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Limiting absorption principle for contractions ^{*}

Joachim Asch [†] ; Olivier Bourget [‡]

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Abstract

We establish limiting absorption principles for contractions on a Hilbert space. Our sufficient conditions are based on positive commutator estimates. We discuss the dynamical implications of this principle to the corresponding discrete-time semigroup and provide several applications. Notably to Toeplitz operators and contractive quantum walks.

1 Introduction

The limiting absorption principle states that there exists a topology in which the resolvent can be continuously extended to parts of the essential spectrum. It was originally developed for resolvents of selfadjoint Schrödinger operators [1] and is widely used to establish propagation properties of the associated strongly continuous group and perturbations thereof, see [23].

In particular it provides information on the absolutely continuous subspace and plays an important role in the proof of asymptotic completeness of the quantum mechanical N -body problem and in the study of the dynamics of embedded eigenvalues.

In the present contribution we are interested in the dynamics of non-isolated systems modeled by a discrete semigroup $(V^n)_{n \in \mathbb{N}}$ for a contraction V .

We consider \mathcal{H} a separable Hilbert space, its bounded operators $\mathcal{B}(\mathcal{H})$ and a contraction

$$V \in \mathcal{B}(\mathcal{H}), \quad \|V\| = 1.$$

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$1 \in \mathcal{B}(\mathcal{H})$ denotes the identity operator. By a limiting absorption principle for V with respect to a weight $W \in \mathcal{B}(\mathcal{H})$ we mean that :

$$\{z \in \mathbb{C}; |z| < 1\} =: \mathbb{D} \ni z \mapsto W(1 - zV^*)^{-1}W^*$$

extends to a norm continuous function on a suitable subset $\mathbf{D} \subset \overline{\mathbb{D}}$.

In particular

$$\sup_{z \in \mathbf{D}} \|W(1 - zV^*)^{-1}W^*\| < \infty.$$

For $\psi \in \overline{\text{Ran } W^*}$ this implies square summable decay of the correlations $\langle \psi, V^n \psi \rangle$, more precisely ψ is an element of

$$\mathcal{H}_{ac}(V) := \overline{\left\{ \psi \in \mathcal{H} : \exists C_\psi \geq 0, \sup_{\|\varphi\|=1} \sum_{n=0}^{\infty} |\langle \varphi, V^n \psi \rangle|^2 \leq C_\psi \right\}},$$

see Proposition 3.1. $\mathcal{H}_{ac}(V)$ is called the absolutely continuous subspace ; for unitary V , it is the space of vectors of absolutely continuous spectral measure.

While there exists a large body of literature concerning limiting absorption principles, let us just briefly mention some work which concerns the non selfadjoint case :

The absolutely continuous subspace for the generator of a continuous contraction semigroup was introduced by Davies [9], see also [14], and used in his non-unitary scattering theory, [10] see [11] for a recent development.

Limiting absorption principles in a non-selfadjoint settings has been developed by Royer [20, 21], see also [8] for interesting information.

Limiting absorption principles for unitary operators and the related propagation properties for the corresponding discrete group have been established in [2], [22], [3], see also Kato [15].

The theory of characteristic functions can be used to obtain complementary spectral information in the case of trace class perturbations of unitary operators, see [16].

1.1 Main results

We now state our conditions on the contraction V and our results. For the proofs we use the positive commutator method pioneered by Mourre [17], see also [6], which makes use of an escape observable i.e.: an unbounded selfadjoint operator which we consistently call A in the sequel.

Definition 1.1. Let A be a selfadjoint operator. For a bounded operator B denote $Ad_{e^{-iAt}}(B) := e^{-iAt} B e^{iAt}$ ($t \in \mathbb{R}$). We say:

1. $B \in C^k(A)$ for $k \in \mathbb{N}$ if $Ad_{e^{-iAt}}(B)$ is strongly C^k ;
2. $B \in \mathcal{C}^{1,1}(A)$ if:

$$\int_0^1 \|Ad_{e^{-iA\tau}}(B) + Ad_{e^{iA\tau}}(B) - 2B\| \frac{d\tau}{|\tau|^2} < \infty .$$

3. For $B \in C^1(A)$, define the commutator $[A, B] := ad_A(B) := i\partial_t Ad_{e^{-iAt}}(B)|_{t=0}$.

Remarks 1.2. 1. One has $C^2(A) \subset \mathcal{C}^{1,1}(A) \subset C^1(A)$ and $\mathcal{C}^{1,1}(A)$ is a $*$ -algebra, see [6].

2. If $U \in C^1(A)$ is unitary then $U^*AU - A$ extends from the domain of A to $\mathbb{B}(\mathcal{H})$ and $U^*AU - A = U^*ad_A(U)$, see [3].

3. For U unitary $U \in \mathcal{C}^{1,1}(A)$ is the minimal regularity assumption needed for proving the limiting absorption principle to hold using Mourre methods [3, 6].

Our first result is a global limiting absorption principle under the rather strong assumption of existence of a positive commutator. We use the notation $\langle A \rangle := (A^2 + 1)^{1/2}$.

Theorem 1.3. Let $V \in \mathcal{B}(\mathcal{H})$, $\|V\| = 1$. Assume: $V \in C^{1,1}(A)$ and there exists an $a_0 > 0$ such that $\Re(V^*ad_A V) \geq a_0\mathbb{I}$.

Then for $s > \frac{1}{2}$ the map $\mathbb{D} \ni z \mapsto \langle A \rangle^{-s}(1 - zV^*)^{-1}\langle A \rangle^{-s}$ extends continuously in the uniform topology to $\overline{\mathbb{D}}$; in particular

$$\sup_{z \in \mathbb{D}} \|\langle A \rangle^{-s}(1 - zV^*)^{-1}\langle A \rangle^{-s}\| < \infty \text{ and}$$

$$\overline{\text{Ran}\langle A \rangle^{-s}} \subset \mathcal{H}_{ac}(V) \cap \mathcal{H}_{ac}(V^*).$$

Remark 1.4. Remark that a contraction can always be decomposed in a direct sum of its unitary and completely non-unitary parts [19]. Theorem 1.3 and Theorem 1.9 below are known to hold for a unitary V_u whereas for a completely non-unitary V_{cnu} it is known that $\mathcal{H}_{ac}(V_{cnu}) \cap \mathcal{H}_{ac}(V_{cnu}^*) = \mathcal{H}$, see Proposition 3.2 below. This decomposition and information on the unitary part may be difficult to obtain in applications, we will not make use of it in the present contribution.

For our second main result it is sufficient to assume positivity of the commutator locally in the spectrum of a unitary reference operator which we always call U .

We denote $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and use its identification with the unit circle $\partial\mathbb{D} = \exp(i\mathbb{T})$. Denote the spectral family of U by $E(\Theta)$, where Θ belongs to the Borel sets of \mathbb{T} . Thus for bounded Borel functions Φ on $\partial\mathbb{D}$, we have

$$\Phi(U) = \int_{\mathbb{T}} \Phi(e^{i\theta}) dE(\theta).$$

Definition 1.5. *Let $U \in C^1(A)$. We say that:*

U satisfies a Mourre estimate w.r.t. A on the Borel set Θ if there exist $a > 0$ and a compact operator K such that

$$E(\Theta)(UAU^* - A)E(\Theta) \geq aE(\Theta) + K, \quad (1)$$

The Mourre estimate is called strict if $K = 0$.

For an operator B denote $|B| := \sqrt{B^*B}$. We assume

Hypotheses (H). *For a selfadjoint operator A ,*

(H1): there exists a unitary operator U , $U \in C^1(A)$ such that a Mourre estimate (1) holds on an open subset $\Theta \subset \mathbb{T}$;

(H2): $V = PUQ$ for P, Q such that $0 \leq P \leq 1$, $0 \leq Q \leq 1$;

(H3): $V \in C^{1,1}(A)$;

(H4): Let a be the Mourre constant defined in (1). For $W = UV^$ and for $W = U^*V$, it holds: there exists a compact selfadjoint operator K_W such that*

$$\|iad_A(\Im(W)) - K_W\| < a \quad (2)$$

and there exists $\alpha > 0$, such that

$$2\Re(1 - W) - (1 + \alpha)|1 - W|^2 \geq 0. \quad (3)$$

Remarks 1.6. *Hypothesis (H4) is technical and used in Section 4.3.2.*

1. *(H4) is not a restriction if in (H2) $P = 1$ or $Q = 1$ c.f. Remark 2.2 below.*
2. *(H4) is not a restriction if in (H2) $P = Q$ and P is an orthogonal projection such that $[U, P]$ is compact, c.f. Remark 2.5 below.*

Concerning eigenvalues on the unit circle, we will prove

Proposition 1.7. *Assume (H1), (H2). Then*

1. *For $\mu \in \partial\mathbb{D}, \psi \in \mathcal{H} \setminus \{0\}$: $V\psi = \mu\psi \implies U\psi = \mu\psi$.*
2. *If a strict Mourre estimate (1) holds with $K = 0$ in Θ , then V has no eigenvalues in $e^{i\Theta}$.*

We now state our second main result, a local limiting absorption principle in the subset Θ of the spectrum of U where the Mourre estimate (1) holds:

We denote by $\mathcal{E}(B)$ the set of eigenvalues of an operator B .

Theorem 1.8. *Assume (H1 – 4). Then for any $s > 1/2$ the map*

$z \mapsto \langle A \rangle^{-s} (1 - zV^)^{-1} \langle A \rangle^{-s}$ extends continuously from \mathbb{D} to $\mathbb{D} \cup e^{i\Theta} \setminus \mathcal{E}(U)$ in the operator norm topology; in particular*

$$\sup_{z \in [0,1] \cdot e^{i\Theta_0}} \|\langle A \rangle^{-s} (1 - zV^*)^{-1} \langle A \rangle^{-s}\| < \infty$$

for any closed set Θ_0 such that $e^{i\Theta_0} \subset e^{i\Theta} \setminus \mathcal{E}(U)$

Our third version of the limiting absorption principle is :

Theorem 1.9. *Assume (H1 – 4). Then for any $s > 1/2$ and any closed set Θ_0 such that $e^{i\Theta_0} \subset e^{i\Theta} \setminus \mathcal{E}(U)$, any $\Phi \in C^\infty(\partial\mathbb{D}; \mathbb{R})$ supported on $e^{i\Theta_0}$:*

$$\sup_{z \in \mathbb{D}} \|\langle A \rangle^{-s} \Phi(U) (1 - zV^*)^{-1} \Phi(U) \langle A \rangle^{-s}\| < \infty \text{ and}$$

$$\overline{\text{Ran } \Phi(U) \langle A \rangle^{-s}} \subset \mathcal{H}_{ac}(V) \cap \mathcal{H}_{ac}(V^*).$$

We will illustrate our abstract results in Section 2 through various examples. In particular, we discuss the role played by the hypotheses. The following sections are dedicated to the proofs. In Section 3, we prove Proposition 1.7 and relate the limiting absorption principle to the control of the absolutely continuous subspace as stated in Theorems 1.3 and 1.9. The proof of the limiting absorption principles is developed in Section 4. We start with some auxiliary results on Mourre inequalities in Section 4.1. The proof of our limiting absorption principle under the $\mathcal{C}^{1,1}(A)$ regularity condition, which is optimal on the scale of $\mathcal{C}^{s,p}(A)$ spaces as developed in [6], requires the technical developments of Section 4.2. In Section 4.3 we then proceed with some a priori estimates on a weighted version of a suitably deformed resolvent. In Section 4.4, we establish some differential inequalities on this weighted deformed resolvent, which, once combined with the a priori estimates, allows to conclude the proofs of Theorems 1.3 and 1.8. Some complementary aspects related to the proof of Theorem 1.9 are developed in Section 5.

2 Applications and discussion of the hypotheses

We will use freely:

Remark 2.1. $B \in C^1(A)$ if and only if the sesquilinear form

$$\mathcal{D}(A) \times \mathcal{D}(A) \ni (\varphi, \psi) \mapsto \langle A\varphi, B\psi \rangle - \langle \varphi, BA\psi \rangle$$

extends continuously to $\mathcal{H} \times \mathcal{H}$. In this case, the bounded operator associated to its extension is $\text{ad}_A(B)$.

2.1 Fundamental example

We illustrate Theorem 1.9 with the rather basic but instructive example of a specific rank 1 perturbation of the shift operator.

With the normalized Lebesgue measure $d\ell$, let $\mathcal{H} := L^2(\partial\mathbb{D}, d\ell)$, $U\psi(z) := z\psi(z)$, $Q\psi(z) := \psi(z) - \int_{\partial\mathbb{D}} \psi d\ell$, $V := UQ$. Remark that

$$Vz^n = \begin{cases} z^{n+1} & n \neq 0 \\ 0 & n = 0 \end{cases}$$

so $\ker(V) = \text{span}\{z^0\}$ and on $\ker V^\perp$ the contraction decouples to the forward shift and a unitary equivalent to the backward shift. In particular

$$\text{spectrum}(V) = \overline{\mathbb{D}} \quad \text{and} \quad \mathcal{E}(V) = \mathbb{D}.$$

While one can see explicitly from the definition of V that $\mathcal{H}_{ac}(V) \cap \mathcal{H}_{ac}(V^*) = \mathcal{H}$, our hypothesis are satisfied and we can apply our theorem. Indeed, for

$$A := z\partial_z \text{ on } D(A) := \left\{ \sum_{n \in \mathbb{Z}} a_n z^n; \sum_{n \in \mathbb{Z}} (1 + n^2) |a_n|^2 < \infty \right\}$$

it holds

$$e^{-iAt} U e^{iAt} = e^{-it} U, \quad e^{-iAt} Q e^{iAt} = Q,$$

in particular U and Q are in $C^\infty(A)$ and $(U^*AU - A) = 1$ so a strict Mourre estimate (1) holds with constant $a = 1$ on any measurable subset Θ .

Concerning hypothesis (H4), remark that $V^*U = Q$, $UV^* = UQU^*$, thus for $W \in \{V^*U, UV^*\}$: $\Im(W) = 0$ so (2) always holds with $K_W = 0$. Also in both cases $0 \leq \Re(W) \leq 1$, thus $0 \leq 1 - \Re(W) \leq 1$, which implies $1 - \Re(W) \geq |1 - \Re(W)|^2$ and (3) holds with $\alpha = 1$.

So (H1–4) are satisfied and we can apply Theorem 1.9 for $\Phi = id$ and conclude that for $s > \frac{1}{2}$, $\mathcal{H}_s := \left\{ \sum_{n \in \mathbb{Z}} a_n z^n; \sum_{n \in \mathbb{Z}} (1 + n^2)^s |a_n|^2 < \infty \right\}$:

$$\overline{\mathcal{H}_s} = \mathcal{H} = \mathcal{H}_{ac}(V) \cap \mathcal{H}_{ac}(V^*).$$

Remark 2.2. *It follows from this argument that in general if $P = 1$, i.e. $V = UQ$, then Hypothesis (H2) implies (H4). The same is true if V is of the form $V = PU$.*

Remark 2.3. *The unilateral forward shift $V\psi(z) := z\psi(z)$ on the Hardy space*

$$\mathcal{H} := \left\{ \sum_{n \in \mathbb{N}} a_n z^n; \sum_{n \in \mathbb{N}} |a_n|^2 < \infty \right\}$$

is an example for Theorem 1.3.

*Indeed, with $A := z\partial_z$ on $D(A) := \left\{ \sum_{n \in \mathbb{N}} a_n z^n; \sum_{n \in \mathbb{Z}} (1 + n^2) |a_n|^2 < \infty \right\}$, it holds $V \in C^\infty(A)$ and $V^*ad_A(V) = 1$, so 1.3 applies.*

2.2 Contractive convolution operators, quantum walks

In $\mathcal{H} = \ell^2(\mathbb{Z}^d; \mathbb{C}^{d'})$ consider $V = UP$ for $U = C_0C_1$, $P = P_0P_1P_0$ where C_1, P_1 are convolution operators and P_0, C_0 matrix valued multiplication operators with C_j unitary and $0 \leq P_j \leq 1$.

More specifically, denote $U(d') \subset \mathbb{M}_{d', d'}$ the unitary group and $\mathcal{P}_{01}(d') = \{Q \in \mathbb{M}_{d', d'}; 0 \leq Q \leq 1\}$. For the symbols $f \in L^\infty(\mathbb{T}^d, U(d'))$, $g \in \ell^\infty(\mathbb{Z}^d, U(d'))$, and $p \in L^\infty(\mathbb{T}^d, \mathcal{P}_{01}(d'))$, $q \in \ell^\infty(\mathbb{Z}^d, \mathcal{P}_{01}(d'))$ consider

$$C_1\psi(x) = \sum_{y \in \mathbb{Z}} \hat{f}(x - y)\psi(y), \quad C_0\psi(x) = g(x)\psi(x)$$

$$P_1\psi(x) = \sum_{y \in \mathbb{Z}} \hat{p}(x - y)\psi(y), \quad P_0\psi(x) = q(x)\psi(x).$$

Suppose that

$$f \text{ is analytic and } \int_1^\infty \sup_{r \leq |x| \leq 2r} \|g(x) - 1\| dr < \infty.$$

$$p \in C^3(\mathbb{T}^d, \mathcal{P}_{01}(d')) \text{ and } \int_1^\infty \sup_{r \leq |x| \leq 2r} \|q(x) - 1\| dr < \infty.$$

Remark that contractive quantum walks with asymptotically periodic coins and local absorption are a particular example of the above [5, 13].

We argue that hypotheses (H1 – 4) are satisfied:

Denote the selfadjoint operator $X\psi(x) := x\psi(x)$ defined on $D(X) = \{\psi \in \mathcal{H}; \sum_x (1+x^2)|\psi(x)|^2 < \infty\}$. Then there exists a selfadjoint propagation observable A such that $U \in \mathcal{C}^{1,1}(A)$ and a discrete subset $\tau_f \subset \mathbb{T}$ such that a Mourre estimate (1) holds on every open Θ such that $\bar{\Theta} \subset \mathbb{T} \setminus \tau_f$.

Furthermore A is relatively bounded, $A^s \langle X \rangle^{-s} \in \mathcal{B}(\mathcal{H})$, $s \in \{1, 2\}$ and $P \in \mathcal{C}^{1,1}(A)$, we again refer to [5, section 3] for proofs.

Also $0 \leq P \leq 1$; Hypotheses (H1) – (H3) are satisfied and by Remark 2.2 also (H4). Thus Theorem 1.9 applies.

2.3 Toeplitz operators

Let $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{C})$, $V = PUP$ with P the multiplication by the characteristic function of the half line: $P\psi(x) = \chi(x \geq 0)\psi(x)$, and U a unitary convolution operator with symbol $f \in L^\infty(\mathbb{T}, U(1))$, $U\psi(x) := \sum_{y \in \mathbb{Z}} \widehat{f}(x-y)\psi(y)$.

Remark that the restriction of V to $\text{Ran } P$ is equivalent to the Toeplitz operator with symbol f .

We now show that Hypotheses (H1) – (H4) are satisfied if f is smooth enough and Theorem 1.9 applies.

If $f \in C^3(\mathbb{T}, U(1))$ and f' has no zeros in $\Theta \subset \mathbb{T}$, then there exists a propagation observable A such that $U \in \mathcal{C}^{1,1}(A)$ and such that a Mourre estimate (1) holds in Θ , see [5, section 3] and [4].

In order to prove smoothness of the Hardy projection P , we recall the construction of A .

Denote L_g be the convolution operator with symbol g on \mathcal{H} . For $g := if\bar{f}'$ the selfadjoint operator $A := \frac{1}{2}(L_gX + XL_g) = L_gX + \frac{i}{2}L_{g'}$ is defined by extension from $D(X)$ which is a core for A .

It holds

Proposition 2.4. *If $f \in C^6(\mathbb{T}, U(1))$ then $P \in \mathcal{C}^{1,1}(A)$.*

Proof. We show first that $\langle X \rangle^\alpha \text{ad}_A P$ is a bounded operator for $\alpha \in \{0, 1\}$.

Let $P^\perp := 1 - P$. For a symbol $h \in C^4(\mathbb{T}, \mathbb{C})$, we can estimate the matrix elements of $P^\perp L_h P$ on the canonical basis of $\ell^2(\mathbb{Z})$:

$$\begin{aligned} |\langle e_k, P^\perp L_h P e_l \rangle| &= \chi(k < 0)\chi(l \geq 0) \left(|\langle e_k, L_h e_l \rangle| = |\widehat{h}_{l+|k|}| \right) \\ &\leq \frac{\text{const}}{(|k| + |l|)^4} \leq \frac{\text{const}}{\langle k \rangle^2 \langle l \rangle^2} \end{aligned}$$

For $\alpha, \beta \in \{0, 1\}$ it follows

$$|\langle e_k, X^\alpha P^\perp L_h P X^\beta e_l \rangle| \leq \frac{\text{const} \langle k \rangle^\alpha \langle l \rangle^\beta}{\langle k \rangle^2 \langle l \rangle^2}$$

which implies that the Hilbert-Schmidt norm of $X^\alpha P^\perp L_h P X^\beta$ is finite. It holds:

$$P^\perp A P = P^\perp L_g P X + \frac{i}{2} P^\perp L_{g'} P$$

with $g, g' \in C^4(\mathbb{T}, \mathbb{C})$. It follows that $\text{ad}_A P = P^\perp A P - P A P^\perp$ and $X \text{ad}_A P$ are bounded operators.

For $\chi \in C^\infty(\mathbb{R}; [0, \infty))$ supported on an interval (a, b) , $0 < a < b$ one has :

$$\begin{aligned} \int_1^\infty \|\chi(\langle X \rangle / r) \text{ad}_A P\| \frac{dr}{r} &\leq \int_1^\infty \|\chi(\langle X \rangle / r) \langle X \rangle^{-1}\| \|\langle X \rangle \text{ad}_A P\| \frac{dr}{r} \\ &\leq \|\langle X \rangle \text{ad}_A P\| \int_1^\infty \frac{dr}{ar^2} < \infty. \end{aligned}$$

In view of the criterion provided by Theorem 7.5.8 [6] (also Theorem 3.2 in [4]), we deduce that $P \in \mathcal{C}^{1,1}(A)$. \square

It is a corollary that $[U, P]$ is Hilbert Schmidt for $f \in C^4(\mathbb{T}, \mathbb{C})$ and compact for continuous f .

Remark 2.5. *Concerning hypothesis (H4). We have $UV^* = UPU^*P$, so $2i\Im UV^* = P^\perp[U, P]U^*P - PU[P, U^*]P^\perp$.*

$[U, P]$ compact and $U, P \in \mathcal{C}^{1,1}(A)$ so (2) holds for $W = UV^$ with $K_W = i\text{ad}_A \Im W$ for any $a > 0$. On the other hand,*

$$\begin{aligned} 2\Re(1 - UV^*) &= 2(1 - PUPU^*P) - P^\perp UPU^*P - PUPU^*P^\perp \\ |1 - UV^*|^2 &= 1 - PUPU^*P - P^\perp UPU^*P - PUPU^*P^\perp \end{aligned}$$

Now observe

$$\begin{aligned} 2\Re(1 - UV^*) - \frac{3}{2}|1 - UV^*|^2 &= \frac{1}{2}(1 - PUPU^*P) + \frac{1}{2}(P^\perp UPU^*P + PUPU^*P^\perp) \\ &= \frac{1}{2}(P^\perp + PUP^\perp U^*P) - \frac{1}{2}(P^\perp U P^\perp U^*P + PUP^\perp U^*P^\perp) \\ &= \frac{1}{2}|P^\perp - UP^\perp U^*P|^2. \end{aligned}$$

which is positive so (3) holds with $\alpha = 1/2$. Similarly one shows that (2) and (3) hold for $W = U^*V = U^*PUP$ and we conclude that the hypotheses are satisfied and we can apply our theorem.

3 Dynamics

To prove the dynamical implications we extend the techniques known for unitary operators.

3.1 Absolutely continuous subspace

We prove that the limiting absorption principle implies that certain vectors belong to \mathcal{H}_{ac} which proves the corresponding assertions in Theorems 1.3 and 1.9.

Proposition 3.1. *For a contraction V and $W \in \mathcal{B}(\mathcal{H})$ it holds: if*

$$\sup_{z \in \mathbb{D}} \|W(1 - zV^*)^{-1}W^*\| < \infty,$$

then

$$\overline{\text{Ran}W^*} \subset \mathcal{H}_{ac}(V) \cap \mathcal{H}_{ac}(V^*).$$

Proof. The result is known for the case where V is unitary, [2, Theorem 2]. Consider the operator

$$P_z(V) := 2\Re((1 - zV^*)^{-1}) - 1 = (1 - zV^*)^{-1}(1 - |z|^2V^*V)(1 - zV^*)^{-1*}.$$

The assumption implies

$$c := \sup_{z \in \mathbb{D}} \|WP_z(V)W^*\| < \infty.$$

Let \widehat{V} be a unitary dilation on a Hilbert space $\widehat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}^\perp$ as described in [19].

For the orthogonal projection $P_{\mathcal{H}}$ onto \mathcal{H} , it then holds for all $\psi \in \mathcal{H}$:

$$P_{\mathcal{H}}(\widehat{V}^n\psi \oplus 0) = V^n\psi \text{ and } P_{\mathcal{H}}(\widehat{V}^{*n}\psi \oplus 0) = V^{*n}\psi$$

and thus $P_{\mathcal{H}}(P_z(\widehat{V})\psi \oplus 0) = P_z(V)\psi$. Define $\mathbf{W} := W \oplus 0 \in \mathcal{B}(\widehat{\mathcal{H}})$. Then, for $\widehat{\varphi}, \widehat{\psi} \in \widehat{\mathcal{H}}$,

$$\left| \left\langle P_z(\widehat{V})\mathbf{W}^*\widehat{\varphi}, \mathbf{W}^*\widehat{\psi} \right\rangle_{\widehat{\mathcal{H}}} \right| = |\langle P_z(V)W^*\varphi, W^*\psi \rangle| \leq c\|\varphi\|\|\psi\| \leq c\|\widehat{\varphi}\|_{\widehat{\mathcal{H}}}\|\widehat{\psi}\|_{\widehat{\mathcal{H}}}$$

so

$$\sup_{z \in \mathbb{D}} \|\mathbf{W}P_z(V)\mathbf{W}^*\| = c.$$

Now by [2, Theorem 2]

$$\sum_{n \in \mathbb{Z}} \|\mathbf{W}\widehat{V}^n\widehat{\varphi}\|_{\widehat{\mathcal{H}}}^2 \leq c\|\widehat{\varphi}\|_{\widehat{\mathcal{H}}}^2.$$

Taking $\widehat{\varphi} = \varphi \oplus 0$ for any $\varphi \in \mathcal{H}$, it follows

$$\sum_{n=0}^{\infty} \|WV^{*n}\varphi\|^2 + \sum_{n=0}^{\infty} \|WV^n\varphi\|^2 \leq c\|\varphi\|^2.$$

Now for $\psi = W^*\eta$

$$|\langle \varphi, V^n\psi \rangle|^2 = |\langle WV^{*n}\varphi, \eta \rangle|^2 \leq \|WV^{*n}\varphi\|^2\|\eta\|^2,$$

which implies $\psi \in \mathcal{H}_{ac}(V)$. Repeating the argument with V^* replacing V , we conclude that $\text{Ran}(W^*) \subset \mathcal{H}_{ac}(V) \cap \mathcal{H}_{ac}(V^*)$, which finishes the proof as the latter set is closed. \square

The result on completely non-unitary contractions mentioned in Remark 1.4 follows from the well known result of Nagy-Foias [18] by the same reasoning :

Proposition 3.2. *Let V be completely non-unitary. Then $\mathcal{H}_{ac}(V) \cap \mathcal{H}_{ac}(V^*) = \mathcal{H}$.*

Proof. Let \widehat{V} be the minimal unitary dilation on a Hilbert space $\widehat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}^\perp$ as described in [19]. For the orthogonal projection $P_{\mathcal{H}}$ onto \mathcal{H} , it then holds for all $\psi \in \mathcal{H}$: $P_{\mathcal{H}}(\widehat{V}^n \psi \oplus 0) = V^n \psi$ and $P_{\mathcal{H}}(\widehat{V}^{*n} \psi \oplus 0) = V^{*n} \psi$. The spectrum of \widehat{V} is absolutely continuous, see [18], which is equivalent to $\widehat{\mathcal{H}} = \mathcal{H}_{ac}(\widehat{V})$.

For $\widehat{\varphi} = \varphi \oplus 0$ $\widehat{\psi} = \psi \oplus 0$ one has

$$\sum_{n \in \mathbb{Z}} \left| \langle \widehat{\varphi}, \widehat{V}^n \widehat{\psi} \rangle \right|^2 = \sum_{n \geq 0} |\langle \varphi, V^n \psi \rangle|^2 + \sum_{n > 0} |\langle \varphi, V^{*n} \psi \rangle|^2$$

from which the assertion follows. \square

3.2 Eigenvalues, proof of Proposition 1.7

Lemma 3.3. *For two bounded operators A, B suppose that $\|A\| \leq 1$, $-1 \leq B \leq 1$ and $\ker(1 + B) = \{0\}$. Then for $\mu \in \partial\mathbb{D}$, $\varphi \in \mathcal{H}$,*

$$AB\varphi = \mu\varphi \Rightarrow A\varphi = \mu\varphi \text{ and } B\varphi = \varphi.$$

Proof. First observe that:

$$0 = \|\varphi\|^2 - \|AB\varphi\|^2 = \langle B\varphi, (1 - A^*A)B\varphi \rangle + \langle \varphi, (1 - B^2)\varphi \rangle.$$

Both terms on the right hand side are non negative. It follows $\varphi \in \ker(\sqrt{1 - B^2})$, which implies

$$B\varphi = \varphi \text{ and } AB\varphi = A\varphi = \mu\varphi \text{ as } -1 \notin \mathcal{E}(B).$$

\square

Now, we prove Proposition 1.7.

Proof. (of Proposition 1.7)

1. $V = PUQ$ with U unitary and $0 \leq P \leq 1, 0 \leq Q \leq 1$. We can apply lemma 3.3 to $A := PU$ and $B := Q$ and we get for $\mu \in \partial\mathbb{D}$,

$$V\psi = PUQ\psi = \mu\psi \Rightarrow PU\psi = U(U^*PU)\psi = \mu\psi \text{ and } Q\psi = \psi.$$

Observe that $0 \leq U^*PU \leq 1$ so we can apply Lemma 3.3 again to $A := U$ and $B := U^*PU$ to conclude that $U\psi = \mu\psi$ and in addition that

$$U^*PU\psi = \psi \text{ and thus } P\psi = \psi.$$

2. Suppose for $\mu \in e^{i\Theta}$, $V\psi = \mu\psi$ thus $U\phi = \mu\psi$ and

$$\langle \psi, E(\Theta)(U^*AU - A)E(\Theta)\psi \rangle = \langle \psi, (U^*AU - A)\psi \rangle$$

which was proven to equal zero in [3] Section 4.1. So $E(\Theta)(U^*AU - A)E(\Theta)$ cannot be positive.

□

4 Proof of Theorems 1.3 and 1.8

We will prove Theorems 1.3 and 1.8 in parallel. We dub 1.3 the global case and 1.8 the local case.

In the local case of Theorem 1.8, we will show that the maps $z \mapsto \langle A \rangle^{-s}(1 - zV^*)^{-1}\langle A \rangle^{-s}$ can be continuously extended to a neighborhood of any $e^{i\theta} \in e^{i\Theta} \setminus \mathcal{E}(U)$. In Section 4.1, we also have to rewrite the Mourre inequality in this neighborhood.

In Section 4.2, we show how Hypothesis (H3) can be translated into the existence of suitable approximations for V and $\text{ad}_A V$. The construction of a deformed resolvent for V , denoted by $G_\epsilon(z)$, is based on this approximation. At this point, the proofs of Theorems 1.3 and 1.8 differ, see Sections 4.3.1 and 4.3.2 respectively. For convenience, we group the hypotheses as follows:

- (Glo) for the set of Hypotheses (H1') and (H3),
where (H1') stands for: there exists $a_0 > 0$ such that $\Re(V^*ad_A V) \geq a_0\mathbb{I}$.
- (Loc) for the set of Hypotheses (H1), (H2), (H3) and (H4).

In both cases, we establish some a priori estimates on the deformed resolvent and on weighted versions of this deformed resolvent, denoted by $F_{s,\epsilon}(z)$, see (43).

Next, we develop Mourre's differential inequality strategy in Section 4.4. We deduce the continuous extension of the weighted resolvent at any point of $\partial\mathbb{D}$ under Assumption (Glo) and at any points $e^{i\theta} \in e^{i\Theta} \setminus \mathcal{E}(U)$ under Assumption (Loc) (Proposition 4.22).

For any $S_1 \subset [0, \infty)$ and any $S_2 \subset \mathbb{T}$, we write

$$S_1 \cdot e^{iS_2} = \{z \in \mathbb{C}, |z| \in S_1, \arg z \in S_2\}.$$

4.1 Reduction to a strict Mourre estimate

If a Mourre estimate holds in a set not containing any eigenvalues, then a strict Mourre estimate holds in an open neighborhood of each point of this set. This remains true if one adds a compact operator to the commutator.

More precisely,

Lemma 4.1. *Suppose (H1). Let $e^{i\theta} \in e^{i\Theta} \setminus \mathcal{E}(U)$, $0 < c_1 < a$ and \mathbf{K} a compact selfadjoint operator then there exists an open connected neighborhood Θ' of θ such that:*

$$\begin{aligned} E(\Theta')(U^*AU - A + \mathbf{K})E(\Theta') &\geq c_1 E(\Theta') \\ E(\Theta')(A - UAU^* + \mathbf{K})E(\Theta') &\geq c_1 E(\Theta'). \end{aligned}$$

Proof. Since $e^{i\theta} \in e^{i\Theta} \setminus \mathcal{E}(U)$ and K, \mathbf{K} are compact operators, we may find an open connected neighborhood Θ' containing θ such that $\|E(\Theta')(K + \mathbf{K})E(\Theta')\| \leq a - c_1$. \square

Proposition 4.2. *Suppose (H1), (H2) and (H4). Let $e^{i\theta} \in e^{i\Theta} \setminus \mathcal{E}(U)$. There exist $0 < a_0 < a$, $a_1 > 0$ and an open connected neighborhood Θ' of θ such that:*

$$\begin{aligned} (U^*AU - A) - iad_A \mathfrak{S}UV^* &\geq a_0 - a_1 E(\Theta')^\perp \\ (A - UAU^*) - iad_A \mathfrak{S}V^*U &\geq a_0 - a_1 E(\Theta')^\perp. \end{aligned}$$

Proof. Let $m < c_2 < c_1 < a$. where

$$m := \max \{ \|iad_A \mathfrak{S}(UV^*) - K_{UV^*}\|, \|iad_A \mathfrak{S}(V^*U) - K_{V^*U}\| \}.$$

First, we apply Lemma 4.1 and fix the neighborhood Θ' accordingly. Next, in view of Proposition 6.1, we deduce there exists $a_1 > 0$ such that:

$$(U^*AU - A) + K_{UV^*} \geq c_2 - a_1 E(\Theta')^\perp.$$

Writing

$$(U^*AU - A) - iad_A \mathfrak{S}UV^* = (U^*AU - A) - (iad_A \mathfrak{S}UV^* - K_{UV^*}) + K_{UV^*}$$

the first inequality follows with $a_0 = c_2 - m$ and the second equality analogously. \square

4.2 Properties of V

Now, we enumerate the properties which are implied by our hypothesis $V \in \mathcal{C}^{1,1}(A)$ and which are used in the proofs of Theorems 1.3 and 1.8. We refer to [6, Lemma 7.3.6] and [7] for a proof of the following results which are used in the proof the priori estimates and in the differential inequality procedure.

Proposition 4.3. *Let $V \in \mathcal{C}^{1,1}(A)$. Then, there exists a map $S \in C^1((0, 1); C^1(A))$ (equipped with the operator norm topology), such that for $B := \text{ad}_A S$, $B \in C^1((0, 1); C^1(A))$ and :*

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-1} \|S_\epsilon - V\| = 0,$$

$$\lim_{\epsilon \rightarrow 0} \|B_\epsilon - \text{ad}_A V\| = 0,$$

$$\sup_{\epsilon \in (0, \epsilon_0)} \|B_\epsilon\| < \infty \text{ for some } \epsilon_0 \in (0, 1).$$

In addition,

$$\int_0^1 \frac{\|\partial_\epsilon S_\epsilon\|}{\epsilon} d\epsilon + \int_0^1 \|\text{ad}_A B_\epsilon\| d\epsilon + \int_0^1 \|\partial_\epsilon B_\epsilon\| d\epsilon < \infty.$$

Remark 4.4. *In particular, there exists $C > 0$ so that $\|S_\epsilon - V\| \leq C\epsilon$ for any $\epsilon \in (0, 1)$; also the functions S , B and $\partial_\epsilon S$ extend continuously to $[0, 1)$ by setting $S_0 = V$, $B_0 = \text{ad}_A V$, so that $(\partial_\epsilon S)(0) = 0$.*

Corollary 4.5. *Let $V \in \mathcal{C}^{1,1}(A)$. For $\epsilon \in (0, 1)$ define the map*

$$\mathcal{Q}(\epsilon) = \partial_\epsilon S_\epsilon - \epsilon \partial_\epsilon B_\epsilon - \epsilon \text{ad}_A B_\epsilon. \quad (4)$$

Then,

$$\epsilon \mapsto \frac{\|\mathcal{Q}(\epsilon)\|}{\epsilon} \in L^1((0, 1)).$$

Remark 4.6. *If $V \in C^2(A)$, we can define for all ϵ : $S_\epsilon \equiv V$ and $B_\epsilon \equiv \text{ad}_A V$.*

4.3 Deformed resolvents and first estimates

In this section, we define the deformed resolvent $(1 - zV^*)^{-1}$, then establish its invertibility and finally prove some a priori estimates, see Proposition 4.16 below.

In view of Section 4.2, we use the following shortcuts :

$$\begin{aligned} q_\epsilon &:= \frac{S_\epsilon^* - V^*}{\epsilon} - B_\epsilon^* - \text{ad}_A V^*, \\ Q_\epsilon &:= \frac{S_\epsilon^* - V^*}{\epsilon} - B_\epsilon^*, \end{aligned} \quad (5)$$

and $q_0 := 0$, $Q_0 := \text{ad}_A V^*$. Note that $\lim_{\epsilon \rightarrow 0^+} \|q_\epsilon\| = 0$.

For $\epsilon \in [0, \epsilon_0)$, $z \in \overline{\mathbb{D}}$, define

$$\begin{aligned} V_\epsilon &:= S_\epsilon - \epsilon B_\epsilon \\ T_\epsilon(z) &:= 1 - zV_\epsilon^* \\ &= T_0(z) - z\epsilon Q_\epsilon, \end{aligned} \quad (6)$$

To sum up:

Lemma 4.7. *Suppose (H3), then with*

$$b := \sup_{\epsilon \in [0, \epsilon_0)} \|Q_\epsilon\| < \infty, \quad (7)$$

for any $\epsilon \in [0, \epsilon_0)$ and any $z \in \overline{\mathbb{D}}$ it holds

$$\|T_\epsilon(z) - T_0(z)\| \leq b\epsilon. \quad (8)$$

4.3.1 First estimates supposing (Glo)

Proposition 4.8. *Suppose (Glo). There exists $0 < \epsilon_1 < \epsilon_0$ such that for any $\epsilon \in [0, \epsilon_1)$, $z \in \overline{\mathbb{D}} \setminus \{0\}$,*

$$T_\epsilon(z)^* + \bar{z}V_\epsilon T_\epsilon(z) \geq d(\epsilon, z) \quad (9)$$

$$T_\epsilon(z) + zV_\epsilon^* T_\epsilon(z)^* \geq d(\epsilon, z), \quad (10)$$

where

$$d(\epsilon, z) := 1 - |z|^2 + a_0\epsilon|z|^2. \quad (11)$$

Proof. Fix $z \in \overline{\mathbb{D}} \setminus \{0\}$. Using the Mourre estimate and the contraction property for V , we have:

$$\begin{aligned} T_\epsilon(z)^* + \bar{z}V_\epsilon T_\epsilon(z) &= 1 - |z|^2 V_\epsilon V_\epsilon^* = 1 - |z|^2 (V + (V_\epsilon - V)) (V^* + (V_\epsilon - V)^*) \\ &= 1 - |z|^2 |V^*|^2 - 2|z|^2 \Re(V(V_\epsilon - V)^*) - |z|^2 |V_\epsilon^* - V^*|^2 \\ &\geq 1 - |z|^2 - 2|z|^2 \Re(V(V_\epsilon - V)^*) - |z|^2 |V_\epsilon^* - V^*|^2 \end{aligned}$$

Mind that: $(V_\epsilon - V)^* = \epsilon(\text{ad}_A V^* + q_\epsilon) = \epsilon Q_\epsilon$, so $|(V_\epsilon - V)^*|^2 = \epsilon^2 |Q_\epsilon|^2$ and

$$\Re(V(V_\epsilon - V)^*) = \epsilon \Re(V \text{ad}_A V^*) + \epsilon \Re(V q_\epsilon).$$

It follows :

$$T_\epsilon(z)^* + \bar{z}V_\epsilon T_\epsilon(z) \geq 1 - |z|^2 - 2|z|^2 \epsilon \Re(V \text{ad}_A V^*) - 2|z|^2 \epsilon \Re(V q_\epsilon) - |z|^2 \epsilon^2 |Q_\epsilon|^2.$$

We observe that: $0 \leq |Q_\epsilon|^2 \leq b^2$ and $-\|q_\epsilon\| \leq \Re(V q_\epsilon) \leq \|q_\epsilon\|$. This yields:

$$T_\epsilon(z)^* + \bar{z}V_\epsilon T_\epsilon(z) \geq d(|z|, \epsilon) + |z|^2 \epsilon (a_0 - b^2 \epsilon - 2\|q_\epsilon\|).$$

Pick $0 < \epsilon_1 < \epsilon_0$ such that

$$b^2 \epsilon_1 + 2 \sup_{\epsilon \in [0, \epsilon_1]} \|q_\epsilon\| \leq a_0, \quad (12)$$

and (9) follows. The proof of (10) can be done analogously . \square

For d and ϵ_1 respectively defined in (11) and (12), we define:

$$\Omega_{\mathbb{T}} = \{(\epsilon, z) \in [0, \epsilon_1] \times \overline{\mathbb{D}} \setminus \{0\}; d(\epsilon, z) > 0\} \quad (13)$$

We deduce that:

Proposition 4.9. *Suppose (Glo). For $(\epsilon, z) \in \Omega_{\mathbb{T}}$, the operator $T_\epsilon(z)$ is boundedly invertible.*

Proof. Fix ϵ, z as stated. Making explicit (9) and (10) in terms of quadratic forms shows that $T_\epsilon(z)$ and $T_\epsilon(z)^*$ are injective from \mathcal{H} into itself and has closed range. Since

$$\overline{\text{Ran } T_\epsilon(z)} = (\text{Ker } T_\epsilon(z)^*)^\perp,$$

we deduce that $T_\epsilon(z)$ and $T_\epsilon(z)^*$ are actually linear and bijective hence boundedly invertible by the Inverse Mapping Theorem. \square

4.3.2 First estimates supposing (Loc)

We now look for an analog of (9) and (10) under Assumptions (Loc).

In this subsection, we fix $\theta \in \Theta \setminus \mathcal{E}(U)$ and look at the local properties of the resolvent on some open subset Θ_0 , such that $\overline{\Theta_0} \subset \Theta'$, where the open set Θ' has been defined in Proposition 4.2.

We introduce some shortcuts:

$$\begin{aligned} R &= UV^* \\ L &= V^*U \\ \overline{R} &= 1 - R \\ \overline{L} &= 1 - L. \end{aligned} \quad (14)$$

We also set: $E = E(\Theta')$.

Lemma 4.10. *Suppose (H1), (H2), (H3). Fix an open neighborhood Θ_0 of θ , such that $\overline{\Theta_0} \subset \Theta'$ and denote $d_0 := \text{dist}(\overline{\Theta_0}, \mathbb{T} \setminus \Theta') > 0$. Then, we have for any $\epsilon \in [0, \epsilon_0)$, $z \in (0, 1] \cdot e^{i\overline{\Theta_0}}$,*

$$E^\perp \leq \frac{3\pi^2}{4d_0^2|z|} (|T_\epsilon(z)|^2 + |z|^2|\overline{R}|^2 + b^2 \epsilon^2|z|^2), \quad (15)$$

$$E^\perp \leq \frac{3\pi^2}{4d_0^2|z|} (|T_\epsilon(z)^*|^2 + |z|^2|\overline{L}^*|^2 + b^2 \epsilon^2|z|^2), \quad (16)$$

with b defined by (7).

Proof. For $\epsilon \in [0, \epsilon_0)$, we use shortcuts

$$\begin{aligned} R_\epsilon &:= UV_\epsilon^*, \\ &= R + \epsilon UQ_\epsilon, \\ L_\epsilon &:= V_\epsilon^*U, \\ &= L + \epsilon Q_\epsilon U. \end{aligned} \tag{17}$$

For $\epsilon \in [0, \epsilon_0)$, $z \in (0, 1] \cdot e^{i\overline{\Theta}_0}$, we have

$$\begin{aligned} E^\perp &= (1 - zU^*)^{-1}E^\perp (T_\epsilon(z) - zU^*\overline{R} + zU^*(R_\epsilon - R)) \\ &= (1 - zU^*)^{-1}E^\perp (T_\epsilon(z) - zU^*\overline{R} + z\epsilon Q_\epsilon), \end{aligned}$$

which entails the first estimates. We conclude analogously for the second one. \square

In Lemma 4.11 and Proposition 4.13, we show how the local estimates obtained in Proposition 4.2 can be used to derive some estimates on $\Re(V\text{ad}_A V^*)$ and $\Re(V^*\text{ad}_A V)$ respectively.

Lemma 4.11. *Suppose (H1), (H2). With*

$$\begin{aligned} C &:= (A - UAU^*) \\ C_U &:= U^*CU = (U^*AU - A) \end{aligned}$$

it holds:

$$2\Re(V\text{ad}_A V^*) = -2C - 2\overline{R^*}C\overline{R} + 4\Re(\overline{R^*}C) + 2\Re(R^*\text{ad}_A R) \tag{18}$$

$$2\Re(V^*\text{ad}_A V) = 2C_U + 2\overline{L}C_U\overline{L^*} - 4\Re(C_U\overline{L^*}) + 2\Re(L\text{ad}_A L^*). \tag{19}$$

Proof. Since U and V belong to $C^1(A)$, R and L also belong to $C^1(A)$. Also, $V \in C^1(A)$ iff $V^* \in C^1(A)$ and $\text{ad}_A V^* = -(\text{ad}_A V)^*$. So, $\text{ad}_A V^* = \text{ad}_A(U^*R) = U^*(\text{ad}_A R) + (\text{ad}_A U^*)R$ and

$$2\Re(V\text{ad}_A V^*) = 2\Re(R^*\text{ad}_A R) - 2\Re(R^*CR),$$

from which (18) follows. Similarly, (19) follows from $\text{ad}_A V = \text{ad}_A(UL^*) = U(\text{ad}_A L^*) + (\text{ad}_A U)L^*$ and

$$2\Re(V^*\text{ad}_A V) = 2\Re(L\text{ad}_A L^*) + 2\Re(LC_U L^*).$$

\square

Remark 4.12. We will use several times the following estimates. Let $A, B \in \mathcal{B}(\mathcal{H})$. Then, for any $p > 0$ and any $\varphi \in \mathcal{H}$,

$$|\langle \varphi, 2\Re(A^*B)\varphi \rangle| \leq \|B\| \left(\frac{\|A\|\|\varphi\|^2}{p} + p\|\varphi\|^2 \right). \quad (20)$$

We also have that

$$|\langle \varphi, A^*BA\varphi \rangle| \leq \|B\| \|A\|\|\varphi\|^2. \quad (21)$$

Proposition 4.13. Suppose (Loc). There exists $c_0 > 0$, such that:

$$\begin{aligned} -2\Re(V\text{ad}_A V^*) &\geq \frac{3a_0}{2} - 2a_1 E^\perp - c_0 |\overline{R}|^2 \\ 2\Re(V^*\text{ad}_A V) &\geq \frac{3a_0}{2} - 2a_1 E^\perp - c_0 |\overline{L}^*|^2. \end{aligned}$$

Proof. Once noted that $2\Re(\text{ad}_A R) = 2i\text{ad}_A \Im R$, Lemma 4.11 writes:

$$-2\Re(V\text{ad}_A V^*) = 2(C - i\text{ad}_A \Im R) + 2\overline{R}^* C \overline{R} - 4\Re(\overline{R}^* C) + 2\Re(\overline{R}^*(\text{ad}_A R)).$$

Applying inequality (20) to the couples $(A, B) = (\overline{R}, C)$ and $(\overline{R}, \text{ad}_A R)$ yields for any $p > 0$,

$$\begin{aligned} -2\Re(\overline{R}^* C) &\geq -\|C\| (p^{-1}|\overline{R}|^2 + p) \\ 2\Re(\overline{R}^*(\text{ad}_A R)) &\geq -\|\text{ad}_A R\| (p^{-1}|\overline{R}|^2 + p) \end{aligned}$$

while: $\overline{R}^* C \overline{R} \geq -\|C\| |\overline{R}|^2$. Summing up, for any $p > 0$,

$$-2\Re((\text{ad}_A V)V^*) \geq 2(C - i(\text{ad}_A \Im R)) - p(2\|C\| + \|\text{ad}_A R\|) - F_R(p) |\overline{R}|^2,$$

where $F_R(p) = (2\|C\| + \|\text{ad}_A R\|)p^{-1} + 2\|C\|$. Fix $p = p_R$ where $2p_R(2\|C\| + \|\text{ad}_A R\|) = a_0$ and apply Proposition 4.2; we get the first estimate with $c_{0,R} = F_R(p_R)$ instead of c_0 .

Back to Lemma 4.11, we can derive the second estimate analogously by defining the function F_L , the positive number p_L and $c_{0,L} = F_L(p_L)$. We conclude by setting $c_0 := \max\{c_{0,L}, c_{0,R}\}$. \square

Proposition 4.14. Suppose (Loc). There exists $0 < \epsilon_1 < \epsilon_0$ such that for any $\epsilon \in [0, \epsilon_1]$, $z \in (0, 1] \cdot e^{i\overline{\Theta}_0}$,

$$T_\epsilon(z)^* + \bar{z}V_\epsilon T_\epsilon(z) + \frac{3\pi^2 a_1}{2d_0^2} \epsilon |z| |T_\epsilon(z)|^2 \geq d(\epsilon, z), \quad (22)$$

$$T_\epsilon(z) + zV_\epsilon^* T_\epsilon(z)^* + \frac{3\pi^2 a_1}{2d_0^2} \epsilon |z| |T_\epsilon(z)^*|^2 \geq d(\epsilon, z), \quad (23)$$

where

$$d(\epsilon, z) = 1 - |z|^2 + a_0 \epsilon |z|^2. \quad (24)$$

Proof. Fix $z \in (0, 1] \cdot e^{i\bar{\Theta}_0}$. We have:

$$\begin{aligned} T_\epsilon(z)^* + \bar{z}V_\epsilon T_\epsilon(z) &= 1 - |z|^2 R_\epsilon^* R_\epsilon = 1 - |z|^2 (R^* + (R_\epsilon - R)^*) (R + (R_\epsilon - R)) \\ &= 1 - |z|^2 |R|^2 - 2|z|^2 \Re(R^*(R_\epsilon - R)) - |z|^2 |R_\epsilon - R|^2. \end{aligned} \quad (25)$$

Mind (5). We have that: $R_\epsilon - R = \epsilon U(\text{ad}_A V^* + q_\epsilon) = \epsilon U Q_\epsilon$, so $|R_\epsilon - R|^2 = \epsilon^2 |Q_\epsilon|^2$ and

$$\Re(R^*(R_\epsilon - R)) = \epsilon \Re(V \text{ad}_A V^*) + \epsilon \Re(V^* q_\epsilon).$$

Rewriting $|R|^2 = |1 - \bar{R}|^2 = 1 - 2\Re \bar{R} + |\bar{R}|^2$, (25) yields

$$\begin{aligned} T_\epsilon(z)^* + \bar{z}V_\epsilon T_\epsilon(z) &= 1 - |z|^2 + |z|^2 X_R + |z|^2 \epsilon Y - |z|^2 \epsilon Z_\epsilon \\ \text{where } X_R &= 2\Re \bar{R} - |\bar{R}|^2 \\ Y &= -2\Re(V \text{ad}_A V^*) \\ Z_\epsilon &= 2\Re(V^* q_\epsilon) + \epsilon |Q_\epsilon|^2. \end{aligned}$$

We estimate the term Y from below. Combining Proposition 4.13 with (15) allows us to derive for $\epsilon \in [0, \epsilon_0)$, $z \in (0, 1] \cdot e^{i\bar{\Theta}_0}$,

$$b_\epsilon + \frac{3\pi^2 a_1}{2d_0^2 |z|} |T_\epsilon(z)|^2 \geq \left(\frac{3a_0}{2} - \frac{3\pi^2 a_1}{2d_0^2} |z| b^2 \epsilon \right) - |\bar{R}|^2 \left(c_0 + \frac{3\pi^2 a_1}{2d_0^2} |z| \right). \quad (26)$$

Taking advantage of the fact that $|z| \leq 1$, we get

$$\begin{aligned} T_\epsilon(z)^* + \bar{z}V_\epsilon T_\epsilon(z) + \frac{3\pi^2 a_1}{2d_0^2} \epsilon |z| |T_\epsilon(z)|^2 &\geq \\ d(\epsilon, z) + |z|^2 \epsilon \left(\frac{a_0}{2} - c_\epsilon - \frac{3\pi^2 a_1}{2d_0^2} b^2 \epsilon^2 \right) &+ |z|^2 \mathfrak{F}_\epsilon(\bar{R}) \end{aligned}$$

where for $\epsilon \geq 0$

$$\mathfrak{F}_\epsilon(\bar{R}) := 2\Re \bar{R} - (1 + \gamma_\epsilon) |\bar{R}|^2, \quad (27)$$

$$\text{with } \gamma_\epsilon = \epsilon \left[c_0 + \frac{3\pi^2 a_1}{2d_0^2} \right]. \quad (28)$$

Now, we observe that: $0 \leq |Q_\epsilon|^2 \leq b^2$ and $-\|q_\epsilon\| \leq \Re(V^* q_\epsilon) \leq \|q_\epsilon\|$. So, $\|c_\epsilon\| = o(\epsilon)$. This allows us to pick $0 < \mu < \epsilon_0$ such that

$$b^2 \mu^2 \frac{3\pi^2 a_1}{2d_0^2} + \sup_{\epsilon \in [0, \mu]} \|c_\epsilon\| \leq \frac{a_0}{2}, \quad (29)$$

$$\text{and } \gamma_\mu \leq \alpha. \quad (30)$$

From (31), we obtain for all $\epsilon \in [0, \mu]$ and $z \in (0, 1] \cdot e^{i\overline{\Theta}_0}$,

$$T_\epsilon(z)^* + \bar{z}V_\epsilon T_\epsilon(z) + \frac{3\pi^2 a_1}{2d_0^2} \epsilon |z| |T_\epsilon(z)|^2 \geq d(\epsilon, z) + |z|^2 (2\Re \bar{R} - (1 + \alpha) |\bar{R}|^2). \quad (31)$$

At this point of the proof, the condition (3) of Hypothesis (H4), comes into scene to estimate the last terms on the RHS. This concludes the proof of (22). The proof of (23) is analogous. □

For d and ϵ_1 respectively defined in (24) and (29), let us write

$$\Omega_0 := \{(\epsilon, z) \in [0, \epsilon_1] \times (0, 1] \cdot e^{i\overline{\Theta}_0}; d(\epsilon, z) > 0\}. \quad (32)$$

We deduce that:

Proposition 4.15. *Assume (Loc). For $(\epsilon, z) \in \Omega_0$, the operator $T_\epsilon(z)$ is boundedly invertible.*

Proof. Fix ϵ, z as stated. Reinterpreting (22) (resp. (23)) in terms of quadratic forms shows that $T_\epsilon(z)$ (resp. $T_\epsilon(z)^*$) is injective from \mathcal{H} into itself and has closed range. Since

$$\overline{\text{Ran } T_\epsilon(z)} = (\text{Ker } T_\epsilon(z)^*)^\perp,$$

we deduce that $T_\epsilon(z)$ and $(T_\epsilon(z))^*$ are actually linear and bijective hence boundedly invertible by the Inverse Mapping Theorem. □

4.3.3 Synthesis

Aside from the local vs global aspects, the remaining components of the proofs of Theorems 1.8 and 1.3 are identical. From now, we unify notations with the introduction of

$$\Omega := \begin{cases} \Omega_{\mathbb{T}} & \text{if (Glo) holds} \\ \Omega_0 & \text{if (Loc) holds} \end{cases} \quad \mathbf{S} := \begin{cases} \partial\mathbb{D} & \text{if (Glo) holds} \\ e^{i\overline{\Theta}_0} & \text{if (Loc) holds} \end{cases}. \quad (33)$$

Accordingly, assuming (Glo) or (Loc), Propositions 4.9 and 4.15 allow us to define for any $(\epsilon, z) \in \Omega$,

$$G_\epsilon(z) := (1 - zV_\epsilon^*)^{-1}. \quad (34)$$

We have:

Proposition 4.16. *Suppose (Glo) or (Loc). Then*

$$C_0 := \sup_{(\epsilon, z) \in \Omega} d(\epsilon, z) \|G_\epsilon(z)\| < \infty. \quad (35)$$

In addition, for any $\epsilon \in (0, \epsilon_1]$, $z \in (0, 1] \cdot \mathbf{S}$,

$$\max \{ \|G_\epsilon(z)\varphi\|, \|G_\epsilon^*(z)\varphi\| \} \leq \sqrt{\frac{2|\Re\langle\varphi, G_\epsilon(z)\varphi\rangle|}{a_0\epsilon|z|^2}} \quad (36)$$

Proof. For $(\epsilon, z) \in \Omega$, denote

$$H_\epsilon(z) := \frac{1 + zV_\epsilon^*}{1 - zV_\epsilon^*} = 2G_\epsilon(z) - 1. \quad (37)$$

We observe that

$$\begin{aligned} \Re H_\epsilon(z) &= G_\epsilon(z)^* (T_\epsilon(z)^* + \bar{z}V_\epsilon T_\epsilon(z)) G_\epsilon(z) \\ &= G_\epsilon(z) (T_\epsilon(z) + zV_\epsilon^* T_\epsilon(z)^*) G_\epsilon(z)^*, \end{aligned} \quad (38)$$

hence for any $\varphi \in \mathcal{H}$,

$$\Re\langle\varphi, H_\epsilon(z)\varphi\rangle \leq 2\|\varphi\| \|G_\epsilon(z)\varphi\| + \|\varphi\|^2. \quad (39)$$

Case (Glo): (9) and (38) yield

$$\Re\langle\varphi, H_\epsilon(z)\varphi\rangle \geq d(\epsilon, z) \|G_\epsilon(z)\varphi\|^2 \quad (40)$$

for any $(\epsilon, z) \in \Omega_{\mathbb{T}}$. We deduce from (39) and (40) that

$$2d(\epsilon, z) \|G_\epsilon(z)\varphi\| + 1 \geq d(\epsilon, z)^2 \|G_\epsilon(z)\varphi\|^2$$

and (35) follows as the region where $2X + 1 - X^2$ is positive is bounded.

Inequality (40) also implies for any $(\epsilon, z) \in \Omega_{\mathbb{T}}$,

$$\|G_\epsilon(z)\varphi\|^2 \leq \frac{\Re\langle\varphi, H_\epsilon(z)\varphi\rangle}{a_0\epsilon|z|^2} \leq \frac{2\Re\langle\varphi, G_\epsilon(z)\varphi\rangle}{a_0\epsilon|z|^2}$$

hence (36) for $\|G_\epsilon(z)\varphi\|$. The proof of (36) for $\|G_\epsilon^*(z)\varphi\|$ is analogous.

Case (Loc): (22) and (38) yield

$$\Re\langle\varphi, H_\epsilon(z)\varphi\rangle + \frac{3\pi^2 a_1}{2d_0^2} \epsilon|z| \|\varphi\|^2 \geq d(\epsilon, z) \|G_\epsilon(z)\varphi\|^2 \quad (41)$$

for any $(\epsilon, z) \in \Omega_0$. Taking into account (11), (28) and (30), we note that:

$$0 \leq d(\epsilon, z) \leq 1 + a_0\epsilon_1 \quad \text{and} \quad \frac{3\pi^2 a_1}{2d_0^2} \epsilon \leq \gamma_\epsilon \leq \alpha \leq 1. \quad (42)$$

We deduce from (39) and (41) that for any $(\epsilon, z) \in \Omega_0$,

$$2d(\epsilon, z)\|G_\epsilon(z)\| + 2(1 + a_0\epsilon_1) \geq d(\epsilon, z)^2\|G_\epsilon(z)\|^2,$$

and (35) follows from the region of positivity of the polynomial $2X + 2(1 + a_0\epsilon_1) - X^2$.

Inequality (41) also yields for any $(\epsilon, z) \in \Omega_0$,

$$\begin{aligned} \|G_\epsilon(z)\varphi\|^2 &\leq \frac{\Re\langle\varphi, H_\epsilon(z)\varphi\rangle}{a_0\epsilon|z|^2} + \frac{3\pi^2 a_1}{2d_0^2 a_0|z|}\|\varphi\|^2, \\ &\leq \frac{2\Re\langle\varphi, G_\epsilon(z)\varphi\rangle}{a_0\epsilon|z|^2} + \frac{1}{a_0\epsilon|z|^2} \left(\frac{3\pi^2 a_1|z|}{2d_0^2}\epsilon - 1 \right) \|\varphi\|^2 \leq \frac{2\Re\langle\varphi, G_\epsilon(z)\varphi\rangle}{a_0\epsilon|z|^2} \end{aligned}$$

where we have finally used (38) and (42). This proves (36) for $\|G_\epsilon(z)\varphi\|$. The proof for $\|G_\epsilon^*(z)\varphi\|$ is analogous. \square

Suppose either (Glo) or (Loc). For any $(\epsilon, z) \in \Omega$ and $1/2 < s \leq 1$, we define the weighted deformed resolvent:

$$F_{s,\epsilon}(z) := W_s(\epsilon)G_\epsilon(z)W_s(\epsilon) \quad (43)$$

$$W_s(\epsilon) := \langle A \rangle^{-s} \langle \epsilon A \rangle^{s-1}. \quad (44)$$

Note that for any $s \in (1/2, 1]$, $(W_s(\epsilon))_{\epsilon \in [0, \epsilon_0]}$ is a family of bounded selfadjoint operators. In particular, $\sup_{\epsilon \in [0, 1]} \|W_s(\epsilon)\| \leq 1$. Proposition 4.16 entails:

Corollary 4.17. *Suppose either (Glo) or (Loc). We have for d defined in (11) and (24) resp.,*

$$\sup_{(\epsilon, z) \in \Omega} d(\epsilon, z)\|F_{s,\epsilon}(z)\| \leq C_0 < \infty.$$

In addition, given $r \in (0, 1)$, for any $(\epsilon, z) \in (0, \epsilon_1] \times [r, 1] \cdot \mathbf{S}$,

$$\max \{ \|G_\epsilon(z)W_s(\epsilon)\varphi\|, \|G_\epsilon(z)^*W_s(\epsilon)\varphi\| \} \leq \sqrt{\frac{2|\Re\langle\varphi, F_{s,\epsilon}(z)\varphi\rangle|}{a_0\epsilon r^2}}. \quad (45)$$

4.4 Differential Inequalities

Next, we derive some differential inequalities for the weighted deformed resolvents $F_{s,\epsilon}(z)$. The first ingredient is

Proposition 4.18. *Suppose (Glo) or (Loc). Then, for any $\epsilon \in (0, \epsilon_1)$, $z \in (0, 1) \cdot \mathbf{S}$, the map $\epsilon \mapsto G_\epsilon(z)$ is continuously differentiable on $(0, \epsilon_1)$ in the operator norm topology with*

$$\partial_\epsilon G_\epsilon(z) = \text{ad}_A G_\epsilon(z) + zG_\epsilon(z)\mathcal{Q}(\epsilon)^*G_\epsilon(z),$$

where \mathcal{Q} is defined in (4).

Proof. For any fixed $z \in (0, 1) \cdot \mathbf{S}$, the map $\epsilon \mapsto G_\epsilon(z)$ is continuous on the interval $[0, \epsilon_1)$ and continuously differentiable on $(0, \epsilon_1)$ in the operator norm topology, and

$$\partial_\epsilon G_\epsilon(z) = zG_\epsilon(z)(\partial_\epsilon S_\epsilon^* - \epsilon \partial_\epsilon B_\epsilon^* - B_\epsilon^*)G_\epsilon(z).$$

Now, for any $\epsilon \in [0, \epsilon_1)$, $z \in (0, 1] \cdot \mathbf{S}$, we have $G_\epsilon(z) \in C^1(A)$ with

$$\text{ad}_A G_\epsilon(z) = zG_\epsilon(z)(\text{ad}_A S_\epsilon^* - \epsilon \text{ad}_A B_\epsilon^*)G_\epsilon(z), \quad (46)$$

which concludes the proof. \square

Now, for any fixed $1/2 < s < 1$, the map $\epsilon \mapsto W_s(\epsilon)$ is strongly continuous on $[0, \epsilon_0)$ and converges strongly to $\langle A \rangle^{-s}$ as ϵ tends to zero.

We have:

Proposition 4.19. *Suppose (Glo) or (Loc). Let $1/2 < s \leq 1$. For any fixed $z \in (0, 1) \cdot \mathbf{S}$, the map $\epsilon \mapsto F_{s,\epsilon}(z)$ is weakly continuously differentiable on $(0, \epsilon_1)$ and for any $\varphi \in \mathcal{H}$, any $\epsilon \in (0, \epsilon_1)$,*

$$|\langle \varphi, \partial_\epsilon F_{s,\epsilon}(z)\varphi \rangle| \leq h_1(\epsilon)\|\varphi\|\sqrt{|\langle \varphi, F_{s,\epsilon}(z)\varphi \rangle|} + h_2(\epsilon)|\langle \varphi, F_{s,\epsilon}(z)\varphi \rangle|. \quad (47)$$

where:

$$h_1(\epsilon) = \frac{2\sqrt{2}(2-s)\epsilon^{s-1}}{\sqrt{a_0}\epsilon r}, \quad h_2(\epsilon) = \frac{2\|\mathcal{Q}(\epsilon)\|}{a_0 r^2 \epsilon}. \quad (48)$$

Proof. First, note that the map $\epsilon \mapsto W_s(\epsilon)$ is strongly continuously differentiable on the interval $(0, \epsilon_0)$ and that for any $\epsilon \in (0, \epsilon_0)$, any $\varphi \in \mathcal{H}$, we have

$$\|\partial_\epsilon W_s(\epsilon)\varphi\| \leq (1-s)\epsilon^{s-1}\|\varphi\|. \quad (49)$$

Note also that for $1/2 < s \leq 1$,

$$\|AW_s(\epsilon)\| = \|W_s(\epsilon)A\| \leq \epsilon^{s-1}. \quad (50)$$

Fix $z \in (0, 1) \cdot \mathbf{S}$. Due to Proposition 4.18, the map $\epsilon \mapsto F_{s,\epsilon}(z)$ is weakly continuously differentiable on $(0, \epsilon_0)$ and we have for any $\epsilon \in (0, \epsilon_0)$,

$$\begin{aligned} |\langle \varphi, \partial_\epsilon F_{s,\epsilon}(z)\varphi \rangle| &\leq t_{1,\epsilon}(z) + |\langle W_s(\epsilon)\varphi, (\partial_\epsilon G_\epsilon(z))W_s(\epsilon)\varphi \rangle| \\ &\leq t_{1,\epsilon}(z) + t_{2,\epsilon}(z) + t_{3,\epsilon}(z) \end{aligned}$$

where

$$\begin{aligned} t_{1,\epsilon}(z) &= |\langle (\partial_\epsilon W_s(\epsilon))\varphi, G_\epsilon(z)W_s(\epsilon)\varphi \rangle + \langle W_s(\epsilon)\varphi, G_\epsilon(z)(\partial_\epsilon W_s(\epsilon))\varphi \rangle| \\ t_{2,\epsilon}(z) &= |\langle W_s(\epsilon)\varphi, \text{ad}_A(G_\epsilon(z))W_s(\epsilon)\varphi \rangle| \\ t_{3,\epsilon}(z) &= \epsilon|z|\|\langle \mathcal{Q}(\epsilon)G_\epsilon(z)^*W_s(\epsilon)\varphi, G_\epsilon(z)W_s(\epsilon)\varphi \rangle|. \end{aligned}$$

Invoking (36), (49) and (50), we deduce that:

$$\begin{aligned}
t_{1,\epsilon}(z) &\leq \|(\partial_\epsilon W_s(\epsilon))\varphi\| (\|G_\epsilon(z)W_s(\epsilon)\varphi\| + \|G_\epsilon(z)^*W_s(\epsilon)\varphi\|) \\
&\leq \frac{2\sqrt{2}(1-s)\epsilon^{s-1}}{\sqrt{a_0\epsilon}r} \|\varphi\| \sqrt{|\langle \varphi, F_{s,\epsilon}(z)\varphi \rangle|}, \\
t_{2,\epsilon}(z) &\leq \|AW_s(\epsilon)\varphi\| (\|G_\epsilon(z)W_s(\epsilon)\varphi\| + \|G_\epsilon(z)^*W_s(\epsilon)\varphi\|) \\
&\leq \frac{2\sqrt{2}\epsilon^{s-1}}{\sqrt{a_0\epsilon}r} \|\varphi\| \sqrt{|\langle \varphi, F_{s,\epsilon}(z)\varphi \rangle|},
\end{aligned}$$

while

$$t_{3,\epsilon}(z) \leq \|\mathcal{Q}(\epsilon)\| \|G_\epsilon(z)^*W_s(\epsilon)\varphi\| \|G_\epsilon(z)W_s(\epsilon)\varphi\| \leq \frac{2\|\mathcal{Q}(\epsilon)\|}{a_0 r^2 \epsilon} |\langle \varphi, F_{s,\epsilon}(z)\varphi \rangle|.$$

Estimate (47) follows. \square

The next result combines Proposition 4.19 with Gronwall's Lemma.

Proposition 4.20. *Suppose (Glo) or (Loc). Fix $s \in (1/2, 1]$. Then,*

$$C_1 := \sup_{(\epsilon, z) \in (0, \epsilon_1] \times [r, 1] \cdot \mathbf{S}} \|F_{s,\epsilon}(z)\| < \infty. \quad (51)$$

In addition, there exists $\mathfrak{H}_s \in L^1((0, \epsilon_1])$, such that for any $(\epsilon, z) \in (0, \epsilon_1] \times [r, 1] \cdot \mathbf{S}$,

$$\|\partial_\epsilon F_{s,\epsilon}(z)\| \leq \mathfrak{H}_s(\epsilon). \quad (52)$$

Proof. With h_1 and h_2 defined in (48), $h_2 \in L^1((0, \epsilon_1])$. By inspection, h_1 also belongs to $L^1((0, \epsilon_1])$. For any $(\epsilon, z) \in (0, \epsilon_1] \times [r, 1] \cdot \mathbf{S}$ and any $\varphi \in \mathcal{H}$,

$$|\langle \varphi, F_{s,\epsilon}(z)\varphi \rangle| \leq |\langle \varphi, F_{s,\epsilon_1}(z)\varphi \rangle| + \int_\epsilon^{\epsilon_1} |\langle \varphi, \partial_\mu F_{s,\mu}(z)\varphi \rangle| d\mu,$$

which combined with Corollary 4.17 and Proposition 4.19 yields:

$$\begin{aligned}
|\langle \varphi, F_{s,\epsilon}(z)\varphi \rangle| &\leq \frac{C_0}{a_0\epsilon_1} \|\varphi\|^2 + \|\varphi\| \int_\epsilon^{\epsilon_1} h_1(\mu) \sqrt{|\langle \varphi, F_{s,\mu}(z)\varphi \rangle|} d\mu \\
&\quad + \int_\epsilon^{\epsilon_1} h_2(\mu) |\langle \varphi, F_{s,\mu}(z)\varphi \rangle| d\mu.
\end{aligned}$$

Using Gronwall's Lemma as stated in e.g. [6] Lemma 7.A.1, we deduce

$$\begin{aligned}
|\langle \varphi, F_{s,\epsilon}(z)\varphi \rangle| &\leq \left[\sqrt{\frac{C_0}{a_0\epsilon_1}} + \frac{1}{2} \int_\epsilon^{\epsilon_1} h_1(\mu) \exp\left(-\frac{1}{2} \int_\mu^{\epsilon_1} h_2(x) dx\right) d\mu \right]^2 \|\varphi\|^2 \\
&\quad \times \exp\left(\int_\epsilon^{\epsilon_1} h_2(\mu) d\mu\right)
\end{aligned}$$

Since the functions h_1 and h_2 are integrable on $(0, \epsilon_1]$, we deduce that:

$$\sup_{\|\varphi\|=1} \sup_{(\epsilon, z) \in (0, \epsilon_1] \times [r, 1] \cdot \mathbf{S}} |\langle \varphi, F_{s, \epsilon}(z) \varphi \rangle| < \infty.$$

Recall that $\|F_{s, \epsilon}(z)\| = \sup_{\|\varphi\|=1, \|\psi\|=1} |\langle \varphi, F_{s, \epsilon}(z) \psi \rangle|$; so, (51) follows by polarisation. Incorporating it into (47) entails,

$$|\langle \varphi, \partial_\epsilon F_{s, \epsilon}(z) \varphi \rangle| \leq \left(\sqrt{C_1} h_1(\epsilon) + C_1 h_2(\epsilon) \right) \|\varphi\|^2,$$

for any $(\epsilon, z) \in (0, \epsilon_1] \times [r, 1] \cdot \mathbf{S}$ and any $\varphi \in \mathcal{H}$. (52) follows by polarisation. \square

Proposition 4.21. *Suppose (Glo) or (Loc). Then for any fixed $s \in (1/2, 1]$ and any fixed $z \in [r, 1] \cdot \mathbf{S}$,*

$$w - \lim_{\epsilon \rightarrow 0^+} F_{s, \epsilon}(z) = F_s(z) \quad (53)$$

where

$$F_s(z) = \langle A \rangle^{-s} (1 - zV^*)^{-1} \langle A \rangle^{-s} \quad \text{for } z \in \mathbb{D}. \quad (54)$$

Proof. Fix $r \in (0, 1)$. Pick two vectors φ, ψ in \mathcal{H} and fix for a moment $z \in [r, 1] \cdot \mathbf{S}$. For any $\epsilon \in (0, \epsilon_0)$,

$$\langle \varphi, \langle A \rangle^{-s} (1 - zV^*)^{-1} \langle A \rangle^{-s} \psi \rangle - \langle \varphi, F_{s, \epsilon}(z) \psi \rangle = t_{1, \epsilon}(z) + t_{2, \epsilon}(z) + t_{3, \epsilon}(z)$$

with

$$\begin{aligned} t_{1, \epsilon}(z) &= \langle W_s(\epsilon) \varphi, ((1 - zV^*)^{-1} - G_\epsilon(z)) W_s(\epsilon) \psi \rangle \\ t_{2, \epsilon}(z) &= \langle W_s(0) \varphi, (1 - zV^*)^{-1} (W_s(0) - W_s(\epsilon)) \psi \rangle \\ t_{3, \epsilon}(z) &= \langle (W_s(0) - W_s(\epsilon)) \varphi, (1 - zV^*)^{-1} W_s(\epsilon) \psi \rangle. \end{aligned}$$

First, we estimate $t_{1, \epsilon}(z)$. For any $(\epsilon, z) \in [0, \epsilon_1] \times (0, 1) \cdot \mathbf{S}$, $G_\epsilon(z) - (1 - zV^*)^{-1} = G_\epsilon(z)(T_0(z) - T_\epsilon(z))(1 - zV^*)^{-1}$. We observe that for $z \in \mathbb{D}$, we have $\|(1 - zV^*)^{-1}\| \leq 1/(1 - |z|)$. Using Lemma 4.7 and Proposition 4.16, we deduce that

$$\|G_\epsilon(z) - (1 - zV^*)^{-1}\| \leq \min \left\{ \frac{C_0 b \epsilon}{(1 + |z|)(1 - |z|)^2}, \frac{C_0 b}{a_0 |z|^2 (1 - |z|)} \right\} \quad (55)$$

hence

$$|t_{1, \epsilon}(z)| \leq \frac{C_0 b \epsilon}{(1 + |z|)(1 - |z|)^2} \|\varphi\| \|\psi\|$$

since $\|W_s(\epsilon)\| \leq 1$ for any $\epsilon \in [0, 1]$. Now, since $\|(1 - zV^*)^{-1}\| \leq 1/(1 - |z|)$ for any $z \in \mathbb{D}$, we also have

$$\begin{aligned} |t_{2, \epsilon}(z)| &\leq \frac{1}{1 - |z|} \|(W_s(0) - W_s(\epsilon)) \varphi\| \|\psi\| \\ |t_{3, \epsilon}(z)| &\leq \frac{1}{1 - |z|} \|\varphi\| \|(W_s(0) - W_s(\epsilon)) \psi\|, \end{aligned}$$

which concludes. □

We have arrived at the heart of the proof of Theorems 1.8 and 1.3:

Proposition 4.22. *Suppose (Glo) or (Loc). For $s \in (1/2, 1]$. Then, F_s defined by (54) admits a continuous extension to $\mathbb{D} \cup \mathbf{S}$. In addition, given $r \in (0, 1)$, it holds*

$$\sup_{z \in [r, 1] \cdot \mathbf{S}} \|F_{s, \epsilon}(z) - F_s(z)\| \rightarrow_{\epsilon \rightarrow 0^+} 0. \quad (56)$$

We denote the extension also by F_s .

Proof. From Proposition 4.20, we get for any $\epsilon, \mu \in (0, \epsilon_1]$, $\epsilon < \mu$, $z \in [r, 1] \cdot \mathbf{S}$,

$$\|F_{s, \epsilon}(z) - F_{s, \mu}(z)\| \leq \int_{\epsilon}^{\mu} \mathfrak{H}_s(\xi) d\xi$$

which vanishes as μ and ϵ tend to 0 (uniformly in $z \in [r, 1] \cdot \mathbf{S}$). So, $(F_{s, \epsilon}(z))$ is Cauchy in $\mathcal{B}(\mathcal{H})$, hence converges to some $F_s^+(z) \in \mathcal{B}(\mathcal{H})$ for any fixed $z \in [r, 1] \cdot \mathbf{S}$. In view of Proposition 4.21, $F_s^+(z) = F_s(z)$ if $z \in [r, 1] \cdot \mathbf{S}$ and

$$\|F_{s, \epsilon}(z) - F_s^+(z)\| \leq \int_0^{\epsilon} \mathfrak{H}_s(\xi) d\xi \quad (57)$$

for any $\epsilon \in (0, \epsilon_1]$. (56) follows. It remains to prove the continuity of F_s^+ . Given $\epsilon \in (0, \epsilon_1]$, $z, z' \in [r, 1] \cdot \mathbf{S}$,

$$\begin{aligned} F_{s, \epsilon}(z) - F_{s, \epsilon}(z') &= W_s(\epsilon)(G_{\epsilon}(z) - G_{\epsilon}(z'))W_s(\epsilon) \\ &= (z - z')W_s(\epsilon)G_{\epsilon}(z)(S_{\epsilon}^* - \epsilon B_{\epsilon}^*)G_{\epsilon}(z')W_s(\epsilon) \end{aligned} \quad (58)$$

where we have used (6) in the final step.

Now, following Corollary 4.17 and Proposition 4.20, we get for any $\epsilon \in (0, \epsilon_1]$ and any $z \in [r, 1] \cdot \mathbf{S}$,

$$\max\{\|G_{\epsilon}(z)^*W_s(\epsilon)\|, \|G_{\epsilon}(z)W_s(\epsilon)\|\} \leq \frac{B_r}{\sqrt{\epsilon}} \quad \text{with} \quad B_r := \frac{\sqrt{2C_1} + 1}{r\sqrt{a_0}}.$$

Combining these estimates with (58) entails for any $\epsilon \in (0, \epsilon_1]$, $z, z' \in [r, 1] \cdot \mathbf{S}$,

$$\|F_{s, \epsilon}(z) - F_{s, \epsilon}(z')\| \leq \frac{B_r^2}{\epsilon}|z - z'|.$$

Due to (57), we obtain

$$\begin{aligned} \|F_s^+(z) - F_s^+(z')\| &\leq \|F_s^+(z) - F_{s, \epsilon}(z)\| + \|F_{s, \epsilon}(z) - F_{s, \epsilon}(z')\| + \|F_s^+(z') - F_{s, \epsilon}(z')\| \\ &\leq 2 \int_0^{\epsilon} \mathfrak{H}_s(\xi) d\xi + \frac{B_r^2}{\epsilon}|z - z'|, \end{aligned}$$

for any $\epsilon \in (0, \epsilon_1]$. The conclusion follows since the function \mathfrak{H}_s is integrable. □

4.5 Proof of Theorem 1.3

Rephrasing Proposition 4.22 under assumptions (Glo), we have shown that the weighted resolvent $\langle A \rangle^{-s}(1 - zV^*)^{-1}\langle A \rangle^{-s}$, which is defined naturally for $z \in \mathbb{D}$, extends continuously to $\overline{\mathbb{D}}$.

4.6 Proof of Theorem 1.8

We rephrase Proposition 4.22 under assumptions (Loc). Given θ such that $e^{i\theta} \in e^{i\Theta} \setminus \mathcal{E}(U)$, we have shown there exists an open neighborhood of θ , $\Theta_0 \subset \Theta$ such that the weighted resolvent $\langle A \rangle^{-s}(1 - zV^*)^{-1}\langle A \rangle^{-s}$, which is defined a priori for $z \in \mathbb{D}$, extends continuously to $\overline{\mathbb{D}} \cup e^{i\Theta_0}$. Since the choice of θ was arbitrary, we have actually shown that this continuous extension holds on $\mathbb{D} \cup e^{i\Theta} \setminus \mathcal{E}(U)$.

5 Proof of Theorem 1.9

In Theorem 1.8, the estimates only hold for $z \in \mathbb{D}$ with $r \leq |z| < 1$ and $\arg z \in \Theta$. In order to obtain an estimate for all $z \in \mathbb{D}$ and to derive Theorem 1.9, we need to localize spectrally. We will do this in Lemmata 5.1 and 5.2, then proceed to the proof of Theorem 1.9.

Lemma 5.1. *Let U be a unitary operator which belongs to $C^1(A)$ and $\phi \in C^1(\mathbb{T})$ be a function such that its derivative ϕ' belongs to the Wiener algebra. With Φ defined by $\Phi(e^{i\theta}) = \phi(\theta)$ for $\theta \in \mathbb{T}$, the operator $\langle A \rangle^{-\delta}\Phi(U)\langle A \rangle^\delta$ extends to a bounded operator in $\mathcal{B}(\mathcal{H})$ for all $\delta \in [-1, 1]$.*

Proof. By hypothesis, we have that: $\Phi(e^{i\theta}) = \phi(\theta) = \sum_{n \in \mathbb{Z}} \hat{\phi}_n e^{in\theta}$ with $\sum_{n \in \mathbb{Z}} |n\hat{\phi}_n| < \infty$. We first prove that $\Phi(U) \in C^1(A)$. First, observe that for any $n \in \mathbb{Z}$, $U^n \in C^1(A)$ with:

$$\text{ad}_A U^n = \sum_{k=0}^{n-1} U^k (\text{ad}_A U) U^{n-1-k}$$

for $n \geq 1$ and $\text{ad}_A U^n = -U^n (\text{ad}_A U^{-n}) U^n$ for $n \leq -1$. So, $\|\text{ad}_A U^n\| \leq |n| \|\text{ad}_A U\|$ for any $n \in \mathbb{Z}$. Given $N \in \mathbb{Z}_+$, let

$$\Phi_N(U) = \sum_{|n| \leq N} \hat{\phi}_n U^n.$$

$(\Phi_N(U))_{N \in \mathbb{N}} \subset C^1(A)$ converges in norm to $\Phi(U)$ and $(\text{ad}_A \Phi_N(U))_{N \in \mathbb{N}}$ is convergent w.r.t the operator norm topology. So, $\Phi(U) \in C^1(A)$ and $\lim_{N \rightarrow \infty} \text{ad}_A \Phi_N(U) = \text{ad}_A \Phi(U)$ [6, 12].

As a consequence, we deduce that for $\delta \in \{-1, 0, 1\}$, $\langle A \rangle^{-\delta}\Phi(U)\langle A \rangle^\delta$ extend to a bounded operator in $\mathcal{B}(\mathcal{H})$. We conclude by interpolation. \square

Lemma 5.2. *For any $z \in \mathbb{D}$ and any $\psi \in \mathcal{H}$, it holds:*

$$\|(U^* - V^*)G_0(z)\psi\|^2 \leq 8\langle \psi, \Re(G_0(z))\psi \rangle.$$

Proof. We decompose $U^* - V^* = U^*\bar{P} + \bar{Q}U^*P$, so

$$\|(U^* - V^*)G_0(z)\psi\|^2 \leq 2\|\bar{P}G_0(z)\psi\|^2 + 2\|\bar{Q}U^*PG_0(z)\psi\|^2.$$

It remains to bound each term on the RHS. Let $\bar{P} = 1 - P$. Since $0 \leq P \leq 1$ and $0 \leq Q \leq 1$, so $P^2 \leq P$ and

$$\begin{aligned} \bar{P}^2 &\leq 1 - P^2 = 1 - PUU^*P \leq 1 - PUQ^2U^*P = 1 - VV^* \\ &\leq 1 - |z|^2VV^* = T_0(z) + T_0(z)^*zV^*, \end{aligned}$$

for any $z \in \mathbb{D}$. It follows that for any $\psi \in \mathcal{H}$,

$$\begin{aligned} \|\bar{P}G_0(z)\psi\|^2 &\leq \langle G_0(z)\psi, \psi \rangle + \langle \psi, zV^*G_0(z)\psi \rangle \\ &= \langle \psi, 2\Re(G_0(z))\psi \rangle - \|\psi\|^2 \leq 2\langle \psi, \Re(G_0(z))\psi \rangle. \end{aligned}$$

Also $\bar{Q}^2 \leq 1 - Q^2$, hence it holds

$$\begin{aligned} PU\bar{Q}^2U^*P &\leq PU(1 - Q^2)U^*P \leq 1 - PUQ^2U^*P \leq 1 - VV^* \\ &\leq 1 - |z|^2VV^* = T_0(z) + T_0(z)^*zV^*, \end{aligned}$$

for any $z \in \mathbb{D}$. As before, we get for any $\psi \in \mathcal{H}$,

$$\|\bar{Q}U^*PG_0(z)\psi\|^2 \leq \langle G_0(z)\psi, \psi \rangle + \langle \psi, zV^*G_0(z)\psi \rangle \leq 2\langle \psi, \Re(G_0(z))\psi \rangle.$$

and the conclusion follows. \square

Proof. (of Theorem 1.9)

Let Θ_1 and Θ_0 be open connected subsets such that $\bar{\Theta}_1 \subset \Theta_0 \subset \bar{\Theta}_0 \subset \Theta$. Let $\phi \in C^\infty(\mathbb{T}; \mathbb{R})$ be supported in $\bar{\Theta}_1$. Combining Theorem 1.8 with Lemma 5.1 yields:

$$\sup_{|z| < 1, \arg z \in \Theta_0} \|\langle A \rangle^{-s} \Phi(U)(1 - zV^*)^{-1} \Phi(U) \langle A \rangle^{-s}\| < \infty.$$

For $z \in \mathbb{D}$, the resolvent identity reads:

$$(1 - zV^*)^{-1} = (1 - zU^*)^{-1} - z(1 - zU^*)^{-1}(U^* - V^*)(1 - zV^*)^{-1}.$$

Given any $z \in \mathbb{D}$, we bound the quantity $\|\langle A \rangle^{-s} \Phi(U)(1 - zV^*)^{-1} \Phi(U) \langle A \rangle^{-s}\|$ by:

$$\begin{aligned} &\|\langle A \rangle^{-s} \Phi(U)(1 - zU^*)^{-1} \Phi(U) \langle A \rangle^{-s}\| \\ &\quad + \|(U^* - V^*)(1 - zV^*)^{-1} \Phi(U) \langle A \rangle^{-s}\| \|(1 - \bar{z}U)^{-1} \Phi(U) \langle A \rangle^{-s}\|. \end{aligned}$$

Unitary functional calculus yields:

$$C_{\phi, \Theta_0} = \sup_{|z| < 1, \arg z \in \mathbb{T} \setminus \Theta_0} \|(1 - \bar{z}U)^{-1}\Phi(U)\| < \infty.$$

From Lemma 5.2, we deduce that for any $z \in \mathbb{D}$ with $\arg z \in \mathbb{T} \setminus \Theta_0$, the quantity $\|\langle A \rangle^{-s}\Phi(U)(1 - zV^*)^{-1}\Phi(U)\langle A \rangle^{-s}\|$ is bounded by:

$$C_{\phi, \Theta_0} \left(\|\Phi\|_\infty + 2\sqrt{2}\sqrt{\|\langle A \rangle^{-s}\Phi(U)(1 - zV^*)^{-1}\Phi(U)\langle A \rangle^{-s}\|} \right),$$

which entails

$$\sup_{|z| < 1, \arg z \in \mathbb{T} \setminus \Theta_0} \|\langle A \rangle^{-s}\Phi(U)(1 - zV^*)^{-1}\Phi(U)\langle A \rangle^{-s}\| < \infty.$$

This completes the proof. □

6 Appendix

Proposition 6.1. *Let $B \in \mathcal{B}(\mathcal{H})$ be symmetric and E an orthogonal projection acting on \mathcal{H} . Denote $E^\perp := 1 - E$. The following statements are equivalent: there exist $c > 0$, $a \in (0, c]$ and $b > 0$ such that:*

- (a) $EBE \geq cE$,
- (b) $B \geq aE - bE^\perp = a - (a + b)E^\perp$.

Proof. Assume (a). Write $B = EBE + 2\Re(EBE^\perp) + E^\perp BE^\perp$ and note that: $E^\perp BE^\perp \geq -\|B\|E^\perp$ and $EBE \geq cE$ by hypothesis. In addition, for any $p > 0$

$$2\Re(EBE^\perp) \leq p\|B\|E + p^{-1}\|B\|E^\perp,$$

so

$$B \geq (c - p\|B\|)E - \|B\|(1 + p^{-1})E^\perp.$$

Once fixed $p > 0$ in such a way that $0 < c - p\|B\| < c$, we obtain (b). The converse implication is immediate. □

Remarks 6.2. 1. *In applications it is convenient to note that: for unitary U one has $U \in C^1(A)$ if and only if*

*there exists a core \mathcal{S} for A such that $U\mathcal{S} \subset \mathcal{S}$ and $U^*AU - A : \mathcal{S} \rightarrow \mathcal{H}$ extends to a bounded operator on \mathcal{H} ; or, equivalently: $U^*\mathcal{S} \subset \mathcal{S}$ and $A - UAU^* : \mathcal{S} \rightarrow \mathcal{H}$ extends to a bounded operator.*

In addition, the bounded extension of $U^*AU - A$ is precisely $U^*(\text{ad}_A U)$. See Propositions 6.3 and 6.4 for details.

Proposition 6.3. *If $U \in C^1(A)$ is unitary, then $U\mathcal{D}(A) \subset \mathcal{D}(A)$ and the operator $U^*AU - A : \mathcal{D}(A) \rightarrow \mathcal{H}$ extends to a bounded operator on \mathcal{H} , denoted $(U^*AU - A)$. It holds: $(U^*AU - A) = U^*\text{ad}_A U = -(\text{ad}_A U^*)U$. Conversely, let \mathcal{S} be a core for A such that $U\mathcal{S} \subset \mathcal{S}$. Assume that the operator $U^*AU - A : \mathcal{S} \rightarrow \mathcal{H}$ extends to a bounded operator on \mathcal{H} and denote by C this extension. Then, U (and U^*) belongs to $C^1(A)$ and $U^*\text{ad}_A U = C$.*

Proof: For the proof of the first statement, see [12] Proposition 2.2. We deduce that for all $(\varphi, \psi) \in \mathcal{H} \times \mathcal{D}(A)$, $\langle \varphi, U^*AU\psi \rangle - \langle \varphi, A\psi \rangle = \langle \varphi, U^*(AU - UA)\psi \rangle$. This implies our claim. Conversely, assume that for all $(\varphi, \psi) \in \mathcal{S} \times \mathcal{S}$, $\langle A\varphi, U\psi \rangle - \langle \varphi, UA\psi \rangle = \langle \varphi, AU\psi \rangle - \langle U^*\varphi, A\psi \rangle = \langle U^*\varphi, U^*AU\psi \rangle - \langle U^*\varphi, A\psi \rangle = \langle U^*\varphi, C\psi \rangle = \langle \varphi, UC\psi \rangle$. The identity extends continuously over $\mathcal{D}(A) \times \mathcal{D}(A)$. This shows that $U \in C^1(A)$ and that: $\text{ad}_A U = UC$. \square

Remark 6.4. *Analog relations can be built between $A - UAU^*$ and $(\text{ad}_A U)U^* = -U\text{ad}_A U^*$.*

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