Numerical Tools for Geometric Optimal Control and the Julia control-toolbox package

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Summary. We present in this article numerical techniques — the Julia `control-toolbox` package — for computing geometric optimal control concepts: Hamiltonian flows associated to the optimal control problem, Jacobi flows, Poisson brackets of Hamiltonians to define the singular control associated to a singular arc, etc. With these tools, it becomes easy to solve an optimal control problem by indirect methods, and to compute conjugate times together with the cut locus. We present the numerical tools on two test bed examples: the surface of revolution of minimum area and the Goddard problem. The first problem comes from calculus of variations and thus is regular. We also compute the conjugate locus for this example. On the other hand, the optimal solution of the Goddard problem contains bang and singular arcs.

1.1 Introduction

When we aim to solve an optimal control problem via the indirect methods, first we have to apply the Pontryagin Maximum Principle [33]. Then, by numerical integration of the underlying Hamiltonian, we obtain the flow of this Hamiltonian system. For obtaining this flow with our Julia `control-toolbox` package, we only have to define the optimal control problem and to give the function which computes the control with respect to the state and the costate (obtained by solving analytically the maximization of the pseudo-Hamiltonian); then, the flow is automatically computed thanks to automatic differentiation. Next, it is easy to define the shooting function and to compute extremals. Still with the use of automatic differentiation, we can compute Jacobi fields and conjugate points in relation with second-order conditions of local optimality. We present the use of our Julia `control-toolbox` package on two test bed examples: the surface of revolution of minimum area from calculus of variations and the well-known Goddard problem.

The article is organized as follows. Section 1.2 is devoted to the theory of geometric optimal control: singular control, weak principle and conjugate point, maximum principle and the Hamiltonian frame, problems affine in the

(scalar) control. In Section 1.3, we present the simple and multiple indirect shooting methods but also the differential homotopy methods in the frame of geometric control. Finally, in Section 1.4, we introduce the JULIA `control-toolbox` package and demonstrate how to use it on the two examples.

Remark 1. In the spirit of the reproducible research, the reader will find at the github repository `https://github.com/control-toolbox/Kupka`, JULIA notebooks for the two numerical examples.

1.2 Geometric optimal control

This section is inspired by references about optimal control theory and more specifically by references about geometric control. We refer to [1, 9, 13, 14, 27, 28, 29, 31, 33, 34, 36, 37, 39, 40] for more details.

1.2.1 Singular control

We consider a $\mathscr{C}^1$ mapping

$$\begin{aligned} f\colon\ \mathbb{R}^n \times \mathbb{R}^m &\longrightarrow \mathbb{R}^n \\ (x,u) &\longmapsto f(x,u) \end{aligned}$$

and define the controlled dynamical system

$$\dot{x}(t) = f(x(t),u(t)). \tag{Σ_u}$$

A control law $t \mapsto u(t)$ is an essentially bounded mapping defined on an interval of the form $[0\,,\tau_u)$, with $\tau_u \in \mathbb{R}^*_+ \cup \{+\infty\}$, and taking values in $\mathbb{R}^m$. We note the set of control laws

$$\mathcal{U} \coloneqq \left\{ u \in L^\infty([0\,,\tau_u),\mathbb{R}^m) \;\middle|\; \tau_u \in \mathbb{R}^*_+ \cup \{+\infty\} \right\}.$$

For any pair $(x_0,u) \in \mathbb{R}^n \times \mathcal{U}$, there exists a unique maximal solution (in Carathéodory sense) of the Cauchy problem $\dot{x}(t) = f(x(t),u(t))$, $x(0) = x_0$. We denote by

$$t \mapsto x(t,x_0,u)$$

this solution and $I(x_0,u)$ its interval of definition. Let $t \geq 0$ and $x_0 \in \mathbb{R}^n$ be fixed. We introduce the set $\mathcal{U}_{t,x_0} \subset \mathcal{U}$ of admissible control laws over $[0\,,t]$ with initial condition x_0, as the subset of control laws $u \in \mathcal{U}_{t,x_0}$ such that $x(\cdot,x_0,u)$ is well defined over $[0\,,t]$. With this notations, the mapping $(t,x_0,u) \mapsto x(t,x_0,u)$ is defined on the set:

$$\mathcal{D} \coloneqq \{(t,x_0,u) \mid t \geq 0,\ x_0 \in \mathbb{R}^n,\ u \in \mathcal{U}_{t,x_0}\}.$$

We introduce the two following partial mappings which are of crucial interest. First, we define the flow mapping at time $t \geq 0$ by:

$$\begin{aligned}\Phi_t\colon \quad \mathcal{D}_t &\longrightarrow \mathbb{R}^n\\ (x_0,u) &\longmapsto \Phi_t(x_0,u) \coloneqq x(t,x_0,u),\end{aligned}$$

with $\mathcal{D}_t \coloneqq \{(x_0,u) \mid (t,x_0,u) \in \mathcal{D}\}$. Then, we introduce the *endpoint* mapping at time $t \geq 0$ from $x_0 \in \mathbb{R}^n$ as:

$$\begin{aligned}E_{t,x_0}\colon\ \mathcal{U}_{t,x_0} &\longrightarrow \mathbb{R}^n\\ u &\longmapsto E_{t,x_0} \coloneqq x(t,x_0,u).\end{aligned}$$

Let $t \geq 0$ and $x_0 \in \mathbb{R}^n$. Then, $\mathcal{U}_{t,x_0}$ is an open subset of $L^\infty([0\,,t],\mathbb{R}^n)$ and the endpoint mapping is of class $\mathscr{C}^1$. The *reachable set* at time $t \geq 0$ from $x_0 \in \mathbb{R}^n$ is defined as $\mathcal{A}(t,x_0) \coloneqq E_{t,x_0}(\mathcal{U}_{t,x_0})$. Let $t > 0$ be a fixed positive time. We say that the system (Σ_u) is *controllable* from $x_0 \in \mathbb{R}^n$ in time t if $\mathcal{A}(t,x_0) = \mathbb{R}^n$, and is *locally controllable* from x_0 in time t around $x_1 \in \mathbb{R}^n$ if x_1 belongs to the interior of $\mathcal{A}(t,x_0)$, that is if $x_1 \in \mathrm{Int}(\mathcal{A}(t,x_0))$. We can notice that if E_{t,x_0} is surjective then (Σ_u) is controllable and if its Fréchet differential $E'_{t,x_0}(u)$ is surjective then, by the following nonlinear open mapping theorem, the system is locally controllable around $E_{t,x_0}(u)$.

Theorem 1. *Let $F\colon U \subset E \to \mathbb{R}^n$ be a function of class $\mathscr{C}^1$ on U, defined on the open set U of a Banach space E. Let $x \in U$ be a regular point of F (i.e. $F'(x)$ surjective), then F is locally open at x.*

A control $u \in \mathcal{U}_{t,x_0}$ is said to be *regular* if $E'_{t,x_0}(u)$ is surjective. Otherwise, it is called *singular*. Hence, if u is regular, then, the system (Σ_u) is locally controllable. By contraposition, if $E_{t,x_0}(u)$ belongs to the boundary of the reachable set, then, the control u is singular. We are now in position to introduce the (pseudo-)Hamiltonian characterization of the singular controls. For this purpose, we introduce the pseudo-Hamiltonian associated to (Σ_u):

$$\begin{aligned}H\colon\ \mathbb{R}^n \times (\mathbb{R}^n)^* \times \mathbb{R}^m &\longrightarrow \mathbb{R}\\ (x,p,u) &\longmapsto H(x,p,u) \coloneqq p \cdot f(x,u).\end{aligned}$$

Let consider now a singular control u over $[0\,,t]$ and denote by $x(\cdot) \coloneqq x(\cdot,x_0,u)$ the associated *state trajectory*. Since $E'_{t,x_0}(u)$ is not surjective, then there exists $\lambda \in (\mathbb{R}^n)^* \setminus \{0\}$ orthogonal to the linear subspace $\mathrm{Im}\, E'_{t,x_0}(u)$, *i.e.* such that $\forall\, \delta u \in L^\infty([0\,,t],\mathbb{R}^m)$:

$$\lambda \cdot (E'_{t,x_0}(u) \cdot \delta u) = \lambda \cdot \int_0^t R(t,s)\, B(s)\, \delta u(s)\, \mathrm{d}s = 0,$$

with $B(s) \coloneqq \partial_u f(x(s),u(s))$ and where $R(t,s)$ is the state transition matrix of the linear differential equation $\dot{X}(\tau) = \partial_x f(x(\tau),u(\tau)) \cdot X(\tau)$, $X(s) = I_n$. Setting[3]

$$p(s) \coloneqq \lambda\, R(t,s) \in (\mathbb{R}^n)^*,$$

[3] The notation $\lambda R(t,s)$ stands for $\lambda \circ R(t,s)$.

we obtain that the *covector* mapping $p\colon [0\,,t] \to (\mathbb{R}^n)^* \setminus \{0\}$ is such that for almost every $s \in [0\,,t]$:

$$\dot{x}(s) = \frac{\partial H}{\partial p}[s], \quad \dot{p}(s) = -\frac{\partial H}{\partial x}[s], \quad 0 = \frac{\partial H}{\partial u}[s],$$

with $[s] \coloneqq (x(s), p(s), u(s))$. We give on Figure 1.1, a two-dimensional illustration of the geometric interpretation of the covector p for a singular control u satisfying $E_{s,x_0}(u) \in \mathrm{Fr}(\mathcal{A}(s, x_0))$ for every $s \in [0\,,t]$.

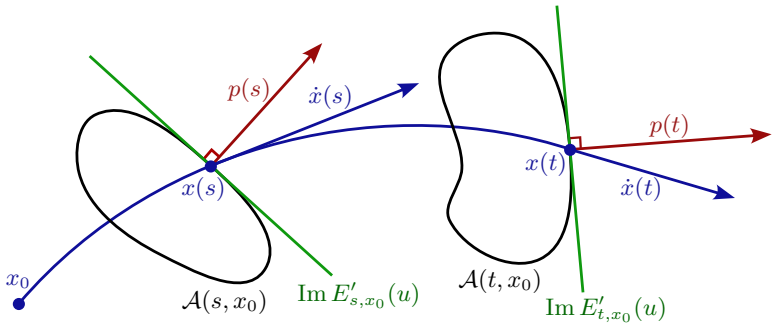

Fig. 1.1. Illustration of the geometric intepretation of the covector p, in the particular case where $E_{s,x_0}(u) \in \mathrm{Fr}(\mathcal{A}(s, x_0))$ for every $s \in [0\,,t]$.

We are interested now in the computation of singular controls, that is in the resolution of the equation $\partial_u H(x, p, u) = 0$. For any given $(\bar{x}, \bar{p}, \bar{u})$, if $\partial_u H(\bar{x}, \bar{p}, \bar{u}) = 0$ and if $\partial^2_{uu} H(\bar{x}, \bar{p}, \bar{u})$ is invertible, then, by the implicit function theorem, one can find an implicit mapping, denoted $u_s(x, p)$, such that locally

$$\frac{\partial H}{\partial u}(x, p, u_s(x, p)) = 0,$$

and such that $u_s(\bar{x}, \bar{p}) = \bar{u}$.

Example 1. Let us consider two examples for which $\partial^2_{uu} H(x, p, u)$ is invertible et compute the singular control.

- Consider a pseudo-Hamiltonian of the form

$$H(x, p, u) \coloneqq H_0(x, p) + u\, p_1 + 0.5\, u^2\, p_2$$

with x, p in $\mathbb{R}^2$, u in $\mathbb{R}$ and where H_0 is a smooth mapping. We have $\partial_u H(x, p, u) = 0$ if and only if $p_1 + up_2 = 0$ so the singular control is of the form

$$u_s(x, p) = -p_1/p_2 \quad \text{if} \quad p_2 \neq 0.$$

- Consider

$$H(x,p,u) = H_0(x,p) + \sum_{i=1}^{m} u_i\, H_i(x,p) + 0.5\,\|u\|^2\, p_n$$

with $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $p \in (\mathbb{R}^n)^*$ and where H_0, H_1, …, H_m are smooth. Let us introduce

$$\Phi \coloneqq (H_1, \ldots, H_m).$$

Then, we have that $\partial_u H(x,p,u) = 0$ if and only if $\Phi + p_n u = 0$ so the singular control is of the form

$$u_s(x,p) = -\Phi(x,p)/p_n \quad \text{if} \quad p_n \neq 0.$$

Example 2. Consider

$$H(x,p,u) = p_1 x_2^2/2 + p_2 u$$

with $u \in \mathbb{R}$, $x \in \mathbb{R}^2$ and $p \in \mathbb{R}^2$. In this example, the matrix $\partial^2_{uu} H(x,p,u)$ is not invertible, the pseudo-Hamiltonian being affine in the control u. Let us assume that along a given extremal of reference: $\partial_u H(x(t),p(t),u(t)) = p_2(t) = 0$ almost everywhere on an interval of non-empty interior. So, for almost every time t we have:

$$\frac{\mathrm{d}}{\mathrm{d}t} p_2(t) = \dot{p}_2(t) = -\frac{\partial H}{\partial x_2}(x(t),p(t),u(t)) = p_1(t)\, x_2(t) = 0.$$

Derivating $t \mapsto p_1(t)\, x_2(t)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(p_1(t)\, x_2(t)) = \dot{p}_1(t) x_2(t) + p_1(t) \dot{x}_2(t) = \alpha u(t) = 0, \quad \alpha = p_1(t) \neq 0.$$

The singular control is thus $u \equiv 0$.

In the previous example, we can do the computations in a more systematic way. We need for that to introduce the following definition.

Definition 1. *Let f and g be two smooth mappings on $\mathbb{R}^n \times (\mathbb{R}^n)^*$. We define for $z \coloneqq (x,p) \in \mathbb{R}^n \times (\mathbb{R}^n)^*$:*

$$\vec{f}(z) \coloneqq \left(\frac{\partial f}{\partial p}(z), -\frac{\partial f}{\partial x}(z) \right),$$

$$\{f,g\}(z) \coloneqq g'(z) \cdot \vec{f}(z) = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}(z) \frac{\partial g}{\partial x_i}(z) - \frac{\partial f}{\partial x_i}(z) \frac{\partial g}{\partial p_i}(z).$$

The bracket $\{f,g\}$ is the Poisson bracket *of f and g, and $\vec{f}$ is the* Hamiltonian vector field *(or Hamiltonian system, or symplectic gradient) associated to f.*

We recall that the Poisson bracket is bilinear, skew-symmetric, and it satisfies the Leibniz rule and the Jacobi identity:

- $\{f,g\} = -\{g,f\}$,
- $\{fh,g\} = f\{h,g\} + h\{f,g\}$,
- $\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$.

We can define the following procedure to compute the singular control (of *minimal order*) when $\partial^2_{uu} H(x,p,u)$ is not invertible.

Example 3. Consider a pseudo-Hamiltonian of the form

$$H(x,p,u) = H_0(x,p) + u\, H_1(x,p),$$

with $u \in \mathbb{R}$, $x \in \mathbb{R}^n$, $p \in (\mathbb{R}^n)^*$ and where H_0 and H_1 are smooth. Then,

$$\frac{\partial H}{\partial u}(x,p,u) = H_1(x,p).$$

We note $z \coloneqq (x,p)$. If $H_1(z(t)) = 0$ over a time interval I not reduced to a point, then, for any time $t \in I$, all the existing derivatives of $t \mapsto H_1(z(t))$ are equal to 0. Derivating as many times as needed, we can make appear the control. Let $t \in I$, then, setting $H_u(z) \coloneqq H(z,u)$, we have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H_1(z(t)) &= H_1'(z(t)) \cdot \dot{z}(t) \\
&= \{H_u, H_1\}(z(t)) && \text{(by definition)} \\
&= \{H_0, H_1\}(z(t)) + u(t)\,\{H_1, H_1\}(z(t)) && \text{(by linearity)} \\
&= \{H_0, H_1\}(z(t)) && \text{(by skew-symmetry)} \\
&\eqqcolon H_{01}(z(t)).
\end{aligned}$$

The control does not appear, we differentiate twice:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H_{01}(z(t)) &= H_{01}'(z(t)) \cdot \dot{z}(t) \\
&= \{H_u, H_{01}\}(z(t)) \\
&= \{H_0, H_{01}\}(z(t)) + u(t)\,\{H_1, H_{01}\}(z(t)) && \text{(by linearity)} \\
&\eqqcolon H_{001}(z(t)) + u(t)\, H_{101}(z(t)).
\end{aligned}$$

Hence, the singular control is of the form

$$u_s(z(t)) = -\frac{H_{001}(z(t))}{H_{101}(z(t))}$$

if $H_{101}(z(t)) \neq 0$. Otherwise, if $H_{101}(z(t)) = 0$, then, we must differentiate again.

1.2.2 Weak principle and conjugate point

To the control system (Σ_u), we add fixed limit conditions and an objective function in integral form involving a smooth mapping $(x,u) \mapsto f^0(x,u)$. We thus consider an optimal control problem in Lagrange form with simple limit conditions, that is the initial and final conditions are respectively of the form $x(0) = x_0$ and $x(t_f) = x_f$, $t_f > 0$ given, and besides, there is no constraint on the control:

$$(P_L) \qquad \begin{cases} \min J(x,u) \coloneqq \displaystyle\int_0^{t_f} f^0(x(t),u(t))\,\mathrm{d}t \\ \dot{x}(t) = f(x(t),u(t)), \quad u(t) \in \mathbb{R}^m, \quad t \in [\,0\,,t_f\,] \text{ a.e.}, \\ x(0) = x_0, \quad x(t_f) = x_f. \end{cases}$$

We introduce the *augmented state* $\widetilde{x} \coloneqq (x, x^0)$ and the *augmented system* $\widetilde{f} \coloneqq (f, f^0)$ defined by:

$$\widetilde{f}(t,\widetilde{x},u) \coloneqq (f(t,x,u), f^0(t,x,u)).$$

Then, setting $\widetilde{x}_0 \coloneqq (x_0, 0)$, one can write Problem (P_L) in the *reduced* form:

$$\min \left\{ (\pi_{x^0} \circ \widetilde{E}_{t_f,\widetilde{x}_0})(u) \;\middle|\; u \in \widetilde{\mathcal{U}}_{t_f,\widetilde{x}_0},\ E_{t_f,x_0}(u) = x_f \right\},$$

where $\widetilde{\mathcal{U}}_{t_f,\widetilde{x}_0}$ is the set of admissible control laws for the augmented system and where $\widetilde{E}_{t_f,\widetilde{x}_0}$ is the endpoint mapping associated to the augmented system. We have also introduced the canonical projection $\pi_{x^0}(\widetilde{x}) = x^0$, with $\widetilde{x} = (x, x^0) \in \mathbb{R}^n \times \mathbb{R}$. Considering the augmented reachable set $\widetilde{\mathcal{A}}(t,\widetilde{x}_0) \coloneqq \widetilde{E}_{t,\widetilde{x}_0}(\widetilde{\mathcal{U}}_{t_f,\widetilde{x}_0})$, then, necessarily

$$\widetilde{E}_{t_f,\widetilde{x}_0}(u) \in \mathrm{Fr}(\widetilde{\mathcal{A}}(t_f,\widetilde{x}_0)),$$

otherwise, we could decrease the cost, see the illustration Figure 1.2. Indeed, if not, then there would exist a neighbourhood of the point $\widetilde{x}(t_f) = \widetilde{E}_{t_f,\widetilde{x}_0}(u) = (x(t_f), x^0(t_f))$ in $\widetilde{\mathcal{A}}(t_f,\widetilde{x}_0)$ containing a point (y, y^0) such that $y^0 < x^0(t_f)$, which would contradict the optimality of the control u. Hence, u is singular for the augmented endpoint mapping and we get the following necessary conditions of optimality.

Proposition 1. *If (x,u) is a solution to Problem (P_L), then, there exists a covector mapping $p\colon [\,0\,,t_f] \to (\mathbb{R}^n)^*$ absolutely continuous, a scalar $p^0 \in \{-1,0\}$, such that $(p,p^0) \neq (0,0)$, and such that the following conditions are satisfied for almost every $t \in [\,0\,,t_f]$:*

$$\begin{aligned} \dot{x}(t) &= \frac{\partial H}{\partial p}(x(t),p(t),p^0,u(t)), \\ \dot{p}(t) &= -\frac{\partial H}{\partial x}(x(t),p(t),p^0,u(t)), \\ 0 &= \frac{\partial H}{\partial u}(x(t),p(t),p^0,u(t)), \end{aligned} \tag{1.1}$$

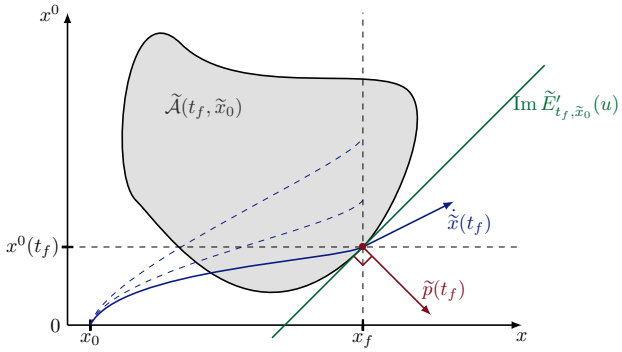

Fig. 1.2. Illustration of the optimality of the control u.

where $H(x,p,p^0,u) \coloneqq p\cdot f(x,u)+p^0\,f^0(x,u)$.

Remark 2. If $E'_{t_f,x_0}(u)$ is surjective, *i.e.* if u is not a singular control for the (non-augmented) endpoint mapping, then $p^0\neq 0$. If $E_{t_f,x_0}(u)\in \mathrm{Fr}(\mathcal{A}(t_f,x_0))$, then u is singular for E_{t_f,x_0} and $p^0=0$, see Figure 1.3.

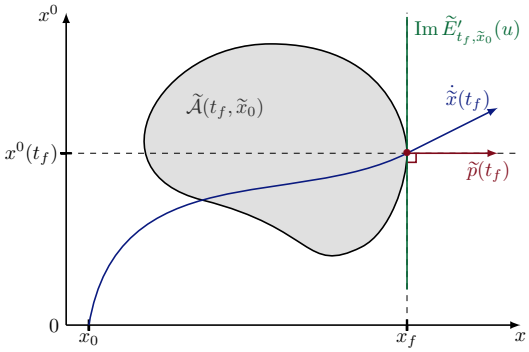

Fig. 1.3. On this illustration, we have $p^0=0$.

Definition 2. *An* extremal *is a quadruplet* (x,p,p^0,u) *solution to the constrained pseudo-Hamiltonian equations* (1.1)*. It is a* BC-extremal *if it satisfies the limit conditions* $x(0)=x_0$ *and* $x(t_f)=x_f$*. An extremal* (x,p,p^0,u) *is said to be* abnormal *if* $p^0=0$ *and* normal *if* $p^0=-1$.

Along an extremal, we have:

$$\frac{\partial H}{\partial u}(x(t),p(t),p^0,u(t))=0.$$

Let us call this condition the *Euler-Hamilton* condition. The Euler-Hamilton condition is obtained through the (first-order) Fréchet differential of the augmented endpoint mapping. The key result was the (first-order) nonlinear open mapping theorem, cf. Theorem 1. Similarly, from the following second-order nonlinear open mapping theorem, see [1, Theorem 20.3], and using the second-order differential of the augmented endpoint mapping, one can obtain a necessary condition of order 2, called the *Legendre-Clebsch condition*. Let us first recall the second-order nonlinear open mapping theorem and then give the condition.

Theorem 2. *Let $F\colon E \to \mathbb{R}^n$ be a smooth function defined on a Banach space E. Let u be a singular point of corank one (*$\operatorname{codim} \operatorname{Im} F'(u) = 1$*). Let $\lambda \in (\mathbb{R}^n)^* \setminus \{0\}$ in $(\operatorname{Im} F'(u))^\perp$. If* [4] *$\lambda F''(u)$ is indefinite on $\operatorname{Ker} F'(u)$, then F is locally open at u.*

Remark 3. The bilinear form

$$\lambda F''(u) \in \mathscr{L}(E, \mathscr{L}(E, \mathbb{R})) \simeq \mathscr{L}_2(E \times E, \mathbb{R})$$

is called the *second-order intrinsic derivative* of F at u, and is defined up to a scalar in the case of corank one.

Along an extremal, the following Legendre-Clebsch condition, see [1, Proposition 20.11], is satisfied:

$$\frac{\partial^2 H}{\partial u^2}(x(t), p(t), p^0, u(t)) \cdot (v, v) \leq 0, \quad \forall\, v \in \mathbb{R}^m, \quad \forall\, t \in [0\,, t_f] \text{ a.e..}$$

Let $t \in [0\,, t_f]$ and write $F_t(u) \coloneqq H(t, x(t), p(t), p^0, u)$. If the Euler-Hamilton and Legendre-Clebsch conditions are satisfied at time t along the extremal, then it means that $u(t)$ satisfies the necessary local optimality conditions of order 1 and 2 of the following unconstrained optimization problem:

$$\max_{u \in \mathbb{R}^m} F_t(u). \tag{1.2}$$

If besides, the Legendre-Clebsch condition is strict, then $u(t)$ satisfies the second-order sufficient condition of strict local optimality for Problem (1.2). But, even if the sufficient condition is satisfied all along the interval $[0\,, t_f]$, *i.e.* if

$$\frac{\partial^2 H}{\partial u^2}(x(t), p(t), p^0, u(t)) \cdot (v, v) < 0, \quad \forall\, v \in \mathbb{R}^m, \quad \forall\, t \in [0\,, t_f] \text{ a.e.,}$$

this does not give us a sufficient condition of local optimality for Problem (P_L). We need an additional condition called the *Jacobi condition* [1], even in the

[4] The notation $\lambda F''(u)$ stands for $\lambda \circ F''(u)$.

frame of calculus of variations [34]. The rest of this section is dedicated to the Jacobi condition, see Definition 4 and Theorem 3.

As mentioned before, if $\bar{u}$ is solution to Problem (P_L), then, necessarily

$$\widetilde{E}_{t_f,\widetilde{x}_0}(\bar{u}) \in \mathrm{Fr}(\widetilde{\mathcal{A}}(t_f,\widetilde{x}_0))$$

and so not only $\bar{u}$ is singular for the augmented endpoint mapping, but also this mapping is not locally open at $\bar{u}$. The fact that $\bar{u}$ is singular ensures the existence of a non-zero linear form $\bar{\tilde{\lambda}} \in (\mathbb{R}^{n+1})^*$ orthogonal to $\operatorname{Im} \widetilde{E}'_{t_f,\widetilde{x}_0}(\bar{u})$, that is

$$\bar{\tilde{\lambda}} \in \left(\operatorname{Im} \widetilde{E}'_{t_f,\widetilde{x}_0}(\bar{u})\right)^{\perp}, \quad \bar{\tilde{\lambda}} \neq 0.$$

The fact that the mapping is not locally open at $\bar{u}$ can be used to obtain new necessary local optimality conditions. For now, we assume that the control $\bar{u}$ is of corank one. Hence, the associated trajectory $\bar{x}$ admits a unique lift (up to a scalar) $(\bar{x},\bar{p},p^0,\bar{u})$ on $[0\,,t_f]$, that we suppose to be normal ($p^0 \neq 0$). In this context, we apply Theorem 2 to the augmented endpoint mapping. We thus obtain that

$$\bar{\tilde{\lambda}}\widetilde{E}''_{t_f,\widetilde{x}_0}(\bar{u}) \text{ is semi-definite on } \operatorname{Ker} \widetilde{E}'_{t_f,\widetilde{x}_0}(\bar{u}). \tag{1.3}$$

Remark 4. Let us relate this to an optimization point of view. Consider Problem (P_L) in its reduced form and define the Lagrangian (we omit indices):

$$L(u,\tilde{\lambda}) = \lambda^0\,\pi_{x^0}(\widetilde{E}(u)) + \lambda\cdot(E(u)-x_f), \quad \tilde{\lambda} =: (\lambda,\lambda^0) \in (\mathbb{R}^n)^*\times\mathbb{R}_-.$$

We note $x^0(u) \coloneqq \pi_{x^0}(\widetilde{E}(u))$ and $x(u) \coloneqq E(u)$ so that $\widetilde{E}(u) = (x(u),x^0(u))$. From the optimization point of view, we have the second-order necessary condition of local optimality: $\partial^2_{uu}L(\bar{u},\bar{\tilde{\lambda}}) = \bar{\tilde{\lambda}}\widetilde{E}''(\bar{u})$ negative semi-definite on the tangent space to the constraints. In the case of qualified constraints, that is $\lambda^0 < 0$, the tangent space is given by $\operatorname{Ker} x'(\bar{u})$ and not by $\operatorname{Ker} \widetilde{E}'(\bar{u})$, as written in Equation (1.3). But these two kernels are the same since for every $v \in \operatorname{Ker} x'(\bar{u})$, we have from the first-order necessary local optimality condition:

$$0 = \frac{\partial L}{\partial u}(\bar{u},\bar{\tilde{\lambda}})\cdot v = \lambda^0\,x^{0\prime}(\bar{u})\cdot v + \lambda\cdot(x'(\bar{u})\cdot v) = \lambda^0\,x^{0\prime}(\bar{u})\cdot v$$

and so $x^{0\prime}(\bar{u})\cdot v = 0$ since $\lambda^0 \neq 0$. Hence, $\operatorname{Ker} x'(\bar{u}) \subset \operatorname{Ker} \widetilde{E}'(\bar{u})$. Besides, $\operatorname{Ker} \widetilde{E}'(\bar{u}) \subset \operatorname{Ker} x'(\bar{u})$ since $\widetilde{E}(u) = (x(u),x^0(u))$. In conclusion, $\operatorname{Ker} \widetilde{E}'(\bar{u}) = \operatorname{Ker} x'(\bar{u})$ and so the second-order necessary local optimality condition is indeed that $\bar{\tilde{\lambda}}\widetilde{E}''(\bar{u})$ has to be negative semi-definite on $\operatorname{Ker} \widetilde{E}'(\bar{u})$. This ends the remark.

For $t \in [0\,,t_f]$, we define the symmetric bilinear form

$$B_t \coloneqq \bar{\tilde{\lambda}}\widetilde{E}''_{t,\widetilde{x}_0}(\bar{u})$$

and we introduce $K_t \coloneqq \operatorname{Ker} \widetilde{E}'_{t,\tilde{x}_0}(\bar{u})$. We have the following result, see [16] for more details.

Proposition 2. *If the extremal $(\bar{x}, \bar{p}, -1, \bar{u})$ satisfies the strong Legendre-Clebsch condition, then there exists $\varepsilon > 0$ such that $B_t|_{K_t}$ is negative definite for every $t \in [0\,, \varepsilon]$.*

Clearly, if $s \leq t$, then $B_t|_{K_t} \prec 0$ (*i.e.* negative definite) implies $B_s|_{K_s} \prec 0$, whence the following definition. We define the *first conjugate time* t_{1c}, along a normal extremal satisfying the strong Legendre-Clebsch condition, as the supremum of times t such that B_t is negative definite:

$$t_{1c} = \sup \left\{ t > 0 \mid B_t|_{K_t} \prec 0 \right\}.$$

From [1], then $B_{t_{1c}}|_{K_{t_{1c}}}$ has a non-trivial kernel. Hence, the conjugate times are defined as the times t_c such that $B_{t_c}|_{K_{t_c}}$ is degenerate.

Let us take a normal extremal of reference $(\bar{x}, \bar{p}, -1, \bar{u})$ and assume that the strong Legendre-Clebsch condition is satisfied:

$$\frac{\partial^2 H}{\partial u^2}(\bar{x}(t), \bar{p}(t), -1, \bar{u}(t)) \cdot (v, v) < 0, \quad \forall\, v \in \mathbb{R}^m, \quad \forall\, t \in [0\,, t_f] \text{ a.e..}$$

Under this assumption, the equation $\partial_u H = 0$ may be solved in a neighbourhood of the reference extremal and we can define the control as a function of the state and the costate, that is in feedback form, that we note $u_s(x, p)$. Setting on this neighbourhood the Hamiltonian $\mathbf{H}(z) \coloneqq H(z, -1, u_s(z))$, $z \coloneqq (x, p)$, we get

$$\mathbf{H}'(z) = \frac{\partial H}{\partial z}(z, -1, u_s(z)) + \frac{\partial H}{\partial u}(z, -1, u_s(z))\, u'_s(z) = \frac{\partial H}{\partial z}(z, -1, u_s(z))$$

since $\partial_u H(z, -1, u_s(z)) = 0$. Hence, on this neighbourhood, the system

$$\dot{x}(t) = \frac{\partial H}{\partial p}[t], \quad \dot{p}(t) = -\frac{\partial H}{\partial x}[t], \quad 0 = \frac{\partial H}{\partial u}[t],$$

with $[t] \coloneqq (x(t), p(t), -1, u(t))$, from Proposition 1, is equivalent to the *Hamiltonian system*

$$\dot{z}(t) = \vec{\mathbf{H}}(z(t)), \quad \vec{\mathbf{H}}(z) = \left(\frac{\partial \mathbf{H}}{\partial p}(z), -\frac{\partial \mathbf{H}}{\partial x}(z) \right).$$

Definition 3. *A solution to the linearized differential equation along z, called* Jacobi equation*,*

$$\dot{\widehat{\delta z}}(t) = \vec{H}'(z(t)) \cdot \delta z(t), \tag{1.4}$$

is called a Jacobi field*. A Jacobi field $\delta z = (\delta x, \delta p) \in \mathbb{R}^n \times (\mathbb{R}^n)^*$, is said to be* vertical at time t *if $\delta x(t) = 0$.*

We can give now a geometric characterization of the first conjugate time.

Proposition 3. *A time $t_c \in (0\,, t_f]$ is a conjugate time along a normal extremal satisfying the strong Legendre-Clebsch condition if and only if there exists a Jacobi field $\delta z = (\delta x, \delta p)$ vertical at 0 and t_c, and such that $\delta x \not\equiv 0$ on $[0\,, t_f]$.*

Definition 4 (Jacobi condition). *For an extremal defined on an interval $[a\,, b]$, we say that the* weak Jacobi condition *is satisfied if the open interval $(a\,, b)$ does not contain any conjugate time. We say that the* strong Jacobi condition *is satisfied if the semi-open interval $(a\,, b]$ does not contain any conjugate time.*

Remark 5. In the previous proposition, it is δx and not $\delta \tilde{x}$ which is under concern. This is possible since along a normal extremal satisfying the strong Legendre-Clebsch condition, we have $\delta \tilde{x} = (\delta x, \delta x^0)$ is vertical at t if and only if δx is vertical at t. To note that, it suffices to notice that $\delta x(t) = E'(u) \cdot \delta u = x'(u) \cdot \delta u$ for a given δu (with the notations of the previous remark). Hence, from the firs-order necessary optimality condition:

$$0 = \lambda^0\, x^{0\prime}(u) \cdot \delta u + \lambda \cdot (x'(u) \cdot \delta u) =: \lambda^0\, \delta x^0(t) + \lambda \cdot \delta x(t)$$

and so, since $\lambda^0 \neq 0$, we get $\delta x(t) = 0$ if and only if $\delta \tilde{x}(t) = (\delta x(t), \delta x^0(t)) = 0$.

The question we can ask now is: does the trajectory become necessarily non optimal after the first conjugate time? To answer yes to this question, the quadratic form Q_t associated to $B_t|_{K_t}$ must be indefinite for $t > t_{1c}$. However, it can happen in degenerate cases, that Q_t stays semi-definite on an interval $[t_{1c}\,, t_{1c} + \eta]$, $\eta > 0$. In the analytic frame, this cannot happen and necessarily for $t > t_{1c}$, Q_t is indefinite. We thus obtain the following (weak) second-order local optimality condition in the analytic case of corank one, see [1] for more details.

Theorem 3. *For a normal extremal satisfying the strong Legendre-Clebsch condition, whose associated analytic control is of corank one, the weak Jacobi condition is a necessary local optimality condition for the L^∞ topology (on u).*

Remark 6. This result is related to the notion of weak local solution. See [1, 8] for more details about this notion. For sufficient weak local conditions, we need to take into account the two-norm discrepancy, see [18]. We prefer to present next a more geometric point of view in relation with strong local optimality, see Theorem 4.

1.2.3 Pontryagin Maximum Principle and Hamiltonian frame

In the weak principle the control is unconstrained. We consider now an optimal control problem in which the control takes its values in any arbitrary set. We consider the following optimal control problem in Bolza form:

$$\text{(OCP)} \qquad \begin{cases} \min\ J(x,u) \coloneqq g(x(t_f)) + \displaystyle\int_0^{t_f} f^0(x(t),u(t))\,\mathrm{d}t \\ \dot{x}(t) = f(x(t),u(t)), \quad u(t) \in U, \quad t \in [0\,,t_f] \text{ a.e.}, \\ x(0) = x_0, \quad c(x(t_f)) = 0_{\mathbb{R}^p}, \end{cases}$$

where f, f^0, g and c are smooth functions. The initial and final times are fixed respectively to 0 and t_f, and the initial condition is simply $x(0) = x_0$, with $x_0 \in \mathbb{R}^n$ given. We have some final conditions of the form $c(x) = 0 \in \mathbb{R}^p$, with $p \leq n$. The set $U \subset \mathbb{R}^m$ is arbitrary. We assume also that c is a submersion on the set $c^{-1}(0)$, that is $c'(x)$ is surjective for any x such that $c(x) = 0$.

From the classical Pontryagin Maximum Principle [33], we have the following. If (x,u) is solution to Problem (OCP), then, there exists a covector mapping $p\colon [0\,,t_f] \to (\mathbb{R}^n)^*$ absolutely continuous,[5] a scalar $p^0 \in \{-1,0\}$ and a linear form $\lambda \in (\mathbb{R}^p)^*$, such that $(p(\cdot),p^0) \neq (0,0)$ and such that the following equations are satisfied for almost every $t \in [0\,,t_f]$:

$$\begin{aligned} \dot{x}(t) &= \quad \frac{\partial H}{\partial p}(x(t),p(t),p^0,u(t)), \\ \dot{p}(t) &= -\frac{\partial H}{\partial x}(x(t),p(t),p^0,u(t)), \\ H(x(t),p(t),p^0,u(t)) &= \max_{w\in U} H(x(t),p(t),p^0,w), \end{aligned} \tag{1.5}$$

where $H(x,p,p^0,u) \coloneqq p \cdot f(x,u) + p^0 f^0(x,u)$. The limit conditions $x(0) = x_0$ and $c(x(t_f))$ are satisfied. Besides, setting

$$\xi(x) \coloneqq p^0\, g(x) + \sum_{i=1}^{p} \lambda_i\, c_i(x),$$

we have the transversality condition:

$$p(t_f) = \xi'(x(t_f)). \tag{1.6}$$

The proof is based upon the construction of needle variations, see Figure 1.4, introduced by Boltyanski [7]. As we can see on the figure, the variations are applied on the control. They are not small in L^∞ norm but in L^1 norm, in comparison with the weak principle. The variation is coming from the constant $\bar{u}$ which replace the reference control on a small time interval of length dt.

The following definition replace Definition 2 in this context.

Definition 5. *A* Pontryagin-Boltyanski extremal *is a quadruplet* (x,p,p^0,u) *solution to the constrained pseudo-Hamiltonian equations* (1.5)*. It is a* BC-extremal*, for Problem* (OCP)*, if it satisfies the limit conditions* $x(0) = x_0$*,* $c(x(t_f)) = 0$ *and the transversality condition* (1.6)*. An extremal* (x,p,p^0,u) *is still said to be* abnormal *if* $p^0 = 0$ *and* normal *if* $p^0 = -1$*.*

[5] Actually, the covector mapping is even Lipschitz in our setting.

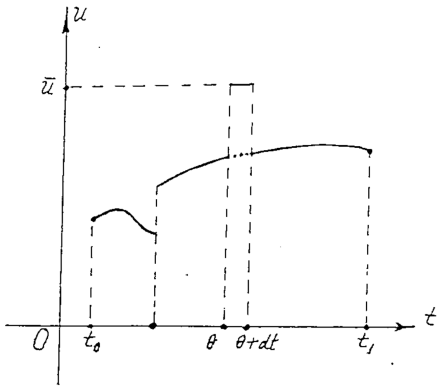

Fig. 1.4. Illustration of the needle variation excerpted from [7].

Let us give a more geometric point of view and reveal the Hamiltonian frame: in the following proposition, $\mathbf{H}$ is a Hamiltonian.

Proposition 4. *Let $(\bar{x}, \bar{p}, p^0, \bar{u})$ be a Pontryagin-Boltyanski extremal. We note $\bar{z} := (\bar{x}, \bar{p})$. If for almost every $t \in [0\,, t_f]$, in a neighbourhood of $\bar{z}(t)$, the maximized pseudo-Hamiltonian defines a smooth mapping*

$$z \mapsto \mathbf{H}(z) := \max_{u \in U} H(z, p^0, u),$$

then, for almost every $t \in [0\,, t_f]$, we have $\dot{\bar{z}}(t) = \overrightarrow{\mathbf{H}}(\bar{z}(t))$.

Proof. We introduce the following notation for a pseudo-Hamiltonian:

$$\overrightarrow{H}(x, p, p^0, u) := \left(\frac{\partial H}{\partial p}(x, p, p^0, u), -\frac{\partial H}{\partial x}(x, p, p^0, u) \right).$$

Since for almost every $t \in [0\,, t_f]$, we have from the Pontryagin Maximum Principle:

$$\dot{\bar{z}}(t) = \overrightarrow{H}(\bar{z}(t), p^0, \bar{u}(t)),$$

it is sufficient to prove that for almost every $t \in [0\,, t_f]$ we have: $\mathbf{H}'(\bar{z}(t)) = \partial_z H(\bar{z}(t), p^0, \bar{u}(t))$. Let $t \in [0\,, t_f]$ for which (1.5) is satisfied and for which $\mathbf{H}$ is well defined and smooth, on an open neighbourhood of $\bar{z}(t)$. On this neighbourhood, we set $F(z) := \mathbf{H}(z) - H(z, p^0, \bar{u}(t))$. Then, we have $F(z) \geq 0$ and $F(\bar{z}(t)) = 0$. So F is minimized on this open neighbourhood at the point $z = \bar{z}(t)$, which implies $F'(\bar{z}(t)) = 0$. The result is proved.

Definition 6 (Hamilton extremal). *A Pontryagin-Boltyanski extremal satisfying the assumptions of Proposition 4 is called a* Hamilton extremal.

Let us consider a smooth pseudo-Hamiltonian and fix p^0 so we do note write p^0 in the pseudo-Hamiltonian. Let us assume that U is an open subset

of $\mathbb{R}^m$ and that $u \mapsto H(z,u)$, $z = (x,p)$, admits a unique maximum over U at $u = u_m(z)$, for any z. We assume also that u_m is smooth. Then, the maximized pseudo-Hamiltonian furnishes a smooth Hamiltonian given by:

$$\mathbf{H}(z) = H(z, u_m(z)).$$

From the Pontryagin Maximum Principle, along a Pontryagin-Boltyanski extremal is satisfied

$$\dot{z}(t) = \overrightarrow{H}(z(t), u_m(z(t))).$$

From the previous corollary, it satisfies also

$$\dot{z}(t) = \overrightarrow{\mathbf{H}}(z(t)),$$

which can be easily checked since $\mathbf{H}'(z(t)) = \partial_z H[t] + \partial_u H[t] \cdot u'_m(z(t)) = \partial_z H[t]$ since $\partial_u H[t] = 0$, with $[t] \coloneqq (z(t), u_m(z(t)))$. Besides, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}(z(t)) &= \mathbf{H}'(z(t)) \cdot \dot{z}(t) \\ &= \mathbf{H}'(z(t)) \cdot \overrightarrow{\mathbf{H}}(z(t)) = \{\mathbf{H}, \mathbf{H}\}(z(t)) = 0,\end{aligned}$$

hence, the Hamiltonian is constant along the extremals.

In this Hamiltonian frame, we can complete Theorem 3 and give a sufficient condition of strong local optimality. We have the following result.

Theorem 4. *For a normal Hamilton BC-extremal of Problem (P_L), the strong Jacobi condition is a sufficient condition of strict local optimality for the $\mathscr{C}^0$ topology (on x).*

Proof. A brief proof is given in [15, Appendix A]. See also [1, Chapter 21].

To summarize, combining Theorems 3 and 4, before the first conjugate time, the local optimality is satisfied in a big neighbourhood of the trajectory for the $\mathscr{C}^0$ topology, while after the first conjugate time, the local optimality is lost, even in small neighbourhoods of the control for the L^∞ topology.

To complete our geometric point of view, let us give a brief insight of the symplectic origins of this Hamiltonian frame. The Hamilton extremals bring us to the theories of Hamiltonian dynamical system and symplectic geometry [1, 3, 6, 30]. Roughly speaking, there are two central objects which play a crucial role in our frame: the Hamiltonian $\mathbf{H}$ and the symplectic form denoted ω. The Hamiltonian is a function defined on the phase space, while the symplectic form is a (differential) two-form on the phase space. It is this symplectic form which permits to define the notion of Hamiltonian vector field (or symplectic gradient) leading to the Hamilton equations. This Hamiltonian together with the symplectic form define how the Hamilton extremals evolve in the phase space, this phase space being the intrinsic geometric (or symplectic) structure in which we can search the solutions of our optimal control problem.

We have already seen from where comes the Hamiltonian. Let us see how to obtain the symplectic form. First, consider that the state space is a differential manifold denoted Ω. We define the phase space, or cotangent space, as the space

$$T^*\Omega \coloneqq \{(x,p) \mid (x,p) \in \Omega \times T_x^*\Omega\},$$

where $T_x^*\Omega$ is the dual space of $T_x\Omega$. In the case where Ω is an open subset of $\mathbb{R}^n$, we have

$$T_x\Omega \simeq \mathbb{R}^n, \quad T_x^*\Omega \simeq \mathbb{R}^n \quad \text{and} \quad T^*\Omega \simeq \Omega \times \mathbb{R}^n.$$

From there, we can define in a canonical way a (differential) one-form on $T^*\Omega$ given by

$$\alpha_{(x,p)} \coloneqq \sum_{i=1}^{n} p_i \mathrm{d}x_i.$$

This one-form is called the *Liouville form*. The symplectic structure of $T^*\Omega$ is given by the exterior derivative of the Liouville form, that is the two-form ω defined by

$$\omega_{(x,p)} \coloneqq -\mathrm{d}\alpha_{(x,p)} = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}p_i.$$

The symplectic form is thus a non-degenerate differential form. The symplectic structure of $T^*\Omega$ leads to the definition of a Hamiltonian vector field. Let H be a function on $T^*\Omega$, that is a Hamiltonian. The associated Hamiltonian vector field of H is the vector field $\overrightarrow{H}$ defined by

$$\omega(\overrightarrow{H}, \cdot) = \mathrm{d}H.$$

Besides, the symplectic structure permits us to define the notion of Hamiltonian flow. Let $\overrightarrow{H}$ be a Hamiltonian vector field. The Hamiltonian flow is the one-parameter family of diffeomorphisms ϕ_t defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(z_0) = \overrightarrow{H}(\phi_t(z_0)), \quad \phi_0(z_0) = z_0.$$

The Hamiltonian flow is thus the solution of the Cauchy problem:

$$\dot{z}(t) = \overrightarrow{H}(z(t)), \quad z(0) = z_0.$$

Remark 7. One of the main results in symplectic geometry is Darboux's theorem which states that, locally, any symplectic form is equivalent to the canonical symplectic form ω_0 given by

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}p_i.$$

This means that, locally, we can find a change of coordinates to pass from ω to ω_0. A change of coordinates on a symplectic structure is called a *symplectomorphism*, it is a diffeomorphism that preserves the symplectic two-form.

In our frame, from a change of coordinates on Ω, we can define canonically a symplectomorphism on $T^*\Omega$. Indeed, if φ is a change of coordinates on the state, then, the canonical change of coordinates on $T^*\Omega$ is given by

$$\begin{aligned}\varphi^*: \; & T^*\Omega \longrightarrow T^*\Omega \\ & (x,p) \longmapsto \varphi^*(x,p) \coloneqq (\varphi(x), p\,\varphi'(x)^{-1}).\end{aligned}$$

One can check that φ^* is a symplectomorphism. Besides, another fundamental theorem of Hamiltonian systems is the Arnold-Liouville theorem and the notion of action-angle variables defined on a torus called the Liouville torus. This theorem permits us to classify the Hamilton extremals with respect to their topological properties and so to study how the extremals describe the torus. In the particular case of classical mechanics, the Morse theory [32] applied to the elevation function given by the mechanical energy leads to the description of the Liouville torus. This is also the case for instance in Riemannian geometry of revolution in dimension 2. The key point being the existence of a non-degenerate potential. This can be extended to Zermelo geometry, see [10]. This ends the remark.

1.2.4 Problems affine in the control

We call *affine control system* any system of the form

$$\dot{x}(t) = F_0(x(t)) + \sum_{i=1}^{m} u_i(t)\, F_i(x(t))$$

where the vector field F_0 is called the *drift* and $D = \{F_1, \ldots, F_m\}$ is called the *control distribution*. The Legendre-Clebsch condition is meaningless in the affine case where $\partial^2_{uu} H \equiv 0$. We thus introduce the following definition. We say that an extremal (x, p, p^0, u) defined on $[0\,, t_f]$ satisfies the *degenerate Legendre-Clebsch condition* if for almost every $t \in [0\,, t_f]$:

$$\frac{\partial^2 H}{\partial u^2}(x(t), p(t), p^0, u(t)) = 0.$$

In this case, for an affine system for instance, we have the following necessary condition. If the control is interior to the constraint, then the *Goh condition* [24] is necessary:

$$\left\{\frac{\partial H_t}{\partial u_i}, \frac{\partial H_t}{\partial u_j}\right\}(z(t)) = 0, \quad i,j = 1, \ldots, m, \quad t \in [0\,, t_f] \text{ a.e..} \tag{1.7}$$

We recall that $z = (x, p)$ and we mention that we have introduced the notations $H_t(z) \coloneqq H(z, p^0, u(t))$ and

$$\frac{\partial H_t}{\partial u_i}(z) \coloneqq \frac{\partial H}{\partial u_i}(z, p^0, u(t)), \quad i = 1, \ldots, m.$$

In addition to the Goh condition, in the case that the extremal satisfies the degenerate Legendre-Clebsch condition, under some additional assumptions, the extremal has to satisfy the *generalized Legendre-Clebsch condition*, see [1]. In the particular case of mono-input and autonomous affine pseudo-Hamiltonian:

$$H(x,p,u) = H_0(x,p) + u\,H_1(x,p),\ u \in \mathbb{R},$$

with $H_i(x,p) \coloneqq p \cdot F_i(x)$ the Hamiltonian lift of F_i, the generalized Legendre-Clebsch condition is given by:

$$H_{101}(z(t)) = \{H_1, \{H_0, H_1\}\}(z(t)) \geq 0, \quad t \in [0\,, t_f] \text{ a.e..}$$

Let us consider a mono-input autonomous affine control system of the form $\dot{x} = F_0(x) + u\,F_1(x)$, where $u \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Let H_0, H_1 be the Hamiltonian lifts of F_0, F_1. Let (x,p,u) be an extremal[6] on $I \subset \mathbb{R}$, I an open interval of non-empty interior. Along this extremal, we have $\partial_u H(z(t),u(t)) = H_1(z(t)) = 0$, $z = (x,p)$. We are in the frame of Section 1.2.1 and we call *singular* this extremal. The value of the control can be obtained differentiating at least twice with respect to the time the mapping H_1 along the extremal, see Example 2:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} H_1(z(t)) = H_{01}(z(t)) = 0 \\ &\frac{\mathrm{d}}{\mathrm{d}t} H_{01}(z(t)) = H_{001}(z(t)) + u(t)\,H_{101}(z(t)) = 0 \end{aligned} \tag{1.8}$$

and (1.8) provides the singular control in feedback form:

$$u_s(z) \coloneqq -\frac{H_{001}(z)}{H_{101}(z)}$$

outside

$$\Sigma_{101} \coloneqq \{z \in T^*\mathbb{R}^n \mid H_{101}(z) = 0\}\,.$$

A singular extremal defined outside Σ_{101} is called a *singular extremal of minimal order*. Denote

$$\Sigma_1 \coloneqq \{z \in T^*\mathbb{R}^n \mid H_1(z) = 0\}\,, \quad \Sigma_{01} \coloneqq \{z \in T^*\mathbb{R}^n \mid H_{01}(z) = 0\}\,,$$

and define the *singular manifold* $\Sigma_s \coloneqq \Sigma_1 \cap \Sigma_{01}$. Plugging u_s inside the pseudo-Hamiltonian, we get the Hamiltonian

$$\mathbf{H_s}(z) \coloneqq H(z, u_s(z)).$$

We have the following result.

[6] We do not mention p^0 in the extremal since it has no role here.

Proposition 5. *Let $\bar{z} \in \Sigma_s \setminus \Sigma_{101}$. Then, there is exactly one singular extremal of minimal order passing through $\bar{z}$. It is contained in Σ_s and it is solution to the Hamiltonian system*

$$\dot{z}(t) = \overrightarrow{\mathbf{H_s}}(z(t)),$$

the control being given by $u_s(z(t))$.

Proof. Denote by $z(\cdot)$ the integral curve of $\overrightarrow{\mathbf{H_s}}$ passing through $\bar{z}$ at time $t = 0$. Let $\varphi(t) \coloneqq \xi(z(t))$, with $\xi(z) \coloneqq (H_1(z), H_{01}(z))$. Then, $\varphi(0) = 0_{\mathbb{R}^2}$ and $\varphi'(t) = \xi'(z(t))\,\overrightarrow{\mathbf{H_s}}(z(t))$.

• To prove that Σ_s is invariant by the Hamiltonian flow of $\mathbf{H_s}$ we prove that $\varphi(t) = 0$ for every t. We have:

$$\begin{aligned}
\varphi_1'(t) &= \{\mathbf{H_s}, H_1\}(z(t)) = \{H_0 + u_s\, H_1, H_1\}(z(t)) \\
&= \Big(\{H_0, H_1\} + u_s\,\{H_1, H_1\} + H_1\,\{u_s, H_1\}\Big)(z(t)) \\
&= \{H_0, H_1\}(z(t)) + H_1(z(t))\,\{u_s, H_1\}(z(t))
\end{aligned}$$

and

$$\begin{aligned}
\varphi_2'(t) &= \{\mathbf{H_s}, H_{01}\}(z(t)) = \{H_0 + u_s\, H_1, H_{01}\}(z(t)) \\
&= \Big(\underbrace{\{H_0, H_{01}\} + u_s\,\{H_1, H_{01}\}}_{=0 \text{ by definition of } u_s} + H_1\,\{u_s, H_{01}\}\Big)(z(t)) \\
&= H_1(z(t))\,\{u_s, H_{01}\}(z(t)).
\end{aligned}$$

Hence, $\varphi'(t) = A(t)\,\varphi(t)$, with

$$A(t) \coloneqq \begin{pmatrix} \{u_s, H_1\}(z(t)) & 1 \\ \{u_s, H_{01}\}(z(t)) & 0 \end{pmatrix}$$

and $\varphi(0) = 0$. Thus, $\varphi(t) = 0$ for every t.

• Now, since

$$\begin{aligned}
\mathbf{H_s}'(z) &= \frac{\partial H}{\partial z}(z, u_s(z)) + \frac{\partial H}{\partial u}(z, u_s(z))\,u_s'(z) \\
&= \frac{\partial H}{\partial z}(z, u_s(z)) + H_1(z)\,u_s'(z),
\end{aligned}$$

we have $\mathbf{H_s}'(z(t)) = \partial_z H(z(t), u_s(z(t)))$ along the integral curve $z(\cdot)$ of $\overrightarrow{\mathbf{H_s}}$. Hence, $(z, u_s(z))$ is a singular extremal (of minimal order). Conversely, let us consider a singular extremal of minimal order passing through $\bar{z}$. Then, by definition, it is contained in Σ_s, the control is given by u_s and it is also solution to the Hamiltonian system $\dot{z}(t) = \overrightarrow{\mathbf{H_s}}(z(t))$ since $\mathbf{H_s}'(z(t)) = \partial_z H(z(t), u_s(z(t)))$ as soon as $H_1(z(t)) = 0$. This ends the proof.

Remark 8. By the previous proposition, Σ_s is invariant by the Hamiltonian flow of $\mathbf{H_s}$. Assume that ξ is a submersion on $\Sigma_s \neq \emptyset$. Then, $\Sigma_s = \xi^{-1}(0)$ is a submanifold of $T^*\mathbb{R}^n$ of codimension 2, and the sets Σ_1 and Σ_{01} are transverse at each point of Σ_s.

Assume now that the control is bounded between -1 and 1. The classification of the bang extremals (of maximal norm) and the singular extremals depends on the order of contact of the bang extremals with the *switching manifold* Σ_1 and the signs of the Poisson brackets H_{001} and H_{101} at the contact points. The local optimality of the extremals depends on this classification and the existence of conjugate points. See [8, 9, 11] for more details. See also [19, Section 1.2] for algorithms to compute conjugate points in the regular and singular cases.

1.3 Indirect numerical methods for geometric control

We present in this part, some indirect numerical methods. An indirect method aims to solve the equations given by the Pontryagin Maximum Principle, that is to compute BC-extremals.

1.3.1 Indirect simple shooting

Let consider Problem (P_L) and assume that for any extremal (z, p^0, u), we can write $u(t) = u(z(t))$, with $z \mapsto u(z)$ at least $\mathscr{C}^1$. Plugging $u(z)$ in $\overrightarrow{H}$, then, finding a BC-extremal amounts to solve the (Two-Point) Boundary Value Problem:

$$\text{(BVP)} \qquad \begin{cases} \dot{z}(t) &= \overrightarrow{H}(z(t), p^0, u(z(t))), \\ 0_{\mathbb{R}^{2n}} &= b(z(0), z(t_f)) \coloneqq (x(0) - x_0, x(t_f) - x_f). \end{cases}$$

We can rewrite (BVP) as a set on nonlinear equations introducing the *simple shooting function*:

$$\begin{aligned} S\colon\ \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ p_0 &\longmapsto S(p_0) \coloneqq \pi(z(t_f, x_0, p_0)) - x_f, \end{aligned}$$

where $\pi(x, p) = x$ and where $z(\cdot, x_0, p_0)$ is the solution of the Cauchy problem $\dot{z}(t) = \overrightarrow{H}(z(t), p^0, u(z(t)))$, $z(0) = (x_0, p_0)$. Solving (BVP) amounts to solve $S(p_0) = 0$. This is the *indirect simple shooting method*, see Figure 1.5.

Remark 9. If $\bar{p}_0$ satisfies $S(\bar{p}_0) = 0$, then, the integral curve $\bar{z}(\cdot) \coloneqq z(\cdot, x_0, \bar{p}_0)$, with the control $\bar{u}(\cdot) \coloneqq u(\bar{z}(\cdot))$ and p^0, is a BC-extremal of Problem (P_L).

Remark 10. From Proposition 4, the integral curve $z(\cdot, x_0, p_0)$ is also solution of $\dot{z}(t) = \overrightarrow{\mathbf{H}}(z(t))$, $z(0) = (x_0, p_0)$, where $\mathbf{H}(z) \coloneqq H(z, p^0, u(z))$.

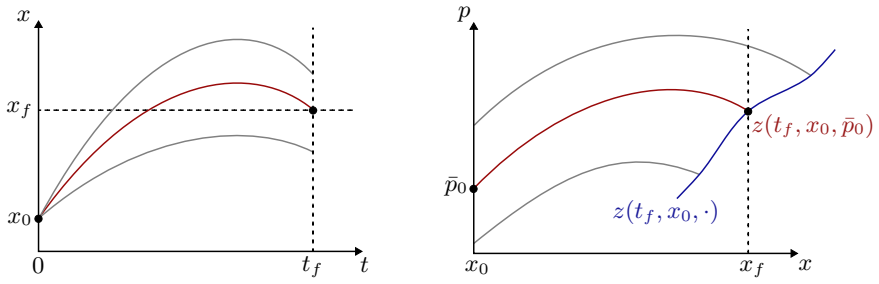

Fig. 1.5. Illustration of the indirect simple shooting method.

To solve the shooting equations, we need to compute $z(t_f, x_0, p_0)$. This is usually computed by Runge-Kutta solvers. Then, to find a zero of the shooting function, we can use Newton-like solvers. The Newton methods are known to be sensitive with respect to the initial iterate. One difficulty is to provide a good enough initial guess to make the Newton solver converge. We recall the Newton iteration:

$$p_0^{(k+1)} = p_0^{(k)} + d^{(k)},$$

with $d^{(k)}$ the solution of the linear system

$$S'(p_0^{(k)}) \cdot d = -S(p_0^{(k)}).$$

The Jacobian of the shooting function is given by:

$$S'(p_0) \cdot d = \pi \left(\frac{\partial z}{\partial p_0}(t_f, x_0, p_0) \cdot d \right) = \pi \left(\frac{\partial z}{\partial z_0}(t_f, x_0, p_0) \cdot (0_{\mathbb{R}^n}, d) \right),$$

where z_0 stands for (x_0, p_0). We need to compute $\partial_{z_0} z(\cdot, x_0, p_0) \cdot \delta z_0$, $\delta z_0 = (0_{\mathbb{R}^n}, d)$, solution of the *variational equations*:

$$\dot{\delta z}(t) = \vec{\mathbf{H}}'(z(t, x_0, p_0)) \cdot \delta z(t), \quad \delta z(0) = \delta z_0$$

that we recognize to be Jacobi equations. Hence, the invertibility of the Jacobian of the shooting function is directly related to the absence of conjugate points.

To compute the directional derivative $\partial_{z_0} z(t_f, z_0) \cdot \delta z_0$, a first possibility is to use finite differences. However, it is crucial to use in practise adaptive step-length Runge-Kutta integrators and in this case, the finite differences are not well suited, since the two grids dynamically evaluated, involved for instance in the computation of $z(t_f, z_0)$ and $z(t_f, z_0 + \delta z_0)$, may be different and could lead to artificial non-differentiability. The key point is thus to force the grid to be the same. This is known as *Internal Numerical Derivative* (IND) [4]. Another possibility is to use *Automatic Differentiation* (AD) on the integration solver code. Because of the adaptive step-length, the code only defines a function

which is piecewise differentiable but DA may lead to the same accuracy that IND for adaptive Runge-Kutta scheme [26]. The last option is to assemble explicitly the variational equations and compute their solutions, that is Jacobi fields. Since $\overrightarrow{H}'$ is evaluated along the current solution, in practise we need to integrate simultaneously the systems in z and δz (the dimension of the full system is $2n + 4n^2$ to get the whole derivative):

$$(\dot{z}(t), \dot{\widehat{\delta z}}(t)) = \left(\overrightarrow{\mathbf{H}}(z(t)), \overrightarrow{\mathbf{H}}'(z(t)) \cdot \delta z(t)\right), \quad (z(0), \delta z(0)) = (z_0, \delta z_0).$$

Considering a one-step explicit adaptive Runge-Kutta scheme, DA on the integration code and VAR (integration of the augmented variational system) are equivalent if the step-length control is done only on the components of z (and not on $(z, \delta z)$), and so in this case the following diagram commutes:

$$\begin{array}{ccc} \text{(IVP)} & \xrightarrow{\text{Numerical integration}} & z(t, z_0) \\ {\scriptstyle\text{Derivation}}\downarrow & & \downarrow{\scriptstyle\text{Derivation}} \\ \text{(VAR)} & \xrightarrow{\text{Numerical integration}} & \dfrac{\partial z}{\partial z_0}(t, z_0) \end{array}$$

where (IVP) stands for the Initial Value Problem: $\dot{z}(t) = \overrightarrow{\mathbf{H}}(z(t))$.

1.3.2 Numerical difficulties of the indirect simple shooting

We present in this section, the illustration of some numerical difficulties from the indirect simple shooting that can we found in [22]. This presentation is a motivation to the introduction to structural multiple shooting and homotopy techniques. We refer to [22] for more details and for a presentation of the homotopy methods in the frame of topological degree.

To illustrate the numerical issues, we consider the double integrator problem with L^1-minimization cost:

$$\begin{cases} \min \displaystyle\int_0^{t_f} |u(t)| \, \mathrm{d}t, \\ \dot{x}_1(t) = x_2(t), \ \dot{x}_2(t) = u(t), \ |u(t)| \leq \gamma, \\ x(0) = x_0, \quad x(t_f) = x_f, \end{cases}$$

with x_0, x_f belonging to $\mathbb{R}^2$, $t_f \geq 0$ and $\gamma > 0$ fixed. The pseudo-Hamiltonian (in normal form) is:

$$H(x, p, u) = p_1 x_2 + p_2 u - |u|.$$

The maximizing control is given by $u = 0$ if $|p_2| < 1$, $u = \gamma \operatorname{sign} p_2$ if $|p_2| > 1$ and $u \in [-\gamma\,, \gamma]$ if $|p_2| = 1$. We denote by $u(p_2)$ this maximizing control. At

the end, we are leading to solve the following non-smooth boundary value problem (even badly defined when $p_2 = \pm 1$):

$$\begin{cases} \dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(p_2(t)), \quad \dot{p}_1(t) = 0, \quad \dot{p}_2(t) = -p_1(t), \\ x(0) = x_0, \quad x(t_f) = x_f. \end{cases}$$

The shooting function is given in Figure 1.6 with the initial covector denoted $p_0 =: (\alpha, \beta)$.

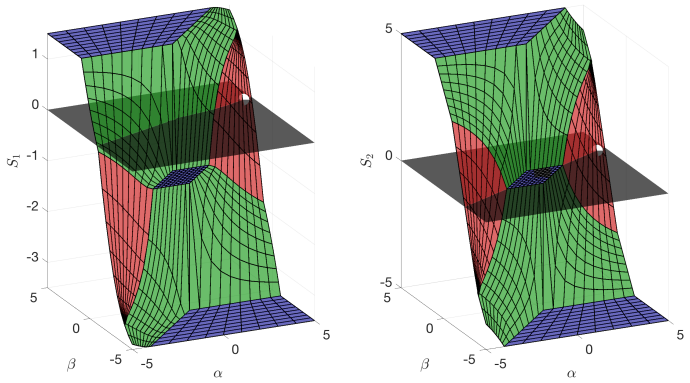

Fig. 1.6. Shooting function for the double integrator problem with L^1-minimization. On this example, $\gamma = 5$, $t_f = 1$, $x_0 = (-1, 0)$ and $x_f = (0, 0)$. The white ball corresponds to the solution, at the intersection between the graph of the shooting function and the planes $S_1 = 0$ and $S_2 = 0$.

We can notice that the shooting function is not continuous at the points $(0, \pm 1)$, it is not differentiable at the interfaces of the different regions, and it is moreover constant on the blue regions. The nine regions are characterized by different control structures, see Figure 1.7. One of the consequences about the use of Newton solver is that if the initial guess p_0 is not chosen in the right region, that is if it does not give a control with the optimal structure, then the algorithm may not converge. On this simple example, the algorithm can generate a point inside the regions where the shooting function is constant. But, in practise, we do not know in advance the optimal structure. One usage of the homotopy methods presented in the next section is to provide the optimal structure together with a good initial guess to make the Newton solver converge. Once the optimal structure is revealed, we can define an appropriate multiple shooting function to get with high accuracy a BC-extremal of the problem, see the next section for examples of this.

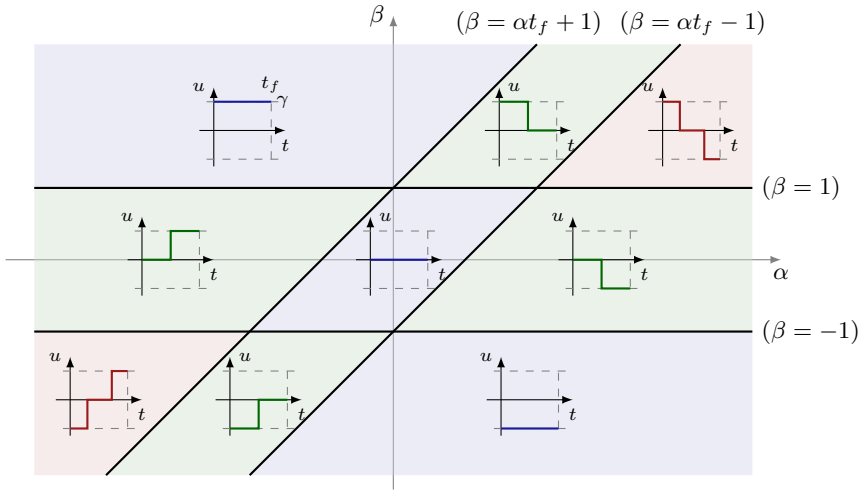

Fig. 1.7. Diagram of the different structures (on $[0\,,t_f]$) in the plane (α, β). Note that the extremals satisfy $p_2(t) = -\alpha t + \beta$ and the control is given by $u(p_2) = 0$ if $p_2 \in (-1\,,1)$, $u(p_2) = \gamma\,\mathrm{sign}(p_2)$ if $|p_2| > 1$ and $u(p_2) \in [-\gamma\,,\gamma]$ otherwise.

Remark 11. An alternative of the use of homotopic methods is the use of direct algorithms to determine the optimal structure together with a good enough initial guess, in order to define and solve the multiple shooting equations.

1.3.3 Structural indirect multiple shooting

Let us consider a scalar and affine control system of the form $H = H_0 + u\,H_1$ with the constraint $|u| \leq 1$. If the optimal extremal is the concatenation of bang arcs (where the control satisfies $|u| = 1$) and singular arcs (where $H_1 = 0$), then, it is needed to use a multiple shooting method that we qualify as *structural indirect multiple method*. Let us give an example. Assume that the solution is composed of three arcs: a bang arc followed by a singular arc followed by another bang arc. We write the structure: bang-singular-bang. In this case, the shooting function must have in addition to the initial covector, two more unknowns: the two switching times. Since there are two more unknowns, we expect to find two more conditions to fulfill. These conditions are given by Proposition 5: since the singular surface Σ_s is invariant by the Hamiltonian flow of $\overrightarrow{\mathbf{H_s}}$, it is sufficient to impose that the first bang arc join Σ_s at the first switching time. Imposing the singular control on the second arc ensures that the extremal stays on Σ_s. Finally, for the third arc, it is sufficient to impose the bang control to leave the singular surface at the second switching time.

Remark 12. The maximization condition of the Pontryagin Maximum Principle gives us a stratification of the cotagent space where some regions are

associated each to a unique Hamiltonian, while other regions like the singular surface are associated to several Hamiltonians. In our case, the region $H_1 < 0$ is associated to the Hamiltonian $H_0 - H_1$, the region $H_1 > 0$ is associated to the Hamiltonian $H_0 + H_1$, while in the region $H_1 = 0$ we have to deal with the two previous Hamiltonians and the Hamiltonian $\mathbf{H_s}$ defined by the singular control. Hence, finding a BC-extremal is a combination of dealing with the competition between Hamiltonians and the research of the optimal path in the cotangent space.

Example 4. Consider the following double integrator problem:

$$\begin{cases} \min \dfrac{1}{2} \displaystyle\int_0^{t_f} (x_1^2(t) + x_2^2(t))\,\mathrm{d}t, \quad t_f = 5, \\ \dot{x}_1(t) = x_2(t),\ \dot{x}_2(t) = u(t),\ u(t) \in [-1, 1], \\ x(0) = (0, 1). \end{cases}$$

Assume that we know that the optimal structure is composed of two arcs: a first arc with $u = -1$ followed by a singular arc. The pseudo-Hamiltonian in normal form is given by:

$$H(x_1, x_2, p_1, p_2, u) = -(x_1^2 + x_2^2)/2 + p_1 x_2 + p_2 u.$$

The singular arcs are of minimal order and given by the condition $p_2 = 0$. The singular surface is given by $\Sigma_s = \{p_2 = x_2 - p_1 = 0\}$ in the cotagent space $T^*\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2$, and the singular control in feedback form is $u = x_1$. We have a competition between three Hamiltonian flows, respectively associated to

$$H_\pm \coloneqq H(x, p, \pm 1), \quad H_s \coloneqq H(x, p, x_1).$$

Knowing the bang-singular optimal structure, we can define the structural indirect multiple shooting function having as unknowns the initial covector $p_0 \in \mathbb{R}^2$ and the switching time $\tau \in \mathbb{R}$. The conditions are given by the transversality condition from Pontryagin Maximum Principle and by Proposition 5:

$$p_1(t_f) = 0, \quad p_2(t_f) = 0, \quad p_2(\tau) = 0, \quad x_2(\tau) - p_1(\tau) = 0.$$

We can notice that the condition $p_2(t_f) = 0$ is redundant since this is automatically checked for a singular extremal. We have three conditions for three unknowns, which defines the shooting function. The solution is given Figure 1.8.

Example 5. We consider the optimal control problem:

$$\begin{cases} \min \displaystyle\int_0^{t_f} x^2(t)\,\mathrm{d}t \\ \dot{x}(t) = u(t), \quad |u(t)| \leq 1, \\ x(0) = 1, \quad x(t_f) = 1/2. \end{cases}$$

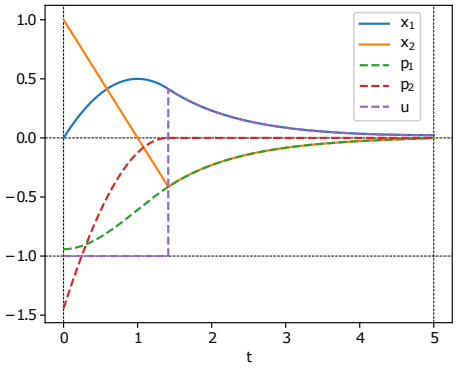

Fig. 1.8. States, covectors and controls for the double integrator problem.

with $t_f = 2$. We assume we know the optimal structure, composed of one bang arc $u = -1$, followed by a singular arc with $u = 0$, and then by another bang arc with $u = 1$. The pseudo-Hamiltonian in normal form is: $H(x, p, u) = -x^2 + pu$. The singular arcs are of minimal order, and given by $p = 0$. The singular surface is $\Sigma_s = \{p = x = 0\}$ in the cotangent space $T^*\mathbb{R} \simeq \mathbb{R} \times \mathbb{R}$, and the singular control in feedback form is given by $u = 0$. We have a competition between three Hamiltonian flows, given by $H_\pm := H(x, p, \pm 1)$, $H_s := H(x, p, 0)$. Knowing that the optimal structure is of the form bang-singular-bang, we can define the structural multiple shooting function having as unknowns the initial covector $p_0 \in \mathbb{R}$ and the two switching times $\tau_1, \tau_2 \in \mathbb{R}$. The conditions are given by the final condition $x(t_f) = 1/2$ and by Proposition 5:

$$x(t_f) = 1/2, \quad p(\tau_1) = 0, \quad x(\tau_1) = 0.$$

We have three conditions for three unknowns, which defines the shooting function. The solution is given Figure 1.9.

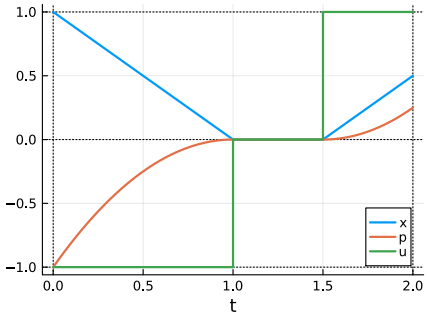

Fig. 1.9. State, covector and control for the second example.

1.3.4 Homotopy methods

Homotopy methods may bu used to solve families of nonlinear equations. One common use consists in adding an artificial parameter to the set of nonlinear equations, embedding the original problem into a one-parameter family of equations, hoping that for a certain value of the parameter the problem is easy to solve in order to compute a sequence of zeros, modifying step by step the value of the parameter, until we get back to the original problem. The homotopic parameter and the nature of the deformation are heuristically chosen in practise, in relation with the constraints of the problem and the physical parameters governing it. This choice is guided on one hand by the simple problem to solve but also by the path of zeros itself, joining the simple problem to the original one, that we wish sufficiently smooth and converging to our target. Another interest of the homotopy is to describe the evolution of solutions with respect to some physical parameters already present inside the problem. In each case, we denote by λ the homotopic parameter, that we can consider in $[0\,,1]$. We talk about continuation when the homotopic parameter is monotone (increasing in our case). The homotopic approach is more general since λ may vary arbitrary. In this paper, we are interested in the homotopy methods in the context of optimal control to compute families of shooting functions or conjugate loci. We present first the homotopy method in the particular frame of geometric control, inspired by [38], and then give some algorithmic tools.

Let consider in a first part, that we have a one-parameter family of optimal control problems parameterized by $\lambda \in [0\,,1]$ of the following form: for a given λ, we minimize the cost

$$J_\lambda(x,u) \coloneqq \int_0^{t_f} f^0(x(t),u(t),\lambda)\,\mathrm{d}t$$

with a fixed final time $t_f > 0$. The state is governed by:

$$\dot{x}(t) = f(x(t),u(t),\lambda).$$

The simple limit conditions are given by:

$$x(0) = x_0, \quad x(t_f) = x_f,$$

with x_0 and x_f fixed. We assume that f and f^0 are smooth on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$. We consider the frame of Section 1.2.2. We can write the problem in its reduced form:

$$\min F_\lambda(u) \coloneqq \pi_{x^0}(\widetilde{E}_\lambda(u)) \text{ under the constraint } E_\lambda(u) = x_f,$$

and we seek u in the open set $\mathcal{U}_\lambda$ (assumed to be non-empty) of the Banach space $L^\infty([0\,,t_f],\mathbb{R}^m)$. The mappings E_λ and $\widetilde{E}_\lambda$ are respectively the endpoint and augmented endpoint mappings. The set $\mathcal{U}_\lambda$ is the admissible set of control

laws. From the Lagrange rule, if the control u_λ is optimal, then, there exists $(\psi_\lambda, \psi_\lambda^0) \neq 0 \in (\mathbb{R}^n)^* \times \mathbb{R}$ such that

$$\psi_\lambda^0 \, F_\lambda'(u_\lambda) + \psi_\lambda \circ E_\lambda'(u_\lambda) = 0.$$

Let us assume that there is no minimizing abnormal. Under this assumption, we can fix $\psi_\lambda^0 = -1$ since $(\psi_\lambda, \psi_\lambda^0)$ is defined up to a scalar factor. We thus seek a pair $(u_\lambda, \psi_\lambda)$ such that $G(\lambda, u_\lambda, \psi_\lambda) = 0$, where G is defined by

$$G(\lambda, u, \psi) = \begin{pmatrix} -F_\lambda'(u) + \psi \circ E_\lambda'(u) \\ E_\lambda(u) - x_f \end{pmatrix} = \begin{pmatrix} \partial_u L_\lambda(u, \psi) \\ E_\lambda(u) - x_f \end{pmatrix},$$

where

$$\begin{aligned} L_\lambda(u, \psi) &= -F_\lambda(u) + \psi \cdot E_\lambda(u) \\ &= -\pi_{x^0}(\widetilde{E}_\lambda(u)) + \psi \cdot E_\lambda(u) = (\psi, -1) \cdot \widetilde{E}_\lambda(u) \end{aligned}$$

is the Lagrangian. Let $(\bar{\lambda}, u_{\bar{\lambda}}, \psi_{\bar{\lambda}})$ be a zero of G. Under our assumptions, G is regular and if the partial derivative of G with respect to (u, ψ) at the point $(\bar{\lambda}, u_{\bar{\lambda}}, \psi_{\bar{\lambda}})$ is invertible, then, from the implicit function theorem, we can solve locally the equation $G(\lambda, u_\lambda, \psi_\lambda) = 0$, and the solution $(u_\lambda, \psi_\lambda)$ depends smoothly on λ. Let us analyze the conditions implying the invertibility of the previously mentioned partial derivative. The Jacobian matrix is

$$\frac{\partial G}{\partial(u, \psi)}(\lambda, u, \psi) = \begin{pmatrix} Q_\lambda & E_\lambda'(u)^* \\ E_\lambda'(u) & 0 \end{pmatrix} \tag{1.9}$$

where Q_λ is the quadratic form associated to the augmented system:

$$Q_\lambda = \frac{\partial^2 L_\lambda}{\partial u^2}(u, \psi).$$

The operator matrix (1.9) is invertible if and only if the linear application $E_\lambda'(u)$ is surjective and the quadratic form Q_λ is non-degenerate on $\operatorname{Ker} E_\lambda'(u)$. The surjectivity of $E_\lambda'(u)$ means that the control u is not a singular point of the non-augmented endpoint mapping. But, for this optimal control problem, the singular controls of E_λ are associated to abnormal extremals, see the illustration Figure 1.3. Hence, the absence of minimizing abnormal trajectories is sufficient to ensure the surjectivity of $E_\lambda'(u)$. The fact that Q_λ is non-degenerate on $\operatorname{Ker} E_\lambda'(u)$ is related to the absence of conjugate times. We can conclude that as long as there is no minimizing abnormal trajectories and no conjugate times along the continuation, the continuation process works locally and the solution $(u_\lambda, \psi_\lambda)$ is smooth with respect to the parameter.

However, we are interested by homotopic methods which do not restrict the homotopic parameter to be monotone. Besides, we want to present the

methods in the Hamiltonian frame and consider techniques in finite dimension. Let us make the following assumption:

(A) For every $\lambda \in [0\,,1]$, there exists a normal Hamilton extremal $(x_\lambda, p_\lambda, u_\lambda)$.

Let $\lambda_0 \in [0\,,1]$. Under our assumptions, the mapping

$$(p_0, \lambda) \mapsto x(t_f, x_0, p_0, \lambda)$$

that maps (p_0, λ) to the value at time t_f of the projection on the state space of the solution $z = (x, p)$ of

$$\dot{z}(t) = \overrightarrow{H}(z(t), \lambda), \quad t \in [0\,, t_f], \quad z(0) = (x_0, p_0),$$

is a smooth implicit function in a neighbourhood of $(p_0(\lambda_0), \lambda_0)$. We define the *homotopic function* (see [21] for instance):

$$h(p_0, \lambda) = x(t_f, x_0, p_0, \lambda) - x_f$$

which is locally smooth. Let us assume that the solution $p_0(\lambda_0)$ of the n-dimensional shooting equation $h(\cdot, \lambda_0) = 0$ gives an extremal along which no Jacobi fields become vertical on $(0\,, t_f]$. In particular, there is no conjugate times on $(0\,, t_f]$ and so we have a solution which is locally $\mathscr{C}^0$-optimal for the optimal control problem with $\lambda = \lambda_0$. Typically, solving the family of problems consists in firstly computing a zero of $h(\cdot, \lambda_0) = 0$ for $\lambda_0 = 0$, then, following the path of zeros of h from $\lambda_0 = 0$ to the given target $\lambda = 1$. We introduce the following frame. Let us assume that the interior of the domain $\Omega \coloneqq h^{-1}(0) \subset \mathbb{R}^n \times [0\,,1]$, is composed only of regular points of h and that the restriction of h on $\lambda = 0$ is a submersion:

$$\operatorname{rank} h'(p_0, \lambda) = n, \quad (p_0, \lambda) \in \operatorname{Int}(\Omega),$$

$$\operatorname{rank} \frac{\partial h}{\partial p_0}(p_0, \lambda)|_{\lambda=0} = n, \quad p_0 \in \mathbb{R}^n.$$

As a consequence, each connected component of the level line $\{h = 0\}$ is a one-dimensional submanifold of $\mathbb{R}^{n+1}$ called a *path of zeros*, starting from $\lambda = 0$. Each path of zeros is diffeomorphic either to $\mathbb{R}$ or $\mathbb{S}^1$, see [2]. For any $c = (p_0, \lambda) \in \Omega$, $\dim \operatorname{Ker} h'(c) = 1$ hence we can define the tangent vector $T(c)$ as the unique — up to the orientation — unitary vector of the kernel of $h'(c)$. The orientation is chosen such that the following determinant:

$$\det \begin{bmatrix} h'(c) \\ {}^t T(c) \end{bmatrix}$$

which never vanishes, has a constant sign along the path. This gives a parameterization by the arc length and the paths are computed integrating the differential equation:

$$c'(s) = T(c(s)), \quad c(0) = c_0 \in \{h = 0\},$$

with $c_0 = (p_0(0), 0)$ obtained by simple shooting for instance. One difficulty is to give sufficient conditions *à la Smale* which ensure the existence of a branch joining $\lambda = 0$ to $\lambda = 1$. Another difficulty is that for each value $\bar{\lambda}$ of the parameter, we must compare the associated cost for each component of $\{h = 0\} \cap \{\lambda = \bar{\lambda}\}$. This global aspect may be responsible of a lack of regularity of the value function which maps λ to the minimal cost.

For a given branch, there exists several possibilities that prevent the path to reach the target $\lambda = 1$, even if for every point $c = (p_0, \lambda)$ of the branch, c is regular. Since c is regular, the rank of $h'(c)$ is n with

$$h'(c) = \left[\frac{\partial h}{\partial p_0}(c)\ \frac{\partial h}{\partial \lambda}(c)\right] = \left[\frac{\partial x}{\partial p_0}(t_f, x_0, p_0, \lambda)\ \frac{\partial x}{\partial \lambda}(t_f, x_0, p_0, \lambda)\right].$$

Let us assume that for all s, $\lambda'(s) \neq 0$, which is equivalent to

$$\operatorname{rank} \frac{\partial x}{\partial p_0}(t_f, x_0, p_0(s), \lambda(s)) = n.$$

The parameter λ is thus increasing monotone (since $\lambda'(0) > 0$) and the only possibilities that prevent to reach $\lambda = 1$ are that the covector p_0, along the path of zeros, converges towards the boundary of Ω if it is bounded or goes to infinity in norm, see [20]. The path may converge to an abnormal extremal for limit value of λ. The following definition permits to deal with the second case when there exists $\bar{s}$ such that $\lambda'(\bar{s}) = 0$.

Definition 7 (Turning point). *A point $c(\bar{s}) = (p_0(\bar{s}), \lambda(\bar{s})) \in \{h = 0\}$ is a* turning point *if $\lambda'(\bar{s}) = 0$. This is equivalent to*

$$\operatorname{rank} \frac{\partial x}{\partial p_0}(t_f, x_0, p_0(\bar{s}), \lambda(\bar{s})) = n - 1.$$

A turning point is a point such that t_f is a conjugate time for the problem with $\lambda = \lambda(\bar{s})$. At an order 1 turning point, that is such that $\lambda''(\bar{s}) \neq 0$, there is a change in the variation of λ, whence the name. We can relate the first turning point to local optimality.

Definition 8. *We define $\bar{c} = c(\bar{s}) \in \{h = 0\}$ as the* first turning point *along the path starting from $c(0)$ if, $\lambda'(\bar{s}) = 0$, and if for all $s \in [0\,, \bar{s})$, the trajectory $t \mapsto x(t, x_0, p_0(s), \lambda(s))$ has no conjugate times on $(0\,, t_f]$.*

Theorem 5 ([17]). *Let $c(\bar{s}) \in \{h = 0\}$ be the first turning point of order 1. Then, for all $s > \bar{s}$, $|s - \bar{s}|$ small enough, there exists a conjugate time on $(0\,, t_f)$.*

Remark 13. In the proof of Theorem 5, cf. [17], is defined the extended homotopy

$$\widetilde{h}(p_0, \lambda, t_c) = (h(t_f, x_0, p_0, \lambda), \det \frac{\partial x}{\partial p_0}(t_c, x_0, p_0, \lambda)).$$

It is proved that the extended homotopy is well defined and regular in a neighbourhood of $(p_0(\bar{s}), \lambda(\bar{s}), t_f)$. We can thus use homotopy techniques to compute paths of zeros of the extended homotopy which provides the additional information of the first conjugate time.

There exists another difficulty for differential path following. When a path is diffeomorphic to $\mathbb{R}$, the extremities (if any) are singular points of h. The classification of such points starts by the following result which is a simple consequence of Morse lemma.

Proposition 6 ([2]). *Let $\bar{c} \in \{h = 0\}$ be an hyperbolic and non-degenerate singular point of h of corank one. Then, there exists coordinates $d_1, \ldots, d_{n+1}$ such that, in a neighbourhood of $\bar{c}$, $\{h = 0\}$ is given by*

$$d_1^2 - d_2^2 = 0, \quad d_3 = \cdots = d_{n+1} = 0.$$

In this case, the intrinsic second-order derivative writes, up to a scalar,

$$\bar{\mu}\, h''(\bar{c})|_{(\operatorname{Ker} h'(\bar{c}))^2} \in \operatorname{Sym}(2, \mathbb{R}) \subset \mathrm{M}_2(\mathbb{R})$$

where $\bar{\mu} \in (\mathbb{R}^n)^*$ is any non-zero covector with kernel $\operatorname{Im} h'(\bar{c})$. The hyperbolicity means that the symmetric matrix of order 2 is non-degenerate and has two eigenvalues of opposite signs. As a consequence, the path of zeros is locally made of two smooth curves intersecting transversally, resulting in a bifurcation at $\bar{c}$.

1.4 Examples solved with the Julia `control-toolbox` package

1.4.1 The Julia `control-toolbox` package

The control-toolbox ecosystem [12] gathers Julia packages for mathematical control and applications. The root package is `OptimalControl.jl` which aims to provide tools to solve optimal control problems by direct and indirect methods. For indirect methods we have developed tools for computing geometric control concepts as the flow of a Hamiltonian system or the Poisson brackets of Hamiltonians. For this we use in particular automatic differentiation (the Julia `ForwardDiff.jl` package).

We suppose here that the `control-toolbox` Julia package has been installed, see [12]. All the numerical experiments are reproducible downloading the codes available online at `https://github.com/control-toolbox/Kupka`.

1.4.2 The surface of revolution of minimum area

The first example is the well-known surface of revolution of minimum area problem which dates back to Euler [34, 31]. This is a problem of calculus of variations for which it is easy to have the analytic solutions. But here, as we use this simple problem to illustrate the use of the `control-toolbox`, we consider the optimal control version:

$$\begin{cases} \min \displaystyle\int_0^1 x(t)\sqrt{1+u^2(t)}\,\mathrm{d}t \\ \dot{x}(t) = u(t), \quad u(t) \in \mathbb{R}, \\ x(0) = 1, \quad x(1) = 2.5. \end{cases}$$

To define this problem in our package we have to type:

```
t0 = 0   # initial time
tf = 1   # final time
x0 = 1   # initial state
xf = 2.5 # final state
@def ocp begin
    t ∈ [ t0, tf ], time
    x ∈ R, state
    u ∈ R, control
    x(t0) == x0
    x(tf) == xf
    ẋ(t) == u(t)
    ∫(x(t)*(1 + u(t)^2)^(1/2)) → min
end
```

Here the maximization of the pseudo-Hamiltonian provides the control with respect to the state and the costate (or covector):

$$u(x,p) = \mathrm{sign}(x)\frac{p}{\sqrt{x^2-p^2}}.$$

Then, we can define the Hamiltonian $\mathbf{H}(x,p) = H(x,p,u(x,p))$ and can compute the flow of the Hamiltonian system by using the `Flow` function of the control-toolbox. At the end we can easily compute and plot this flow for different values of the initial costate, see Figure 1.10.

```
# Control in feedback form
u(x, p) = sign(x) * p / sqrt(x^2-p^2)

# The Flow function computes the Hamiltonian flow
ocp_flow = Flow(ocp, u, reltol=1e-10, abstol=1e-10);
```

Here, the shooting equation given by

$$S(p_0) = \pi(z(t_f, x_0, p_0)) - x_f = 0,$$

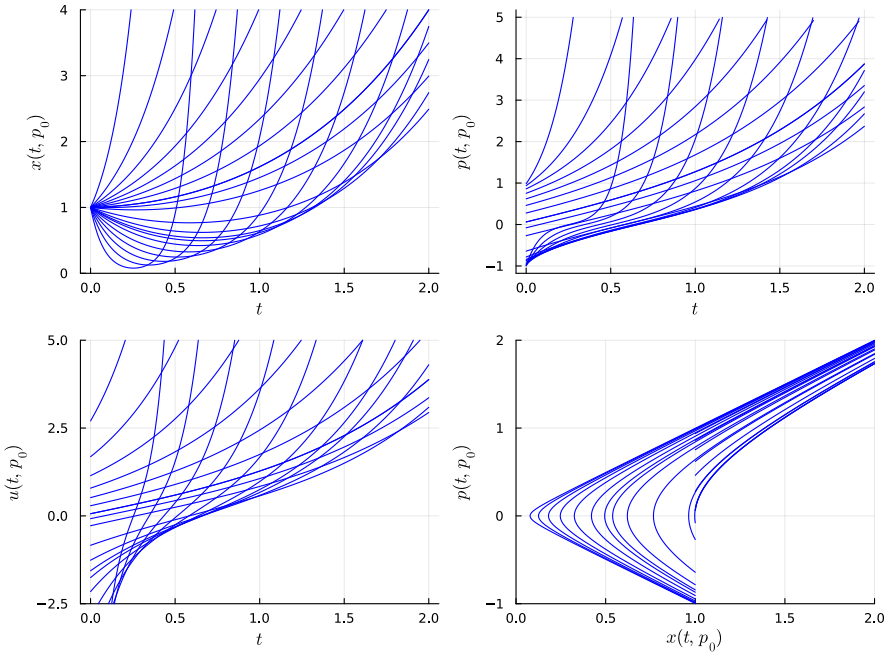

Fig. 1.10. States, costates and controls for $p_0 \in (-1\,, 1)$.

with $\pi(x, p) = x$, has two solutions: $p_0 = -0.9851$ and $p_0 = 0.5126$, see Figure 1.11.

```
# Shooting function
pi((x, p)) = x

tf = 1
xf = 2.5
S(p0) = (pi ∘ ocp_flow)(t0, x0, p0, tf) - xf

# Solve the shooting equation
p0 = -0.985 # First extremal
sol1_p0 = Roots.find_zero(S, (-0.99, -0.97))

p0 = 0.515 # Second extremal
sol2_p0 = Roots.find_zero(S, (0.5, 0.6))
```

Now, we can compute the conjugate points along the two extremals. That's why we have to compute the flow $\delta z(t, p_0)$ of the Jacobi equation with the initial condition $\delta z(0) = (0, 1)$, i.e

$$\delta z(t, p_0) = \frac{\partial}{\partial p_0} z(t, p_0).$$

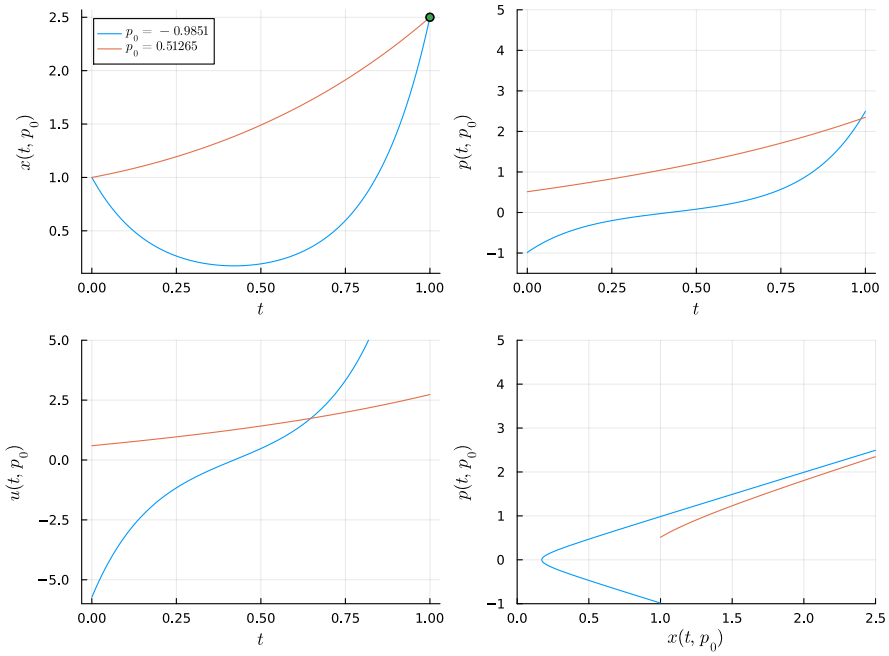

Fig. 1.11. Extremals for the problem of the surface of revolution of minimum area.

To compute conjugate points, we only need the first component:

$$\delta z(t, p_0)_1.$$

```
function jacobi_flow(t, p0)
    x(t, p0) = (pi ∘ ocp_flow)(t0, x0, p0, t)
    return ForwardDiff.derivative(p0 -> x(t, p0), p0)
end
```

The first conjugate time is then the first time τ such that

$$\delta x(\tau, p_0) = \delta z(\tau, p_0)_1 = 0,$$

with p_0 fixed. On Figure 1.12, one can see that the first extremal has a conjugate time smaller than $t_f = 1$ while for the second extremal, there is no conjugate time. Thus, the first extremal cannot be optimal.

```
# Compute the first conjugate time
p0 = sol1_p0
tau0 = Roots.find_zero(tau -> jacobi_flow(tau, p0), (0.4, 0.6))
```

To conclude on this example, we compute the conjugate locus by using a path following algorithm. Define $F(\tau, p_0) = \delta x(\tau, p_0)$ and suppose that

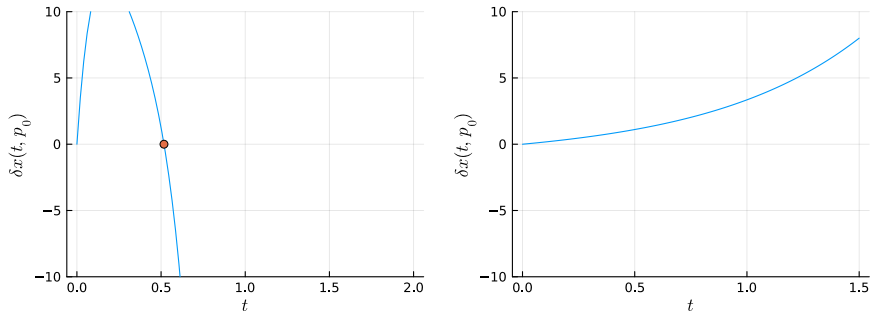

Fig. 1.12. (Left) Conjugate time for $p_0 = -0.9851$. (Right) No conjugate time for $p_0 = -0.51265$.

the partial derivative $\partial_\tau F(\tau, p_0)$ is invertible, then, by the implicit function theorem the conjugate time is a function of p_0. So, since here $p_0 \in \mathbb{R}$, we can compute them (see Figure 1.4.2) by solving the initial value problem for $p_0 \in [\alpha\,, \beta]$:

$$\dot{\tau}(p_0) = -\frac{\partial F}{\partial \tau}(\tau(p_0), p_0)^{-1}\, \frac{\partial F}{\partial p_0}(\tau(p_0), p_0), \quad \tau(\alpha) = \tau_0.$$

For the numerical experiment, we set $\alpha = -0.9995$, $\beta = -0.5$.

```
# conjugate points by path following
function conjugate_times_rhs_path(tau, p0)
    dF = ForwardDiff.gradient(y -> jacobi_flow(y...), [tau, p0])
    return -dF[2]/dF[1]
end
```

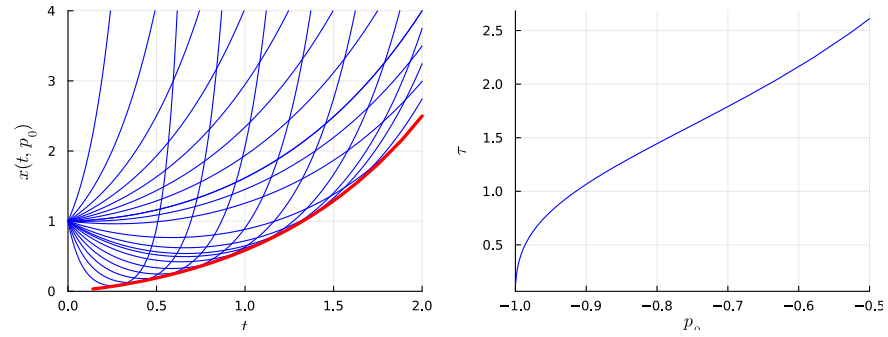

Fig. 1.13. The left graphic represents in blue the geodesic flow for $p_0 \in (-1\,, 1)$, and in red the conjugate locus. The right graphic plots the conjugate time with respect to p_0.

1.4.3 Goddard Problem

For this advanced example, we consider the well-known Goddard problem [23, 34] which models the ascent of a rocket through the atmosphere, and we restrict here ourselves to vertical (one dimensional) trajectories. The state variables are the altitude r, speed v and mass m of the rocket during the flight, for a total dimension of 3. The rocket is subject to gravity g, thrust u and drag force D (function of speed and altitude). The final time t_f is free, and the objective is to reach a maximal altitude with a bounded fuel consumption.

We thus want to solve the optimal control problem in Mayer form

$$r(t_f) \to \max$$

subject to the controlled dynamics

$$\dot{r} = v, \quad \dot{v} = \frac{T_{\max}\, u - D(r,v)}{m} - g, \quad \dot{m} = -u,$$

and subject to the control constraint $u(t) \in [0\,,1]$. The initial state is fixed while only the final mass is prescribed. The dynamics may be written in the form: $\dot{x}(t) = F_0(x(t)) + u(t)\, F_1(x(t))$ with $x = (r, v, m)$. The Julia code to define this problem is simply:

```
t0 = 0      # initial time
r0 = 1      # initial altitude
v0 = 0      # initial speed
m0 = 1      # initial mass
mf = 0.6    # final mass to target

@def ocp begin

    tf, variable           # tf is free
    t ∈ [ t0, tf ], time
    x ∈ R³, state
    u ∈ R, control

    r = x₁
    v = x₂
    m = x₃

    x(t0) == [ r0, v0, m0 ]
    m(tf) == mf,                    (1)
    0 ≤ u(t) ≤ 1
    r(t) ≥ r0

    ẋ(t) == F0(x(t)) + u(t) * F1(x(t))

    r(tf) → max

end
```

```julia
# Dynamics
const Cd = 310
const Tmax = 3.5
const β = 500
const b = 2

F0(x) = begin
    r, v, m = x
    D = Cd * v^2 * exp(-β*(r - 1))
    return [ v, -D/m - 1/r^2, 0 ]
end

F1(x) = begin
    r, v, m = x
    return [ 0, Tmax/m, -b*Tmax ]
end
```

Remark 14. The Hamiltonian is affine with respect to the control, so singular arcs may occur.

We suppose that the optimal solution is composed of a bang arc with maximal control, followed by a singular arc and the final arc is with zero control. Note that the switching function vanishes along the singular. We are in position to solve the problem by an indirect shooting method. We first define the three control laws in feedback form and their associated flows. The control along the *minimal order* singular arcs is obtained as the quotient

$$u_s = -\frac{H_{001}}{H_{101}}$$

of the length three Poisson brackets:

$$H_{001} = \{H_0, \{H_0, H_1\}\}, \quad H_{101} = \{H_1, \{H_0, H_1\}\},$$

see Section 1.2.1 for more details.

Remark 15 (Poisson bracket and Lie derivative). The Poisson bracket $\{H, G\}$ of two Hamiltonians H and G is also given by the Lie derivative of G along the Hamiltonian vector field

$$X_H = (\nabla_p H, -\nabla_x H)$$

of H, that is

$$\{H, G\} = X_H \cdot G$$

which is the reason why we use the "@Lie" macro notation to compute Poisson brackets below.

With the help of the differential geometry primitives from the package `CTBase.jl` these expressions are straightforwardly translated into Julia code:

```
# Controls
u0 = 0                                  # off control
u1 = 1                                  # bang control

H0 = Lift(F0)                           # H0(x, p) = p' * F0(x)
H01  = @Lie { H0, H1 }
H001 = @Lie { H0, H01 }
H101 = @Lie { H1, H01 }
us(x, p) = -H001(x, p) / H101(x, p)     # singular control

# Flows
f0 = Flow(ocp, (x, p, tf) -> u0)
f1 = Flow(ocp, (x, p, tf) -> u1)
fs = Flow(ocp, (x, p, tf) -> us(x, p))
```

Then, we define the shooting function according to the optimal structure we have determined, that is a concatenation of three arcs.

```
x0 = [ r0, v0, m0 ] # initial state

function shoot!(s, p0, t1, t2, tf)

    x1, p1 = f1(t0, x0, p0, t1)
    x2, p2 = fs(t1, x1, p1, t2)
    xf, pf = f0(t2, x2, p2, tf)

    s[1] = constraint(ocp, :eq1)(x0, xf, tf) - mf # constraint (1)
    s[2:3] = pf[1:2] - [ 1, 0 ]                   # transversality
    s[4] = H1(x1, p1)           # H1 = H01 = 0
    s[5] = H01(x1, p1)          # at the entrance of the singular arc
    s[6] = H0(xf, pf)           # since tf is free
end
```

Finally, with a good initialization we can solve the shooting equations thanks to the MINPACK solver.

```
# auxiliary function with aggregated inputs
nle = (s, y) -> shoot!(s, y[1:3], y[4], y[5], y[6])

y = [ p0 ; t1 ; t2 ; tf ]     # initial guess
indirect_sol = MINPACK.fsolve(nle, y) # resolution of S(y) = 0

# we retrieve the costate solution together with the times
p0 = indirect_sol.x[1:3]
t1 = indirect_sol.x[4]
t2 = indirect_sol.x[5]
tf = indirect_sol.x[6]
```

We plot the solution given by the indirect shooting method on Figure 1.14. To do this, a nice feature of the `control-toolbox` is the concatenation of the flows:

```
f = f1 * (t1, fs) * (t2, f0)   # concatenation of the flows
flow_sol = f((t0, tf), x0, p0) # compute x, p and u solution
plot!(plt, flow_sol)           # plot the solution
```

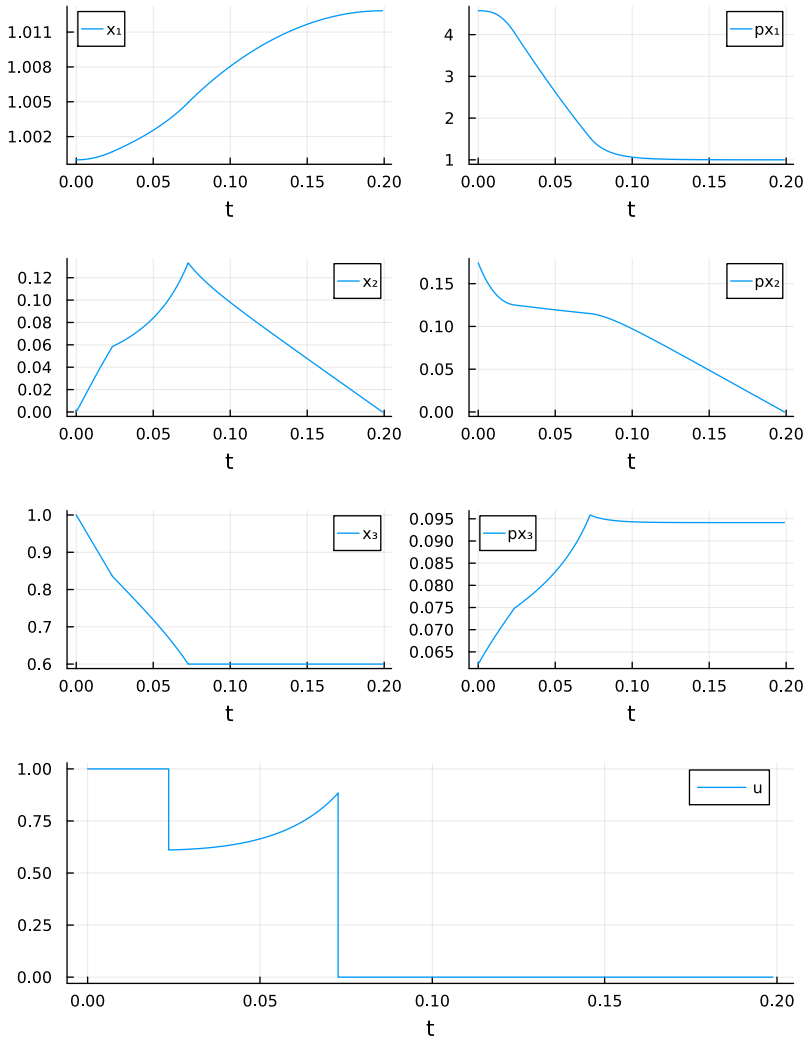

Fig. 1.14. *Solution of the Goddard problem.*

1.5 Conclusion

We have seen here how it is relatively easy with our JULIA `control-toolbox` package to solve optimal control problems and to compute geometric optimal control concepts. The main difficulties for computing the numerical solution of an optimal control problem by indirect methods are to know the optimal control structure and to have a good initial iterate. For the moment, we obtain this by solving the problem by a direct method. But another possibility is to use homotopy methods as described in Section 1.3.4. In the future, we'll develop a path following package for computing the path of zeros of an homotopy $h(z, \lambda) = 0$. Then we'll have in the same environment all the functionalities that the `bocop` [5] and `hampath` [25] softwares provide.

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