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Fabien Leurent

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# Line Ride-Sharing as a bi-sided mobility service with price schedule, transactional protocol and waiting policy: a Time&Money traffic assignment model and its equilibrium

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## Abstract

A line ride-sharing service is supplied along a given roadway path by an operator that matches Users (riders) and Agents (drivers), under specific protocol that involves price schedule on both the U and A sides, waiting policy on either side and transaction times. The resulting time and money items add up over trip legs, yielding trip time and money cost depending on the service role, A or U, compared to Non-commitment, called role N for Neutral. The article brings about a traffic model of people involvement in the service. Service conditions of frequency  $\varphi$  and average number of users per car run  $\omega$  are key factors of the time and money features of the alternative roles A, U and N. Individual choice of role is modeled as a rational behavior of minimizing the generalized cost depending on the individual Value-of-Time (VoT). Aggregation over trip-makers according to the statistical distribution of VoT yields the respective role flows  $(y_A, y_U, y_N)$ , which in turn determine the macroscopic factors  $(\varphi, \omega)$ . Traffic equilibrium is defined as a balance condition between the “supplied flows” and the “demanded flows” of the three roles. A computational scheme is provided, with graphical interpretation in the  $(y_A, y_U)$  plane as well as in the  $(\varphi, \omega)$  plane. A numerical experiment is conducted, showing that two alternative configurations can arise at equilibrium: either  $\{A, U, N\}$  with less wealthy Agents driving wealthier Users, versus  $\{U, A, N\}$  where less wealthy Users are driven by wealthier Agents; in both cases the Neutral role attracts the upper range of the VoT distribution.

## Keywords:

Ride-sharing service; bi-sided platform; traffic equilibrium; multi-sided equilibrium; equilibration algorithm

## Highlights

H1/ Service featured out in terms of run frequency and average car load

H2/ Microeconomic behavior at the individual level to select a role with respect to the service

H3/ Traffic equilibrium as balance conditions between supplied- and demanded- role flows

H4/ Equilibrium computation by graphical method

H5/ Alternative configurations of Agents<Users versus Users<Agents along the VoT axis

## 1/ Introduction

### Background

Line ride-sharing (LRS) is a specific kind of platform-based RS service in which the rides take place along a specific roadway path, thus called the line or more specifically the line link. Rider access to cars may be restricted to special stop points called “stations” (cf. Ecov’s “Line” kind of service) or allowed at any point along the line path (cf. Ecov’s “Line+”).

Then, along the roadway path the car trip-makers fall into either one out of three classes: Service users as Riders (type U), car drivers involved in the service as Agents (type A) and other trip-makers that do not take part to the service, called Neutral (type N). The three types can be seen as specific

travel modes, say sub-modes of the car mode. By time period and flow direction, the respective trip flows of the three roles, denoted as  $y_r$  for each role  $r \in \mathfrak{R} \equiv \{U, A, N\}$ , determine both the service quantity and quality of the RS line:

- $y_A$  amounts to the service frequency  $\varphi$  to potential users – a key component of service quality to them.
- Ratio  $\omega = y_U/y_A$  is the expected passenger load by service-affiliated car – a basic indicator of money incomes to agents based on the principle of trip cost sharing.
- $y_U$  and  $y_A$  are the basic quantities of service production, leading to its market share within the car mode,  $(y_A + y_U)/q$  with  $q = \sum_{r \in \mathfrak{R}} y_r$ , and the overall occupancy rate in persons per car,  $q/(y_A + y_N)$ .

Service participation either on the User or Agent side depends on the attractiveness of the respective modal option to car trip-makers.

The modal specific money and time expenses per trip constitute option attributes in the modal decisions of the individual trip-makers. Money costs  $m_r$  are lower for service participants (A/U) than for non-participants (N). In contrast, times  $t_r$  are higher for service participants A and U than for N because of transaction operations, car dwelling and waiting on one side. Postulating microeconomic rationality, every individual chooses the role yielding minimal generalized cost to him depending on his own value-of-time (VoT)  $\beta$ :  $g_r(\beta) = m_r + \beta t_r$ . Aggregating the individual choices over the statistical distribution of trip-makers (CDF  $B$  of  $\beta$ ), and multiplying by the total number of trip-makers,  $q$ , yield the modal flows  $y_r$  hence frequency  $\varphi$  and passenger load  $\omega$ , and also the average car speed  $v$  on the roadway path.

## Research questions

This article addresses the following research questions:

RQ1/ Given the service conditions in terms of price schedule and usage protocol (including waiting conditions), what are the respective money and time attributes of the three modes?

RQ2/ Having featured out the money and time attributes of the three modes, what are their respective ranges of efficiency along the VoT axis, i.e., according to  $\beta$ ?

RQ3/ Given the modal attributes and the  $(q, B)$  pair of trip-maker flow and statistical distribution of VoT, what are the modal flows [ $y_r: r \in \mathfrak{R}$ ] and the related service performances  $(\varphi, \omega, v)$ ?

RQ4/ Taking the determination of modal conditions from the  $(\varphi, \omega)$  factors as a supply function, and that of trip flows from the modal conditions as a demand function, what is their joint outcome, is there a traffic equilibrium between supply and demand?

## Article objectives

The article is aimed to devise a bicriterion, time-money model of traffic assignment to a line ride-sharing service on a roadway path.

Under theoretical form, not only does the model shed light on service issues such as price schedule, waiting policies either on the User or Agent side and transactional protocol, but also it yields modal flows as analytical formulas of the various parameters and system state variables.

Two companion objectives are also addressed: (i) a mock case study is conducted to provide a first understanding of potential situations, (ii) the computation of equilibrium states: a specific, graphical approach is devised to deal with the multifold interaction between the modal flows and the modal conditions in time and money.

## Approach

Time&money bicriterion assignment was originally devised to study the modal competition between the plane and the train (Abraham & Blanchet, 1973). It was then applied to interurban ground mobility and the train vs. car modal competition (Quandt & Baumol, 1966; Marche, 1973) and to traffic assignment on roadway networks (Dafermos, 1972; Marche & Papon, 1987; Daito et al., 1992; Leurent, 1993; Dial, 1996, etc.). Here we model the individual roles of User, Agent and Neutral as specific travel modes on a roadway path, thereby constituting a specific kind of bicriterion model. The  $(\omega, \varphi)$  variables are specific and give rise to an original kind of bicriterion traffic equilibrium, together with path speed  $v$  that is endogenous as in former time&money models of traffic assignment.

The “line” features allows for simple theory: the issue of path diversion which is of paramount importance in bicriterion network traffic assignment is not involved here. The simple form of the model allows for explicit specification of the ride-sharing service characteristics, especially so to compare two alternative policies either “Agent Waits” (AW) or “User Waits” (UW). Their respective effects on the modal times translate straightforwardly on the model outcomes  $(\omega, \varphi, v)$  and  $[\gamma_r: r \in \mathfrak{R}]$ .

## Article structure

The rest of the article is in five parts. Section Two represents the system of Roadway path, Line RS service and the population of car trip-makers. Section Three provides the traffic assignment model and studies its equilibrium state as a multi-sided traffic equilibrium, considering the two sides of Service Agents and Users and also the Neutral trips as a third tier. Section Four is devoted to a numerical experiment (mock case study). Section Five discusses the model outreach and limitations and also some issues of system design. Section Six concludes and points out to some directions of further research.

Tab.1: Notation.

$\varphi$ service frequency	$L$ link length
$\omega$ rider load of car run	$H$ period duration
$r, s \in \mathfrak{R} \equiv \{A, U, N\}$ service role	$v$ link speed
$\ell \in \mathcal{L} \equiv \{R, S, T, W\}$ trip leg	$V$ speed-flow function
$\delta_r$ indicator variable of wait assignment to role $r$	$q$ population size
$\varepsilon \equiv \delta_A - \delta_U$	$\gamma_r$ person flow of role $r$
$t_r$ trip time of role $r$ , time $t_r^\ell$ by leg $\ell$	$\psi_r$ time flow rate of individuals of role $r$
$\gamma_r$ modal constant of role $r$	$x_r$ car trip flow of role $r$
$\underline{m}$ base car cost per trip	$\beta$ Value-of-Time (VoT)
$\dot{m}$ length factor of car cost	B CDF of VoT
$\tau_r$ base trip award ( $r = A$ ) or fare ( $r = U$ )	$E_r^\#$ efficiency range of role $r$
$\dot{\tau}_r$ length factor of award ( $r = A$ ) or fare ( $r = U$ )	$\underline{\beta}_r^\#$ lower bound of efficiency range
$m_r$ money cost of role $r$	$\bar{\beta}_r^\#$ upper bound of efficiency range
$g_r$ generalized cost of $r$ as function of VoT	$\beta_{rs}^\#$ cut-off VoT between roles $r$ and $s$
$\omega_M$ parameter of money line	$p_r$ population proportion assigned to role $r$
$\omega_\varepsilon(\varphi), \omega_\infty$ parameters of time line	$\mu_{s-r}$ short hand notation of fixed cost difference
$\rho$ parameter of car ridership	$\tilde{t}$ short-hand notation

## 2/ System representation

### 2.1/ Roadway path and traffic scene

A roadway link of network length denoted  $L$ , is considered in either flow direction as a specific “traffic scene”. It is studied on a daily basis, by within-day time period. A specific period of interest is considered, e.g. morning peak or evening peak or in-between, with time duration denoted  $H$ .

### 2.2/ Service roles as travel modes

With respect to a line ride-sharing service on the roadway link, three roles are identified for car trip-makers: service user as Rider (role U), service agent as driver (role A) or non-participant i.e. neutral (role N). The set of roles is denoted  $\mathfrak{R} \equiv \{U, A, N\}$ .

On a per trip basis, the respective money expenses are denoted as  $m_r$ , and time expenses as  $t_r$ . These split into “leg times”  $t_r^\ell$  according to trip phases or legs,  $\ell \in \mathcal{L} \equiv \{R, S, T, W\}$ :

- Leg R of car running on the link, same for all modes,
- Leg S of Stop or Dwelling for rider boarding and alighting, for Users and for those Agents that get “customers”,
- Leg T of Transaction operations: getting information and possibly being assigned, Paying on the User side or Being paid on the Agent side, both using the service digital platform,
- Leg W of Waiting: either the User is required to Wait for the next Agent under the UW policy, or the Agent is asked to wait for an incoming user under the AW policy.

According to role  $r$ , the modal trip times sums up the leg times of the different kinds:

$$t_r = \sum_{\ell \in \mathcal{L}} t_r^\ell \quad (1)$$

Trip travel time is a primary indicator of the quality of the mode as a travel option.

### 2.3/ Service features and policy

Ride-sharing is a bi-sided form of transport service as it involves people in two different ways: not only the service-to-demand form of Users, but also the cooperator-to-service form of Agents. These cooperate with the service coordination (platform) by supplying seats in their vehicles, the driving function including dwelling for rider boarding and alighting, possibly waiting for incoming riders to get to the car and other assistance (e.g. putting a foldable two-wheeler in the car trunk).

Service coordination between the two sides, Agents and Users, relies upon a digital interaction platform that achieves user and agent matching (in other words, the assignment of riders to cars and that of cars to riders) by suitable information collection and delivery, as well as fare collection from users and money compensation to agents.

These transactional operations are assumedly performed efficiently owing to high level of automation and suitable platform customer interface, with total time per trip of  $t_A^T$  and  $t_U^T$  on the agent and user side respectively.

Also given are the service stop times per trip,  $t_U^S$  on the user side giving rise to  $t_A^S$  on the agent side if the agent car run is endowed with rides.

Both the transaction and stopping times are taken as exogenous. By contrast, the major features of service quality are endogenous as they fundamentally depend on people participation to the service. The disaggregate involvement of individuals as Users or Agents, aggregated over the link trips, gives rise both to of rider flow  $y_U$  and frequency  $\varphi = y_A$  of service runs, hence to average time interval  $H/\varphi$  between successive service runs during the period, and to average wait times per trip as follows:

$$\underline{t}_U^W = H/\varphi \text{ and } \underline{t}_A^W = 0 \quad (2-UW)$$

$$\underline{t}_U^W = 0 \text{ and } \underline{t}_A^W = H/\gamma_U, \quad (2-AW)$$

depending on whether the service waiting policy is targeted to Users or to Agents.

While  $\underline{t}_U^W$  applies to all rides on the user side, agent wait time  $\underline{t}_A^W$  only applies to the trips of agents to which customer rides are assigned.

Furthermore, ratio  $\gamma_U/\gamma_A$  determines the average passenger load by agent run, denoted  $\omega$ :

$$\omega = \gamma_U/\gamma_A \quad (3)$$

The following Proposition is demonstrated in Appendix A on postulating that both flows of Users and Agents are independent Poisson processes with respective time rates  $\psi_r \equiv \gamma_r/H$ . Let also  $\rho \equiv \omega/(1 + \omega)$ .

**Proposition 1: Time items and car occupation according to waiting policy.**

1/ Under AW policy: (i) the wait times of users are negligible, (ii) the wait times of agents make a random variable (RV) that is distributed as the probability mixture of 0 and  $\text{EXP}(\psi_A)$ , with respective coefficients  $1 - \rho$  and  $\rho$ , yielding average agent wait time of  $\rho H/\varphi$ .

2/ Under UW policy: (i) the wait times of agents are negligible, (ii) the user wait times are distributed  $\text{EXP}(\psi_A)$ , i.e., exponentially distributed with parameter  $\psi_A$ .

3/ Under both policies: (i) the stop times of agents is a binary variable with values either 0 or  $\underline{t}_A^S$ , the latter with probability  $\rho$ , yielding an average of  $\rho \underline{t}_A^S$ . (ii) the number of users per agent car is a geometric RV with parameter  $\rho$ , hence average load is  $\frac{\rho}{1-\rho} = \omega$ .

Letting  $\delta_A$  be the binary indicator of AW and  $\delta_U$  that of UW, the option times are formulated generically as follows:

$$t_N = t_N^R = t_R, \quad (4-N)$$

$$t_A = t_R + t_A^T + \rho \underline{t}_A^S + \delta_A \rho \frac{H}{\varphi}, \quad (4-A)$$

$$t_U = t_R + t_U^T + t_U^S + \delta_U \frac{H}{\varphi}. \quad (4-U)$$

Beside the waiting policy, service policy also includes (i) roadway link selection in relation to the local Mobility Organizing Authority, (ii) Access conditions of Rides to Car runs, assumed here at link endpoints only, (iii) the price schedule.

By assumption, for each car trip offered by an agent as a service run, a twofold reward (fee) is awarded to the agent: a base fee denoted  $\tau_A$  applies whatever the number of riders (including modality Zero), plus a per ride contribution that depends on ride length, say  $L\hat{t}_A$  with  $\hat{t}_A$  the ride award rate per unit length. On average per service run, the money reward to the agent is thus

$$\hat{t}_A = \tau_A + \omega L\hat{t}_A \quad (5-A)$$

On the user side, we similarly distinguish between a fixed fare  $\tau_U$  and a length-variable fare  $L\hat{t}_U$ : the ride fare is thus

$$\hat{t}_U = \tau_U + L\hat{t}_U \quad (5-U)$$

There certainly is a relationship between  $\hat{t}_A$  and  $\hat{t}_U$ : some financial balance between  $\gamma_A \hat{t}_A$  and  $\gamma_U \hat{t}_U$  may be expected, up to external subsidizes e.g. from the local mobility organizing authority on a per run basis, to taxes and of course to platform remuneration. Such issues of crucial interest of the service business model are left aside in this article.

## 2.4/ Modal money items and generalized costs

The ride fare constitutes the basic money cost of the link trip to a service user:

$$M_U \equiv \hat{t}_U = \tau_U + \omega L \dot{t}_U \quad (6-U)$$

Service-neutral trip-makers are assumed to spend a fixed money cost of  $\underline{m}$  plus a length-variable cost at unit length rate of  $\dot{m}$ , yielding link cost of

$$M_N \equiv \underline{m} + L \dot{m} \quad (6-N)$$

To a service agent, the link cost amounts to car expense minus run income, yielding

$$M_A \equiv M_N - \hat{t}_A = \underline{m} + L \dot{m} - \tau_A - \omega L \dot{t}_A \quad (6-A)$$

Some trip features remain unobserved in the model, notably the sub-paths up- and down-stream the roadway link.

The unobserved features of the alternative modes  $r$  are called “modal constants” measured as basic generalized costs and denoted  $\gamma_r$ . Denoting now the individual VoT as  $\beta$  (the change rate of travel time to money), at the trip level the modal option  $r$  induces a “travel generalized cost” of

$$g_r^\#(\beta) \equiv m_r + t_r \cdot \beta \quad (7)$$

Wherein  $m_r \equiv M_r + \gamma_r$ .

The value of  $\beta$  is likely to be modulated according to trip legs – for instance, wait time may be perceived by the individual as more costly than the run time. We keep such modulations implicit – it may be denoted by role- and leg-dependent VoTs  $\beta_r^\ell$  such that

$$\beta \cdot t_r = \sum_{\ell \in \mathcal{L}} \beta_r^\ell \cdot t_r^\ell.$$

## 2.5/ Population composition and trip flows

By assumption, each individual in the population of interest makes one and one trip only on the roadway link during the period under study. Thus the population size and the trip flow are equal and both are denoted as  $q$ .

Within this population, the individual VoTs are distributed statistically, with CDF denoted  $B$  and complementary function  $\bar{B} \equiv 1 - B$ .

The modal flow  $y_r$  by travel mode  $r \in \mathfrak{R}$  gives rise to a car flow  $x_r$  that is equal to  $y_r$  for  $r \in \{A, N\}$  or null otherwise, i.e.  $x_U = 0$ . Total car flow amounts to

$$x = x_A + x_N = y_A + y_N \quad (8)$$

## 2.6/ Traffic issues

We model the influence of car trip flow  $x$  onto link speed  $v$  as a traffic law  $V$  between speed and hourly flow rate:

$$v = V\left(\frac{x}{H}\right) \quad (9)$$

This traffic law induces the link run time by trip whatever its mode:

$$t_r^R = \frac{L}{v}, \quad \forall r \in \mathfrak{R} \quad (10)$$

Both flow volume  $x$  and speed  $v$  are traffic conditions determining the local emissions of noise and of air pollutants, including GHG emissions of global outreach.

### 3/ Multi-sided traffic equilibrium

#### 3.1/ Flow relationships

We have already introduced the relationships linking the endogenous service features  $(\omega, \varphi)$  to the service flows  $(y_A, y_U)$  in (2) and (3) respectively, as well as the composition of car flow in (8) and the speed-flow relationship (9). The modal split in the population of individuals makes up another flow relation:

$$q = \sum_{r \in \mathfrak{R}} y_r \quad (11)$$

#### 3.2/ Microeconomic behaviors and the configuration of modal costs

According to the service Waiting policy  $\# \in \{AW, UW\}$ , the modal generalized costs are functions of individual VoT  $\beta$  – linear affine functions under the previous assumptions.

The fundamental model postulate about individual behavior is that each individual selects the option of minimum cost to him or her, meaning minimum disutility hence maximum utility, i.e., rational microeconomic behavior.

Depending on  $\beta$ , the modal option yielding minimum cost is denoted  $\hat{r}^\#(\beta)$ , with associate generalized cost of

$$\hat{g}^\#(\beta) = \min\{g_r^\#(\beta): r \in \mathfrak{R}\} \quad (12)$$

The variations of option costs with respect to  $\beta$  determine the efficiency domains of the mode: in the classical Time&Money model, lower price modes are selected by individuals with low values of  $\beta$ , whereas shorter time ones are selected by higher  $\beta$  (Figure 1).

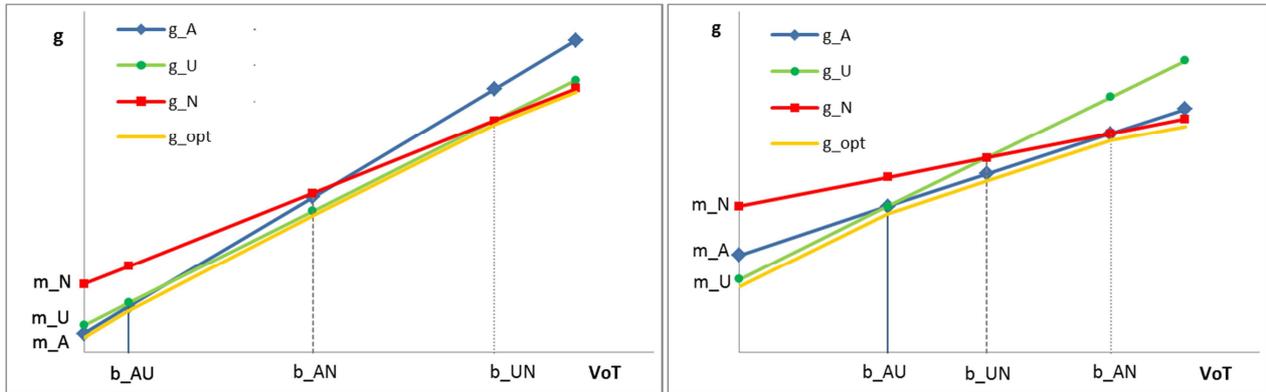


Fig.1: Generalized cost functions of the service roles, yielding configuration (A) AUN, (B) UAN.

As regards modal prices, based on (6) it holds that

$$M_A \leq M_N \quad (13a)$$

Under realistic values satisfying  $\tau_A \geq 0$ ,  $\hat{\tau}_A \geq 0$  and  $\omega \geq 0$ .

We may safely assume that  $\gamma_A \approx \gamma_N$ , leading to  $m_A = M_A + \gamma_A \leq M_N + \gamma_N = m_N$ .

On the user side, we postulate that  $\gamma_U$  does not change the ordering of money expenses: so that

$$m_U \leq m_N \quad (13b)$$

*NB. This postulate is a strong assumption about the up- and down-stream parts of individual trips.*

In a diagram of modal cost functions with respect to VoT, modal prices make the origin ordinates and the modal times make the slopes of their respective cost lines.

As transaction and stop times are null for Neutral option,  $t_N^T = t_N^S = 0$ , but have positive values for types U and A, whatever the Waiting policy we have that

$$t_N \leq t_U^\# \quad (14a)$$

$$t_N \leq t_A^\# \quad (14b)$$

This makes “Neutral” the best value option for high VoTs.

The comparison between  $t_U^\#$  and  $t_A^\#$  involves the waiting policy and the frequency  $\varphi$ , as well as the  $\omega$  factor through  $\rho$ :

- Under AW and  $\omega \geq 1$ , then  $t_U^\# < t_A^\#$  up to significant discrepancy between  $t_U^T$  and  $t_A^T$  (unlikely) or between  $t_U^S$  and  $t_A^S$  (unlikely, too).
- Under UW and  $\omega < 1$ , then  $t_A^\# < t_U^\#$ .
- There are other cases such as AW with  $\omega < 1$ , or UW with  $\omega \geq 1$ .

The two specific cases yield opposite outcomes: the former making User a higher quality option than Agent, the latter making Agent a higher quality option than User.

The joint condition AW and  $\omega \geq 1$  looks consistent: under high patronage, having agents to wait for riders is not much requiring. A contrasted condition AW and  $\omega < 1$  yields no obvious outcomes.

The joint condition UW and  $\omega < 1$  looks consistent, too: making Users to wait for Agents that are more numerous. Yet a contrasted condition UW and  $\omega \geq 1$  would put both waiting requirements and in-vehicle crowding as burdens on the users.

### 3.3/ Efficiency domains and flow outcomes

The efficiency set of mode  $r$  under policy  $\#$  is

$$E_r^\# \equiv \{\beta \in [0, +\infty[ : g_r^\#(\beta) \leq g_s^\#(\beta), \forall s \in \mathfrak{R}\}$$

By definition,

$$\underline{\beta}_r^\# \equiv \inf\{\beta \in \mathbb{R}^+ : g_r^\#(\beta) \leq g_s^\#(\beta), \forall s \in \mathfrak{R}\} = \inf E_r^\#$$

$$\bar{\beta}_r^\# \equiv \sup\{\beta \in \mathbb{R}^+ : g_r^\#(\beta) \leq g_s^\#(\beta), \forall s \in \mathfrak{R}\} = \sup E_r^\#$$

From the affine linear nature of modal generalized costs as functions of VoT  $\beta$ , the non-empty efficiency domains on  $[0, +\infty[$  are simple intervals  $[\underline{\beta}_r^\#, \bar{\beta}_r^\#)$  from “lower bound”  $\underline{\beta}_r^\#$  to “upper bound”  $\bar{\beta}_r^\#$ : we expect that  $\underline{\beta}_r^\# \leq \bar{\beta}_r^\#$  if the interval is non-empty, but values  $\bar{\beta}_r^\# = 0$  and  $\underline{\beta}_r^\# = +\infty$  indicate an empty domain.

The modal market share is (Figure 2):

$$p_r^\# = \Pr\{\beta \in E_r^\#\} = (B(\bar{\beta}_r^\#) - B(\underline{\beta}_r^\#))^+. \quad (15)$$

It gives rise to modal volume

$$y_r^\# = q \cdot p_r^\#. \quad (16)$$

The configuration of efficiency domains along the VoT axis has a deep outreach. Two basic configurations are {A,U,N} meaning  $E_A^\# \leq E_U^\# \leq E_N^\#$  versus {U,A,N} meaning  $E_U^\# \leq E_A^\# \leq E_N^\#$ .

Degenerate cases include {A,N} i.e. empty set  $E_U^\#$  - a service with agents but no customers, and {U,N} a service with potential users but no agents, entailing no available runs.

Non-degeneracy with positive  $p_A^\#$  and  $p_U^\#$  may come with  $p_N^\# = 0$ , under bounded VoT distribution such that  $\underline{\beta}_N^\# > 0$  but  $B(\underline{\beta}_N^\#) = 1$ .

The comparison of two options  $r$  and  $s$  gives rise to “frontier VoT”, denoted  $\beta_{rs}^\#$  and also called “cut-off VoT”, at which point the two modes are equivalent, i.e., their generalized costs are equal:

$$g_r^\#(\beta) = g_s^\#(\beta). \quad (17)$$

The equivalency condition straightforwardly entails that

$$\beta_{rs}^\# = \frac{m_s - m_r}{t_r - t_s} (= \beta_{sr}^\#). \quad (18)$$

Provided that  $t_r \neq t_s$ .

Non-degenerate situation {U,A,N} involves the following inequality conditions between the cut-off VoTs:

$$0 < \beta_{UA}^\# \leq \beta_{UN}^\# \leq \beta_{AN}^\#$$

And more precisely the stronger conditions that

$$0 < B(\beta_{UA}^\#) < B(\beta_{UN}^\#) < B(\beta_{AN}^\#) \quad (19a)$$

Non-degenerate configuration {A,U,N} arises when:

$$0 < B(\beta_{UA}^\#) < B(\beta_{AN}^\#) < B(\beta_{UN}^\#) \quad (19b)$$

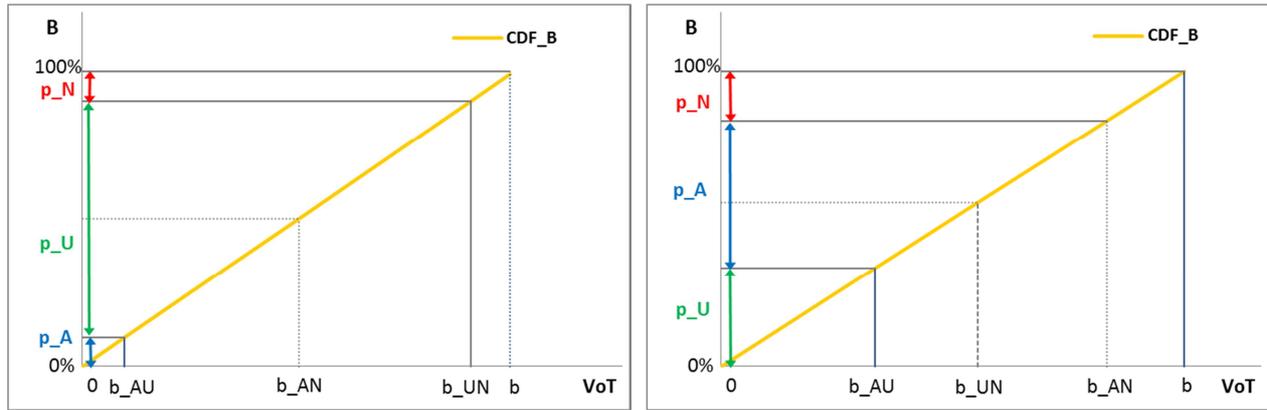


Fig.2: Modal shares of the service roles, under configuration (A) AUN, (B) UAN.

### 3.4/ Traffic equilibrium

**Definition.** The system is in equilibrium if all relations (1-18) linking its endogenous state variables are satisfied jointly, i.e. they do hold true simultaneously.

Such equilibrium state is a traffic equilibrium since it involves flows  $[y_r: r \in \mathfrak{R}]$  and related variables  $(\varphi, \omega, v)$ . It is a multi-sided equilibrium as it involves individual behaviors on both sides of the cooperative mobility service: agents and users.

The system of influences between the variables is depicted in Figure 3. A “basic” state vector is a vector of selected endogenous variables from which all of the other endogenous variables are derived straightforwardly: two possible selections are exhibited in the Figure, namely the vector  $\mathbf{y}_{\mathfrak{R}} = [y_r: r \in \mathfrak{R}]$  of modal flow volumes in part A, or the vector  $(\varphi, \omega)$  of service conditions in part B.

The derivation of “demanded flows” (or “people flows”)  $[y_r^\#: r \in \mathfrak{R}]$  is straightforward from  $(\varphi, \omega)$ , though lengthier than that of “supplied flows” (or “service flows”)  $[\hat{y}_r: r \in \mathfrak{R}]$ .



$$\hat{y}_A = q \cdot \underline{B}_A^\# \text{ and } \hat{y}_U = \hat{y}_S - \hat{y}_A. \quad (24\_AU)$$

Configuration U<A involves  $\underline{\beta}_U^\# = 0$  and  $\bar{\beta}_U^\# = \underline{\beta}_A^\# = \beta_{UA}^\# < \bar{\beta}_A^\# = \underline{\beta}_N^\# = \beta_{AN}^\#$ , yielding that

$$\hat{y}_U = q \cdot \underline{B}_U^\# \text{ and } \hat{y}_A = \hat{y}_S - \hat{y}_U. \quad (24\_UA)$$

Thus, any non-degenerate tri-modal equilibrium is characterized by a threefold condition – essentially a twofold condition since the 3<sup>rd</sup> one is obvious:

$$A<U: \begin{cases} \hat{y}_A = q \cdot B(\beta_{UA}^\#) \\ \hat{y}_A + \hat{y}_U = q \cdot B(\beta_{UN}^\#) \\ \hat{y}_N = q - \hat{y}_A - \hat{y}_U \end{cases} \text{ versus } U<A: \begin{cases} \hat{y}_U = q \cdot B(\beta_{UA}^\#) \\ \hat{y}_A + \hat{y}_U = q \cdot B(\beta_{AN}^\#) \\ \hat{y}_N = q - \hat{y}_A - \hat{y}_U \end{cases}. \quad (25)$$

The first condition may be called an “inner service split” (ISS) function, as it involves the two roles A and U, and the second one an “outer service split” (OSS) function, as it delimits  $\hat{y}_S$  from  $\hat{y}_N$ .

**Under A<U configuration**,  $\beta_{UN}^\#$  only involves  $\varphi$  and not  $\omega$ , so that the OSS is simply

$$\bar{\omega} = \frac{q}{\varphi} B(\beta_{UN}^\#) - 1.$$

The OSS condition is stated as  $\bar{\omega}(\varphi)$  or, about equivalently, as  $\bar{y}_U(y_A)$  with respect to modal flows. The ISS condition also relates  $\omega$  to  $\varphi$ , or about equivalently  $y_U$  to  $y_A$ . If the CDF B is invertible at  $\hat{y}_A/q$  then the ISS is equivalent to

$$m_A + (t_A - t_U)B^{(-1)}\left(\frac{y_A}{q}\right) = m_U,$$

From the time items in eqn. (4), it holds that

$$t_A - t_U = \rho(t_A^S + \delta_A \frac{H}{\varphi}) + t_A^T - (t_U^T + t_U^S + \delta_U \frac{H}{\varphi}). \quad (26)$$

Denoting  $\tilde{t} \equiv t_U^T + t_U^S - t_A^T$  and  $\mu_{A-U} \equiv \gamma_A + \underline{m} + L\dot{m} - \tau_A - \gamma_U - \tau_U - L\dot{t}_U$ , to:

$$\mu_{A-U} - \omega L\dot{t}_A + \left(\rho(t_A^S + \delta_A \frac{H}{\varphi}) - \tilde{t} - \delta_U \frac{H}{\varphi}\right) \cdot B^{(-1)}\left(\frac{y_A}{q}\right) = 0.$$

Multiplying by  $\varphi(1 + \omega)$ , it makes  $\omega$  an obvious function of  $\varphi$ , denoted by  $\underline{\omega}(\varphi)$ , and about equivalently  $y_U$  an obvious function  $\underline{y}_U(y_A)$ .

The feasible domain of  $(\varphi, \omega)$  is  $[0, q] \times \mathbb{R}^+$ .

**Proposition: non-degenerate traffic equilibrium under A<U.** A feasible pair  $(\varphi, \omega)$  is a non-degenerate tri-modal equilibrium with A<U configuration if it satisfies that

$$\omega = \bar{\omega}(\varphi) = \underline{\omega}(\varphi).$$

**Under U<A configuration** the direction of influence is reversed. The ISS condition on  $\hat{y}_U$ , assuming invertible B at  $\hat{y}_U/q$ , is

$$m_A = m_U + (t_U - t_A)B^{(-1)}\left(\frac{y_U}{q}\right), \text{ yielding}$$

$$\mu_{A-U} - \omega L\dot{t}_A - \left(\rho(t_A^S + \delta_A \frac{H}{\varphi}) - \tilde{t} - \delta_U \frac{H}{\varphi}\right) \cdot B^{(-1)}\left(\frac{y_U}{q}\right) = 0.$$

Multiplying by  $\varphi^2(1 + \omega)$ , it makes  $y_A$  an obvious function of  $y_U$ , denoted by  $\underline{y}_A(y_U)$ , and about equivalently  $\varphi$  a function  $\underline{\varphi}(\omega)$ .

Then, the condition on  $\hat{y}_S$  is taken as an OSS  $\bar{\varphi}(\omega)$ , about equivalent to an OSS  $\bar{y}_A(y_U)$ .

**Proposition: non-degenerate traffic equilibrium under U<A.** A feasible pair  $(\varphi, \omega)$  is a non-degenerate tri-modal equilibrium with U<A configuration if it satisfies that

$$\varphi = \bar{\varphi}(\omega) = \underline{\varphi}(\omega).$$

Appendix B provides a more comprehensive analysis of the equilibrium configurations: not only the non-degenerate ones but also the degenerate ones bi-modal or uni-modal. The OSS-ISS pairs are further studied in Appendix C. A fully explicit solution is provided in Appendix D for uniformly distributed VoT.

### 3.6/ Time and Money relations and the configurations of roles

The feasible domain of  $(\varphi, \omega)$ ,  $D_{(\varphi, \omega)} \equiv [0, q] \times \mathbb{R}^+$ , or about equivalently that  $D_y$  of pairs  $(y_A, y_U)$  such that  $y_A \geq 0$ ,  $y_U \geq 0$  and  $y_A + y_U \leq q$ , is useful to locate the graphs of the OSS and ISS and to determine equilibrium states graphically at the intersection points between the function graphs. Each domain includes specific regions of interest to compare options A and U according to their respective money costs (called the money-related regions), or according to their times (hence the time-related regions). By combining the money and time conditions, specific regions come out that locate the alternative configurations of the two roles along the VoT axis.

**Money-related regions:** condition  $m_A \leq m_U$  is equivalent to  $\mu_{A-U} - \omega L t_A \leq 0$ , hence to

$$\omega \geq \omega_M \equiv \mu_{A-U} / (L t_A)$$

Thus, region  $\{m_A \leq m_U\}$  lies above the “money line”  $\{m_A = m_U\}$  which is a horizontal line in the  $(\varphi, \omega)$  domain. Region  $\{m_A \geq m_U\}$  lies below the money line. In the  $(y_A, y_U)$  domain, the money line is a diagonal line  $\{y_U = \omega_M y_A\}$ , with region  $\{m_A \leq m_U\}$  on its upper side and region  $\{m_A \geq m_U\}$  on the lower side.

**Time-related regions:** as for times, condition  $t_A \geq t_U$  corresponds to  $t_A - t_U \geq 0$  in (26), hence to pairs  $(\varphi, \omega)$  and the related  $\rho$  such that

$$\rho \geq \rho_\varphi \equiv \frac{\tilde{t} + \delta_U \frac{H}{\varphi}}{t_A^S + \delta_A \frac{H}{\varphi}}$$

As function  $\omega \mapsto \rho \equiv \frac{\omega}{1+\omega}$  is increasing, we consider the inverse function  $\Omega: \rho \mapsto \omega$  and define it as  $+\infty$  for  $\rho \geq 1$ . Thus, in the  $(\varphi, \omega)$  domain, condition  $t_A \geq t_U$  is equivalent to  $\omega \geq \Omega(\rho_\varphi)$ . Region  $\{t_A \geq t_U\}$  lies above the time line  $\{t_A = t_U\} \Leftrightarrow \{\omega = \Omega(\rho_\varphi)\}$  and  $\{t_A < t_U\}$  below it.

An  $\{A < U < N\}$  configuration requires both that  $\{m_A \leq m_U\}$  and  $\{t_A \geq t_U\}$ : hence it is located above both the money line and the time line, in the upper side of both domains  $D_{(\varphi, \omega)}$  and  $D_y$ . Conversely an  $\{U < A < N\}$  configuration, requiring both that  $\{m_A \geq m_U\}$  and  $\{t_A \leq t_U\}$ , is located in the lower side. Over an intermediary region  $\{m_A \leq m_U\}$  and  $\{t_A \leq t_U\}$ , option A fully dominates option U, whereas over  $\{m_A > m_U\}$  and  $\{t_A > t_U\}$  it is option U that fully dominates option A – making an unsustainable state of service traffic with zero agents.

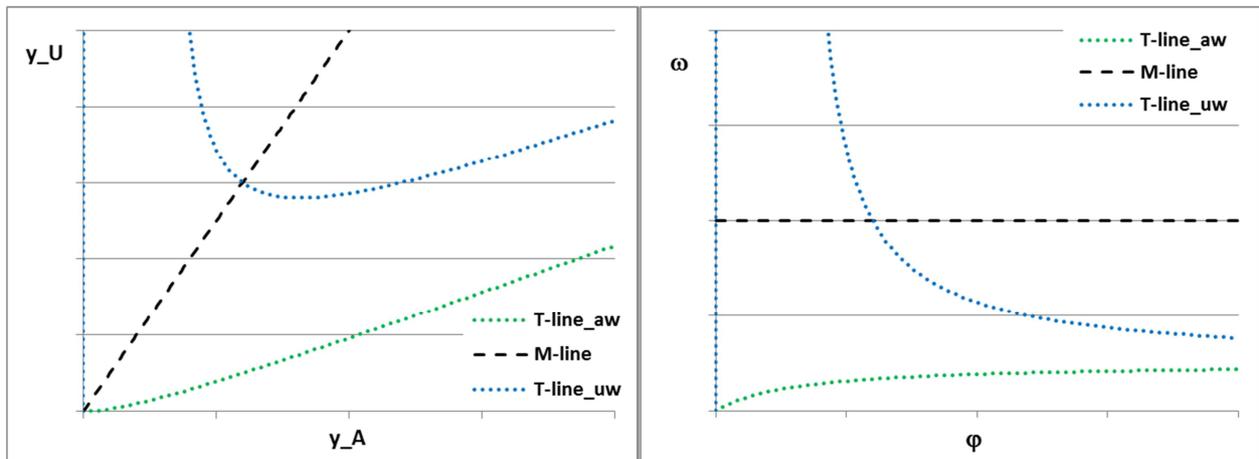


Fig.4: Configurations when  $\tilde{t} < t_A^S$ : (A) in the  $(y_A, y_U)$  domain, (B) in the  $(\varphi, \omega)$  domain.

Figure 4 exhibits the money line and time line under either waiting policy when  $\tilde{t} < \underline{t}_A^S$ , whereas Figure 5 addresses  $\tilde{t} > \underline{t}_A^S$ .

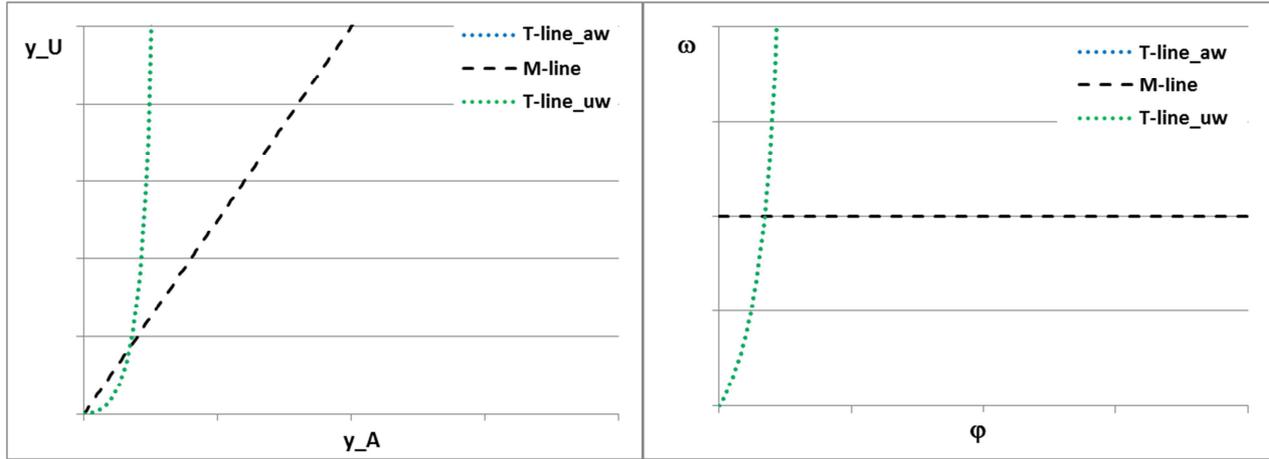


Fig. 5: Configurations when  $\tilde{t} > \underline{t}_A^S$ : (A) in the  $(y_A, y_U)$  domain, (B) in the  $(\varphi, \omega)$  domain.

## 4/ Numerical experiment

The experiment is aimed to demonstrate the model outcomes depending on specific conditions, most notably the Waiting policy and the effect of trip flow rate low or high.

### 4.1/ Case setting

An interurban road link with  $L = 10$  km is considered.

The ride-sharing service has the following price schedule:

- On the agent side,  $\tau_A = 0.5$  € and  $\dot{t}_A = 0.10$  €/km,
- On the user side,  $\tau_U = 1$  € and  $\dot{t}_U = 0.20$  €/km.

The base costs of car holding and utilization are set up to  $\underline{m} = 1$  € per trip and  $\dot{m} = 0.30$  €/km.

The modal constants are set up to  $\gamma_N = 1$  €,  $\gamma_A = 1$  € and  $\gamma_U = 0$  €.

Thus, the per-trip costs amount to:

- Neutral:  $M_N = 4$  € and  $m_N = 5$  €,
- User:  $M_U = 2$  € and  $m_U = 2$  €,
- Agent:  $M_A = 3.5 - \omega$  € and  $m_A = 4.5 - \omega$  €.

The  $\omega_M$  parameter amounts to 2.5 (riders/car run).

As for time items, base stop times of  $\underline{t}_A^S = 1$  min and  $\underline{t}_U^S = 1$  min are assumed, together with transaction times of  $t_A^T = 1$  min and  $t_U^T = 1.5$  min. This induces  $\omega_\infty = 1.5$  (riders/car run).

In the population of trip-makers, individual VoTs follow a uniform distribution from 0 to  $b = 30$ €/h, hence average VoT of 15€/h. During a period of  $H = 1$  hour, contrasted flow levels are considered: “low flow” with  $q = 20$  trips, versus “high flow” with  $q = 200$  trips.

### 4.2/ Low flow

Assuming UW policy, the OSS and ISS relationships are depicted in Figure 6 both in the  $(y_A, y_U)$  domain (part A) and in the  $(\varphi, \omega)$  domain (part B), along with the money line and the time line. All intersection points between OSS and ISS counterparts lie below the money line, hence also below the

timeline that is above the money line in the feasible domain under UW and the parametric set-ups. Thus the equilibrium configuration is  $\{U,A,N\}$  so that only the intersection point of  $OSS_{U<A}$  and  $ISS_{U<A}$  is in equilibrium, making it the unique state of equilibrium.

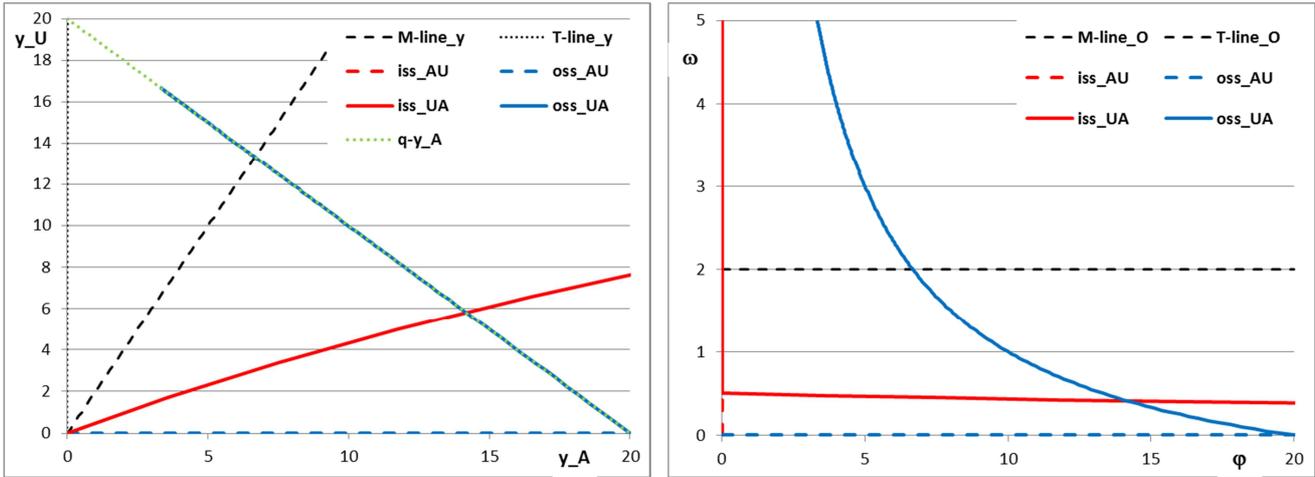


Fig.6: Low-flow situation under UW in (A) the  $(y_A, y_U)$  domain, (B) the  $(\varphi, \omega)$  domain.

Among the equilibrium features (Table 2), salient points are:

- (i) involvement of all individuals in the service:  $y_N = 0$  since the VoT distribution has null upper tail,
- (ii) the  $y_A$  and  $y_U$  flows are close together, with occupancy rate  $\omega$  close to 1,
- (iii) service frequency about 10 per hour is relatively high,
- (iv) Users have lower VoTs than Agents.

Under the same settings save for AW policy (Figure 7), there is only one equilibrium point between OSS and ISS pairs: this time it is between the  $A<U$  functions, meaning a single equilibrium with  $\{A<U<N\}$  configuration. While the neutral flow  $y_N$  is still null, occupancy rate  $\omega \approx 4$  is very high for cars and indicates  $y_U \approx 4$ .  $y_A$ : about 20% of less wealthy people are drivers to the 80% wealthier individuals. The high value of  $\omega$  corresponds to fully occupied cars, yielding both a factor 5 reduction in the car flow of flexible trip-makers and relatively important money incomes to the agents, going beyond the principle of just sharing costs. Service frequency of about 4 car runs per hour is low.

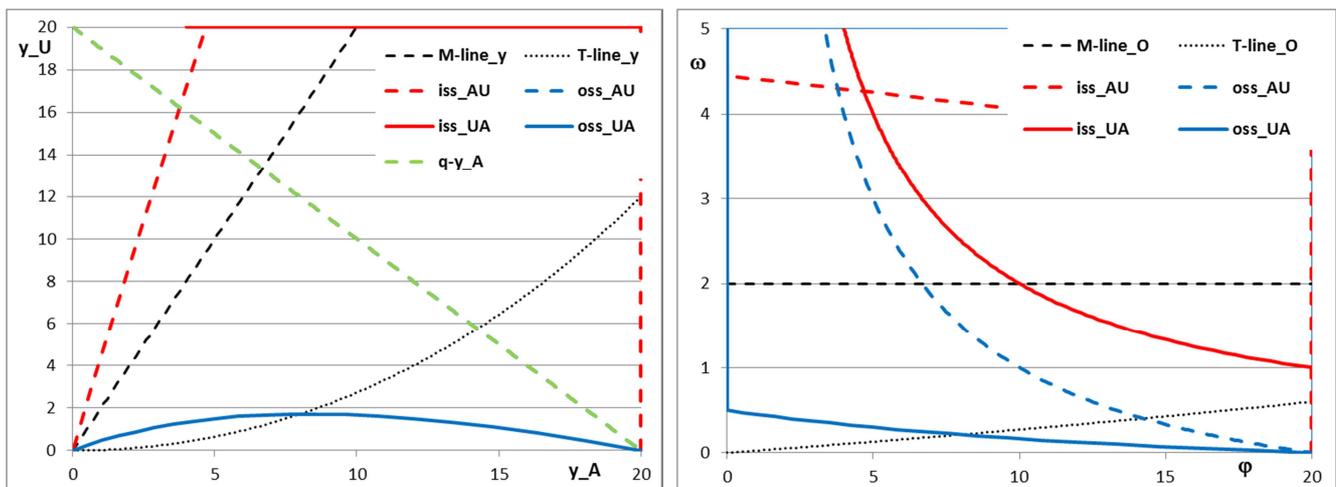


Fig.7: Low-flow situation under AW in (A) the  $(y_A, y_U)$  domain, (B) the  $(\varphi, \omega)$  domain.

Tab. 2: Equilibrium states under low flow (left part) versus high flow (right part).

Indicator (unit)	LOW	FLOW	HIGH	FLOW
	UW	AW	UW	AW
$\omega$ (persons/car)	0.412	4.305	1.142	1.795
$\varphi = y_A$ (trips)	14.166	3.770	93.373	71.558
$y_U$ (trips)	5.834	16.230	106.627	128.442
$y_N$ (trips)	0	0	0	0
$m_A$ (€)	2.794	0.848	2.429	2.103
$t_A$ (hour, without run time)	0.022	<b>0.245</b>	0.026	0.036
$t_U$ (hour, without run time)	0.112	0.042	0.052	0.042
$\beta_{AU}^\#$ (€/h)	8.752	5.655	15.994	19.266
Equilibrium Configuration	U<A<N	A<U<N	U<A<N	U<A<N

### 4.3/ High flow

Under high flow, again a single point of equilibrium is found for each waiting policy UW (fig. 8) or AW (fig. 9). This time, both UW and AW yield an {U<A<N} configuration of equilibrium. Again we have null neutral flow  $y_N = 0$  in either case (table 2, right hand part). Service conditions  $(\varphi, \omega)$  are relatively close between the two waiting policies with  $\omega$  between 2 and 3, meaning  $y_U \approx (2 - 3) \cdot y_A$ , and service frequency of about 55 car runs per hour – very high, indeed.

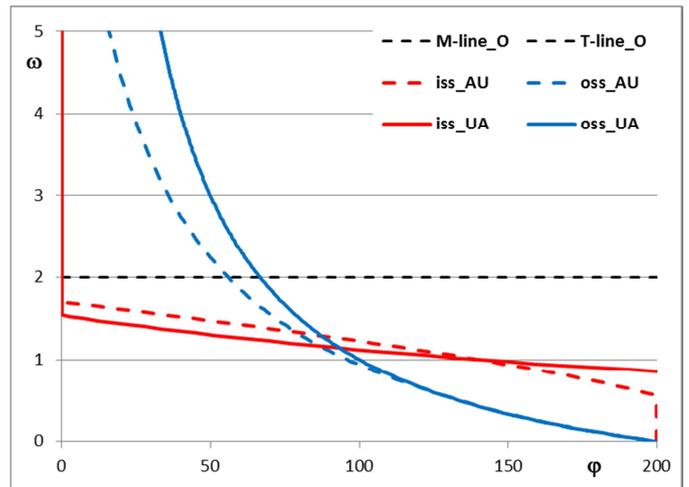
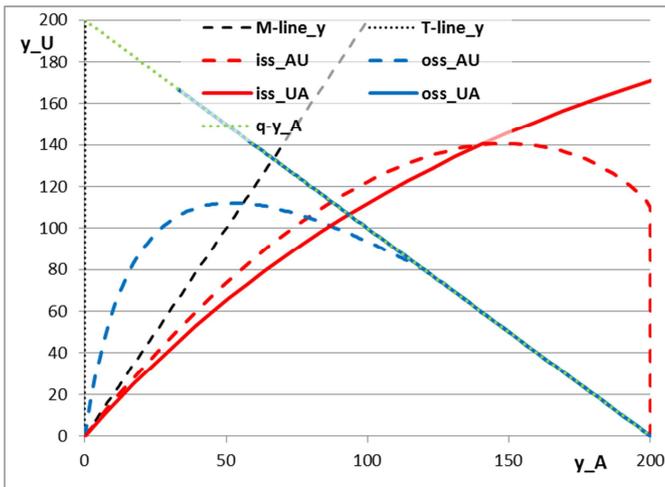


Fig.8: High-flow situation under UW in (A) the  $(y_A, y_U)$  domain, (B) the  $(\varphi, \omega)$  domain.

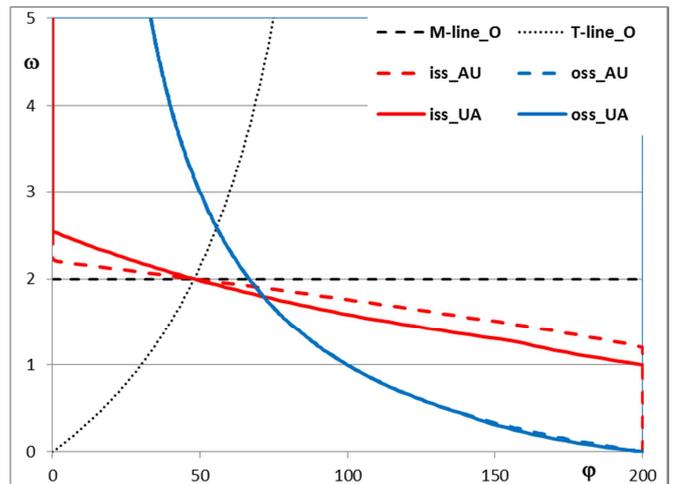
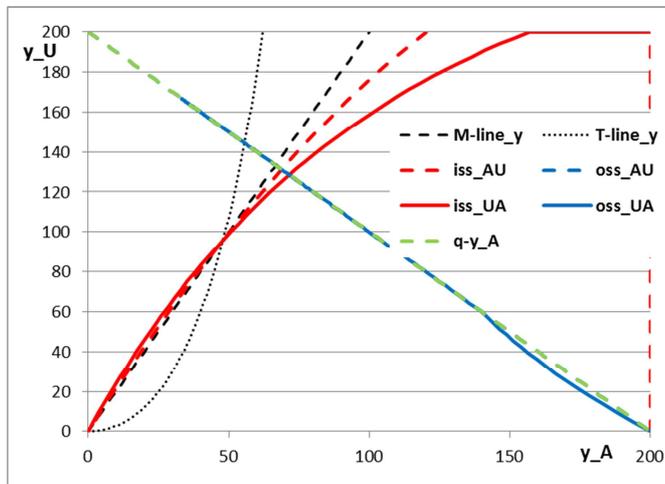


Fig.9: High-flow situation under AW in (A) the  $(y_A, y_U)$  domain, (B) the  $(\varphi, \omega)$  domain.

## 5/ Discussion

### 5.1/ Lessons from the numerical experiment

The graphical determination of equilibrium constitutes a robust computational scheme. It is essentially a bicriterion assignment algorithm as in Dial (1996) and Leurent (1996). Demand disaggregation along the VoT axis is the source of computational robustness that is all the more useful here since the traffic effects between modal flows and modal conditions are much involved and highly intertwined, yielding asymmetry effects in addition to nonlinearities.

The numerical experiment suffices to demonstrate that different equilibrium configurations can arise between the service rules, A versus U and also N. Both configuration {AUN} and configuration {UAN} arose under low flow, whereas only {UAN} came out under high flow. Flow volumes as aggregate indicators are insufficient to determine the configuration, owing to the complex interaction between the roles.

The experiment also demonstrates the correlation between AW policy and {AUN} configuration, on one hand, and UW and {UAN}, on the other hand. Yet, under high flow configuration {UAN} arose under AW as well as UW.

Whatever the configuration of U and A roles, the Neutral role stands at the high end of the VoT distribution. Service participation as U or A basically involves money savings at the expense of some additional times: the associated trade-offs are fruitful for lower VoTs, while higher VoTs would keep to the Neutral role (no involvement in the service).

Lastly, the modal proportions varied greatly among the four instances: such variations call for truly behavioral modeling of ride-sharing services rather than making educated guesses on the respective modal proportions (e.g. Yin et al. 2018).

### 5.2/ About model assumptions: outreach and limitations

The numerical experiment also emphasized the key places of the “money line” (parameter  $\omega_M$  standing for money items and service price schedule) and of the “tree line” according to the waiting policy (parameters  $\varepsilon$  and  $\omega_\infty$ ). The temporal features of wait times and their assignment to either Users or Agents, of stop times and of transaction times are as much influential in service usage as the money items from the price schedule.

Both the money items and the “time items” aside from car run times constitute key features in the “user experience” and the “customer journey” as evocated in marketing literature. Our model captures these features as well as the run times and the other physical times that are commonly modeled for passenger traffic in transit networks (dwell times, wait times, in addition to run times).

On the demand side, the disaggregation of individual preferences along the VoT axis mirrors the disaggregation of individual incomes in microeconomic theory. It is a cornerstone in the microeconomic foundations of the traffic model. The individual trade-offs between time and money items lie at the heart of the model, which derives the aggregate consequences on the supply side in terms of service frequency  $\varphi$  and car occupancy rate  $\omega$ .

In the current stage of the model, individual behavior conditionally to the VoT is a deterministic one with all-or-nothing assignment to one option only. Small differences in time or money can induce modal changes between the roles (car sub-modes) in an abrupt way – a fairly dramatic change at the individual level. In reality, we can expect individual behaviors to be less clear cut and more fuzzy: the modal generalized cost functions are a first step towards the modeling of modal utility functions in the frame of random utility theory and discrete choice models. Such frame will enable one not only to model unobserved effects as residual random variables (cf. the modal constants) but also to represent comfort issues, notably VoT modulation according to service role and trip leg. Such modulation is especially important about waiting policies, since Agent wait times take place in-

vehicles hence at sitting comfortably, whereas User wait times occur out of cars, possibly at standing, maybe so outdoor and subjected to meteorological circumstances.

Beyond comfort come the issues of individual mobility practices, routines and attitudes. Here, service awareness is postulated, so that parameter  $q$  standing for population size represents in fact the number of service-aware people. The prior mobility routines of individuals may also induce captivities of some sort: some car drivers have specific requirements of their private cars in their day activity programs, possibly with professional constraints. Such constraints may come with company rules about access to company cars and usage conditions, departing from our assumptions about the microeconomic situation of the trip-maker.

Yet, other features of individual travel routines may also align with our modeling assumptions. Home to work or study commuting is a recurrent purpose for the working people on a daily basis or so (at least on worked days, up to working from home). Frequent reiteration at the individual level of the travel situation is likely to even out the fluctuations between the occurrences, giving ground to considering the expected times and money outcomes of each role as statistical means over the population of occurrences. In other words, commuting frequency certainly is a factor of alignment between the real-world conditions and the model assumptions.

### 5.3/ Some hints about service design

As an abstract representation, the model brings about concepts, state variables and notations to describe a real-world system, to indicate its conditions and to monitor its performances. It is inspired from early field experiments of ride-sharing lines along some roadway paths and also from pioneer implementations in France by the Ecov company. Our numerical experiment, though simple, reveals density economies and supports the techno-economic principle of lines for ride-sharing.

Path selection and identification is a primary topic of service design, along with station identification and the allowance or not to stop at any place along the path. Up to now, the waiting policy in Ecov's implementations is an "Agent waits" one. Our numerical experiment suggests that the alternative policy "User waits" may be fruitful on some path and demand conditions.

The model also emphasizes the importance of the price schedule and of the transactional conditions. Transaction times, though expectedly short, are nonetheless significant compared to dwell times and even to run times on one or two tens of km. While wait times are even more important under low flows, under high flows they are likely to fall down to a level similar to dwell times and transaction times. The ease and comfort of transaction operations are important, too, so as to make transaction times enjoyable rather than hard-felt. The "customer journey" has to be addressed in all of its respects by any LRS operator.

We have not considered the business model of the LRS operator, but only the demand and traffic facets of the service as a system. Such model of demand and traffic can be a useful tool to design a business model, by considering path features  $(L, v)$  as well as population characteristics  $(q, B)$  and by specifying service scenarios in terms of Waiting policy, price schedule and transactional conditions, up to access conditions.

While the model is focused on service attractiveness compared to the Neutral option, we have not addressed the issue of service awareness. Awareness rising in the population living near the line corridor is especially important to develop the service basis both of users and of agents. Among the solutions under way, displaying RS lines in local MaaS systems of multimodal service information is a prominent one, since they are purported for general use by all of the local population. MaaS integration will likely not be limited to service and traffic information, it will also foster commercial integration and multimodal synergies between the different kinds of public transit services.

## 6/ Conclusion

To understand and simulate a line ride-sharing service, we devised a model that is both a traffic model and a microeconomic model of the service on its two sides of Users and Agents. The economic conditions in money and time of the User and Agent roles and also of the service-Neutral role, are modeled at the trip level from the expected conditions of trip legs (Run, Stop, Wait, Transaction), depending on the service protocol that includes price schedule, waiting policy and transaction times, together with macroscopic service conditions in run frequency  $\varphi$  and average car rider load  $\omega$ . This sensitive and parametric representation of service roles answers to the first Research Question.

Then, on the demand side, role choice is modeled at the individual level as a rational microeconomic behavior of cost minimization, considering the generalized costs of the roles according to the individual VoT. This microeconomic model answers to the second Research Question. Integration along the VoT axis according to the statistical distribution of VoT in the population of car trip-makers yields the modal flows, which in turn determine the  $(\varphi, \omega)$  variables. This answers to the third RQ.

The “service facet” and the “people facet” of the LRS as a traffic system interact in two ways: from service to people (RQ1&2) and conversely from people to service (RQ2&3). The resulting equilibrium was defined mathematically as a set of balance conditions by role A, U and N, between “supplied flow” (from  $\varphi$  and  $\omega$ ) and “demanded flow”. This answers to the fourth Research Question, while raising a further question of equilibrium determination. A computational scheme was devised that locates the two essential balance conditions as a pair of so-called Outer- and Inner- Split Service functions in both the  $(\varphi, \omega)$  plane and the plane of role flows  $(y_A, y_U)$ .

Depending on  $(\varphi, \omega)$ , the system state lies in either one of two basic configurations of roles along the VoT axis: either  $\{U < A\}$  where less wealthy users are driven by wealthier agents, or  $\{A < U\}$  where less wealthy agents drive wealthier users. In both cases the higher range of VoT keeps to the Neutral option.

The formulation of traffic equilibrium as a fixed point problem depending on the configuration ensures the existence of an equilibrium state. The issue of uniqueness is still under study.

In its current stage, the model of LRS as a bi-sided mobility service with traffic equilibrium and configuration issues lies at the interface of economic modeling and traffic modeling. Further research may be invested along three avenues. The first avenue consists in developing the physical and economic content. On the demand side, multiple classes may be considered with different conditions regarding travel modes, for instance car dependency or transit dependency. By class, random utility functions would represent a significant advance over generalized cost functions, as explained in the Discussion. The individual frequency of the travel situation in a multi-day timeframe also deserves specific investigation, as does the consideration of individual mobility at the level of the day chain of activities and trips. On the supply side, the interaction of people through and with the service operator may be modeled in a dynamic way rather than reduced to average conditions – cf. hyperpath motivation in traffic assignment to transit networks. Furthermore, the economic conditions of the service operator deserve to be modeled, not only commercial revenues but also specific production costs and the relationship between the operator and the mobility authority – the potential rewarding of runs and rides in line with the service impacts on the society and the environment. Potential applications may be targeted to the design of price schedules.

The second avenue of research is to develop the spatial scope of the model by introducing the LRS concept into OR problems of network design and service planning. Potential applications include (i) path selection for LRS design, (ii) optimal development of LRS network.

Empirical matters constitute the third avenue of research. The model may be applied to LRS experiment so as to characterize the service conditions and assess its performances. Conversely, such application would enable for econometric estimation of the model parameters, from transaction times to VoT depending on service role and trip leg.

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## Appendix A: Car occupation & time items depending on waiting policy

It is postulated that the flows of Users and Agents are independent Poisson processes with respective time rates  $\psi_r \equiv y_r/H$ . On average over  $H$ , the expected number of individuals with role  $r$  is then  $\psi_r \cdot H = y_r$ . In each flow, the individuals arrive successively, with inter-arrival times between two successive individuals that are independent and distributed  $\text{EXP}(\psi_r)$ . The average inter-arrival time is thus  $1/\psi_r = H/y_r$ .

Recall that  $\rho \equiv \omega/(1 + \omega)$ .

### A1/ Under User Waits: User wait time

The seat capacity in cars is neglected, so that every user can board any car as soon as it arrives after the user own time of arrival at the origin station. Whatever the user time of arrival, from it the wait time up to an agent arrival is distributed  $\text{EXP}(\psi_A)$ , from the memory-less property of the exponential distribution of agent inter-arrival times.

As for Agents, there is no wait time on their side. In practice, an exception would be the arrival of additional users during the stop time of a agent dwelling for a first user to board in it.

## A2/ Under Agent Waits: Agent wait time

Let us consider any agent conditionally to the delay  $\theta$  from his time of arrival to that of the next agent. During  $\theta$ , the number  $N_\theta$  of user arrivals is an RV distributed Poisson with parameter  $\psi_U\theta$ , yielding

$$\Pr\{N_\theta = n\} = e^{-\psi_U\theta} \cdot \frac{(\psi_U\theta)^n}{n!}, \forall n \in \mathbb{N} \quad (\text{A.1})$$

From the properties of the Poisson process, the  $n$  instants of user arrivals are independent RVs denoted  $t_i$  with identical uniform distribution on  $[0, \theta]$ , hence with CDF as follows:

$$F_\theta(z) \equiv \Pr\{t_i \leq z \mid \theta\} = \min(1, \frac{z}{\theta}), \forall z \in \mathbb{R}^+.$$

The agent wait time  $t_{W|\theta,n}$  conditionally to  $\theta$  and  $n$  is the maximum of the  $n$  RVs  $t_i$ :

$$t_{\theta,n} \equiv \max\{t_i : i \in \{1, \dots, n\}\}$$

Thus  $\{t_{W|\theta,n} \leq z \mid \theta\} = \cap_{i=1}^n \{t_i \leq z \mid \theta\}$ . From the independence between the  $t_i$ , the CDF of  $t_{W|\theta,n}$  is

$$\Pr\{t_{W|\theta,n} \leq z \mid \theta\} = \prod_{i=1}^n \Pr\{t_i \leq z \mid \theta\} = (F_{z\theta})^n, \text{ wherein } F_{z\theta} \equiv F_\theta(z).$$

Deconditioning over  $n$ ,

$$\begin{aligned} \Pr\{t_{W|\theta} \leq z \mid \theta\} &= \sum_{n \geq 0} \Pr\{N_\theta = n\} \Pr\{t_{W|\theta,n} \leq z \mid \theta, n\} \\ &= e^{-\psi_U\theta} \sum_{n \geq 0} \frac{(\psi_U\theta F_{z\theta})^n}{n!} \\ &= \exp(-\psi_U\theta(1 - F_{z\theta})) = \exp(-\psi_U(\theta - z)^+) \end{aligned}$$

Deconditioning now over  $\theta$ ,

$$\begin{aligned} \Pr\{t_W \leq z\} &= \int_0^\infty \Pr\{t_{W|\theta} \leq z \mid \theta\} d \Pr\{\theta\} \\ &= \int_0^\infty \exp(-\psi_U(\theta - z)^+) e^{-\psi_A\theta} \psi_A d\theta \\ &= \int_0^z e^{-\psi_A\theta} \psi_A d\theta + e^{\psi_U z} \int_z^\infty e^{-\psi_{U+A}\theta} \psi_A d\theta \\ &= 1 - e^{-\psi_A z} + e^{\psi_U z} e^{-\psi_{U+A} z} \frac{\psi_A}{\psi_{U+A}} \\ &= 1 - e^{-\psi_A z} \frac{\psi_U}{\psi_{U+A}} \\ &= \frac{\psi_A}{\psi_{U+A}} + (1 - e^{-\psi_A z}) \frac{\psi_U}{\psi_{U+A}} \end{aligned}$$

This CDF is that of the probabilistic mixture of a Dirac variable at 0 and a variable  $\text{EXP}(\psi_A)$ , with respective probabilities  $\psi_A/\psi_{U+A}$  and  $\psi_U/\psi_{U+A}$ , i.e., of the probabilities of having null or strictly positive number of users on board the Agent's car.

Thus, to the agent the probability of having to stop amounts to  $\frac{\psi_U}{\psi_{U+A}} = \frac{\omega}{1+\omega} = \rho$ , yielding average Agent stop time of  $t_A^W = \rho t_A^S$ .

**N.B.** To satisfy the distributional assumptions, it requires that (i) user flow to be not so large compared to agent flow, so that the seat capacity in the car is mostly sufficient, (ii) the service coordination collects information about passage times from both the users and the agents, so that the platform can know  $\theta$  and  $n$  "in advance" compared to the individuals: thus, the platform is able to tell to each agent whether to stop or not, and how many users will board the car.

### A3/ Number of Users per Agent car run (of riders per driver)

Under policy AW, conditionally to  $\theta$  the number of Users boarding the car,  $N_\theta$ , is a Poisson RV with parameter  $\psi_U\theta$ , hence the  $\Pr\{N_\theta = n\}$  in (A.1).

Deconditioning over  $\theta$ ,

$$\begin{aligned}\Pr\{N = n\} &= \int_0^\infty \Pr\{N_\theta = n\} d\Pr\{\theta\} \\ &= \int_0^\infty e^{-\psi_U\theta} \frac{(\psi_U\theta)^n}{n!} e^{-\psi_A\theta} \psi_A d\theta \\ &= \left(\frac{\psi_U}{\psi_{U+A}}\right)^n \frac{\psi_A}{\psi_{U+A}} \int_0^\infty \frac{(\psi_{U+A}\theta)^n}{n!} e^{-\psi_{U+A}\theta} \psi_{U+A} d\theta\end{aligned}$$

The latter integral is easily integrated by parts, yielding say  $u_n$  such that

$$u_n \equiv \int_0^\infty \frac{(\psi\theta)^n}{n!} e^{-\psi\theta} \psi d\theta = \left[ -\frac{(\psi\theta)^n}{n!} e^{-\psi\theta} \right]_0^\infty + \int_0^\infty \frac{(\psi\theta)^{n-1}}{(n-1)!} e^{-\psi\theta} \psi d\theta = u_{n-1}$$

So that  $u_n = u_0 = 1$ . Thus,

$$\Pr\{N = n\} = \left(\frac{\psi_U}{\psi_{U+A}}\right)^n \frac{\psi_A}{\psi_{U+A}} = (1 - \rho) \cdot \rho^n$$

We recognize a geometric distribution with parameter  $\rho$ . Its average is thus  $\frac{\rho}{(1-\rho)} = \omega = \frac{\gamma_U}{\gamma_A}$ .

Put in words, the number of users per agent is an RV that only depends on the ratio of their respective flows.

**N.B.** In the initial version of the article, on expecting variable  $\omega$  to be small, it was taken not only as the ratio of their flows but also as the agent probability of getting users – hence the diverse dependencies on  $\omega$  in that simplistic model. In the Poisson stochastic model,  $\omega$  is just the number of users per agent car. Its effect on the agent stop time is just to multiply the base stop time  $t_A^S$  by  $\rho$  that is a probability. In case of several users boarding the car, we can expect them to board simultaneously under UW, or under AW maybe to board in turn – yet, if so then the stop time associated to all of them save the last one are embedded in the agent wait time.

Under policy UW, considering now  $\theta$  to be the time interval from the previous agent instant of passage, the number of users boarding on the car is also a Poisson RV with parameter  $\psi_U\theta$ , leading to the same outcome of  $N$  having geometric distribution with parameter  $\rho$ .

Under either policy, conditionally to  $\theta$  the probability of No users riding in the agent car is  $\Pr\{N_\theta = 0\}$ , i.e.  $\exp(-\psi_U\theta)$ . Deconditioning over  $\theta$ , we get

$$\Pr\{N = 0\} = \frac{\psi_A}{\psi_{U+A}} = 1 - \rho = \frac{1}{1 + \omega}$$

This outcome can be obtained more straightforwardly by considering the respective inter-arrival times  $\theta_U$  and  $\theta_A$  of users and agents: the probability of an agent not to get any user is the probability that there is no user arriving during the inter-arrival time between himself and the next agent (AW) or between himself and the previous agent (UW), so that

$$\Pr\{N = 0\} = \Pr\{\theta_A \leq \theta_U\} = \Pr\{\theta_A = \min\{\theta_A, \theta_U\}\} = \frac{\psi_A}{\psi_A + \psi_U} = \frac{1}{1 + \omega} = 1 - \rho$$

## Appendix B: Traffic equilibrium

### B1/ System state and state vector

A line ride-sharing service as a system has a system state determined by the values of its endogenous variables. These include (cf. Figure 3):

- (i) people trip flows ( $y_r$ ) according to service roles  $r \in \mathfrak{R}$ ,
- (ii) service quality variables  $(\varphi, \omega, v)$ ,
- (iii) the money and time items of the roles, denoted  $t_r^\ell$  and  $m_r^\ell$  by leg  $\ell \in \{R, S, T, W\}$ , depending on waiting policy  $\# \in \{AW, UW\}$ ,
- (iv) the generalized cost functions  $\beta \mapsto g_r^\#(\beta)$  according to role  $r$  and waiting policy  $\#$ ,
- (v) the efficiency domains  $E_r^\#$  and the related bounds  $\underline{\beta}_r^\# = \inf E_r^\#$  and  $\bar{\beta}_r^\# = \sup E_r^\#$ ,
- (vi) the role probabilities  $p_r^\#$  among the  $q$  individuals.

These probabilities sum up the disaggregate preferences of the individuals, by assigning each of them to the optimal role according to his or her  $\beta$  and taking into account the probability distribution of  $\beta$  among the population. Thus,  $p_r^\# = \Pr\{\beta \in E_r^\#\} = \int_0^{+\infty} 1_{\{\beta \in E_r^\#\}} dB(\beta)$ .

### B2/ Structure of influences and basic state vector

The endogenous variables are related by the following set of conditions:

- a/  $y_r \geq 0$  and  $\sum_{r \in \mathfrak{R}} y_r = q$ .
- b/ quality formation:  $\varphi = y_A$  and  $\omega$  such that  $y_U = \omega \cdot y_A$ , denoted as  $(\varphi, \omega) = (\hat{\varphi}, \hat{\omega})(\mathbf{y}_{\mathfrak{R}})$ .
- c/ traffic law  $v = V(\frac{y_A + y_N}{H})$ .
- d/ time and money item composition on the basis of  $(\varphi, \omega, v)$  and according to waiting policy.
- e/ generalized cost function composition from money and time items.
- f/ optimal choices: choice probabilities  $p_r^\#$  stem from generalized cost functions.
- g/ flow assignment  $y_r^\# = q \cdot p_r^\#, \forall r \in \mathfrak{R}$ .

The flow vector  $[y_r: r \in \mathfrak{R}]$  can make a basic state vector from which all of the other variables are derived. Such flow vector constitutes an equilibrium state if it satisfies that:

$$\forall r \in \mathfrak{R}: y_r = y_r^\#(\mathbf{y}_{\mathfrak{R}}) \quad (\text{B.1})$$

It is a condition of quantitative balance (market clearing) between supply ( $y_r$ ) and demand ( $y_r^\#$ ).

Yet, as basic state vector it is more convenient to consider the  $(\varphi, \omega)$  pair, from which stem  $(\hat{y}_A, \hat{y}_U)$ , service flow  $\hat{y}_S = \hat{y}_A + \hat{y}_U$ , neutral flow  $\hat{y}_N = q - \hat{y}_S$ , hence  $v$  and the rest of endogenous variables. This basic state vector is feasible if  $\varphi \geq 0$ ,  $\omega \geq 0$  and  $\varphi \leq q$ .

A feasible vector  $(\varphi, \omega)$  is an equilibrium state iff

$$\forall r \in \mathfrak{R}: \hat{y}_r(\varphi, \omega) = y_r^\#(\varphi, \omega) \quad (\text{B.2})$$

### B3/ State properties and the configurations of Pareto frontiers

Any feasible state  $(\varphi, \omega)$  gives rise to roles' features in money, time and generalized cost function, hence to a competition between the roles with respect to the VoTs of the individuals. The Pareto frontier is the sequence of efficiency domains along the  $\beta$ -axis from zero to infinity.

From the affine-linear functions of generalized costs, the efficiency domains are intervals (perhaps void with  $\underline{\beta}_r^\# = +\infty$  and  $\bar{\beta}_r^\# = 0$ ) so that

$$p_r^\# = (B(\bar{\beta}_r^\#) - B(\underline{\beta}_r^\#))^+ \quad (\text{B.3})$$

Together the efficiency domains cover  $\mathbb{R}^+$ . The intersections between different domains  $E_r^\#, E_s^\#$  are point-wise or void, except for degenerate cases with  $t_r = t_s$  and  $m_r = m_s$ . At junction point between  $r$  and  $s$ , the domain belonging conditions on both sides combine into

$$\begin{aligned} m_s + t_s \beta_{rs}^\# &= m_r + t_r \beta_{rs}^\# \\ \beta_{rs}^\# &= \frac{m_s - m_r}{t_r - t_s}, \end{aligned} \quad (\text{B.4})$$

Provided that  $t_r \neq t_s$ .

Now, for positive transaction times and dwell times, then it holds that  $t_N < t_U$ , making N the quickest option hence that of best value to high VoTs. Then  $\bar{\beta}_N^\# = +\infty$  and  $B(\bar{\beta}_N^\#) = 1$ , so that

$$p_N^\# = 1 - B(\underline{\beta}_N^\#). \quad (\text{B.5})$$

The value of  $\underline{\beta}_N^\#$  depends on which role A or U is competitive with N at that point. If  $m_r \geq m_N$  then  $\beta_{rN}^\# \leq 0$  yielding  $E_r^\# = \emptyset$  hence  $\bar{\beta}_r^\# = 0$ : then role  $r$  is fully dominated by N on range  $\mathbb{R}^+$  of  $\beta$ .

Conversely, if  $m_N > m_r$  then  $\bar{\beta}_r^\# \leq \underline{\beta}_N^\#$  and it takes a non-negative, finite value.

The competition between A and U is less pre-determined. If  $r$  fully dominates  $s$  with  $t_r < t_s$  and  $m_r \leq m_s$ , hence  $\beta_{rs}^\# < 0$ , then only  $r$  is competitive with N and the Pareto frontier is  $\{r, N\}$ . But if  $\beta_{AU}^\# > 0$  then each role has a non-empty competitive range with respect to the other one, leaving the issue of which set has lower values – that of lower money cost.

If  $m_U \leq m_A$  then U's range comes prior to A's one, denoted  $\{U, A\}$ , whereas if  $m_A \leq m_U$  then A's range is lower than U's one i.e.  $\{A, U\}$ .

Joining with N, the  $\{U, A\}$  configuration leads to either:

- Type 1 outcome, N dominates over  $\mathbb{R}^+$ .
- Type 2 with  $\{U, N\}$  if  $\beta_{AU}^\# \geq \beta_{UN}^\#$ ,
- Type 3 with  $\{U, A, N\}$  if  $\beta_{AU}^\# < \beta_{UN}^\#$ ,

Whereas an  $\{A, U\}$  configuration leads to either:

- Type 1 outcome with  $\{N\}$  only as the Pareto frontier of the generalized costs,
- Type 2 with  $\{A, N\}$  if  $\beta_{AU}^\# \geq \beta_{AN}^\#$ ,
- Type 3 with  $\{A, U, N\}$  if  $\beta_{AU}^\# < \beta_{AN}^\#$ .

### B4/ Configurational characterization of multi-sided equilibrium

The configurations of states  $(\varphi, \omega)$  combine with the demand-supply balance condition of multi-sided equilibrium to yield the following configurational properties:

Proposition: Equilibrium properties. under positive transaction and dwell times, if  $(\varphi, \omega)$  is an equilibrium state then, depending on the Pareto frontier:

- $\{N\}$  with  $m_N \leq \min\{m_U, m_A\}$ : then  $p_N^\# = 1$ ,  $p_A^\# = p_U^\# = 0$ ,  $\hat{y}_N = q$ ,  $\hat{y}_A = \hat{y}_U = 0$ ,  $\varphi = 0$ .
- $\{U, N\}$ : either  $\beta_{AU}^\# \leq 0$  with  $m_U \leq m_A$  and  $t_U \leq t_A$ , or  $\beta_{AU}^\# \geq \beta_{UN}^\#$  with  $m_U \leq m_A$  and  $t_U > t_A$ : in either case  $\underline{\beta}_U^\# = 0$ ,  $\bar{\beta}_U^\# = \beta_{UN}^\# = \underline{\beta}_N^\#$ ,  $\varphi = y_A^\# = 0$ .
- $\{A, N\}$ : either  $\beta_{AU}^\# \leq 0$  with  $m_A \leq m_U$  and  $t_A \leq t_U$ , or  $\beta_{AU}^\# \geq \beta_{AN}^\#$  with  $m_A \leq m_U$  and  $t_U < t_A$ : in either case  $\underline{\beta}_A^\# = 0$ ,  $\bar{\beta}_A^\# = \beta_{AN}^\# = \underline{\beta}_N^\#$ ,  $y_U^\# = 0$ .
- $\{A, U, N\}$ : with  $m_A \leq m_U \leq m_N$  and  $t_N \leq t_A \leq t_U$ ,  $\beta_{AU}^\# < \beta_{AN}^\#$ ,  $\underline{\beta}_A^\# = 0$ ,  $\bar{\beta}_A^\# = \underline{\beta}_U^\# = \beta_{UA}^\# < \beta_{UN}^\# = \bar{\beta}_U^\# = \underline{\beta}_N^\#$ : thus  $\varphi = qB(\beta_{UA}^\#)$  and  $\omega\varphi = q(B(\underline{\beta}_N^\#) - B(\beta_{UA}^\#))$ .
- $\{U, A, N\}$ : with  $m_U \leq m_A \leq m_N$  and  $t_N \leq t_U \leq t_A$ ,  $\beta_{AU}^\# < \beta_{UN}^\#$ ,  $\underline{\beta}_U^\# = 0$ ,  $\bar{\beta}_U^\# = \underline{\beta}_A^\# = \beta_{UA}^\# < \beta_{AN}^\# = \bar{\beta}_A^\# = \underline{\beta}_N^\#$ : thus  $\omega \cdot \varphi = qB(\beta_{UA}^\#)$  and  $\varphi = q(B(\underline{\beta}_N^\#) - B(\beta_{UA}^\#))$ .

**Proposition: Equilibrium characterization.** If a feasible state  $(\varphi, \omega)$  satisfies the characteristic conditions of its configuration, then it is a state of equilibrium.

This is because the configuration-specific condition determines the efficiency domains and yields the associated balance conditions. Then, the ‘‘supplied’’ flows  $\hat{y}_r(\varphi, \omega)$  and their ‘‘demanded’’ counterparts  $y_r^\#(\varphi, \omega)$  are balanced.

It is thus easy to test whether a state is in equilibrium or not, according to the configuration of its Pareto frontier.

## B5/ Fixed-point problem and iterative solution scheme

**FPP in  $(\varphi, \omega)$ .** A basic state vector  $(\varphi, \omega)$  is in equilibrium if it is a fixed point for the mapping  $(\varphi, \omega) \mapsto (\varphi', \omega')$  such that  $\varphi' = y_A^\#(\varphi, \omega)$  and  $\omega'$  satisfies that  $y_U^\#(\varphi, \omega) = \omega' y_A^\#(\varphi, \omega)$ .

**FPP in  $(y_A, y_U)$ .** A basic state vector  $(y_A, y_U)$  is in equilibrium if it is a fixed point for the chained mapping  $(y_A, y_U) \mapsto (\varphi, \omega)$  such that  $\varphi = y_A$  and  $\omega$  satisfies that  $y_U = \omega y_A$ , followed by  $(\varphi, \omega) \mapsto (y_A^\#, y_U^\#)$ .

**Iterative solution scheme.** The basic strategy to solve an FPP is to progressively adapt a current state vector by combining it with its image through the mapping so as to obtain the next value. At step  $k$ , current state  $(\varphi_k, \omega_k)$  induces image  $(\hat{\varphi}_k, \hat{\omega}_k)$ : then the next step can be obtained as

$$\varphi_{k+1} \leftarrow \text{Fn}[\varphi_k, \hat{\varphi}_k],$$

$$\omega_{k+1} \leftarrow \text{Fn}[\omega_k, \hat{\omega}_k].$$

For instance, a convex combination scheme with step sizes  $\zeta_k$  that decrease to zero:

$$\varphi_{k+1} \leftarrow \varphi_k + \zeta_k(\hat{\varphi}_k - \varphi_k) = (1 - \zeta_k)\varphi_k + \zeta_k\hat{\varphi}_k$$

$$\omega_{k+1} \leftarrow \omega_k + \zeta_k(\hat{\omega}_k - \omega_k) = (1 - \zeta_k)\omega_k + \zeta_k\hat{\omega}_k.$$

A similar strategy on the  $(y_A, y_U)$  flow vector constitutes an equilibration algorithm well-known in network traffic assignment. The convex combination scheme is an instance of the ‘‘Method of Successive Averages’’.

Yet a more straightforward strategy is introduced hereafter.

## Appendix C: Configurational analysis and equilibrium determination

### C1/ Time and Money relations and the configurations of roles

The feasible domain of  $(\varphi, \omega)$ ,  $D_{(\varphi, \omega)} \equiv [0, q] \times \mathbb{R}^+$ , or about equivalently that  $D_y$  of pairs  $(y_A, y_U)$  such that  $y_A \geq 0$ ,  $y_U \geq 0$  and  $y_A + y_U \leq q$ , is useful to locate the graphs of the OSS and ISS and to determine equilibrium states graphically at the intersection points between the function graphs. Each domain includes specific regions of interest to compare options A and U according to their respective money costs (called the money-related regions), or according to their times (hence the time-related regions). By combining the money and time conditions, specific regions come out that locate the alternative configurations of the two roles along the VoT axis.

#### C.1.1 Money-related regions in the feasible domain

Condition  $m_A \leq m_U$  is equivalent to  $\mu_{A-U} - \omega L \dot{t}_A \leq 0$ , hence to

$$\omega \geq \omega_M \equiv \mu_{A-U} / (L \dot{t}_A)$$

Thus, region  $\{m_A \leq m_U\}$  lies above the “money line”  $\{m_A = m_U\}$  which is a horizontal line in the  $(\varphi, \omega)$  domain. Region  $\{m_A \geq m_U\}$  lies below the money line. In the  $(y_A, y_U)$  domain, the money line is a diagonal line  $\{y_U = \omega_M y_A\}$ , with region  $\{m_A \leq m_U\}$  on its upper side and region  $\{m_A \geq m_U\}$  on the lower side.

#### C.1.2 Time-related regions in the feasible domain

As for times, condition  $t_A \geq t_U$  corresponds to  $t_A - t_U \geq 0$  in (26), hence to pairs  $(\varphi, \omega)$  and the related  $\rho$  such that

$$\rho \geq \rho_\varphi \equiv \frac{\tilde{t} + \delta_U \frac{H}{\varphi}}{\underline{t}_A^S + \delta_A \frac{H}{\varphi}}$$

As function  $\omega \mapsto \rho \equiv \frac{\omega}{1+\omega}$  is increasing, if  $\rho_\varphi < 1$  then the inequality is equivalent to  $\omega \geq \omega_\varphi$  with  $\omega_\varphi \equiv \rho_\varphi / (1 - \rho_\varphi)$ . When  $\varphi \rightarrow +\infty$  then  $\rho_\varphi \rightarrow \rho_\infty \equiv \tilde{t} / \underline{t}_A^S$ . Furthermore, if  $\rho_\infty < 1$ , i.e.,  $\underline{t}_A^S > \tilde{t}$ , then  $\omega_\varphi \rightarrow \omega_\infty \equiv \frac{\rho_\infty}{1 - \rho_\infty} = \frac{\tilde{t}}{\underline{t}_A^S - \tilde{t}}$ .

Under UW, then  $\rho_\varphi^{UW} = (\tilde{t} + \frac{H}{\varphi}) / \underline{t}_A^S$  is a decreasing function of  $\varphi \geq 0$ , down to  $\rho_\infty$ . If  $\tilde{t} \geq \underline{t}_A^S$  then  $\rho_\varphi^{UW}$  is always greater than 1, making condition  $t_A \geq t_U$  impossible to meet, so that region  $\{t_A < t_U\}$  fills up the feasible domain. This may be represented by a timeline  $\{\omega = +\infty\}$ .

But if  $\tilde{t} < \underline{t}_A^S$  then an upper region  $\{t_A \geq t_U\}$  appears from  $\varphi$  such that  $\rho_\varphi^{UW} = 1$ , i.e., from  $\tilde{\varphi}_{UW} \equiv H / (\underline{t}_A^S - \tilde{t})$ : the decreasing function  $\varphi \mapsto \rho_\varphi^{UW}$  yields a timeline function  $\{t_A = t_U\}$  in the  $(\varphi, \omega)$  plane that is decreasing with  $\varphi$  from  $\tilde{\varphi}_{UW}$  to  $+\infty$ , down to limit value  $\omega_\infty$ . It may be extended to  $[0, \tilde{\varphi}_{UW}]$  by  $\{\varphi \mapsto \omega = +\infty\}$ . Region  $\{t_A \geq t_U\}$  lies above the time line and  $\{t_A < t_U\}$  below it.

Under AW, then  $\rho_\varphi^{AW} = \tilde{t} / (\underline{t}_A^S + \frac{H}{\varphi})$  is an increasing function of  $\varphi \geq 0$ , from 0 at 0 up to  $\rho_\infty$ . Region  $\{t_A \geq t_U\}$  lies above the time line  $\{t_A = t_U\} \Leftrightarrow \{\rho = \rho_\varphi^{AW}\}$ , and  $\{t_A < t_U\}$  below it.

If  $\tilde{t} \leq \underline{t}_A^S$  then  $\rho_\varphi$  keeps below 1, so that the time line traverses the feasible domain along  $\{\varphi \geq 0\}$ , with region  $\{t_A \geq t_U\}$  above it and  $\{t_A < t_U\}$  below it. The asymptote  $\rho_\varphi^{AW} \rightarrow \rho_\infty$  on  $\rho$  translates in an asymptote  $\omega_\varphi^{AW} \rightarrow \omega_\infty$ .

But if  $\tilde{t} > \underline{t}_A^S$  then  $\rho_\varphi$  reaches value 1 at  $\tilde{\varphi}_{AW}$  such that  $\underline{t}_A^S + H/\varphi = \tilde{t}$ , i.e.,  $\tilde{\varphi}_{AW} \equiv H / (\tilde{t} - \underline{t}_A^S)$ : the opposite definition of  $\tilde{\varphi}_{UW}$ . The vertical line  $\{\varphi = \tilde{\varphi}_{AW}\}$  makes a vertical asymptote to the time line  $\{\rho = \rho_\varphi^{AW}\}$ , above which lies region  $\{t_A \geq t_U\}$ , while region  $\{t_A < t_U\}$  lies below the time line. All

pairs  $(\varphi, \omega)$  with  $\varphi \geq \tilde{\varphi}_{AW}$  belong to region  $\{t_A < t_U\}$ . Thus the time line may be extended to  $[\tilde{\varphi}_{UW}, +\infty]$  by  $\{\varphi \mapsto \omega = +\infty\}$ .

The following table summarizes the four cases that combine either one of two waiting policies to the respective ordering of parameters  $t_A^S$  and  $\tilde{t}$ .

**Tab. 3: Time-related regions and the configurations of service roles' efficiency ranges.**

	UW: function $\rho_\varphi$ decreases with $\varphi$	AW: function $\rho_\varphi$ increases with $\varphi$
$t_A^S < \tilde{t}$	No $\{t_A \geq t_U\}$ region. Region $\{t_A < t_U\}$ everywhere.	Upper region $\{t_A \geq t_U\}$ from $\varphi = 0$ and $\omega = 0$ , up to $\tilde{\varphi}_{AW} \equiv \frac{H}{\tilde{t} - t_A^S}$ Vertical asymptote $\{\varphi = \tilde{\varphi}_{AW}\}$
$t_A^S > \tilde{t}$	Upper region $\{t_A \geq t_U\}$ from $\tilde{\varphi}_{UW}$ with $\tilde{\varphi}_{UW} \equiv \frac{H}{t_A^S - \tilde{t}}$ , with horizontal asymptote at $\omega_\infty \equiv \frac{\tilde{t}}{t_A^S - \tilde{t}}$	Upper region $\{t_A \geq t_U\}$ throughout feasible domain, above increasing timeline with horizontal asymptote at $\omega_\infty$

### C.1.3 Joining the Money and Time conditions

Between roles A and U, the respective ranges of efficiency along the VoT axis are determined by the roles' times and money costs:

- if  $\{m_A < m_U\}$  and  $\{t_A < t_U\}$  then A prevails everywhere,
- if  $\{m_A < m_U\}$  and  $\{t_A > t_U\}$ , then A prevails up to  $\beta_{AU}^\# > 0$ , from which U prevails,
- if  $\{m_A > m_U\}$  and  $\{t_A < t_U\}$ , then U prevails up to  $\beta_{AU}^\# > 0$ , from which A prevails,
- if  $\{m_A > m_U\}$  and  $\{t_A > t_U\}$  then U prevails everywhere.

Graphically, to locate the different configurations of efficiency it suffices to combine the money-related regions and the time-related ones. The combination depends on the waiting policy. If  $t_A^S < \tilde{t}$ , the money regions combine straightforwardly to their time counterparts. If  $t_A^S > \tilde{t}$  then the comparison involves the ordering between  $\omega_M$  and  $\omega_\infty$ .

**Tab. 4: Money- and Time-related regions and the configurations of service roles' efficiency ranges.**

	UW: function $\rho_\varphi$ decreases with $\varphi$	AW: function $\rho_\varphi$ increases with $\varphi$
$t_A^S < \tilde{t}$	{A} above money line, {U,A} below it	{A,U} above max(M-line, T-line) {U,A} below min(M-line, T-line) Between them: {U} up to $\tilde{\varphi}_{AW}$ , {A} from it
$t_A^S > \tilde{t}$ , $\omega_\infty > \omega_M$	T-line above M-line {U,A} below M-line {A,U} above T-line {A} between the M- and the T-line	T-line crosses M-line at $\varphi_M^{AW}$ Up to $\varphi_M^{AW}$ , {A} between M- & T-lines. From it, {U} between T- & M-. {A,U} above max(T-line, M-line) {U,A} below min(T-line, M-line)
$t_A^S > \tilde{t}$ , $\omega_\infty \leq \omega_M$	T-line crosses M-line at $\varphi_M^{UW}$ Up to $\varphi_M^{UW}$ : {A} above M-line From $\varphi_M^{UW}$ : {U} between M- and T-lines, {A,U} above max(T-line, M-line) {U,A} below min(T-line, M-line)	M-line above T-line {A,U} above M-line {U,A} below T-line {U} between the T- and the M-line

## C2/ Straightforward solution strategy

Under positive times of transaction and dwelling, the lower bound  $\underline{\beta}_N^\#$  of role N must satisfy that

$$\varphi(1 + \omega) = q \cdot B(\underline{\beta}_N^\#) \quad (\text{C.1})$$

When  $\omega = 0$ , then role U is not competitive so that either N dominates A yielding  $\underline{\beta}_N^\# = 0$  and  $\varphi = 0$ , or  $\underline{\beta}_N^\# = \beta_{AN}^\# \geq 0$ , with  $\beta_{AN}^\#$  a function of  $\varphi$  only. Then eqn. (C.1) constitutes an equation in the unknown  $\varphi$  only,

$$\varphi = q \cdot B(\underline{\beta}_N^\#) = q \cdot B\left(\frac{\mu_{N-A} + L\dot{t}_A\omega}{t_A^T + \rho(t_A^S + \delta_A \frac{H}{\varphi})}\right).$$

Wherein  $\delta_A = 1$  under AW or 0 under UW. In both cases, at  $\omega = 0$  then  $\rho = 0$  and  $\varphi$  comes out directly.

More generally, given  $\omega$  we consider the lowest solution of eqn. (C.1) as a function  $\bar{\varphi}(\omega)$ : it satisfies

$$\bar{\varphi} = \frac{q}{1+\omega} B(\underline{\beta}_N^\#) \quad (\text{C.2})$$

Function  $\bar{\varphi}$  is conditional on waiting policy # and on current configuration {A,U} or {U,A} that determines whether  $\underline{\beta}_N^\# = \beta_{UN}^\#$  or  $\underline{\beta}_N^\# = \beta_{AN}^\#$ . The configuration also implies a second condition linking  $\varphi$  and  $\omega$ , either

$$\underline{\varphi} = q \cdot B(\beta_{AU}^\#) \quad (\text{C.3\_AU})$$

$$\underline{\varphi} = \frac{q}{\omega} \cdot B(\beta_{AU}^\#) \quad (\text{C.3\_UA})$$

Then, the straightforward strategy is to look for an intersection point of the two functions  $\bar{\varphi}$  and  $\underline{\varphi}$  of  $\omega$ . It is further specified in Appendix D in the case of uniformly distributed VoTs.

The discrepancy between C3\_AU and C3\_UA highlights the key influence of configuration.

## C3/ Specific properties of OSS functions

### C.3.1 OSS under {A,U}

Under configuration {U} or {A,U}, then, denoting  $\mu_{N-U} \equiv m_N - m_U$ ,

$$\underline{\beta}_N^\# = \beta_{UN}^\# = \frac{m_N - m_U}{t_U - t_N} = \frac{\mu_{N-U}}{t_U^{T+S} + \delta_U \frac{H}{\varphi}}$$

Thus the OSS  $\bar{\omega}$  is stated as

$$\bar{\omega} = \frac{q}{\varphi} B\left(\frac{\mu_{N-U}}{t_U^{T+S} + \delta_U \frac{H}{\varphi}}\right) - 1.$$

Under AW hence  $\delta_U = 0$ , then  $\beta_{UN}^\#$  is constant, yielding constant  $B_{UN}^\# \equiv B(\beta_{UN}^\#)$  and simple OSS

$$\bar{\omega} = \frac{q}{\varphi} B_{UN}^\# - 1$$

It is obviously a decreasing function of  $\varphi > 0$ .

### C.3.2 OSS under {U,A}

Under configuration {A} or {U,A} then

$$\underline{\beta}_N^\# = \beta_{AN}^\# = \frac{m_N - m_A}{t_A - t_N} = \frac{\mu_{N-A} + L\dot{t}_A\omega}{t_A^T + \rho(t_A^S + \delta_A \frac{H}{\varphi})}$$

Under UW hence  $\delta_A = 0$ , the OSS  $\bar{\varphi}(\omega)$  is straightforward:

$$\bar{\varphi} = \frac{q}{1 + \omega} B\left(\frac{\mu_{N-A} + L\dot{t}_A\omega}{t_A^T + \rho t_A^S}\right)$$

Under AW the OSS is somewhat more involved.

## C4/ Specific properties of ISS functions

ISS functions involve the frontier VoT  $\beta_{AU}^\#$  that depends on  $\omega$  and  $\varphi$  as follows:

$$\beta_{AU}^\# = \frac{m_U - m_A}{t_A - t_U} = \frac{L\dot{t}_A\omega - \mu_{A-U}}{\rho\left(t_A^S + \delta_A \frac{H}{\varphi}\right) - \tilde{t} - \delta_U \frac{H}{\varphi}}$$

### C.4.1 ISS under {A,U}

Under configuration {A,U}, then  $m_A \leq m_U$  and  $t_A \geq t_U$ , implying that  $\omega \geq \omega_M$  and  $\omega \geq \Omega(\rho_\varphi)$ . Furthermore,  $y_A$  is stated as a simple function of  $\beta_{AU}^\#$ :  $y_A = q \cdot B(\beta_{AU}^\#)$ .

If  $0 < \varphi < q$ , we can state the ISS function  $\underline{\omega}$  with respect to  $\varphi$  in a simple way: as  $\varphi = y_A$ ,

$$B^{(-1)}\left(\frac{\varphi}{q}\right) = \beta_{AU}^\#$$

$$L\dot{t}_A(\omega - \omega_M) = \left\{ \rho\left(t_A^S + \delta_A \frac{H}{\varphi}\right) - \tilde{t} - \delta_U \frac{H}{\varphi} \right\} \cdot B^{(-1)}\left(\frac{\varphi}{q}\right)$$

Denoting  $\eta_\varphi \equiv B^{(-1)}\left(\frac{\varphi}{q}\right) / (L\dot{t}_A)$ ,

$$(\omega - \omega_M)(1 + \omega) = \eta_\varphi \left\{ \omega\left(t_A^S + \delta_A \frac{H}{\varphi}\right) - (1 + \omega)\left(\tilde{t} + \delta_U \frac{H}{\varphi}\right) \right\}$$

$$(\omega - \omega_M)(1 + \omega) = \eta_\varphi \left\{ \omega\left(t_A^S + \delta_A \frac{H}{\varphi}\right) - (1 + \omega)\left(\tilde{t} + \delta_U \frac{H}{\varphi}\right) \right\}$$

$$\omega^2 + \omega \left\{ 1 - \omega_M - \eta_\varphi\left(t_A^S - \tilde{t} + \delta_{A-U} \frac{H}{\varphi}\right) \right\} = \omega_M - \eta_\varphi\left(\tilde{t} + \delta_U \frac{H}{\varphi}\right)$$

It is a second degree equation in  $\omega$ , parameterized by  $\varphi$ .

### C.4.2 ISS under {U,A}

Under configuration {U,A} then  $m_A \geq m_U$  and  $t_A \leq t_U$ , implying that  $\omega \leq \omega_M$  and  $\omega \leq \Omega(\rho_\varphi)$ . Furthermore, the ISS is more involved with more complex influence of  $\omega$ :

$$\underline{\varphi}\omega = q \cdot B(\beta_{AU}^\#)$$

## Appendix D: Uniformly distributed VoT

Here a uniform distribution of VoT is considered, over range  $[0, b]$ . (A positive lower bound  $a > 0$  may be considered by transferring  $a \cdot t$  from time items to money items.)

The CDF function satisfies:

$$B(\beta) = (\min\{1, \frac{\beta}{b}\})^+ \quad (C.1)$$

The inverse function is

$$B^{(-1)}(\alpha) = b \cdot (\min\{1, \alpha\})^+ \quad (C.2)$$

### D1/ Degenerate configuration {A,N}

A degenerate configuration {A,N} with no service users has  $\omega = 0$  and  $\underline{\beta}_N^\# = \beta_{AN}^\#$ . The OSS function is

$$\varphi = q \cdot B(\beta_{AN}^\#) = \frac{q}{b} (\min\{b, \frac{\mu_{N-A}}{t_A^T + \delta_A H / \varphi}\})^+ \quad (C.3)$$

Wherein:  $\mu_{N-A} \equiv \gamma_N - \gamma_A + \tau_A$ .

The solution satisfies that

$$\begin{aligned} \varphi t_A^T + \delta_A H &= \frac{q}{b} \mu_{NA} \\ \varphi &= \frac{\frac{q}{b} \mu_{NA} - \delta_A H}{t_A^T} \end{aligned} \quad (C.4)$$

It is feasible only if it belongs to  $[0, q]$ , i.e. iff  $0 \leq \mu_{NA} - \delta_A H b / q \leq b t_A^T$ .

If  $\mu_{NA} > b(t_A^T + \frac{\delta_A H}{q})$  then  $\varphi = q$  and the A option prevails upon the Neutral alternative over the VoT range with positive probability density.

Furthermore, for the configuration to hold it requires U to be a dominated option. It occurs if

$$\begin{cases} m_A \leq m_U \\ t_A \leq t_U \end{cases} \text{ i.e. if } \begin{cases} \gamma_A + \underline{m} + L\dot{m} - \tau_A \leq m_U \gamma_U + \tau_U + L\dot{t}_U \\ t_A^T + \delta_A H / \varphi \leq t_U^T + t_U^S + \delta_U H / \varphi \end{cases} \Leftrightarrow \begin{cases} \mu_{A-U} \leq 0 \\ 0 \leq \tilde{\tau} - \varepsilon H / \varphi \end{cases}$$

(recalling  $\varepsilon \equiv \delta_A - \delta_U$ ), or if

$$\begin{cases} m_A \leq m_U \\ t_A > t_U \\ \beta_{AU}^\# \geq \min\{b, \beta_{AN}^\#\} \end{cases} \text{ i.e. if } \begin{cases} \gamma_A + \underline{m} + L\dot{m} - \tau_A \leq m_U \gamma_U + \tau_U + L\dot{t}_U \\ t_A^T \leq t_U^T + t_U^S \\ \beta_{AU}^\# \geq \min\{b, \beta_{AN}^\#\} \end{cases} \Leftrightarrow \begin{cases} \mu_{A-U} \leq 0 \\ \tilde{\tau} - \frac{\varepsilon H}{\varphi} \leq 0 \\ \beta_{AU}^\# \geq \min\{b, \beta_{AN}^\#\} \end{cases}$$

### D2/ Configuration {A,U,N}

#### D.2.1 OSS under A<U

The basic condition on  $y_S$  yields that, recalling  $\mu_{N-U} \equiv m_N - m_U$ ,

$$y_A + y_U = qB(\beta_{UN}^\#) = qB(\frac{\mu_{N-U}}{t_U^{T+S} + \delta_U H / y_A}), \text{ hence that}$$

$$\bar{y}_U = qB(\frac{\mu_{N-U}}{t_U^{T+S} + \delta_U H / y_A}) - y_A.$$

If  $t_U^{T+S} + \frac{\delta_U H}{y_A} > \mu_{N-U} b$ , then

$$\bar{y}_U = \frac{q}{b} \frac{\mu_{N-U}}{t_U^{T+S} + \delta_U H / y_A} - y_A.$$

Dividing by  $y_A$  and replacing  $y_U / y_A$  by  $\omega$  yields the OSS function in the  $(\varphi, \omega)$  domain:

$$\bar{\omega} = \frac{q}{b} \beta_{UN}^{\#} = \frac{q}{b} \frac{\mu_{N-U}}{t_U^{T+S} \varphi + \delta_U H} - 1$$

In the  $\{m_A < m_U\}$  region, it is a decreasing function of  $\varphi$ .

It is also easy to consider the inverse relationship that determines  $\varphi$  from  $\omega$ :

$$\varphi(1 + \omega) = \frac{q}{b} \frac{\mu_{N-U}}{t_U^{T+S} + \delta_U H / \varphi}$$

$$t_U^{T+S} \varphi + \delta_U H = \frac{q}{b} \frac{\mu_{N-U}}{1 + \omega}$$

$$\bar{\varphi} = \frac{1}{t_U^{T+S}} \left( \frac{q}{b} \frac{\mu_{N-U}}{1 + \omega} - \delta_U H \right).$$

### D.2.2 ISS under A<U

Under AUN the ISS condition is

$$y_A = qB(\beta_{AU}^{\#})$$

If  $0 < \beta_{AU}^{\#} < b$  then we can invert  $\beta_{AU}^{\#} = B^{(-1)}\left(\frac{y_A}{q}\right) = b \frac{\varphi}{q}$ , yielding that

$$\frac{m_U - m_A}{t_A - t_U} = \frac{L\dot{t}_A \omega - \mu_{A-U}}{\rho \left( \underline{t}_A^S + \delta_A \frac{H}{\varphi} \right) - \tilde{t} - \delta_U \frac{H}{\varphi}} = \frac{b}{q} \varphi$$

$$q(L\dot{t}_A \omega - \mu_{A-U})(1 + \omega) = b \left\{ \omega (\underline{t}_A^S \varphi + \delta_A H) - (1 + \omega)(\tilde{t} \varphi + \delta_U H) \right\}$$

$$\omega^2 q L \dot{t}_A + \omega \{ q(L\dot{t}_A - \mu_{A-U}) - b(\varphi(\underline{t}_A^S - \tilde{t}) + \delta_{A-U} H) \} = q \mu_{A-U} - b(\tilde{t} \varphi + \delta_U H)$$

It is a second degree equation in  $\omega$  parameterized by  $\varphi$ . Yet, it is simpler to consider the inverse relationship that determines  $\varphi$  from  $\omega$ :

$$b \{ \rho(\underline{t}_A^S \varphi + \delta_A H) - \tilde{t} \varphi - \delta_U H \} = q(L\dot{t}_A \omega - \mu_{A-U})$$

$$\varphi b(\rho \underline{t}_A^S - \tilde{t}) = q(L\dot{t}_A \omega - \mu_{A-U}) - bH(\rho \delta_A - \delta_U)$$

Hence

$$\underline{\varphi} = \frac{q(L\dot{t}_A \omega - \mu_{A-U})(1 + \omega) - bH(\omega \delta_{A-U} - \delta_U)}{b(\omega(\underline{t}_A^S - \tilde{t}) - \tilde{t})}.$$

### D.2.3 Equilibrium characterization

An equilibrium state is characterized by feasible points  $(\varphi, \omega)$  such that  $\omega = \bar{\omega}(\varphi) = \underline{\omega}(\varphi)$  or, equivalently, such that  $\varphi = \bar{\varphi}(\omega) = \underline{\varphi}(\omega)$ . The conjunction  $\bar{\varphi} = \underline{\varphi}$  amounts to the following equation in  $\omega$ :

$$\frac{1}{t_U^{T+S}} \left( \frac{q}{b} \frac{\mu_{N-U}}{1 + \omega} - \delta_U H \right) = \frac{q(L\dot{t}_A \omega - \mu_{A-U})(1 + \omega) - bH(\omega \delta_{A-U} - \delta_U)}{b(\omega(\underline{t}_A^S - \tilde{t}) - \tilde{t})}$$

It is a 3<sup>rd</sup> degree equation in  $\omega$ .

## D3/ Configuration {U,A,N}

### D.3.1 OSS under U<A

Under configuration UAN, if  $\beta_{AN}^\# < b$  then the OSS function satisfies that

$$\varphi(1 + \omega) = \frac{q}{b} \beta_{AN}^\# = \frac{q}{b} \frac{\mu_{N-A} + L\dot{t}_A\omega}{\rho \left( \underline{t}_A^S + \delta_A \frac{H}{\varphi} \right) + t_A^T}$$

$$\omega(\underline{t}_A^S \varphi + \delta_A H) + \varphi(1 + \omega)t_A^T = \frac{q}{b} (\mu_{N-A} + L\dot{t}_A\omega)$$

Yielding that

$$\bar{\varphi} = \frac{\frac{q}{b}(\mu_{N-A} + L\dot{t}_A\omega) - \omega\delta_A H}{\omega\underline{t}_A^S + (1+\omega)t_A^T} \quad (\text{C.5})$$

### D.3.2 ISS under U<A

Under U<A, if  $\beta_{AU}^\# < b$  then the ISS function satisfies that

$$\varphi\omega = \frac{q}{b} \beta_{AU}^\# = \frac{q}{b} \frac{L\dot{t}_A\omega - \mu_{A-U}}{\rho \left( \underline{t}_A^S + \delta_A \frac{H}{\varphi} \right) - \tilde{t} - \delta_U \frac{H}{\varphi}}$$

So that

$$\frac{\omega}{1 + \omega} (\underline{t}_A^S \varphi + \delta_A H) - \tilde{t}\varphi - \delta_U H = \frac{q}{b} \frac{L\dot{t}_A\omega - \mu_{A-U}}{\omega}$$

$$\varphi(\omega\underline{t}_A^S - (1 + \omega)\tilde{t}) + \omega\delta_A H - (1 + \omega)\delta_U H = (1 + \omega) \frac{q}{b} \frac{L\dot{t}_A\omega - \mu_{A-U}}{\omega}$$

Yielding that

$$\underline{\varphi} = \frac{\frac{q}{b}(L\dot{t}_A\omega - \mu_{A-U})(1+\omega) - \omega H(\omega\delta_{A-U} - \delta_U)}{\omega[\omega\underline{t}_A^S - (1+\omega)\tilde{t}]} \quad (\text{C.6})$$

### D.3.3 Equilibrium characterization

The conjunction  $\bar{\varphi} = \underline{\varphi}$  amounts to the following equation in  $\omega$ :

$$\frac{\frac{q}{b}(\mu_{N-A} + L\dot{t}_A\omega) - \omega\delta_A H}{\omega\underline{t}_A^S + (1 + \omega)t_A^T} = \frac{\frac{q}{b}(L\dot{t}_A\omega - \mu_{A-U})(1 + \omega) - \omega H(\omega\delta_{A-U} - \delta_U)}{\omega[\omega\underline{t}_A^S - (1 + \omega)\tilde{t}]}$$

It is a 3rd degree equation in  $\omega$ .