &= \sum_{k=\lceil (n-1)x\rceil}^{+\infty}\sum_{k'=\lceil (n-1)y\rceil}^{+\infty} e^{-k(t+iv)-k'(s+iw)} \\ &= \Big(\frac{e^{-\lceil (n-1)x\rceil(t+iv)}}{1-e^{-(t+iv)}}\Big)\Big(\frac{e^{-\lceil (n-1)y\rceil(s+iw)}}{1-e^{-(s+iw)}}\Big).\end{aligned}$$

In the following, we choose $t = t_x$ and $s = t_y$. One can observe that t_x and t_y are both positive since $x, y > 1/2$. Consequently, we deduce from Lemma 6.1 and (6.6), together with the fact that $I(x) = xt_x - L(t_x)$ and $\{x_n\} = nx - \lceil (n-1)x\rceil$, that

$$\mathbb{P}\Big(\frac{D_n}{n-1} \geq x, \frac{D'_n}{n-1} \geq y\Big) = \Big(\frac{e^{-nI(x)-t_x\{x_n\}}}{2\pi}\Big)\Big(\frac{e^{-nI(y)-t_y\{y_n\}}}{2\pi}\Big)I_n \tag{6.7}$$

where

$$I_n = \int_{[-\pi,\pi]^2} e^{-n\varphi_x(v)-n\varphi_y(w)} f_n^1(v,w)dvdw,$$

with

$$\varphi_x(v) = -(L(t_x + iv) - L(t_x) - ixv)$$

and, for $a \in \{0, 1\}$,

$$f_n^a(v, w) = \Big(\frac{e^{i\{x_n\}v}}{t_x + iv}\Big)\Big(\frac{e^{i\{y_n\}w}}{t_y + iw}\Big) S_n^a(t_x + iv, t_y + iw). \tag{6.8}$$

By using Lemmas 6.2 and 6.3, we easily obtain that

$$\big|f_n^a(v, w)\big| \leq \frac{e^{t_x t_y}}{t_x t_y}(1 + aC_{t_x})(1 + aC_{t_y}). \tag{6.9}$$

Hereafter, we split the integral I_n into two parts, $I_n = J_n + K_n$ where

$$\begin{aligned} J_n &= \int_{[-\pi/2,\pi/2]^2} e^{-n\varphi_x(v) - n\varphi_y(w)} f_n^1(v, w) dv dw, \\ K_n &= \int_{[-\pi,\pi]^2 \setminus [-\pi/2,\pi/2]^2} e^{-n\varphi_x(v) - n\varphi_y(w)} f_n^1(v, w) dv dw. \end{aligned}$$

It follows from the last part of Lemma 5.1 in [3] that for all $v \in \mathbb{R}$ such that $|v| \leq \pi$,

$$-\Re(\varphi_x(v)) = \Re(L(t_x + iv) - L(t_x)) \leq -\lambda_x \frac{|v|^2}{2} \tag{6.10}$$

where

$$\lambda_x = \frac{t_x^2 \sigma_x^2}{t_x^2 + \pi^2}.$$

Then, by symmetry, we obtain from (6.9) and (6.10) that

$$\begin{aligned} |K_n| &\leq \frac{e^{t_x t_y}}{t_x t_y}(1 + C_{t_x})(1 + C_{t_y}) \int_{[-\pi,\pi]^2 \setminus [-\pi/2,\pi/2]^2} e^{-n\Re(\varphi_x(v) + \varphi_y(w))} dv dw, \\ &\leq \frac{4e^{t_x t_y}}{t_x t_y}(1 + C_{t_x})(1 + C_{t_y})\Big(K_n^1 + K_n^2 + K_n^3\Big) \qquad (6.11) \end{aligned}$$

where, via standard Gaussian calculations, K_n^1, K_n^2 and K_n^3 are such that

$$K_n^1 = \int_{\pi/2}^{\pi} \exp\Big(-\frac{n\lambda_x v^2}{2}\Big) dv \int_{\pi/2}^{\pi} \exp\Big(-\frac{n\lambda_y w^2}{2}\Big) dw \leq \frac{4}{n^2\pi^2\lambda_x\lambda_y} \exp\Big(-\frac{n\pi^2}{8}\big(\lambda_x + \lambda_y\big)\Big),$$

$$K_n^2 = \int_{0}^{\pi/2} \exp\Big(-\frac{n\lambda_x v^2}{2}\Big) dv \int_{\pi/2}^{\pi} \exp\Big(-\frac{n\lambda_y w^2}{2}\Big) dw \leq \frac{\sqrt{2}}{n\lambda_y\sqrt{n\pi\lambda_x}} \exp\Big(-\frac{n\pi^2}{8}\lambda_y\Big),$$

$$K_n^3 = \int_{\pi/2}^{\pi} \exp\Big(-\frac{n\lambda_x v^2}{2}\Big) dv \int_{0}^{\pi/2} \exp\Big(-\frac{n\lambda_y w^2}{2}\Big) dw \leq \frac{\sqrt{2}}{n\lambda_x\sqrt{n\pi\lambda_y}} \exp\Big(-\frac{n\pi^2}{8}\lambda_x\Big).$$

Consequently, we clearly deduce from (6.11) that K_n goes exponentially fast to zero. It only remains to study the integral J_n through the following three quantities,

$$\begin{aligned} J_n &= \int_{[-\pi/2,\pi/2]^2} e^{-n\varphi_x(v)-n\varphi_y(w)}(f_n^1(v,w) - f_n^0(v,w))dvdw \\ &+ \int_{[-\pi/2,\pi/2]^2} e^{-n\varphi_x(v)-n\varphi_y(w)}(f_n^0(v,w) - f_n^0(0,0))dvdw \\ &+ f_n^0(0,0)\int_{[-\pi/2,\pi/2]^2} e^{-n\varphi_x(v)-n\varphi_y(w)}dvdw = J_n^1 + J_n^2 + J_n^3. \end{aligned}$$

On the one hand, we have from (6.3) that

$$f_n^0(0,0) = \frac{1}{t_x t_y}\sum_{k=0}^{n-1} a_{n,k} \qquad \text{where} \qquad a_{n,k} = \begin{bmatrix} n \\ n-k \end{bmatrix}\left(\frac{(n-k)!}{n!}\right)^2 (t_x t_y)^k.$$

We obtain from Lemma 6.2 that

$$\lim_{n\to+\infty} a_{n,k} = \frac{t_x^k t_y^k}{2^k k!} \qquad \text{and} \qquad 0 \leq a_{n,k} \leq \frac{t_x^k t_y^k}{k!}.$$

Therefore, we can apply the dominated convergence theorem to get that

$$\lim_{n\to+\infty} f_n^0(0,0) = \frac{e^{\frac{t_x t_y}{2}}}{t_x t_y}.$$

On the other hand, recalling (6.10) and using a slight extension of the usual Laplace method given by Lemma 5.2 in [3], we obtain that

$$\int_{[-\pi/2,\pi/2]^2} e^{-n\varphi_x(v)-n\varphi_y(w)}dvdw = \frac{2\pi}{n\sigma_x\sigma_y}(1+o(1)),$$

which ensures that

$$J_n^3 = \frac{2\pi}{n\sigma_x\sigma_y}\frac{e^{\frac{t_x t_y}{2}}}{t_x t_y}(1+o(1)). \tag{6.12}$$

Furthermore, one can observe that $S_n^0(t_x+iv, t_y+iw)$ is a bounded Lipschitz function on $[-\pi/2,\pi/2]^2$, uniformly in n. Indeed, by using Lemma 6.2, we have

$$\left|S_n^0(t_x+iv,t_y+iw)\right| \leq \sum_{k=0}^{n-1}\frac{1}{k!}\big(t_x+|v|\big)^k\big(t_y+|w|\big)^k \leq \exp\Big(\Big(t_x+\frac{\pi}{2}\Big)\Big(t_y+\frac{\pi}{2}\Big)\Big),$$

and

$$\begin{aligned} \left|\frac{\partial S_n^0(t_x+iv,t_y+iw)}{\partial v}\right| &\leq \sum_{k=1}^{n-1}\frac{k\big(t_x+|v|\big)^{k-1}\big(t_y+|w|\big)^k}{k!}, \\ &\leq \Big(t_y+\frac{\pi}{2}\Big)\exp\Big(\Big(t_x+\frac{\pi}{2}\Big)\Big(t_y+\frac{\pi}{2}\Big)\Big). \end{aligned}$$

Since $S_n^0(t_x+iv,t_y+iw) = S_n^0(t_y+iw,t_x+iv)$, it implies that there exists a positive constant C which does not depend on n, such that

$$|f_n^0(v,w) - f_n^0(0,0)| \leq C(|v|+|w|).$$

Applying this bound, inequality (6.10) and the usual Laplace method, we obtain

$$|J_n^2| \leq C \int_{[-\pi/2,\pi/2]^2} e^{-n\Re(\varphi_x(v)+\varphi_y(w))}(|v|+|w|)dvdw. \tag{6.13}$$

Hence, as for K_n, J_n^2 goes exponentially fast to zero. Hereafter, we just have to study the last term J_n^1 in order to conclude. By using Lemma 6.3, we have, for all $v \in [-\pi/2, \pi/2]$ and for all $n \geq 1$,

$$|r_n(t_x + iv)| \leq C_{t_x}(q_x)^{n+1} \qquad \text{where} \qquad q_x = \left(\frac{t_x^2 + (\pi/2)^2}{t_x^2 + (3\pi/2)^2}\right)^{1/2}.$$

One can observe that $0 < q_x < 1$, which means that $r_n(t_x + iv)$ goes exponentially fast to zero. Then, by using once again Lemma 6.2, we obtain from (6.3) that for all $(v, w) \in [-\pi/2, \pi/2]^2$,

$$\begin{aligned}\left|f_n^1(v,w) - f_n^0(v,w)\right| &\leq \frac{1}{t_x t_y}\sum_{k=0}^{n-1}\frac{(t_x+|v|)^k(t_y+|w|)^k}{k!}\left|R_{n-k}(t_x+v, t_y+iw)-1\right|,\\
&\leq \frac{C_{t_x}(q_x)^{n+1}}{t_x t_y}\exp\Big(\frac{1}{q_x}\Big(t_x+\frac{\pi}{2}\Big)\Big(t_y+\frac{\pi}{2}\Big)\Big) + \frac{C_{t_y}(q_y)^{n+1}}{t_x t_y}\exp\Big(\frac{1}{q_y}\Big(t_x+\frac{\pi}{2}\Big)\Big(t_y+\frac{\pi}{2}\Big)\Big)\\
&\qquad + \frac{C_{t_x}C_{t_y}(q_x q_y)^{n+1}}{t_x t_y}\exp\Big(\frac{1}{q_x q_y}\Big(t_x+\frac{\pi}{2}\Big)\Big(t_y+\frac{\pi}{2}\Big)\Big).\end{aligned}$$

Consequently, we deduce from inequality (6.10) that J_n^1 goes exponentially fast to zero. Finally, the only contribution for the integral I_n is given by (6.12) and (6.7) clearly leads to (4.1). From now on, we shall carry out the proof of (4.3). Equation (6.6) becomes, for all $t, s < 0$,

$$\mathbb{P}\Big(\frac{D_n}{n-1} \leq 1-x, \frac{D'_n}{n-1} \leq 1-y\Big) = \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} m_n(t+iv, s+iw)\Sigma_n(t+iv, s+iw)dvdw$$

where

$$\begin{aligned}\Sigma_n(t+iv, s+iw) &= \sum_{k=-\infty}^{\lfloor (n-1)(1-x)\rfloor}\ \sum_{k'=-\infty}^{\lfloor (n-1)(1-y)\rfloor} e^{-k(t+iv)-k'(s+iw)}\\
&= \Big(\frac{e^{-(\lfloor (n-1)(1-x)\rfloor+1)(t+iv)}}{e^{-(t+iv)}-1}\Big)\Big(\frac{e^{-(\lfloor (n-1)(1-y)\rfloor+1)(s+iw)}}{e^{-(s+iw)}-1}\Big).\end{aligned}$$

In the following, we choose $t = t_{1-x}$ and $s = t_{1-y}$. In contrast with the previous case, we have $t_{1-x} = -t_x < 0$, $t_{1-y} = -t_y < 0$ and $I(1-x) = I(x)$, $I(1-y) = I(y)$. Moreover, a simple computation shows that for any $z \in \mathbb{R}$,

$$n(1-z) - \lfloor (n-1)(1-z)\rfloor - 1 = -(nz - \lceil (n-1)z\rceil).$$

Then, via the same arguments as in the proof of (6.7), we find that

$$\begin{aligned}\mathbb{P}\Big(\frac{D_n}{n-1} \leq 1-x, \frac{D'_n}{n-1} \leq 1-y\Big) &= \Big(\frac{e^{-nI(1-x)-t_{1-x}\{x_n\}}}{2\pi}\Big)\Big(\frac{e^{-nI(1-y)-t_{1-y}\{y_n\}}}{2\pi}\Big)I_n,\\
&= \Big(\frac{e^{-nI(x)+t_x\{x_n\}}}{2\pi}\Big)\Big(\frac{e^{-nI(y)+t_y\{y_n\}}}{2\pi}\Big)I_n\end{aligned}$$

where

$$I_n = \int_{[-\pi,\pi]^2} e^{-n\varphi_{1-x}(v)-n\varphi_{1-y}(w)} f_n^1(v,w)dvdw,$$

with

$$\varphi_{1-x}(v) = -(L(-t_x+iv) - L(-t_x) - i(1-x)v)$$

and, for $a \in \{0,1\}$,

$$f_n^a(v,w) = \Big(\frac{e^{-i\{x_n\}v}}{t_x - iv}\Big)\Big(\frac{e^{-i\{y_n\}w}}{t_y - iw}\Big) S_n^a(-t_x+iv, -t_y+iw).$$

Therefore, we just have to make use of the same calculations as previously done to show (4.3). Finally, we establish (4.4) and (4.5) using the same lines, which completes the proof of Theorem 4.2. □

7. Proof of the construction of the probability space

Recall the construction made in Section 2. We call fiber and we denote by $\mathcal{F}_n(\ell)$ the set of cells $\big\{\mathcal{C}_n(k,\ell)/k \in \{1,2,\dots,n+1\}\big\}$ for $\ell \in \{1,2,\dots,n+1\}$.

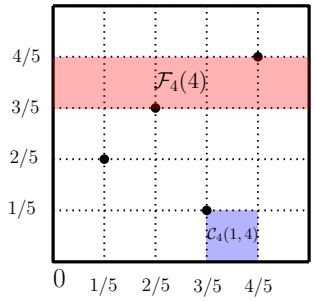


Figure 2. Graphical representation of $\pi_4 = (2314)$ to the left. We colored in light blue the cell $\mathcal{C}_4(1,4)$ and in light red the cells belonging to the fiber $\mathcal{F}_4(4)$.

The following lemma states that the number of cells in each fiber with $\Delta D_{n+1} = +1$ depends only on the number of descent of π_n and has a closed formula.

Lemma 7.1. *For each $\ell \in \{1,2,\dots,n+1\}$, the fiber $\mathcal{F}_n(\ell)$ in the graphical representation has $n - D_n$ cells with increment $+1$ and $D_n + 1$ with increment 0. In other words, Card$\big\{k/\Delta D_{n+1}(k,\ell) = 1\big\} = n - D_n$, Card$\big\{k/\Delta D_{n+1}(k,\ell) = 0\big\} = D_n + 1$.*

Proof. For each point $p = (k/(n+1), \ell/(n+1))$ in $GR(\pi_n)$, we color in red the cells that are adjacent to the left of the line $k/(n+1) \times [0,1]$ and that are bigger than p and we color in blue the cells adjacent to the right of the line $k/(n+1) \times [0,1]$ and that are smaller than the point p.
Let $R_n(k,\ell) = 1$ if the cell $\mathcal{C}_n(k,\ell)$ is colored red and 0 otherwise, and let $B_n(k,\ell) = 1$ if the cell $\mathcal{C}_n(k,\ell)$ is colored blue and 0 otherwise. With our representation, note

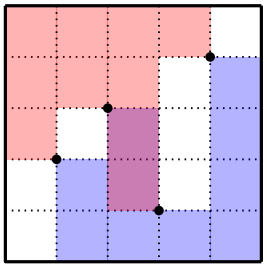


FIGURE 3. Coloring of the graphical representation associated to $\pi_4 = (2314)$.

that every point induce exactly one cell colored in each fiber, that is for all (k, ℓ) in $\{1, \dots, n+1\}^2$,

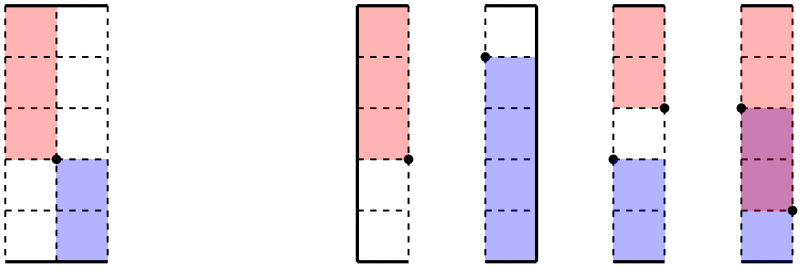
$$R_n(k, \ell) + B_n(k+1, \ell) = 1. \tag{7.1}$$

This property can be easily checked in Figure 4 A). Let $\ell \in \{1, 2, \dots, n+1\}$ be the fiber under study. A deeper analysis shows that there are four cases for each cell.

- L) The cell belongs to the left boundary, i.e. $\mathcal{C}_n(1, \ell)$. In this case, it cannot be colored blue and it has an increment $+1$ if the cell is colored red, otherwise the increment is 0, see Figure 4 B). The function $R_n(1, \ell)$ is equal to the indicator function of the cell having increment $+1$.
- R) The cell belongs to the right boundary, i.e. $\mathcal{C}_n(n+1, \ell)$. In ths case, it cannot be colored red and it has an increment $+1$ if the cell is colored blue, otherwise the increment is 0, see Figure 4 B). The function $B_n(n+1, \ell)$ is equal to the indicator function of the cell having increment $+1$.
- I) The cell belongs to the inner columns, i.e. $\mathcal{C}_n(k, \ell)$ for $k \in \{2, 3, \dots, n\}$. In this case, it depends on whether or not there is a descent at $k-1$ for π_n.
- I0) If $\pi_n(k-1) < \pi_n(k)$, then the cell adds an increment $+1$ if it is colored only red or only blue, and in this case it cannot be simultaneously colored blue and red, see Figure 4 B). The indicator function of the cell having increment $+1$ can be written as $R_n(k, \ell) + B_n(k, \ell)$.
- I1) If $\pi_n(k-1) > \pi_n(k)$, then the cell adds an increment $+1$ if it is colored red and blue, otherwise the increment is 0. Notice that when $D_n(k-1) = 1$, all the cells in $\{\mathcal{C}_n(k, \ell) / \ell \in \{1, 2, \dots, n+1\}\}$ are colored by at least one color, see Figure 4 B). It can easily be checked that the indicator function of the cell having increment $+1$ can be written as $R_n(k, \ell) + B_n(k, \ell) - 1$.

For each $\ell \in \{1, 2, \dots, n+1\}$ and for the fiber $\mathcal{F}_n(\ell)$, we obtain that the sum of cells with increment $+1$ is given by $\Sigma_n^1 + \Sigma_n^2 + \Sigma_n^3$ where $\Sigma_n^1 = R_n(1, \ell) + B_n(n+1, \ell)$,

$$\begin{aligned} \Sigma_n^2 &= \sum_{\substack{k \in \{2, \dots, n\} \\ \pi_n(k-1) < \pi_n(k)}} (R_n(k, \ell) + B_n(k, \ell)), \\ \Sigma_n^3 &= \sum_{\substack{k \in \{2, \dots, n\} \\ \pi_n(k-1) > \pi_n(k)}} (R_n(k, \ell) + B_n(k, \ell) - 1). \end{aligned}$$



(A) (B)

FIGURE 4. Figure A): In each fiber, there is exactly one red or blue cell induced by each point of the permutation, as stated by (7.1). Figure B): From left to right, we show the cases in R), L), I0) and I1), respectively.

Finally, we deduce from (1.1) and (7.1) that

$$\Sigma_n^1 + \Sigma_n^2 + \Sigma_n^3 = \sum_{k=1}^{n} (R_n(k,\ell) + B_n(k+1,\ell)) - D_n = n - D_n.$$

Moreover, as the cells can have either increments $+1$ or 0, we find that the sum of cells with increment 0 is given by $n+1-n+D_n = D_n+1$. □

Corollary 7.1. *For every permutation $\pi_n \in \mathcal{S}_n$, out of the $(n+1)^2$ cells in our graphical representation, there are exactly $(n+1)(n-D_n)$ cells with increment $+1$.*

We are now in the position to prove Theorem 2.1.

Proof of Theorem 2.1. We claim that it is enough to show that

$$N_{n+1}(1,1,\pi_n) = (n-D_n)(n-D'_n) + n. \tag{7.2}$$

As a matter of fact, for a permutation $\pi_n \in \mathcal{S}_n$, denote $\bar{\pi}_n = n - \pi_n$. The permutation $\bar{\pi}_n$ is a bijective involution, which exchanges the descents (resp. ascents) of π_n into ascents (resp. descents) of $\bar{\pi}_n$ and it does the same for π_n^{-1} and $\bar{\pi}_n^{-1}$. Since in total the ascents and descents sum up to $n-1$, we have that the descents for $\bar{\pi}_n$ are equal to D_n+1 and the descents for $\bar{\pi}_n^{-1}$ are equal to D'_n+1. Therefore, we have

$$N_{n+1}(0,0,\pi_n) = N_{n+1}(1,1,\bar{\pi}_n) = (D_n+1)(D'_n+1) + n.$$

Moreover, for $N_{n+1}(1,0,\pi_n)$, it follows from to Corollary 7.1 that

$$N_{n+1}(1,0,\pi_n) + N_{n+1}(1,1,\pi_n) = (n+1)(n-D_n),$$

which clearly leads via (7.2) to

$$\begin{aligned} N_{n+1}(1,0,\pi_n) &= (n+1)(n-D_n) - (n-D_n)(n-D'_n) - n, \\ &= (n-D_n)(D'_n+1) - n. \end{aligned}$$

Consequently, as

$$N_{n+1}(1,1,\pi_n) + N_{n+1}(1,0,\pi_n) + N_{n+1}(0,1,\pi_n) + N_{n+1}(0,0,\pi_n) = (n+1)^2,$$

we find that

$$N_{n+1}(0,1,\pi_n) = (D_n+1)(n-D'_n) - n.$$

Hereafter, we focus our attention on the proof of (7.2) by induction on the size of the permutation.

The base case $n=1$ is easy to check since there is only one possible permutation, the identity for 1 element, which is equal to its inverse. Consequently, we have $N_2(1,1,\pi_1) = 2 = (1-D_1)(1-D'_1)+1$ where $D_1 = D'_1 = 0$.

For the induction step, suppose that (7.2) holds true for $n \geq 1$ and let π_{n+1} be a permutation of size $\mathcal{S}_{n+1}$. We define π_n as the permutation induced by π_{n+1} when one takes out $\pi_{n+1}(n+1)$, see Figure 5 below. More precisely, we see π_n as the permutation $(\pi_{n+1}(k)/k \in \{1,\dots,n\})$ with respect to the relative order of the images present in this vector from left to right.

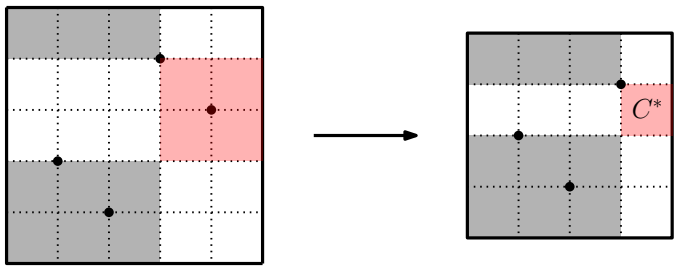


FIGURE 5. Permutation $\pi_3 = (213)$ (to the right) induced from $\pi_4 = (2143)$ (to the left) when taking out $\pi_4(4) = 3$. We keep track of the increment behavior of the red cell during the induction.

We denote by C^* the cell in π_n where the point u belongs to create the whole permutation π_{n+1}. It is important to note that if the cell $C^* = \mathcal{C}_n(n+1,\ell)$ where $\ell \in \{1,\dots,n+1\}$, then the increment behavior of this cell fixes the relative order of $\pi_n(n)$ and the values of the inverse for ℓ and $\ell-1$, that is $\pi_n^{-1}(\ell)$ and $\pi_n^{-1}(\ell-1)$, this are the three points we plot in each case of what follows. It is also important to observe that the increment behavior of the gray cells in Figure 5 remain invariant by this transformation. There are four possible cases depending on $u \in C^*$.

1) $\Delta D_{n+1} = 1$ and $\Delta D'_{n+1} = 1$. In this case, the relative order of $\pi_n(n)$, $\pi_n^{-1}(\ell)$ and $\pi_n^{-1}(\ell-1)$ is given in Figure 6.
   In this case, as $D_{n+1} = D_n + 1$ and $D'_{n+1} = D'_n + 1$, we have from (7.2) that

$$\begin{aligned} N_{n+2}(1,1,\pi_{n+1}) &= N_{n+1}(1,1,\pi_n) - 1 + 2, \\ &= (n-D_n)(n-D'_n) + n + 1 \\ &= (n+1-D_{n+1})(n+1-D'_{n+1}) + n + 1, \end{aligned}$$

   where the first equality follows since the only cell which is simultaneously red and blue becomes two cells that are simultaneously red and blue.
2) $\Delta D_{n+1} = 1$ and $\Delta D'_{n+1} = 0$. In this case, the relative order of $\pi_n(n)$, $\pi_n^{-1}(\ell)$ and $\pi_n^{-1}(\ell-1)$ is given in Figure 7.

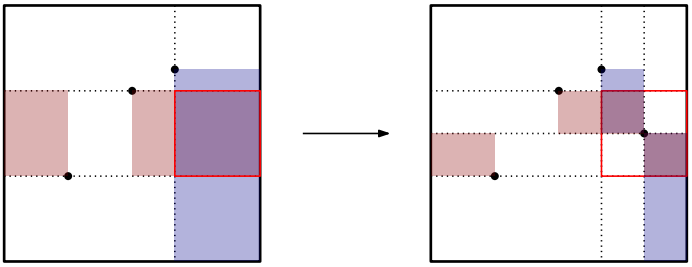


FIGURE 6. Relative order of $\pi_n(n)$, $\pi_n^{-1}(\ell)$ and $\pi_n^{-1}(\ell-1)$ in case 1). The blue cells are cells with $\Delta D_{n+1} = 1$ and the red cells are cells with $\Delta D'_{n+1} = 1$.

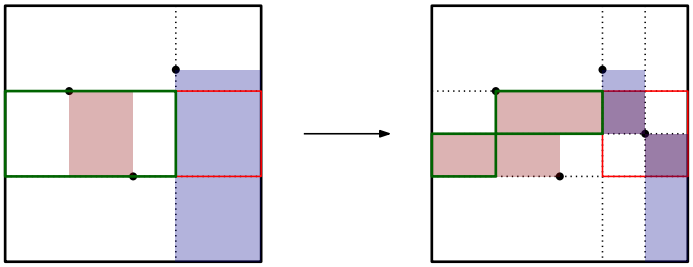


FIGURE 7. Relative order of $\pi_n(n)$, $\pi_n^{-1}(\ell)$ and $\pi_n^{-1}(\ell-1)$ in case 2). The blue cells are cells with $\Delta D_{n+1} = 1$ and the red cells are cells with $\Delta D'_{n+1} = 1$.

What remains is to count the number of cells in the delimited green box in the right of Figure 7 that are also colored in blue. We do this bijectively from the left. The quantity of cells in the delimited green box to the right having $\Delta D_{n+2} = 1$ are in bijection with the ones having $\Delta D_{n+1} = 1$ in the green box to the left. One has from Lemma 7.1 applied to π_n that there are $n - D_n - 1$ cells in the delimited green part, the -1 coming from the cell C^*, which is colored blue, meaning that $\Delta D_{n+1} = 1$. Consequently, as $D_n = D_{n+1} + 1$ and $D'_n = D'_{n+1}$, we have from (7.2) that

$$\begin{aligned} N_{n+2}(1,1,\pi_{n+1}) &= N_{n+1}(1,1,\pi_n) + n - D_n - 1 + 2 \\ &= (n - D_n)(n - D'_n) + n + 1 + (n - D_n) \\ &= (n - D_n)(n + 1 - D'_n) + n + 1 \\ &= (n + 1 - D_{n+1})(n + 1 - D'_{n+1}) + n + 1. \end{aligned}$$

3) $\Delta D_{n+1} = 0$ and $\Delta D'_{n+1} = 1$. It follows from a similar argument as in case 2) using Lemma 7.1 applied to π_n^{-1}.
4) $\Delta D_{n+1} = 0$ and $\Delta D'_{n+1} = 0$. It follows by joining the argument in cases 2) and 3) using once again Lemma 7.1 applied to π_n and π_n^{-1} simultaneously.

□

8. Proof of the martingale results

Finally, we shall proceed with the proof of the martingale results concerning the couple(D_n, D'_n) such as the functional central limit theorem, the quadratic strong law and the law of iterated logarithm.

Proof of Proposition 3.1. We already saw from (3.5) that the predictable quadratic variation $\langle M \rangle_n$ of the locally square integrable martingale (M_n) is given by

$$\langle M \rangle_n = \sum_{k=1}^{n-1} \begin{pmatrix} (k - D_k)(D_k + 1) & k \\ k & (k - D'_k)(D'_k + 1) \end{pmatrix}.$$

It follows from the almost sure convergences

$$\lim_{n\to\infty} \frac{D_n}{n} = \frac{1}{2} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{D'_n}{n} = \frac{1}{2} \tag{8.1}$$

together with the classical Toeplitz lemma that

$$\lim_{n\to\infty} \frac{1}{n^3} \langle M \rangle_n = \frac{1}{12} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{a.s.} \tag{8.2}$$

Consequently, we immediately obtain from (8.2) that for all $t \geq 0$,

$$\lim_{n\to\infty} \frac{1}{n^3} \langle M \rangle_{\lfloor nt \rfloor} = \frac{t^3}{12} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{a.s.} \tag{8.3}$$

It is not hard to see that (M_n) satisfies Lindeberg's condition since for all $n \geq 1$, the increments $||\Delta M_n|| \leq 2n$. Therefore, we deduce from the functional central limit theorem for martingales given e.g. in Theorem 2.5 of [10] that

$$\Big(\frac{1}{\sqrt{n^3}} M_{\lfloor nt \rfloor}, t \geq 0 \Big) \Longrightarrow \big(B_t, t \geq 0 \big) \tag{8.4}$$

where $\big(B_t, t \geq 0 \big)$ is a continuous two-dimensional centered Gaussian process starting from the origin with covariance matrix given, for all $0 < s \leq t$, by

$$\mathbb{E}[B_s B_t^T] = \frac{s^3}{12} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally, (3.4) and (8.4) lead to (3.6) where $W_t = B_t/t^2$, which completes the proof of Proposition 3.1. □

Proof of Proposition 3.2. We shall make use of the quadratic strong law for scalar martingales given e.g. by Theorem 3 in [2]. For any vector u of $\mathbb{R}^2$, denote $M_n(u) = \langle u, M_n \rangle$. It follows from (3.4) that

$$M_n(u) = n \left(\langle u, V_n \rangle - \frac{(n-1)}{2} \langle u, v \rangle \right). \tag{8.5}$$

We obtain from (3.1) and (8.5) that $M_n(u)$ can be rewritten in the additive form

$$M_n(u) = \sum_{k=1}^{n} k \, \varepsilon_k(u) \tag{8.6}$$

where $\varepsilon_{n+1}(u) = \langle u, \xi_{n+1} - \pi_n \rangle$ and π_n is the vector of $\mathbb{R}^2$ given by $\pi_n = (p_n, p'_n)^T$. One can observe that $(M_n(u))$ is a locally square integrable scalar martingale. Since for any vector u of $\mathbb{R}^2$,

$$\begin{aligned}\mathbb{E}\big[\big(M_{n+1}(u) - M_n(u)\big)^2 | \mathcal{F}_n\big] &= u^T \mathbb{E}\big[(M_{n+1} - M_n)(M_{n+1} - M_n)^T | \mathcal{F}_n\big] u, \\ &= u^T (\langle M \rangle_{n+1} - \langle M \rangle_n) u,\end{aligned}$$

we deduce from (3.5) and (8.6) that

$$\mathbb{E}\big[\varepsilon_{n+1}^2(u) | \mathcal{F}_n\big] = \frac{1}{(n+1)^2} u^T \begin{pmatrix} (n - D_n)(D_n + 1) & n \\ n & (n - D'_n)(D'_n + 1) \end{pmatrix} u.$$

Consequently, we obtain from (8.1) that

$$\lim_{n \to \infty} \mathbb{E}\big[\varepsilon_{n+1}^2(u) | \mathcal{F}_n\big] = \frac{1}{4} ||u||^2 \qquad \text{a.s.} \tag{8.7}$$

Moreover, we already saw that for all $n \geq 1$, the increments $||\Delta M_n|| \leq 2n$ which implies that $|\varepsilon_n(u)| \leq 2||u||$, leading to

$$\sup_{n \geq 0} \mathbb{E}\big[\varepsilon_{n+1}^4(u) | \mathcal{F}_n\big] < \infty \qquad \text{a.s.}$$

Therefore, it follows from the quadratic strong law for scalar martingales that

$$\lim_{n \to \infty} \frac{1}{\log s_n} \sum_{k=1}^{n} f_k \Big(\frac{M_k^2(u)}{s_k} \Big) = \frac{1}{4} ||u||^2 \qquad \text{a.s.} \tag{8.8}$$

where

$$f_n = \frac{n^2}{s_n} \qquad \text{and} \qquad s_n = \sum_{k=1}^{n} k^2.$$

We clearly have that nf_n converges to 3 which immediately implies that f_n goes to zero. Consequently, the quadratic strong law (8.8) reduces to

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{M_k^2(u)}{k^4} = \frac{1}{12} ||u||^2 \qquad \text{a.s.} \tag{8.9}$$

Therefore, we deduce from (8.5) and (8.9) that for any vector u of $\mathbb{R}^2$,

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k^2} u^T \begin{pmatrix} \frac{D_k}{k} - \frac{1}{2} \\ \frac{D'_k}{k} - \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{D_k}{k} - \frac{1}{2} \\ \frac{D'_k}{k} - \frac{1}{2} \end{pmatrix}^T u = \frac{1}{12} ||u||^2 \qquad \text{a.s.} \tag{8.10}$$

By virtue of the second part of Proposition 4.2.8 in [9], we obtain from (8.10) that

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k^2} \begin{pmatrix} \frac{D_k}{k} - \frac{1}{2} \\ \frac{D'_k}{k} - \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{D_k}{k} - \frac{1}{2} \\ \frac{D'_k}{k} - \frac{1}{2} \end{pmatrix}^T = \frac{1}{12} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{a.s.} \tag{8.11}$$

Furthermore, we obviously obtain (3.8) by taking the trace on both sides of (8.11). It only remains to prove the law of iterated logarithm (3.9). We clearly have

$$\sum_{n=1}^{\infty} f_n^2 < \infty.$$

Hence, it follows from the law of iterated logarithm for martingales due to Stout [22], see also Corollary 6.4.25 in [9], that for any vector u of $\mathbb{R}^2$,

$$\begin{aligned} \limsup_{n\to\infty}\Big(\frac{1}{2n^3\log\log n}\Big)^{1/2} M_n(u) &= -\liminf_{n\to\infty}\Big(\frac{1}{2n^3\log\log n}\Big)^{1/2} M_n(u) \\ &= \frac{1}{\sqrt{12}}||u|| \qquad \text{a.s.} \end{aligned} \tag{8.12}$$

Therefore, we obtain from (8.12) that for any vector u of $\mathbb{R}^2$,

$$\limsup_{n\to\infty}\frac{1}{2n^3\log\log n}\langle u, M_n\rangle^2 = \frac{1}{12}||u||^2 \qquad \text{a.s.}$$

which leads to

$$\limsup_{n\to\infty}\frac{1}{2n^3\log\log n}||M_n||^2 = \frac{1}{6} \qquad \text{a.s.} \tag{8.13}$$

Finally, we deduce (3.9) from (8.13), which achieves the proof of Propostion 3.2. □

References

[1] A. Baxter and D. Zeilberger. The number of inversions and the major index of permutations are asymptotically joint-independently normal (second edition). arXiv:1004.1160, 2010.

[2] B. Bercu. On the convergence of moments in the almost sure central limit theorem for martingales with statistical applications. *Stochastic Process. Appl.*, 111(1):157–173, 2004.

[3] B. Bercu, M. Bonnefont, and A. Richou. Sharp large deviations and concentration inequalities for the number of descents in a random permutation. *J. Appl. Probab.*, To appear.

[4] W. Bryc, D. Minda, and S. Sethuraman. Large deviations for the leaves in some random trees. *Adv. in Appl. Probab.*, 41(3):845–873, 2009.

[5] L. Carlitz, D. P. Roselle, and R. A. Scoville. Permutations and sequences with repetitions by number of increases. *J. Combinatorial Theory*, 1:350–374, 1966.

[6] S. Chatterjee. A new method of normal approximation. *Ann. Probab.*, 36(4):1584–1610, 2008.

[7] S. Chatterjee and P. Diaconis. A central limit theorem for a new statistic on permutations. *Indian J. Pure Appl. Math.*, 48(4):561–573, 2017.

[8] A. Dembo and O. Zeitouni. *Large deviations techniques and applications*, volume 38 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 1998.

[9] M. Duflo. *Random iterative models*, volume 34 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1997.

[10] R. Durrett and S. I. Resnick. Functional limit theorems for dependent variables. *Ann. Probab.*, 6(5):829–846, 1978.

[11] D. Foata and M.-P. Schützenberger. Major index and inversion number of permutations. *Math. Nachr.*, 83:143–159, 1978.

[12] D. A. Freedman. Bernard Friedman's urn. *Ann. Math. Statist.*, 36:956–970, 1965.

[13] O. Garet. A central limit theorem for the number of descents and some urn models. *Markov Process. Related Fields*, 27(5):789–801, 2021.

[14] A. M. Garsia and I. Gessel. Permutation statistics and partitions. *Adv. in Math.*, 31(3):288–305, 1979.

[15] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, eighth edition, 2015. Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll.

[16] P. A. MacMahon. The Indices of Permutations and the Derivation Therefrom of Functions of a Single Variable Associated with the Permutations of any Assemblage of Objects. *Amer. J. Math.*, 35(3):281–322, 1913.

[17] P.-L. Méliot and A. Nikeghbali. Asymptotics of the major index of a random standard tableau. arXiv:2210.13022, 2022.
[18] F. W. Olver. *NIST handbook of mathematical functions hardback and CD-ROM*. Cambridge university press, 2010.
[19] A. Özdemir. Martingales and descent statistics. *Adv. in Appl. Math.*, 140:Paper No. 102395, 34, 2022.
[20] T. K. Petersen. Two-sided Eulerian numbers via balls in boxes. *Math. Mag.*, 86(3):159–176, 2013.
[21] D. P. Roselle. Coefficients associated with the expansion of certain products. *Proc. Amer. Math. Soc.*, 45:144–150, 1974.
[22] W. F. Stout. Maximal inequalities and the law of the iterated logarithm. *Ann. Probability*, 1:322–328, 1973.
[23] J. P. Swanson. On a theorem of Baxter and Zeilberger via a result of Roselle. *Ann. Comb.*, 26(1):87–95, 2022.
[24] D. Wood. The computation of polylogarithms. Technical Report 15-92*, University of Kent, Computing Laboratory, University of Kent, Canterbury, UK, June 1992.