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# Line Ride-Sharing as a bi-sided mobility service with price schedule, transactional protocol and waiting policy: a Time&Money traffic assignment model and its equilibrium

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## Abstract

A line ride-sharing service is supplied along a given roadway path by an operator that matches Users (riders) and Agents (drivers), under specific protocol that involves price schedule on both the U and A sides, waiting policy on either side and transaction times. The resulting time and money items add up over trip legs, yielding trip time and money cost depending on the service role, A or U, compared to Non-commitment, called role N for Neutral. The article brings about a traffic model of people involvement in the service. Service conditions of frequency  $\varphi$  and average number of users per car run  $\omega$  are key factors of the time and money features of the alternative roles A, U and N. Individual choice of role is modeled as a rational behavior of minimizing the generalized cost depending on the individual Value-of-Time (VoT). Aggregation over trip-makers according to the statistical distribution of VoT yields the respective role flows  $(y_A, y_U, y_N)$ , which in turn determine the macroscopic factors  $(\varphi, \omega)$ . Traffic equilibrium is defined as a balance condition between the “supplied flows” and the “demanded flows” of the three roles. A computational scheme is provided, with graphical interpretation in the  $(y_A, y_U)$  plane as well as in the  $(\varphi, \omega)$  plane. A numerical experiment is conducted, showing that two alternative configurations can arise at equilibrium: either  $\{A, U, N\}$  with less wealthy Agents driving wealthier Users, or  $\{U, A, N\}$  where less wealthy Users are driven by wealthier Agents: in both cases the Neutral role attracts the upper range of the VoT distribution.

## Keywords:

Ride-sharing service; bi-sided platform; traffic equilibrium; multi-sided equilibrium; equilibration algorithm

## Highlights

H1/ Service featured out in terms of run frequency and average car load

H2/ Microeconomic behavior at the individual level to select a role with respect to the service

H3/ Traffic equilibrium as balance conditions between supplied- and demanded- role flows

H4/ Equilibrium computation by graphical method

H5/ Alternative configurations of Agents<Users versus Users<Agents along the VoT axis

## 1/ Introduction

### Background

Line ride-sharing (LRS) is a specific kind of platform-based RS service in which the rides take place along a specific roadway path, thus called the line or more specifically the line link. Rider access to cars may be restricted to special stop points called “stations” (cf. Ecov’s “Line” kind of service) or allowed at any point along the line path (cf. Ecov’s “Line+”).

Then, along the roadway path the car trip-makers fall into either one out of three classes: Service users as Riders (type U), car drivers involved in the service as Agents (type A) and other trip-makers that do not take part to the service, called Neutral (type N). The three types can be seen as specific

travel modes, say sub-modes of the car mode. By time period and flow direction, the respective trip flows of the three roles, denoted as  $y_r$  for each role  $r \in \mathfrak{R} \equiv \{U, A, N\}$ , determine both the service quantity and quality of the RS line:

- $y_A$  amounts to the service frequency  $\varphi$  to potential users – a key component of service quality to them.
- Ratio  $\omega = y_U/y_A$  is the expected passenger load by service-affiliated car – a basic indicator of money incomes to agents based on the principle of trip cost sharing.
- $y_U$  and  $y_A$  are the basic quantities of service production, leading to its market share within the car mode,  $(y_A + y_U)/q$  with  $q = \sum_{r \in \mathfrak{R}} y_r$ , and the overall occupancy rate in persons per car,  $q/(y_A + y_N)$ .

Service participation either on the User or Agent side depends on the attractiveness of the respective modal option to car trip-makers.

The modal specific money and time expenses per trip constitute option attributes in the modal decisions of the individual trip-makers. Money costs  $m_r$  are lower for service participants (A/U) than for non-participants (N). In contrast, times  $t_r$  are higher for service participants A and U than for N because of transaction operations, car dwelling and waiting on one side. Postulating microeconomic rationality, every individual chooses the role yielding minimal generalized cost to him depending on his own value-of-time (VoT)  $\beta$ :  $g_r(\beta) = m_r + \beta t_r$ . Aggregating the individual choices over the statistical distribution of trip-makers (CDF  $B$  of  $\beta$ ), and multiplying by the total number of trip-makers,  $q$ , yield the modal flows  $y_r$  hence frequency  $\varphi$  and passenger load  $\omega$ , and also the average car speed  $v$  on the roadway path.

## Research questions

This article addresses the following research questions:

RQ1/ Given the service conditions in terms of price schedule and usage protocol (including waiting conditions), what are the respective money and time attributes of the three modes?

RQ2/ Having featured out the money and time attributes of the three modes, what are their respective ranges of efficiency along the VoT axis, i.e., according to  $\beta$ ?

RQ3/ Given the modal attributes and the  $(q, B)$  pair of trip-maker flow and statistical distribution of VoT, what are the modal flows [ $y_r: r \in \mathfrak{R}$ ] and the related service performances  $(\varphi, \omega, v)$ ?

RQ4/ Taking the determination of modal conditions from the  $(\varphi, \omega)$  factors as a supply function, and that of trip flows from the modal conditions as a demand function, what is their joint outcome, is there a traffic equilibrium between supply and demand?

## Article objectives

The article is aimed to devise a bicriterion, time-money model of traffic assignment to a line ride-sharing service on a roadway path.

Under theoretical form, not only does the model shed light on service issues such as price schedule, waiting policies either on the User or Agent side and transactional protocol, but also it yields modal flows as analytical formulas of the various parameters and system state variables.

Two companion objectives are also addressed: (i) a mock case study is conducted to provide a first understanding of potential situations, (ii) the computation of equilibrium states: a specific, graphical approach is devised to deal with the multifold interaction between the modal flows and the modal conditions in time and money.

## Approach

Time&money bicriterion assignment was originally devised to study the modal competition between the plane and the train (Blanchet & Abraham, 1973). It was then applied to interurban ground mobility and the train vs. car modal competition (Quandt & Baumol, 1966; Marche, 1973) and to traffic assignment on roadway networks (Dafermos, 1972; Marche & Papon, 1987; Daito et al., 1992; Leurent, 1993; Dial, 1996, etc.). Here we model the individual roles of User, Agent and Neutral as specific travel modes on a roadway path, thereby constituting a specific kind of bicriterion model. The  $(\omega, \varphi)$  variables are specific and give rise to an original kind of bicriterion traffic equilibrium, together with path speed  $v$  that is endogenous as in former time&money models of traffic assignment.

The “line” features allows for simple theory: the issue of path diversion which is of paramount importance in bicriterion network traffic assignment is not involved here. The simple form of the model allows for explicit specification of the ride-sharing service characteristics, especially so to compare two alternative policies either “Agent Waits” (AW) or “User Waits” (UW). Their respective effects on the modal times translate straightforwardly on the model outcomes  $(\omega, \varphi, v)$  and  $[y_r: r \in \mathfrak{R}]$ .

## Article structure

The rest of the article is in five parts. Section Two represents the system of Roadway path, Line RS service and the population of car trip-makers. Section Three provides the traffic assignment model and studies its equilibrium state as a multi-sided traffic equilibrium, considering the two sides of Service Agents and Users and also the Neutral trips as a third tier. Section Four is devoted to a numerical experiment (mock case study). Section Five discusses the model outreach and limitations and also some issues of system design. Section Six concludes and points out to some directions of further research.

**Tab.1: Notation.**

$\varphi$ service frequency	$L$ link length
$\omega$ rider load of car run	$H$ period duration
$r, s \in \mathfrak{R} \equiv \{A, U, N\}$ service role	$v$ link speed
$\ell \in \mathcal{L} \equiv \{R, S, T, W\}$ trip leg	$V$ speed-flow function
$\delta_r$ indicator variable of wait assignment to role $r$	$q$ population size
$\varepsilon \equiv \delta_A - \delta_U$	$y_r$ person flow of role $r$
$t_r$ trip time of role $r$ , time $t_r^\ell$ by leg $\ell$	$x_r$ car trip flow of role $r$
$\gamma_r$ modal constant of role $r$	$\beta$ Value-of-Time (VoT)
$\underline{m}$ base car cost per trip	$B$ CDF of VoT
$\dot{m}$ length factor of car cost	$E_r^\#$ efficiency range of role $r$
$\tau_r$ base trip award ( $r = A$ ) or fare ( $r = U$ )	$\underline{\beta}_r^\#$ lower bound of efficiency range
$\dot{\tau}_r$ length factor of award ( $r = A$ ) or fare ( $r = U$ )	$\bar{\beta}_r^\#$ upper bound of efficiency range
$m_r$ money cost of role $r$	$\beta_{rs}^\#$ cut-off VoT between roles $r$ and $s$
$g_r$ generalized cost of $r$ as function of VoT	$p_r$ population proportion assigned to role $r$
$\omega_M$ parameter of money line	$\mu_{s-r}$ short hand notation of fixed cost difference
$\omega_\varepsilon(\varphi), \omega_\infty$ parameters of time line	$\tilde{t}$ short-hand notation

## 2/ System representation

### 2.1/ Roadway path and traffic scene

A roadway link of network length denoted  $L$ , is considered in either flow direction as a specific “traffic scene”. It is studied on a daily basis, by within-day time period. A specific period of interest is considered, e.g. morning peak or evening peak or in-between, with time duration denoted  $H$ .

### 2.2/ Service roles as travel modes

With respect to a line ride-sharing service on the roadway link, three roles are identified for car trip-makers: service user as Rider (role U), service agent as driver (role A) or non-participant i.e. neutral (role N). The set of roles is denoted  $\mathfrak{R} \equiv \{U, A, N\}$ .

On a per trip basis, the respective money expenses are denoted as  $m_r$ , and time expenses as  $t_r$ . These split into “leg times”  $t_r^\ell$  according to trip phases or legs,  $\ell \in \mathcal{L} \equiv \{R, S, T, W\}$ :

- Leg R of car running on the link, same for all modes,
- Leg S of Stop or Dwelling for rider boarding and alighting, for Users and for those Agents that get “customers”,
- Leg T of Transaction operations: getting information and possibly being assigned, Paying on the User side or Being paid on the Agent side, both using the service digital platform,
- Leg W of Waiting: either the User is required to Wait for the next Agent under the UW policy, or the Agent is asked to wait for an incoming user under the AW policy.

According to role  $r$ , the modal trip times sums up the leg times of the different kinds:

$$t_r = \sum_{\ell \in \mathcal{L}} t_r^\ell \quad (1)$$

Trip travel time is a primary indicator of the quality of the mode as a travel option.

### 2.3/ Service features and policy

Ride-sharing is a bi-sided form of transport service as it involves people in two different ways: not only the service-to-demand form of Users, but also the cooperator-to-service form of Agents. These cooperate with the service coordination (platform) by supplying seats in their vehicles, the driving function including dwelling for rider boarding and alighting, possibly waiting for incoming riders to get to the car and other assistance (e.g. putting a foldable two-wheeler in the car trunk).

Service coordination between the two sides, Agents and Users, relies upon a digital interaction platform that achieves user and agent matching (in other words, the assignment of riders to cars and that of cars to riders) by suitable information collection and delivery, as well as fare collection from users and money compensation to agents.

These transactional operations are assumedly performed efficiently owing to high level of automation and suitable platform customer interface, with total time per trip of  $t_A^T$  and  $t_U^T$  on the agent and user side respectively.

Also given are the service stop times per trip,  $t_U^S$  on the user side giving rise to  $t_A^S$  on the agent side if the agent car run is endowed with rides.

Both the transaction and stopping times are taken as exogenous. By contrast, the major features of service quality are endogenous as they fundamentally depend on people participation to the service. The disaggregate involvement of individuals as Users or Agents, aggregated over the link trips, gives rise both to of rider flow  $y_U$  and frequency  $\varphi = y_A$  of service runs, hence to average time interval  $H/\varphi$  between successive service runs during the period, and to average wait times per trip as follows:

$$\underline{t}_U^W = H/\varphi \text{ and } \underline{t}_A^W = 0 \quad (2-UW)$$

$$\underline{t}_U^W = 0 \text{ and } \underline{t}_A^W = H/y_U, \quad (2-AW)$$

depending on whether the service waiting policy is targeted to Users or to Agents.

While  $\underline{t}_U^W$  applies to all rides on the user side, agent wait time  $\underline{t}_A^W$  only applies to the trips of agents to which customer rides are assigned.

Furthermore, ratio  $y_U/y_A$  determines the average passenger load by agent run, denoted  $\omega$ :

$$\omega = y_U/y_A \quad (3)$$

On average per agent trip, the stop time is  $t_A^S = \omega t_A^S$ . Under AW the average agent wait time is  $t_A^W = \omega t_A^W = H/\varphi$ . Letting  $\delta_A$  be the binary indicator of AW and  $\delta_U$  that of UW, the option times are formulated generically as follows:

$$t_N = t_N^R = t_R, \quad (4-N)$$

$$t_A = t_R + t_A^T + \omega t_A^S + \delta_A \frac{H}{\varphi}, \quad (4-A)$$

$$t_U = t_R + t_U^T + t_U^S + \delta_U \frac{H}{\varphi}. \quad (4-U)$$

Beside the waiting policy, service policy also includes (i) roadway link selection in relation to the local Mobility Organizing Authority, (ii) Access conditions of Rides to Car runs, assumed here at link endpoints only, (iii) the price schedule.

By assumption, for each car trip offered by an agent as a service run, a twofold reward (fee) is awarded to the agent: a base fee denoted  $\tau_A$  applies whatever the number of riders (including modality Zero), plus a per ride contribution that depends on ride length, say  $L\hat{t}_A$  with  $\hat{t}_A$  the ride award rate per unit length. On average per service run, the money reward to the agent is thus

$$\hat{t}_A = \tau_A + \omega L\hat{t}_A \quad (5-A)$$

On the user side, we similarly distinguish between a fixed fare  $\tau_U$  and a length-variable fare  $L\hat{t}_U$ : the ride fare is thus

$$\hat{t}_U = \tau_U + L\hat{t}_U \quad (5-U)$$

There certainly is a relationship between  $\hat{t}_A$  and  $\hat{t}_U$ : some financial balance between  $y_A\hat{t}_A$  and  $y_U\hat{t}_U$  may be expected, up to external subsidizes e.g. from the local mobility organizing authority on a per run basis, to taxes and of course to platform remuneration. Such issues of crucial interest of the service business model are left aside in this article.

## 2.4/ Modal money items and generalized costs

The ride fare constitutes the basic money cost of the link trip to a service user:

$$M_U \equiv \hat{t}_U = \tau_U + \omega L\hat{t}_U \quad (6-U)$$

Service-neutral trip-makers are assumed to spend a fixed money cost of  $\underline{m}$  plus a length-variable cost at unit length rate of  $\dot{m}$ , yielding link cost of

$$M_N \equiv \underline{m} + L\dot{m} \quad (6-N)$$

To a service agent, the link cost amounts to car expense minus run income, yielding

$$M_A \equiv M_N - \hat{t}_A = \underline{m} + L\dot{m} - \tau_A - \omega L\hat{t}_A \quad (6-A)$$

Some trip features remain unobserved in the model, notably the sub-paths up- and down-stream the roadway link.

The unobserved features of the alternative modes  $r$  are called “modal constants” measured as basic generalized costs and denoted  $\gamma_r$ . Denoting now the individual VoT as  $\beta$  (the change rate of travel time to money), at the trip level the modal option  $r$  induces a “travel generalized cost” of

$$g_r^\#(\beta) \equiv m_r + t_r \cdot \beta \quad (7)$$

Wherein  $m_r \equiv M_r + \gamma_r$ .

The value of  $\beta$  is likely to be modulated according to trip legs – for instance, wait time may be perceived by the individual as more costly than the run time. We keep such modulations implicit – it may be denoted by role- and leg-dependent VoTs  $\beta_r^\ell$  such that

$$\beta \cdot t_r = \sum_{\ell \in \mathcal{L}} \beta_r^\ell \cdot t_r^\ell.$$

## 2.5/ Population composition and trip flows

By assumption, each individual in the population of interest makes one and one trip only on the roadway link during the period under study. Thus the population size and the trip flow are equal and both are denoted as  $q$ .

Within this population, the individual VoTs are distributed statistically, with CDF denoted  $B$  and complementary function  $\bar{B} \equiv 1 - B$ .

The modal flow  $y_r$  by travel mode  $r \in \mathfrak{R}$  gives rise to a car flow  $x_r$  that is equal to  $y_r$  for  $r \in \{A, N\}$  or null otherwise, i.e.  $x_U = 0$ . Total car flow amounts to

$$x = x_A + x_U = y_A + y_U \quad (8)$$

## 2.6/ Traffic issues

We model the influence of car trip flow  $x$  onto link speed  $v$  as a traffic law  $V$  between speed and hourly flow rate:

$$v = V\left(\frac{x}{H}\right) \quad (9)$$

This traffic law induces the link run time by trip whatever its mode:

$$t_r^R = \frac{L}{v}, \quad \forall r \in \mathfrak{R} \quad (10)$$

Both flow volume  $x$  and speed  $v$  are traffic conditions determining the local emissions of noise and of air pollutants, including GHG emissions of global outreach.

## 3/ Multi-sided traffic equilibrium

### 3.1/ Flow relationships

We have already introduced the relationships linking the endogenous service features  $(\omega, \varphi)$  to the service flows  $(y_A, y_U)$  in (2) and (3) respectively, as well as the composition of car flow in (8) and the speed-flow relationship (9). The modal split in the population of individuals makes up another flow relation:

$$q = \sum_{r \in \mathfrak{R}} y_r \quad (11)$$

### 3.2/ Microeconomic behaviors and the configuration of modal costs

According to the service Waiting policy  $\# \in \{AW, UW\}$ , the modal generalized costs are functions of individual VoT  $\beta$  – linear affine functions under the previous assumptions.

The fundamental model postulate about individual behavior is that each individual selects the option of minimum cost to him or her, meaning minimum disutility hence maximum utility, i.e., rational microeconomic behavior.

Depending on  $\beta$ , the modal option yielding minimum cost is denoted  $\hat{r}^\#(\beta)$ , with associate generalized cost of

$$\hat{g}^\#(\beta) = \min\{g_r^\#(\beta) : r \in \mathfrak{R}\} \quad (12)$$

The variations of option costs with respect to  $\beta$  determine the efficiency domains of the mode: in the classical Time&Money model, lower price modes are selected by individuals with low values of  $\beta$ , whereas shorter time ones are selected by higher  $\beta$  (Figure 1).

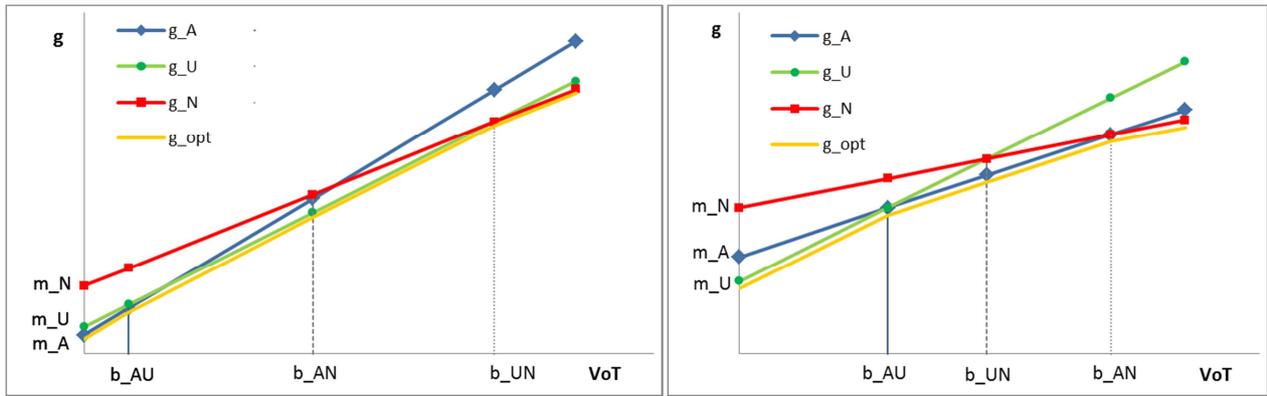


Fig.1: Generalized cost functions of the service roles, yielding configuration (A) AUN, (B) UAN.

As regards modal prices, based on (6) it holds that

$$M_A \leq M_N \quad (13a)$$

Under realistic values satisfying  $\tau_A \geq 0$ ,  $\dot{\tau}_A \geq 0$  and  $\omega \geq 0$ .

We may safely assume that  $\gamma_A \approx \gamma_N$ , leading to  $m_A = M_A + \gamma_A \leq M_N + \gamma_N = m_N$ .

On the user side, we postulate that  $\gamma_U$  does not change the ordering of money expenses: so that

$$m_U \leq m_N \quad (13b)$$

*NB. This postulate is a strong assumption about the up- and down-stream parts of individual trips.*

In a diagram of modal cost functions with respect to VoT, modal prices make the origin ordinates and the modal times make the slopes of their respective cost lines.

As transaction and stop times are null for Neutral option,  $t_N^T = t_N^S = 0$ , but have positive values for types U and A, whatever the Waiting policy we have that

$$t_N \leq t_U^\# \quad (14a)$$

$$t_N \leq t_A^\# \quad (14b)$$

This makes “Neutral” the best value option for high VoTs.

The comparison between  $t_U^\#$  and  $t_A^\#$  involves the waiting policy and the frequency  $\varphi$ , as well as the  $\omega$  factor:

- Under AW and  $\omega \geq 1$ , then  $t_U^\# < t_A^\#$  up to significant discrepancy between  $t_U^T$  and  $t_A^T$  (unlikely) or between  $t_U^S$  and  $t_A^S$  (unlikely, too).
- Under UW and  $\omega < 1$ , then  $t_A^\# < t_U^\#$ .

The two cases yield opposite outcomes: the former making User a higher quality option than Agent, the latter making Agent a higher quality option than User.

The joint condition AW and  $\omega \geq 1$  looks consistent: under high patronage, having agents to wait for riders is not much requiring. A contrasted condition AW and  $\omega < 1$  yields no obvious outcomes.

The joint condition UW and  $\omega < 1$  looks consistent, too: making Users to wait for Agents that are more numerous. Yet a contrasted condition UW and  $\omega \geq 1$  would put both waiting requirements and in-vehicle crowding as burdens on the users.

### 3.3/ Efficiency domains and flow outcomes

The efficiency set of mode  $r$  under policy  $\#$  is

$$E_r^\# \equiv \{\beta \in [0, +\infty[ : g_r^\#(\beta) \leq g_s^\#(\beta), \forall s \in \mathfrak{R}\}$$

By definition,

$$\underline{\beta}_r^\# \equiv \inf\{\beta \in [0, +\infty[ : g_r^\#(\beta) \leq g_s^\#(\beta), \forall s \in \mathfrak{R}\} = \inf E_r^\#$$

$$\bar{\beta}_r^\# \equiv \sup\{\beta \in [0, +\infty[ : g_r^\#(\beta) \leq g_s^\#(\beta), \forall s \in \mathfrak{R}\} = \sup E_r^\#$$

From the affine linear nature of modal generalized costs as functions of VoT  $\beta$ , the non-empty efficiency domains on  $[0, +\infty[$  are simple intervals  $[\underline{\beta}_r^\#, \bar{\beta}_r^\#]$  from “lower bound” to “upper bound”  $\bar{\beta}_r^\#$ : we expect that  $\underline{\beta}_r^\# \leq \bar{\beta}_r^\#$  if the interval is non-empty, but values  $\bar{\beta}_r^\# = 0$  and  $\underline{\beta}_r^\# = +\infty$  indicate an empty domain.

The modal market share is (Figure 2):

$$p_r^\# = \Pr\{\beta \in E_r^\#\} = (B(\bar{\beta}_r^\#) - B(\underline{\beta}_r^\#))^+. \quad (15)$$

It gives rise to modal volume

$$y_r^\# = q \cdot p_r^\#. \quad (16)$$

The configuration of efficiency domains along the VoT axis has a deep outreach. Two basic configurations are  $\{A,U,N\}$  meaning  $E_A^\# \leq E_U^\# \leq E_N^\#$  versus  $\{U,A,N\}$  meaning  $E_U^\# \leq E_A^\# \leq E_N^\#$ .

Degenerate cases include  $\{A,N\}$  i.e. empty set  $E_U^\#$  - a service with agents but no customers, and  $\{U,N\}$  a service with potential users but no agents, entailing no available runs.

Non-degeneracy with positive  $p_A^\#$  and  $p_U^\#$  may come with  $p_N^\# = 0$ , under bounded VoT distribution such that  $\underline{\beta}_N^\# > 0$  but  $B(\underline{\beta}_N^\#) = 1$ .

The comparison of two options  $r$  and  $s$  gives rise to “frontier VoT”, denoted  $\beta_{rs}^\#$  and also called “cut-off VoT”, at which point the two modes are equivalent, i.e., their generalized costs are equal:

$$g_r^\#(\beta) = g_s^\#(\beta). \quad (17)$$

The equivalency condition straightforwardly entails that

$$\beta_{rs}^\# = \frac{m_s - m_r}{t_r - t_s} (= \beta_{sr}^\#). \quad (18)$$

Provided that  $t_r \neq t_s$ .

Non-degenerate situation  $\{U,A,N\}$  involves the following inequality conditions between the cut-off VoTs:

$$0 < \beta_{UA}^{\#} \leq \beta_{UN}^{\#} \leq \beta_{AN}^{\#}$$

And more precisely the stronger conditions that

$$0 < B(\beta_{UA}^{\#}) < B(\beta_{UN}^{\#}) < B(\beta_{AN}^{\#}) \quad (19a)$$

Non-degenerate configuration {A,U,N} arises when:

$$0 < B(\beta_{UA}^{\#}) < B(\beta_{AN}^{\#}) < B(\beta_{UN}^{\#}) \quad (19b)$$

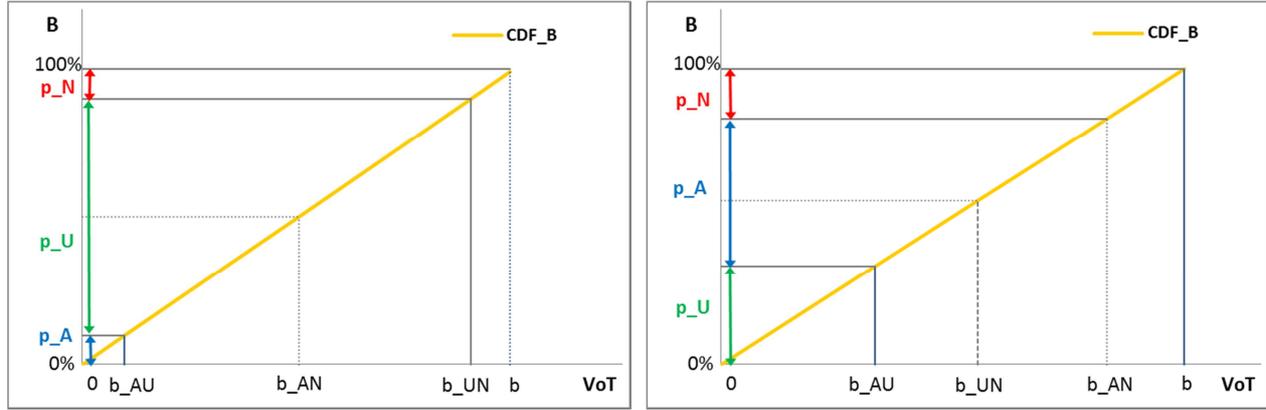


Fig.2: Modal shares of the service roles, under configuration (A) AUN, (B) UAN.

### 3.4/ Traffic equilibrium

**Definition.** The system is in equilibrium if all relations (1-18) linking its endogenous state variables are satisfied jointly, i.e. they do hold true simultaneously.

Such equilibrium state is a traffic equilibrium since it involves flows  $[y_r: r \in \mathfrak{R}]$  and related variables  $(\varphi, \omega, v)$ . It is a multi-sided equilibrium as it involves individual behaviors on both sides of the cooperative mobility service: agents and users.

The system of influences between the variables is depicted in Figure 3. A “basic” state vector is a vector of selected endogenous variables from which all of the other endogenous variables are derived straightforwardly: two possible selections are exhibited in the Figure, namely the vector  $\mathbf{y}_{\mathfrak{R}} = [y_r: r \in \mathfrak{R}]$  of modal flow volumes in part A, or the vector  $(\varphi, \omega)$  of service conditions in part B.

The derivation of “demanded flows” (or “people flows”)  $[y_r^{\#}: r \in \mathfrak{R}]$  is straightforward from  $(\varphi, \omega)$ , though lengthier than that of “supplied flows” (or “service flows”)  $[\hat{y}_r: r \in \mathfrak{R}]$ .

**Proposition.** A  $(\varphi, \omega)$  pair of service conditions is a traffic equilibrium if the supplied flows and the demanded flows are balanced, i.e.,

$$\forall r \in \mathfrak{R}: \hat{y}_r(\varphi, \omega) = y_r^{\#}(\varphi, \omega) \quad (20)$$

These balance conditions are also the characteristic equilibrium conditions characterizing a basic state vector of flows  $\mathbf{y}_{\mathfrak{R}}$ :

$$\forall r \in \mathfrak{R}: y_r = y_r^{\#}((\hat{\varphi}, \hat{\omega})(\mathbf{y}_{\mathfrak{R}})) \quad (21)$$

This mathematical condition is obviously a Fixed Point Problem in the flow vector  $\mathbf{y}_{\mathfrak{R}}$ .

In Appendix A, the balance conditions are made more explicit depending on the multimodal configuration, in other words on the shape of the Pareto frontier of the modes along the VoT axis.

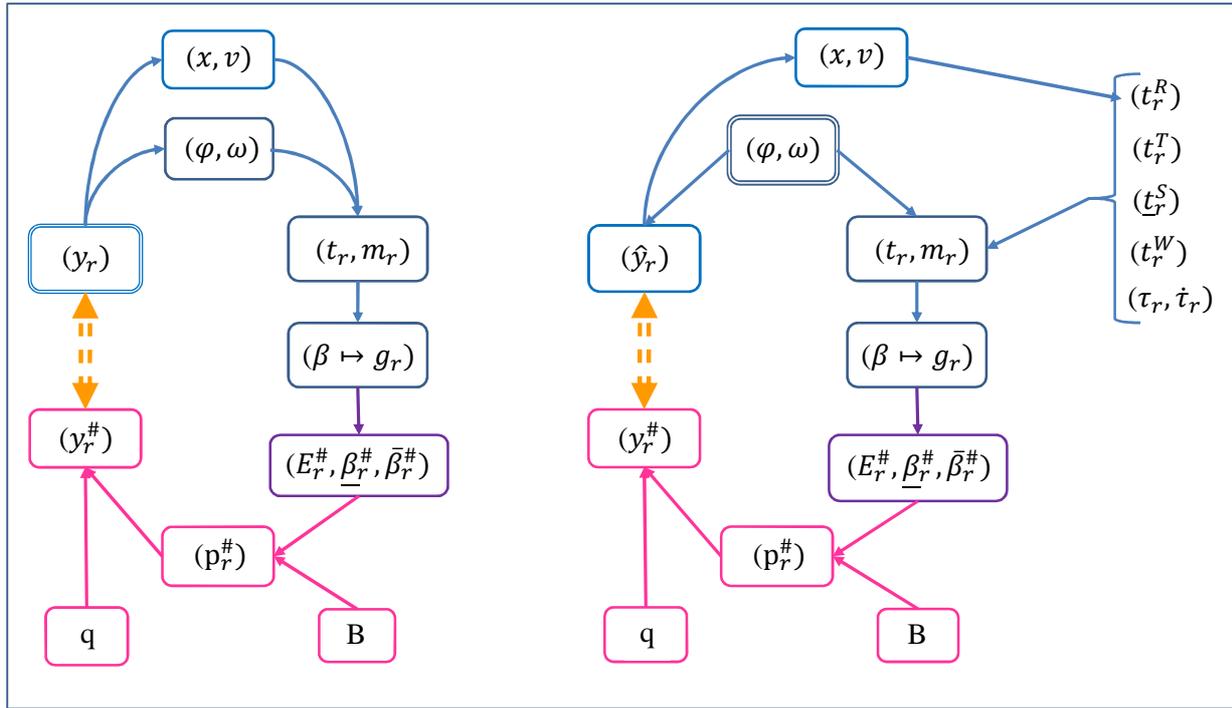


Fig.3: Model architecture, with basic state vector of (A) flows, (B) service conditions.

### 3.5/ Equilibrium determination

The efficiency range bounds  $\underline{\beta}_r^\#$  and  $\bar{\beta}_r^\#$  are useful variables to restate the balance conditions as

$$\forall r \in \mathfrak{R}: \hat{y}_r = q \cdot (\bar{B}_r^\# - \underline{B}_r^\#)^+ \quad (22)$$

Wherein  $\underline{B}_r^\# \equiv B(\underline{\beta}_r^\#)$  and  $\bar{B}_r^\# \equiv B(\bar{\beta}_r^\#)$ .

From the composition of the modal money and time items, it comes out that mode N, having lowest time  $t_N$  and highest cost  $m_N$ , is the best quality options to high VoTs, meaning that  $\bar{\beta}_N^\# = +\infty$  and  $\underline{\beta}_N^\# = 1$ , hence that

$$\hat{y}_N = q \cdot (1 - \underline{B}_N^\#) \quad (23\_N)$$

As  $\sum_{r \in \mathfrak{R}} \hat{y}_r = q$ , it is a condition on the service flow  $\hat{y}_S \equiv \hat{y}_A + \hat{y}_U$ :

$$\hat{y}_A + \hat{y}_U = q \cdot \underline{B}_N^\# \quad (23\_S)$$

Then, if both modes A and U have non-empty efficiency ranges, their respective configuration is either A<U or U<A. Configuration A<U involves  $\underline{\beta}_A^\# = 0$  and  $\bar{\beta}_A^\# = \underline{\beta}_U^\# = \beta_{UA}^\# < \bar{\beta}_U^\# = \underline{\beta}_N^\# = \beta_{UN}^\#$ , yielding that

$$\hat{y}_A = q \cdot \underline{B}_A^\# \text{ and } \hat{y}_U = \hat{y}_S - \hat{y}_A. \quad (24\_AU)$$

Configuration U<A involves  $\underline{\beta}_U^\# = 0$  and  $\bar{\beta}_U^\# = \underline{\beta}_A^\# = \beta_{UA}^\# < \bar{\beta}_A^\# = \underline{\beta}_N^\# = \beta_{AN}^\#$ , yielding that

$$\hat{y}_U = q \cdot \underline{B}_U^\# \text{ and } \hat{y}_A = \hat{y}_S - \hat{y}_U. \quad (24\_UA)$$

Thus, any non-degenerate tri-modal equilibrium is characterized by a threefold condition – essentially a twofold condition since the 3<sup>rd</sup> one is obvious:

$$A < U: \begin{cases} \hat{y}_A = q \cdot B(\beta_{UA}^\#) \\ \hat{y}_A + \hat{y}_U = q \cdot B(\beta_{UN}^\#) \\ \hat{y}_N = q - \hat{y}_A - \hat{y}_U \end{cases} \text{ versus } U < A: \begin{cases} \hat{y}_U = q \cdot B(\beta_{UA}^\#) \\ \hat{y}_A + \hat{y}_U = q \cdot B(\beta_{AN}^\#) \\ \hat{y}_N = q - \hat{y}_A - \hat{y}_U \end{cases}. \quad (25)$$

The first condition may be called a “lower hand function” (LHF) and the second one an “upper hand function” (UHF).

**Under A<U configuration**,  $\beta_{UN}^\#$  only involves  $\varphi$  and not  $\omega$ , so that the UHF is simply

$$\bar{\omega} = \frac{q}{\varphi} B(\beta_{UN}^\#) - 1.$$

The UHF condition is stated as  $\bar{\omega}(\varphi)$  or, about equivalently, as  $\bar{y}_U(y_A)$  with respect to modal flows. The LHF condition also relates  $\omega$  to  $\varphi$ , or about equivalently  $y_U$  to  $y_A$ . If the CDF B is invertible at  $\hat{y}_A/q$  then the LHF is equivalent to

$$m_A + (t_A - t_U)B^{(-1)}\left(\frac{y_A}{q}\right) = m_U,$$

And, denoting  $\tilde{t} \equiv t_U^T + t_U^S - t_A^T$  and  $\mu_{A-U} \equiv \gamma_A + \underline{m} + L\dot{m} - \tau_A - \gamma_U - \tau_U - Lt_U$ , to:

$$y_A \left( \mu_{A-U} - \tilde{t} \cdot B^{(-1)}\left(\frac{y_A}{q}\right) \right) - y_U \left( Lt_A - \underline{t}_A^S \cdot B^{(-1)}\left(\frac{y_A}{q}\right) \right) + B^{(-1)}\left(\frac{y_A}{q}\right) \varepsilon_\# H = 0.$$

It makes  $y_U$  an obvious function of  $y_A$ , denoted by  $\underline{y}_U(y_A)$ , and  $\omega$  an obvious function  $\underline{\omega}(\varphi)$ .

The feasible domain of  $(\varphi, \omega)$  is  $[0, q] \times \mathbb{R}^+$ .

**Proposition: non-degenerate traffic equilibrium under A<U.** A feasible pair  $(\varphi, \omega)$  is a non-degenerate tri-modal equilibrium with A<U configuration if it satisfies that

$$\omega = \bar{\omega}(\varphi) = \underline{\omega}(\varphi).$$

**Under U<A configuration** the direction of influence is reversed. The LHF condition on  $\hat{y}_U$ , assuming invertible B at  $\hat{y}_U/q$ , is

$$m_A = m_U + (t_U - t_A)B^{(-1)}\left(\frac{y_U}{q}\right), \text{ yielding}$$

$$y_A \left( \mu_{A-U} - \tilde{t} \cdot B^{(-1)}\left(\frac{y_U}{q}\right) \right) - y_U \left( Lt_A - \underline{t}_A^S \cdot B^{(-1)}\left(\frac{y_U}{q}\right) \right) + B^{(-1)}\left(\frac{y_U}{q}\right) \varepsilon_\# H = 0,$$

Wherein an LHF  $\underline{y}_A(y_U)$  appears clearly, yielding an LHF  $\underline{\varphi}(\omega)$ .

Then, the condition on  $\hat{y}_S$  is taken as an UHF  $\bar{y}_A(y_U)$ , about equivalent to an UHF  $\bar{\varphi}(\omega)$ .

**Proposition: non-degenerate traffic equilibrium under U<A.** A feasible pair  $(\varphi, \omega)$  is a non-degenerate tri-modal equilibrium with U<A configuration if it satisfies that

$$\varphi = \bar{\varphi}(\omega) = \underline{\varphi}(\omega).$$

Appendix A provides a more comprehensive analysis of the equilibrium configurations: not only the non-degenerate ones but also the degenerate ones bi-modal or uni-modal. The UHF-LHF pairs are further studied in Appendix B. A fully explicit solution is provided in Appendix C for uniformly distributed VoT.

### 3.6/ Graphical solution

The feasible domain of  $(\varphi, \omega)$ ,  $D_{(\varphi, \omega)} \equiv [0, q] \times \mathbb{R}^+$ , or about equivalently that  $D_y$  of pairs  $(y_A, y_U)$  such that  $y_A \geq 0$ ,  $y_U \geq 0$  and  $y_A + y_U \leq q$ , is useful to locate the graphs of the UHF and LHF and to determine equilibrium states graphically at the intersection points between the function graphs. Each domain includes specific regions of interest. Region  $\{m_A \leq m_U\}$  is delimited from region  $\{m_A \geq m_U\}$  by the “money line”  $\{m_A = m_U\}$ , hence

$$\omega = \omega_M \text{ and } y_U = \omega_M \cdot y_A, \text{ wherein: } \omega_M \equiv \mu_{A-U}/(Lt_A).$$

The condition that  $m_A \leq m_U$  is equivalent to  $\omega \geq \omega_M$ , locating region  $\{m_A \leq m_U\}$  on the upper side of the money line in  $D_{(\varphi, \omega)}$  and also in  $D_y$ , whereas region  $\{m_A \geq m_U\}$ , i.e.,  $\omega \leq \omega_M$ , is located on the lower side in  $D_{(\varphi, \omega)}$  and  $D_y$ .

Region  $\{t_A \geq t_U\}$  is delimited from region  $\{t_A \leq t_U\}$  by the “time line”  $\{t_A = t_U\}$ , which amounts to a condition

$$\omega = \omega_\varepsilon(\varphi) \equiv \omega_\infty - \frac{\varepsilon_\# H}{\varphi t_A^S}, \text{ and } y_U = \omega_\infty \cdot y_A - \frac{\varepsilon_\# H}{t_A^S}.$$

Wherein:  $\omega_\infty \equiv \tilde{t}/t_A^S$  and  $\varepsilon_\# = +1$  under AW or  $-1$  under UW. The condition that  $t_A \geq t_U$  is equivalent to  $\omega \geq \omega_\varepsilon(\varphi)$ , locating region  $\{t_A \geq t_U\}$  on the upper side of  $\omega_\varepsilon(\varphi)$  in  $D_{(\varphi, \omega)}$  and also in  $D_y$ , whereas region  $t_A \leq t_U$ , i.e.,  $\omega \leq \omega_\varepsilon(\varphi)$ , is located on the lower side in  $D_{(\varphi, \omega)}$  and also in  $D_y$ .

An  $\{A < U < N\}$  configuration requires both that  $\{m_A \leq m_U\}$  and  $\{t_A \geq t_U\}$ , hence it is located in the upper side. Conversely an  $\{U < A < N\}$  configuration, requiring both that  $\{m_A \geq m_U\}$  and  $\{t_A \leq t_U\}$ , is located in the lower side. Over an intermediary region  $\{m_A \leq m_U\}$  and  $\{t_A \leq t_U\}$ , option A fully dominates option U, whereas over  $\{m_A > m_U\}$  and  $\{t_A > t_U\}$  it is option U that fully dominates option A – making an unsustainable state of service traffic with zero agents.

Figure 4 exhibits the money line and time line under the “User Waits” policy, whereas Figure 5 addresses “Agent Waits”.

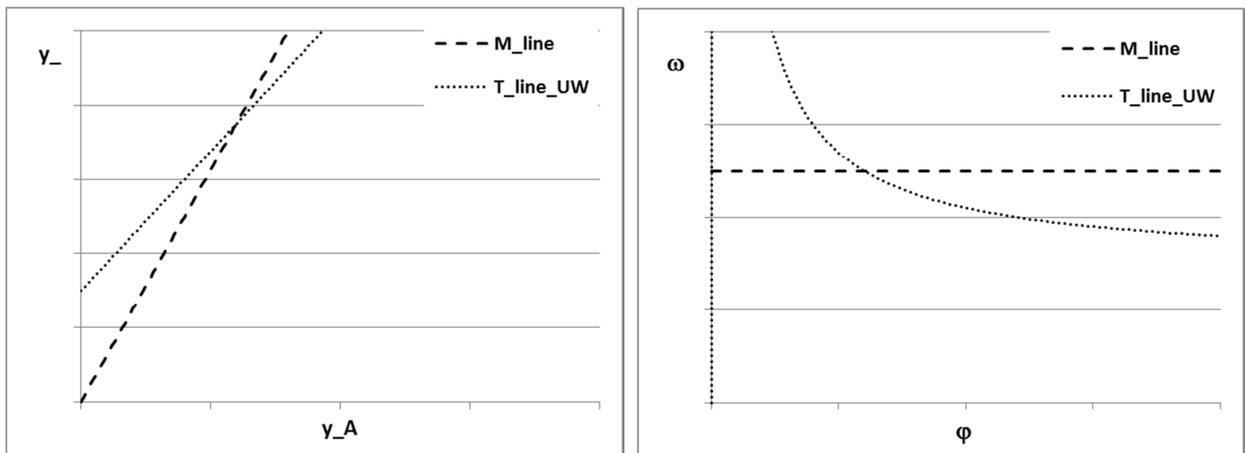


Fig.4: Configurational analysis under UW in (A) the  $(y_A, y_U)$  domain, (B) the  $(\varphi, \omega)$  domain.

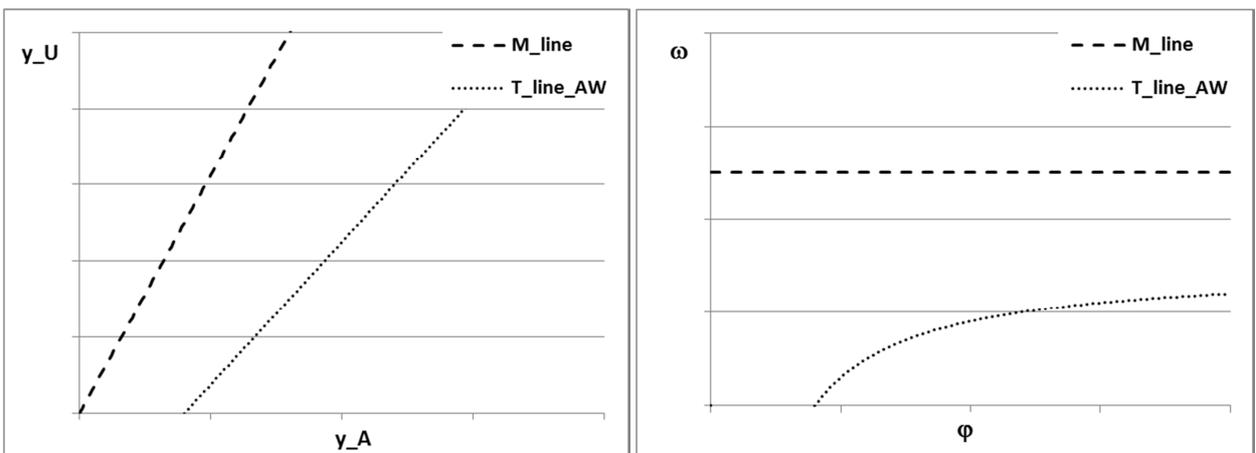


Fig.5: Configurational analysis under AW in (A) the  $(y_A, y_U)$  domain, (B) the  $(\varphi, \omega)$  domain.

## 4/ Numerical experiment

The experiment is aimed to demonstrate the model outcomes depending on specific conditions, most notably the Waiting policy and the effect of trip flow rate low or high.

### 4.1/ Case setting

An interurban road link with  $L = 10$  km is considered.

The ride-sharing service has the following price schedule:

- On the agent side,  $\tau_A = 0.5$  € and  $\dot{t}_A = 0.10$  €/km,
- On the user side,  $\tau_U = 1$  € and  $\dot{t}_U = 0.20$  €/km.

The base costs of car holding and utilization are set up to  $\underline{m} = 1$  € per trip and  $\dot{m} = 0.30$  €/km.

The modal constants are set up to  $\gamma_N = 1$  €,  $\gamma_A = 1$  € and  $\gamma_U = -1$  €.

Thus, the per-trip costs amount to:

- Neutral:  $M_N = 4$  € and  $m_N = 5$  €,
- User:  $M_U = 2$  € and  $m_U = 2$  €,
- Agent:  $M_A = 3.5 - \omega$  € and  $m_A = 4.5 - \omega$  €.

The  $\omega_M$  parameter amounts to 2.5 (riders/car run).

As for time items, base stop times of  $t_A^S = 1$  min and  $t_U^S = 1$  min are assumed, together with transaction times of  $t_A^T = 1$  min and  $t_U^T = 1.5$  min. This induces  $\omega_\infty = 1.5$  (riders/car run).

In the population of trip-makers, individual VoTs follow a uniform distribution from 0 to  $b = 30$ €/h, hence average VoT of 15€/h. During a period of  $H = 1$  hour, contrasted flow levels are considered: “low flow” with  $q = 20$  trips, versus “high flow” with  $q = 200$  trips.

### 4.2/ Low flow

Assuming UW policy, the UHF and LHF relationships are depicted in Figure 6 both in the  $(y_A, y_U)$  domain (part A) and in the  $(\varphi, \omega)$  domain (part B), along with the money line and the time line. All intersection points between UHF and LHF counterparts lie below the money line, hence also below the timeline that is above the money line under UW and the parametric set-ups in the feasible domain. Thus the equilibrium configuration is  $\{U, A, N\}$  so that only the intersection point of  $UHF_{U<A}$  and  $LHF_{U<A}$  is in equilibrium, making it the unique state of equilibrium.

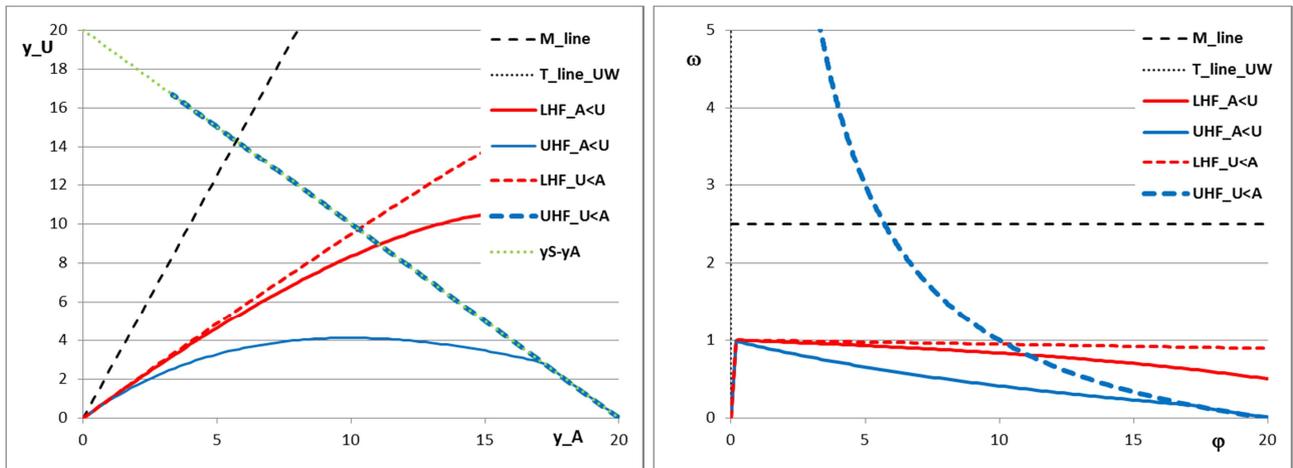


Fig.6: Low-flow situation under UW in (A) the  $(y_A, y_U)$  domain, (B) the  $(\varphi, \omega)$  domain.

Tab. 2: Equilibrium states under low flow (left part) versus high flow (right part).

Indicator (unit)	LOW FLOW		HIGH FLOW	
	UW	AW	UW	AW
$\omega$ (persons/car)	0.946	4.262	2.48	2.82
$\varphi = y_A$ (trips)	10.277	3.800	57.53	52.31
$y_U$ (trips)	9.723	16.200	142.47	147.69
$y_N$ (trips)	0	0	0	0
$m_A$ (€)	3.554	0.238	2.024	1.677
$t_A$ (hour, without run time)	0.032	0.351	0.05794	0.08283
$t_U$ (hour, without run time)	0.139	0.042	0.05905	0.04167
$\beta_{AU}^\#$ (€/h)	14.585	5.701	21.4	7.85
Equilibrium Configuration	U<A<N	A<U<N	U<A<N	A<U<N

Among the equilibrium features (Table 2), salient points are:

- (i) the involvement of all of the population in the service:  $y_N = 0$  because the VoT distribution has null upper tail,
- (ii) the  $y_A$  and  $y_U$  flows are close together, with occupancy rate  $\omega$  close to 1,
- (iii) service frequency about 10 per hour is relatively high,
- (iv) Users have lower VoTs than Agents.

Under the same settings save for AW policy (Figure 7), there is only one equilibrium point between UHF and LHF pairs: this time it is between the A<U functions, meaning a single equilibrium with {A<U<N} configuration. While the neutral flow  $y_N$  is still null, occupancy rate  $\omega \approx 4$  is very high for cars and indicates  $y_U \approx 4 \cdot y_A$ : about 20% of less wealthy people are drivers to the 80% wealthier individuals. The high value of  $\omega$  corresponds to fully occupied cars, yielding both a factor 5 reduction in the car flow of flexible trip-makers and relatively important money incomes to the agents, going beyond the principle of just sharing costs. Service frequency of about 40 car runs per hour is very high.

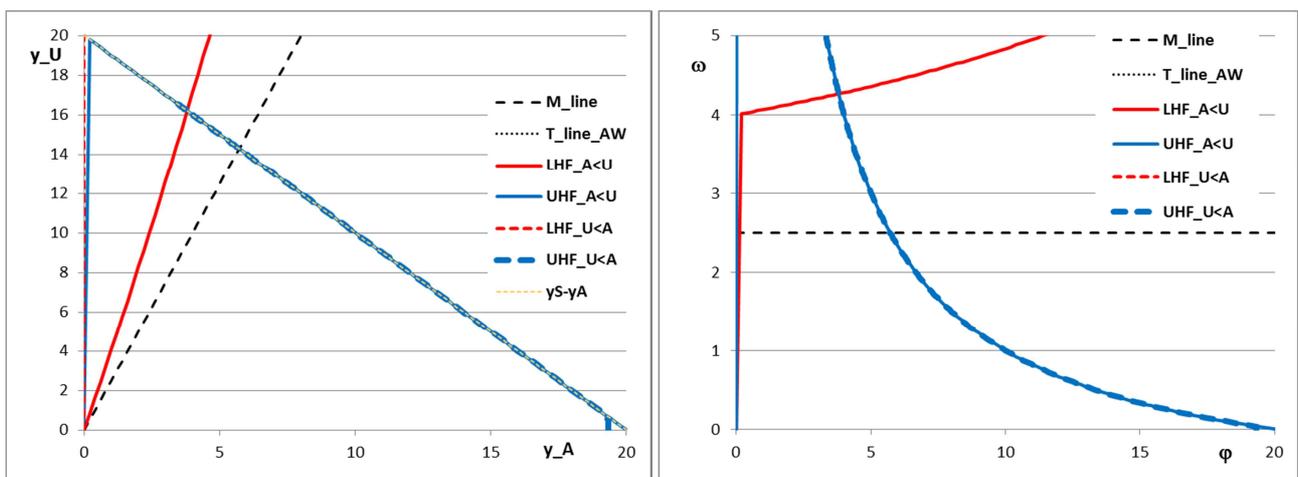


Fig.7: Low-flow situation under AW in (A) the  $(y_A, y_U)$  domain, (B) the  $(\varphi, \omega)$  domain.

### 4.3/ High flow

Under high flow and either UW or AW, again a single point of equilibrium is found for each waiting policy UW (figure 8) or AW (figure 9).

Again UW comes out with  $\{U < A < N\}$  configuration of equilibrium and AW with an  $\{A < U < N\}$  one. Again we have null neutral flow  $y_N = 0$  in either case (table 2, right hand part). Service conditions  $(\varphi, \omega)$  are relatively close between the two waiting policies with  $\omega$  close to 3, meaning  $y_U \approx 3y_A$  and service frequency of about 50 car runs per hour –very high frequency indeed.

But the similarities between the waiting policies extend only to the aggregate indicators of modal flows: the configurations are different, with fairly different modal time and money components, hence markedly different microeconomic situations of the individual trip-makers. Thus, significant differences are expected in the respective cost-benefit assessment of the two waiting policies.

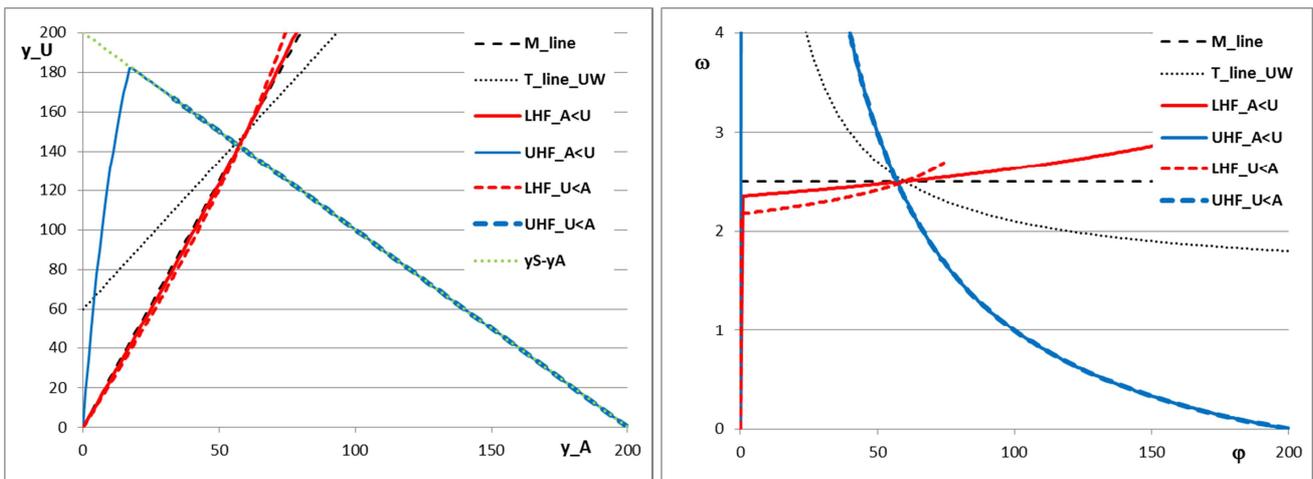


Fig.8: High-flow situation under UW in (A) the  $(y_A, y_U)$  domain, (B) the  $(\varphi, \omega)$  domain.

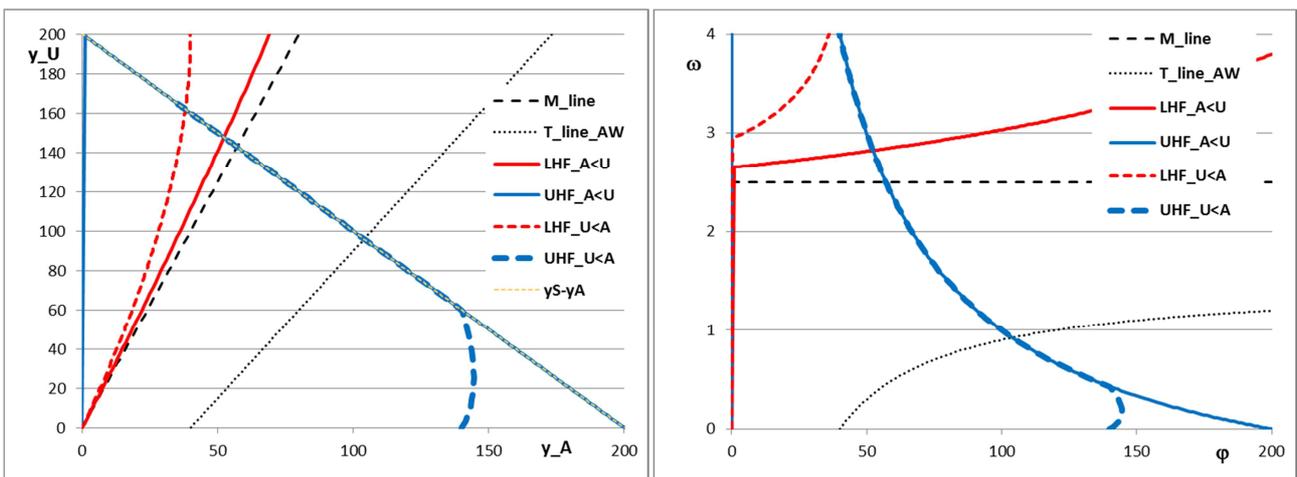


Fig.9: High-flow situation under AW in (A) the  $(y_A, y_U)$  domain, (B) the  $(\varphi, \omega)$  domain.

## 5/ Discussion

### 5.1/ Lessons from the numerical experiment

The graphical determination of equilibrium constitutes a robust computational scheme. It is essentially a bicriterion assignment algorithm as in Dial (1996) and Leurent (1996). Demand disaggregation along the VoT axis is the source of computational robustness that is all the more useful here since the traffic effects between modal flows and modal conditions are much involved and highly intertwined, yielding asymmetry effects in addition to nonlinearities.

The numerical experiment suffices to demonstrate that different equilibrium configurations can arise between the service rules, A versus U and also N. Configuration {AUN} and configuration {UAN} arose under low flow as well as high flow. Flow volumes as aggregate indicators are insufficient to determine the configuration, owing to the complex interaction between the roles.

The experiment also demonstrates the correlation between AW policy and {AUN} configuration, on one hand, and UW and {UAN}, on the other hand. Yet, in other instances we found some occurrences of configuration {UAN} at traffic equilibrium under AW.

Whatever the configuration of U and A roles, the Neutral role stands at the high end of the VoT distribution. Service participation as U or A basically involves money savings at the expense of some additional times: the associated trade-offs are fruitful for lower VoTs, while higher VoTs would keep to the Neutral role (no involvement in the service).

Lastly, the modal proportions varied greatly among the four instances: such variations call for truly behavioral modeling of ride-sharing services rather than making educated guesses on the respective modal proportions (e.g. Yin et al. 2018).

### 5.2/ About model assumptions: outreach and limitations

The numerical experiment also emphasized the key places of the “money line” (parameter  $\omega_M$  standing for money items and service price schedule) and of the “tree line” according to the waiting policy (parameters  $\varepsilon$  and  $\omega_\infty$ ). The temporal features of wait times and their assignment to either Users or Agents, of stop times and of transaction times are as much influential in service usage as the money items from the price schedule.

Both the money items and the “time items” aside from car run times constitute key features in the “user experience” and the “customer journey” as evocated in marketing literature. Our model captures these features as well as the run times and the other physical times that are commonly modeled for passenger traffic in transit networks (dwell times, wait times, in addition to run times).

On the demand side, the disaggregation of individual preferences along the VoT axis mirrors the disaggregation of individual incomes in microeconomic theory. It is a cornerstone in the microeconomic foundations of the traffic model. The individual trade-offs between time and money items lie at the heart of the model, which derives the aggregate consequences on the supply side in terms of service frequency  $\varphi$  and car occupancy rate  $\omega$ .

In the current stage of the model, individual behavior conditionally to the VoT is a deterministic one with all-or-nothing assignment to one option only. Small differences in time or money can induce modal changes between the roles (car sub-modes) in an abrupt way – a fairly dramatic change at the individual level. In reality, we can expect individual behaviors to be less clear cut and more fuzzy: the modal generalized cost functions are a first step towards the modeling of modal utility functions in the frame of random utility theory and discrete choice models. Such frame will enable one not only to model unobserved effects as residual random variables (cf. the modal constants) but also to represent comfort issues, notably VoT modulation according to service role and trip leg. Such modulation is especially important about waiting policies, since Agent wait times take place in-

vehicles hence at sitting comfortably, whereas User wait times occur out of cars, possibly at standing, maybe so outdoor and subjected to meteorological circumstances.

Beyond comfort come the issues of individual mobility practices, routines and attitudes. Here, service awareness is postulated, so that parameter  $q$  standing for population size represents in fact the number of service-aware people. The prior mobility routines of individuals may also induce captivities of some sort: some car drivers have specific requirements of their private cars in their day activity programs, possibly with professional constraints. Such constraints may come with company rules about access to company cars and usage conditions, departing from our assumptions about the microeconomic situation of the trip-maker.

Yet, other features of individual travel routines may also align with our modeling assumptions. Home to work or study commuting is a recurrent purpose for the working people on a daily basis or so (at least on worked days, up to working from home). Frequent reiteration at the individual level of the travel situation is likely to even out the fluctuations between the occurrences, giving ground to considering the expected times and money outcomes of each role as statistical means over the population of occurrences. In other words, commuting frequency certainly is a factor of alignment between the real-world conditions and the model assumptions.

### 5.3/ Some hints about service design

As an abstract representation, the model brings about concepts, state variables and notations to describe a real-world system, to indicate its conditions and to monitor its performances. It is inspired from early field experiments of ride-sharing lines along some roadway paths and also from pioneer implementations in France by the Ecov company. Our numerical experiment, though simple, reveals density economies and supports the techno-economic principle of lines for ride-sharing.

Path selection and identification is a primary topic of service design, along with station identification and the allowance or not to stop at any place along the path. Up to now, the waiting policy in Ecov's implementations is an "Agent waits" one. Our numerical experiment suggests that the alternative policy "User waits" may be fruitful on some path and demand conditions.

The model also emphasizes the importance of the price schedule and of the transactional conditions. Transaction times, though expectedly short, are nonetheless significant compared to dwell times and even to run times on one or two tens of km. While wait times are even more important under low flows, under high flows they are likely to fall down to a level similar to dwell times and transaction times. The ease and comfort of transaction operations are important, too, so as to make transaction times enjoyable rather than hard-felt. The "customer journey" has to be addressed in all of its respects by any LRS operator.

We have not considered the business model of the LRS operator, but only the demand and traffic facets of the service as a system. Such model of demand and traffic can be a useful tool to design a business model, by considering path features  $(L, v)$  as well as population characteristics  $(q, B)$  and by specifying service scenarios in terms of Waiting policy, price schedule and transactional conditions, up to access conditions.

While the model is focused on service attractiveness compared to the Neutral option, we have not addressed the issue of service awareness. Awareness rising in the population living near the line corridor is especially important to develop the service basis both of users and of agents. Among the solutions under way, displaying RS lines in local MaaS systems of multimodal service information is a prominent one, since they are purported for general use by all of the local population. MaaS integration will likely not be limited to service and traffic information, it will also foster commercial integration and multimodal synergies between the different kinds of public transit services.

## 6/ Conclusion

To understand and simulate a line ride-sharing service, we devised a model that is both a traffic model and a microeconomic model of the service on its two sides of Users and Agents. The economic conditions in money and time of the User and Agent roles and also of the service-Neutral role, are modeled at the trip level from the expected conditions of trip legs (Run, Stop, Wait, Transaction), depending on the service protocol that includes price schedule, waiting policy and transaction times, together with macroscopic service conditions in run frequency  $\varphi$  and average car rider load  $\omega$ . This sensitive and parametric representation of service roles answers to the first Research Question.

Then, on the demand side, role choice is modeled at the individual level as a rational microeconomic behavior of cost minimization, considering the generalized costs of the roles according to the individual VoT. This microeconomic model answers to the second Research Question. Integration along the VoT axis according to the statistical distribution of VoT in the population of car trip-makers yields the modal flows, which in turn determine the  $(\varphi, \omega)$  variables. This answers to the third RQ.

The “service facet” and the “people facet” of the LRS as a traffic system interact in two ways: from service to people (RQ1&2) and conversely from people to service (RQ2&3). The resulting equilibrium was defined mathematically as a set of balance conditions by role A, U and N, between “supplied flow” (from  $\varphi$  and  $\omega$ ) and “demanded flow”. This answers to the fourth Research Question, while raising a further question of equilibrium determination. A computational scheme was devised that locates the two essential balance conditions as a pair of so-called Upper Hand- and Lower Hand-functions in both the  $(\varphi, \omega)$  plane and the plane of role flows  $(y_A, y_U)$ .

Depending on  $(\varphi, \omega)$ , the system state lies in either one of two basic configurations of roles along the VoT axis: either  $\{U < A\}$  where less wealthy users are driven by wealthier agents, or  $\{A < U\}$  where less wealthy agents drive wealthier users. In both cases the higher range of VoT keeps to the Neutral option.

The formulation of traffic equilibrium as a fixed point problem depending on the configuration ensures the existence of an equilibrium state. The issue of uniqueness is still under study.

In its current stage, the model of LRS as a bi-sided mobility service with traffic equilibrium and configuration issues lies at the interface of economic modeling and traffic modeling. Further research may be invested along three avenues. The first avenue consists in developing the physical and economic content. On the demand side, multiple classes may be considered with different conditions regarding travel modes, for instance car dependency or transit dependency. By class, random utility functions would represent a significant advance over generalized cost functions, as explained in the Discussion. The individual frequency of the travel situation in a multi-day timeframe also deserves specific investigation, as does the consideration of individual mobility at the level of the day chain of activities and trips. On the supply side, the interaction of people through and with the service operator may be modeled in a dynamic way rather than reduced to average conditions – cf. hyperpath motivation in traffic assignment to transit networks. Furthermore, the economic conditions of the service operator deserve to be modeled, not only commercial revenues but also specific production costs and the relationship between the operator and the mobility authority – the potential rewarding of runs and rides in line with the service impacts on the society and the environment. Potential applications may be targeted to the design of price schedules.

The second avenue of research is to develop the spatial scope of the model by introducing the LRS concept into OR problems of network design and service planning. Potential applications include (i) path selection for LRS design, (ii) optimal development of LRS network.

Empirical matters constitute the third avenue of research. The model may be applied to LRS experiment so as to characterize the service conditions and assess its performances. Conversely, such application would enable for econometric estimation of the model parameters, from transaction times to VoT depending on service role and trip leg.

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## Appendix A: Traffic equilibrium

### A1/ System state and state vector

A line ride-sharing service as a system has a system state determined by the values of its endogenous variables. These include (cf. Figure 3):

- (i) people trip flows ( $y_r$ ) according to service roles  $r \in \mathfrak{R}$ ,
- (ii) service quality variables ( $\varphi, \omega, v$ ),
- (iii) the money and time items of the roles, denoted  $t_r^\ell$  and  $m_r^\ell$  by leg  $\ell \in \{R, S, T, W\}$ , depending on waiting policy  $\# \in \{AW, UW\}$ ,
- (iv) the generalized cost functions  $\beta \mapsto g_r^\#(\beta)$  according to role  $r$  and waiting policy  $\#$ ,
- (v) the efficiency domains  $E_r^\#$  and the related bounds  $\underline{\beta}_r^\# = \inf E_r^\#$  and  $\bar{\beta}_r^\# = \sup E_r^\#$ ,
- (vi) the role probabilities  $p_r^\#$  among the  $q$  individuals.

These probabilities sum up the disaggregate preferences of the individuals, by assigning each of them to the optimal role according to his or her  $\beta$  and taking into account the probability distribution of  $\beta$  among the population. Thus,  $p_r^\# = \Pr\{\beta \in E_r^\#\} = \int_0^{+\infty} 1_{\{\beta \in E_r^\#\}} dB(\beta)$ .

## A2/ Structure of influences and basic state vector

The endogenous variables are related by the following set of conditions:

a/  $y_r \geq 0$  and  $\sum_{r \in \mathfrak{R}} y_r = q$ .

b/ quality formation:  $\varphi = y_A$  and  $\omega$  such that  $y_U = \omega \cdot y_A$ , denoted as  $(\varphi, \omega) = (\hat{\varphi}, \hat{\omega})(\mathbf{y}_{\mathfrak{R}})$ .

c/ traffic law  $v = V(\frac{y_A + y_N}{H})$ .

d/ time and money item composition on the basis of  $(\varphi, \omega, v)$  and according to waiting policy.

e/ generalized cost function composition from money and time items.

f/ optimal choices: choice probabilities  $p_r^\#$  stem from generalized cost functions.

g/ flow assignment  $y_r^\# = q \cdot p_r^\#, \forall r \in \mathfrak{R}$ .

The flow vector  $[y_r: r \in \mathfrak{R}]$  can make a basic state vector from which all of the other variables are derived. Such flow vector constitutes an equilibrium state if it satisfies that:

$$\forall r \in \mathfrak{R}: y_r = y_r^\#(\mathbf{y}_{\mathfrak{R}}) \quad (\text{A.1})$$

It is a condition of quantitative balance (market clearing) between supply ( $y_r$ ) and demand ( $y_r^\#$ ).

Yet, as basic state vector it is more convenient to consider the  $(\varphi, \omega)$  pair, from which stem  $(\hat{y}_A, \hat{y}_U)$ , service flow  $\hat{y}_S = \hat{y}_A + \hat{y}_U$ , neutral flow  $\hat{y}_N = q - \hat{y}_S$ , hence  $v$  and the rest of endogenous variables. This basic state vector is feasible if  $\varphi \geq 0$ ,  $\omega \geq 0$  and  $\varphi \leq q$ .

A feasible vector  $(\varphi, \omega)$  is an equilibrium state iff

$$\forall r \in \mathfrak{R}: \hat{y}_r(\varphi, \omega) = y_r^\#(\varphi, \omega) \quad (\text{A.2})$$

## A3/ State properties and the configurations of Pareto frontiers

Any feasible state  $(\varphi, \omega)$  gives rise to roles' features in money, time and generalized cost function, hence to a competition between the roles with respect to the VoTs of the individuals. The Pareto frontier is the sequence of efficiency domains along the  $\beta$ -axis from zero to infinity.

From the affine-linear functions of generalized costs, the efficiency domains are intervals (perhaps void with  $\underline{\beta}_r^\# = +\infty$  and  $\bar{\beta}_r^\# = 0$ ) so that

$$p_r^\# = (B(\bar{\beta}_r^\#) - B(\underline{\beta}_r^\#))^+ \quad (\text{A.3})$$

Together the efficiency domains cover  $\mathbb{R}^+$ . The intersections between different domains  $E_r^\#, E_s^\#$  are point-wise or void, except for degenerate cases with  $t_r = t_s$  and  $m_r = m_s$ . At junction point between  $r$  and  $s$ , the domain belonging conditions on both sides combine into

$$\begin{aligned} m_s + t_s \beta_{rs}^\# &= m_r + t_r \beta_{rs}^\# \\ \beta_{rs}^\# &= \frac{m_s - m_r}{t_r - t_s}, \end{aligned} \quad (\text{A.4})$$

Provided that  $t_r \neq t_s$ .

Now, for positive transaction times and dwell times, then it holds that  $t_N < t_U$ , making N the quickest option hence that of best value to high VoTs. Then  $\bar{\beta}_N^\# = +\infty$  and  $B(\bar{\beta}_N^\#) = 1$ , so that

$$p_N^\# = 1 - B(\underline{\beta}_N^\#). \quad (\text{A.5})$$

The value of  $\underline{\beta}_N^\#$  depends on which role A or U is competitive with N at that point. If  $m_r \geq m_N$  then  $\beta_{rN}^\# \leq 0$  yielding  $E_r^\# = \emptyset$  hence  $\bar{\beta}_r^\# = 0$  : then role  $r$  is fully dominated by N on range  $\mathbb{R}^+$  of  $\beta$ .

Conversely, if  $m_N > m_r$  then  $\bar{\beta}_r^\# \leq \underline{\beta}_N^\#$  and it takes a non-negative, finite value.

The competition between A and U is less pre-determined. If  $r$  fully dominates  $s$  with  $t_r < t_s$  and  $m_r \leq m_s$ , hence  $\beta_{rs}^\# < 0$ , then only  $r$  is competitive with N and the Pareto frontier is  $\{r, N\}$ . But if  $\beta_{AU}^\# > 0$  then each role has a non-empty competitive range with respect to the other one, leaving the issue of which set has lower values – that of lower money cost.

If  $m_U \leq m_A$  then U's range comes prior to A's one, denoted  $\{U, A\}$ , whereas if  $m_A \leq m_U$  then A's range is lower than U's one i.e.  $\{A, U\}$ .

Joining with N, the  $\{U, A\}$  configuration leads to either:

- Type 1 outcome, N dominates over  $\mathbb{R}^+$ .
- Type 2 with  $\{U, N\}$  if  $\beta_{AU}^\# \geq \beta_{UN}^\#$ ,
- Type 3 with  $\{U, A, N\}$  if  $\beta_{AU}^\# < \beta_{UN}^\#$ ,

Whereas an  $\{A, U\}$  configuration leads to either:

- Type 1 outcome with  $\{N\}$  only as the Pareto frontier of the generalized costs,
- Type 2 with  $\{A, N\}$  if  $\beta_{AU}^\# \geq \beta_{AN}^\#$ ,
- Type 3 with  $\{A, U, N\}$  if  $\beta_{AU}^\# < \beta_{AN}^\#$ .

#### A4/ Configurational characterization of multi-sided equilibrium

The configurations of states  $(\varphi, \omega)$  combine with the demand-supply balance condition of multi-sided equilibrium to yield the following configurational properties:

Proposition: Equilibrium properties. under positive transaction and dwell times, if  $(\varphi, \omega)$  is an equilibrium state then, depending on the Pareto frontier:

- $\{N\}$  with  $m_N \leq \min\{m_U, m_A\}$ : then  $p_N^\# = 1$ ,  $p_A^\# = p_U^\# = 0$ ,  $\hat{y}_N = q$ ,  $\hat{y}_A = \hat{y}_U = 0$ ,  $\varphi = 0$ .
- $\{U, N\}$ : either  $\beta_{AU}^\# \leq 0$  with  $m_U \leq m_A$  and  $t_U \leq t_A$ , or  $\beta_{AU}^\# \geq \beta_{UN}^\#$  with  $m_U \leq m_A$  and  $t_U > t_A$ : in either case  $\underline{\beta}_U^\# = 0$ ,  $\bar{\beta}_U^\# = \beta_{UN}^\# = \underline{\beta}_N^\#$ ,  $\varphi = y_A^\# = 0$ .
- $\{A, N\}$ : either  $\beta_{AU}^\# \leq 0$  with  $m_A \leq m_U$  and  $t_A \leq t_U$ , or  $\beta_{AU}^\# \geq \beta_{AN}^\#$  with  $m_A \leq m_U$  and  $t_U < t_A$ : in either case  $\underline{\beta}_A^\# = 0$ ,  $\bar{\beta}_A^\# = \beta_{AN}^\# = \underline{\beta}_N^\#$ ,  $y_U^\# = 0$ .
- $\{A, U, N\}$ : with  $m_A \leq m_U \leq m_N$  and  $t_N \leq t_A \leq t_U$ ,  $\beta_{AU}^\# < \beta_{AN}^\#$ ,  $\underline{\beta}_A^\# = 0$ ,  $\bar{\beta}_A^\# = \underline{\beta}_U^\# = \beta_{UA}^\# < \beta_{UN}^\# = \bar{\beta}_U^\# = \underline{\beta}_N^\#$ : thus  $\varphi = qB(\beta_{UA}^\#)$  and  $\omega\varphi = q(B(\underline{\beta}_N^\#) - B(\beta_{UA}^\#))$ .
- $\{U, A, N\}$ : with  $m_U \leq m_A \leq m_N$  and  $t_N \leq t_U \leq t_A$ ,  $\beta_{AU}^\# < \beta_{UN}^\#$ ,  $\underline{\beta}_U^\# = 0$ ,  $\bar{\beta}_U^\# = \underline{\beta}_A^\# = \beta_{UA}^\# < \beta_{AN}^\# = \bar{\beta}_A^\# = \underline{\beta}_N^\#$ : thus  $\omega \cdot \varphi = qB(\beta_{UA}^\#)$  and  $\varphi = q(B(\underline{\beta}_N^\#) - B(\beta_{UA}^\#))$ .

Proposition: Equilibrium characterization. If a feasible state  $(\varphi, \omega)$  satisfies the characteristic conditions of its configuration, then it is a state of equilibrium.

This is because the configuration-specific condition determines the efficiency domains and yields the associated balance conditions. Then, the ‘‘supplied’’ flows  $\hat{y}_r(\varphi, \omega)$  and their ‘‘demanded’’ counterparts  $y_r^\#(\varphi, \omega)$  are balanced.

It is thus easy to test whether a state is in equilibrium or not, according to the configuration of its Pareto frontier.

## Appendix B: Equilibrium computation

### B1/ Fixed-point problem and iterative solution scheme

FPP in  $(\varphi, \omega)$ . A basic state vector  $(\varphi, \omega)$  is in equilibrium if it is a fixed point for the mapping  $(\varphi, \omega) \mapsto (\varphi', \omega')$  such that  $\varphi' = y_A^\#(\varphi, \omega)$  and  $\omega'$  satisfies that  $y_U^\#(\varphi, \omega) = \omega' y_A^\#(\varphi, \omega)$ .

FPP in  $(y_A, y_U)$ . A basic state vector  $(y_A, y_U)$  is in equilibrium if it is a fixed point for the chained mapping  $(y_A, y_U) \mapsto (\varphi, \omega)$  such that  $\varphi = y_A$  and  $\omega$  satisfies that  $y_U = \omega y_A$ , followed by  $(\varphi, \omega) \mapsto (y_A^\#, y_U^\#)$ .

Iterative solution scheme. The basic strategy to solve an FPP is to progressively adapt a current state vector by combining it with its image through the mapping so as to obtain the next value. At step  $k$ , current state  $(\varphi_k, \omega_k)$  induces image  $(\hat{\varphi}_k, \hat{\omega}_k)$ : then the next step can be obtained as

$$\begin{aligned}\varphi_{k+1} &\leftarrow \text{Fn}[\varphi_k, \hat{\varphi}_k], \\ \omega_{k+1} &\leftarrow \text{Fn}[\omega_k, \hat{\omega}_k].\end{aligned}$$

For instance, a convex combination scheme with step sizes  $\zeta_k$  that decrease to zero:

$$\begin{aligned}\varphi_{k+1} &\leftarrow \varphi_k + \zeta_k(\hat{\varphi}_k - \varphi_k) = (1 - \zeta_k)\varphi_k + \zeta_k\hat{\varphi}_k \\ \omega_{k+1} &\leftarrow \omega_k + \zeta_k(\hat{\omega}_k - \omega_k) = (1 - \zeta_k)\omega_k + \zeta_k\hat{\omega}_k.\end{aligned}$$

A similar strategy on the  $(y_A, y_U)$  flow vector constitutes an equilibration algorithm well-known in network traffic assignment. The convex combination scheme is an instance of the ‘‘Method of Successive Averages’’.

Yet a more straightforward strategy is introduced hereafter.

### B2/ Straightforward strategy

Under positive times of transaction and dwelling, the lower bound  $\underline{\beta}_N^\#$  of role N must satisfy that

$$\varphi(1 + \omega) = q \cdot \text{B}(\underline{\beta}_N^\#) \quad (\text{B.1})$$

When  $\omega = 0$ , then role U is not competitive so that either N dominates A yielding  $\underline{\beta}_N^\# = 0$  and  $\varphi = 0$ , or  $\underline{\beta}_N^\# = \beta_{AN}^\# \geq 0$ , with  $\beta_{AN}^\#$  a function of  $\varphi$  only. Then eqn. (B.1) constitutes an equation in the unknown  $\varphi$  only,

$$\varphi = q \cdot \text{B}(\underline{\beta}_N^\#) = q \cdot \text{B}\left(\frac{\mu_{N-A} + L\hat{t}_A\omega}{\omega\hat{t}_A^S + \hat{t}_A^T + \delta_A \frac{H}{\varphi}}\right).$$

Wherein  $\delta_A = 1$  under AW or 0 under UW. If  $\delta_A = 0$  then  $\varphi$  comes out directly from  $\omega = 0$ . If  $\delta_A = 1$ , the equation is a simple fixed point problem with continuous LHS function that increases from 0 to  $q$  and continuous RHS function that keeps its value in  $[0, q]$ . By the intermediary value theorem, there exists at least one solution in  $[0, q]$ , which is an equilibrium state if the resulting  $t_A$  satisfies that  $t_A - t_U \leq 0$ .

More generally, given  $\omega$  we consider the lowest solution of eqn. (B.5) as a function  $\bar{\varphi}(\omega)$ : it satisfies

$$\bar{\varphi} = \frac{q}{1+\omega} \text{B}(\underline{\beta}_N^\#) \quad (\text{B.2})$$

Function  $\bar{\varphi}$  is conditional on waiting policy # and on current configuration {A,U} or {U,A} that determines whether  $\underline{\beta}_N^\# = \beta_{UN}^\#$  or  $\underline{\beta}_N^\# = \beta_{AN}^\#$ . The configuration also implies a second condition linking  $\varphi$  and  $\omega$ , either

$$\underline{\varphi} = q \cdot \text{B}(\beta_{AU}^\#) \quad (\text{B.3}_{AU})$$

$$\underline{\varphi} = \frac{q}{\omega} \cdot B(\beta_{AU}^\#) \quad (\text{B.3\_UA})$$

Then, the straightforward strategy is to look for an intersection point of the two functions  $\overline{\varphi}$  and  $\underline{\varphi}$  of  $\omega$ . It is further specified in Appendix C in the case of uniformly distributed VoTs.

The discrepancy between B3\_AU and B3\_UA highlights the key influence of configuration.

### B3/ On configurations in the feasible domain

Between roles U and A, what is the range of efficiency of each role with respect to the other one? Does one role  $r$  fully dominates the other role  $s$ , with  $m_r \leq m_s$  and  $t_r \leq t_s$ ? Or is there a true sharing of  $\mathbb{R}^+$  between the respective efficiency ranges, when  $\beta_{AU}^\# > 0$ ?

Let us consider in turn each of the two conditions with respect to  $(\varphi, \omega)$ :  $m_A \leq m_U$  iff

$$\gamma_A + \underline{m} + L\dot{m} - \tau_A - L\dot{t}_A\omega \leq \gamma_U + \tau_U + L\dot{t}_U$$

$$L\dot{t}_A\omega \geq \mu_{A-U} \equiv \gamma_A + \underline{m} + L\dot{m} - \tau_A - \gamma_U - \tau_U - L\dot{t}_U$$

Assuming  $\dot{t}_A > 0$ , it amounts to

$$\omega \geq \omega_M \equiv \mu_{A-U} / (L\dot{t}_A). \quad (\text{B.4})$$

Thus,  $m_U - m_A = L\dot{t}_A(\omega - \omega_M)$  and condition  $\{m_A \leq m_U\}$  is equivalent to  $\{\omega \geq \omega_M\}$ , a constant.

Turning to the respective times,

$$\begin{aligned} t_A - t_U &= \omega \underline{t}_A^S + t_A^T + \delta_A \frac{H}{\varphi} - t_U^S - t_U^T - \delta_U \frac{H}{\varphi} \\ &= \omega \underline{t}_A^S - \tilde{t} + \varepsilon \frac{H}{\varphi} \end{aligned}$$

Wherein  $\tilde{t} \equiv t_U^S + t_U^T - t_A^T$ ,  $\delta_A$  is the binary indicator of AW and  $\delta_U$  that of UW, so that  $\delta_A + \delta_U = 1$  and  $\delta_A \cdot \delta_U = 0$ , and  $\varepsilon \equiv \delta_A - \delta_U \in \{-1, +1\}$  takes value +1 under AW and -1 under UW. Now,

$t_A - t_U \geq 0$  iff

$$\begin{aligned} \omega \underline{t}_A^S &\geq \tilde{t} - \varepsilon \frac{H}{\varphi} \\ \omega &\geq \omega_\varepsilon(\varphi) \equiv \frac{1}{\underline{t}_A^S} \left( \tilde{t} - \varepsilon \frac{H}{\varphi} \right) \end{aligned}$$

Thus  $t_A - t_U = \underline{t}_A^S(\omega - \omega_\varepsilon(\varphi))$ .

Function  $\omega_\varepsilon(\varphi)$  increases with  $\varphi$  if  $\varepsilon = +1$  i.e. under AW but it decreases with  $\varphi$  if  $\varepsilon = -1$  i.e. under UW. In both cases,  $\omega_\varepsilon(\varphi) \rightarrow \omega_\infty \equiv \tilde{t} / \underline{t}_A^S$  when  $\varphi \rightarrow +\infty$ .

Functions  $\omega_\varepsilon(\varphi)$  involve only dwelling times and transaction times. We expect it to be positive in practical instances, with  $t_U^T \approx t_A^T$  of the same magnitude as  $t_U^S$ . We may also expect  $\underline{t}_A^S \approx 1 - 2$  times  $t_U^S$ .

Given  $\varepsilon$  hence  $\# \in \{UW, AW\}$ , the respective positions of  $\omega_M$ ,  $\omega_\infty$  and 0 along the  $\omega$ -axis determine the respective locations of the different configurations in the  $(\omega, \varphi)$  domain.

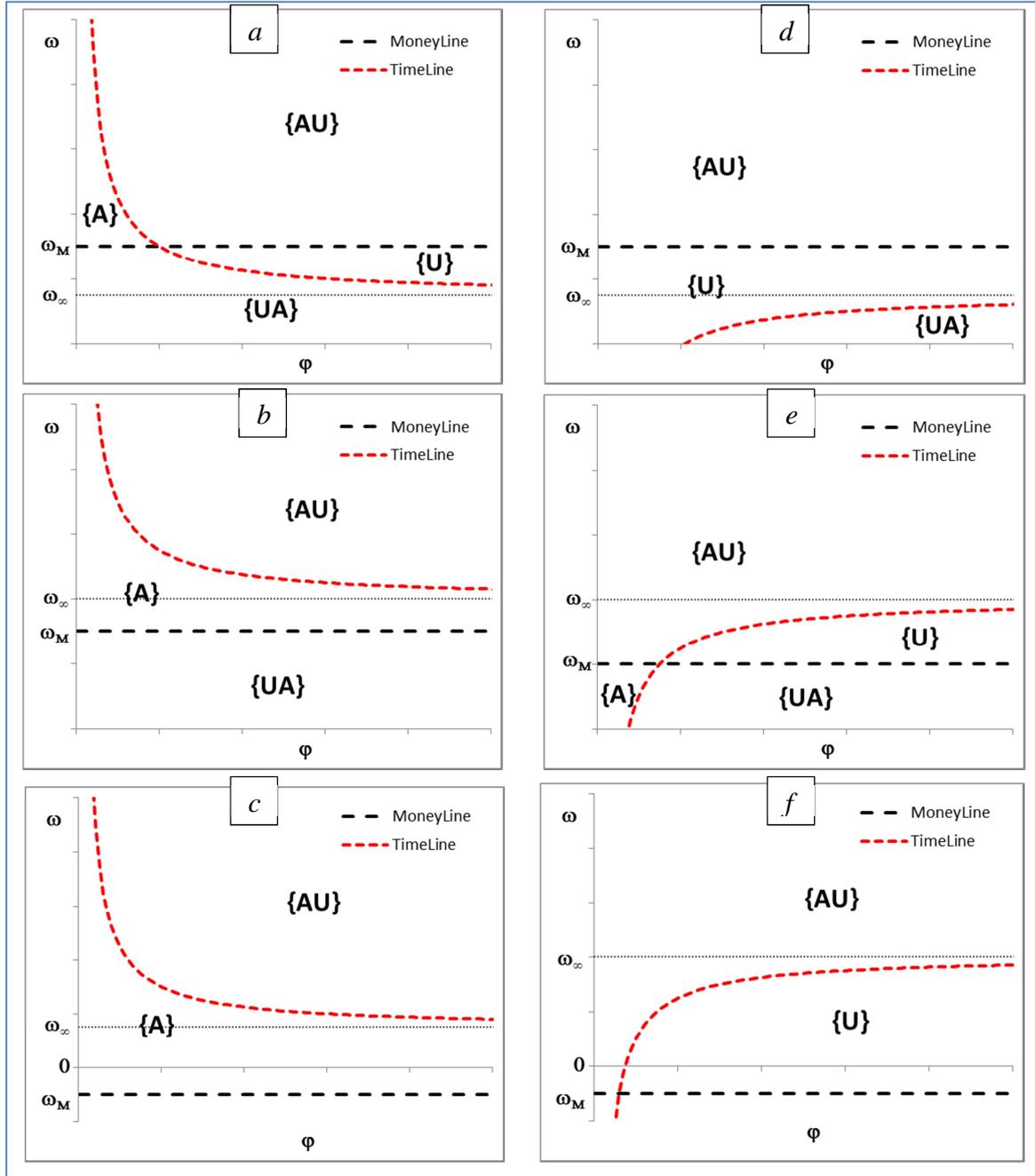


Fig.10: Configurations in the feasible domain. (a-c) under UW, (d-e) under AW.

Under UW i.e.  $\varepsilon = +1$ , three cases are relevant:

- $0 \leq \omega_\infty \leq \omega_M$  : in decreasing order along the  $\omega$ -axis, there are 4 regions (i) {A,U} where  $\omega \geq \omega_M$  and  $\omega \geq \omega_\varepsilon(\varphi)$  so that  $m_A \leq m_U$  and  $t_A \geq t_U$ , (ii) {A} where  $\omega_M \leq \omega \leq \omega_\varepsilon(\varphi)$  so that  $m_A \leq m_U$  and  $t_A \leq t_U$ , (iii) {U} where  $\omega_\varepsilon(\varphi) \leq \omega \leq \omega_M$  i.e.  $m_A \geq m_U$  and  $t_A \geq t_U$ , (iv) {U,A} where  $\omega \leq \omega_M$  and  $\omega \leq \omega_\varepsilon(\varphi)$  (figure 10a).
- $0 \leq \omega_M \leq \omega_\infty$  : with 3 regions (i) {A,U} where  $\omega \geq \omega_\varepsilon(\varphi)$  hence  $\omega \geq \omega_\infty \geq \omega_M$ , hence  $m_A \leq m_U$  along with  $t_A \geq t_U$ , (ii) {A} where  $\omega_M \leq \omega \leq \omega_\varepsilon(\varphi)$  hence  $m_A \leq m_U$  and  $t_A \leq t_U$ , (iii) {U,A} where  $\omega \leq \omega_M$  hence  $\omega \leq \omega_\varepsilon(\varphi)$ , hence  $m_A \geq m_U$  and  $t_A \leq t_U$  (figure 10b).
- $\omega_M \leq 0 \leq \omega_\infty$  is a degenerate version of the former one: only sub-cases (i) and (ii) can exist in the  $\{\omega \geq 0, \varphi \geq 0\}$  domain (figure 10c).

Under AW i.e.  $\varepsilon = -1$ , the same three cases are relevant but with specific outcomes:

- $0 \leq \omega_\infty \leq \omega_M$ : in decreasing order along the  $\omega$ -axis, there are 3 regions (figure 10d):  
 (i) Region {A,U} where  $\omega \geq \omega_M$  hence  $\omega \geq \omega_\varepsilon(\varphi) \forall \varphi \geq 0$ , so that  $m_A \leq m_U$  and  $t_A \geq t_U$ ,  
 (ii) Region {U} where  $\omega < \omega_M$  and  $\omega \geq \omega_\varepsilon(\varphi)$ , hence  $m_A > m_U$  and  $t_A \geq t_U$ ,  
 (iii) Region {U,A} associated to  $\omega < \omega_\varepsilon(\varphi)$ , so that  $m_A > m_U$  and  $t_A < t_U$ .
- $0 \leq \omega_M \leq \omega_\infty$  (figure 10e): there are 4 regions in decreasing order of  $\omega$ , namely (i) {A,U} where ( $\omega \geq \omega_M$ ) and ( $\omega \geq \omega_\varepsilon(\varphi)$ ) i.e.  $m_A \leq m_U$  and  $t_A \geq t_U$ , (ii) {A} where  $\omega_M \leq \omega \leq \omega_\varepsilon(\varphi)$  hence  $m_A \leq m_U$  and  $t_A \leq t_U$ , (iii) {U} where ( $\omega \leq \omega_\varepsilon(\varphi)$ ) hence ( $\omega \leq \omega_M$ ), i.e.  $m_A \geq m_U$  and  $t_A \leq t_U$ .
- $\omega_M \leq 0 \leq \omega_\infty$  is a degenerate version of the former one: only sub-cases (i) and (ii) can exist in the  $\{\omega \geq 0, \varphi \geq 0\}$  domain (figure 10f).

In each case the feasible domain is split between 2 to 4 regions. Condition  $\omega_M \leq 0$  is more comfortable since it makes A competitive everywhere.

But under  $\omega_M > 0$  then both configurations {A,U} and {U,A} take place in the feasible domain, meaning different pairs of upper- and lower- hand functions to stand for the characteristic conditions of multi-sided equilibrium.

## B4/ Specific properties of upper hand functions

### B.4.1 UHF under {A,U}

Under configuration {U} or {A,U}, then

$$\underline{\beta}_N^\# = \beta_{UN}^\# = \frac{m_N - m_U}{t_U - t_N} = \frac{\mu_{N-U}}{t_U^{T+S} + \delta_U \frac{H}{\varphi}}$$

Thus the UHF  $\bar{\omega}$  is stated as

$$\bar{\omega} = \frac{q}{\varphi} B \left( \frac{\mu_{N-U}}{t_U^{T+S} + \delta_U \frac{H}{\varphi}} \right) - 1.$$

Under AW hence  $\delta_U = 0$ , then  $\beta_{UN}^\#$  is constant, yielding constant  $B_{UN}^\# \equiv B(\beta_{UN}^\#)$  and simple UHF

$$\bar{\omega} = \frac{q}{\varphi} B_{UN}^\# - 1$$

It is obviously a decreasing function of  $\varphi > 0$ .

### B.4.2 UHF under {U,A}

Under configuration {A} or {U,A} then

$$\underline{\beta}_N^\# = \beta_{AN}^\# = \frac{m_N - m_A}{t_A - t_N} = \frac{\mu_{N-A} + L\dot{t}_A\omega}{\omega t_A^S + t_A^T + \delta_A \frac{H}{\varphi}}$$

Under UW hence  $\delta_A = 0$ , the UHF  $\bar{\varphi}(\omega)$  is straightforward:

$$\bar{\varphi} = \frac{q}{1 + \omega} B \left( \frac{\mu_{N-A} + L\dot{t}_A\omega}{\omega t_A^S + t_A^T} \right)$$

Under AW the UHF is much more involved.

## B5/ Specific properties of lower hand functions

Lower hand functions involve the frontier VoT  $\beta_{AU}^\#$  that depends on  $\omega$  and  $\varphi$  as follows:

$$\beta_{AU}^{\#} = \frac{m_U - m_A}{t_A - t_U} = \frac{L\dot{t}_A(\omega - \omega_M)}{\underline{t}_A^S(\omega - \omega_\varepsilon(\varphi))}$$

Wherein  $\omega_M$  is a constant and function  $\omega_\varepsilon(\varphi) = \frac{1}{\underline{t}_A^S}(\tilde{t} - \varepsilon \frac{H}{\varphi})$ .

### B.5.1 LHF under {A,U}

Under configuration {A,U}, then  $m_A \leq m_U$  and  $t_A \geq t_U$ , implying that  $\omega \leq \omega_M$  and  $\omega \leq \omega_\varepsilon(\varphi)$ . Furthermore,  $y_A$  is stated as a simple function of  $\beta_{AU}^{\#}$ :  $y_A = q \cdot B(\beta_{AU}^{\#})$ .

If  $0 < \varphi < q$ , we can state the LHF function  $\underline{\omega}$  with respect to  $\varphi$  in a simple way: as  $\varphi = y_A$ ,

$$\begin{aligned} B^{(-1)}\left(\frac{\varphi}{q}\right) &= \beta_{AU}^{\#} \\ L\dot{t}_A(\omega - \omega_M) &= \underline{t}_A^S(\omega - \omega_\varepsilon(\varphi)) \cdot B^{(-1)}\left(\frac{\varphi}{q}\right) \\ \underline{\omega} &= \frac{L\dot{t}_A\omega_M - \underline{t}_A^S\omega_\varepsilon(\varphi)B^{(-1)}\left(\frac{\varphi}{q}\right)}{L\dot{t}_A - \underline{t}_A^S B^{(-1)}\left(\frac{\varphi}{q}\right)}. \end{aligned}$$

### B.5.2 LHF under {U,A}

Under configuration {U,A} then  $m_A \geq m_U$  and  $t_A \leq t_U$ , implying that  $\omega \geq \omega_M$  and  $\omega \geq \omega_\varepsilon(\varphi)$ . Furthermore, the LHF is more involved with more complex influence of  $\omega$ :

$$\underline{\varphi}\omega = q \cdot B(\beta_{AU}^{\#})$$

## Appendix C: Uniformly distributed VoT

Here a uniform distribution of VoT is considered, over range  $[0, b]$ . (A positive lower bound  $a > 0$  may be considered by transferring  $a \cdot t$  from time items to money items.)

The CDF function satisfies:

$$B(\beta) = (\min\left\{1, \frac{\beta}{b}\right\})^+ \quad (C.1)$$

The inverse function is

$$B^{(-1)}(\alpha) = b \cdot (\min\{1, \alpha\})^+ \quad (C.2)$$

### C.1/ Degenerate configuration {A,N}

A degenerate configuration {A,N} with no service users has  $\omega = 0$  and  $\underline{\beta}_N^{\#} = \beta_{AN}^{\#}$ . The upper hand function is

$$\varphi = q \cdot B(\beta_{AN}^{\#}) = \frac{q}{b} (\min\left\{b, \frac{\mu_{NA}}{t_A^T + \delta_A H / \varphi}\right\})^+ \quad (C.3)$$

Wherein:  $\mu_{NA} \equiv \gamma_N - \gamma_A + \tau_A$ .

The solution satisfies that

$$\varphi t_A^T + \delta_A H = \frac{q}{b} \mu_{NA}$$

Letting  $\eta \equiv Hb/q$ , then

$$\varphi = \frac{q}{b} \frac{\mu_{NA} - \delta_A \eta}{t_A^T} \quad (C.4)$$

It is feasible only if it belongs to  $[0, q]$ , i.e. iff  $0 \leq \mu_{NA} - \delta_A \eta \leq bt_A^T$ .

If  $\mu_{NA} > \delta_A \eta + bt_A^T$  then  $\varphi = q$  and the A option prevails upon the Neutral alternative over the VoT range with positive probability density.

Furthermore, for the configuration to hold it requires U to be a dominated option. It occurs if

$$\begin{cases} m_A \leq m_U \\ t_A \leq t_U \end{cases} \text{ i.e. if } \begin{cases} \gamma_A + \underline{m} + L\dot{m} - \tau_A \leq m_U \gamma_U + \tau_U + L\dot{t}_U \\ t_A^T + \delta_A H / \varphi \leq t_U^T + t_U^S + \delta_U H / \varphi \end{cases} \Leftrightarrow \begin{cases} \mu_{A-U} \leq 0 \\ 0 \leq \tilde{t} - \varepsilon H / \varphi \end{cases}$$

(recalling  $\varepsilon \equiv \delta_A - \delta_U$ ), or if

$$\begin{cases} m_A \leq m_U \\ t_A > t_U \\ \beta_{AU}^\# \geq \min\{b, \beta_{AN}^\#\} \end{cases} \text{ i.e. if } \begin{cases} \gamma_A + \underline{m} + L\dot{m} - \tau_A \leq m_U \gamma_U + \tau_U + L\dot{t}_U \\ t_A^T \leq t_U^T + t_U^S \\ \beta_{AU}^\# \geq \min\{b, \beta_{AN}^\#\} \end{cases} \Leftrightarrow \begin{cases} \mu_{A-U} \leq 0 \\ \tilde{t} - \frac{\varepsilon H}{\varphi} \leq 0 \\ \beta_{AU}^\# \geq \min\{b, \beta_{AN}^\#\} \end{cases}$$

## C.2/ Configuration {A,U,N}

### C.2.1 UHF under A<U

The basic condition on  $y_S$  yields that

$$y_A + y_U = qB(\beta_{UN}^\#) = qB\left(\frac{\mu_{N-U}}{t_U^{T+S} + \delta_U H / \gamma_A}\right), \text{ hence that}$$

$$\bar{y}_U = qB\left(\frac{\mu_{N-U}}{t_U^{T+S} + \delta_U H / \gamma_A}\right) - y_A.$$

If  $t_U^{T+S} + \frac{\delta_U H}{\gamma_A} > \mu_{N-U} b$ , then

$$\bar{y}_U = \frac{q}{b} \frac{\mu_{N-U}}{t_U^{T+S} + \delta_U H / \gamma_A} - y_A.$$

Dividing by  $y_A$  and replacing  $y_U / y_A$  by  $\omega$  yields the upper hand function in the  $(\varphi, \omega)$  domain:

$$\bar{\omega} = \frac{q}{b} \beta_{UN}^\# = \frac{q}{b} \frac{\mu_{N-U}}{t_U^{T+S} \varphi + \delta_U H} - 1$$

In the  $\{m_A < m_U\}$  region, it is a decreasing function of  $\varphi$ .

### C.2.2 LHF under A<U

Under AUN the LHF condition is

$$y_A = qB(\beta_{AU}^\#)$$

If  $0 < \beta_{AU}^\# < b$  then  $\beta_{AU}^\# = B^{(-1)}\left(\frac{y_A}{q}\right) = B^{(-1)}\left(\frac{\varphi}{q}\right)$ , yielding that

$$\frac{m_U - m_A}{t_A - t_U} = \frac{L\dot{t}_A \omega - \mu_{A-U}}{\omega \underline{t}_A^S - \tilde{t} + \varepsilon \frac{H}{\varphi}} = \frac{b}{q} \varphi$$

$$q(L\dot{t}_A \omega - \mu_{A-U}) = b\varepsilon H + b(\underline{t}_A^S \omega \varphi - \tilde{t} \varphi)$$

$$\underline{\omega} = \frac{q\mu_{A-U} + b(\varepsilon H - \tilde{t} \varphi)}{qL\dot{t}_A - b\underline{t}_A^S \varphi}$$

Multiplying by  $y_A$  gives that

$$\underline{y}_U = y_A \frac{q\mu_{A-U} + b(\varepsilon H - \tilde{t} y_A)}{qL\dot{t}_A - b\underline{t}_A^S y_A}$$

### C.2.3 Equilibrium characterization

An equilibrium state is characterized by feasible points  $(\varphi, \omega)$  such that  $\omega = \bar{\omega}(\varphi) = \underline{\omega}(\varphi)$

And about equivalently by  $(y_A, y_U)$  such that  $y_U = \bar{y}_U(y_A) = \underline{y}_U(y_A)$  and  $y_A > 0$ .

If  $\beta_{AU}^\# < b$  then the equilibrium condition is

$$\frac{q}{b} \frac{\mu_{N-U}}{t_U^{T+S} + \delta_U H / y_A} - y_A = y_A \frac{q\mu_{A-U} + b(\varepsilon H - \tilde{t}y_A)}{qL\dot{t}_A - bt_A^S y_A}$$

It is a 2<sup>nd</sup> degree equation in  $y_A$ .

With respect to  $(\varphi, \omega)$ , the conjunction  $\bar{\omega} = \underline{\omega}$  amounts to the following equation in  $\omega$ :

$$\frac{q}{b} \frac{\mu_{N-U}}{t_U^{T+S} \varphi + \delta_U H} - 1 = \frac{q\mu_{A-U} + b(\varepsilon H - \tilde{t}\varphi)}{qL\dot{t}_A - bt_A^S \varphi}$$

It is a 2<sup>nd</sup> degree equation in  $\varphi$ .

## C.3/ Configuration {U,A,N}

### C.3.1 UHF under U<A

Under configuration UAN, if  $\beta_{AN}^\# < b$  then the upper hand function satisfies that

$$\bar{\varphi}(1 + \omega) = \frac{q}{b} \beta_{AN}^\# = \frac{q}{b} \frac{\mu_{N-A} + L\dot{t}_A \omega}{t_A^T + \omega t_A^S + \delta_A \frac{H}{\varphi}}$$

Hence

$$\bar{\varphi}(t_A^T + \omega t_A^S) + \delta_A H = \frac{q}{b} \frac{\mu_{N-A} + L\dot{t}_A \omega}{1 + \omega}$$

Yielding that

$$\bar{\varphi} = \frac{q}{b} \frac{\mu_{N-A} - \delta_A \eta + (L\dot{t}_A - \delta_A \eta) \omega}{(1 + \omega)(t_A^T + \omega t_A^S)} \quad (C.5)$$

### C.3.2 LHF under U<A

Under U<A the lower hand function satisfies that

$$\underline{\varphi} \omega = \frac{q}{b} \beta_{AU}^\# = \frac{q}{b} \frac{L\dot{t}_A \omega - \mu_{A-U}}{\omega t_A^S - \tilde{t} + \varepsilon \frac{H}{\varphi}}$$

So that

$$\underline{\varphi}(\omega t_A^S - \tilde{t}) + \varepsilon H = \frac{q}{b} \frac{L\dot{t}_A \omega - \mu_{A-U}}{\omega}$$

Yielding that

$$\underline{\varphi} = \frac{q}{b} \frac{(L\dot{t}_A - \varepsilon \eta) \omega - \mu_{A-U}}{\omega(\omega t_A^S - \tilde{t})} \quad (C.6)$$

### C.3.3 Equilibrium characterization

The conjunction  $\bar{\varphi} = \underline{\varphi}$  amounts to the following equation in  $\omega$ : denoting  $\tilde{\gamma} \equiv \mu_{N-A} - \delta_A \eta$ ,

$$\omega(\omega t_A^S - \tilde{t})(\tilde{\gamma} + (L\dot{t}_A - \delta_A \eta) \omega) = (1 + \omega)(t_A^T + \omega t_A^S)[(L\dot{t}_A - \varepsilon \eta) \omega - \mu_{A-U}]$$

It is a 3<sup>rd</sup> degree equation in  $\omega$ . Under ‘‘Agent Waits’’ i.e.  $\delta_A = 1$  and  $\varepsilon = 1$ , then the 3<sup>rd</sup> degree term vanishes and the equation boils down to a 2<sup>nd</sup> degree equation in  $\omega$ .