What Are "Good" Rhythms? Generating Rhythms Based on the Properties Set Out in The Geometry of Musical Rhythm

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Abstract. We propose in this article to generate good rhythms from geometric properties. This approach is based on the work by Toussaint, who investigated the properties that make a "good" rhythm good in his book *The Geometry of Musical Rhythm*. To do this, he analyzed the shapes of polygons corresponding to certain rhythms to derive geometric properties of a good rhythm. In this article, we propose to quantify these properties using original mathematical formulas. This scores each rhythm against several properties to measure how good a rhythm is, resulting in the generation of rhythms with k onsets and n pulses. When $k = 5$ and $n = 16$, we reveal that the *son* rhythm obtains the best score, as predicted by Toussaint at the end of his book. Applying the method for other values of k and n, we demonstrate that some of the rhythms with the highest scores are musically important rhythms, such as the *tresillo* rhythm with $k = 3$ and $n = 8$, the *fume-fume* rhythm with $k = 5$ and $n = 12$, the *samba* rhythm with $k = 7$ and $n = 16$ or the *Steve Reich's signature* rhythm with $k = 8$ and $n = 12$.

Keywords: Musical rhythm · Geometry of music · Generating music · Geometrical modeling of rhythms.

1 Introduction

What Makes a "Good" Rhythm Good? is the central theme of Toussaint's book *The Geometry of Musical Rhythm* [14]. To address this question, he investigated the properties of six distinguished rhythms, each with $k = 5$ onsets and $n = 16$ pulses: *son*, *bossa-nova*, *gahu*, *rumba*, *shiko* and *soukous* rhythms. These "good" rhythms were chosen given their musical importance. They are represented in polygon notation in Figure 1, which helps to understand the cyclical aspect of a rhythm. The originality of Toussaint's work is the geometric approach he uses to describe the properties that make these rhythms good. In particular, only

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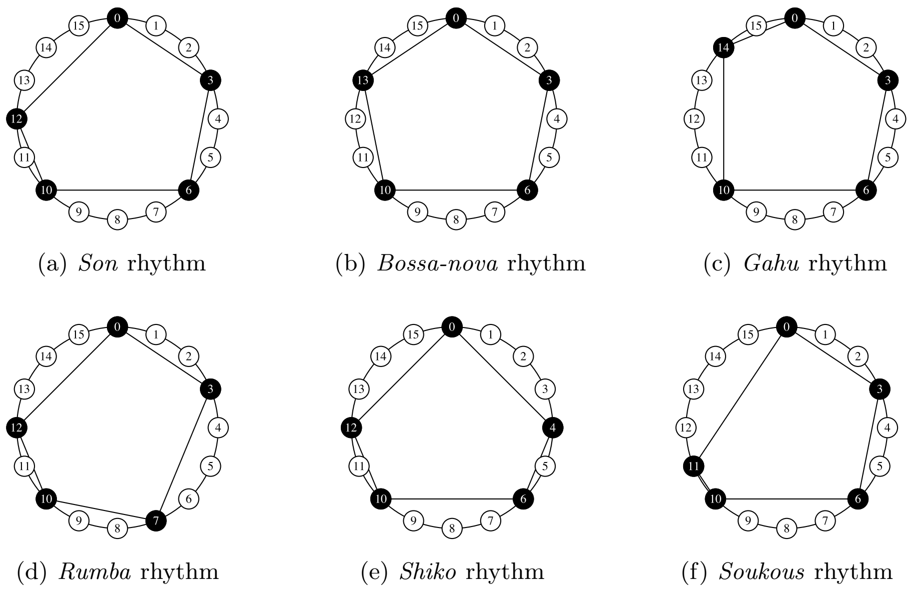


(a) *Son* rhythm (b) *Bossa-nova* rhythm (c) *Gahu* rhythm

(d) *Rumba* rhythm (e) *Shiko* rhythm (f) *Soukous* rhythm

Fig. 1. The six distinguished rhythms in polygon notation. The n pulses are indicated by numbers in the cyclic representation (from 1 to $n = 16$ here), and black points correspond to the k onsets ($k = 5$ here).

the shape of polygons associated with rhythms is analyzed, instead of using a more traditional representation such as a score. The properties of a "good" rhythm he proposed are therefore described using a mathematical, rather than musical, language. Among the six distinguished rhythms, he concluded that the *son* rhythm is the best one, as it satisfies the most properties [14]. But what about other rhythms that have $k = 5$ onsets and $n = 16$ pulses? Is there a rhythm as good or even better than the *son* rhythm? What are the "good" rhythms for other values of k onsets and n pulses? In this article, we propose mathematical models of the properties obtained by Toussaint to answer these questions and generate "good" rhythms with k onsets and n pulses.

We inherit from Toussaint's book the challenging term "good" rhythm, with the understanding that it underlines the theoretical interest of studying certain rhythmic properties in relation to the importance and pertinency of rhythms in their musical context. It is by no means an evaluation in an ethical sense. Rather, it points to the desire to have landmarks and measures of theoretical interest in the otherwise mindnumbing list of all combinatorially possible rhythms. In the following, we will omit the quotation marks for "good" with the understanding that the attribute stands for the value of fulfilment of rhythmic properties. This also includes the possibility for dissense among music theorists about the relevance of certain properties (see for example the criticism of some of Toussaint's featured properties [6]). Therefore, the purpose of this article is not to generate the best possible rhythms, which is a highly complex problem, but to extend Toussaint's work by providing non-binary mathematical formulas that measure each of these properties.

Previous works have already used the geometric properties of polygons associated with rhythms to generate good rhythms, such as balance [11], eveness [12] or interlocking reflection [13]. However, to our knowledge, there is no method that combines several properties with a non-binary score to generate good rhythms. To this end, in Section 2, we propose mathematical formulas for quantifying certain properties that characterize a good rhythm, in order to determine the level at which a rhythm with k onsets and n pulses satisfies each property, and to assign it a score aggregating all properties. Then, in Section 3, we apply this method to rhythms with $k = 5$ onsets and $n = 16$ pulses to generate good rhythms according to several properties. Finally, we apply the method to other values of k onsets and n pulses in Section 4 and demonstrate that some of the rhythms with the highest scores are musically important rhythms.

2 Characterizing the Properties of Good Rhythms

In this section, we present the properties we have considered in order to generate good rhythms. Most of these properties are from the book *The Geometry of Musical Rhythm* by Toussaint [14]. The main contribution of this article is to propose mathematical formulas to measure how a given rhythm satisfies each property. This quantification makes it possible to generalize the analysis to rhythms other than the six distinguished rhythms illustrated in Figure 1. In particular, this enables to analyze a large database of rhythms, which would be impossible to do manually. Among the properties mentioned in [14], we have selected 12 that are particularly relevant for obtaining good rhythms. For each property, we normalize the value obtained by the mathematical formula with the *min-max normalization*[3] with the exception of the symmetry, shadow and fractal properties, to avoid overemphasizing these properties because we noticed that the min-max normalization gave too much importance to these properties (the values for rhythms with $k = 5$ onsets and $n = 16$ pulses lie in the interval $[0.6, 1]$, $[0.4, 1]$ and $[0.2, 1]$, respectively). We therefore sum the 12 values of the 12 properties to obtain a score between 0 and 12 for each rhythm (the sum was chosen for its compensation effect among the properties). By computing the score of all rhythms that have, for example, 5 onsets and 16 pulses, this allows to generate good rhythms with 5 onsets and 16 pulses based on geometric properties only.

In the remainder of this article, we define a rhythm R with k onsets and n pulses by $R = \{\overline{x_1}, \ldots, \overline{x_k}\}$ with $\overline{x_i} \in \mathbb{Z}_n$, where k and n are non-zero integers such that $k \leq n$. In this case, we set $x_i \in \mathbb{Z}$ the representative of $\overline{x_i} \in \mathbb{Z}_n$ such that $0 \leq x_i \leq n - 1$. For example, the *son* rhythm displayed in polygon notation in Figure 1(a) is $\{\overline{0}, \overline{3}, \overline{6}, \overline{10}, \overline{12}\}$. In addition, we require in this article that $\overline{x_1} = \overline{0}$ in order to identify an onset as the beginning of the rhythm. We

[3] The min-max normalization of x is defined by the formula $\frac{x-m}{M-m}$ where m and M are respectively the minimum and maximum values of the data.

define $d_j = x_{j+1} - x_j$ the *duration* between two successive onsets x_j and x_{j+1} with $x_{k+1} = x_1 + n$, and the *interval* as the minimum duration between two onsets which are not necessarily successive. In the rest of this section, we present each of the 12 properties we have considered to generate good rhythms, explain the relevance of each property from a musical point of view and use existing or newly proposed mathematical formulas to quantify how much a rhythm satisfies each property in order to give it a score out of all 12 properties.

2.1 Eveness

The *eveness* of a rhythm characterizes how well the onsets are temporally spaced. If k divides n, then the polygon associated with the rhythm that maximizes eveness is a regular polygon. Otherwise it is the rhythm that best approximates the regular polygon. For example, with $k = 5$ and $n = 16$, the rhythm that maximizes eveness is the *bossa-nova* rhythm (or its rotations) represented in polygon notation in Figure 1(b), which is close to a regular pentagon. Rhythms that maximize eveness given k onsets and n pulses are called *Euclidean* rhythms. There exists an algorithm for generating the Euclidean rhythms with k onsets and n pulses up to a rotation [12]. To quantify the eveness of a rhythm, Milne et al. have proposed the following formula [8], by identifying $\mathbb{Z}_n$ with vectors of norm 1 in the complex space:

$$\text{Eveness} = \frac{1}{k}\left|\sum_{j=1}^{k} e^{2i\pi\left(\frac{x_j}{n}-\frac{j}{k}\right)}\right|$$

This formula can be interpreted as the sum of the deviations between the polygon associated with the rhythm $\{e^{2i\pi\frac{x_j}{n}}\}_{0\leq j\leq k}$ and the regular polygon $\{e^{2i\pi\frac{j}{k}}\}_{0\leq j\leq k}$.

2.2 Balance

The *balance* of a rhythm measures how close the centroid of the associated polygon is to the center of the circle. This concept was introduced by Milne et al. and has also been generalized to musical scales [8, 9]. The formula for quantifying the balance of a rhythm is as follows:

$$\text{Balance} = 1 - \frac{1}{k}\left|\sum_{j=1}^{k} e^{2i\pi\frac{x_j}{n}}\right|$$

Among rhythms with $k = 5$ onsets and $n = 16$ pulses, the rhythm that maximizes the balance is not the *bossa-nova* rhythm but the *son* rhythm (Figure 1(a)), which demonstrates that eveness and balance are two different concepts.

2.3 Area

Another way of quantifying the dispersion of the onsets is to consider the *area* of the polygon associated with the rhythm. The larger the polygon area, the more dispersed the onsets, and vice versa. This idea has already been mentioned in [14], but rhythms with the largest area are barely addressed. We propose to use this property to characterize a good rhythm, which can be quantified as:

$$\text{Area} = \frac{1}{2}\left|\sum_{j=1}^{k} \sin\left(2\pi \frac{d_j}{n}\right)\right|$$

This formula can be interpreted as a sum of isosceles triangles at the center of the circle. Among rhythms with $k = 5$ onsets and $n = 16$ pulses, the *bossa-nova* rhythm is the one that maximizes area. It is important to note that this property is close to eveness and balance but is different.

2.4 Oddity

The *oddity* characterizes rhythms with an even number of pulses that do not have two onsets that are opposite to each other, i.e. two onsets that divide the rhythm into two sub-rhythms of equal length [2]. We can therefore quantify the oddity of a rhythm as its proportion of onsets that do not divide the rhythm in half, as follows for n even:

$$\text{Oddity} = 1 - \frac{1}{k}\left|R \cap (R + n/2)\right|,$$

where $R + n/2 = \{\overline{x_1} + \overline{n/2}, \ldots, \overline{x_k} + \overline{n/2}\}$ is the rhythm R shifted in time by half its length. This concept does not make sense for n odd, for which the oddity is not defined. Not all rhythms that maximize oddity are good. For example, the rhythm $\{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$ maximizes oddity but is not musically relevant, which argues for the combination of different properties to generate good rhythms.

2.5 Closure

For rhythms with $n = 16$ pulses, Toussaint has defined the *closure* of a rhythm if the last onset is equal to 12, meaning that the last onset is located on the last strong beat [14]. Here, we propose to generalize this notion to give it more nuance. First, we propose to measure how an onset is a strong beat by computing its *orbit*, defined as: $\text{Orb}(\overline{x_i}) = \{\overline{x_i} \times \overline{y} \mid \overline{y} \in \mathbb{Z}_n\}$. For example, if $n = 16$, we have $\text{Orb}(\overline{4}) = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}\}$ and $\text{Orb}(\overline{5}) = \mathbb{Z}_{16}$. With fewer elements in the orbit, the onset is a stronger beat. Therefore, we define the closure of a rhythm to quantify the level at which the last onset $\overline{x_k}$ of the rhythm is on a strong beat:

$$\text{Closure} = 1 - \frac{1}{n}\left|\text{Orb}\left(\overline{x_k}\right)\right|$$

This formula generalizes the idea introduced by Toussaint, but is useless if n is prime, as the strong and weak beats are more complex to determine.

2.6 Aperture

Inspired by the closure property, we propose here the *aperture* property. A rhythm satisfies this property if its second onset $\overline{x_2}$ is on a weak beat. We propose the following formula to quantify this property:

$$\text{Aperture} = \frac{1}{n} \left|\text{Orb}\left(\overline{x_2}\right)\right|$$

For example, if $n = 16$ and $\overline{x_2} = \overline{3}$, the aperture is maximal. The aperture creates tension while the closure makes a resolution. This new property partly explains why the *tresillo* rhythm $\{\overline{0}, \overline{3}, \overline{6}\}$ is one of the most important rhythms with $k = 3$ onsets and $n = 8$ pulses. In this case, the second onset is on a weak beat ($\overline{x_2} = \overline{3}$) which creates tension, while the last onset is not on a weak beat ($\overline{x_3} = \overline{6}$) which can be interpreted as a resolution.

2.7 Deepness

The *deepness* property characterizes rhythms where each interval between onsets occurs with a unique multiplicity, such that these multiplicities are the set $\{1, 2, \ldots, k-1\}$ [5]. The formula that quantifies deepness is as follows:

$$\text{Deepness} = \frac{k(k-1)}{2} - \min_{0 \leq p \leq \lfloor n/2 \rfloor} \left(\sum_{j=1}^{\lfloor n/2 \rfloor} h_i - \max\left(0, j+1-p\right) \right),$$

where $H = \{h_1, \ldots, h_{\lfloor n/2 \rfloor}\}$ is the ordered full interval-content histogram, and $\lfloor x \rfloor$ is the integer part of x. Rhythms that maximize the value of deepness are called *deep* rhythms. There are some links between Euclidean and deep rhythms. If a rhythm is Euclidean then it is deep, but the converse is false [5]. There is also an algorithm for generating deep rhythms [3]. Finally, some deep rhythms are also minimally even (unlike Euclidean rhythms, which are maximally even) which reveals that these concepts are related but also very different.

2.8 Entropy of Full Histogram

The *entropy of full histogram* ensures that interval multiplicities between onsets are similar [14]. This property can be quantified using the following formula:

$$\text{Entropy of Full Histogram} = -\sum_{j=1}^{\lfloor n/2 \rfloor} p_j \log_2\left(p_j\right),$$

where $P = \{p_1, \ldots, p_{\lfloor n/2 \rfloor}\}$ is the normalized full interval-content histogram. Unlike the deepness property, this produces a rhythm with a flat histogram if zero-multiplicity intervals are not considered.

2.9 Entropy of Durations

The *entropy of durations* ensures that the durations between two successive onsets are similar to each other. This property is quantified with the following formula:

$$\text{Entropy of Durations} = -\sum_{j=1}^{k} d_j \log_2 (d_j)$$

We propose this property because it provides another method of obtaining a rhythm with temporally dispersed onsets.

2.10 Symmetry

One of the properties that characterizes a good rhythm is that the associated polygon has an axis of symmetry [14]. To quantify this property, we propose to compute the maximal proportion of symmetrical onsets according to the different axes with the following formula:

$$\text{Symmetry} = \max_{0 \leq j \leq n'} \frac{1}{k} \left| S_j(R) \cap R \right|,$$

where $S_j(R)$ is the symmetry of R with respect to the axis passing through the points $\{0, e^{i\pi \frac{j}{n}}\}$, i.e.: $S_j(R) = \{\overline{j} - \overline{x_i} \mid \overline{x_i} \in R\}$, and $n' = n - 1$ if n even or n if n odd.

2.11 Shadow

The *shadow* property characterizes rhythms that have a cyclic rhythmic contour that is the same as the cyclic rhythmic contour of the shadow of the rhythm [14]. We first introduce a function N of normalization such that $N(x) = 1$ if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$, and we set $x_{k+j} = x_j + n$. The *cyclic rhythmic contour* CR of a rhythm R is defined by the cyclic contour of the rhythm durations: $CR = \{N(x_{i+2} - 2x_{i+1} + x_i)\}_{1 \leq i \leq k}$. Next, the *shadow* of a rhythm R is defined by the set $\{\frac{1}{2}(x_{i+1} - x_i)\}_{1 \leq i \leq k}$, and the cyclic rhythmic contour CS of the shadow of a rhythm is given by: $CS = \{N(x_{i+3} - x_{i+2} - x_{i+1} + x_i)\}_{1 \leq i \leq k}$. Consequently, the shadow property quantifies the similarity of CR and CS with respect to rotation, which is given by the following formula:

$$\text{Shadow} = \max_{j \in \{1,\ldots,k\}} \left| i \in \{1, \ldots, k\} : cr_i = cs_{i+j} \right|,$$

where $CR = \{cr_1, \ldots, cr_k\}$ is the cyclic rhythmic contour of the rhythm and $CS = \{cs_1, \ldots, cs_k\}$ is the cyclic rhythmic contour of the shadow of the rhythm. Among the six distinguished rhythms, only the *son* rhythm maximizes it.

2.12 Fractal

The *fractal* property characterizes the inclusion of a dilated R sub-rhythm in R. We propose the following formula to quantify this property:

$$\text{Fractal} = \frac{1}{k} \max_{R' \subseteq R/l, lR' \subseteq R} |R' \cup lR'| \, ,$$

where $R/l = R \cap \{\overline{0}, \dots, \overline{x_{\lfloor n/l \rfloor}}\}$ is the rhythm R until time $x_{\lfloor n/l \rfloor}$, and $lR' = \{\overline{lx_1}, \dots, \overline{lx_k}\}$ is the rhythm dilated by l such that $2 \leq l \leq n$. For example, the rhythm $R = \{\overline{0}, \overline{5}, \overline{6}, \overline{10}, \overline{12}\}$ displayed in Figure 2(e) in polygon notation maximizes the value of fractal because $R' = \{\overline{0}, \overline{5}, \overline{6}\}$ is included in R and $2R' = \{\overline{0}, \overline{10}, \overline{12}\}$ is also included in R and we have $\frac{1}{5}|R' \cup lR'| = 1$.

3 Generating Good Rhythms for $k = 5$ and $n = 16$

In this section, we present the results we obtained by applying the proposed method to rhythms with $k = 5$ onsets and $n = 16$ pulses. First, Table 1 provides the values of the six distinguished rhythms for all 12 properties described in Section 2. The last line of this table is the sum of the 12 values, and measures how good a rhythm is. The *son* rhythm is the one with the highest score among the six distinguished rhythms. However, what about other rhythms with $k = 5$ onsets and $n = 16$ pulses? We have applied the method to these rhythms such that the first onset is located at the origin ($\overline{x_1} = \overline{0}$). This generates many different good rhythms by selecting those with the best score. We have represented the twenty rhythms with the highest scores in Figure 2 using polygon notation.

Table 1. Values of the six distinguished rhythms for all 12 properties.

	Son	Bossa-nova	Gahu	Rumba	Shiko	Soukous
Eveness	0.99	1.0	0.97	0.98	0.97	0.94
Balance	1.0	0.94	0.85	0.91	0.92	0.92
Area	0.97	1.0	0.97	0.97	0.93	0.87
Oddity	1.0	1.0	0.5	1.0	0.5	0.5
Closure	0.86	0.0	0.57	0.86	0.86	0.0
Aperture	1.0	1.0	1.0	1.0	0.14	1.0
Deepness	0.33	1.0	0.0	0.33	1.0	0.0
Entropy H	0.97	1.0	0.97	0.97	0.94	0.87
Entropy D	0.37	0.0	0.81	0.56	0.0	0.81
Symmetry	1.0	1.0	0.6	0.8	1.0	0.8
Shadow	1.0	0.6	0.6	0.4	0.6	0.8
Fractal	0.8	0.6	0.6	0.6	0.6	0.6
Total	10.29	9.14	8.44	9.37	8.46	8.11

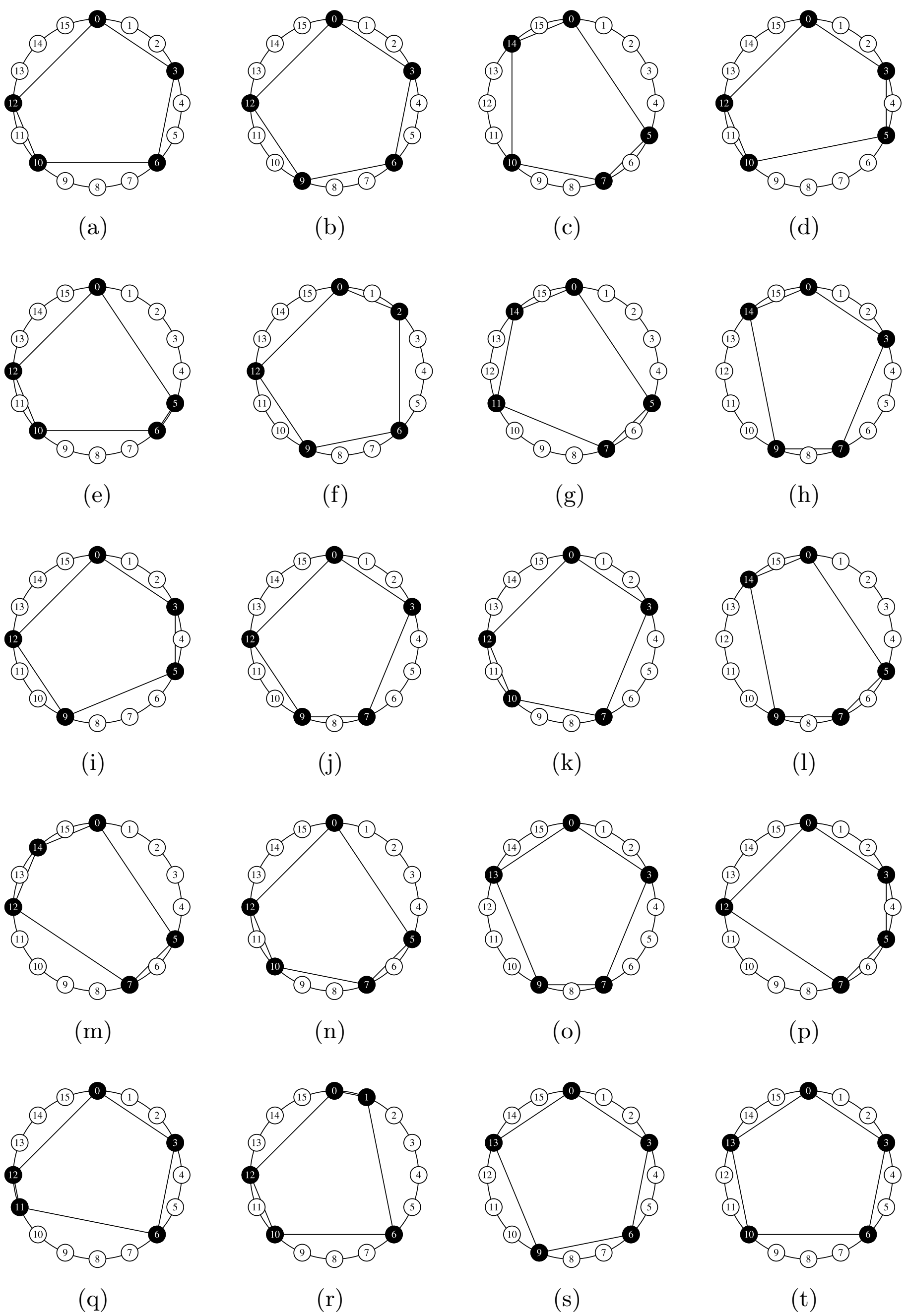


Fig. 2. Generation of the twenty rhythms with the best score for $k = 5$ and $n = 16$.

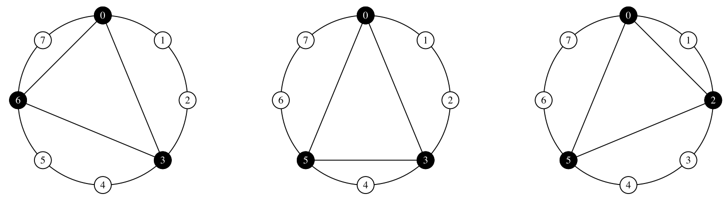
The *son* rhythm is the rhythm with the highest score, as predicted by Toussaint [14]. This can be explained by the fact that most of the properties come from his book, for which the *son* was already on top. Also, among these twenty rhythms, (k) is the *rumba* rhythm and (t) the *bossa-nova* rhythm. The other rhythms represented in Figure 2 are rhythms that are close to the six distinguished rhythms, in the sense that they share common properties because they have a high score. It is also possible to generate "bad" rhythms by selecting rhythms with the worst score, as already done in [7]. For example, $\{\overline{0},\overline{12},\overline{13},\overline{14},\overline{15}\}$, $\{\overline{0},\overline{2},\overline{3},\overline{4},\overline{15}\}$ and $\{\overline{0},\overline{2},\overline{3},\overline{14},\overline{15}\}$ are the three rhythms with the worst scores.

4 Generating Good Rhythms for Other Values of k and n

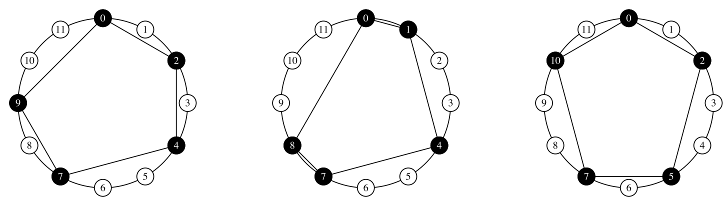
In this section, we apply the method for other values of k onsets and n pulses and demonstrate that some rhythms with the best score are musically important. The three rhythms with the highest scores for onset and pulse values (k,n) equal to $(3,8)$, $(5,12)$, $(7,16)$ and $(8,12)$ are illustrated in Figure 3 using polygon notation. For $k=3$ and $n=8$, the rhythm with the highest score is the *tresillo* rhythm, shown on the left in Figure 3(a), used in part in Latin and Afro-Cuban music. For $k=5$ and $n=12$, the rhythm with the highest score is the *fume-fume* rhythm, shown on the left in Figure 3(b), frequently employed in West African music. For $k=7$ and $n=16$, the rhythm with the highest score is the *samba* rhythm necklace from Brazil, shown on the left in Figure 3(c). Note that the actual *samba* rhythm is a rotation of it by 2 pulses: $\{\overline{0},\overline{2},\overline{5},\overline{7},\overline{9},\overline{12},\overline{14}\}$. For $k=8$ and $n=12$, the rhythm with the highest score is the *Steve Reich's signature* rhythm [15], shown on the left in Figure 3(d), used in *Clapping Music* and in many of his compositions. However, with our method, the rhythm with the highest score is not always the most popular rhythm. For $k=7$ and $n=12$, one of the most popular rhythms is the *bembé* rhythm: $\{\overline{0},\overline{2},\overline{4},\overline{5},\overline{7},\overline{9},\overline{11}\}$. However, the *bembé* rhythm is not the one with the best score for $k=7$ and $n=12$ because it fails to satisfy the closure and aperture properties. The rhythm with the highest score is a rotation of the *bembé* rhythm by 1 pulse: $\{\overline{0},\overline{1},\overline{3},\overline{5},\overline{6},\overline{8},\overline{10}\}$.

5 Conclusion and Future Work

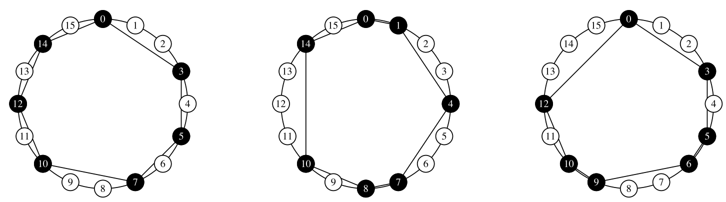
We have proposed in this paper a simple method for generating good rhythms based on a geometric approach. Despite the simplicity of this method, we have demonstrated that the results are the ones expected from Toussaint's findings, as some of the best generated rhythms are musically significant. Furthermore, defining scores, i.e. gradual satisfaction properties, instead of binary values allows, to some extent, to link qualitative concepts with concrete geometrical measures. In our future work, we will add other properties to the 12 we have chosen, and study their relevance. We also plan to go further in linking formal properties and musical meaning, for instance, high degree of deepness leads to good phase canons as done by Steve Reich [4]. In addition, perceptual analyses can be performed



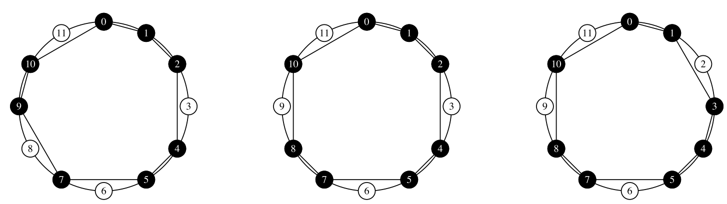
(a) $k = 3$ and $n = 8$



(b) $k = 5$ and $n = 12$



(c) $k = 7$ and $n = 16$



(d) $k = 8$ and $n = 12$

Fig. 3. Generation of the three rhythms with the best score (from left to right) for different values of k onsets and n pulses. In each case, the rhythm with the best score is a musically important rhythm: (a) the *tresillo* rhythm, (b) the *fume-fume* rhythm, (c) the *samba* rhythm and (d) the *Steve Reich's signature* rhythm.

to determine whether the generated rhythms are indeed good and to determine whether certain properties are more important than others, as already done for eveness, balance or entropy [10]. We also believe that the case where n is odd or prime is worth studying, by adapting certain properties and analyzing the obtained results. Moreover, the proposed approach can be framed within the theory of *formal concept analysis*. Rhythms are objects O, satisfying a series of properties P, to some degree. This satisfaction relation can be seen as a fuzzy binary relation I, thanks to the gradual nature of our formulas, in $O \times P$, i.e. for $r \in O$ and $p \in P$, $I(r, p)$ is the degree to which rhythm r satisfies property p (score computed according to the relevant formula for p). From the formal context (O, P, I), fuzzy formal concepts can be derived, and organized in a lattice (see [1] for details on formal concept analysis). This structured way of representing rhythms and their properties could then be used for classification, or more generally as a support for the music-theoretical discourse about rhythms. Similar reasoning can be employed to generate musical scales by adapting certain geometric properties to the associated polygons, e.g. based on the work in [3]. Finally, we hope that these results, developed from a purely mathematical approach, can be applied in a musical context for composition and improvisation using the rhythms generated by this type of method.

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