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Quantum roots for Kac-Moody root systems and finiteness properties of the Kac-Moody affine Bruhat order

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Abstract

Let G be a split Kac-Moody group over a local field. In their study of the Iwahori-Hecke algebra of G , A.Braverman, D. Kazhdan and M. Patnaik defined a partial order - called the affine Bruhat order - on the extended affine Weyl semi-group W^+ of G . In this paper, we study finiteness questions for covers and co-covers of W^+ , generalizing results of A. Welch. In particular we prove that the intervals for this order are finite. Our results rely on the finiteness of the set of quantum roots of arbitrary Kac-Moody root systems, which we prove. We also obtain a classification of quantum roots.

1 Introduction

Affine Weyl semi group W^+ Let \mathbb{G} be a split reductive group scheme with the data of a Borel subgroup \mathbb{B} containing a maximal torus \mathbb{T} . Let $W^v = N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$ be its vectorial Weyl group and Y be its coweight lattice: $Y = \text{Hom}(\mathbb{G}_m, \mathbb{T})$. The action of W^v on \mathbb{T} induces an action of W^v on Y and allows to form the semidirect product $W^a = Y \rtimes W^v$. This group, called the extended affine Weyl group of (\mathbb{G}, \mathbb{T}) , naturally appears in the geometry and the representation theory of \mathbb{G} over discretely valued fields, for example when one studies the Iwahori-Hecke of $\mathbb{G}(\mathcal{K})$, where \mathcal{K} is a non-Archimedean local field; it is a finite extension of an affine Coxeter group and it is naturally equipped with a Bruhat order.

Kac-Moody groups are natural infinite dimensional generalizations of reductive groups, and we are interested by the structure of these groups over discretely valued fields. Assume now that (\mathbb{G}, \mathbb{T}) is a split Kac-Moody group (à la Tits, as defined in [Tit87]). Braverman,

Kazhdan and Patnaik and Bardy-Panse, Gaussent and Rousseau associated to $\mathbb{G}(\mathcal{K})$ an Iwahori-Hecke algebra \mathcal{H} (see [BKP16] and [BPGR16]). The analog of W^a in this context is then $W^+ = W^v \rtimes Y^+$, where Y^+ is the integral Tits cone. Unless \mathbb{G} is reductive, W^v is infinite and Y^+ is a proper sub-semi group of Y . Therefore W^+ is only a semi-subgroup of $W^v \rtimes Y$ in general.

Affine Bruhat order on W^+ In [BKP16, Appendix B2], A. Braverman, D. Kazhdan and M. Patnaik propose the definition of a preorder on W^+ which would replace the Bruhat order of W^a and they conjecture that it is a partial order. In [Mut18], D. Muthiah extends the definition of this preorder to any Kac-Moody group G and proves that it is actually an order. Muthiah and D. Orr have proved that W^+ admits a length $\ell^a : W^+ \rightarrow \mathbb{Z}$ strictly compatible with the affine Bruhat order ([MO19]).

Covers for the affine Bruhat order If \mathbf{x}, \mathbf{y} lie in W^+ , we say that \mathbf{y} covers \mathbf{x} (and that \mathbf{x} co-covers \mathbf{y}) if $\mathbf{y} > \mathbf{x}$ and $\{\mathbf{z} \in W^+ \mid \mathbf{x} < \mathbf{z} < \mathbf{y}\} = \emptyset$. The affine Bruhat length gives a characterization of covers: Muthiah and Orr (in the affine ADE case) and the second named author (in the general case) have proved that \mathbf{y} covers \mathbf{x} if and only if $\mathbf{y} > \mathbf{x}$ and $\ell^a(\mathbf{y}) = \ell^a(\mathbf{x}) + 1$ ([MO19], [Phi23]).

Quantum roots Let Φ denote the root system of (\mathbb{G}, \mathbb{T}) . The notion of quantum root comes up in our work through the study of the affine Bruhat order. Let $\mathbf{x}, \mathbf{y} \in W^+$ be such that \mathbf{y} covers \mathbf{x} . Then we can write $\mathbf{y} = \mathbf{s}\mathbf{x}$, where \mathbf{s} is a reflection of $W^v \rtimes Y$. The linear part of \mathbf{s} is the reflection s_β associated to a root β of Φ . Suppose that $\mathbf{y} \notin W^v \mathbf{x} W^v$ and, to simplify, assume that \mathbf{x} has dominant coweight. Then β has to satisfy certain conditions (see Lemma 3.1), a root satisfying these conditions is called a **quantum root** (see Definition 2.4). We prove that the set of quantum roots of a Kac-Moody root system is finite (see Theorem 2.27). We also obtain a complete classification of quantum roots through Dynkin diagrams (see Theorem 2.36), which may be insightful for a future construction of quantum Bruhat graphs associated to infinite root systems.

In the reductive setting, quantum roots first came up in the construction of the quantum Bruhat graph by F. Brenti, S. Fomin and A. Postnikov in 1999 ([BFP99]). This graph is used to study the quantum cohomology of flag varieties, which justifies the name. The specific terminology of quantum root seems to come from the more recent studies of the quantum Bruhat graph by F. Schremmer (see [Sch24]). Note that in his work, Schremmer already relates quantum roots to covers for the affine Bruhat order, but in the reductive setting.

Finiteness results for covers and co-covers. Using our study of quantum roots, we obtain finiteness results for the affine Bruhat order. More precisely, denote by $\varpi^\lambda w \in W^+$ the element of W^+ corresponding to (λ, w) for $\lambda \in Y^+$ and $w \in W^v$.

We say that $\varpi^\lambda w \in W^+$ is spherical if λ has finite stabilizer in W^v . Our main results on the affine Bruhat order are the following:

Theorem 1.1. (see Theorem 3.2 and Theorem 3.6) *Let $\mathbf{x} \in W^+$. Then \mathbf{x} admits finitely many covers. If furthermore \mathbf{x} is spherical, then \mathbf{x} also admits finitely many co-covers.*

Corollary 1.2. *Let $\mathbf{x}, \mathbf{y} \in W^+$. Then the interval $[\mathbf{x}, \mathbf{y}]$ is finite.*

In the reductive case, these theorems are easily obtained by considering reduced decompositions. In the Kac-Moody frameworks these theorems were first proved by A. Welch in the affine ADE case using different methods in [Wel22].

When W^v admits an infinite proper parabolic subgroup (this implies that \mathbb{G} is not affine), we also prove the existence of elements of W^+ admitting infinitely many co-covers, see Lemma 3.8.

Our main motivation for the study of the combinatorics of W^+ is the construction of an affine Kazhdan-Lusztig theory for Kac-Moody groups. This theory was initiated by Muthiah in [Mut19], where he uses Corollary 1.2 to give a conjectural definition of spherical R -polynomials in the affine ADE case. We plan to use our results to define an Iwahori version of Muthiah’s polynomials in the general Kac-Moody frameworks.

Organization of the paper In section 2, we define the quantum roots and we prove that the set of quantum roots of a Kac-Moody datum is finite (see Theorem 2.27). We also obtain a general classification of quantum roots (see Theorem 2.36). We then describe the set of quantum roots in particular cases.

In section 3, we explain the link between covers for the affine Bruhat order and quantum roots. We then prove Theorem 1.1. We also show that the spherical condition in Theorem 1.1 is necessary by constructing non-spherical elements with infinitely many co-covers (see Proposition 3.10). We end the section with an explicit construction of covers associated to any given quantum root, thus showing that the notion of quantum root is well-suited.

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2 Quantum roots

2.1 General definitions and notations

Generalized Cartan matrices and Dynkin Diagrams

Definition 2.1. Let I be a finite set. A **generalized Cartan matrix** over I is a matrix $A = (a_{i,j})_{(i,j) \in I \times I} \in M_I(\mathbb{Z})$ with integral coefficients satisfying the following properties:

1. $\forall i \in I \ a_{i,i} = 2$
2. $\forall (i,j) \in I^2, \ i \neq j \implies a_{i,j} \leq 0$
3. $\forall (i,j) \in I^2, \ a_{i,j} = 0 \implies a_{j,i} = 0$.

Definition 2.2. A **Dynkin diagram** is a tuple (I, E, w) where I is a finite set (its vertices), E is a symmetric subset of $I \times I \setminus \{(i,i) \mid i \in I\}$ (its edges) and w is an application $E \rightarrow \mathbb{Z}_{>0}$ (its weight function). In other words, it is a positively weighted oriented graph such that $(i,j) \in E \iff (j,i) \in E$.

Let $\Gamma = (I, E, w)$ be a Dynkin diagram. Then any subset $J \subset I$ admits an induced structure of Dynkin diagram: $(J, E \cap (J \times J), w|_J)$, where $w|_J$ is the restriction of w to $E \cap (J \times J)$. By subdiagram (or subgraph) of Γ , we mean any Dynkin diagram of this form.

A **leaf** in a Dynkin diagram is any vertex which lies in at most one edge. That is to say, it is any $i \in I$ for which there is at most one $j \in I$ such that $(i,j) \in E$.

Dynkin diagrams are graphic representations of generalized Cartan matrices: Suppose that $A = (a_{i,j})_{(i,j) \in I \times I}$ is a generalized Cartan matrix indexed by I . Then, setting $E = \{(i,j) \in I^2 \mid a_{i,j} < 0\}$ and $w : (i,j) \mapsto -a_{i,j}$, we obtain a Dynkin diagram (I, E, w) . Conversely, to any Dynkin diagram (I, E, w) we associate a generalized Cartan matrix

$$(a_{i,j})_{(i,j) \in I \times I} \text{ indexed by } I, \text{ setting } a_{i,j} = \begin{cases} 2 & \text{if } i = j \\ -w(i,j) & \text{if } (i,j) \in E. \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.3. *Connectivity and 1-star-convexity* A Dynkin diagram (I, E, w) is said **connected** if it is connected as a graph. That is to say if, for any $i, j \in I$ there is a sequence $i = i_0, i_1, \dots, i_n = j$ such that $(i_k, i_{k+1}) \in E$ for all $k \in \llbracket 0, n-1 \rrbracket$.

For a general Dynkin diagram Γ , a **connected component** of Γ is any connected subdiagram which is maximal for the inclusion. Therefore a Dynkin diagram is connected if and only if it has exactly one connected component.

A connected Dynkin diagram is a **Dynkin tree** if it does not admit any closed circuit: For any sequence i_0, \dots, i_n such that $(i_k, i_{k+1}) \in E$ for all $k \in \llbracket 0, n-1 \rrbracket$ and $i_{k-1} \neq i_{k+1}$ for all $k \in \llbracket 1, n-1 \rrbracket$, we have $i_0 \neq i_n$. A Dynkin tree is a **Dynkin segment** if there are at most two edges going out of any given vertex.

We introduce a refinement of the notion of connectivity: We say that a Dynkin diagram $\Gamma = (I, E, w)$ is **1-star-convex** if there exists a vertex $i_0 \in I$ such that, for any $j \in I$, there is a sequence (i_1, \dots, i_n) with $i_n = j$ such that, for all $k \in \llbracket 0, n-1 \rrbracket$,

$$(i_k, i_{k+1}) \in E \text{ and } w(i_k, i_{k+1}) = 1. \quad (2.1)$$

In this case, we say that Γ is 1-star-convex at i_0 .

By symmetry of E , a 1-star-convex Dynkin diagram is always connected, but the converse is not true.

Note that given a Dynkin tree Γ , we can check whether it is 1-star-convex by applying the following procedure. Pick one leaf j of Γ . Let i be the vertex of Γ such that $\{i, j\}$ is an edge of Γ . If $w(i, j) \neq 1$, then Γ is not 1-star-convex and we stop here. Otherwise, we consider the subtree of Γ obtained by deleting the edge $\{i, j\}$ and the vertex j and we iterate the procedure. If the procedure stops when the graph has 2 or more vertices, then Γ is not 1-star-convex. Otherwise, Γ is 1-star-convex at the last vertex.

Kac-Moody root systems Let $\mathcal{D} = (A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$ be a Kac-Moody root datum in the sense of [Rém02, §8]. It is a quintuplet such that:

- I is a finite indexing set and $A = (a_{ij})_{(i,j) \in I \times I}$ is a generalized Cartan matrix.
- X and Y are two dual free \mathbb{Z} -modules of finite rank, we write \langle, \rangle the duality bracket.
- $(\alpha_i)_{i \in I}$ (resp. $(\alpha_i^\vee)_{i \in I}$) is a family of linearly independent elements of X (resp. Y), the simple roots (resp. simple coroots).
- For all $(i, j) \in I^2$ we have $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$.

Let (I, E, w) be the Dynkin diagram corresponding to A , with vertices I , which is fixed once and for all. In this article, every Dynkin diagram we will consider will be a subdiagram of (I, E, w) . Since they are identified with subsets of I , any $J \subseteq I$ now designates indistinctively the subset J of I , or the corresponding subdiagram of (I, E, w) .

Height function We define a height function on Y as follows: Let $\rho \in X$ be any element such that $\langle \alpha_i^\vee, \rho \rangle = 1$ for all $i \in I$, then, for any $\lambda \in Y$, the **height** of λ is: $\text{ht}(\lambda) = \langle \lambda, \rho \rangle$. This definition depends on the choice of ρ , but its restriction to $Q^\vee = \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee$ does not:

$$\text{ht}(\sum_{i \in I} n_i \alpha_i^\vee) = \sum_{i \in I} n_i.$$

Vectorial Weyl group For every $i \in I$ set $r_i \in \text{Aut}_{\mathbb{Z}}(X) : x \mapsto x - \langle \alpha_i^\vee, x \rangle \alpha_i$. The generated group $W^v = \langle r_i \mid i \in I \rangle$ is the **vectorial Weyl group** of the Kac-Moody root datum. The duality bracket $\langle Y, X \rangle$ induces a contragredient action of W^v on Y , explicitly $r_i(y) = y - \langle y, \alpha_i \rangle \alpha_i^\vee$. By construction the duality bracket is then W^v -invariant.

The group W^v is a Coxeter group, in particular it has a Bruhat order $<$ and a length function ℓ compatible with the Bruhat order.

Real roots Let $\Phi = W^v \cdot \{\alpha_i \mid i \in I\}$ be the set of real roots of \mathcal{D} , it is a, possibly infinite, root system (see [Kum02, 1.2.2 Definition]). In particular let $\Phi_+ = \Phi \cap \bigoplus_{i \in I} \mathbb{N} \alpha_i$ be the set of positive real roots, then $\Phi = \Phi_+ \sqcup -\Phi_+$, we write $\Phi_- = -\Phi_+$ the set of negative roots.

The set $\Phi^\vee = W^v \cdot \{\alpha_i^\vee \mid i \in I\}$ is the set of **coroots**, and its subset $\Phi_+^\vee = \Phi^\vee \cap \bigoplus_{i \in I} \mathbb{N} \alpha_i^\vee$ is the set of **positive coroots**.

To each root β corresponds a unique coroot β^\vee : if $\beta = w(\alpha_i)$ then $\beta^\vee = w(\alpha_i^\vee)$. This map $\beta \mapsto \beta^\vee$ is well defined, bijective between Φ and Φ^\vee and sends positive roots to positive coroots. Note that $\langle \beta^\vee, \beta \rangle = 2$ for all $\beta \in \Phi$.

Moreover to each root β one associates a reflection $s_\beta \in W^v$: if $\beta = w(\pm \alpha_i)$ then $s_\beta := wr_iw^{-1}$. It is well-defined, independently of the choices of w and i . Explicitly it is the map $x \mapsto x - \langle \beta^\vee, x \rangle \beta$. We have $s_\beta = s_{-\beta}$ and the map $\beta \mapsto s_\beta$ forms a bijection between the set of positive roots and the set $\{wr_iw^{-1} \mid (w, i) \in W^v \times I\}$ of reflections of W^v .

Dynkin sequence associated to a positive root Let $\beta \in \Phi_+$. Since $(\alpha_i^\vee)_{i \in I}$ are linearly independent, there is a unique decomposition $\beta^\vee = \sum_{i \in I} N_i(\beta) \alpha_i^\vee$, and for all $i \in I$, $N_i(\beta) \in \mathbb{Z}_{\geq 0}$. For any $n \in \mathbb{Z}_{\geq 0}$, let us define

$$I_n(\beta) = \{i \in I \mid N_i(\beta) \geq n\}. \quad (2.2)$$

We consider $I_n(\beta)$ as a subdiagram of Γ . We call $(I_n(\beta))_{n \geq 1}$ the **Dynkin sequence** of β . The Dynkin sequence associated to a root is, by definition, non-increasing (for the inclusion) and finitely supported (in the sense that $I_n = \emptyset$ for n large enough). Note that the Dynkin sequence of a simple root α_i is given by $I_1(\alpha_i) = \{i\}$ and $I_n(\alpha_i) = \emptyset$ for $n \geq 2$. We will provide a classification of quantum roots using Dynkin sequences.

Note that the Dynkin sequence $(I_n(\beta))_{n \geq 1}$ fully determines β . Indeed, we have $\beta^\vee = \sum_{n=0}^{+\infty} \sum_{i \in I_n(\beta)} \alpha_i^\vee$, which proves that β^\vee and thus β is determined by its Dynkin sequence.

Inversion sets For any $w \in W^v$, let $\text{Inv}(w) = \Phi_+ \cap w^{-1} \cdot \Phi_- = \{\alpha \in \Phi_+ \mid w(\alpha) < 0\}$. These sets are strongly connected to the Bruhat order, as by [Kum02, 1.3.13],

$$\forall \alpha \in \Phi_+, \alpha \in \text{Inv}(w) \iff ws_\alpha < w \iff s_\alpha w^{-1} < w^{-1}.$$

Moreover, they are related to the length ℓ on W : $\ell(w) = |\text{Inv}(w)|$ ([Kum02, 1.3.14]).

2.2 Quantum roots: definition, motivation and notations

Definition 2.4. Let $\beta \in \Phi_+$. We say that β is **quantum** if for every $\gamma \in \text{Inv}(s_\beta) \setminus \{\beta\}$, we have $\langle \beta^\vee, \gamma \rangle = 1$. We denote by $\mathcal{Q}(\Phi_+)$ the set of quantum roots of Φ_+ .

Note that $\text{Inv}(r_i) = \{\alpha_i\}$ for every $i \in I$ and thus every simple root is quantum.

The aim of this section is to study quantum roots and in particular to prove that for every Kac-Moody datum, the number of quantum roots is finite (see Theorem 2.27). This result is a key step to prove our finiteness results on covers and co-covers (see Theorems 3.2 and 3.6). We then classify the quantum roots. We end up this section with examples: we describe the set of quantum roots for particular choices of Kac-Moody matrices.

Let us describe the main steps of the proofs of Theorem 2.27 and Theorem 2.36. Let $\beta \in \mathcal{Q}(\Phi_+)$. Let $\beta \in \mathcal{Q}(\Phi_+)$. We write

$$\beta = r_{i_L} \dots r_{i_2} \cdot \alpha_{i_1}, \quad (2.3)$$

with $i_1, \dots, i_L \in I$ and L minimal possible.

1. We begin by proving that up to renumbering, we can assume that if $L_1 = |I_1(\beta)|$, we have $\{i_1, \dots, i_{L_1}\} = I_1(\beta)$ (see Lemma 2.9).
2. We prove that $I_1(\beta)$ is a 1-star convex tree (see Proposition 2.11).
3. We prove that the "orders of appearance" in (2.3) are very constrained. If $i, j \in I_2(\beta)$ belong to the same connected component and if the second appearance of i (when we run the right hand side of (2.3) from the right) precedes the second appearance of j , then for $n \in \mathbb{Z}_{\geq 3}$,

$$j \in I_n(\beta) \Rightarrow i \in I_n(\beta),$$

and the n -th appearance of i precedes the n -th appearance of j (see Lemma 2.15).

4. We prove that the connected components of $I_2(\beta)$ are as follows. Let \mathcal{C} be a connected of $I_2(\beta)$ and j be the element of \mathcal{C} whose second appearance in (2.3) is the most precocious. Then \mathcal{C} is the union of at most three segments based at j (see Lemma 2.12 and Proposition 2.13).
5. Using the results described above, we obtain restrictions on the form of the Dynkin sequence $(I_n(\beta))$. We deduce that if $k = \max\{n \in \mathbb{Z} \mid I_n(\beta) \neq \emptyset\}$, then $k \leq \max(6, k' + 1)$ where k' is the maximal length of a segment appearing in point 4 (see Proposition 2.26). As k' is bounded by the size of I , we deduce a majoration of the height of β^\vee and then deduce that $\mathcal{Q}(\Phi_+)$ is finite.
6. Refining the results of 5., we describe all the possible forms of the Dynkin sequences. We then construct for all these Dynkin sequences a quantum root to which it is associated (see Theorem 2.36).

Let $\beta \in \Phi_+$. We write $\beta = w.\alpha_{i_1}$, with $i_1 \in I$ and $w \in W^v$ and we assume that $L := \ell(w) + 1$ is the minimal possible. We fix a reduced decomposition $\underline{w} = (r_{i_L}, \dots, r_{i_2})$ of w , with $i_L, \dots, i_2 \in I$.

By [Kum02, 1.3.14 Lemma] we have

$$\text{Inv}(s_\beta) = \{\alpha_{i_L}, r_{i_L}.\alpha_{i_{L-1}}, \dots, r_{i_L} \dots r_{i_2}.\alpha_{i_1}, r_{i_L} \dots r_{i_1}.\alpha_{i_2}, \dots, r_{i_L} \dots r_{i_1} \dots r_{i_{L-1}}.\alpha_{i_L}\}. \quad (2.4)$$

Set

$$\gamma_1^\vee = \alpha_{i_1}^\vee, \gamma_2^\vee = r_{i_2}.\alpha_{i_1}^\vee, \dots, \gamma_L^\vee = r_{i_L} \dots r_{i_2}.\alpha_{i_1}^\vee = \beta^\vee.$$

Lemma 2.5. *Let $\beta \in \Phi_+$. Then $\ell(s_\beta) \leq 2\text{ht}(\beta^\vee) - 1$.*

Proof. By [Phi23, Corollary 1.10], we have $2\text{ht}(\beta^\vee) = 2 + \sum_{\gamma \in \text{Inv}(s_\beta) \setminus \{\beta\}} \langle \beta^\vee, \gamma \rangle$ and all the terms in the right hand side are positive integers. Therefore $2\text{ht}(\beta^\vee) - 2 \geq |\text{Inv}(s_\beta)| - 1 = 2\ell(s_\beta) - 1$. \square

Lemma 2.6. *Let $\beta \in \Phi_+$. Using the same notation as above, set $\beta_1 = r_{i_L} \dots r_{i_3}.\alpha_{i_2}$, $\beta_2 = r_{i_L} \dots r_{i_4}.\alpha_{i_3}, \dots, \beta_{L-1} = \alpha_{i_L}$. Then the following conditions are equivalent:*

1. β is quantum,
2. for every $t \in \llbracket 1, L-1 \rrbracket$, $\langle \beta^\vee, \beta_t \rangle = 1$,
- 3.

$$\forall t \in \llbracket 1, L \rrbracket, \gamma_t^\vee = \sum_{k=1}^t \alpha_{i_k}^\vee \text{ and } \langle \gamma_{t-1}^\vee, \alpha_{i_t} \rangle = -1, \text{ if } t \neq 1, \quad (2.5)$$

4. $\ell(s_\beta) = 2\text{ht}(\beta^\vee) - 1$.

Proof. Let $t \in \llbracket 2, L \rrbracket$. We have:

$$\langle \beta^\vee, \beta_{t-1} \rangle = \langle r_{i_L} \dots r_{i_2}.\alpha_{i_1}^\vee, r_{i_L} \dots r_{i_{t+1}}.\alpha_{i_t} \rangle = \langle r_{i_t} \dots r_{i_2}.\alpha_{i_1}^\vee, \alpha_{i_t} \rangle = -\langle r_{i_{t-1}} \dots r_{i_2}.\alpha_{i_1}^\vee, \alpha_{i_t} \rangle.$$

We thus have:

$$\begin{aligned} \langle r_{i_L} \dots r_{i_2}.\alpha_{i_1}^\vee, r_{i_L} \dots r_{i_1} \dots r_{i_{t-1}}.\alpha_{i_t} \rangle &= -\langle \alpha_{i_1}^\vee, r_{i_1} \dots r_{i_{t-1}}.\alpha_{i_t} \rangle \\ &= -\langle r_{i_{t-1}} \dots r_{i_2}.\alpha_{i_1}^\vee, \alpha_{i_t} \rangle = \langle \beta^\vee, \beta_{t-1} \rangle. \end{aligned}$$

Therefore

$$\{\langle \beta^\vee, \gamma \rangle \mid \gamma \in \text{Inv}(s_\beta) \setminus \{\beta\}\} = \{\langle \beta^\vee, \beta_t \rangle \mid t \in \llbracket 2, L \rrbracket\},$$

which proves the equivalence between (1) and (2).

Let $t \in \llbracket 1, L-1 \rrbracket$. We have:

$$\langle \beta^\vee, \beta_t \rangle = \langle r_{i_L} \dots r_{i_2}.\alpha_{i_1}^\vee, r_{i_L} \dots r_{i_{t+2}}.\alpha_{i_{t+1}} \rangle = \langle r_{i_{t+1}} \dots r_{i_2}.\alpha_{i_1}^\vee, \alpha_{i_{t+1}} \rangle = -\langle \gamma_t^\vee, \alpha_{i_{t+1}} \rangle.$$

Therefore

$$\gamma_{t+1}^\vee = r_{i_{t+1}}.\gamma_t^\vee = \gamma_t^\vee - \langle \gamma_t^\vee, \alpha_{i_{t+1}} \rangle \alpha_{i_{t+1}}^\vee = \gamma_t^\vee + \langle \beta^\vee, \beta_t \rangle \alpha_{i_{t+1}}^\vee.$$

The equivalence between (1) and (3) follows.

Let $\beta \in \Phi_+$ satisfying (3). We have $\gamma_L^\vee = \beta^\vee$ and $\text{ht}(\gamma_L^\vee) = L$. As $\ell(s_\beta) = 2L - 1$, we deduce that (3) implies (4). Let $\beta^\vee \in \Phi_+^\vee$. Assume that β^\vee satisfies (4). Then by [Phi23, Corollary 1.10], we have $2\text{ht}(\beta^\vee) = \sum_{\gamma \in \text{Inv}(s_\beta)} \langle \beta^\vee, \gamma \rangle = 2 + \sum_{\gamma \in \text{Inv}(s_\beta) \setminus \{\beta^\vee\}} \langle \beta^\vee, \gamma \rangle = \ell(s_\beta) + 1$ and all the terms of this sum are positive. Therefore $\langle \beta^\vee, \gamma \rangle = 1$ for all $\gamma \in \text{Inv}(s_\beta) \setminus \{\beta^\vee\}$, which proves that β is quantum. \square

Remark 2.7. *By Lemma 2.6, for all $t \in \llbracket 1, L \rrbracket$, $\gamma_t = r_{i_t} \dots r_{i_2}(\alpha_{i_1})$ is quantum and $\text{ht}(\gamma_t^\vee) = t$.*

The following proposition indicates that the set $\mathcal{Q}(\Phi_+)$ can be constructed recursively. This will be used extensively in Section 2.6 in order to prove Theorem 2.36.

Proposition 2.8. *1. If $\beta \in \mathcal{Q}(\Phi_+)$ is a quantum root which is not simple, then there exists $i \in I$ such that $r_i(\beta) \in \mathcal{Q}(\Phi_+)$ and $\text{ht}(r_i(\beta^\vee)) = \text{ht}(\beta^\vee) - 1$.*

2. Let $\beta \in \mathcal{Q}(\Phi_+)$ and $i \in I$. Then $r_i(\beta) \in \mathcal{Q}(\Phi_+)$ if and only if $|\langle \beta^\vee, \alpha_i \rangle| \leq 1$.

Proof. 1. We simply take $i = i_L$ with the notation of Formula (2.4), then by Remark 2.7, $r_i(\beta) = \gamma_{L-1}$ is a quantum root.

2. Let $\beta \in \mathcal{Q}(\Phi_+)$ and $i \in I$. Let $\tilde{\beta} = r_i(\beta)$. We have $s_{\tilde{\beta}} = r_i s_\beta r_i$ and thus:

$$\ell(s_{\tilde{\beta}}) \in \{\ell(s_\beta) - 2, \ell(s_\beta), \ell(s_\beta) + 2\}. \quad (2.6)$$

We have $\text{ht}(\tilde{\beta}^\vee) = \text{ht}(\beta^\vee) - \langle \beta^\vee, \alpha_i \rangle$. If $r_i(\beta) \in \mathcal{Q}(\Phi_+)$, then by Lemma 2.6, we have $\ell(s_{\tilde{\beta}}) = 2\text{ht}(\tilde{\beta}^\vee) - 1 = 2\text{ht}(\beta^\vee) - 1 - 2\langle \beta^\vee, \alpha_i \rangle = \ell(s_\beta) - 2\langle \beta^\vee, \alpha_i \rangle$. Using (2.6) we deduce that if $r_i(\beta) \in \mathcal{Q}(\Phi_+)$, then $|\langle \beta^\vee, \alpha_i \rangle| \leq 1$.

Let now $\beta \in \mathcal{Q}(\Phi_+)$ and $i \in I$. If $\langle \beta^\vee, \alpha_i \rangle = 0$, then $r_i(\beta) = \beta \in \mathcal{Q}(\Phi_+)$. Assume now $\langle \beta^\vee, \alpha_i \rangle = 1$. Set $\tilde{\beta} = r_i(\beta)$. We have $\text{ht}(\tilde{\beta}^\vee) = \text{ht}(\beta^\vee) - 1$ and by Lemma 2.5,

$$\ell(s_{\tilde{\beta}}) \leq 2\text{ht}(\tilde{\beta}^\vee) - 1 = 2\text{ht}(\beta^\vee) - 3 = \ell(s_\beta) - 2.$$

By (2.6) we deduce $\ell(s_{\tilde{\beta}}) = \ell(s_\beta) - 2 = 2\text{ht}(\tilde{\beta}^\vee) - 1$ which proves that $\tilde{\beta} = r_i(\beta)$ is quantum by Lemma 2.6.

Eventually, assume that $\langle \beta^\vee, \alpha_i \rangle = -1$. Let $\tilde{\beta} := r_i(\beta)$. Note that the associated coroot is $\tilde{\beta}^\vee = r_i(\beta^\vee) = \beta^\vee + \alpha_i^\vee$. Therefore $\text{ht}(\tilde{\beta}^\vee) = \text{ht}(\beta^\vee) + 1$ and, by Lemma 2.6, $\tilde{\beta}$ is quantum if and only if

$$\ell(s_{\tilde{\beta}}) = \ell(r_i s_\beta r_i) = \ell(s_\beta) + 2.$$

Since r_i is a simple reflection, $\{\ell(r_i s_\beta), \ell(s_\beta r_i)\} \subset \{\ell(s_\beta) + 1, \ell(s_\beta) - 1\}$. Moreover since s_β is a reflection, $\ell(r_i s_\beta) = \ell((r_i s_\beta)^{-1}) = \ell(s_\beta r_i)$.

We have $\langle \beta^\vee, \alpha_i \rangle < 0$, so $s_\beta(\alpha_i) \in \Phi_+$. Hence $\alpha_i \notin \text{Inv}(s_\beta)$, $s_\beta r_i > s_\beta$ and thus $\ell(s_\beta r_i) = \ell(r_i s_\beta) = \ell(s_\beta) + 1$.

Note that $s_\beta(\alpha_i) = \alpha_i - \langle \beta^\vee, \alpha_i \rangle \beta = \beta + \alpha_i$. Moreover since $\langle \beta^\vee, \alpha_i \rangle < 0$, by [Bar96, Lemma 1.1.10] we also have $\langle \alpha_i^\vee, \beta \rangle < 0$, that is to say $\langle \alpha_i^\vee, \beta \rangle + 1 \leq 0$ (because it is an integer). Therefore,

$$r_i s_\beta(\alpha_i) = r_i(\beta + \alpha_i) = r_i(\beta) - \alpha_i = \beta - (\langle \alpha_i^\vee, \beta \rangle + 1)\alpha_i \in \Phi_+.$$

So $\alpha_i \notin \text{Inv}(r_i s_\beta)$ and $\ell(r_i s_\beta r_i) = \ell(r_i s_\beta) + 1 = \ell(s_\beta) + 2$, which proves the proposition. \square

From now on we assume that β is quantum.

For $i \in I$ and $t \in \llbracket 1, L \rrbracket$, set $N_i(t) = |\{t' \in \llbracket 1, t \rrbracket \mid i_{t'} = i\}|$. Then by (2.5),

$$\gamma_t^\vee = \sum_{i \in I_1} N_i(t) \alpha_i^\vee. \quad (2.7)$$

Note that by (2.5), N_i is non-decreasing for every $i \in I_1(\beta)$, and for every $t \in \llbracket 1, L-1 \rrbracket$, $N_i(t) = N_i(t+1)$ for all but one element k of $I_1(\beta)$, and we have $N_k(t+1) = N_k(t) + 1$.

Note that the Dynkin sequence of β can be defined by $I_n(\beta) := \{i \in I \mid \max N_i \geq n\}$, with the induced diagram structure from I . The set $I_n(\beta)$ is also the set of elements which appear at least n times in (\underline{w}, r_{i_1}) .

In this section, we have fixed a quantum root β , therefore to simplify notation we set

$$I_n := I_n(\beta). \quad (2.8)$$

For $j \in I$ and $n \in \mathbb{Z}_{\geq 0}$, we set

$$\mathbf{t}(j, n) = \min\{t \in \llbracket 1, L \rrbracket \mid N_j(t) = n\} - 1 \in \llbracket 0, L-1 \rrbracket \cup \{\infty\}. \quad (2.9)$$

We then have $N_j(\mathbf{t}(j, n)) = n - 1$ and $N_j(\mathbf{t}(j, n) + 1) = n$, if $\mathbf{t}(j, n) \neq \infty$.

Lemma 2.9. *Let $L_1 = |I_1|$. Then up to changing the reduced decomposition of w , we may assume $I_1 = \{i_1, \dots, i_{L_1}\}$.*

Proof. Assume $\{i_1, \dots, i_{L_1}\} \neq I_1$. Then $(i_j)_{j \in \llbracket 1, L_1 \rrbracket}$ is not injective. Let $k \in \llbracket 1, L_1 \rrbracket$ be minimal such that $i_k \in \{i_1, \dots, i_{k-1}\}$. Let $k' \in \llbracket k+1, L \rrbracket$ be minimal such that $i_{k'} \notin \{i_1, \dots, i_{k'-1}\}$. Then for every $k'' \in \llbracket k, k'-1 \rrbracket$, $N_{i_{k''}}(k') = N_{i_{k''}}(k'-1) \geq 2$ and $N_{i_{k'}}(k'-1) = 0 = N_{i_{k'}}(k') - 1$. By assumption on β , we have $\langle r_{i_{k'-1}} \dots r_{i_2} \alpha_{i_1}^\vee, \alpha_{i_{k'}} \rangle = -1$. Using (2.7), we deduce $\sum_{i \in I_1} N_i(k'-1) a_{i, i_{k'}} = -1$. As $N_{i_{k'}}(k'-1) = 0$, all the terms appearing in the left hand side are non positive integers. Therefore only one is nonzero and equal to -1 . Therefore $a_{i_{k''}, i_{k'}} = 0$ for every $k'' \in \llbracket k, k'-1 \rrbracket$. In other words, $r_{i_{k''}}$ commutes with $r_{i_{k'}}$ for every $k'' \in \llbracket k, k'-1 \rrbracket$. Therefore we can replace $r_{i_{k'}} r_{i_{k'-1}} \dots r_{i_k} r_{i_{k-1}} \dots r_{i_2}$ by $r_{i_{k'-1}} \dots r_{i_k} r_{i_{k'}} r_{i_{k-1}} \dots r_{i_2}$ in the reduced decomposition of w . Reiterating this process as many times as necessary, we end up with a reduced decomposition \underline{w} such that $(i_j)_{j \in \llbracket 1, L_1 \rrbracket}$ is injective and then $\{i_1, \dots, i_{L_1}\} = I_1$. \square

From now on, we assume that the reduced decomposition \underline{w} is chosen so that $\{i_1, \dots, i_{L_1}\} = I_1$, which is possible by Lemma 2.9.

2.3 General structure of $I_1(\beta)$

In this section, we study the Dynkin diagram structure of $I_1(\beta)$ when β is a quantum root. We have fixed a quantum root β and we have set $I_1 = I_1(\beta)$.

For $i \in I$, set

$$\deg(i) = - \sum_{j \in I_1 \setminus \{i\}} a_{j,i} \in \mathbb{Z}_{\geq 0} \quad (2.10)$$

and

$$\text{Supp}(i) = \{j \in I_1 \mid a_{j,i} \neq 0\}.$$

Lemma 2.10. *The graph I_1 is a tree: it is connected and without circuit.*

Proof. By [Kac94, Lemma 1.6], I_1 is connected (I_1 is the support of β for the terminology of [Kac94]). Assume by contradiction that I_1 contains a circuit. Let $n \in \mathbb{Z}_{>0}$ and $\tau : \llbracket 0, n \rrbracket \rightarrow I_1$ be such that $\tau(0) = \tau(n)$ and $\{\tau(i), \tau(i+1)\}$ is an edge of I_1 for all $i \in \llbracket 0, n-1 \rrbracket$. We assume moreover that $\tau(i) \neq \tau(0)$ for $i \in \llbracket 1, n-1 \rrbracket$. Then every element of $\tau(\llbracket 0, n \rrbracket)$ has at least two neighbours in I_1 . Let $i \in \llbracket 0, n-1 \rrbracket$ be such that $\mathbf{t}(\tau(i), 1) = t := \max\{\mathbf{t}(\tau(j), 1) \mid j \in \llbracket 0, n-1 \rrbracket\}$. Then there exist $i_1, i_2 \in I_1 \setminus \{\tau(i)\}$ such that $a_{i_1, \tau(i)}, a_{i_2, \tau(i)} \neq 0$ and $\gamma_t^\vee = \alpha_{i_1}^\vee + \alpha_{i_2}^\vee + x$, for some $x \in \bigoplus_{j \in I_1 \setminus \{\tau(i)\}} \mathbb{N}\alpha_j^\vee$. Then $\alpha_{\tau(i)}(\gamma_t^\vee) \leq \alpha_{\tau(i)}(\alpha_{i_1}^\vee + \alpha_{i_2}^\vee) \leq -2 < -1$: a contradiction. Lemma follows. \square

A first step in order to determine $\mathcal{Q}(\Phi_+)$ is to understand what the support of quantum roots can be. That is to say, to determine $\{I_1(\beta) \mid \beta \in \mathcal{Q}(\Phi_+)\}$. Proposition 2.11 below characterizes this set.

Proposition 2.11. *Let T be a subset of I . Then the following properties are equivalent:*

1. *As a subdiagram of I , T is a 1-star-convex Dynkin tree.*
2. *There exists $\beta \in \mathcal{Q}(\Phi_+)$ such that $T = I_1(\beta)$.*
3. *There exists $\beta \in \mathcal{Q}(\Phi_+)$ such that $T = I_1(\beta)$ and $I_2(\beta) = \emptyset$. More precisely, β is the unique element of Φ such that $\beta^\vee = \sum_{i \in T} \alpha_i^\vee$.*

Proof. Let $\beta \in \mathcal{Q}(\Phi_+)$ and $T = I_1(\beta)$. By Remark ??, we can assume $\beta = \sum_{i \in T} \alpha_i$. Write $\beta = r_{i_L} \dots r_{i_2} \cdot \alpha_{i_1}$, where $L = |I_1(\beta)|$ and $\{i_L, \dots, i_1\} = I_1(\beta)$. Let us prove that T is 1-star-convex at i_1 . Let $t \in \llbracket 1, L-1 \rrbracket$ and assume that, as a subdiagram of T , $\{i_1, \dots, i_t\}$ is 1-star-convex at i_1 . Then we have $r_{i_t} \dots r_{i_2} \cdot \alpha_{i_1}^\vee = \sum_{i''=1}^t \alpha_{i''}^\vee$ and $r_{i_{t+1}} \cdot (r_{i_t} \dots r_{i_2} \cdot \alpha_{i_1}^\vee) = (r_{i_t} \dots r_{i_2} \cdot \alpha_{i_1}^\vee) + \alpha_{i_{t+1}}^\vee$, by Lemma 2.6. Therefore $\langle \sum_{i''=1}^t \alpha_{i''}^\vee, \alpha_{i_{t+1}} \rangle = \sum_{i''=1}^t \langle \alpha_{i''}^\vee, \alpha_{i_{t+1}} \rangle = -1$. As this is a sum of non-positive integers, exactly one term is -1 and the others are 0. Let $t_1 \in \llbracket 1, t \rrbracket$ be such that $a_{i_{t_1}, i_{t+1}} = -1$. By assumption, there exist $t' \in \llbracket 1, t \rrbracket$ and a sequence $(u_i)_{i \in \llbracket 1, t' \rrbracket} \in \{i_1, \dots, i_t\}^{\llbracket 1, t' \rrbracket}$ such that $a_{u_i, u_{i+1}} = -1$ for all $i \in \llbracket 1, t'-1 \rrbracket$, $u_1 = i_1$ and $u_{t'} = i_{t_1}$. Then $((u_i)_{i \in \llbracket 1, t' \rrbracket}, i_{t+1})$ is a path in I_1 relating i_1 and i_{t+1} , which proves the

1-star-convexity of $\{i_1, \dots, i_{t+1}\}$ at i_1 . By induction, we deduce that T is 1-star-convex. Using Lemma 2.10, we deduce that I_1 is a 1-star-convex tree.

Let T be a subdiagram of I which is 1-star-convex and which is a tree. We assume that $|T| \geq 2$ and that for all $T' \subset I$ such that $|T'| < |T|$ and for which T' is 1-star-convex, there exists $\beta' \in \mathcal{Q}(\Phi_+)$ such that $T' = I_1(\beta')$. Let j be a leaf of T and write $\{j, k\}$ the edge of T containing j . Then by 1-star-convexity of T , we have $a_{k,j} = -1$. Moreover, $T \setminus \{j\}$ is 1-star-convex. Let $\beta_1 \in \mathcal{Q}(\Phi_+)$ be such that $(\beta_1)^\vee = \sum_{i \in T \setminus \{j\}} \alpha_i^\vee$. We have $\langle (\beta_1)^\vee, \alpha_j \rangle = -1$, thus $\beta^\vee = r_j(\beta_1^\vee)$ and thus by Proposition 2.8, $\beta = r_j(\beta_1)$ is quantum. \square

2.4 Order of appearance and general structure of $I_2(\beta)$

Let $n \in \mathbb{N}$. For $i, j \in I_n$, we write $i \triangleleft_n j$ if (i, j) is an edge of I_n , and if $\mathbf{t}(i, n) < \mathbf{t}(j, n)$. The first condition means that $a_{i,j} \neq 0$ and the second condition simply means that if we run the decomposition of w from the right, we meet i for the n -th times before we meet j for the n -th times. We denote by \leq_n the preorder generated by \triangleleft_n . Note that for $i, j \in I_n$, if $i \leq_n j$ and $j \leq_n i$, then i and j appears at the same time for the n -th times when we run (\underline{w}, i_1) from the right, and thus $i = j$. Therefore \leq_n is an order on I_n . Note that although this is not obvious from the definition, this order is independent of the chosen decomposition of w , as we will see in Remark 2.16. As I_n is finite, all the chains are contained in a maximal one.

In this section, we use \leq_2 to characterize the diagram I_2 . In Lemma 2.15 we then describe chains for \leq_n . This will permit to characterize the diagrams I_n in Section 2.6.2.

Lemma 2.12. *Assume $I_2 \neq \emptyset$. Let $j_1, \dots, j_k \in I_2$ be a maximal chain for \leq_2 , i.e:*

1. j_1 is minimal in I_2 for \leq_2 ,
2. j_k is maximal in I_2 for \leq_2 ,
3. $j_1 \triangleleft_2 j_2 \triangleleft_2 \dots \triangleleft_2 j_k$.

Then:

1. $\deg(j_1) = 3$, $|\text{Supp}(j_1)| \leq 4$.
2. for every $t \in \llbracket 2, k-1 \rrbracket$, $\text{Supp}(j_t) = \{j_{t-1}, j_t, j_{t+1}\}$, $\deg(j_t) = 2$ and $a_{j_{t-1}, j_t} = a_{j_{t+1}, j_t} = -1$.
3. We have $\deg(j_k) \leq 2$, $|\text{Supp}(j_k)| \leq 3$ and $\text{Supp}(j_k) \supset \{j_{k-1}, j_k\}$.
 - (a) If $|\text{Supp}(j_k)| = 3$, then $\text{Supp}(j_k) = \{j_{k-1}, j_k, j_{k+1}\}$ for some $j_{k+1} \in I_1 \setminus I_2$, and we have $a_{j_{k-1}, j_k} = a_{j_{k+1}, j_k} = -1$.
 - (b) If $|\text{Supp}(j_k)| = \{j_{k-1}, j_k\}$, then

$$a_{j_{k-1}, j_k} = -1 \text{ and } \mathbf{t}(j_{k-1}, 3) < \mathbf{t}(j_k, 2) < \mathbf{t}(j_{k-1}, 4). \quad (2.11)$$

In particular, if $j \in I_2$ is such that $|\text{Supp}(j)| = 2$, then j is either minimal or maximal for \leq_2 .

Proof. Let $t_1 = \mathbf{t}(j_1, 2)$. By (2.7), we have: $\gamma_{t_1}^\vee = \sum_{i \in I_1} N_i(t_1) \alpha_i^\vee$ and $N_i(t_1) \geq 1$ for every $i \in I_1$ by choice of \underline{w} . Then $\alpha_{i_{t_1}}(\gamma_{t_1}^\vee) = -1$. Let $i \in I_1$ and assume that $N_i(t_1) \geq 2$. Then as $i \not\leq_2 j_1$, we have $a_{i, j_1} = 0$. Therefore

$$\begin{aligned} \alpha_{j_1}(\gamma_{t_1}^\vee) &= -1 \\ &= \alpha_{j_1} \left(\sum_{i \in \text{Supp}(j_1)} N_i(t_1) \alpha_i^\vee \right) \\ &= \alpha_{j_1} \left(\sum_{i \in \text{Supp}(j_1)} \alpha_i^\vee \right) \\ &= 2 + \sum_{i \in \text{Supp}(j_1) \setminus \{j_1\}} a_{i, j_1} \\ &= 2 - \deg(j_1), \end{aligned}$$

and thus $\deg(j_1) = 3$.

Let now $x \in \llbracket 2, k \rrbracket$. Let $t_x = \mathbf{t}(j_x, 2)$. We have:

$$\alpha_{j_x}(\gamma_{t_x}^\vee) = -1 = \alpha_{j_x} \left(\sum_{i \in \text{Supp}(j_x)} N_i(t_x) \alpha_i^\vee \right) = 2 + \sum_{i \in \text{Supp}(j_x) \setminus \{j_x\}} N_i(t_x) a_{i, j_x}.$$

Therefore $\sum_{i \in \text{Supp}(j_x) \setminus \{j_x\}} N_i(t_x) a_{i, j_x} = -3$, and this is a sum of non positive integers. Moreover, as $N_{j_{x-1}}$ is non decreasing and $t_x > t_{x-1}$, we have $N_{j_{x-1}}(t_x) \geq 2$. Consequently, $|\text{Supp}(j_x)| \leq 3$. Assume $x \leq k-1$. As $j_{x-1} \triangleleft_2 j_x \triangleleft_2 j_{x+1}$, we have $\text{Supp}(j_x) = \{j_{x-1}, j_x, j_{x+1}\}$ and $a_{j_{x+1}, j_x} = a_{j_{x-1}, j_x} = -1$. In particular, $\deg(j_x) = 2$.

Assume now $x = k$. Then if $|\text{Supp}(j_k)| = 3$, we have $a_{j_{k+1}, j_k} = -1$, for some $j_{k+1} \in I_1$. If $j_{k+1} \in I_2$, then we would have $j_k \triangleleft_2 j_{k+1}$, which would contradict our maximality assumption. Therefore, $j_{k+1} \in I_1 \setminus I_2$. Assume $|\text{Supp}(j_k)| = 2$. Then $\text{Supp}(j_k) = \{j_{k-1}, j_k\}$. Let $t = \mathbf{t}(j_k, 2)$. Then

$$\alpha_{j_k}(\gamma_t^\vee) = -1 = \alpha_{j_k}(N_{j_{k-1}}(t) \alpha_{j_{k-1}}^\vee + \alpha_{j_k}^\vee) = 2 - 3,$$

which proves (2.11). \square

In terms of Dynkin diagrams, Lemma 2.12 implies that the form of the connected components of I_2 are very constrained, see Proposition 2.13 and Figure 2.1.

Proposition 2.13. *Let \mathcal{C} be a connected component of I_2 . Then:*

1. \mathcal{C} admits a minimum j_1 and $m := |\text{Supp}(j_1)| - 1 \leq 3$.
2. Write $\text{Supp}(j_1) \setminus \{j_1\} = \{j_2^{(i)} \mid i \in \llbracket 1, m \rrbracket\}$. For $i \in \llbracket 1, m \rrbracket$, $\{x \in I_2 \mid x \geq_2 j_2^{(i)}\}$ is a segment: we can write it $\{j_2^{(i)}, j_3^{(i)}, \dots, j_{k(i)}^{(i)}\}$, for some $j_2^{(i)}, \dots, j_{k(i)}^{(i)} \in I_2$ and the edges are the $(j_t^{(i)}, j_{t+1}^{(i)})$ for $t \in \llbracket 1, k(i) - 1 \rrbracket$.

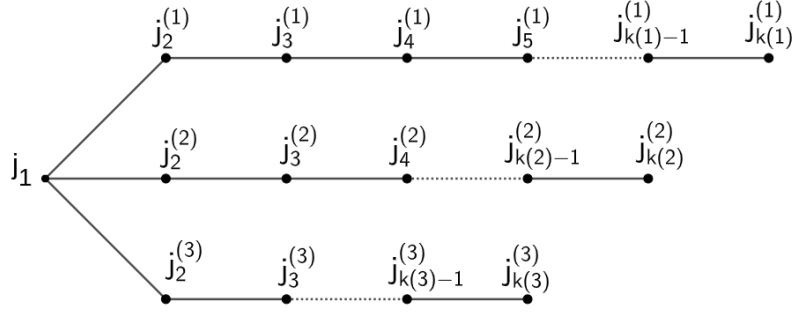


Figure 2.1: Generic form of a connected component of I_2 , when $m = 3$. Actually, as we shall see in Lemma 2.22, if $j_1 \in I_3$, then we have $\min\{k(1), k(2), k(3)\} \leq 2$.

3. We have $\mathcal{C} = \{j_1\} \sqcup \bigsqcup_{i=1}^m [j_2^{(i)}, j_{k(i)}^{(i)}]$.

4. If $j \in \mathcal{C} \setminus \{j_1\}$ then $|\text{Supp}(j)| \leq 3$ and $a_{j',j} = -1$ for all $j' \in \text{Supp}(j)$.

Proof. Let $j_1 \in \mathcal{C}$ be minimum for \leq_2 . By Lemma 2.12, $m := |\text{Supp}(j_1)| - 1 \leq 3$. Write $\text{Supp}(j_1) = \{j_1\} \cup \{j_2^{(i)} \mid i \in \llbracket 1, m \rrbracket\}$. Let $i \in \llbracket 1, m \rrbracket$. By Lemma 2.12, the set E_i of the elements of I_2 which are bigger than $j_2^{(i)}$ for \leq_2 is of the form $\{j_2^{(i)}, j_3^{(i)}, \dots, j_{k(i)}^{(i)}\}$ for some $k(i) \in \mathbb{Z}_{\geq 2}$, and as a subgraph of I_2 , E_i is the line segment $[j_2^{(i)}, j_{k(i)}^{(i)}]$. In particular, if $\mathcal{C}' = \{j_1\} \cup \bigcup_{i=1}^m E_i$, then for each $x \in \mathcal{C}'$, we have $\text{Supp}(x) \cap I_2 \subset \mathcal{C}'$. Therefore \mathcal{C}' is connected and $\mathcal{C} = \mathcal{C}'$. Moreover \mathcal{C} is a connected subgraph of the tree I_1 (by Proposition 2.11) and thus it is a tree. Therefore the E_i are disjoint. Finally, by Lemma 2.12 2-3, for all $j \in \mathcal{C}'$ we have $|\text{Supp}(j)| \leq 3$ and, for all $j' \in \text{Supp}(j)$, $a_{j',j} = -1$. Lemma follows. \square

Remark 2.14. The order \leq_2 does not depend on the choice of the reduced decomposition made in the beginning of subsection 2.2, provided that it satisfies Lemma 2.9. Indeed, if 2 elements of I_2 are comparable for \leq_2 , they belong to the same connected component of I_2 . Thus it suffices to prove that for each connected component \mathcal{C} of I_2 the restriction of \leq_2 to \mathcal{C} is independent of the choice of the reduced decomposition. By Lemma 2.12 and Proposition 2.13, the minimum j_1 for \leq_2 is the unique element of \mathcal{C} whose degree is 3. Then the elements $j_2^{(1)}, \dots, j_2^{(m)}$ (where $m = |\text{Supp}(j_1) - 1|$) that cover j_1 are the elements of $\text{Supp}(j_1) \setminus \{j_1\}$. If $i \in \llbracket 1, m \rrbracket$, the only element (if it exists) covering $j_2^{(i)}$ is the element of $\text{Supp}(j_2^{(i)}) \setminus \{j_1, j_2^{(i)}\}$ etc. All these elements are defined uniquely from the graph I_2 , which is independent of the choice of the reduced decomposition.

With the above notation, we set:

$$k' = k, \text{ if } |\text{Supp}(j_k)| = 2 \text{ and } k' = k + 1 \text{ if } \text{Supp}(j_k) = \{j_{k-1}, j_k, j_{k+1}\}. \quad (2.12)$$

The following lemma implies that

$$N_{j_1}(t) \geq N_{j_2}(t) \geq \dots \geq N_{j_{k'}}(t), \forall t \in \llbracket L_1 + 1, L \rrbracket. \quad (2.13)$$

Lemma 2.15. *Let $j_1, \dots, j_k \in I_2$ be a maximal chain for \leq_2 . Let $n \in \mathbb{Z}_{\geq 3}$. Then:*

1. *there exists $k_n \in \llbracket 0, k \rrbracket$ such that $\{i \in \llbracket 1, k \rrbracket \mid j_i \in I_n\} = \llbracket 1, k_n \rrbracket$,*
2. *we have $j_1 \triangleleft_n j_2 \triangleleft_n \dots \triangleleft_n j_{k_n}$,*
3. *the sequence (k_n) is non-increasing.*

Suppose moreover that for all $n \geq 2$, $\mathbf{t}(j_1, n) \geq \mathbf{t}(j_2, n - 1)$. Then:

4. *we have $\text{Supp}(j_k) = \{j_{k-1}, j_k, j_{k+1}\}$,*
5. *for all $i \in \llbracket 1, k \rrbracket$ and $n \geq 2$*

$$\mathbf{t}(j_i, n) \geq \mathbf{t}(j_{i+1}, n - 1), \quad (2.14)$$

6. *the sequence (k_n) is decreasing until it reaches 0. In particular $\max N_{j_i} \leq k + 1$.*

Proof. We assume that $E := \{x \in \llbracket 1, k \rrbracket \mid j_x \in I_n\}$ is non-empty. Let $y = |E|$. For $x \in \llbracket 1, y \rrbracket$, let $\tilde{t}_x \in \llbracket 2, L \rrbracket$ be such that $|\{x' \in \llbracket 1, k \rrbracket \mid N_{j_{x'}}(\tilde{t}_x) \geq n\}| \geq x$ and such that \tilde{t}_x is minimal for this property. By definition, $\tilde{t}_1 < \tilde{t}_2 < \dots < \tilde{t}_y$.

Let $x \in \llbracket 1, k \rrbracket$ be such that $N_{j_x}(\tilde{t}_1) = n$. We then have $N_{j_x}(\tilde{t}_1 - 1) = n - 1$ and $\alpha_{j_x}(\gamma_{\tilde{t}_1 - 1}^\vee) = -1$. Assume $x \in \llbracket 2, k' - 1 \rrbracket$. Then by Lemma 2.12 we have $\text{Supp}(j_x) = \{j_{x-1}, j_x, j_{x+1}\}$ and $a_{j_{x-1}, j_x} = -1 = a_{j_{x+1}, j_x}$. We have $N_{j_{x-1}}(\tilde{t}_1 - 1), N_{j_{x+1}}(\tilde{t}_1 - 1) \leq n - 1$. Then:

$$\begin{aligned} \alpha_{j_x}(\gamma_{\tilde{t}_1 - 1}^\vee) &= \alpha_{j_x} \left(N_{j_{x-1}}(\tilde{t}_1 - 1) \alpha_{j_{x-1}}^\vee + N_{j_x}(\tilde{t}_1 - 1) \alpha_{j_x}^\vee + N_{j_{x+1}}(\tilde{t}_1 - 1) \alpha_{j_{x+1}}^\vee \right) \\ &\geq 2(n - 1) - 2(n - 1) = 0 : \end{aligned}$$

a contradiction. Therefore $x \notin \llbracket 2, k' - 1 \rrbracket$. Assume $k = k'$ and $x = k$. Then $N_{j_{x-1}}(\tilde{t}_1 - 1) \leq n - 1$. By (2.11), $a_{j_{k-1}, j_k} = -1$ and we reach a contradiction again. Therefore, $x = 1$.

Let $x \in \llbracket 2, y \rrbracket$ and $i \in \llbracket 1, k \rrbracket$ be such that $N_{j_i}(\tilde{t}_x - 1) = n - 1 = N_{j_i}(\tilde{t}_x) - 1$. We assume that for all $x' \in \llbracket 1, x - 1 \rrbracket$,

$$\{z \in \llbracket 1, k \rrbracket \mid N_{j_z}(\tilde{t}_{x'}) \geq n\} = \llbracket 1, x' \rrbracket \text{ and } \{z \in \llbracket 1, k \rrbracket \mid N_{j_z}(\tilde{t}_{x'}) \leq n - 1\} = \llbracket x' + 1, k \rrbracket.$$

Then $i \geq x$. Assume $i \in \llbracket x + 1, y \rrbracket$ and $i < k'$. Then:

$$\alpha_{j_i}(\gamma_{\tilde{t}_x - 1}^\vee) = -1 = \alpha_{j_i} \left(N_{j_{i-1}}(\tilde{t}_x - 1) \alpha_{j_{i-1}}^\vee + N_{j_i}(\tilde{t}_x - 1) \alpha_{j_i}^\vee + N_{j_{i+1}}(\tilde{t}_x - 1) \alpha_{j_{i+1}}^\vee \right). \quad (2.15)$$

By choice of i , $N_{j_i}(\tilde{t}_x - 1) = n - 1$ and by our assumption $N_{j_{i-1}}(\tilde{t}_x - 1), N_{j_{i+1}}(\tilde{t}_x - 1) \leq n - 1$. Therefore $\alpha_{j_i}(\gamma_{\tilde{t}_x - 1}^\vee) \geq 2(n - 1) - 2(n - 1) = 0$: a contradiction. Therefore $i = x$ (the case $i = k' = k$ is similar). Thus we proved that for every $x \in \llbracket 1, y \rrbracket$, we have:

$$\{z \in \llbracket 1, k \rrbracket \mid N_{j_z}(\tilde{t}_x) \geq n\} = \llbracket 1, x \rrbracket \text{ and } \{z \in \llbracket 1, k \rrbracket \mid N_{j_z}(\tilde{t}_x) \leq n - 1\} = \llbracket x + 1, k \rrbracket.$$

In other words, $\mathbf{t}(j_1, n) < \mathbf{t}(j_2, n) < \dots < \mathbf{t}(j_y, n)$. As $a_{j_z, j_{z+1}} \neq 0$ for every $z \in \llbracket 1, k-1 \rrbracket$, we have $j_1 \triangleleft_n j_2 \triangleleft_n \dots \triangleleft_n j_y$. We thus set $k_n = y = |\{i \in \llbracket 1, k \rrbracket \mid j_i \in I_n\}|$ and we have 1) and 2). Then 3) is clear since $I_n \subset I_{n-1}$.

We prove (2.14) by iteration on i , as the assumption corresponds to the case $i = 1$. Let $i \in \llbracket 1, k-1 \rrbracket$ and suppose that (2.14) is satisfied for $(i-1, n+1)$, that is to say $\mathbf{t}(j_{i-1}, n+1) \geq \mathbf{t}(j_i, n)$. Let $t = \mathbf{t}(j_i, n)$. If it is infinite, then (2.14) is clear, so we suppose it is finite. Then $n \leq N_{j_{i-1}}(t)$, since $j_{i-1} \triangleleft_n j_i$ by 2). We also have $N_{j_{i-1}}(t) < n+1$ by assumption, so $N_{j_{i-1}}(t) = n$. Therefore $-1 = \alpha_{j_i}(\gamma_t^\vee) = 2N_{j_i}(t) - N_{j_{i-1}}(t) - N_{j_{i+1}}(t) = 2(n-1) - n - N_{j_{i+1}}(t)$, so $N_{j_{i+1}}(t) = n-1$. Thus $t \geq \mathbf{t}(j_{i+1}, n-1)$, which ends the recursion. For $(i, n) = (k, 2)$, since $\mathbf{t}(j_k, 2)$ is finite (because $j_k \in I_2$) the same computation proves that $N_{j_{k+1}}(\mathbf{t}(j_k, 2)) = 1$, hence $|\text{Supp}(j_k)| = 3$. Thus we have proved 4) and 5). In particular, if $\mathbf{t}(j_{i+1}, n-1) = \infty$ then $\mathbf{t}(j_i, n) = \infty$. If $k_n \neq 0$, applying this to $(i, n) = (k_n, n+1)$ we deduce that $j_{k_n} \notin I_{n+1}$, so $k_{n+1} < k_n$. Therefore the sequence (k_n) is decreasing until it reaches zero, $k_{k+2} = 0$ and $j_1 \notin I_{k+2}$, which proves 5). \square

Remark 2.16. Let \mathcal{C} be a connected component of I_2 . Write $\mathcal{C} = \{j_1\} \sqcup \bigsqcup_{i=1}^m \{j_2^{(i)}, \dots, j_{k(i)}^{(i)}\}$, with the same notation as in Proposition 2.13. Then Lemma 2.15 implies that if $I_n \cap \mathcal{C} \neq \emptyset$, then there exists $(k_m(i))_{i \in \llbracket 1, m \rrbracket} \in \prod_{i=1}^m \llbracket 1, k(i) \rrbracket$ such that $\mathcal{C} \cap I_n = \{j_1\} \sqcup \bigsqcup_{i=1}^m \{j_2^{(i)}, \dots, j_{k_m(i)}^{(i)}\}$ (where $\{j_2^{(i)}, \dots, j_{k_m(i)}^{(i)}\} = \emptyset$, if $k_m(i) = 1$). Moreover, the order \leq_n is simply the restriction of \leq_2 to \leq_n , and it is independent on the choice of a reduced decomposition for β , by Remark 2.14.

2.5 Optimal bound on $\max N_{j_1}$

In this section, we fix $j_1 \in I_2$ minimal for \leq_2 , and we fix a maximal chain $j_1 \triangleleft_2 j_2 \triangleleft_2 \dots \triangleleft_2 j_k$ as in Lemma 2.12. In particular, k always denote the length of a maximal \leq_2 -chain starting at j_1 .

By Lemma 2.12 1), we have $\deg(j_1) = 3$, therefore $|\text{Supp}(j_1)| \in \{2, 3, 4\}$. We treat each case separately, to obtain a bound on $\max N_{j_1}$ which depends on k . Along the way, we obtain more precise information on the \leq_n -chains starting at j_1 . Beforehand, let us give a technical lemma on the difference $N_{j_k} - N_{j_{k-1}}$.

Lemma 2.17. Let $t \in \llbracket L_1 + 1, L \rrbracket$ be such that $N_{j_k}(t) \geq 2$. Then:

1. If $\text{Supp}(j_k) = \{j_{k-1}, j_k, j_{k+1}\}$ (i.e if $k' = k+1$), then $2N_{j_k}(t) \leq N_{j_{k-1}}(t) + 2$. If moreover $N_{j_k}(t) < \max N_{j_k}$, then $N_{j_{k-1}}(t) \leq 2N_{j_k}(t)$.
2. If $\text{Supp}(j_k) = \{j_{k-1}, j_k\}$ (i.e if $k' = k$), then $2N_{j_k}(t) \leq N_{j_{k-1}}(t) + 1$. If moreover $N_{j_k}(t) < \max N_{j_k}$, then $2N_{j_k}(t) \geq N_{j_{k-1}}(t) - 1$. In particular, if $N_{j_{k-1}}(t)$ is even, $N_{j_k}(t) = \frac{1}{2}N_{j_{k-1}}(t)$ and if $N_{j_{k-1}}(t)$ is odd, $N_{j_k}(t) \in \left\{ \frac{N_{j_{k-1}}(t)-1}{2}, \frac{N_{j_{k-1}}(t)+1}{2} \right\}$.

Proof. Let $t' = \mathbf{t}(j_k, N_{j_k}(t))$ and let $t'' = \mathbf{t}(j_k, N_{j_k}(t) + 1)$, if $N_{j_k}(t) < \max N_{j_k}$. We have $t' < t \leq t''$.

1. If $\text{Supp}(j_k) = \{j_{k-1}, j_k, j_{k+1}\}$, by Lemma 2.12 we have $j_{k+1} \in I_1 \setminus I_2$ and thus $N_{j_{k+1}}(t') = 1$. We also have $2N_{j_k}(t') = N_{j_{k-1}}(t') + N_{j_{k+1}}(t') - 1 = N_{j_{k-1}}(t')$ and thus $2N_{j_k}(t) - 2 = N_{j_{k-1}}(t') \leq N_{j_{k-1}}(t)$, since $N_{j_{k-1}}$ is non-decreasing. We also have $2N_{j_k}(t'') = 2N_{j_k}(t) = N_{j_{k-1}}(t'') \geq N_{j_{k-1}}(t)$, which proves 1).

2. If $\text{Supp}(j_k) = \{j_{k-1}, j_k\}$, we have $2N_{j_k}(t') = N_{j_{k-1}}(t') - 1$ and thus $2(N_{j_k}(t) - 1) = 2N_{j_k}(t') = N_{j_{k-1}}(t') - 1 \leq N_{j_{k-1}}(t) - 1$. Therefore $2N_{j_k}(t) \leq N_{j_{k-1}}(t) + 1$.

We also have $2N_{j_k}(t'') = N_{j_{k-1}}(t'') - 1$ and thus $2N_{j_k}(t) \geq N_{j_{k-1}}(t) - 1$. Consequently we have

$$N_{j_{k-1}}(t) - 1 \leq 2N_{j_k}(t) \leq N_{j_{k-1}}(t) + 1,$$

which proves 2).

□

We can now obtain a bound on N_{j_1} .

2.5.1 Case where $|\text{Supp}(j_1)| \leq 3$

Lemma 2.18. *Let $j_1 \triangleleft_2 j_2 \triangleleft_2 \cdots \triangleleft_2 j_k \in I_2$ be as in Lemma 2.12. We assume that $\text{Supp}(j_1) = \{j_1, j_2\}$. Then $\max N_{j_1} \leq 2$.*

Proof. By Lemma 2.12, $\deg(j_1) = -a_{j_2, j_1} = 3$. Assume $j_1 \in I_3$ and set $t = \mathbf{t}(j_1, 3)$. Then

$$\alpha_{j_1}(\gamma_t^\vee) = \alpha_{j_1}(2\alpha_{j_1}^\vee + N_{j_2}(t)\alpha_{j_2}^\vee) = 4 - 3N_{j_2}(t) = -1 :$$

this is impossible. Lemma follows. □

Lemma 2.19. *Let $j_1 \in I_2$ be minimal for \leq_2 such that $|\text{Supp}(j_1)| = 3$. Then $\max N_{j_1} \leq k + 1$.*

More precisely, write $\text{Supp}(j_1) = \{j_1, j_2, j_2'\}$ with $a_{j_2', j_1} = -1$ and $a_{j_2, j_1} = -2$. Then:

1. if $\mathbf{t}(j_2', 2) \leq \mathbf{t}(j_1, 3)$ then $\mathbf{t}(j_1, 3) = \infty$,
2. if $\mathbf{t}(j_1, 3) < \mathbf{t}(j_2', 2) \neq \infty$ then $\text{Supp}(j_2') = \{j_1, j_2'\}$ and $\mathbf{t}(j_1, 4) = \mathbf{t}(j_2', 3) = \infty$,
3. else $j_2' \notin I_2$ and, for all $n \geq 2$, $\mathbf{t}(j_2, n-1) \leq \mathbf{t}(j_1, n) \leq \mathbf{t}(j_2, n)$. The chain $j_1 \triangleleft_2 j_2 \triangleleft_2 \cdots \triangleleft_2 j_k$ then satisfies properties 4-6 of Lemma 2.15 and in particular $\max N_{j_1} \leq k+1$.

Proof. Note that, by minimality of j_1 for \leq_2 , $\mathbf{t}(j_1, 2) \leq \mathbf{t}(j_2', 2)$ and $\mathbf{t}(j_1, 2) \leq \mathbf{t}(j_2, 2)$. Moreover by Lemma 2.15, the same holds replacing 2 by $n \geq 2$.

Suppose first that $\mathbf{t}(j_2', 2) \leq \mathbf{t}(j_1, 3)$. If $\mathbf{t}(j_2', 2) = \infty$ then $\mathbf{t}(j_1, 3) = \infty$ is clear. Else, for $t \geq \mathbf{t}(j_2', 2) + 1$ such that $N_{j_1}(t) = 2$, $\alpha_{j_1}(\gamma_t^\vee)$ is even and therefore $\mathbf{t}(j_1, 3) = \infty$.

Suppose now that $\mathbf{t}(j_1, 3) < \mathbf{t}(j_2', 2) \neq \infty$. Then at time $t = \mathbf{t}(j_2', 2)$, we have $-1 = \alpha_{j_2'}(\gamma_t^\vee) \geq 2 - N_{j_1}(t) \geq 2 - 3$. Therefore $N_{j_1}(t) = 3$ and there is no other element in $\text{Supp}(j_2')$. In particular $\mathbf{t}(j_2', 2) < \mathbf{t}(j_1, 4) \leq \mathbf{t}(j_2', 3)$. Let $\mathbf{t}(j_2', 2) < t \leq \mathbf{t}(j_1, 4)$ be a finite time. Then

$N_{j'_2}(t) = 2$ and $\alpha_{j_1}(\gamma_t^\vee) = 2N_{j_1}(t) - 2N_{j_2}(t) - 2$ is even, so $\mathbf{t}(j_1, 4) = \infty$. Therefore $\max N_{j_1} \leq 3$. Moreover $k \geq 2$, since $|\text{Supp}(j_1)| > 1$, which proves that $\max N_{j_1} \leq k + 1$.

Finally, suppose that $\mathbf{t}(j'_2, 2) = \infty$, that is to say $j'_2 \notin I_2$. Let $n \geq 2$ and suppose that $t := \mathbf{t}(j_1, n)$ is finite. Then $-1 = \alpha_{j_1}(\gamma_t^\vee) = 2(n-1) - 2N_{j_2}(t) - 1$, so $N_{j_2}(t) = n-1$. Hence $\mathbf{t}(j_2, n-1) \leq \mathbf{t}(j_1, n)$. The second inequality is given by Lemma 2.15 2). By Lemma 2.15 6), we deduce $\max N_{j_1} \leq k + 1$. \square

2.5.2 Case where $|\text{Supp}(j_1)| = 4$

Proposition 2.20. *Let $j_1 \in I_2$ be minimal for \leq_2 such that $\text{Supp}(j_1) = 4$. Let k denote the maximal length of a \leq_2 chain starting at j_1 . Then $\max N_{j_1} \leq \max(6, k + 1)$.*

Proof. If $j_1 \notin I_3$ then $\max N_{j_1} \leq 2$ and the result is clear, so we can assume $j_1 \in I_3$. If $|\text{Supp}(j_1)| = 4$ and $\text{Supp}(j_1) \not\subset I_2$, then by Lemma 2.21 $\max N_{j_1} \leq k + 1$. Else, since we assume $j_1 \in I_3$, either there is $j \in \text{Supp}(j_1)$ such that $\text{Supp}(j) \not\subset I_2$ and, by Lemma 2.23, $\max N_{j_1} \leq k + 1$; either $\text{Supp}(j) \subset I_2$ for all $j \in \text{Supp}(j_1)$ and, by Lemma 2.24, we have $\max N_{j_1} \leq 6$. \square

Lemma 2.21. *Let $j_1 \in I_2$ be minimal for \leq_2 such that $|\text{Supp}(j_1)| = 4$ and suppose $\text{Supp}(j_1) \not\subset I_2$. Then, any maximal chain $j_1 \triangleleft_2 j_2 \cdots \triangleleft_2 j_k$ with $k \geq 2$ satisfies properties 4-5-6 of Lemma 2.15, and in particular $\max N_{j_1} \leq k + 1$.*

Proof. Let us write $\text{Supp}(j_1) = \{j_1, j_2, j'_2, j''_2\}$ with $j''_2 \notin I_2$. By Lemma 2.15, to prove the result it suffices to prove that $\mathbf{t}(j_1, n) \geq \mathbf{t}(j_2, n-1)$ for all $n \geq 2$.

If $\mathbf{t}(j_1, n) = \infty$ then the inequality is clear, so suppose $t = \mathbf{t}(j_1, n) \neq \infty$. Since $N_{j''_2}(t) = 1$ we have

$$-1 = \alpha_{j_1}(\gamma_t^\vee) = 2N_{j_1}(t) - N_{j_2}(t) - N_{j'_2}(t) - N_{j''_2}(t) = 2(n-1) - 1 - N_{j_2}(t) - N_{j'_2}(t),$$

so $N_{j_2}(t) + N_{j'_2}(t) = 2(n-1)$. Moreover, $\max(N_{j_2}(t), N_{j'_2}(t)) \leq n-1$, since, by Lemma 2.15 2), j_1 is also minimal for \leq_n . Therefore $N_{j_2}(t) + N_{j'_2}(t) = 2(n-1)$ implies $N_{j_2}(t) = N_{j'_2}(t) = n-1$, which proves $\mathbf{t}(j_1, n) = t > \mathbf{t}(j_2, n-1)$. \square

Lemma 2.22. *Let $j_1 \in I_2$ be minimal for \leq_2 such that $|\text{Supp}(j_1)| = 4$, $\text{Supp}(j_1) \subset I_2$ and $j_1 \in I_3$. Then there is a unique $j \in \text{Supp}(j_1)$ such that $|\text{Supp}(j)| = 2$, and for any $j_2 \in \text{Supp}(j_1) \setminus \{j_1, j\}$, we have $\text{Supp}(j_2) = 3$ and*

$$\mathbf{t}(j_2, 2) < \mathbf{t}(j_1, 3) < \mathbf{t}(j, 2) \leq \mathbf{t}(j_1, 4). \quad (2.16)$$

Proof. Write $\text{Supp}(j_1) = \{j_2^{(1)}, j_2^{(2)}, j_2^{(3)}\}$. Fix $i \in \llbracket 1, 3 \rrbracket$. Let $t = \mathbf{t}(j_1, 3) \neq \infty$. Since j_1 is minimal for \leq_2 , it is also minimal for \leq_3 . Then $N_{j_2^{(1)}}(t) + N_{j_2^{(2)}}(t) + N_{j_2^{(3)}}(t) = 5$ and $N_{j_2^{(i)}}(t) \leq 2$ for all i , so upon relabelling, we can suppose $(N_{j_2^{(1)}}(t), N_{j_2^{(2)}}(t), N_{j_2^{(3)}}(t)) = (2, 2, 1)$.

For $i \in \{1, 2\}$, $t_i := \mathbf{t}(j_2^{(i)}, 2)$ satisfies $t_i < t$ and $-1 = \alpha_{j_2^{(i)}}(\gamma_{t_i}^\vee) < 0 = 2N_{j_2^{(i)}}(t_i) - N_{j_1}(t_i)$. Therefore $\text{Supp}(j_2^{(i)})$ admits exactly one other element, hence is of cardinality 3. Let now $t = \mathbf{t}(j_2^{(3)}) < \mathbf{t}(j_1, 3)$. Then $\alpha_{j_2^{(3)}}(\gamma_t^\vee) = -1$ and $2N_{j_2^{(3)}}(t) - N_{j_1}(t) \leq 2 - 3 = -1$ so $N_{j_1}(t) = 3$ and $\text{Supp}(j_2^{(3)}) = \{j_1, j_2^{(3)}\}$, which concludes the proof. \square

Lemma 2.23. *Let $j_1 \in I_2$ be minimal for \leq_2 such that $|\text{Supp}(j_1)| = 4$, $\text{Supp}(j_1) \subset I_2$ and $j_1 \in I_3$. Let j_2'' be the unique element of $\text{Supp}(j_1)$ with $|\text{Supp}(j_2'')| = 2$, given by Lemma 2.22.*

Suppose that there exists $j_2' \in \text{Supp}(j_1)$ with $\text{Supp}(j_2') \not\subset I_2$. Then the following properties are satisfied:

1. For any $n \geq 1$,

$$\mathbf{t}(j_1, 2n) \leq \mathbf{t}(j_2', n+1) \leq \mathbf{t}(j_1, 2n+1) \leq \mathbf{t}(j_2'', n+1) \leq \mathbf{t}(j_1, 2n+2). \quad (2.17)$$

2. The unique maximal chain $j_1 \triangleleft_2 j_2 \cdots \triangleleft_2 j_k$ such that $j_2 \notin \{j_2', j_2''\}$ satisfies properties 4-5-6 of Lemma 2.15, and in particular $\max N_{j_1} \leq k+1$.

Proof. By assumption, $\text{Supp}(j_2'') = \{j_1, j_2''\}$ and $\text{Supp}(j_2') \not\subset I_2$, hence both j_2' , j_2'' are maximal for \leq_2 . Let j_2 denote the remaining element of $\text{Supp}(j_1)$.

Fix $n \geq 2$. Since j_2' is maximal for \leq_2 and has support of cardinality 3, Lemma 2.17 applied to $t := \mathbf{t}(j_2', n+1) + 1$ (if it is finite, otherwise it is clear) gives $\mathbf{t}(j_1, 2n) \leq \mathbf{t}(j_2', n+1)$. Similarly, point 2 of the same Lemma applied to $t := \mathbf{t}(j_2'', n+1) + 1$ gives $\mathbf{t}(j_1, 2n+1) \leq \mathbf{t}(j_2'', n+1)$.

Let $t = \mathbf{t}(j_1, 2n+1)$, which we suppose finite. Then $N_{j_1}(t) = 2n$, $N_{j_2'}(t) \leq n$ and, since j_1 is minimal for \triangleleft_{2n+1} , $N_{j_2}(t) \leq 2n$. Therefore $-1 = \alpha_{j_1}(\gamma_t^\vee) \geq 4n - n - 2n - N_{j_2}(t) = n - N_{j_2}(t)$, so $N_{j_2}(t) \geq n+1$ and $t > \mathbf{t}(j_2', n+1)$.

Let $t = \mathbf{t}(j_1, 2n+2)$ which we suppose finite, then $N_{j_1}(t) = 2n+1$, $N_{j_2'}(t) \leq n+1$ (by Lemma 2.17) and $N_{j_2}(t) \leq 2n+1$. Therefore $-1 = \alpha_{j_1}(\gamma_t^\vee) \geq 4n+2 - (2n+1) - (n+1) - N_{j_2}(t) = n - N_{j_2}(t)$, so $N_{j_2}(t) \geq n+1$ and $\mathbf{t}(j_2'', n+1) \leq t$. This concludes the proof of Formula (2.17).

Let us check that $\mathbf{t}(j_1, n) \geq \mathbf{t}(j_2, n-1) \geq \mathbf{t}(j_1, n-1)$ for any $n \geq 3$. The inequality $\mathbf{t}(j_2, n-1) \geq \mathbf{t}(j_1, n-1)$ holds because $j_1 \triangleleft_2 j_2 \implies j_1 \triangleleft_{n-1} j_2$ by point 2) of Lemma 2.15 (if they belong to I_{n-1}). For the remaining inequality, suppose $t := \mathbf{t}(j_1, n) \neq \infty$:

- If n is even, write $n = 2p$. Then by Formula (2.17), $\max(N_{j_2'}(t), N_{j_2''}(t)) \leq p$, so $-1 = \alpha_{j_1}(\gamma_t^\vee) \geq 2(2p-1) - 2p - N_{j_2}(t)$. We deduce $N_{j_2}(t) \geq n-1$.
- Else, write $n = 2p+1$. Then similarly $N_{j_2''}(t) \leq p$ and $N_{j_2'}(t) \leq p+1$, so $-1 = \alpha_{j_1}(\gamma_t^\vee) \geq 2(2p) - 2p - 1 - N_{j_2}(t)$, so $N_{j_2}(t) \geq 2p = n-1$.

Either way, we have proved $\mathbf{t}(j_1, n) \geq \mathbf{t}(j_2, n-1)$. Then 2) follows from Lemma 2.15. \square

Lemma 2.24. *Let $j_1 \in I_2$ be minimal for \leq_2 with $|\text{Supp}(j_1)| = 4$, $j_1 \in I_3$ and, for all $j \in \text{Supp}(j_1)$, $\text{Supp}(j) \subset I_2$. Then $\max N_{j_1} \leq 6$.*

More precisely, using Lemma 2.22, we write $\text{Supp}(j_1) = \{j_1, j_2, j'_2, j''_2\}$ such that $|\text{Supp}(j''_2)| = 2$ and $\mathbf{t}(j_2, 3) \leq \mathbf{t}(j'_2, 3)$, and write $\text{Supp}(j'_2) = \{j_1, j'_2, j'_3\}$. Then:

1. *If $\mathbf{t}(j'_2, 3) \leq \mathbf{t}(j_1, 4)$, then $j_1 \notin I_4$ and $j''_2 \notin I_3$*
2. *If $\mathbf{t}(j_1, 4) \neq \infty$ and $\mathbf{t}(j'_2, 3) = \infty$, then $j_1 \notin I_5$ and $j''_2 \notin I_3$*
3. *Else if $\mathbf{t}(j_1, 4) < \mathbf{t}(j'_2, 3) < \infty$, then $\mathbf{t}(j_2, n - 1) \leq \mathbf{t}(j_1, n)$ for all $n \in \llbracket 2, 5 \rrbracket$, $\text{Supp}(j'_3) = \{j'_2, j'_3\}$ and $j_1 \notin I_7$, $j'_2 \notin I_5$, $j'_3 \notin I_3$.*

Proof. Suppose first that $\mathbf{t}(j'_2, 3) \leq \mathbf{t}(j_1, 4)$ and assume by contradiction that $t := \mathbf{t}(j_1, 4)$ is finite. By Lemma 2.22, we have $N_{j''_2}(t) \geq 2$, and since $\mathbf{t}(j_2, 3) \leq \mathbf{t}(j'_2, 3) \leq \mathbf{t}(j_1, 4)$, we have $N_{j_2}(t) \geq 3$, $N_{j'_2}(t) \geq 3$. Therefore $N_{j_2}(t) + N_{j'_2}(t) + N_{j''_2}(t) \geq 8$, so $-1 = \alpha_{j_1}(\gamma_t^\vee) \leq -2$, a contradiction. Hence in this case, $\mathbf{t}(j_1, 4) = \infty$ and $j_1 \notin I_4$.

Suppose now that $\mathbf{t}(j_1, 4) \neq \infty$ and $\mathbf{t}(j'_2, 3) = \infty$ and let $t \geq \mathbf{t}(j_1, 4) + 1$. Then $N_{j_1}(t) = 4$ so $N_{j_2}(t) \leq 4$ and $N_{j''_2}(t) \leq 2$ by Lemma 2.17. Moreover, by assumption $N_{j'_2}(t) \leq 2$ and thus $\alpha_{j_1}(\gamma_t^\vee) \geq 8 - 4 - 2 - 2 = 0$ can not be equal to -1 , so N_{j_1} is blocked at 4: $j_1 \notin I_5$. Moreover, in these two cases, $N_{j_1} \leq 4$ and thus by Lemma 2.17, $N_{j''_2} \leq 2$.

Finally, assume that $\mathbf{t}(j_1, 4) < \mathbf{t}(j'_2, 3) \neq \infty$. Note that $\mathbf{t}(j_2, 3) < \mathbf{t}(j_1, 4)$ is necessary in order to have $\alpha_{j_1}(\gamma_{\mathbf{t}(j_1, 4)}^\vee) = -1$. Let $t = \mathbf{t}(j'_2, 3)$, by assumption $N_{j_1}(t) \geq 4$ and $N_{j'_3}(t) \geq 1$. So $\alpha_{j'_2}(\gamma_t^\vee) = -1$ enforces that $N_{j_1}(t) = 4$ and $N_{j'_3}(t) = 1$. The first consequence is that $\mathbf{t}(j'_2, 3) < \mathbf{t}(j'_3, 2)$. Since we suppose $\text{Supp}(j'_2) \subset I_2$, computation at time $\mathbf{t}(j'_3, 2)$ then proves that

$$\text{Supp}(j'_3) = \{j'_2, j'_3\}.$$

We then have

$$j'_3 \in I_3 \implies j'_2 \in I_5 \implies j_1 \in I_7.$$

Indeed, the first implication is a direct consequence of Lemma 2.17 2). Now assume that $t := \mathbf{t}(j'_2, 5) < \infty$. Then by Lemma 2.17, we have $N_{j'_3}(t) \leq 2$ and thus $\alpha_{j'_2}(\gamma_t^\vee) = -1 = 8 - N_{j_1}(t) - N_{j'_3}(t)$ and so $N_{j_1}(t) \geq 7$ and

$$\mathbf{t}(j_1, 7) \leq \mathbf{t}(j'_2, 5). \tag{2.18}$$

We now show that, under the assumptions of the statement, $j_1 \notin I_7$, which is enough to conclude.

We computed $N_{j_1}(\mathbf{t}(j'_2, 3)) = 4$ and $N_{j'_3}(\mathbf{t}(j'_2, 3)) = 1$, so $\mathbf{t}(j'_2, 3) < \mathbf{t}(j_1, 5)$. Let $t = \mathbf{t}(j_1, 5)$ and suppose it is finite. For $\mathbf{t}(j'_2, 3) < t' \leq t$ we have $\alpha_{j'_2}(\gamma_{t'}^\vee) \geq 6 - 4 - 2 \geq 0$, so $N_{j'_2}(t) = 3$. Moreover since $N_{j_1}(t) = 4$, we have $N_{j''_2}(t) = 2$ and $N_{j_2}(t) \leq 4$. Therefore $\alpha_{j_1}(\gamma_t^\vee) = -1$ enforces that $N_{j_2}(t) = 4$. This proves $\mathbf{t}(j_2, n - 1) \leq \mathbf{t}(j_1, n)$ for $n = 5$. For $n = 2$ it is clear, for $n = 3$ it is given by Lemma 2.22 and we have already checked it for $n = 4$, hence the first statement of point 3 is checked. Moreover at time $t' = t + 1$ we have

$$(N_{j_1}(t'), N_{j_2}(t'), N_{j'_2}(t'), N_{j''_2}(t')) = (5, 4, 3, 2). \tag{2.19}$$

Now set $t = \mathbf{t}(j_1, 6)$. If it is finite, then at this time exactly two out of the three values $N_{j_2}, N_{j'_2}, N_{j''_2}$ have increased by one. Indeed $N_{j_2} + N_{j'_2} + N_{j''_2}$ need to increase by two. By Lemma 2.17, $N_{j_2} \leq N_{j_1}$ and $2N_{j''_2} \leq N_{j_1} + 1$ so these two terms can not increase by two. Moreover, by the same lemma, as long as $N_{j'_2} \leq 4$, $N_{j'_3} \leq 2$. Thus if $t' = \mathbf{t}(j'_2, 5)$, we have $\alpha_{j'_2}(\gamma_{t'}^\vee) = -1 = 8 - N_{j_1}(t') - N_{j'_3}(t') \geq 6 - N_{j_1}(t')$, so $t' \geq \mathbf{t}(j_1, 7) \geq t$ and $N_{j'_2}(t) \leq 4$: $N_{j'_2}$ can not increase by two either. Thus two out of the three values $N_{j_2}, N_{j'_2}, N_{j''_2}$ have increased by one, we can actually check that all the three cases are possible. We deal with each case separately.

1. Suppose that $(N_{j_2}(t), N_{j'_2}(t), N_{j''_2}(t)) = (5, 3, 3)$. Then at any time t' such that $\mathbf{t}(j'_2, 4) \geq t' > t$ we have $\alpha_{j'_2}(\gamma_{t'}^\vee) \leq 6 - 6 - 2 < -1$ by Lemma 2.17), so $\mathbf{t}(j'_2, 4) = \infty$. Therefore if $t' = \mathbf{t}(j_1, 7)$ is finite, necessarily $(N_{j_1}(t'), N_{j'_2}(t')) = (6, 4)$, which contradicts Lemma 2.17 2). Hence $\mathbf{t}(j_1, 7)$ is infinite and $j_1 \notin I_7$, which implies $j'_2 \notin I_5$, $j'_3 \notin I_3$.
2. Suppose that $(N_{j_2}(t), N_{j'_2}(t), N_{j''_2}(t)) = (5, 4, 2)$. The situation is similar: in this case $N_{j''_2}$ is stuck at 2, so $\mathbf{t}(j'_2, 5) \leq \mathbf{t}(j_1, 7)$. However by Lemma 2.17 2), $\mathbf{t}(j'_2, 5) \leq \mathbf{t}(j'_3, 3)$. We deduce that, for any $\mathbf{t}(j_1, 6) < t' \leq \mathbf{t}(j'_2, 5)$, $\alpha_{j'_2}(\gamma_{t'}^\vee) = 0$, so $\mathbf{t}(j'_2, 5) = \mathbf{t}(j_1, 7) = \mathbf{t}(j'_3, 3) = \infty$.
3. Suppose that $(N_{j_2}(t), N_{j'_2}(t), N_{j''_2}(t)) = (4, 4, 3)$. Then by (2.18) and Lemma 2.17, we have

$$\mathbf{t}(j_2, 6) \leq \mathbf{t}(j_1, 7) \leq \min(\mathbf{t}(j'_2, 5), \mathbf{t}(j''_2, 4)).$$

Hence if $t := \mathbf{t}(j_1, 7) < \infty$, we have $N_{j_2}(t) + N_{j'_2}(t) + N_{j''_2}(t) = 13$, with $N_{j'_2}(t) + N_{j''_2}(t) \leq 7$. Consequently $\mathbf{t}(j_2, 6) < \mathbf{t}(j_1, 7) < \infty$. But this contradicts Lemma 2.25 below. Therefore $j_1 \notin I_7$, $j'_2 \notin I_5$ and $j'_3 \notin I_3$.

□

Lemma 2.25. *Let j_1, j_2 be the first terms of a maximal chain for \leq_2 . Then there exists no couple $(x, y) \in \llbracket 1, L \rrbracket^2$ such that $(N_{j_1}(x), N_{j_2}(x)) = (6, 4)$ and $(N_{j_1}(y), N_{j_2}(y)) = (6, 6)$.*

Proof. We assume that there exists $(x, y) \in \llbracket 1, L \rrbracket^2$ such that $(N_{j_1}(x), N_{j_2}(x)) = (6, 4)$ and $(N_{j_1}(y), N_{j_2}(y)) = (6, 6)$. Let $z \in \llbracket 1, L \rrbracket$. Assume $(N_{j_1}(z), N_{j_2}(z)) = (6, 4)$. If we had $\text{Supp}(j_2) = \{j_1, j_2\}$, then j_2 would be maximal for \leq_2 and Lemma 2.17 (2) would lead to a contradiction. Therefore we have $\text{Supp}(j_2) = \{j_1, j_2, j_3\}$, for some $j_3 \in I_1$, and by Lemma 2.12, we have $a_{j_1, j_2} = a_{j_3, j_1} = -1$.

Let $t = \mathbf{t}(j_2, 6)$. We have $N_{j_1}(t) = 6$ by assumption. We have $10 - 6 - N_{j_3}(t) = -1$ and thus

$$\mathbf{t}(j_3, 4) < \mathbf{t}(j_3, 5) < \mathbf{t}(j_2, 6). \quad (2.20)$$

Let $t = \mathbf{t}(j_2, 5)$. We have $t < \mathbf{t}(j_1, 7)$ and thus we have $N_{j_3}(t) = 3$ (in order to have $8 - 6 - 3 = -1$). Therefore $(N_{j_1}(t+1), N_{j_2}(t+1), N_{j_3}(t+1)) = (6, 5, 3)$.

Let now $t \in \llbracket 1, L \rrbracket$ be such that $(N_{j_1}(t), N_{j_2}(t), N_{j_3}(t)) = (6, 5, 3)$. As $\mathbf{t}(j_3, 4) < \infty$ and $6 - 5 \neq -1$, we have $\text{Supp}(j_3) \neq \{j_2, j_3\}$. So by Lemma 2.12, we can write $\text{Supp}(j_3) =$

$\{j_2, j_3, j_4\}$, for some $j_4 \in I_1$, and we have $a_{j_2, j_3} = a_{j_4, j_2} = -1$. If $t = \mathbf{t}(j_3, 4)$, we have $N_{j_4}(t) = 2$ (since $6 - 5 - 2 = -1$). So we have

$$(N_{j_1}(t), N_{j_2}(t), N_{j_3}(t), N_{j_4}(t)) = (6, 5, 3, 2) \quad (2.21)$$

and

$$(N_{j_1}(t+1), N_{j_2}(t+1), N_{j_3}(t+1), N_{j_4}(t+1)) = (6, 5, 4, 2). \quad (2.22)$$

Let $t = \mathbf{t}(j_3, 5)$. By (2.20), we have $\gamma_t^\vee = 6\alpha_{j_1}^\vee + 5\alpha_{j_2}^\vee + 4\alpha_{j_3}^\vee + a\alpha_{j_4}^\vee + ?$, for some $a \in \llbracket 1, 4 \rrbracket$. We have $\alpha_{j_3}(\gamma_t^\vee) = 8 - 5 + a\alpha_{j_3}(\alpha_{j_4}^\vee) = -1$ and thus $a\alpha_{j_3}(\alpha_{j_4}^\vee) = a\alpha_{j_4, j_3} = -4$ and thus $a = 4$. Consequently

$$\mathbf{t}(j_4, 4) < \mathbf{t}(j_3, 5) < \infty.$$

Let $\tilde{t} = \mathbf{t}(j_4, 4)$. We have $2N_{j_4}(\tilde{t}) = 6 > N_{j_3}(\tilde{t}) + 1$ since $N_{j_3}(\tilde{t}) \leq 4$. By Lemma 2.17, this implies $\text{Supp}(j_4) \neq \{j_3, j_4\}$. Let j_5 be such that $\text{Supp}(j_4) = \{j_3, j_4, j_5\}$. Then $6 - N_{j_3}(\tilde{t}) - N_{j_5}(\tilde{t}) = -1 = 6 - 4 - N_{j_5}(\tilde{t})$ and thus

$$\mathbf{t}(j_5, 3) < \mathbf{t}(j_4, 4) < \infty. \quad (2.23)$$

Let $t = \mathbf{t}(j_4, 3)$. By (2.22), we have $4 - 4 - N_{j_5}(t) = -1$ and thus $N_{j_5}(t) = 1$. We thus have $(N_{j_1}(t), N_{j_2}(t), N_{j_3}(t), N_{j_4}(t), N_{j_5}(t)) = (6, 5, 4, 2, 1)$ and then $(N_{j_1}(t+1), N_{j_2}(t+1), N_{j_3}(t+1), N_{j_4}(t+1), N_{j_5}(t+1)) = (6, 5, 4, 3, 1)$.

By (2.23), $t := \mathbf{t}(j_5, 2) < \infty$. Then $\gamma_t^\vee = z\alpha_{j_4}^\vee + \alpha_{j_5}^\vee + ?$, for some $z \in \mathbb{Z}_{\geq 3}$. By Lemma 2.12 $a_{j_4, j_5} = -1$, thus $2 - z - \alpha_{j_5}(\alpha_{j_4}^\vee) = -1$ and hence $z = 3$ and $\alpha_{j_5}(\alpha_{j_4}^\vee) = 0$. This proves that $\text{Supp}(j_5) = \{j_4, j_5\}$. Then if $t = \mathbf{t}(j_5, 2)$, we have

$$(N_{j_1}(t), N_{j_2}(t), N_{j_3}(t), N_{j_4}(t), N_{j_5}(t)) = (6, 5, 4, 3, 1)$$

and

$$(N_{j_1}(t+1), N_{j_2}(t+1), N_{j_3}(t+1), N_{j_4}(t+1), N_{j_5}(t+1)) = (6, 5, 4, 3, 2).$$

But then $2N_{j_i}(t+1) - N_{j_{i+1}}(t+1) - N_{j_{i-1}}(t+1) = 0$ for every $i \in \llbracket 2, 4 \rrbracket$, and $2N_{j_5}(t+1) - N_{j_4}(t+1) = 1$. Therefore the situation can no longer evolve, which proves the lemma. \square

2.5.3 Conclusion: Finiteness of the set of quantum roots

With the preceding results, we are already able to give an upper bound on the number of quantum roots in a given Kac-Moody root system. In particular, we prove that the set $\mathcal{Q}(\Phi_+)$ is finite for any Kac-Moody root system.

Proposition 2.26. *Let k_{\max} be the length of a maximal chain for \leq_2 and $n = |I|$ (we have $k_{\max} \leq n$). Let β be a quantum root. Write $\beta = \sum_{i \in I} x_i \alpha_i$. Then we have $x_i \leq \max(k_{\max} + 1, 6)$ for every $i \in I$. In particular, $\text{ht}(\beta^\vee) \leq n(\max(6, k_{\max} + 1))$.*

Proof. Let $i \in I$. If $i \notin I_2$, we have $x_i \leq 1$. Assume $i \in I_2$. Let $j \in I_2$ be such that $j \leq_2 i$ and such that j is minimal for \leq_2 . Then by (2.13), we have $x_i \leq x_j$. Moreover, by Lemma 2.19, Lemma 2.18 and Proposition 2.20, $x_j \leq \max(6, k_{\max} + 1)$. Therefore $\text{ht}(\beta^\vee) = \sum_{i' \in I_1} x_{i'} \leq |I_1| \max(k_{\max} + 1, 6) \leq n \max(k_{\max} + 1, 6)$. \square

Theorem 2.27. *Let $n \in \mathbb{Z}_{\geq 2}$. Then for every Kac-Moody matrix A of size n , if Φ is the real root system associated to A , then the set of quantum roots is finite, with cardinality at most n^{n+5} .*

Proof. Let $\mathcal{Q}(\Phi_+)$ be the set of quantum roots. By Proposition 2.26, for every $\beta \in \mathcal{Q}(\Phi_+)$, we can write $\beta^\vee = \sum_{i \in I} x_i \alpha_i$, with $x_i \in \llbracket 0, n+5 \rrbracket$ for every $i \in I$. Therefore $|\mathcal{Q}(\Phi_+)| = |\{\beta^\vee \mid \beta \in \mathcal{Q}(\Phi_+)\}| \leq n^{n+5}$. \square

Remark 2.28. 1. *Note that the bound given in Theorem 2.27 is very rough and is never sharp. A way to improve it would be to first count the number of possible 1-star convex subtrees of I (whose cardinality is at most $2^{|I|}$). Then for a given such subtree I_1 of I , the constraints of Lemma 2.12, Lemma 2.15 and Proposition 2.26 should enable to give a much better majoration of the number of quantum β satisfying $I_1(\beta) = I_1$. Moreover if a Kac-Moody matrix has few -1 coefficients, then it will have few quantum roots, see Lemma 2.41 for example.*

2. *Note that in the reductive case, in ADE type, the set of quantum root is the entire set Φ_+ (see Proposition 2.37). On the contrary, when Φ_+ is infinite, “almost all” the roots are not quantum, by the theorem above.*

2.6 Classification of quantum roots through Dynkin sequences

In Subsections 2.5.1 and 2.5.2 we have obtained finer structure results on quantum roots than what we used to prove finiteness of $\mathcal{Q}(\Phi_+)$. In this section, we reformulate these results in terms of Dynkin sequences. We then obtain a complete classification of quantum roots through Dynkin sequences (see Theorem 2.36). This section is independent of Section 3.

2.6.1 Standard Dynkin diagrams

There is a name for certain classes of Dynkin diagrams, which we now give. Most of our notation follow the standard classification of Dynkin diagrams (see e.g [Bou81]). In this section, Γ is a Dynkin diagram with set of edges E and weight function w . Recall that, if $A = (a_{i,j})_{i,j \in I}$ is a generalized Cartan matrix, then the associated Dynkin diagram is $\Gamma = (I, E, w)$ with $E = \{(i, j) \in I^2 \mid a_{i,j} < 0\}$ and $w(i, j) = -a_{i,j}$ for all $(i, j) \in E$. We identify a Dynkin diagram Γ with its set of vertices.

Amongst Dynkin segments, we have the following classes:

- (A) For $n \geq 1$, we say that Γ is a Dynkin diagram of type A_n if it is a segment with n vertices such that $w(i, j) = 1$ for all $(i, j) \in E$.
- (C) For $n \geq 2$, we say that Γ is a Dynkin diagram of type C_n if it is a segment with a leaf j_0 such that $\Gamma \setminus \{j_0\}$ is a diagram of type A_{n-1} and, for j_1 the unique vertex adjacent to j_0 , $(w(j_0, j_1), w(j_1, j_0)) = (1, 2)$. We call j_0 the **special vertex** of Γ .

- (F) For $n \geq 4$, we say that Γ is a Dynkin diagram of type F_n if it is a Dynkin segment with a leaf j_{-1} such that $\Gamma \setminus \{j_{-1}\}$ is a Dynkin diagram of type C_{n-1} and its special vertex j_0 is the unique neighbour of j_{-1} .
- (G) For $n \geq 2$, we say that Γ is a Dynkin diagram of type G_n if it is a segment with a leaf j_0 such that $\Gamma \setminus \{j_0\}$ is a diagram of type A_{n-1} and, for j_1 the unique vertex adjacent to j_0 , $(w(j_0, j_1), w(j_1, j_0)) = (1, 3)$.

We also have classes of Dynkin trees which are not Dynkin segments, but Dynkin trees with a unique branching point:

- (D) For $n \geq 4$, we say that Γ is a Dynkin diagram of type D_n if it has two distinct leaves j'_2, j''_2 sharing their unique neighbour j_1 , such that $\Gamma \setminus \{j'_2, j''_2\}$ is a Dynkin diagram of type A_{n-2} with leaf j_1 .
- (E) For $n \geq 6$, we say that Γ is a Dynkin diagram of type E_n if it has a unique branching point j_1 with three neighbours, it is not of type D_n but there exists $j'_3 \in \Gamma$ such that $\Gamma \setminus \{j'_3\}$ is a Dynkin diagram of type D_{n-1} .

Note that the Dynkin diagrams associated to finite root systems are exactly the Dynkin diagrams of type $(A_n)_{n \geq 1}$, $(B_n)_{n \geq 2}$ (which is dual to type C), $(C_n)_{n \geq 2}$, $(D_n)_{n \geq 4}$ and the exceptional types E_6, E_7, E_8, F_4 and G_2 .

2.6.2 Dynkin sequences associated to quantum roots

In this section, we fix a quantum root β . For $n \geq 2$ and $j \in I_2(\beta)$ such that $\deg(j) = 3$, let $I_n(\beta, j)$ denote the connected component of $I_n(\beta)$ containing j (if $j \notin I_n(\beta)$ then we set $I_n(\beta, j) = \emptyset$). We have the following classification for $(I_n(\beta, j))_{n \geq 2}$.

Proposition 2.29. *Let $j \in I_2(\beta)$ be such that $\deg(j) = 3$ and $|\text{Supp}(j)| = 2$. Then*

- (2G) $I_2(\beta, j)$ is a Dynkin diagram of type G , j is the only leaf of $I_2(\beta, j)$ which is a leaf in $I_1(\beta)$. Moreover $I_3(\beta, j)$ is empty.



Figure 2.2: Dynkin diagram of type (2G)

Proof. By Lemma 2.18, $j \notin I_3$, so $I_3(\beta, j)$ is empty. Moreover, since $|\text{Supp}(j)| = 2$, by Lemma 2.12 $I_2(\beta, j)$ is a Dynkin segment of type G , with j as special vertex. If $I_2(\beta, j) \neq \{j\}$, by Lemma 2.15 4), the other leaf of $I_2(\beta, j)$ has support of size three, hence is not a leaf in $I_1(\beta)$. \square

Proposition 2.30. *Let $j \in I_2(\beta)$ be such that $\deg(j) = 3$ and $|\text{Supp}(j)| = 3$. Then, one of the following is satisfied:*

- (3S) $I_3(\beta, j) = \emptyset$ and $I_2(\beta, j) \setminus \{j\}$ is the disjoint union of two non-empty Dynkin diagrams of type A. Moreover the leaves of $I_2(\beta, j)$ are not leaves in $I_1(\beta)$.
- (3C) $(I_n(\beta, j))_{n \geq 2}$ is a decreasing sequence of Dynkin diagrams of type C with leaf j . Leaves of $I_2(\beta, j)$ are not leaves of $I_1(\beta)$.
- (3F) Let $j'_2 \in \text{Supp}(j)$ be such that $a_{j'_2, j} = -1$. Then j'_2 is a leaf of both $I_1(\beta)$ and $I_2(\beta, j)$, $I_2(\beta, j) \setminus \{j'_2\}$ is of type C, $I_3(\beta, j)$ is of type C and $I_4(\beta, j) = \emptyset$. The leaves of $I_3(\beta, j)$ (resp. $I_2(\beta, j) \setminus \{j'_2\}$) are not leaves in $I_2(\beta, j)$ (resp. $I_1(\beta)$).

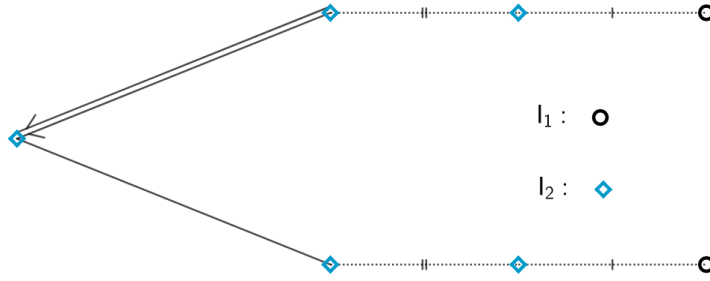


Figure 2.3: Dynkin diagram of type (3S)

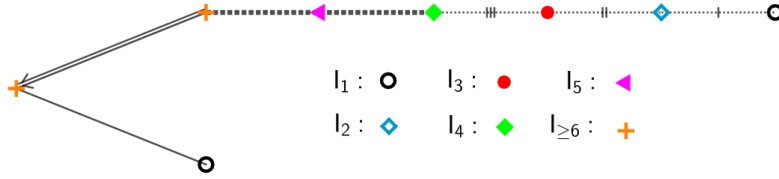


Figure 2.4: Dynkin diagram of type (3C)

Proof. This is a reformulation of Lemma 2.19. More precisely, write $\text{Supp}(j) = \{j, j_2, j'_2\}$ with $a_{j_2, j_1} = -2$ and $a_{j'_2, j_1} = -1$. By Proposition 2.13, $I_2(\beta, j) \setminus \{j\}$ is the disjoint union of two diagrams of type A. The case 1) of Lemma 2.19 therefore corresponds to type (3S), the case 2) corresponds to the type (3F), and the case 3) corresponds to type (3C). Moreover if $j_2 \in I_2(\beta, j)$, in every case we can apply Lemma 2.15 4-5) to the chain starting by $j_1 \triangleleft_2 j_2$ to deduce that, for every $n \geq 2$ leaves of $I_n(\beta, j)$ are not leaves of $I_{n-1}(\beta, j)$ (except j'_2 for $n = 2$ in type (3F), and j for $n \geq 3$ in type (3C)). \square

Proposition 2.31. *Let $j_1 \in I_2(\beta)$ be such that $\deg(j_1) = 3$ and $|\text{Supp}(j_1)| = 4$. Then the Dynkin sequence $(I_n(\beta, j_1))_{n \geq 2}$ is of one of the following forms.*

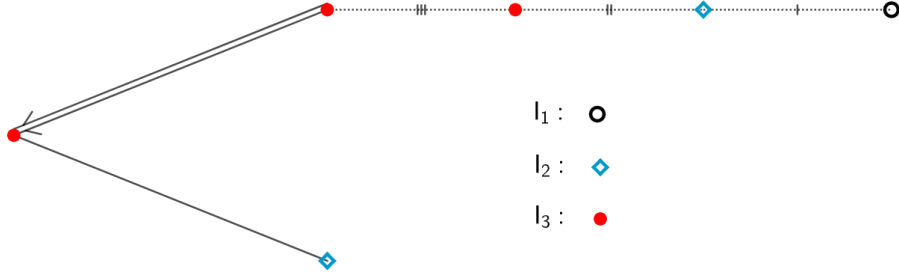


Figure 2.5: Dynkin diagram of type (3F)

- (4S) The diagram $I_2(\beta, j_1) \setminus \{j_1\}$ is the disjoint union of three non-empty diagrams of type A. Moreover the leaves of $I_2(\beta, j_1)$ are not leaves in $I_1(\beta)$, and $I_3(\beta, j_1) = \emptyset$.
- (4A) The Dynkin sequence $(I_n(\beta, j_1))_{n \geq 2}$ is a decreasing sequence of diagrams of type A such that, for all $n \geq 2$, the leaves of $I_n(\beta, j_1)$ are not leaves of $I_{n-1}(\beta, j_1)$.
- (4D) Let $K = N_{j_1}(\beta) = \max\{n \in \mathbb{Z}_{\geq 0} \mid j_1 \in I_K(\beta)\}$. Then $K \geq 5$ and there exist $j'_2, j''_2 \in \text{Supp}(j_1)$ such that:
- j''_2 is a leaf in $I_1(\beta)$ and j'_2 is a leaf of $I_2(\beta, j_1)$ but not of $I_1(\beta)$
 - $(I_n(\beta, j_1))_{n \in \llbracket 1, K \rrbracket}$ is a decreasing sequence of diagrams. If K is even, then $I_n(\beta, j_1)$ is of type D for $n \in \llbracket 1, K/2 \rrbracket$ and $I_n(\beta, j_1)$ is of type A for $n \in \llbracket K/2 + 1, K \rrbracket$. If K is odd, there exists $\eta \in \{0, 1\}$ such that $I_n(\beta, j_1)$ is of type D for $n \in \llbracket 1, (K-1)/2 + \eta \rrbracket$ and $I_n(\beta, j_1)$ is of type A for $n \in \llbracket (K-1)/2 + \eta + 1, K \rrbracket$.
 - j'_2 and j''_2 are leaves of the diagrams containing them. For $n \in \llbracket 1, K-1 \rrbracket$, if \tilde{j} is a leaf of $I_n(\beta, j_1)$ and $\tilde{j} \notin \{j_1, j'_2, j''_2\}$, then $\tilde{j} \notin I_{n+1}(\beta, j_1)$.
 - $I_{K-1}(\beta, j_1) \neq \{j_1\}$.
- (4EA) $I_2(\beta, j_1)$ is a Dynkin diagram of type E, and there exist $j'_2, j''_2 \in \text{Supp}(j_1)$ such that:
- j''_2 is a leaf in $I_1(\beta)$ and does not lie in $I_3(\beta, j_1)$
 - $\text{Supp}(j'_2) = \{j_1, j'_2, j'_3\}$ for some leaf j'_3 of both $I_1(\beta)$, $I_2(\beta)$ and $j'_3 \notin I_3(\beta, j_1)$
 - $(I_n(\beta, j_1))_{n \in \{3, 4, 5\}}$ is a decreasing sequence of diagrams of type A, j'_2 is a leaf of $I_3(\beta, j_1)$ and $j'_2 \notin I_5(\beta, j_1)$
 - $I_7(\beta, j_1) = \emptyset$ and, if $I_5(\beta, j_1) = \emptyset$ then $j'_2 \notin I_4(\beta, j_1)$
 - For $n \geq 1$, if j is a leaf of $I_n(\beta, j_1)$ and $j \notin \{j_1, j'_2, j'_3, j''_2\}$ then $j \notin I_{n+1}(\beta)$.
- (4ED) $I_2(\beta, j_1)$ is a Dynkin diagram of type E, and there exist $j'_2, j''_2 \in \text{Supp}(j_1)$ such that:
- $\text{Supp}(j'_2) = \{j_1, j'_2, j'_3\}$ for some leaf j'_3 of $I_1(\beta)$ and $j'_3 \notin I_3(\beta, j_1)$
 - $I_3(\beta, j_1)$ is a diagram of type D, two of its leaves are j'_2, j''_2 and $j''_2 \notin I_4(\beta, j_1)$

- $I_4(\beta, j_1), I_5(\beta, j_1)$ are non empty diagrams of type A, j_1 is a leaf of $I_5(\beta, j_1)$
- $I_7(\beta, j_1) = \emptyset$
- Except in the case (4ED2) below, for $n \geq 1$ if j is a leaf of $I_n(\beta, j_1)$ and $j \notin \{j_1, j'_2, j'_3, j''_2\}$ then $j \notin I_{n+1}(\beta, j_1)$.

Moreover, exactly one of the following holds:

(4ED1) $j'_2 \in I_4(\beta, j_1), I_6(\beta, j_1) = \emptyset$.

(4ED2) $j'_2 \in I_4(\beta, j_1), I_6(\beta, j_1) = \{j_1\}$ and there exists $n_0 \in \llbracket 2, 6 \rrbracket$ such that $|I_{n-1}(\beta, j_1) \setminus (I_n(\beta, j_1) \cup \{j'_2, j''_2, j'_3\})| = 1$ for $n \in \llbracket n_0 + 1, 5 \rrbracket$, and $I_{n_0-1}(\beta, j_1) \setminus \{j'_2, j''_2, j'_3\} = I_{n_0}(\beta, j_1) \setminus \{j'_2, j''_2, j'_3\}$. Moreover for $n < n_0$, the leaf of $I_n(\beta, j_1)$ not in $\{j_1, j'_2, j''_2, j'_3\}$ is not a leaf of $I_{n-1}(\beta, j_1)$. In particular if $n_0 = 2$ then β is the highest root of an E_8 -subsystem of Φ .

(4ED3) $j'_2 \notin I_4(\beta, j_1)$ and $I_5(\beta, j_1) \neq I_6(\beta, j_1)$.

(4SA) There is a unique $j''_2 \in \text{Supp}(j_1)$ which is a leaf of $I_2(\beta, j_1)$ and it is the unique leaf of $I_2(\beta, j_1)$ which is also a leaf of $I_1(\beta, j_1)$; $I_3(\beta, j_1)$ is a non-empty diagram of type A. Moreover exactly one of the following holds:

(4SA1) The vertex j_1 is not a leaf of $I_3(\beta, j_1)$. Then $I_4(\beta, j_1) = \emptyset$, and leaves of $I_3(\beta, j_1)$ are not leaves of $I_2(\beta, j_1)$.

(4SA2) The vertex j_1 is a leaf of $I_3(\beta, j_1)$. Then $I_4(\beta, j_1)$ is a diagram of type A, either empty either having j_1 as a leaf, and $I_5(\beta, j_1) = \emptyset$. Moreover j_1 is the unique leaf of $I_4(\beta, j_1)$ which is also a leaf of $I_3(\beta, j_1)$, and leaves of $I_3(\beta, j_1)$ are not leaves of $I_2(\beta, j_1)$.

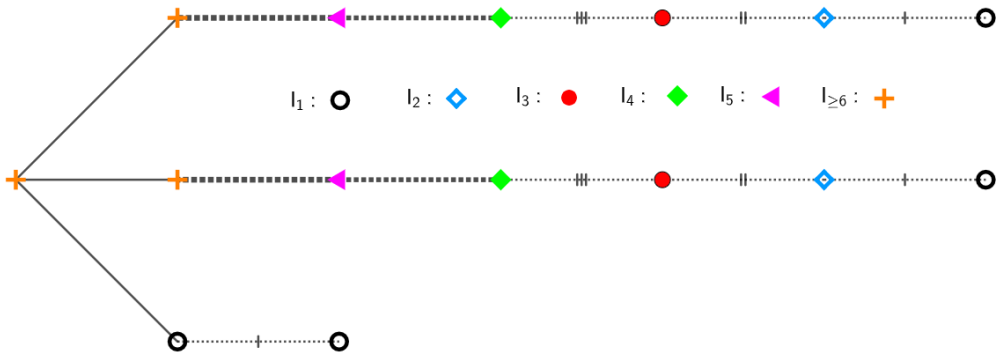


Figure 2.6: Dynkin diagram of type (4A)

Proof. Fix $j_1 \in I_2(\beta)$ such that $|\text{Supp}(j_1)| = 4$ and $\deg(j_1) = 3$. Then the connected components $(I_n(\beta, j_1))_{n \geq 2}$ of j_1 in $I_n(\beta)$ form a sequence of Dynkin diagrams ordered

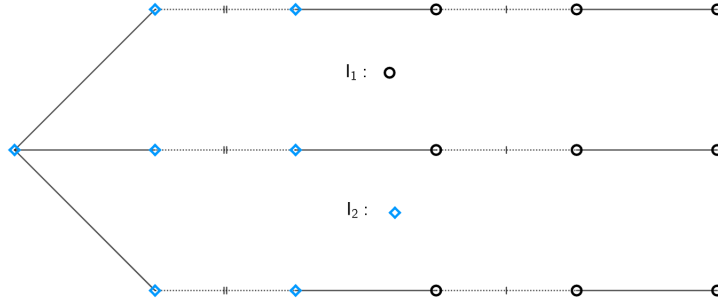


Figure 2.7: Dynkin diagram of type (4S)

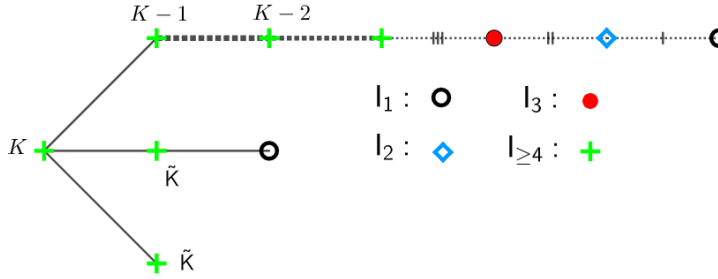


Figure 2.8: Dynkin diagram of type (4D)

for the inclusion. By Proposition 2.13, $I_2(\beta, j_1) \setminus \{j_1\}$ (and therefore by Lemma 2.15, $I_n(\beta, j_1) \setminus \{j_1\}$ for all $n \geq 2$) is a disjoint union of 3 diagrams of type A (possibly empty), each one starting at a distinct element of $\text{Supp}(j_1) \setminus \{j_1\}$. Moreover, Lemmas 2.21 to 2.25 give more information on their forms:

1. If $\text{Supp}(j_1) \not\subset I_2$, then by Lemma 2.21, the sequence of Dynkin diagrams $(I_n(\beta, j_1))_{n \geq 2}$ is of the form (4A).
2. Else, if $|\text{Supp}(j_1)| = 3$ for all $j \in \text{Supp}(j_1) \subset I_2$, by Lemma 2.22 it is of the form (4S). Moreover, by Lemma 2.15 4-6), the leaves of $I_2(\beta, j_1)$ are not leaves of $I_1(\beta)$.
3. Else, if there is $j \in \text{Supp}(j_1)$ such that $\text{Supp}(j_1) \not\subset I_2$, by Lemma 2.23, it is of the form (4D). Indeed, let $K = N_{j_1}(\beta)$. First assume that K is even. Let $\tilde{K} = K/2$. Then by (2.17), we have

$$\mathbf{t}(j_2', \tilde{K}) \leq \mathbf{t}(j_2'', \tilde{K}) \leq \mathbf{t}(j_1, 2\tilde{K}) \leq \mathbf{t}(j_2', \tilde{K} + 1) \leq \mathbf{t}(j_1, 2\tilde{K} + 1) = \infty = \mathbf{t}(j_2'', \tilde{K} + 1),$$

with $\mathbf{t}(j_1, 2\tilde{K}) < \infty$.

We have $I_{\tilde{K}}(\beta, j_1) \supset \{j_2', j_2''\}$ and $j_2'' \notin I_{\tilde{K}+1}(\beta, j_1)$. Therefore $I_{\tilde{K}}(\beta, j_1)$ is of type D and $I_{\tilde{K}+1}(\beta, j_1)$ is of type A . By Lemma 2.23, $j_2 \in I_{2\tilde{K}-1}(\beta, j_1) \neq \{j_1\}$.

Assume now that K is odd. Set $\tilde{K} = (K - 1)/2$. Then by (2.17), we have

$$\mathbf{t}(j_2'', \tilde{K}) \leq \mathbf{t}(j_2', \tilde{K} + 1) \leq \mathbf{t}(j_1, 2\tilde{K} + 1) \leq \mathbf{t}(j_2'', \tilde{K} + 1) \leq \mathbf{t}(j_1, 2\tilde{K} + 2) = \infty,$$

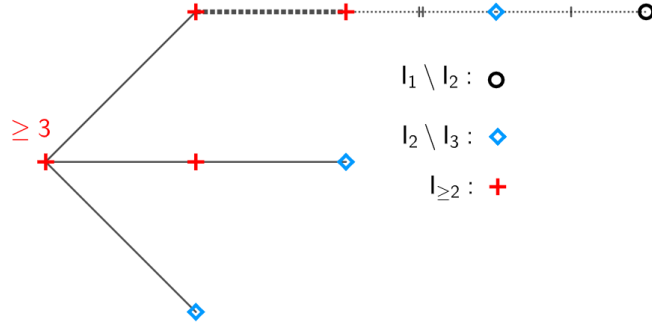


Figure 2.9: Dynkin diagram of type (4EA)

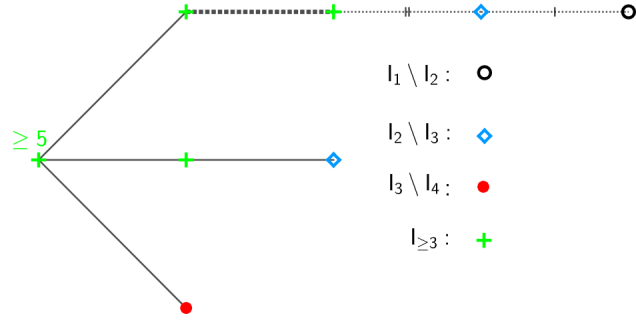


Figure 2.10: Dynkin diagram of type (4ED)

$$\mathbf{t}(j_1, 2\tilde{K} + 1) < \infty \text{ and } \infty = \mathbf{t}(j_1, 2\tilde{K} + 2) = \mathbf{t}(j'_1, \tilde{K} + 2) = \mathbf{t}(j''_1, \tilde{K} + 2).$$

Therefore $j'_2 \in I_{\tilde{K}+1}(\beta, j_1)$, $j'_2, j''_2 \notin I_{\tilde{K}+2}(\beta, j_1)$ and thus $I_{\tilde{K}+1}(\beta, j_1)$ is of type D or A and $I_{\tilde{K}+2}(\beta, j_1)$ is of type A . By Lemma 2.21, $j_2 \in I_{2\tilde{K}}(\beta, j_1) \neq \{j_1\}$.

4. Else, we can apply Lemma 2.24. If $j_1 \notin I_4(\beta, j_1)$ we are in the case 1) of Lemma 2.24 and we obtain the form (4SA1), else if $j'_2 \notin I_3(\beta, j_1)$ we are in the cases 1) or 2) of Lemma 2.24 and we obtain the form (4SA2).

Finally case 3) of Lemma 2.24 corresponds to the forms (4EA) or (4ED), according to whether j''_2 lies in I_3 . We obtain the more specific forms (4ED1) to (4ED3) through a more careful study in these cases, as it is done in the proofs of Lemma 2.24 and Lemma 2.25, we now develop these cases.

Recall the notation of Lemma 2.24 for j'_3 . Assume by contradiction that $j_1 \notin I_5(\beta, j_1)$ and $j'_2 \in I_4(\beta, j_1)$. Then as we are in the case 3) of Lemma 2.24, $j'_3 \notin I_3(\beta, j_1)$. But then if $t = \mathbf{t}(j'_2, 4)$, we have $\alpha_{j'_2}(\gamma_t^\vee) = 6 - N_{j_1}(\gamma_t^\vee) - N_{j'_3}(\gamma_t^\vee) = -1$, with $N_{j'_3}(\gamma_t^\vee) \leq 2$ and thus $N_{j_1}(\gamma_t^\vee) \geq 5$: $j_1 \in I_5$, a contradiction. Therefore if $j_1 \notin I_5$, we have $j_2 \notin I_4(\beta, j_1)$. Moreover by Lemma 2.17, we have $j''_2 \notin I_3(\beta, j_1)$. Thus if $j_1 \notin I_5$, we have $j'_2 \notin I_4(\beta, j_1)$, $j'_3 \notin I_3(\beta, j_1)$, $j''_2 \notin I_3(\beta, j_1)$ so the sequence is of the form (4EA).

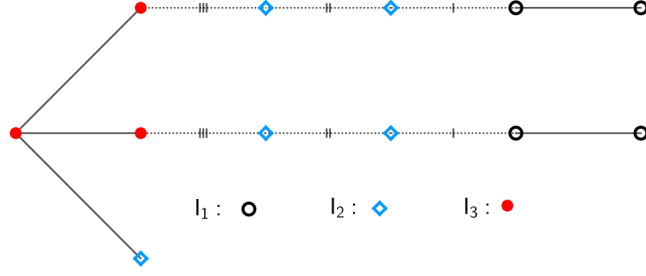


Figure 2.11: Dynkin diagram of type (4SA1)

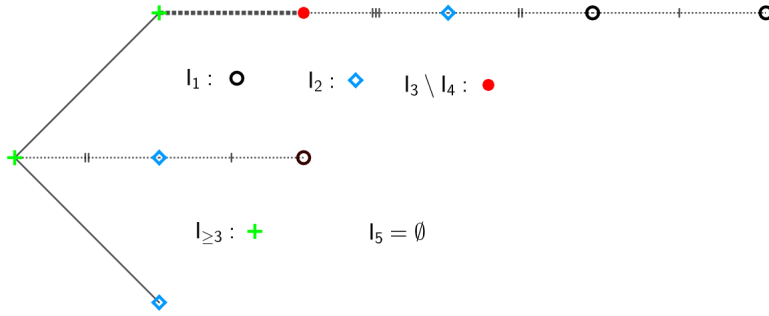


Figure 2.12: Dynkin diagram of type (4SA2)

Else, $j_1 \in I_5(\beta, j_1)$ and by Formula (2.19), there is a time t such that

$$(N_{j_1}(t), N_{j_2}(t), N_{j'_2}(t), N_{j''_2}(t)) = (5, 4, 3, 2).$$

We can also suppose that $N_{j'_3}(t) = 2$. Indeed, $j'_3 \in I_2$ so $\mathbf{t}(j'_3, 2)$ is finite. Since $\text{Supp}(j'_3) = \{j'_2, j'_3\}$, by Lemma 2.17 $\mathbf{t}(j'_2, 3) < \mathbf{t}(j'_3, 2) < \mathbf{t}(j'_2, 4)$, $r_{j'_3}$ commutes with every other reflection applied between $\mathbf{t}(j'_2, 3)$ and $\mathbf{t}(j'_2, 4)$, so we can suppose that $\mathbf{t}(j'_3, 2) = \mathbf{t}(j'_2, 3) + 1$.

- If $N_{j''_2}$ no longer increases, the sequence is of the form (4EA).
- Else, by Lemma 2.17, $N_{j''_2}$ increases before N_{j_1} , and if $N_{j'_2}$ does not increase the sequence is of the form (4ED3). Indeed, by Lemma 2.24 3), $\mathbf{t}(j_2, n-1) \leq \mathbf{t}(j_1, n)$ for $n \in \llbracket 2, 5 \rrbracket$ and it is also satisfied for $n \geq 7$, since $\mathbf{t}(j_1, 7) = \infty$. If $t := \mathbf{t}(j_1, 6)$ is finite, since $N_{j'_2}(t) + N_{j''_2}(t) \leq 6$ necessarily $N_{j_2}(t) \geq 5$ hence $\mathbf{t}(j_2, 5) \leq \mathbf{t}(j_1, 6)$. Therefore by Lemma 2.15 6), the corresponding sequence $(k_n)_{n \geq 2}$ is decreasing, in particular $I_5(\beta, j_1) \neq I_6(\beta, j_1)$.
- Else if $N_{j'_2}$ increases and $j_1 \notin I_6$, we obtain the form (4ED1).
- The final case is if $N_{j'_2}$ increases and $j_1 \in I_6(\beta, j_1)$, then there exists a time t such that $(N_{j_1}(t), N_{j_2}(t), N_{j'_2}(t), N_{j''_2}(t)) = (6, 4, 4, 3)$. Following the analysis done in Lemma 2.25, we obtain the form (4ED2): if N_{j_2} stays at 4, it is of the form (4ED2) with $n_0 = 6$ (since $j'_3 \notin I_3$ and $\max N_{j_1} = 6$, we have $j'_2 \notin I_5$).

- Else (using the same reasoning as in the proof of Lemma 2.25 to obtain Formula (2.21)) there is a time t such that

$$(N_{j_1}(t), N_{j_2}(t), N_{j_3}(t), N_{j_4}(t)) \in \{(6, 5, 3, 2), (6, 5, 3, 3)\}.$$

If $j_3 \notin I_4(\beta, j_1)$ we obtain the form (4ED2) with $n_0 = 5$, else there is a time for which Equality (2.22) holds, and the argument can be repeated. This necessarily ends for $n_0 = 2$, in which case $j_5 \in I_2(\beta, j_1) \setminus I_3(\beta, j_1)$ has support of size 2, and the root is the highest root of E_8 .

□

2.6.3 General classification of quantum roots

Definition 2.32. Let I be a Dynkin diagram, and let $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ be a sequence of Dynkin subdiagram. We say that $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ is a **quantum Dynkin sequence** if it is the Dynkin sequence of a quantum root. That is to say, $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ is a quantum Dynkin sequence if and only if there exists $\beta \in \mathcal{Q}(\Phi_+)$ such that $\beta^\vee = \sum_{n \geq 1} \sum_{i \in I_n} \alpha_i^\vee$.

In this section, we obtain a classification of quantum Dynkin sequences, and therefore a classification of quantum roots.

Definition 2.33. Let I be a Dynkin diagram and let $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ and $(I'_n)_{n \in \mathbb{Z}_{\geq 1}}$ be two Dynkin sequences of I . We say that (I_n) and (I'_n) are **mergeable** if $I_1 = I'_1$ and if I_2, I'_2 are disconnected. That is to say, if $I_2 \cap I'_2 = \emptyset$ and if there is no edge (i, i') such that $i \in I_2, i' \in I'_2$.

Lemma 2.34. Suppose that $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ and $(I'_n)_{n \in \mathbb{Z}_{\geq 1}}$ are two quantum Dynkin sequences. Then, if $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ and $(I'_n)_{n \in \mathbb{Z}_{\geq 1}}$ are mergeable, $(I_n \cup I'_n)_{n \in \mathbb{Z}_{\geq 1}}$ is also quantum.

Proof. Let $\beta \in \mathcal{Q}(\Phi_+)$ (resp. $\beta' \in \mathcal{Q}(\Phi_+)$) be the quantum root such that $(I_n)_{n \in \mathbb{Z}_{\geq 1}} = (I_n(\beta))_{n \in \mathbb{Z}_{\geq 1}}$ (resp. $(I'_n)_{n \in \mathbb{Z}_{\geq 1}} = (I_n(\beta'))_{n \in \mathbb{Z}_{\geq 1}}$), which exists since by assumption the two sequences are quantum. Since the two sequences are also mergeable, we have $I_1 = I'_1$ and this is a 1-star convex tree by Proposition 2.11.

Let $\beta_0 \in \mathcal{Q}(\Phi_+)$ be the quantum root such that $I_1(\beta_0) = I_1$ and $I_2(\beta_0) = \emptyset$, which is given by Proposition 2.11. Note that, since $I_1 = I'_1$ and I_n, I'_n are disjoint for $n \geq 2$, we have $\sum_{n \geq 1} \sum_{i \in I_n \cup I'_n} \alpha_i^\vee = \beta^\vee + (\beta')^\vee - \beta_0^\vee$. It therefore suffices to prove that this is the coroot associated to a quantum root.

By Lemma 2.9, we can write $\beta = w(\beta_0)$ (resp. $\beta' = w'(\beta_0)$) where $w \in \langle r_i \mid i \in I_2 \rangle \subset W^v$ (resp. $w' \in \langle r_i \mid i \in I'_2 \rangle \subset W^v$) is minimal. Fix a reduced decomposition $r_{i_1} \dots r_{i_n}$ of w' . Note that for all $k \in \llbracket 1, n \rrbracket$, $r_{i_k} \dots r_{i_n}(\beta_0)$ is quantum with coroot $\beta_0^\vee + \sum_{j=k}^n \alpha_{i_j}^\vee$ (in particular $(\beta')^\vee = \beta_0^\vee + \sum_{j=1}^n \alpha_{i_j}^\vee$). Since I_2 and I'_2 are disconnected, we have $w^{-1}(\alpha_i) = \alpha_i$ for all $i \in \{i_1, \dots, i_n\} \subset I'_2$. Therefore, by applying Lemma 2.35 to the triple (β_0, i_n, w) , $r_{i_n} w(\beta_0)$ is quantum with coroot $\beta^\vee + \alpha_{i_n}^\vee$. Since $r_{i_n} w(\beta_0) = w r_{i_n}(\beta_0)$ we can iterate the application of Lemma 2.35, to the triple $(r_{i_n}(\beta_0), i_{n-1}, w)$ and we obtain that $r_{i_{n-1}} r_{i_n} w(\beta_0)$

is quantum with coroot $\beta^\vee + \alpha_{i_{n-1}}^\vee + \alpha_{i_n}^\vee$. After n successive applications of Lemma 2.35, we obtain that $w'w(\beta_0) = r_{i_1} \dots r_{i_n} w(\beta_0)$ is quantum with coroot $\beta^\vee + \sum_{j=1}^n \alpha_{i_j}^\vee = \beta^\vee + (\beta')^\vee - \beta_0^\vee$. Therefore $(I_n \cup I'_n)_{n \in \mathbb{Z}_{\geq 1}} = (I_n(w'w(\beta_0)))_{n \in \mathbb{Z}_{\geq 1}}$ is a quantum Dynkin sequence. \square

Lemma 2.35. *Suppose that $\beta \in \Phi_+$, $i \in I$ and $w \in W^v$ are such that $w^{-1}(\alpha_i) = \alpha_i$. Then if β , $r_i(\beta)$, $w(\beta)$ are quantum, $r_i w(\beta)$ is also quantum. Moreover $r_i w(\beta^\vee) = w(\beta^\vee) + r_i(\beta^\vee) - \beta^\vee$*

Proof. Suppose that β and $w(\beta)$ are quantum. By Proposition 2.8, $r_i w(\beta)$ is quantum if and only if $|\langle w(\beta^\vee), \alpha_i \rangle| = |\langle \beta^\vee, w^{-1}(\alpha_i) \rangle| \leq 1$. However, $w^{-1}(\alpha_i) = \alpha_i$. Therefore this is satisfied if and only if $|\langle \beta^\vee, \alpha_i \rangle| \leq 1$ that is to say, by Proposition 2.8, if and only if $r_i(\beta)$ is quantum.

Moreover since $r_i(\beta^\vee) - \beta^\vee = \langle \beta^\vee, \alpha_i \rangle \alpha_i^\vee$, we check that $r_i w(\beta^\vee) = w(\beta^\vee) - \langle w(\beta^\vee), \alpha_i \rangle \alpha_i^\vee = w(\beta^\vee) + r_i(\beta^\vee) - \beta^\vee$. \square

Theorem 2.36. *Let I be a Dynkin diagram with associated root system Φ , and let $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ be a sequence of Dynkin subdiagrams of I . Then there exists a quantum root $\beta \in \mathcal{Q}(\Phi_+)$ such that $(I_n)_{n \geq 1} = (I_n(\beta))_{n \geq 1}$ (see Definition (2.2)) if and only if the sequence $(I_n)_{n \in \mathbb{Z}_{\geq 1}}$ satisfies the following properties:*

1. For any $n \in \mathbb{Z}_{\geq 1}$, I_{n+1} is a subdiagram of I_n .
2. The diagram I_1 is a 1-convex Dynkin tree.
3. For any $n \geq 2$, for any vertex $j \in I_n$ we have $\deg(j) \leq 3$ (see Formula (2.10)). Moreover each connected component of I_n admits a unique vertex of degree 3.
4. For any vertex $j \in I_2$ such that $\deg(j) = 3$, let $I_n(j)$ denote the corresponding connected component (with $I_n(j) = \emptyset$ if $j \notin I_n$), given by point 3. Then:
 - (a) If $|\text{Supp}(j)| = 2$, then $(I_n(j))_{n \geq 2}$ is a sequence of type (2G) (see Proposition 2.29).
 - (b) If $|\text{Supp}(j)| = 3$, then $(I_n(j))_{n \geq 2}$ is a sequence of type (3S), (3C) or (3F) (see Proposition 2.30).
 - (c) If $|\text{Supp}(j)| = 4$, then $(I_n(j))_{n \geq 2}$ is a sequence of type (4S), (4A), (4D), (4EA), (4ED) or (4SA) (see Proposition 2.31).

Proof. The definition of quantum Dynkin sequences is such that it is clear, with the results of the preceding sections, that $(I_n(\beta))_{n \geq 1}$ is a quantum Dynkin sequence when $\beta \in \mathcal{Q}(\Phi_+)$ (see Proposition 2.11 for point 2) of Definition 2.32, Proposition 2.13 for point 3) and Propositions 2.29, 2.30 and 2.31 for point 4). Therefore it remains to associate a quantum root β to each Dynkin sequence sequence satisfying the properties 1 to 4.

Let $(I_n)_{n \geq 1}$ be a quantum Dynkin sequence. Let $J_3 := \{j \in I_2 \mid \deg(j) = 3\}$. For any $j \in J_3$ and $n \geq 1$, as in Definition 2.32 let $I_n(j)$ denote the connected component of j in I_n , which is of one of the type given by Definition 2.32 4), and either $I_n(j) = \emptyset$,

either $I_n(j) \cap J_3 = \{j\}$. We have $I_1(j) = I_1$ for every $j \in J_3$. The graphs $I_2(j)$ and $I_2(j')$ are disconnected for $j, j' \in J_3$ such that $j \neq j'$, and thus $(\bigcup_{j \in E_1} I_n(j))_{n \in \mathbb{Z}_{\geq 1}}$ and $(\bigcup_{j \in E_2} I_n(j))_{n \in \mathbb{Z}_{\geq 1}}$ are mergeable for every subsets E_1, E_2 of J_3 such that $E_1 \cap E_2 = \emptyset$. Therefore using Lemma 2.34, we can assume $|J_3| \leq 1$. We already proved the case where J_3 is empty in Proposition 2.11 and thus we now assume that J_3 is a singleton $\{j\}$. We have $I_n(j) = I_n$ for all $n \in \mathbb{Z}_{\geq 1}$.

By Proposition 2.11 there exists $\beta_\emptyset \in \mathcal{Q}(\Phi_+)$ such that $I_1(\beta_\emptyset) = I_1$ and $I_n(\beta_\emptyset) = \emptyset$ for $n \geq 2$. Note that, since $(I_n)_{n \geq 2}$ is amongst the types given in Propositions 2.29, 2.30 and 2.31, it has the same properties as the connected component \mathcal{C} in Proposition 2.13 and Remark 2.16. That is to say, with $m = |I_2 \cap \text{Supp}(j)| - 1$, $I_2 \setminus \{j\}$ is the disjoint union of m non-empty Dynkin segments of type A , one for each neighbour of j in I_1 . As in Proposition 2.13, let us write $(I_2 \cap \text{Supp}(j)) \setminus \{j\} = \{j_2^{(i)} \mid i \in \llbracket 1, m \rrbracket\}$ and, for all $i \in \llbracket 1, m \rrbracket$ write $j_2^{(i)}, j_3^{(i)}, \dots, j_{k(2,i)}^{(i)}$ the ordered elements of the segment starting at $j_2^{(i)}$. Therefore, $k(2, i) - 1$ is the length of the segment of type A starting at $j_2^{(i)}$ in $I_2 \setminus \{j\}$. Upon relabelling, we may suppose $k(2, i) \geq k(2, i + 1)$ for all $i \in \llbracket 1, m - 1 \rrbracket$. Moreover by Remark 2.16, $[j_2^{(i)}, j_{k(2,i)}^{(i)}] \cap I_n = [j_2^{(i)}, j_{k(n,i)}^{(i)}]$ for some non-increasing sequence $(k(n, i))_{n \geq 2}$ (with the convention that $k(n, i) = 1$ if $j_2^{(i)} \notin I_n$, and $[j_2^{(i)}, j_1^{(i)}] = \emptyset$).

Finally, for each $i \in \llbracket 1, m \rrbracket$, $n \geq 2$, let us define the following element of W^v :

$$w_n^{(i)} = \begin{cases} r_{j_{k(n,i)}^{(i)}} \dots r_{j_2^{(i)}} & \text{if } k(n, i) \geq 2 \\ 1_{W^v} & \text{otherwise.} \end{cases}, \quad w_n = \left(\prod_{i=1}^m w_n^{(i)} \right) r_j \quad (2.24)$$

With these notations, we can now construct β from β_\emptyset .

Suppose for now that $(I_n)_{n \geq 2}$ is of the form (2G), (3S), (3C), (4S) or (4A). In all these cases, $j_{k(n,i)}^{(i)}$ is not a leaf of I_{n-1} so the sequences $(k(n, i))_{n \geq 2}$ are decreasing until they reach 1, and $|\text{Supp}(j_{k(2,i)}^{(i)})| = 3$.

By Proposition 2.8, as $\deg(j) = 3$, $r_j(\beta_\emptyset)$ is quantum, and $I_2(r_j(\beta_\emptyset)) = \{j\}$. If $m = 0$, $\text{Supp}(j) \cap I_2 = \{j\}$, so we are done. Else, recursively applying Proposition 2.8 $\ell(w_2)$ times, we similarly check that $w_2(\beta_\emptyset)$ is quantum (we crucially use that $|\text{Supp}(j_{k(2,i)}^{(i)})| = 3$ when we apply $r_{j_{k(2,i)}^{(i)}}$). Indeed, for any $i \in \llbracket 1, m \rrbracket$ and $p \in \llbracket 2, k(2, i) \rrbracket$, at time $t := \mathbf{t}(j_p^{(i)}, 2)$, we have $(N_{j_{p-1}^{(i)}}(t), N_{j_p^{(i)}}(t), N_{j_{p+1}^{(i)}}(t)) = (2, 1, 1)$ therefore we can apply Proposition 2.8. Moreover $I_2(w_2(\beta_\emptyset)) = I_2$ and $I_3(w_2(\beta_\emptyset)) = \emptyset$. We iterate the process: if $j \in I_n$ we apply w_n to $w_{n-1} \dots w_2(\beta_\emptyset)$, which is quantum, and we obtain a quantum root (we need to use that $k(n, i) < k(n - 1, i)$ if $w_n^{(i)} \neq 1_{W^v}$). When we reach n_0 such that $I_{n_0+1} = \emptyset$ and $I_{n_0} \neq \emptyset$, we stop the process, and $\beta := w_{n_0} \dots w_2(\beta_\emptyset)$ is a quantum root such that $I_n(\beta) = I_n$ for all $n \geq 2$.

It remains to deal with the forms (3F), (4D), (4EA), (4ED), (4SA). Let us start by the form (4D), it is the only one amongst these for which I_n may be non empty for arbitrary large n . Let $K = \max\{n \in \mathbb{Z} \mid I_n \neq \emptyset\}$. Let $\tilde{K} = \lfloor (K - 1)/2 \rfloor$. In this case $m = 3$ but

$k(2, 2) = k(2, 3) = 2$, and upon relabelling we can suppose that $j_2^{(3)}$ is a leaf of I_1 , but $j_2^{(2)}$ is not. Set $\beta^0 = \beta_\emptyset$. Suppose that $p \in \llbracket 0, \tilde{K} - 1 \rrbracket$ is such that β^p is constructed and quantum, and satisfies (H_p) :

$$N_{j_k^{(1)}}(\beta^p) = N_j(\beta^p) = 2p + 1 \text{ for } k \in \llbracket 1, k(2p + 1, 1) \rrbracket, N_{j_{k(2p+1,1)+1}^{(1)}}(\beta^p) = 2p \quad (H_p)$$

$$\text{and } N_{j_2^{(i)}}(\beta^p) = p + 1 \text{ for } i \in \{2, 3\},$$

where the condition " $N_{j_{k(2p+1,1)+1}^{(1)}}(\beta_p) = 2p$ " is deleted when $p = 0$.

Then using the fact that $(k(n, 1))$ is decreasing and applying Proposition 2.8 we check that $\beta^{p+1} := r_{j_2^{(3)}} w_{2p+3}^{(1)} r_j r_{j_2^{(2)}} w_{2p+2}^{(1)} r_j(\beta^{2p+1})$ is a quantum root which satisfies (H_{p+1}) . We obtain $\beta^{\tilde{K}}$ satisfying $(H_{\tilde{K}})$. Then if K is odd we are done. Otherwise, we set $\beta = r_j(\beta^{\tilde{K}})$, which concludes the case (4D).

In the remaining cases, $I_n(j) = \emptyset$ for all $n \geq 7$, so they can be checked by hand in a similar fashion, following the arguments in the proofs of Lemma 2.19 (for the case (3F)) or Lemma 2.24 (for the cases (4EA), (4ED), (4SA)). To have a complete proof, we now explicit these verifications.

In the case (3F), $m = 2$ and upon relabelling, suppose that $(a_{j_2^{(1)}, j}, a_{j_2^{(2)}, j}) = (-2, -1)$. Then we check that $\beta := r_{j_2^{(2)}} w_3^{(1)} r_j w_2^{(1)} r_j(\beta_\emptyset)$ is a quantum root such that $I_n(\beta_{J \cup \{j\}}) = I_n(\beta_J) \sqcup I_n(j)$ for all $n \geq 2$.

For the case (4SA), $m = 3$ and denote $j_2^{(3)}$ the unique leaf of I_2 in $\text{Supp}(j)$, by definition it is also a leaf of I_1 . In the case (4SA2), since j is a leaf of I_3 , there is another $j_2^{(2)} \in \text{Supp}(j)$ which does not belong to I_3 . We set $w = r_{j_2^{(3)}} w_3^{(2)} w_3^{(1)} r_j w_2^{(2)} w_2^{(1)} r_j$. Then, in the case (4SA1), $\beta := w(\beta_\emptyset)$ is the required quantum root. In the case (4SA2), we instead set $\beta := w_4^{(1)} r_j w(\beta_\emptyset)$ (note that $w_3^{(2)} = 1_{W^v}$ hence $\langle w(\beta_\emptyset^\vee), \alpha_j \rangle = -1$ as required to apply Proposition 2.8).

It remains to deal with the cases (4EA) and (4ED). Again $m = 3$ and by definition we can suppose that $j_2^{(3)}, j_3^{(2)}$ are leaves of I_1 (they correspond to j_2'' and j_3' respectively, with the notation of Proposition 2.31). Suppose that $(I_n)_{n \geq 2}$ is not of the form (4ED2).

Let

$$\beta^1 := r_{j_3^{(2)}} r_{j_2^{(2)}} w_4^{(1)} r_j r_{j_2^{(3)}} w_3^{(1)} r_j r_{j_2^{(2)}} w_2^{(1)} r_j(\beta_\emptyset),$$

then by Proposition 2.8, β^1 is a quantum root and satisfies $(N_j, N_{j_2^{(1)}}, N_{j_2^{(2)}}, N_{j_2^{(3)}}, N_{j_3^{(2)}}) = (4, 4, 3, 2, 2)$. If $(I_n)_{n \geq 2}$ is of the form (4EA) and $j \notin I_5$, then we set $\beta := \beta^1$. If $(I_n)_{n \geq 2}$ is of the form (4EA) but $j \in I_5$, we instead set $\beta := w_5^{(1)} r_j(\beta^1)$. It remains to treat the cases (4ED).

Let $\beta^2 := w_4^{(2)} r_{j_2^{(3)}} w_5^{(1)} r_j(\beta^1)$. If $j \notin I_5$ (hence in the case (4ED1) or a particular case of (4ED3)), we set $\beta := \beta^2$. Else $j \in I_5$, we set $\beta := r_j(\beta^2)$ (note that this is indeed a quantum root by Proposition 2.8 because either $w_4^{(2)} = 1_{W^v}$ either or $w_5^{(1)} = 1_{W^v}$).

The only remaining case is the form (4ED2). It corresponds to the analysis done in the proof of Lemma 2.25, and we construct β following this proof: Recall the notation for

$n_0 \in \llbracket 2, 6 \rrbracket$, such that the leaf $j_{k(n_0,1)}^{(1)}$ of I_{n_0} is also a leaf of I_{n_0-1} , note that $k(n_0, 1) = 7 - n_0$. Let us start by β_0 such that $(I_n(\beta_0))_{n \geq 0}$ is of the form (4ED1), with $I_k(\beta_0) = I_k$ for $k \leq n_0 - 1$ and $I_k(\beta_0) \setminus \{j_2^{(2)}, j_3^{(2)}, j_2^{(3)}\} = [j, j_{6-k}^{(1)}]$ for $k \in \llbracket n_0, 6 \rrbracket$ (in particular $I_6(\beta_0) = \emptyset$ and $I_5(\beta_0) = \{j\}$). Then $\alpha_j(\beta_0^\vee) = -1$ so $r_j(\beta_0)$ is quantum with Dynkin sequence of the form (4ED2). If $n_0 = 6$, we can set $\beta := r_{j_{7-n_0}^{(1)}} \dots r_{j_2^{(1)}} r_j(\beta_0)$ is a quantum root with Dynkin sequence $(I_n)_{n \geq 1}$.

This concludes the proof of Theorem 2.36. \square

2.7 Examples of sets of quantum roots

As shown by Lemma 3.1 and Proposition 3.11, quantum roots naturally appears in the study of covers for the affine Bruhat order. In this subsection, we determine the sets of quantum roots in some particular cases (type A_n , $A_n^{(1)}$, $D_n^{(1)}$, etc.). Besides giving examples, we want to illustrate that the methods developed in this section to prove Theorem 2.27 are constructive and enable to determine the set of quantum roots in particular examples.

Let $\beta \in \Phi_+$. Write $\beta = r_{i_L} \dots r_{i_2} \alpha_{i_1}$, with L minimal, $i_1, \dots, i_L \in I$. For $\ell \in \mathbb{Z}_{\geq 0}$, set

$$I_\ell(\beta) = \{j \in I \mid |\{t \in \llbracket 1, L \rrbracket \mid i_t = j\}| \geq \ell\}.$$

2.7.1 Finite type

The following result was explained to us by Dinakar Muthiah.

Proposition 2.37. *Assume that the root system Φ is of type A , D or E . Then the set of quantum roots of Φ is Φ_+ .*

Proof. By [Bou81, Planche I to X], we can embed Φ in \mathbb{R}^n , for some $n \in \mathbb{N}$, equipped with the canonical scalar product $(\cdot | \cdot)$ and Φ is a subset of elements which have exactly two non-zero coordinates, both equal to ± 1 and their norm is thus $\sqrt{2}$. Let $\alpha \in \Phi$. We have $\alpha^\vee = 2\alpha/(\alpha|\alpha) = \alpha$ for every $\alpha \in \Phi$. Thus if $\alpha, \beta \in \Phi$, we have $\langle \alpha^\vee, \beta \rangle = (\alpha|\beta) \in \mathbb{Z}$ and $|(\alpha|\beta)| \leq |\alpha||\beta| \leq 2$, so $\langle \alpha^\vee, \beta \rangle \in \llbracket -2, 2 \rrbracket$. Now if α and β are not proportional, we have $|(\alpha|\beta)| < |\alpha||\beta|$ and thus $\langle \alpha^\vee, \beta \rangle \in \llbracket -1, 1 \rrbracket$. For every $\alpha \in \text{Inv}(s_\beta)$, we have $\langle \alpha^\vee, \beta \rangle > 0$ (see [Phi23, Corollary 1.10] for example) and thus $\langle \alpha^\vee, \beta \rangle = 1$ for all $\alpha \in \text{Inv}(s_\beta)$, which proves that β is quantum. \square

Remark 2.38. *Proposition 2.37 can also be proved using Theorem 2.36: Dynkin diagrams of type ADE are trees which are 1-star-convex at any given vertex. Fix a root system Φ of type ADE , and $\beta \in \Phi_+$. Then the Dynkin diagram $I_1(\beta)$ is of type ADE aswell. If it is of type A , it has no vertex of degree 3 or larger, so $I_2(\beta) = \emptyset$. Hence $(I_n(\beta))_{n \geq 1}$ is quantum and $\beta \in \mathcal{Q}(\Phi_+)$. If $I_1(\beta)$ is of type D or E , it admits a unique vertex j of degree 3 and no vertex of higher degree, so $I_2(\beta)$ is non-empty if and only if it contains j . We check that if this is the case, $(I_n(\beta))_{n \geq 1}$ needs to be of the form (4A), with $I_3(\beta) = \emptyset$, when $I_1(\beta)$ is of type D ; or of the forms (4A), (4D), (4EA), (4ED) when $I_1(\beta)$ is of type E .*

However, for every other indecomposable Cartan matrix (of finite type), there exists a positive root which is not quantum. Indeed, there exist $i, j \in I$ such that $\langle \alpha_i^\vee, \alpha_j \rangle \leq -2$. Set $\beta = r_i \cdot \alpha_j = -\langle \alpha_i^\vee, \alpha_j \rangle \alpha_i + \alpha_j$. Then $\alpha_i^\vee \in \text{Inv}(s_\beta)$ and $\langle \alpha_i^\vee, r_i \cdot \alpha_j \rangle = -2\langle \alpha_i^\vee, \alpha_j \rangle - 1 \geq 3$ and thus $r_i \cdot \alpha_j$ is not quantum. Note that when Φ is infinite, then Φ_+ strictly contains the set of quantum roots, which is finite.

2.7.2 Type $A_n^{(1)}$, $n \geq 2$

Proposition 2.39. *We assume that the Kac-Moody matrix A is of type $A_n^{(1)}$, with $n \in \mathbb{Z}_{\geq 2}$ (see [Kac94, Chapter 4] for the classification). We write $I = \mathbb{Z}/(n+1)\mathbb{Z}$, with $\langle \alpha_{\bar{i}}^\vee, \alpha_{\bar{i}+1} \rangle = \langle \alpha_{\bar{i}}^\vee, \alpha_{\bar{i}-1} \rangle = -1$ for all $\bar{i} \in \mathbb{Z}/(n+1)\mathbb{Z}$.*

Let $k, m \in \llbracket 0, n \rrbracket$ with $k \leq m$. Set $\alpha_{\bar{k}, \bar{m}} = \sum_{i=k}^m \alpha_{\bar{i}}$ and $\alpha_{\bar{m}, \bar{k}} = \sum_{i=m}^{k+n+1} \alpha_{\bar{i}}$. Then the set of quantum roots is $\{\alpha_{\bar{k}, \bar{m}} \mid (k, m) \in \llbracket 0, n \rrbracket^2\}$.

Proof. Let $\beta \in \Phi_+$ be a quantum root. We have $\deg(j) = 2$ for every $j \in \mathbb{Z}/(n+1)\mathbb{Z}$ (we defined the degree in (2.10)). By Lemma 2.12, we deduce that $I_2(\beta) = \emptyset$. Therefore we can write $\beta = \sum_{i \in \mathbb{Z}/(n+1)\mathbb{Z}} n_i \alpha_i$, where $n_i \in \{0, 1\}$ for $i \in \mathbb{Z}/(n+1)\mathbb{Z}$. We conclude with Proposition 2.11.

Conversely, take $k, m \in \llbracket 0, n \rrbracket$ such that $k < m$. Then since $\langle \alpha_{\bar{i}+1}^\vee, \alpha_{\bar{i}} \rangle = \langle \alpha_{\bar{i}-1}^\vee, \alpha_{\bar{i}} \rangle = -1$ for every $\bar{i} \in \mathbb{Z}/(n+1)\mathbb{Z}$, we have $\alpha_{\bar{k}, \bar{m}} = r_{\bar{k}} \dots r_{\bar{m}-1} \cdot \alpha_{\bar{m}} = \alpha_{\bar{k}, \bar{m}}$ and $r_{\bar{k}'} \dots r_{\bar{m}-1} \cdot \alpha_{\bar{m}} = \alpha_{\bar{k}'}^\vee + \alpha_{\bar{k}'+1}^\vee + \dots + \alpha_{\bar{m}}^\vee$ for every $k' \in \llbracket k, m \rrbracket$. Then by Lemma 2.6, we deduce that $\alpha_{\bar{k}, \bar{m}}$ is quantum. Similarly, $\alpha_{\bar{m}, \bar{k}}$ is quantum, which proves the lemma. \square

Note that the case $A_1^{(1)}$ is a particular case of subsection 2.7.4.

2.7.3 Type $D_n^{(1)}$, $n \geq 4$

Let $n \in \mathbb{Z}_{\geq 4}$. Let $I = I_{D_n^{(1)}}$ be the Dynkin diagram of $D_n^{(1)}$ (see Figure 2.13). For Γ a subgraph of I , we set $\beta_\Gamma = \sum_{i \in \Gamma} \alpha_i$ and for (Γ, Γ') a pair of subgraphs of I we set $\beta_{(\Gamma, \Gamma')} = \sum_{i \in \Gamma} \alpha_i + \sum_{i \in \Gamma'} \alpha_i$.

Let $\iota : I \rightarrow I$ be the automorphism defined by $\iota(j_i) = j_{m-i}$ for all $i \in \llbracket 0, m \rrbracket$, $\iota(j_{m'}) = j_{0'}$ and $\iota(j_{0'}) = j_{m'}$.

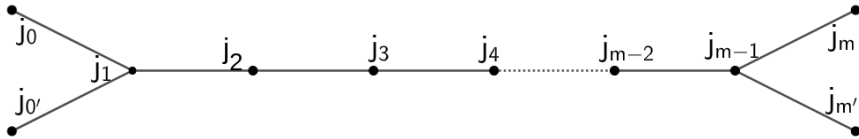


Figure 2.13: Dynkin diagram of type $D_n^{(1)}$

Proposition 2.40. *1. Let Γ be a connected subgraph of I . Then β_Γ is a quantum root.*

2. Let (Γ, Γ') be a pair of connected subgraphs of I such that $\Gamma' \subset \Gamma$, $\{j_0, j_{0'}\} \subset \Gamma$, $\{j_0, j_{0'}, j_1, j_2\} \subset \Gamma'$, $j_m, j_{m'} \notin \Gamma'$ and such that $|\{j_m, j_{m'}\} \cap \Gamma| \leq 1$. Then $\beta_{(\Gamma, \Gamma')}$ and $\beta_{(\iota(\Gamma), \iota(\Gamma'))}$ are quantum roots.
3. Let $k, \ell \in \llbracket 2, m-1 \rrbracket$ be such that $k+1 < \ell$. Then $\sum_{i \in I} \alpha_i + \sum_{i=1}^k \alpha_{j_i} + \sum_{i=m-1}^{\ell} \alpha_{j_i}$ is a quantum root.
4. Let β be a quantum root of Φ_+ . Then $I_3(\beta) = \emptyset$ and β belongs to the set of roots described in (1), (2) and (3).

Proof. Note that since the Kac-Moody matrix defining Γ is symmetric, if $\beta \in \Phi$ is written $\beta = \sum_{i \in I} n_i \alpha_i$, with $(n_i) \in \mathbb{Z}^I$, then $\beta^\vee = \sum_{i \in I} n_i \alpha_i^\vee$.

(1) Follows from Proposition 2.11, since $a_{i_j} = -1$ for all $i, j \in I$ with $i \neq j$.

(2) Set $\beta_\emptyset = \sum_{i \in \Gamma} \alpha_i$. Then β_\emptyset is quantum by (1). Let $k = \max\{i \in \llbracket 2, m-1 \rrbracket \mid j_i \in \Gamma'\}$. Then by successive applications of Proposition 2.8,

$$\beta_{(\Gamma, \Gamma')} = r_{j_{0'}} r_{j_0} r_{j_k} r_{j_{k-1}} \cdots r_{j_2} r_{j_1} (\beta_\emptyset)$$

is quantum. By symmetry, $\beta_{(\iota(\Gamma), \iota(\Gamma'))}$ is quantum.

(3) Set $\beta_\emptyset = \sum_{i \in I} \alpha_i$. By successive applications of Proposition 2.8,

$$\sum_{i \in I} \alpha_i + \sum_{i=1}^k \alpha_{j_i} + \sum_{i=m-1}^{\ell} \alpha_{j_i} = \left(\prod_{i=\ell}^{m-1} r_{j_i} \right) \left(\prod_{t=k}^1 r_{j_t} \right) (\beta_\emptyset) = r_{j_\ell} r_{j_{\ell+1}} \cdots r_{j_{m-2}} r_{j_{m-1}} r_{j_k} r_{j_{k-1}} \cdots r_{j_1} (\beta_\emptyset)$$

is quantum.

(4) Let β be a quantum root. If $I_2(\beta) = \emptyset$, then β is as in (1) by Proposition 2.11. Assume now that $I_2(\beta)$ is non-empty. Let E be the set of elements of I which are minimal for \leq_2 . Then by Lemma 2.12, $E \subset \{j_1, j_{m-1}\}$ and $|E|$ is the number of connected components of $I_2(\beta)$. First assume that $|E| = 1$. Up to replacing β by $\iota(\beta)$ (where $\iota(\beta) = \sum_{i \in I} k_{i(i)} \alpha_i$, if $\beta = \sum_{i \in I} k_i \alpha_i$), we may assume that $E = \{j_1\}$. Then as $\deg(j_1) = 3$, we have $j_0, j_{0'}, j_2 \in I_1(\beta)$. By Lemma 2.17 2) applied with $(j_{k-1}, j_k) = (j_1, j_0)$, we have $\infty \geq \mathbf{t}(j_0, 2) \geq \mathbf{t}(j_1, 3)$ and similarly $\mathbf{t}(j_{0'}, 2) \geq \mathbf{t}(j_1, 3)$. By Lemma 2.15, $\mathbf{t}(j_2, 3) \geq \mathbf{t}(j_1, 3)$. Suppose $t := \mathbf{t}(j_1, 3) < \infty$. Then $\gamma_t^\vee = \alpha_{j_0}^\vee + \alpha_{j_{0'}}^\vee + 2\alpha_{j_1}^\vee + x\alpha_{j_2}^\vee + ?$, with $x \in \{1, 2\}$. Then $\alpha_{j_1}(\gamma_t^\vee) = 4 - 2 - x \geq 0$ and thus $\alpha_{j_1}(\gamma_t^\vee) \neq -1$: a contradiction. Consequently, $j_1 \notin I_3(\beta)$ and by Lemma 2.15, $I_3(\beta) = \emptyset$. Therefore $j_0, j_{0'} \notin I_2(\beta)$. As $\text{Supp}(j_1) = \{j_0, j_{0'}, j_2\}$, if $I_2(\beta) \neq \{j_1\}$, then $j_2 \triangleright_2 j_1$. By Proposition 2.13, $I_2(\beta) = \{j_1, j_2, \dots\}$ is totally ordered for \leq_2 , with $j_1 \leq_2 j_2 \leq_2 j_3 \leq_2 \dots$. Let x be the maximal element of $I_2(\beta)$. By Lemma 2.12, $\deg(x) \leq 2$ and thus if $x = j_{m-1}$, then $|\{j_m, j_{m'}\} \cap I_1(\beta)| \leq 1$. By (2.11), $j_m, j_{m'} \notin I_2(\beta)$. Then β belongs to the set of roots described in 2).

We now assume $|E| = 2$. Then $E = \{j_1, j_{m-1}\}$. Similarly as above, $I_3(\beta) = \emptyset$. Similarly as above, $j_0, j_{0'}, j_m, j_{m'} \in I_1(\beta)$. As $I_1(\beta)$ is connected, we deduce that $I_1(\beta) = I$. By Proposition 2.13, the unique maximal chains for \leq_2 containing j_1 and j_{m-1} respectively are

$$j_1 \triangleleft_2 \dots \triangleleft_2 j_k, \text{ and } j_m \triangleleft_2 j_{m-1} \triangleleft_2 \dots \triangleleft_2 j_\ell,$$

for some $k \in \llbracket 1, m-2 \rrbracket$ and some $\ell \in \llbracket 2, m-1 \rrbracket$. By Lemma 2.12 (3), $j_{k+1}, j_{\ell-1} \in I_2 \setminus I_1$. Therefore $k+1 \leq \ell-1$, and the proposition follows. \square

2.7.4 Case where the Kac-Moody matrix has no coefficient -1

Lemma 2.41. *Let $\beta \in \Phi_+$ be a quantum root. Write $\beta = r_{i_L} \dots r_{i_2} \alpha_{i_1}$, with $i_1, \dots, i_L \in I$ and $L \in \mathbb{N}$ minimal. Then we have $a_{i_2, i_1} = -1$ (where $A = (a_{i,j})_{i,j \in I}$ is the Kac-Moody matrix of the Kac-Moody datum). In particular, if $a_{i,j} \neq -1$ for every $i, j \in I$, the set of quantum roots is exactly the set of simple roots.*

Proof. We have $s_\beta = r_{i_L} \dots r_{i_2} r_{i_1} r_{i_2} \dots r_{i_L}$. We have $r_{i_L} \dots r_{i_3} \alpha_{i_2} \in \text{Inv}(s_\beta) \setminus \{\beta\}$ and

$$\langle r_{i_L} \dots r_{i_2} \alpha_{i_1}^\vee, r_{i_L} \dots r_{i_3} \alpha_{i_2} \rangle = -\langle \alpha_{i_1}^\vee, \alpha_{i_2} \rangle = -a_{i_2, i_1} = 1.$$

\square

3 From quantum roots to covers for the affine Bruhat order

In this section, we use the finiteness of the set of quantum roots to obtain finiteness results on the affine Bruhat order on the Weyl semi-group associated to a Kac-Moody datum. We start this section by the introduction of this object and of the affine Bruhat order.

3.1 Definition and notations: Affine Weyl semi-group

Let $\mathcal{D} = (A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$ be a Kac-Moody root datum. Let W^v be the vectorial Weyl group and Φ the root system attached to it.

Dominant coweights and Tits cone Elements of Y are called coweights, a coweight λ is **regular** if $\langle \lambda, \alpha_i \rangle \neq 0$ for all $i \in I$, or equivalently if its fixator in W^v is trivial. We say that a coweight $\lambda \in Y$ is **dominant** if $\langle \lambda, \alpha_i \rangle \geq 0$ for all $i \in I$, or equivalently if $\langle \lambda, \beta \rangle \geq 0$ for all $\beta \in \Phi_+$. The set of dominant coweights is denoted as $Y^{++} = \{\lambda \in Y \mid \langle \lambda, \alpha_i \rangle \geq 0 \forall i \in I\}$ and the set of dominant regular coweights is denoted $Y_{reg}^{++} = \{\lambda \in Y^{++} \mid \langle \lambda, \alpha_i \rangle > 0 \forall i \in I\}$.

The orbit of Y^{++} by W^v is the **integral Tits cone**, $Y^+ = \bigcup_{w \in W^v} w \cdot Y^{++}$; it is a sub-cone of Y and it is a proper sub-cone if and only if W^v is infinite, if and only if A is not of finite type (see [Kum02, 1.4.2]).

The set Y^{++} is a fundamental domain for the action of W^v on Y^+ , and for any coweight $\lambda \in Y^+$, we denote λ^{++} to be the unique element of Y^{++} in its W^v -orbit.

Parabolic subgroups and minimal coset representatives Let $\lambda \in Y^{++}$ be a dominant coweight. Then the fixator of λ in W^v is a standard parabolic subgroup, which we denote by W_λ . Explicitly, if $J = \{i \in I \mid \langle \lambda, \alpha_i \rangle = 0\}$, then $W_\lambda = W_J := \langle r_j \mid j \in J \rangle$.

Any class $[w] \in W^v/W_\lambda$ admits a unique element of minimal length, and we denote by W^λ the set of coset representatives of minimal length. Therefore $w \in W^\lambda \iff \ell(w) \leq \ell(\tilde{w}) \forall \tilde{w} \in wW_\lambda$.

Suppose now that $\lambda \in Y^+$ (not necessarily dominant), we denote by v^λ the unique element of minimal length such that $\lambda = v^\lambda \lambda^{++}$, it is an element of $W^{\lambda^{++}}$. The coweight λ also admits a parabolic fixator in W^v , which is $W_\lambda = v^\lambda W_{\lambda^{++}} (v^\lambda)^{-1}$.

A coweight λ is **spherical** if its fixator W_λ is finite, and it is **regular** if W_λ is trivial. These notions only depend on the dominant part λ^{++} of λ . We denote by Y_{sph}^+ the set of spherical coweights.

Affine Weyl semi-group Through the left action $W^v \curvearrowright Y$ we can form the semidirect product $Y \rtimes W^v$, for $(\lambda, w) \in Y \rtimes W^v$ we denote by $\varpi^\lambda w$ the corresponding element of $Y \rtimes W^v$. If W^v is finite, this semi-direct product is (a finite extension of) a Coxeter group, hence admits a Bruhat order, a Bruhat length and a set of simple reflections. However when W^v is infinite, it does not have a Coxeter structure, and there is no known analog of the Bruhat order on it.

By definition, $Y^+ \subset Y$ is stable by the action of W^v , therefore we can form $W^+ = Y^+ \rtimes W^v$ which is a sub semi-group of $Y \rtimes W^v$. This semi-group is called **the affine Weyl semi-group**, it is sometimes referred as "the affine Weyl group" in the literature, even though it is not a group.

Eventhough $Y \rtimes W^v$ is a priori not a Coxeter group when W^v is infinite, Braverman, Kazhdan and Patnaik have defined a preorder on W^+ in [BKP16, B.2] using an affine root system on which $Y \rtimes W^v$ acts.

We denote by proj^{Y^+} the projection function $W^+ \rightarrow Y^+$ and $\text{proj}^{Y^{++}}$ the projection on the dominant coweight, so if $\mathbf{x} = \varpi^\lambda w \in W^+$, then $\text{proj}^{Y^+}(\mathbf{x}) = \lambda$ and $\text{proj}^{Y^{++}}(\mathbf{x}) = \lambda^{++}$. We say that an element $\mathbf{x} \in W^+$ is **spherical** if its coweight $\text{proj}^{Y^+}(\mathbf{x})$ is spherical.

Affine roots Let $\Phi^a = \Phi \times \mathbb{Z}$, it is **the set of affine roots** of \mathcal{D} . Let $(\beta, n) \in \Phi^a$, we say that (β, n) is positive if $n > 0$ or $(n = 0 \text{ and } \beta \in \Phi_+)$; we write Φ_+^a for the set of positive affine roots. We also have $\Phi^a = \Phi_+^a \sqcup -\Phi_+^a$. We refer to [Phi23, 1.2] for a geometric interpretation of affine roots and of the affine Bruhat order.

The semidirect product $Y \rtimes W^v$ acts on Φ^a by:

$$\varpi^\lambda w.(\beta, n) = (w\beta, n + \langle \lambda, w\beta \rangle). \quad (3.1)$$

For any $n \in \mathbb{Z}$, its sign is denoted $\text{sgn}(n) \in \{-1, +1\}$, with the convention that $\text{sgn}(0) = +1$.

For $n \in \mathbb{Z}$ and $\beta \in \Phi_+$, we set:

$$\beta[n] = (\text{sgn}(n)\beta, |n|\varpi) \in \Phi_+^a \quad (3.2)$$

$$s_{\beta[n]} = \varpi^{n\beta^\vee} s_\beta \in Y \rtimes W^v. \quad (3.3)$$

If $n \neq 0$ we also define $\beta[n] \in \Phi_+^a$ for $\beta \in \Phi_-$, by $\beta[n] = (-\beta)[-n]$. Note that the element $s_{\beta[n]}$ does not belong to W^+ in general, but it is an element of order 2 in $Y \rtimes W^v$, the **affine reflection associated to $\beta[n]$** .

Affine Bruhat order The **affine Bruhat order** on W^+ is the transitive closure of the relation $<$ defined by, for $\mathbf{x} \in W^+$ and $\beta[n] \in \Phi_+^a$: $s_{\beta[n]}\mathbf{x} < \mathbf{x} \iff \mathbf{x}^{-1}(\beta[n]) \notin \Phi_+^a$. In some sense this means that $\beta[n]$ belongs to the inversion set of \mathbf{x}^{-1} . Note in particular that if $\mathbf{x}^{-1}(\beta[n]) \notin \Phi_+^a$ and \mathbf{x} lies in W^+ then so does $s_{\beta[n]}\mathbf{x}$.

Explicitly, if $\mathbf{x} = \varpi^\lambda w \in W^+$, then $\mathbf{x}^{-1}(\beta[n]) \notin \Phi_+^a$ if and only if $|n| < \text{sgn}(n)\langle \lambda, \beta \rangle$, or $|n| = \text{sgn}(n)\langle \lambda, \beta \rangle$ and $\text{sgn}(n)w^{-1}\beta \in \Phi_-$.

D. Muthiah (see [Mut18]) has proved that the transitive closure of $<$ is anti-symmetric, hence it is an order. Moreover, together with D. Orr, they define in [MO19] a \mathbb{Z} -valued length function on W^+ strictly compatible with the affine Bruhat order which we now introduce.

Affine Bruhat length If $\varpi^\lambda w \in W^+$, the **affine Bruhat length** is defined by:

$$\ell^a(\varpi^\lambda w) = 2\text{ht}(\lambda^{++}) + \ell_{v^\lambda}(w). \quad (3.4)$$

The function ℓ_{v^λ} is the relative Bruhat length on W^v : $\ell_v(w) = \ell(v^{-1}w) - \ell(v)$ for all $v, w \in W^v$. The equivalence between this definition of the affine Bruhat length and its definition in [MO19] is given by [Phi23, Proposition 2.10]. This length is strictly compatible with the affine Bruhat order by [MO19, Theorem 3.3].

Covers and gradation of partially ordered sets Let $(X, <)$ be a partially ordered set, for $x, y \in X$, we say that y **covers** x if: $x < y$ and $\{z \in X \mid x < z < y\} = \emptyset$. We write $x \triangleleft y$ if y covers x .

A length function $\ell : X \rightarrow \mathbb{Z}$ is said to be **strictly compatible** with $<$ if, for a comparable pair $(x, y) \in X^2$, we have $x < y \iff \ell(x) < \ell(y)$. Moreover it is a **grading** of $(X, <)$ if: y covers x if and only if $y > x$ and $\ell(y) = \ell(x) + 1$.

The Bruhat length on W^v is an example of grading (actually \mathbb{N} valued) of $(W^v, <)$, where $<$ is the Bruhat order, we refer to [BB05, Chapter 2] for general properties of the Bruhat order.

The affine Bruhat length is a \mathbb{Z} grading of $(W^+, <)$ (by [MO19, Theorem 6.2] in particular cases and [Phi23] in the general case): for all $\mathbf{x}, \mathbf{y} \in W^+$ such that $\mathbf{x} < \mathbf{y}$,

$$\mathbf{x} \triangleleft \mathbf{y} \iff \ell^a(\mathbf{y}) = \ell^a(\mathbf{x}) + 1. \quad (3.5)$$

The next lemma is a weaker reformulation of [Phi23, Proposition 3.20], which is a more explicit classification of covers in W^+ . In the reductive case, this is a reformulation of the classification of covers given by Schremmer in terms of quantum Bruhat graph (see [Sch24, Proposition 4.5]).

Lemma 3.1. *Let $\mathbf{x}, \mathbf{y} \in W^+$ and suppose that \mathbf{y} covers \mathbf{x} . Then there are unique elements $\lambda \in Y^{++}$, $v, w \in W^v$, $\beta \in \Phi_+$ and $n \in \mathbb{Z}$ such that:*

- v is a minimal coset representative: $v \in W^\lambda$
- $\mathbf{x} = \varpi^{v\lambda}w$
- $\mathbf{y} = s_{v(\beta)[n]}\mathbf{x} = \varpi^{v(s_\beta\lambda+n\beta^\vee)}s_{v(\beta)}w$.

Moreover $n \in \{0, \langle \lambda, \beta \rangle, -1, \langle \lambda, \beta \rangle + 1\}$, and $\text{proj}^{Y^{++}}(\mathbf{y}) = \text{proj}^{Y^{++}}(\mathbf{x})$ if and only if $n \in \{0, \langle \lambda, \beta \rangle\}$.

Furthermore:

- i) If $n = \langle \lambda, \beta \rangle$ (possibly equal to zero), then $\ell(s_\beta v^{-1}w) = \ell(v^{-1}w) + 1$, so $v^{-1}w \triangleleft s_\beta v^{-1}w$*
- ii) If $n = 0 \neq \langle \lambda, \beta \rangle$, then $\ell(vs_\beta) = \ell(v) - 1$, so $vs_\beta \triangleleft v$.*
- iii) If $n \in \{-1, \langle \lambda, \beta \rangle + 1\}$ then β is a quantum root and there exists an element $u \in W^v$ such that $\text{proj}^{Y^+}(\mathbf{y}) = vu.\text{proj}^{Y^{++}}(\mathbf{y})$ and $\ell(vu) = \ell(v) + \ell(u)$, in particular $v \leq vu$.*

The first part is a direct consequence of the definition of the affine Bruhat order and of [Phi23, Prop. 3.1], items *i)* and *ii)* correspond to items 1) and 2) of [Phi23, Proposition 3.20] whereas *iii)* corresponds to subitems (a), (c) of items 3) and 4) of the same reference.

In particular, we use finiteness of the set of quantum roots to show that any element $\mathbf{x} \in W^+$ has a finite number of covers, and if it is spherical, then it also has a finite number of co-covers. This extends to the general Kac-Moody case a result obtained by Welch in the affine ADE case ([Wel22, Theorem 1.1]). We provide examples of non-spherical elements which admit an infinite number of co-covers, and we give a characterization of the irreducible Kac-Moody root systems for which every element admits a finite number of co-covers. At the end of this section, we construct explicit covers in W^+ associated to any given quantum root, therefore showing that the notion of quantum root is the most refined one in this context.

3.2 Finiteness results for covers and co-covers

Theorem 3.2. *Suppose that \mathbf{x} is an element of an affine Weyl semi-group W^+ . Then \mathbf{x} has a finite number of covers for the affine Bruhat order.*

Proof. For any $u \in W^v$, let $n_{cov}(u)$ (resp. $n_{cocov}(u)$) denote the number of covers (resp. cocovers) of u in W^v . Since W^v is a Coxeter group of finite rank, these numbers are finite.

Let \mathbf{x} be an element of W^+ . Then, by item *iii)* of Lemma 3.1, the number of covers \mathbf{y} of \mathbf{x} such that $\text{proj}^{Y^{++}}(\mathbf{y}) \neq \text{proj}^{Y^{++}}(\mathbf{x})$ is bounded by twice the number $N_{asr}(\Phi)$ of quantum roots of Φ , which is finite by Theorem 2.27. Moreover, write $\mathbf{x} = \varpi^{v\lambda}w$ with $\lambda = \text{proj}^{Y^{++}}(\mathbf{x})$ and $v \in W^v$ minimal, then the number of covers \mathbf{y} such that $\text{proj}^{Y^{++}}(\mathbf{y}) = \text{proj}^{Y^{++}}(\mathbf{x})$ is exactly $n_{cov}(v^{-1}w) + n_{cocov}(v)$ using items *i), ii)* of the same lemma. Hence \mathbf{x} has at most $2N_{asr}(\Phi) + n_{cov}(v^{-1}w) + n_{cocov}(v)$ covers. \square

Corollary 3.3. *Let $n \in \mathbb{Z}_{\geq 0}$ and $\mathbf{x} \in W^+$. Then*

$$\{\mathbf{y} \in W^+ \mid \exists (\mathbf{x}_1, \dots, \mathbf{x}_n) \in (W^+)^n, \mathbf{x} = \mathbf{x}_1 \triangleleft \mathbf{x}_2 \triangleleft \dots \triangleleft \mathbf{x}_n = \mathbf{y}\}$$

is finite.

Corollary 3.4. *Let $\mathbf{x}, \mathbf{y} \in W^+$. Then $[\mathbf{x}, \mathbf{y}] := \{\mathbf{z} \in W^+ \mid \mathbf{x} \leq \mathbf{z} \leq \mathbf{y}\}$ is finite.*

Proof. Assume $\mathbf{x} \leq \mathbf{y}$. As the Bruhat order is strictly compatible with ℓ^a (by [MO19, Theorem 3.3]), for every $\mathbf{z} \in [\mathbf{x}, \mathbf{y}]$, there exist $k \in \llbracket 0, \ell^a(\mathbf{y}) - \ell^a(\mathbf{x}) \rrbracket$ and $\mathbf{x}_1, \dots, \mathbf{x}_k \in W^+$ such that $\mathbf{x} = \mathbf{x}_1 \triangleleft \mathbf{x}_2 \triangleleft \dots \triangleleft \mathbf{x}_k = \mathbf{z}$ and thus the result follows from Corollary 3.3. \square

Lemma 3.5. *Let $Y_{in} = \{\lambda \in Y \mid \langle \lambda, \alpha_i \rangle = 0, \forall i \in I\}$. Let $\lambda \in Y_{in}$, $\mu \in Y^+$ and $v, w \in W^v$. Then $\varpi^\mu w \leq \varpi^\lambda v$ if and only if $\mu = \lambda$ and $v \leq w$ (for the Bruhat order on W^v).*

Proof. Let $\beta \in \Phi_+$ and $n \in \mathbb{Z}$ be such that $s_{\beta[n]} \varpi^\lambda v < \varpi^\lambda v$, note that $s_{\beta[n]} \varpi^\lambda v = \varpi^{\lambda+n\beta^v} s_\beta v$. Then by Lemma [Phi23, Lemma 1.7] we have $\lambda + n\beta^v \in [\lambda, s_\beta \lambda] = \{\lambda\}$ and hence $n = 0$.

As $\lambda \in Y_{in}$, we have $W^\lambda = \{1\}$. Now $s_\beta \varpi^\lambda w = \varpi^\lambda s_\beta w < \varpi^\lambda w$ if and only if $\ell^a(\varpi^\lambda s_\beta w) < \ell^a(\varpi^\lambda w)$ if and only if $\ell(s_\beta w) < \ell(w)$ if and only if $s_\beta w < w$ for the Bruhat order on W^v . Lemma follows. \square

Theorem 3.6. *Suppose that $\mathbf{y} \in W^+$ is spherical, or that $\text{proj}^{Y^+}(\mathbf{y}) \in Y_{in}$. Then it admits a finite number of co-covers.*

Proof. Suppose that \mathbf{y} is spherical, let us write $\mathbf{y} = \varpi^\mu w$ with $\mu \in Y_{sph}^+$. Let $C(\mathbf{y})$ denote the set of co-covers of \mathbf{y} in W^+ . We have $C(\mathbf{y}) = C_1(\mathbf{y}) \sqcup C_2(\mathbf{y})$, where $C_1(\mathbf{y}) = \{\mathbf{x} \in C(\mathbf{y}) \mid \text{proj}^{Y^{++}}(\mathbf{x}) = \text{proj}^{Y^{++}}(\mathbf{y}) = \mu^{++}\}$ and $C_2(\mathbf{y}) = C(\mathbf{y}) \setminus C_1(\mathbf{y})$. We show separately that both of these sets are finite. Recall that v^μ denotes the minimal element of $\{v \in W^v \mid \mu = v(\mu^{++})\}$.

- Suppose that $\mathbf{x} \in C_1(\mathbf{y})$, then by Lemma 3.1, we can write $\mathbf{x} = \varpi^{v(\mu^{++})} \tilde{w}$ and $\mathbf{y} = s_{v(\beta)[n]}(\mathbf{x})$ for some $\beta \in \Phi_+$, $v \in W^{\mu^{++}}$, $\tilde{w} \in W^v$ and $n \in \{0, \langle \mu^{++}, \beta \rangle\}$. The element \tilde{w} is entirely determined by (v, β) since $\tilde{w} = s_{v(\beta)} w$. So we need to prove that there is a finite number of viable choices for (v, β) . We have:
 - If $n = 0 \neq \langle \mu^{++}, \beta \rangle$, then $\mu = vs_\beta(\mu^{++})$, so vs_β lies in $v^\mu.W_{\mu^{++}}$, where $W_{\mu^{++}}$ is the stabilizer of μ^{++} in W^v . By assumption, μ is spherical so $W_{\mu^{++}}$ is finite, and so is $v^\mu.W_{\mu^{++}}$. Moreover, by item *ii*) of Lemma 3.1, v covers vs_β . Since W^v is a Coxeter group of finite rank, any element of $v^\mu.W_{\mu^{++}}$ admits a finite number of covers, hence the pair (v, β) can only take a finite number of values.
 - If $n \neq 0$, then \mathbf{x} and \mathbf{y} have the same co-weight μ , so $v = v^\mu$. Moreover, $\tilde{w} = s_{v(\beta)} w = vs_\beta v^{-1} w$ and therefore by item *i*) of Lemma 3.1, $v^{-1} w = s_\beta v^{-1} \tilde{w}$ covers $v^{-1} \tilde{w} = s_\beta v^{-1} w$. So the number of viable choices of β is equal to the number of co-covers of $v^{-1} w$, which is finite.

Either way, we thus have shown that $C_1(\mathbf{y})$ is finite, let us now turn to $C_2(\mathbf{y})$.

- Suppose that $\mathbf{x} \in C_2(\mathbf{y})$, by Lemma 3.1 we can write $\mathbf{x} = \varpi^{v\lambda}\tilde{w}$ with $\lambda \in Y^{++}$, $\tilde{w} \in W^v$, $v \in W^\lambda$ and $\mathbf{y} = s_{v(\beta)[n]}\mathbf{x}$ for $\beta \in \Phi_+$ and $n \in \{-1, \langle \lambda, \beta \rangle + 1\}$. The coweight λ is either equal to $v^{-1}\mu - \beta^\vee$ (if $n \neq 1$) or to $s_\beta v^{-1}\mu - \beta^\vee$ (if $n = -1$). On top of that, $\tilde{w} = s_{v(\beta)}w$ and thus it suffices to show that the set of pairs (v, β) such that $\mathbf{y} \in \{\varpi^{v(\lambda+\beta^\vee)}w, \varpi^{vs_\beta(\lambda+\beta^\vee)}w\}$ and is a cover of $\varpi^{v\lambda}s_{v(\beta)}w$ (for one of these two choices of λ) is finite. Note that, since μ is spherical, the set $\{\tilde{v} \in W^v \mid \mu = \tilde{v}\mu^{++}\} = v^\mu.W_{\mu^{++}}$ is finite so we can define $L = \max\{\ell(\tilde{v}) \mid \tilde{v} \in W^v, \tilde{v}\mu^{++} = \mu\}$. Then by item *iii*) of Lemma 3.1, we have $\ell(v) < L$, and β is a quantum root. Therefore, since $\{v \in W^v \mid \ell(v) < L\}$ is finite and the set of quantum roots is finite as well by Theorem 2.27, we deduce that $C_2(\mathbf{y})$ is finite.

If $\mathbf{y} \in W^+$ is such that $\text{proj}^{Y^+}(\mathbf{y}) \in Y_{in} = \bigcap_{i \in I} \ker(\alpha_i)$, then by Lemma 3.5 $\text{proj}^{Y^+}(\mathbf{x}) = \text{proj}^{Y^+}(\mathbf{y})$ for all $\mathbf{x} \in W^+$ such that $\mathbf{x} < \mathbf{y}$, and there is as many co-covers of \mathbf{y} as co-covers of w in W^v , hence a finite number. \square

Remark 3.7. *When Y is associated with an affine Kac-Moody matrix, then every element of Y^+ is either spherical or in Y_{in} (by [Kac94, Proposition 5.8 b])). Thus the theorem above generalizes [Wel22, Corollary 24].*

Recall that $A = (\alpha_j(\alpha_i^\vee))_{i,j \in I}$ is the Kac-Moody matrix of the root datum \mathcal{D} . Following [Kac94, 1], we say that the matrix A is **indecomposable** if for every non-empty subsets $J, K \subset I$ such that $J \sqcup K = I$, there exists $(j, k) \in J \times K$ such that $\alpha_j(\alpha_k^\vee) \neq 0$.

Lemma 3.8. *Assume that A is indecomposable and that there exists $\mu \in Y^{++}$ such that W_μ is infinite and W^v does not fix μ . Then there exists $\mathbf{x} \in W^+$ admitting infinitely many co-covers.*

Proof. Let $J = \{i \in I \mid \alpha_i(\mu) = 0\}$. Then $J \subsetneq I$. Let $J' \subset J$ be such that $W_{J'}$ is infinite and for every $J'' \subsetneq J'$, $W_{J''}$ is finite. Let $\nu \in Y$ be such that $\alpha_j(\nu) = 3$ for every $j \in I \setminus J'$ and $\alpha_j(\nu) = 0$ for every $j \in J'$. Then $\nu \in Y^{++}$ and $W_\nu = W_{J'}$.

As A is indecomposable, there exists $(i, j) \in (I \setminus J') \times J'$ such that $\alpha_i(\alpha_j^\vee) \neq 0$. Set $\lambda = \nu - \alpha_i^\vee \in]\nu, r_i.\nu[$. Then $\alpha_j(\lambda) = \alpha_j(\nu) - \alpha_j(\alpha_i^\vee) = -\alpha_j(\alpha_i^\vee) > 0$, $\alpha_i(\lambda) = \alpha_i(\nu) - 2 > 0$ and for every $i' \in I \setminus \{j\}$, we have $\alpha_{i'}(\lambda) = \alpha_{i'}(\nu) - \alpha_{i'}(\alpha_j^\vee) \geq \alpha_{i'}(\nu) \geq 0$. Therefore $\lambda \in Y^{++}$ and $\{k \in I \mid \alpha_k(\lambda) = 0\} \subset J' \setminus \{j\}$. Thus $W_\lambda \subsetneq W_\nu$ and by assumption on J' , W_λ is finite.

Let $w \in W_\nu \cap W^\lambda$ (we prove below that this set is non-empty). Set $\beta = w.\alpha_i$ and $\mathbf{x}_w = \varpi^{w.\lambda}s_\beta$. We have:

$$s_{\beta[(w.\lambda, \beta)+1]}\mathbf{x}_w = \varpi^{(\langle w.\lambda, \beta \rangle + 1)\beta^\vee} \varpi^{s_\beta w.\lambda} = \varpi^\nu$$

and thus $\mathbf{x}_w < \varpi^\nu$ if and only if $\ell^a(\mathbf{x}_w) < \ell^a(\varpi^\nu)$. Moreover we have:

$$\begin{aligned} \ell^a(\mathbf{x}_w) &= 2\text{ht}(\lambda) + \ell_w(s_{w.\alpha_i}) \\ &= 2(\text{ht}(\nu) - 1) + \ell(w^{-1}wr_iw^{-1}) - \ell(w) \\ &= 2\ell^a(\varpi^\nu) - 2 + \ell(r_iw^{-1}) - \ell(w). \end{aligned}$$

Therefore

$$\ell^a(\mathbf{x}_w) \in \{\ell^a(\varpi^\nu) - 1, \ell^a(\varpi^\nu) - 3\}, \quad (3.6)$$

for every $w \in W^\lambda \cap W_\nu$.

Let $w \in W_\nu$. Then we can write uniquely $w = vu$, with $v \in W^\lambda$, $u \in W_\lambda$. But then $u \in W_\nu$ and hence $v \in W^\lambda \cap W_\nu$. Therefore $W_\nu \cdot \lambda = (W_\nu \cap W^\lambda) \cdot \lambda$. As W_λ is finite, $W_\nu \cdot \lambda$ is infinite and thus $(W^\lambda \cap W_\nu) \cdot \lambda$ and $\{\mathbf{x}_w \mid w \in W^\lambda \cap W_\nu\}$ are infinite.

For $\mathbf{y} \in W^+$ denote by $C'(\mathbf{y})$ the union of the set of co-covers of \mathbf{y} with $\{\mathbf{y}\}$. By compatibility of ℓ^a with $<$ and by (3.6), for every $w \in W^\nu \cap W^\lambda$, we have $\mathbf{x}_w \in C'(C'(C'(\varpi^\nu)))$. Therefore

$$C'(C'(C'(\varpi^\nu))) \supset \{\mathbf{x}_w \mid w \in W^\lambda \cap W_\nu\},$$

and the right hand side is infinite. Consequently there exists $\mathbf{x} \in W^+$ for which $C'(\mathbf{x})$ is infinite. \square

Example 3.9. Assume that I has cardinality 3 and write $I = \{i, j, k\}$. Assume that $\alpha_m(\alpha_{m'}^\vee)\alpha_{m'}(\alpha_m^\vee) \geq 4$ for every $m, m' \in \{i, j, k\}$. By [Kum02, 1.3.21 Proposition], this implies that $\langle r_i, r_j \rangle$, $\langle r_j, r_k \rangle$ and $\langle r_i, r_k \rangle$ are infinite. Take $\nu \in Y$ such that $\alpha_i(\nu) \geq 3$, $\alpha_j(\nu) = \alpha_k(\nu) = 0$. Set $\lambda = \nu - \alpha_i^\vee \in Y^{++}$. Then $W_\lambda = \{1\}$ and with the same proof as in Lemma 3.8, we can prove that $\varpi^{w \cdot \lambda} w r_i w^{-1}$ is covered by ϖ^ν for every $w \in \langle r_j, r_k \rangle$.

Let $\mathbb{A} = Y \otimes \mathbb{R}$ be the **standard apartment**. The elements of Φ can naturally be regarded as elements of the dual \mathbb{A}^* of \mathbb{A} . The Tits cone \mathcal{T} is then $W^v \cdot \overline{C_f^v}$, where $\overline{C_f^v} = \{x \in \mathbb{A} \mid \langle \lambda, \alpha_i \rangle \geq 0, \forall i \in I\}$.

Proposition 3.10. We assume that $(\alpha_i(\alpha_j^\vee))_{i, j \in I}$ is indecomposable. Let $\mathbb{A}_{in} = \bigcap_{i \in I} \ker(\alpha_i)$. The following properties are equivalent:

1. The set $\mathcal{T} \setminus \mathbb{A}_{in}$ is open.
2. Every proper parabolic subgroup of W^v is finite.
3. For every $\mathbf{x} \in W^+$, the set of co-covers of \mathbf{x} is finite.

Proof. (1) \Rightarrow (2). Assume (1). Let J be a proper subset of I . Let $\lambda \in Y$ be such that $\alpha_j(\lambda) = 0$ for all $j \in J$ and $\alpha_j(\lambda) > 0$ for all $j \in I \setminus J$. Then $\lambda \in \overline{C_f^v} \setminus \mathbb{A}_{in} \subset \mathcal{T} \setminus \mathbb{A}_{in} \subset \overset{\circ}{\mathcal{T}}$. Then $W_\lambda = W_J$ and by [Kac94, Proposition 3.12 f)], W_J is finite.

(2) \Rightarrow (1). Assume (2). Let $x \in \mathcal{T} \setminus \mathbb{A}_{in}$. We can write $x = w \cdot y$, for some $w \in W^v$ and $y \in \overline{C_f^v} \setminus \mathbb{A}_{in}$. Let $J = \{i \in I \mid \alpha_i(y) = 0\}$. Then $J \subsetneq I$ and thus $W_y = W_J$ is finite. By [Kac94, Proposition 3.12 f)], there exists an open neighbourhood V of y contained in \mathcal{T} . Then the fixator of every element of V is finite and thus $V \subset \mathcal{T} \setminus \mathbb{A}_{in}$. Then $x \in w^{-1} \cdot V \subset \mathcal{T} \setminus \mathbb{A}_{in}$, which proves that $\mathcal{T} \setminus \mathbb{A}_{in}$ is open.

(3) \Rightarrow (1) Assume (3). Then by Lemma 3.8, for every $\mu \in Y^+$, we have either W_μ is finite or $W_\mu = W^v$. Let $x \in \mathcal{T}$. Write $x = w \cdot y$, with $w \in W^v$ and $y \in \overline{C_f^v}$. Let $J = \{i \in I \mid \alpha_i(y) = 0\}$. Let $\mu \in Y$ be such that for $i \in I$, $\alpha_i(\mu) = 1$ if $i \in I \setminus J$ and $\alpha_i(\mu) = 0$ if $i \in J$. Then $W_\mu = W_x$. If $W_x = W^v$, then $x \in \mathbb{A}_{in}$ and if W_x is finite, then $x \in \overset{\circ}{\mathcal{T}}$ by [Kac94, Proposition 3.12 f)], which proves that $\mathcal{T} = \mathbb{A}_{in} \sqcup \overset{\circ}{\mathcal{T}}$. Thus we have 1).

(1) \Rightarrow (3) is a consequence of Theorem 3.6. \square

3.3 Explicit covers for a given quantum root

In this section, we construct, for any quantum root, explicit covers where it appears, hence the restriction on $\beta \in \Phi_+$ appearing in Lemma 3.1 is optimal.

Proposition 3.11. *Let β be a quantum root, and let $\lambda \in Y_{reg}^{++}$ be such that $\lambda + \beta^\vee \in Y_{reg}^{++}$ (explicitly, $\langle \lambda, \alpha_i \rangle > -\langle \beta^\vee, \alpha_i \rangle$ for all $i \in I$). Then:*

1. *Let v be any element of W^v and let $w \in W^v$ be such that $\ell(s_\beta w) = \ell(s_\beta) + \ell(w)$ (in particular this is always verified for $w = 1_{W^v}$). Then $\varpi^{v(\lambda+\beta^\vee)}vw$ covers $\varpi^{v(\lambda)}vs_\beta w$.*
2. *Let $v \in W^v$ be such that $\ell(vs_\beta) = \ell(v) + \ell(s_\beta)$ and let w be any element of W^v . Then $\varpi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$ covers $\varpi^{v(\lambda)}w$.*

In particular, for any quantum root β there exist covers of the form given in Lemma 3.1 with varying dominance class.

Proof. Recall the expression of the affine Bruhat length obtained in [Phi23, Prop. 2.10]: For any $\lambda \in Y_{reg}^{++}$, $v, w \in W^v$:

$$\ell^a(\varpi^{v(\lambda)}w) = 2\text{ht}(\lambda) + \ell(v^{-1}w) - \ell(v). \quad (3.7)$$

Thus, for any $\lambda \in Y_{reg}^{++}$ such that $\lambda + \beta^\vee \in Y_{reg}^{++}$:

$$\begin{aligned} \ell^a(s_{v(\beta)[\langle \lambda, \beta \rangle + 1]} \varpi^{v(\lambda)} vs_\beta w) &= \ell^a(\varpi^{v(\lambda+\beta^\vee)}vw) \\ &= 2\text{ht}(\lambda + \beta^\vee) + \ell(w) - \ell(v) \\ &= \ell^a(\varpi^{v(\lambda)}vs_\beta w) + 2\text{ht}(\beta^\vee) + \ell(w) - \ell(s_\beta w). \\ \ell^a(s_{v(\beta)[-1]} \varpi^{v(\lambda)} w) &= \ell^a(\varpi^{vs_\beta(\lambda+\beta^\vee)}vs_\beta v^{-1}w) \\ &= 2\text{ht}(\lambda + \beta^\vee) + \ell(v^{-1}w) - \ell(vs_\beta) \\ &= \ell^a(\varpi^{v(\lambda)}w) + 2\text{ht}(\beta^\vee) + \ell(v) - \ell(vs_\beta). \end{aligned}$$

Therefore, since quantum root are characterized by the fact that $2\text{ht}(\beta^\vee) = \ell(s_\beta) + 1$ (by Lemma 2.6), we obtained that the length difference $\ell^a(s_{v(\beta)[\langle \lambda, \beta \rangle + 1]} \varpi^{v(\lambda)} vs_\beta w) - \ell^a(\varpi^{v(\lambda)}vs_\beta w)$ (resp. $\ell^a(s_{v(\beta)[-1]} \varpi^{v(\lambda)} w) - \ell^a(\varpi^{v(\lambda)}w)$) is exactly one if and only if condition 1. (resp. condition 2.) is satisfied. \square

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