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# A NONPARAMETRIC MEDIAN TEST FOR FUNCTIONAL DATA

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**Résumé.** Il a été prouvé que le test de la médiane de Mood est plus puissant que les tests de Student et de Wilcoxon-Mann-Whitney dans le cas des distributions à queues lourdes pour des données univariées. L'extension multivariée du test de la médiane, pour des données multidimensionnelles, s'est également montrée plus efficace que les tests Hotelling  $T^2$  et Wilcoxon-Mann-Whitney dans les cas des distributions à queues lourdes. En se basant sur ces postulats, dans ce travail, nous construisons un test de la médiane basé sur les rangs spatiaux pour données fonctionnelles et obtenons des résultats asymptotiques. Ensuite, nous le comparons avec plusieurs tests concurrents en utilisant des données fonctionnelles simulées et réelles: comme dans les cas univariés et multivariés, nos résultats montrent que le test proposé est plus adapté aux distributions à queues lourdes.

**Mots-clés.** Distributions à queues lourdes, Données fonctionnelles, Espace de Hilbert séparable, Test de localisation pour deux échantillons, . . .

**Abstract.** The Mood median test has been proven to be more powerful than the Student t-test and the Wilcoxon-Mann-Whitney test in heavy-tailed cases for univariate data. The multivariate extension of the median test, for multidimensional data, was shown to be more efficient than the Hotelling  $T^2$  and the Wilcoxon-Mann-Whitney tests for high dimensions and in very heavy-tailed cases. Based on these postulates, in this work, we construct a median type test based on spatial ranks for functional data and we obtain asymptotic results. Then, we compare the proposed functional test with a wide range of competing tests using simulated and real functional data: as in the univariate and multivariate cases, our results show that the proposed test is more adapted to heavy-tailed distributions.

**Keywords.** Functional data, Heavy-tailed Distributions, Separable Hilbert space, Two-sample location test, . . .

## 1 Introduction

Statistical hypothesis testing plays a fundamental role in statistics. In nonparametric statistics, tests of hypotheses are known as nonparametric or distribution-free tests. In this setting, it is not necessary to assume hypotheses on the shape of the distribution and estimate its parameters. These tests can be used to verify that two or more datasets come from identical

populations. Here, we will focus on this type of tests to solve the two-sample location problem.

For univariate data, Wilcoxon (1945) and Mann and Whitney (1947) proposed nonparametric tests based on ranks. Each of them defined their own test statistic which leads to the same test named Wilcoxon-Mann-Whitney. This test is more powerful than the Student's t-test for various non-Gaussian distributions (Blair and Higgins, 1980). Another nonparametric test designed by Mood (1950) and is called the median test. Another version of this test based on ranks was presented in Van der Vaart (1998). It is asymptotically optimal in the case of a double exponential distribution. Nowadays, the median test is not often used because it is less powerful than the Wilcoxon-Mann-Whitney test when applied to Gaussian distributions (Mood, 1954). However, when dealing with symmetrical distributions with heavy-tails, this test is more efficient than the Wilcoxon-Mann-Whitney one (Capéraà and Cutsem, 1988).

For multivariate data, several versions of the Hotelling, Wilcoxon-Mann-Whitney and median tests have been studied (Puri and Sen, 1971; Chakraborty and Chaudhuri, 1999; Oja, 2010). The extension of univariate two-sample Mood test has the best efficiency compared to the Hotelling test in heavy-tailed cases and for high dimensions (Oja and Randles, 2004).

Currently, the development of the sensing and computing tools allows us to work with huge datasets. So, we have more and more access to data of functional type (Ramsay and Silverman, 2005). The main particularity of such data is the infinite dimension of the data space such as Banach and Hilbert spaces. Appropriate statistical tools are necessary to handle these types of data. Two-sample tests for functional data have also been proposed by several authors using either parametric or nonparametric techniques, such as in the univariate and multivariate settings. To decide whether two samples of curves are issued from the same distribution, Horváth et al. (2013) proposed two test statistics for testing the equality of mean functions. One of them is the same as the Hotelling statistic in finite dimensional space. Moreover, Cuevas et al. (2004) introduced an analog of the classical one-way analysis of variance (ANOVA) problem for functional data. In addition, in a nonparametric setting, Chakraborty and Chaudhuri (2015) proposed a Wilcoxon-Mann-Whitney test based on spatial ranks.

In this work, we propose a median test statistic based on spatial ranks in Banach space and especially in separable Hilbert space. We study the asymptotic behavior of the proposed test statistic under the null hypothesis and under some shrinking location shift models. Then, we implement the test using its asymptotic distribution on the one hand and using the random permutation method on the other hand. To illustrate our theoretical results, we compare the performance of the proposed test with various other tests using simulated and real functional datasets.

## 2 Construction of the test

### 2.1 Introduction of the median test statistics

First we recall what median tests look like in the univariate case.

### 2.1.1 Univariate case

Let  $X$  and  $Y$  be two  $\mathbb{R}$ -valued random variables. We consider  $X_1, \dots, X_m$  and  $Y_1, \dots, Y_n$  two independent random samples of  $X$  and  $Y$  with distribution functions  $F$  and  $F_\theta$  respectively, such that  $\forall x \in \mathbb{R}; F_\theta(x) = F(x - \theta)$ . The constant  $\theta$  is called *the translation parameter*.

The median test statistic based on ranks for testing the hypothesis

$$H_0 : \theta = 0 \quad \text{against} \quad H_1 : \theta \neq 0$$

is defined as

$$T_{\text{Mo}} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{R_i > 0\}},$$

where  $R_i = 1 + (\sum_{j=1}^m \mathbb{1}_{\{Y_i > X_j\}} + \sum_{k=1}^n \mathbb{1}_{\{Y_i > Y_k\}} - \frac{N+1}{2})$  is the centered rank of  $Y_i$  when  $X_1, \dots, X_m$  and  $Y_1, \dots, Y_n$  are ordered together in the same sample of size  $N = n + m$ . To make things easier afterwards, we introduce a test statistic which counts the number of the observations  $Y_1, \dots, Y_n$  that are greater than the median of the observations  $X_1, \dots, X_m$  instead of the global median. In other words, in the univariate case it is equal to

$$T_{\text{MED}} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{\hat{F}_m(Y_i) > \frac{1}{2}\}},$$

where  $\hat{F}_m(x) = 1/m \sum_{j=1}^m \mathbb{1}_{\{X_j \leq x\}}$  is the empirical distribution function of  $X_1, \dots, X_m$ . This statistic is inspired from the work of Koul and Staudte (1972).

Our goal here is to construct an extension of these tests in infinite dimensional space.

### 2.1.2 Functional case

Now, let  $X$  and  $Y$  be two independent random elements in a Banach space  $\chi$ . We denote by  $\chi^*$  its dual space, i.e., the space of the linear continuous functions on  $\chi$  with values in  $\mathbb{R}$ , and  $\chi^{**}$  its bidual space, i.e., the space of the linear continuous functions on  $\chi^*$  with values in  $\mathbb{R}$ . We denote by  $\|\cdot\|_\chi$  (resp.  $\|\cdot\|_{\chi^*}$ ) a norm on  $\chi$  (resp. on  $\chi^*$ ). Moreover, we consider  $X_1, \dots, X_m$  and  $Y_1, \dots, Y_n$  independent random samples of  $X$  and  $Y$  from two probability measures  $P$  and  $Q$  on  $\chi$ . We suppose that  $P$  and  $Q$  differ by a shift  $\Delta \in \chi$ .

Then, we want to test

$$H_0 : \Delta = 0 \quad \text{against} \quad H_1 : \Delta \neq 0.$$

To construct our test statistics, we suppose that :

- The space  $\chi$  is smooth, i.e., the norm function  $\|\cdot\|_\chi$  is Gateaux differentiable at each  $x \neq 0, x \in \chi$ . We denote by  $\mathbf{SGN}_x \in \chi^*$  its Gateaux derivative.
- The space  $\chi^*$  is smooth, i.e, the norm function  $\|\cdot\|_{\chi^*}$  is Gateaux differentiable at each  $y \neq 0, y \in \chi^*$ . We denote by  $\mathbf{SGN}_y^* \in \chi^{**}$  its Gateaux derivative.

1.  $T_{\text{MED}}$  in the functional case: the test statistic  $T_{\text{MED}}$  becomes

$$\text{MED} = \frac{1}{n} \sum_{i=1}^n \mathbf{SGN}^* \left\{ \frac{1}{m} \sum_{j=1}^m \mathbf{SGN}_{\{Y_i - X_j\}} \right\} = \frac{1}{n} \sum_{i=1}^n \mathbf{SGN}_{F_m(Y_i)}^*, \quad (1)$$

where  $F_m(y) = 1/m \sum_{j=1}^m \mathbf{SGN}_{\{y - X_j\}}$  is the empirical spatial distribution associated to the iid observations  $X_1, \dots, X_m$  (Chakraborty and Chaudhuri, 2014). Remark that, in the univariate case, the empirical spatial distribution is equal to  $2\hat{F}_m(y) - 1$  where  $\hat{F}_m$  is the empirical distribution function of  $X_1, \dots, X_m$ .

2.  $T_{\text{Mo}}$  in the functional case: the test statistic  $T_{\text{Mo}}$  becomes

$$\text{Mo} = \frac{1}{n} \sum_{i=1}^n \mathbf{SGN}^* \left\{ \sum_{k=1}^n \mathbf{SGN}_{Y_i - Y_k} + \sum_{j=1}^m \mathbf{SGN}_{Y_i - X_j} \right\}.$$

**Remark:** When the space  $\chi$  is assumed to be an Hilbert one, the sign function become simpler (Chakraborty and Chaudhuri, 2015) so that  $\mathbf{SGN}_x = \frac{x}{\|x\|_\chi}$ . Thus, the MED statistic (1) can be rewritten as:

$$\text{MED} = \frac{1}{n} \sum_{i=1}^n \frac{\sum_{j=1}^m \frac{Y_i - X_j}{\|Y_i - X_j\|_\chi}}{\left\| \sum_{j=1}^m \frac{Y_i - X_j}{\|Y_i - X_j\|_\chi} \right\|_\chi},$$

and the statistic Mo becomes

$$\text{Mo} = \frac{1}{n} \sum_{i=1}^n \frac{\sum_{k=1, k \neq i}^n \frac{Y_i - Y_k}{\|Y_i - Y_k\|_\chi} + \sum_{j=1}^m \frac{Y_i - X_j}{\|Y_i - X_j\|_\chi}}{\left\| \sum_{k=1, k \neq i}^n \frac{Y_i - Y_k}{\|Y_i - Y_k\|_\chi} + \sum_{j=1}^m \frac{Y_i - X_j}{\|Y_i - X_j\|_\chi} \right\|_\chi}.$$

## 2.2 Asymptotic distribution of the proposed test statistic

In this subsection, we study the asymptotic normality of MED. First, we introduce the following notations which will be used later :

- We denote by  $G(\mathbf{m}, \mathbf{C})$  the distribution of a Gaussian random element in a separable Banach space  $\chi$  with mean  $\mathbf{m} \in \chi$  and covariance  $\mathbf{C}$ , where  $\mathbf{C} : \chi^* \times \chi^* \rightarrow \mathbb{R}$  is a symmetric nonnegative definite continuous bilinear function.

- For all  $x, y \in \chi$ , we define

$$F_X(y) = \mathbb{E}[\mathbf{SGN}_{\{y-X\}}|y], \quad (2)$$

$$F_Y(x) = \mathbb{E}[\mathbf{SGN}_{\{Y-x\}}|x]. \quad (3)$$

These two functions are also used in Chakraborty and Chaudhuri (2015) to prove the theorem of the asymptotic normality of the Wilcoxon-Mann-Whitney test statistic. Moreover, we define

$$\mu = \mathbb{E}[\mathbf{SGN}_{\{F_X(Y)\}}^*]$$

and

$$\tilde{\mu} = \mathbb{E}[\mathbf{SGN}_{\{F_Y(X)\}}^*].$$

- Let  $\Gamma_1, \Gamma_2 : \chi^{***} \times \chi^{***} \rightarrow \mathbb{R}$  be the symmetric positive definite continuous bilinear operators defined as :

$$\Gamma_1(f, g) = \mathbb{E} [f(\mathbf{SGN}_{\{F_X(Y)\}}^*) g(\mathbf{SGN}_{\{F_X(Y)\}}^*)] - f(\mu)g(\mu) \quad (4)$$

and

$$\Gamma_2(f, g) = \mathbb{E} [f(\mathbf{SGN}_{\{F_Y(X)\}}^*) g(\mathbf{SGN}_{\{F_Y(X)\}}^*)] - f(\tilde{\mu})g(\tilde{\mu}), \quad (5)$$

where  $f, g \in \chi^{***}$ .

For our next theorem, we shall also consider the following assumptions:

**Assumption 1** *We assume that the norm in  $\chi^*$  is twice Gateaux differentiable at every  $x \neq 0$ .*

**Assumption 2** *From Assumption 1, we suppose that the Hessian of the function  $g : x \mapsto \mathbb{E}[\|F_X(Y) + x\|_{\chi^*} | X_1, \dots, X_m]$ ,  $x \in \chi^*$ , denoted by  $\mathbf{J}_x : \chi^* \rightarrow \chi^{**}$ , exists for all  $x \in \chi^*$  and there is a constant  $c > 0$  such that*

$$\|\mathbf{J}_0\| \leq c.$$

Hence, using the previous notations and assumptions, the asymptotic normality of MED is given by the following theorem.

**Theorem 1** *(Asymptotic Gaussianity of MED)*

*Let  $N = m + n$  and  $m/N \rightarrow \lambda \in (0, 1)$  as  $m, n \rightarrow \infty$ . Assume that the bidual  $\chi^{**}$  space is a separable and type 2 Banach space. Then, under assumptions (1) and (2), for any two probability measures  $P$  and  $Q$  on  $\chi$ ,*

$$(mn/N)^{1/2}(MED - \mu) \text{ converges weakly to } G(0, \lambda\Gamma_1 + (1 - \lambda)\Gamma_2)$$

*as  $m, n \rightarrow \infty$ .*

See Smida et al (2022) for the proof of Theorem 1 and more mathematical details.

## 2.3 Computing the significance

In this subsection, we propose two methods to compute the significance of the MED test statistic: the first one is based on Theorem 1 and the second one is based on Monte-Carlo simulations.

- **Method 1: using the asymptotic distribution** The significance of the test based on the MED statistic can rely on Theorem 1. Since  $\mu = 0$  under  $H_0$ , we shall reject the null hypothesis if  $\|(mn/N)^{1/2}\text{MED}\| > q_\alpha$  where  $q_\alpha$  denotes the  $(1 - \alpha)$  quantile of the limiting distribution  $\|G(0, \lambda\Gamma_1 + (1 - \lambda)\Gamma_2)\|$  and  $\alpha$  is the asymptotic size of the test. Thus, we need to derive the covariance operators  $\Gamma_1$  and  $\Gamma_2$  and the norm of the asymptotic distribution given by Theorem 1.
- **Method 2: using random permutations** The method based on the asymptotic distribution suffers two limitations: the need to estimate the covariance operators  $\Gamma_1$  and  $\Gamma_2$  and the distance to the asymptotic distribution when  $m$  and  $n$  are quite small, which is often the case when comparing two samples of functions. Consequently, we may consider a test procedure based on Monte-Carlo simulations allowing to give an approximation of the null distribution (Dwass, 1957).

## 2.4 Asymptotic power under shrinking location shifts

In this section, we give the asymptotic distribution of the test statistic MED under appropriate sequences of shrinking location shifts. In order to do that, we suppose that  $Y$  is distributed as  $X + \Delta_N$ , where

$$\Delta_N = \delta \left( \frac{mn}{N} \right)^{-1/2} \quad (6)$$

for some nonzero fixed  $\delta \in \chi$  and  $N$  is the total size of the two samples. In order to derive the distribution of the median statistic under these alternative hypotheses, we need two more assumptions.

**Assumption 3** *We assume that the norm in  $\chi$  is twice Gateaux differentiable at every  $y \neq 0$ . In addition, we suppose that the Hessian of the map  $y \mapsto \mathbb{E}[\|y + Y - Z\|_\chi | Y]$ , at  $y \in \chi$ , denoted by  $\tilde{\mathbf{J}}_y : \chi \rightarrow \chi^*$ , exists where  $Z$  is an independent copy of  $Y$ .*

**Assumption 4** *Since we have assumed that the norm in  $\chi^*$  is twice Gateaux differentiable at every  $x \neq 0$  (Assumption 1), we suppose here that the Hessian of the function  $u \mapsto \mathbb{E} \left[ \left\| u + \mathbb{E}[\mathbf{SGN}_{\{Y-Z\}} | Y] \right\|_{\chi^*} \right]$ , denoted by  $\mathbf{H}_u : \chi^* \rightarrow \chi^{**}$ , exists where  $Z$  is an independent copy of  $Y$ .*

Thus, we obtain the following theorem.

**Theorem 2** *Let  $N = m + n$  and  $m/N \rightarrow \lambda \in (0, 1)$  as  $m, n \rightarrow \infty$ . Suppose that  $\chi^{**}$  is a separable and type 2 Banach space. Assume that the distributions of  $X$  and  $Y$  are nonatomic. Then, under assumptions (1), (2), (3) and (4) and under the sequence of shrinking location shifts defined as (6),*

$$(mn/N)^{1/2}MED \text{ converges weakly to } G\left(\mathbf{H}_0(\tilde{\mathbf{J}}_0(\delta)), \lambda\Gamma_1 + (1 - \lambda)\Gamma_2\right)$$

as  $m, n \rightarrow \infty$ .

See Smida et al (2022) for the proof of Theorem 2.

## 3 Applications

In this section, we aim to compare the power of the two proposed median statistics with those of the tests of WMW (Chakraborty and Chaudhuri, 2015), CFF (Cuevas et al., 2004) and HKR1 and HKR2 (Horváth et al., 2013).

### 3.1 Simulation study

We set the separable Hilbert space  $\chi = L^2[0, 1]$ . Now, let us consider the decomposition

$$X = \sum_{k=1}^{\infty} Z_k e_k,$$

where for all  $k \geq 0$ ,  $e_k = \sqrt{2}\sin(t/\sigma_k)$  is an orthonormal basis of  $\chi$ ,  $\sigma_k = ((k - 0.5)\pi)^{-1}$  and the  $Z_k$ 's are independent random variables which correspond to the projection of  $X$  on the Karhunen-Loève basis. We have considered four scenarios:  $Z_k/\sigma_k \sim \mathcal{N}(0, 1)$ ,  $Z_k/\sigma_k \sim t(5)$ ,  $Z_k/\sigma_k \sim \mathcal{C}(0, 1)$  and  $Z_k/\sigma_k \sim \mathcal{Dexp}(0, 1)$ .

#### 3.1.1 Finite-size powers

Assume that  $Y$  is distributed as  $X + \Delta$  and under the alternative hypotheses  $H_1 : \Delta \neq 0$ . Three choices are considered, namely :  $\Delta_1(t) = c$ ,  $\Delta_2(t) = ct$  and  $\Delta_3(t) = ct(1 - t)$  where  $c > 0$  for all  $t \in [0, 1]$ . Before computing the powers of the tests, we have derived the size of each of test, i.e. the probability of rejecting the null hypothesis when it is true. We have chosen 1000 random simulations of  $(X, Y)$ . For each simulated dataset, all test statistics and their critical values are derived in the same way as described in subsection 2.3:

- To apply the asymptotic method: critical values of the test statistic MED are derived as described above. Similarly, those of WMW, HKR1 and CFF are calculated using their associated asymptotic theorems.



- To apply the permutation procedure: we have used 999 random permutations. The hypothesis  $H_0$  is rejected if  $p_{value} < \alpha$ , where  $\alpha$  is the significance level which is chosen equal to 0.05.

**Results:** Using the asymptotic method and when the sample sizes are small, the sizes of all the tests are different from the 5% nominal level whatever the distribution. However, the ones obtained using the permutation method are close to 5%: this is not surprising since the asymptotic method is more adapted for large sample sizes.

Because of these results, we decided to first focus on simulations with limited sample sizes, using the permutation method to derive the corresponding statistical power of the different tests. We set  $m = n = 10$  and each sample curve is observed at 100 equidistant points on  $[0, 1]$ .

**Results:** With  $\Delta_3$ , the test using Mo statistic outperforms the test based on WMW against heavy-tailed distributions  $t(5)$ ,  $\mathcal{C}(0, 1)$  and  $\mathcal{Dexp}(0, 1)$  for large values of  $c$  and it has a similar power against  $\mathcal{N}(0, 1)$  distribution. We also remark that the power of the test based on MED is very similar to the one based on WMW or Mo for small values of the shift  $c$  and for all the distributions. We also deduce that the tests based on spatial ranks MED, Mo and WMW performs better than the mean-based ones CFF, HKR1 and HKR2 in all heavy-tailed distributions cases.

### 3.1.2 Asymptotic powers

In this subsection, we compare the asymptotic powers of the tests based on MED and WMW since their asymptotic distributions are known under the sequence of shrinking location shifts. To do so, recall that  $Y$  is distributed as  $X + \Delta_N$ , where  $\Delta_N$  is given by (6). We have considered three choices of  $\delta \in \chi$ , namely  $\delta_1(t) = c$ ,  $\delta_2(t) = ct$  and  $\delta_3(t) = ct(1 - t)$ , where  $t \in [0, 1]$  and  $c > 0$ . For evaluating the asymptotic powers of these tests, we have used 1000 sample functions from the different distributions described as above.

**Results:**

- Both tests achieve the 5% nominal level asymptotically using different location shift models and different types of distributions and their asymptotic powers are equal when  $c = 0$ .
- Under  $\delta_1(t)$  and  $\delta_3(t)$ , our test based on MED and the one based on WMW have similar asymptotic powers in the case of the Gaussian distribution. However, our test outperforms the WMW one with  $t(5)$  distribution and even more powerful with  $\mathcal{C}(0, 1)$  and  $\mathcal{Dexp}(0, 1)$  which are more heavy-tailed.

## 3.2 Application to real data

In this subsection, we compare the two median tests based on MED and Mo with those based on WMW, CFF, HKR1 and HKR2 using numerous datasets analysed by Chakraborty and Chaudhuri (2015). One of this datasets is available in the R package `fa` and contains the heights of 39 boys and 54 girls measured at 31 time points from age 1 to 18.

**Results:** All the p-values based on 999 random permutations and the asymptotic method are 0 up to two decimal places and are similar to the ones given by Chakraborty and Chaudhuri (2015). These p-values exhibit a strong difference between the two distributions.

So, we decided to evaluate the proportion of rejection of the null hypothesis to compare the behaviour of the different statistics when the level  $\alpha$  is equal to 0.05. We have randomly chosen 20% subsamples of the 2 classes from the complete dataset and this subsampling was repeated 100 times.

**Results:** Our tests based on Mo and MED statistics have the highest rate of rejection of the null hypothesis, close to the one WMW of Chakraborty and Chaudhuri (2015) and much larger than the tests proposed by Cuevas et al. (2004) and Horváth et al. (2013).

## 4 Discussion

We show that the proposed median tests have good performance compared to the Wilcoxon-Mann-Whitney test proposed by Chakraborty and Chaudhuri (2015), the ANOVA test of Cuevas et al. (2004) and the mean-based tests of Horváth et al. (2013). Moreover, when the distribution of the processes is heavy-tailed, the median test is as powerful as the WMW test for moderate sample size and asymptotically more powerful than any other test. For more mathematical details and other real data examples, see Smida et al (2022). In addition, R codes of all the tests used here are available in the R package proposed by Smida et al (2023).

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