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Generalized possibilistic Theories : entanglement, steering and Bell non-locality

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Abstract

Inspired by the operational quantum logic program, we have the contention that probabilities can be viewed as a derived concept, even in a reconstruction program of Quantum Mechanics. In a series of paper, the author has proposed an operational description of physical theories based on a possibilistic semantic (in a sense, the 'probabilities' are replaced by statements associated to a semantic domain made of three values 'indeterminate', 'definitely YES', 'definitely NO'). After a careful analysis of the notions of incompatibility between measurements and of the associated notion of contextuality, proposed in our previous paper, the present paper intends to develop one step further our operational formalism. We show the existence of entangled states and exploit this result to show the existence of the steering of some states by a pair of incompatible measurements. Then, we show the existence of states which are Bell non-local with respect to two pairs of incompatible measurements.

2020 *Mathematics subject classification.* 81P10, 18C50, 18B35

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1 Introduction

Quantum theory appears roughly as a framework allowing to predict probabilities of different outcomes in different experimental settings. The necessity of the use of probabilities in the description of a physical experiment arises generally from the incompleteness of our knowledge about the ultimate parameters of the described system. Disposing of a huge number of copies of the same system, all prepared in the same way, we are able to estimate the probability distribution of a given measurement for this preparation by performing this measurement in several copies, and count the relative frequencies of each outcome. For most of the measurements, at least two exclusive outcomes have non-trivial probability. Two possible explanations for this indeterminacy on the outcomes of the measurements are a priori conceivable.

- First, our description of the prepared system is incomplete : different states are consistent with it. These states can be distinguished by "supplementary hidden variables". We could then separate the available copies of the system in sub-ensembles, each corresponding to a definite state leading to definite outcomes for all measurements. The probabilistic character of the experiments is, in this case, explained by our lack of information
- Second, all individual systems are in the same pure state, but different outcomes are possible even when we perform the same measurement on two identically prepared systems.

This second option is proved to be true for the description of quantum systems and no hidden-variables description can be proposed to describe their behaviour [14][4].

This strange reality of quantum theory with respect to standard classical perspective has several "strange" consequences. Two of these strange characteristics of quantum theory are "contextuality" and "non-locality." Contextuality is the fact that we cannot think about a measurement on a quantum system as revealing a property which is independent of the set of measurements we choose to make. Non-locality is the fact that measurements made by spatially separated observers in a multipartite quantum system can exhibit strong correlations. Contextuality and non-locality in quantum mechanics appear to be intrinsically related to the question of "incompatibility of measurements", as shown in [1].

The origin of quantum incompatibility goes back to Heisenberg's uncertainty principle [13] and Bohr's notion of complementarity [5]. The basic fact emerging from these precursor studies is that there exist measurements that cannot be realized simultaneously, and it is in this sense that they are 'incompatible'. Incompatibility of measurements may seem more like an obstacle than an advantage. However, it appears that only incompatible quantum measurements enable the violation of a Bell inequality [10][15][20]. This suggests to see incompatibility as a useful resource and to explore its properties.

Compatibility is usually defined as a property of a family of measurements, but the concept can be easily generalized to families of channels. The concept of incompatibility gives then a foundation for different impossibility statements within quantum theory or other no-signalling theories. This general approach has been summarized in the report paper [12].

Many physicists have tried to abstract these features of quantum theory to other probabilistic descriptions in order to clarify the deep origins of the singular properties of quantum systems. In order to develop such a high-level perspective on probabilistic theories, a dedicated framework has been elaborated called *General Probabilistic Theory* (GPT) (see [17] for a recent review of the abundant literature and an axiomatic construction of GPTs). This framework appeared initially in the context of axiomatizations of quantum theory. In the current days, this study is oriented towards operational properties of GPTs, the main skill being to identify what structure is needed to realize certain protocols or constructions known from quantum information theory

or classical information theory. One uses GPTs to get better understanding of what makes different things in quantum information theory work.

According to GPT, a physical state (corresponding to a class of operationally equivalent preparation procedures) is defined by a vector of probabilities associated with the outcomes of a maximal and irredundant set of fiducial tests that can be effectuated on collections of samples produced by any of these preparation procedures. It is a basic fact in GPT that this approach is the same as starting with an abstract state space, but instead of using vectors we would describe states in terms of all of the probabilities they can produce. In GPTs, ensembles of objects, conditional probabilities and conditional states can be represented by their respective state spaces and so we can treat them as any other state space and we can use known results, instead of having to prove them *ab initio*. Representing all transformations by "channels" allows us to use the constructions from frameworks based on category theory, since one can interpret state spaces as objects and channels as morphisms.

Alternative categorical formalisms, adapted to the study of quantum theory, have been developed recently [3]. In [2, Theorem 3.15], S. Abramsky makes explicit the fact that the *Projective quantum symmetry groupoid* $PSymmH$ ^[1] is fully and faithfully represented by the category $bmChu_{[0,1]}$, i.e., by the sub-category of the category of bi-extensional Chu spaces associated with the evaluation set $[0, 1]$ obtained by restricting it to Chu morphisms (f_*, f^*) for which f_* is injective. This result suggests that Chu categories could have a central role in the construction of axiomatic quantum mechanics as they provide a natural characterization of the automorphisms of the theory. More surprisingly, and interestingly for us, S. Abramsky shows that the aforementioned representation of $PSymmH$ is 'already' full and faithful if we replace the evaluation space of the Chu category by a three-element set, where the three values represent "definitely yes", "definitely no" and "maybe" [2, Theorem 4.4]. S. Abramsky did not affirm that a three valued semantic is sufficient to found a complete axiomatic quantum theory and allows a complete reconstruction of the usual Hilbert formalism, although its result was clearly leading to this prospect. It was the purpose of our first paper on this subject [7] to explore this question. This paper was devoted to present the basic elements of this 'possibilistic'^[2] semantic formalism. In our second paper [8], we tried to reexamine the basic axioms of our construction and clarified the notion of States/Effects Chu spaces which play a central role in our construction. In a second part of [8], we tried to elaborate on the description of multipartite experiments and were led to define the different tensor products of our spaces of states.

Equipped with our formalism for *Generalized possibilistic Theories* (GpT) developed in these papers, we are now ready to address more specific questions relative to the entanglement, the steering and the Bell non-locality. In the present paper, after a preliminary section recalling the basic elements of GpT gathered along our previous papers, we analyze the existence of entangled states (subsection 3.1). This result allow us to explore the problem of steering of an entangled states by a pair of incompatible measurements (subsection 3.2). Then, we adress the problem of the existence of Bell non-local states (subsection3.3).

¹The objects of this category are the Hilbert spaces of dimension greater than two, and the morphisms are the orbits on semi-unitary maps (i.e. unitary or anti-unitary) under the $U(1)$ group action, which are the relevant symmetries of Hilbert spaces from the point of view of quantum mechanics.

²In the rest of this paper we refer to this construction, based on a three-valued Chu space, as a 'possibilistic' approach to distinguish it from the 'probabilistic' one.

2 Preliminaries

2.1 Preamble

The set \mathfrak{B} will then be equipped with the following poset structure, characterizing the 'information' gathered by the observer:

$$\forall u, v \in \mathfrak{B}, \quad (u \leq v) :\Leftrightarrow (u = \perp \text{ or } u = v). \quad (1)$$

(\mathfrak{B}, \leq) is also an Inf semi-lattice which infima will be denoted \wedge . We have

$$\forall x, y \in \mathfrak{B}, \quad x \wedge y = \begin{cases} x & \text{if } x = y \\ \perp & \text{if } x \neq y \end{cases} \quad (2)$$

We will also introduce a commutative monoid law denoted \bullet and defined by

$$\forall x \in \mathfrak{B}, \quad x \bullet \mathbf{Y} = x, \quad x \bullet \mathbf{N} = \mathbf{N}, \quad \perp \bullet \perp = \perp. \quad (3)$$

$x \bullet y$ will be called *the product of the determinations x and y* .

This law verifies the following properties

$$\forall x \in \mathfrak{B}, \forall B \subseteq \mathfrak{B} \quad x \bullet \bigwedge B = \bigwedge_{b \in B} (x \bullet b), \quad (4)$$

$$\forall x \in \mathfrak{B}, \forall C \subseteq_{Chain} \mathfrak{B} \quad x \bullet \bigvee C = \bigvee_{b \in C} (x \bullet b). \quad (5)$$

(\mathfrak{B}, \leq) will be also equipped with the following involution map :

$$\overline{\perp} := \perp \quad \overline{\mathbf{Y}} := \mathbf{N} \quad \overline{\mathbf{N}} := \mathbf{Y}. \quad (6)$$

2.2 States/Effects Chu spaces

Adopting the operational perspective on quantum experiments, we will introduce the following definitions.

A *preparation process* is an objectively defined, and thus 'repeatable', experimental sequence that allows singular samples of a certain physical system to be produced, in such a way that we are able to submit them to tests. We will denote by \mathfrak{P} the set of preparation processes (each element of \mathfrak{P} can be equivalently considered as the collection of samples produced through this preparation procedure).

For each *property*, that the observer aims to test macroscopically on *any particular sample* of the considered micro-system, it will be assumed that the observer is able to define (i) some detailed 'procedure', in reference to the modes of use of some experimental apparatuses chosen to perform the operation/test, and (ii) a 'rule' allowing the answer 'yes' to be extracted if the macroscopic outcome of the experiment conforms with the expectation of the observer, when the test is performed on any input sample (as soon as this experimental procedure can be opportunely applied to this particular sample). These operations/tests, designed to determine the occurrence of a given property for a given sample, will be called *yes/no tests associated with this property*. The set of 'yes/no tests' at the disposal of the observer will be denoted by \mathfrak{T} .

A yes/no test $t \in \mathfrak{T}$ will be said to be *positive with certainty* (resp. *negative with certainty*) relatively to a preparation process $p \in \mathfrak{P}$ iff the observer is led to affirm that the result of this test, realized on any of the particular samples that could be prepared according to this preparation process, would be 'positive with certainty' (resp. would be 'negative with certainty'), 'should'

this test be effectuated. If the yes/no test can not be stated as 'certain', this yes/no test will be said to be *indeterminate*.

Concretely, the observer can establish the 'certainty' of the result of a given yes/no test on any given sample issued from a given preparation procedure, by running the same test on a sufficiently large (but finite) collection of samples issued from this same preparation process: if the outcome is always the same, the observer will be led to claim that similarly prepared 'new' samples would also produce the same result, if the experiment was effectuated. To summarize, for any preparation process p and any yes/no test t , the evaluation $e(p, t)$ is an element of $\mathfrak{B} := \{\perp, \mathbf{Y}, \mathbf{N}\}$ defined to be \perp (alternatively, \mathbf{Y} or \mathbf{N}) if the outcome of the yes/no test t on any sample prepared according to the preparation procedure p is judged as 'indeterminate' ('positive with certainty' or 'negative with certainty', respectively) by the observer. We note that the order placed on \mathfrak{B} by equation (1) characterizes the amount of information gathered by the observer. Precisely, when the determinacy of a yes/no test is established for an observer, we can consider that this observer possesses some elementary 'information' about the state of the system, whereas, in the 'indeterminate case', the observer has none.

This definition leads to a pre-order structure denoted by $\sqsubseteq_{\mathfrak{P}}$ on the space of preparations \mathfrak{P} and to a pre-order structure denoted by $\sqsubseteq_{\mathfrak{T}}$ on the space of tests \mathfrak{T} as shown in [7] :

$$\forall p_1, p_2 \in \mathfrak{P}, (p_1 \sqsubseteq_{\mathfrak{P}} p_2) :\Leftrightarrow (\forall t \in \mathfrak{T}, e(p_1, t) \leq e(p_2, t)) \quad (7)$$

$$\forall t_1, t_2 \in \mathfrak{T}, (t_1 \sqsubseteq_{\mathfrak{T}} t_2) :\Leftrightarrow (\forall p \in \mathfrak{P}, e(p, t_1) \leq e(p, t_2)). \quad (8)$$

If we consider a collection of preparation processes, we can define a new preparation procedure called *mixture* as follows. The samples produced from the mixed preparation procedure are obtained by a random mixing of the samples issued from the preparation processes of the considered collection indiscriminately. As a consequence, the statements that the observer can establish after a sequence of tests on these samples produced through the mixed procedure is given as the infimum of the statements that the observer can establish for the elements of the collection separately.

We will also assume that there exists a preparation process, unique from the point of view of the statements that can be produced about it, that can be interpreted as a 'randomly-selected' collection of 'un-prepared samples'. This element leads to complete indeterminacy for any yes/no test realized on it.

An equivalence relation can be defined from the previously defined pre-order on the set of preparations \mathfrak{P} . Two preparation processes are identified iff the statements established by the observer about the corresponding prepared samples are identical. A *state* of the physical system is an equivalence class of preparation processes corresponding to the same informational content. The set of equivalence classes will be called *space of states* and denoted \mathfrak{S} .

$$\forall p_1, p_2 \in \mathfrak{P}, (p_1 \sim_{\mathfrak{P}} p_2) :\Leftrightarrow (\forall t \in \mathfrak{T}, e(p_1, t) = e(p_2, t)) \quad (9)$$

$$[p] := \{p' \in \mathfrak{P} \mid p' \sim_{\mathfrak{P}} p\} \quad (10)$$

$$\mathfrak{S} := \{[p] \mid p \in \mathfrak{P}\} \quad (11)$$

The space of states inherits a partial order structure denoted by $\sqsubseteq_{\mathfrak{S}}$.

If we consider a collection of tests, we can define a new test called *mixture* as follows. The result obtained for the mixed test is obtained by a random mixing of the results issued from the tests of the considered collection indiscriminately. As a consequence, the statements that the observer can establish after a sequence of tests is given as the infimum of the statements that the observer can establish for each test separately.

An equivalence relation can be defined from the previously defined pre-order on the set of yes/no tests \mathfrak{T} . An *effect* of the physical system is an equivalence class of yes/no tests, i.e., a class of yes/no tests that are not distinguished from the point of view of the statements that the observer can produce by using these yes/no tests on finite collections of samples. The set of equivalence classes of yes/no tests will be denoted by \mathfrak{E} .

$$\forall t_1, t_2 \in \mathfrak{T}, (t_1 \sim_{\mathfrak{T}} t_2) :\Leftrightarrow (\forall p \in \mathfrak{P}, \epsilon(p, t_1) = \epsilon(p, t_2)) \quad (12)$$

$$[t] := \{t' \in \mathfrak{T} \mid t' \sim_{\mathfrak{T}} t\} \quad (13)$$

$$\mathfrak{E} := \{[t] \mid t \in \mathfrak{T}\}. \quad (14)$$

The space of effects inherits a partial order structure denoted by $\sqsubseteq_{\mathfrak{E}}$.

We then derive a map ϵ according to the following definition :

$$\begin{aligned} \epsilon^{\mathfrak{S}} : \mathfrak{E} &\rightarrow \mathfrak{B}^{\mathfrak{S}} \\ l &\mapsto \epsilon_l^{\mathfrak{S}} \quad | \quad \epsilon_{[t]}^{\mathfrak{S}}([p]) := \epsilon(p, t), \forall p \in \mathfrak{P}, \forall t \in \mathfrak{T}. \end{aligned} \quad (15)$$

According to previous considerations (see [7, 8]), we will assume :

- the space of states \mathfrak{S} is a down-complete Inf semi-lattice (i.e. $\forall S \subseteq \mathfrak{S}$ the infimum $(\prod^{\mathfrak{S}} S)$ exists in \mathfrak{S}), and admits a bottom element denoted $\perp_{\mathfrak{S}}$;
- the space of effects \mathfrak{E} is a down-complete Inf semi-lattice (i.e. $\forall E \subseteq \mathfrak{E}$ the infimum $(\prod^{\mathfrak{E}} E)$ exists in \mathfrak{E});
- the evaluation map $\epsilon^{\mathfrak{S}}$ is a map from \mathfrak{E} to $\mathfrak{B}^{\mathfrak{S}}$ satisfying

$$\forall l \in \mathfrak{E}, \forall S \subseteq \mathfrak{S}, \quad \epsilon_l^{\mathfrak{S}}(\prod^{\mathfrak{S}} S) = \bigwedge_{\sigma \in S} \epsilon_l^{\mathfrak{S}}(\sigma), \quad (16)$$

$$\forall \sigma \in \mathfrak{S}, \forall E \subseteq \mathfrak{E}, \quad \epsilon_{\prod^{\mathfrak{E}} E}^{\mathfrak{S}}(\sigma) = \bigwedge_{l \in E} \epsilon_l^{\mathfrak{S}}(\sigma), \quad (17)$$

and

$$\forall l, l' \in \mathfrak{E}, \quad (\forall \sigma \in \mathfrak{S}, \epsilon_l^{\mathfrak{S}}(\sigma) = \epsilon_{l'}^{\mathfrak{S}}(\sigma)) \Leftrightarrow (l = l'), \quad (18)$$

$$\forall \sigma, \sigma' \in \mathfrak{S}, \quad (\forall l \in \mathfrak{E}, \epsilon_l^{\mathfrak{S}}(\sigma) = \epsilon_l^{\mathfrak{S}}(\sigma')) \Leftrightarrow (\sigma = \sigma') \quad (19)$$

- the following properties are satisfied by the space of effects \mathfrak{E} :

$$\forall l \in \mathfrak{E}, \exists \bar{l} \in \mathfrak{E} \quad | \quad \forall \sigma \in \mathfrak{S}, \epsilon_l^{\mathfrak{S}}(\sigma) = \overline{\epsilon_{\bar{l}}^{\mathfrak{S}}(\sigma)}, \quad (20)$$

$$\exists \mathfrak{Y}_{\mathfrak{E}} \in \mathfrak{E} \quad | \quad \forall \sigma \in \mathfrak{S}, \epsilon_{\mathfrak{Y}_{\mathfrak{E}}}^{\mathfrak{S}}(\sigma) = \mathbf{Y}, \quad (21)$$

$$\exists \perp_{\mathfrak{E}} \in \mathfrak{E} \quad | \quad \forall \sigma \in \mathfrak{S}, \epsilon_{\perp_{\mathfrak{E}}}^{\mathfrak{S}}(\sigma) = \perp. \quad (22)$$

The partial order on \mathfrak{S} (resp. on \mathfrak{E}) will be denoted $\sqsubseteq_{\mathfrak{S}}$ (resp. $\sqsubseteq_{\mathfrak{E}}$).

As soon as all these properties are satisfied, we will say that the triple $(\mathfrak{S}, \mathfrak{E}, \epsilon^{\mathfrak{S}})$ is a *States/Effects Chu space*.

Moreover, we will say that *the space of states admits a description in terms of pure states* iff we have

- the set of completely meet-irreducible elements of \mathfrak{S} , denoted \mathfrak{S}^{pure} and called *set of pure states*, is equal to the set of maximal elements $Max(\mathfrak{S})$ and it is a generating set for \mathfrak{S} , i.e.

$$\forall \sigma \in \mathfrak{S}, \sigma = \prod_{\mathfrak{S}}^{\sigma} \underline{\sigma}_{\mathfrak{S}}, \text{ where } \underline{\sigma}_{\mathfrak{S}} := (\mathfrak{S}^{pure} \cap (\uparrow^{\mathfrak{S}} \sigma)) \text{ and } \mathfrak{S}^{pure} = Max(\mathfrak{S}). \quad (23)$$

Here and in the following, $\widehat{\sigma\sigma'}$ means that σ and σ' have a common upper-bound in \mathfrak{S} , and $\neg\widehat{\sigma\sigma'}$ means they have none). We will also adopt the following notation $\uparrow^{\mathfrak{S}} \sigma$ for the upper subset $\{\sigma' \in \mathfrak{S} \mid \sigma' \sqsupseteq_{\mathfrak{S}} \sigma\}$. We note that, as long as σ and σ' have a common upper-bound in \mathfrak{S} , they have also a supremum which will be denoted $\sigma \sqcup_{\mathfrak{S}} \sigma'$. Endly, we will write $\sigma \parallel_{\mathfrak{S}} \sigma'$ equivalently for $(\sigma \not\sqsubseteq_{\mathfrak{S}} \sigma' \text{ and } \sigma' \not\sqsubseteq_{\mathfrak{S}} \sigma)$.

Now, we intend to describe a natural States/Effects Chu space defined once is given the space of states \mathfrak{S} .

The *generalized space of effects*, denoted $\mathfrak{E}_{\mathfrak{S}}$ is defined to be

$$\mathfrak{E}_{\mathfrak{S}} := \{l_{(\sigma, \sigma')} \mid \sigma, \sigma' \in \mathfrak{S}, \neg\widehat{\sigma\sigma'}\} \cup \{l_{(\sigma, \cdot)} \mid \sigma \in \mathfrak{S}\} \cup \{l_{(\cdot, \sigma)} \mid \sigma \in \mathfrak{S}\} \cup \{l_{(\cdot, \cdot)}\} \quad (24)$$

as a set, with the following Inf semi-lattice law

$$l_{(\sigma_1, \sigma'_1)} \sqcap_{\mathfrak{E}_{\mathfrak{S}}} l_{(\sigma_2, \sigma'_2)} = \begin{cases} l_{(\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2, \sigma'_1 \sqcup_{\mathfrak{S}} \sigma'_2)} & \text{if } \widehat{\sigma_1 \sigma_2} \text{ and } \widehat{\sigma'_1 \sigma'_2} \\ l_{(\cdot, \sigma'_1 \sqcup_{\mathfrak{S}} \sigma'_2)} & \text{if } \neg\widehat{\sigma_1 \sigma_2} \text{ and } \widehat{\sigma'_1 \sigma'_2} \\ l_{(\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2, \cdot)} & \text{if } \widehat{\sigma_1 \sigma_2} \text{ and } \neg\widehat{\sigma'_1 \sigma'_2} \\ l_{(\cdot, \cdot)} & \text{if } \neg\widehat{\sigma_1 \sigma_2} \text{ and } \neg\widehat{\sigma'_1 \sigma'_2} \end{cases} \quad (25)$$

This expression is naturally extended to the whole set of effects (i.e. including the elements of the form $l_{(\sigma, \cdot)}$, $l_{(\cdot, \sigma)}$ and $l_{(\cdot, \cdot)}$) as long as we adopt the convention defining

$$\forall \sigma \in \mathfrak{S}, \neg\widehat{\cdot \sigma} := \text{TRUE} \quad \text{and} \quad \neg\widehat{\sigma \cdot} := \text{TRUE}. \quad (26)$$

Here and in the following, we adopt the following notations

$$(\uparrow^{\mathfrak{E}_{\mathfrak{S}}} l) := \{l' \in \mathfrak{E}_{\mathfrak{S}} \mid l \sqsubseteq_{\mathfrak{E}_{\mathfrak{S}}} l'\} \quad \text{and} \quad (\downarrow_{\mathfrak{E}_{\mathfrak{S}}} l) := \{l' \in \mathfrak{E}_{\mathfrak{S}} \mid l \sqsupseteq_{\mathfrak{E}_{\mathfrak{S}}} l'\}. \quad (27)$$

The evaluation map $\varepsilon^{\mathfrak{S}}$ is defined by

$$\varepsilon_{(\sigma, \sigma')}^{\mathfrak{S}}(\sigma'') := \begin{cases} \mathbf{Y} & \text{if } \sigma \sqsubseteq_{\mathfrak{S}} \sigma'' \\ \mathbf{N} & \text{if } \sigma' \sqsubseteq_{\mathfrak{S}} \sigma'' \\ \perp & \text{otherwise} \end{cases} \quad (28)$$

this expression is naturally extended to the whole set of effects by adopting the following convention

$$(\cdot \sqsubseteq_{\mathfrak{S}} \sigma) := \text{FALSE}. \quad (29)$$

We note that $\perp_{\mathfrak{E}_{\mathfrak{S}}} = l_{(\cdot, \cdot)}$ and $\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}}} = l_{(\perp_{\mathfrak{E}_{\mathfrak{S}}}, \cdot)}$.

To conclude, we have obtained a well-defined States/Effects Chu space $(\mathfrak{S}, \mathfrak{E}_{\mathfrak{S}}, \varepsilon^{\mathfrak{S}})$.

We have to note that we have not necessarily $Irr(\mathfrak{E}_{\mathfrak{S}}) = Max(\mathfrak{E}_{\mathfrak{S}})$. Nevertheless, we will conclude our analysis by characterizing the elements of $Max(\mathfrak{E}_{\mathfrak{S}})$:

$$Max(\mathfrak{E}_{\mathfrak{S}}) = \{I_{(\sigma, \sigma')} \in \mathfrak{E}_{\mathfrak{S}} \mid \sigma, \sigma' \in \mathfrak{S} \text{ with } \sigma \check{\bowtie}_{\mathfrak{S}} \sigma'\} \cup \{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}}}\} \cup \{\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}}}}\}. \quad (30)$$

where we use the following binary relation, denoted $\check{\bowtie}_{\mathfrak{S}}$ and defined on \mathfrak{S} by

$$\forall (\sigma, \sigma') \in \mathfrak{S}^{\times 2}, \quad \sigma \check{\bowtie}_{\mathfrak{S}} \sigma' :\Leftrightarrow ((\forall \sigma'' \sqsubseteq_{\mathfrak{S}} \sigma', \widehat{\sigma\sigma''}^{\mathfrak{S}}) \text{ and } (\forall \sigma'' \sqsubseteq_{\mathfrak{S}} \sigma, \widehat{\sigma'\sigma''}^{\mathfrak{S}}) \text{ and } (\neg \widehat{\sigma\sigma'}^{\mathfrak{S}})). \quad (31)$$

Definition 1. In the following, we will always consider that \mathfrak{E} is a subset of the natural space of states $\mathfrak{E}_{\mathfrak{S}}$ satisfying

$$\forall \sigma, \sigma' \in \mathfrak{S}, \quad (\forall I \in \mathfrak{E}, \varepsilon_I^{\mathfrak{S}}(\sigma) = \varepsilon_I^{\mathfrak{S}}(\sigma')) \Leftrightarrow (\sigma = \sigma') \quad (32)$$

and

$$\forall L \subseteq \mathfrak{E}, \quad (\prod^{\mathfrak{E}_{\mathfrak{S}}} L) \in \mathfrak{E}, \quad (33)$$

$$\forall I \in \mathfrak{E}, \quad \overline{I} \in \mathfrak{E}, \quad (34)$$

$$\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}}} \in \mathfrak{E} \text{ and } \perp_{\mathfrak{E}_{\mathfrak{S}}} \in \mathfrak{E}. \quad (35)$$

We also assume that $\varepsilon^{\mathfrak{S}}$ is the restriction to \mathfrak{E} of the evaluation map defined on $\mathfrak{E}_{\mathfrak{S}}$.

Theorem 1. If the space of effects \mathfrak{E} satisfies the conditions of Definition 1, then $(\mathfrak{S}, \mathfrak{E}, \varepsilon^{\mathfrak{S}})$ is a well-defined States/Effects Chu space. \blacksquare

Proof. Trivial. \square

2.3 Particular spaces of states

Definition 2. A space of states will be said to be a *simplex space of states* (or a *classical space of states*) iff

$$\forall \sigma \in \mathfrak{S}, \exists! U_{\sigma} \subseteq \mathfrak{S}^{pure} \mid \sigma = \prod^{\mathfrak{E}_{\mathfrak{S}}} U_{\sigma}. \quad (36)$$

We note that $U_{\sigma} = \underline{\sigma}_{\mathfrak{S}}$.

Lemma 1. If \mathfrak{S} is a simplex space of states, then for any $\sigma \in \mathfrak{S}$ there exists a unique homomorphism ψ_{σ} from \mathfrak{S} to \mathfrak{B} satisfying $\psi_{\sigma}(\mathfrak{S}^{pure}) \subseteq \{\mathbf{Y}, \mathbf{N}\}$ and $\psi_{\sigma}(\sigma) = \mathbf{Y}$. The reciprocal assertion is also true. \blacksquare

Proof. Let us assume that \mathfrak{S} is a simplex space of states. For any $\sigma \in \mathfrak{S}$, we choose $\sigma' := \prod_{\alpha \in \mathfrak{S}^{pure} \setminus \underline{\sigma}_{\mathfrak{S}}}^{\mathfrak{E}_{\mathfrak{S}}} \alpha$. We then define, for any $\eta \in \mathfrak{S}$, $\psi_{\sigma}(\eta) := \mathbf{Y}$ iff $\sigma \sqsubseteq_{\mathfrak{S}} \eta$, and $\psi_{\sigma}(\eta) := \mathbf{N}$ iff $\sigma' \sqsubseteq_{\mathfrak{S}} \eta$, and $\psi_{\sigma}(\eta) := \perp$ otherwise. The map ψ_{σ} satisfies the requirements. The unicity of ψ_{σ} is clear.

Concerning the reciprocal assertion, it suffices to define $U_{\sigma} := \psi_{\sigma}^{-1}(\mathbf{Y}) \cap \mathfrak{S}^{pure}$. The unicity of U_{σ} is trivial to check from the unicity of ψ_{σ} . \square

Definition 3. According to [11, definition p.117], an Inf semi-lattice \mathfrak{S} is said to be *distributive* iff

$$\forall \sigma, \sigma_1, \sigma_2 \in \mathfrak{S} \mid \sigma \neq \sigma_1, \sigma_2, \quad (\sigma_1 \sqcap_{\mathfrak{S}} \sigma_2) \sqsubseteq_{\mathfrak{S}} \sigma \Rightarrow \\ \exists \sigma'_1, \sigma'_2 \in \mathfrak{S} \mid (\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma'_1, \sigma_2 \sqsubseteq_{\mathfrak{S}} \sigma'_2 \text{ and } \sigma = \sigma'_1 \sqcap_{\mathfrak{S}} \sigma'_2). \quad (37)$$

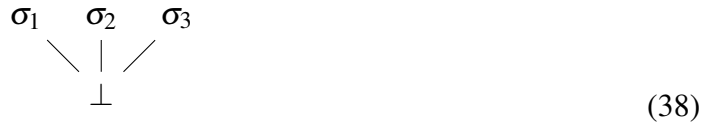
Lemma 2. A simplex space of states is necessarily distributive as an Inf semi-lattice. Conversely, a distributive Inf semi-lattice is also necessarily a simplex space of states. ■

Proof. Let us first assume that \mathfrak{S} is a simplex. Let us consider $\sigma, \sigma_1, \sigma_2 \in \mathfrak{S}$ such that $\sigma \neq \sigma_1, \sigma_2$ and $(\sigma_1 \sqcap_{\mathfrak{S}} \sigma_2) \sqsubseteq_{\mathfrak{S}} \sigma$. We have then $\underline{\sigma}_{\mathfrak{S}} \subseteq (\underline{\sigma}_{1_{\mathfrak{S}}} \cup \underline{\sigma}_{2_{\mathfrak{S}}})$. If we define $\sigma'_1 := \prod^{\mathfrak{S}} (\underline{\sigma}_{1_{\mathfrak{S}}} \cap \underline{\sigma}_{\mathfrak{S}})$ and $\sigma'_2 := \prod^{\mathfrak{S}} (\underline{\sigma}_{2_{\mathfrak{S}}} \cap \underline{\sigma}_{\mathfrak{S}})$ we check immediately that $\sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma'_1$, $\sigma_2 \sqsubseteq_{\mathfrak{S}} \sigma'_2$ and $\sigma = \sigma'_1 \sqcap_{\mathfrak{S}} \sigma'_2$.

Let us now suppose that \mathfrak{S} is distributive. Let us imagine that \mathfrak{S} is not a simplex. It would exist $\sigma \in \mathfrak{S}$ and $U \subsetneq \underline{\sigma}_{\mathfrak{S}}$ such that $\sigma = \prod^{\mathfrak{S}} U$. Let us suppose that U is a minimal subset for inclusion among subsets V satisfying $\sigma = \prod^{\mathfrak{S}} V$. We then introduce $\sigma_3 \in \underline{\sigma}_{\mathfrak{S}} \setminus U$ and $U_1, U_2 \neq \emptyset$ such that $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = U$, we denote $\sigma_1 := \prod^{\mathfrak{S}} U_1$ and $\sigma_2 := \prod^{\mathfrak{S}} U_2$. We have $\sigma \sqsupseteq_{\mathfrak{S}} (\sigma_1 \sqcap_{\mathfrak{S}} \sigma_2)$. Because of the distributivity of \mathfrak{S} , it would exist σ'_1 and σ'_2 such that $\sigma'_1 \sqsupseteq_{\mathfrak{S}} \sigma_1$, $\sigma'_2 \sqsupseteq_{\mathfrak{S}} \sigma_2$ and $\sigma_3 = (\sigma'_1 \sqcap_{\mathfrak{S}} \sigma'_2)$. However, σ_3 being meet-irreducible we must have $\sigma_3 = \sigma'_1$ or $\sigma_3 = \sigma'_2$, and then necessarily $\sigma_3 \in U$. We have then obtained the announced contradiction. \mathfrak{S} is then a simplex. □

Remark 1. We note that the basic example of a simplex admitting a description in terms of two pure states is given by $\mathfrak{S} := \mathfrak{B}$.

Below we give the Hasse diagrams of a non-simplex space of states admitting a description in terms of three pure states :



It will be denoted $\mathcal{S}_3^{(1)}$.

Definition 4. The space of states \mathfrak{S} is said to be *orthocomplemented* iff there exists a map $\star : \mathfrak{S} \setminus \{\perp_{\mathfrak{S}}\} \rightarrow \mathfrak{S} \setminus \{\perp_{\mathfrak{S}}\}$ such that

$$\forall \sigma \in \mathfrak{S} \setminus \{\perp_{\mathfrak{S}}\}, \quad \sigma^{\star\star} = \sigma \quad (39)$$

$$\forall \sigma_1, \sigma_2 \in \mathfrak{S} \setminus \{\perp_{\mathfrak{S}}\}, \quad \sigma_1 \sqsubseteq_{\mathfrak{S}} \sigma_2 \Rightarrow \sigma_2^{\star} \sqsubseteq_{\mathfrak{S}} \sigma_1^{\star} \quad (40)$$

$$\forall \sigma \in \mathfrak{S} \setminus \{\perp_{\mathfrak{S}}\}, \quad \sigma \check{\boxtimes}_{\mathfrak{S}} \sigma^{\star}, \quad (41)$$

(see (31) for the definition of $\check{\boxtimes}$).

Remark 2. Except \mathfrak{B} , the more simple orthocomplemented space of states is



with for example $\sigma_1^* := \sigma_3$ and $\sigma_2^* := \sigma_4$.
It is not a simplex. It will be denoted $\mathcal{S}_4^{(1)}$.

By the way, we note that a simplex space of states \mathfrak{S} is always orthocomplemented, as long as we fix for any $\sigma \in \mathfrak{S}$: $\sigma^* := \prod_{\alpha \in \mathfrak{S}^{pure} \setminus \underline{\sigma}} \alpha$.

2.4 GpT and Quantum theory

Let \mathcal{H} be a finite-dimensional complex Hilbert space. We will denote $\mathcal{H}^* := \mathcal{H} \setminus \{0\}$. We will adopt the bra-ket notation to denote the vectors as $|\psi\rangle$ and the inner product of $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ as $\langle \phi | \psi \rangle$. We will denote by $|\psi\rangle^{ray}$ the ray associated to $|\psi\rangle \in \mathcal{H}^*$, i.e. the set $\{\lambda \cdot |\psi\rangle \mid \lambda \in \mathbb{C}^*\}$. $B(\mathcal{H})$ will denote the real vector space of self-adjoint operators. And for any $U \in B(\mathcal{H})$, $\text{Tr}(U)$ will denote the trace of U . We will denote by $B^+(\mathcal{H})$ the set of positive semi-definite operators, i.e. the set of operators U satisfying $0 \leq \langle \psi | U | \psi \rangle$ for any $|\psi\rangle \in \mathcal{H}^*$.

For any $T \in B(\mathcal{H})$, we will adopt the following notation

$$\text{spec}(T, \mathcal{H}; \lambda) := \{ |\psi\rangle \in \mathcal{H} \mid T|\psi\rangle = \lambda \cdot |\psi\rangle \}. \quad (43)$$

According to Von Neumann's formalism of Quantum mechanics, we will adopt the following choice for the space of preparations \mathfrak{P} :

$$\mathfrak{P} := \mathcal{D}(\mathcal{H}) = \{ P \in B^+(\mathcal{H}) \mid \text{Tr}(P) = 1 \}. \quad (44)$$

$\mathcal{D}(\mathcal{H})$ is the set of density operators on \mathcal{H} .

\mathfrak{P} is a convex set. The set of extremal elements (i.e. the pure states of quantum mechanics), denoted \mathfrak{P}^{pure} , is given by

$$\mathfrak{P}^{pure} = \{ L_{|\psi\rangle} \mid |\psi\rangle \in \mathcal{H}^* \} \cong \{ |\psi\rangle^{ray} \mid |\psi\rangle \in \mathcal{H}^* \}, \quad (45)$$

where $L_{|\psi\rangle}$ denotes the rank-one projector $\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$.

According to Von Neumann's formalism of Quantum mechanics, the space of tests \mathfrak{T} will be chosen as follows

$$\mathfrak{T} := \{ T \in B(\mathcal{H}) \mid \forall P \in \mathfrak{P}, 0 \leq \text{Tr}(PT) \leq 1 \} \quad (46)$$

$$= \{ T \in B(\mathcal{H}) \mid \forall |\psi\rangle \in \mathcal{H}^*, 0 \leq \text{Tr}(L_{|\psi\rangle}T) \leq 1 \} \quad (47)$$

\mathfrak{T} is a convex set. The set of extremal elements, denoted \mathfrak{T}^{pure} , is given by

$$\mathfrak{T}^{pure} = \{ L_{\mathcal{G}} \mid \mathcal{G} \text{ closed subspace of } \mathcal{H} \}, \quad (48)$$

where, for any closed subspace \mathcal{G} of the Hilbert space \mathcal{H} , we denote by $L_{\mathcal{G}}$ the self-adjoint projector whose spectral decomposition is $\mathcal{G} \oplus \mathcal{G}^\perp \cong \mathcal{H}$ (more precisely, $\text{spec}(L_{\mathcal{G}}, \mathcal{H}; 1) = \mathcal{G}$ and $\text{spec}(L_{\mathcal{G}}, \mathcal{H}; 0) = \mathcal{G}^\perp$).

The evaluation map ϵ will be defined as follows

$$\forall P \in \mathfrak{P}, \forall T \in \mathfrak{T}, \epsilon(P, T) := \begin{cases} \mathbf{Y} & \text{iff } \text{Tr}(PT) = 1 \\ \mathbf{N} & \text{iff } \text{Tr}(PT) = 0 \\ \perp & \text{otherwise} \end{cases} \quad (49)$$

We have already

$$\forall L, L' \in \mathfrak{T}^{pure}, \quad (\forall U \in \mathfrak{P}^{pure}, \epsilon(U, L) = \epsilon(U, L')) \Leftrightarrow (L = L') \quad (50)$$

$$\forall U, U' \in \mathfrak{P}^{pure}, \quad (\forall L \in \mathfrak{T}^{pure}, \epsilon(U, L) = \epsilon(U', L)) \Leftrightarrow (U = U'). \quad (51)$$

According to its definition in terms of the space of preparations, the space of states satisfies

$$\mathfrak{S} \cong \{ \text{non-zero closed subspaces of } \mathcal{H} \} \quad (52)$$

\mathfrak{S} is a down-complete Inf semi-lattice with

$$\forall \mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{S}, \quad \mathcal{G}_1 \sqcap_{\mathfrak{S}} \mathcal{G}_2 := \mathcal{G}_1 \oplus \mathcal{G}_2. \quad (53)$$

The evaluation map ϵ is then defined as a map from \mathfrak{T} to $\mathfrak{B}^{\mathfrak{S}}$ by

$$\forall T \in \mathfrak{T}, \forall \mathcal{G} \in \mathfrak{S}, \quad \epsilon_T^{\mathfrak{S}}(\mathcal{G}) := \begin{cases} \mathbf{Y} & \text{iff } \mathcal{G} \subseteq \text{spec}(T, \mathcal{H}; 1) \\ \mathbf{N} & \text{iff } \mathcal{G} \subseteq \text{spec}(T, \mathcal{H}; 0) \\ \perp & \text{otherwise} \end{cases} \quad (54)$$

and the set of pure states

$$\mathfrak{S}^{pure} \cong \{ \text{one-dimensional subspaces of } \mathcal{H} \}. \quad (55)$$

We note that $\perp_{\mathfrak{S}} = \mathcal{H}$.

We also note that \mathfrak{S} is ortho-complemented with $\mathcal{G}^* := \mathcal{G}^{\perp}$ for any \mathcal{G} in \mathfrak{S} (here, \perp denotes the orthogonality of sub-spaces in \mathcal{H}).

The involutive and order-reversing properties of \star are trivial to check. The property (41) is in fact a direct consequence of the property

$$\forall \mathcal{G} \in \mathfrak{S}, \forall x \in \mathcal{H} \setminus \mathcal{G}, \quad (\{x\} \oplus \mathcal{G}) \cap \mathcal{G}^{\perp} \neq \emptyset \quad (56)$$

satisfied by any Hilbert space \mathcal{H} .

According to its definition in terms of the space of tests, the space of effects satisfies

$$\mathfrak{E} \cong \{ (\mathcal{G}_1, \mathcal{G}_2) \mid \mathcal{G}_1, \mathcal{G}_2 \text{ closed subspaces of } \mathcal{H} \text{ with } \mathcal{G}_1 \perp \mathcal{G}_2 \} \quad (57)$$

\mathfrak{E} is equipped with the following down-complete Inf semi-lattice structure

$$\forall (\mathcal{G}_1, \mathcal{G}_2), (\mathcal{G}'_1, \mathcal{G}'_2) \in \mathfrak{E}, \quad (\mathcal{G}_1, \mathcal{G}_2) \sqcap_{\mathfrak{E}} (\mathcal{G}'_1, \mathcal{G}'_2) := (\mathcal{G}_1 \cap \mathcal{G}'_1, \mathcal{G}_2 \cap \mathcal{G}'_2). \quad (58)$$

The evaluation map from \mathfrak{E} to $\mathfrak{B}^{\mathfrak{S}}$ is given by

$$\forall (\mathcal{G}_1, \mathcal{G}_2) \in \mathfrak{E}, \forall \mathcal{G} \in \mathfrak{S}, \quad \epsilon_{(\mathcal{G}_1, \mathcal{G}_2)}^{\mathfrak{S}}(\mathcal{G}) := \begin{cases} \mathbf{Y} & \text{iff } \mathcal{G} \subseteq \mathcal{G}_1 \\ \mathbf{N} & \text{iff } \mathcal{G} \subseteq \mathcal{G}_2 \\ \perp & \text{otherwise} \end{cases} \quad (59)$$

and the set of pure effects

$$\mathfrak{E}^{pure} \cong \{ (\mathcal{G}, \mathcal{G}^{\perp}) \mid \mathcal{G} \text{ closed subspace of } \mathcal{H} \}. \quad (60)$$

We note that we have in particular $\mathcal{G} \oplus \mathcal{G}^{\perp} = \mathcal{H}$.

We note that the space of effects defined in (57) is only a subspace of the space of effects built naturally in the framework of GpT, i.e. it is only a subspace of

$$\mathfrak{E}_{\mathfrak{S}} := \{ (\mathcal{G}_1, \mathcal{G}_2) \mid \mathcal{G}_1, \mathcal{G}_2 \text{ closed subspaces of } \mathcal{H} \text{ with } \mathcal{G}_1 \cap \mathcal{G}_2 = \{0\} \}. \quad (61)$$

Then, the description of quantum theory within GpT necessitates the adjunction of a star structure on the space of states that will lead to a selection of the corresponding effects. This point has been mentioned in [7] and will be precised now. It is clear that the star structure on \mathfrak{S} is obviously given by $\forall \mathcal{G} \in \mathfrak{S}, \mathcal{G}^* := \mathcal{G}^{\perp}$.

2.5 Reduction of the space of effects

Let us come back to the general case. We will introduce the following generic reduction of the space of effects, as soon as the space of states is orthocomplemented :

Definition 5. Let \mathfrak{S} be a space of states, $\mathfrak{E}_{\mathfrak{S}}$ and $\varepsilon^{\mathfrak{S}}$ be the generalized space of effects and evaluation map associated to it.

If the space of states \mathfrak{S} is orthocomplemented, we can define a *reduction* of the space of effects $\mathfrak{E}_{\mathfrak{S}}$, denoted by $\overline{\mathfrak{E}}_{\mathfrak{S}}$, as follows :

$$\overline{\mathfrak{E}}_{\mathfrak{S}} := \{l_{(\sigma, \sigma')} \mid \sigma, \sigma' \in \mathfrak{S} \setminus \{\perp_{\mathfrak{S}}\}, \sigma' \sqsupseteq_{\mathfrak{S}} \sigma^*\} \cup \{l_{(\sigma, \cdot)} \mid \sigma \in \mathfrak{S}\} \cup \{l_{(\cdot, \sigma)} \mid \sigma \in \mathfrak{S}\} \cup \{l_{(\cdot, \cdot)}\} \text{ as a set} \quad (62)$$

and the Inf semi-lattice structure on $\overline{\mathfrak{E}}_{\mathfrak{S}}$ is induced from that defined on $\mathfrak{E}_{\mathfrak{S}}$, i.e. defined by (25). The evaluation map is the restriction of $\varepsilon^{\mathfrak{S}}$ to $\overline{\mathfrak{E}}_{\mathfrak{S}}$.

Remark 3. We note that, in the quantum case, we have effectively a complete equality between the space of effects defined in (57) and the reduced space of effects $\overline{\mathfrak{E}}_{\mathfrak{S}}$.

Lemma 3. We note that the conditions of Definition 1 are fulfilled. As a consequence, $(\mathfrak{S}, \overline{\mathfrak{E}}_{\mathfrak{S}}, \varepsilon^{\mathfrak{S}})$ is a well defined States/Effects Chu space. ■

Proof. Due to the fact that the elements of the form $l_{(\sigma, \cdot)}$ must be in \mathfrak{E} for any $\sigma \in \mathfrak{S}$, we deduce easily that $\forall \sigma, \sigma' \in \mathfrak{S}, (\forall l \in \mathfrak{E}, \varepsilon_l^{\mathfrak{S}}(\sigma) = \varepsilon_l^{\mathfrak{S}}(\sigma')) \Leftrightarrow (\sigma = \sigma')$.

Secondly, let us consider $\sigma_1, \sigma'_1 \in \mathfrak{S} \setminus \{\perp_{\mathfrak{S}}\}, \sigma'_1 \sqsupseteq_{\mathfrak{S}} \sigma_1^*$ and $\sigma_2, \sigma'_2 \in \mathfrak{S} \setminus \{\perp_{\mathfrak{S}}\}, \sigma'_2 \sqsupseteq_{\mathfrak{S}} \sigma_2^*$. We then have immediately $(\sigma'_1 \sqcup_{\mathfrak{S}} \sigma'_2) \sqsupseteq_{\mathfrak{S}} (\sigma_1^* \sqcup_{\mathfrak{S}} \sigma_2^*) \sqsupseteq_{\mathfrak{S}} (\sigma_1^* \sqcap_{\mathfrak{S}} \sigma_2^*) = (\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2)^*$. Hence, $l_{(\sigma_1 \sqcup_{\mathfrak{S}} \sigma_2, \sigma'_1 \sqcup_{\mathfrak{S}} \sigma'_2)} \in \overline{\mathfrak{E}}_{\mathfrak{S}}$.

The other properties are true by assumption. □

Theorem 2. The reduced space of effects $\overline{\mathfrak{E}}_{\mathfrak{S}}$ is defined in terms of pure effects :

$$\forall l \in \overline{\mathfrak{E}}_{\mathfrak{S}}, l = \prod^{\overline{\mathfrak{E}}_{\mathfrak{S}}} \underline{l}_{\overline{\mathfrak{E}}_{\mathfrak{S}}}, \text{ where } \underline{l}_{\overline{\mathfrak{E}}_{\mathfrak{S}}} = (\overline{\mathfrak{E}}_{\mathfrak{S}}^{\text{pure}} \cap (\uparrow^{\overline{\mathfrak{E}}_{\mathfrak{S}}} l)) \quad (63)$$

with

$$\overline{\mathfrak{E}}_{\mathfrak{S}}^{\text{pure}} = \text{Max}(\overline{\mathfrak{E}}_{\mathfrak{S}}) = \{l_{(\sigma, \sigma^*)} \mid \sigma \in \mathfrak{S} \setminus \{\perp_{\mathfrak{S}}\}\} \cup \{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}}}\} \cup \{\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{S}}}}\}. \quad (64)$$

■

Proof. It suffices to note that the set of maximal elements of $\overline{\mathfrak{E}}_{\mathfrak{S}}$ is a generating subset :

$$\forall \sigma, \sigma' \in \mathfrak{S} \mid \sigma^* \sqsubseteq_{\mathfrak{S}} \sigma', \quad l_{(\sigma, \sigma')} = l_{(\sigma, \sigma^*)} \sqcap_{\mathfrak{E}_{\mathfrak{S}}} l_{(\sigma'^*, \sigma')}. \quad (65)$$

This subset of maximal elements is then equal to the set of meet-irreducible elements of $\overline{\mathfrak{E}}_{\mathfrak{S}}$. □

2.6 Channels

We turn the collection of States/Effects Chu spaces into a category by defining the following morphisms.

The channels of the system are defined as Chu morphisms [18] from a States/Effects Chu space $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$ to another States/Effects Chu space $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$, i.e. as pairs of maps $f : \mathfrak{S}_A \rightarrow \mathfrak{S}_B$ and $f^* : \mathfrak{E}_B \rightarrow \mathfrak{E}_A$ satisfying the following property

$$\forall \sigma_A \in \mathfrak{S}_A, \forall l_B \in \mathfrak{E}_B \quad \varepsilon_{l_B}^{\mathfrak{S}_B}(f(\sigma_A)) = \varepsilon_{f^*(l_B)}^{\mathfrak{S}_A}(\sigma_A). \quad (66)$$

As noted in [7, 8], the duality property (66) suffices to deduce the following properties. The left-component f of a channel from $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$ to $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$ satisfies

$$\forall S \subseteq \mathfrak{S}_A, \quad f(\prod^{\mathfrak{S}_A} S) = \prod_{\sigma \in S}^{\mathfrak{S}_B} f(\sigma) \quad (67)$$

$$\forall \mathfrak{C} \subseteq_{Chain} \mathfrak{S}_A, \quad f(\bigsqcup^{\mathfrak{S}_A} \mathfrak{C}) = \bigsqcup_{\sigma \in \mathfrak{C}}^{\mathfrak{S}_B} f(\sigma). \quad (68)$$

As a consequence of (68), f is in particular monotone.

The right-component f^* of a channel from $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$ to $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$ satisfies

$$\forall E \subseteq \mathfrak{E}_B, \quad f^*(\prod^{\mathfrak{E}_B} E) = \prod_{l \in E}^{\mathfrak{E}_A} f^*(l) \quad (69)$$

$$\forall \mathfrak{C} \subseteq_{Chain} \mathfrak{E}_B, \quad f^*(\bigsqcup^{\mathfrak{E}_B} \mathfrak{C}) = \bigsqcup_{l \in \mathfrak{C}}^{\mathfrak{E}_A} f^*(l) \quad (70)$$

$$\forall l \in \mathfrak{E}_B, \quad f^*(\overline{l}) = \overline{f^*(l)} \quad (71)$$

$$f^*(\mathfrak{Y}_{\mathfrak{E}_B}) = \mathfrak{Y}_{\mathfrak{E}_A}. \quad (72)$$

In particular, f^* is monotone.

As noted in [8], as soon as a map f which is monotonic and satisfies (67) is given, we can define unambiguously the map f^* satisfying with f the duality relation (66). Indeed, we have the following theorem

Theorem 3. Let us consider a map $(A : \mathfrak{S} \rightarrow \mathfrak{B}, \sigma \mapsto \mathfrak{a}_\sigma)$ satisfying

$$\forall \sigma, \sigma' \in \mathfrak{S}, \quad (\sigma \sqsubseteq_{\mathfrak{S}} \sigma') \Rightarrow (\mathfrak{a}_\sigma \leq \mathfrak{a}_{\sigma'}), \quad (73)$$

$$\forall \{\sigma_i \mid i \in I\} \subseteq \mathfrak{S}, \quad \mathfrak{a}_{\prod_{i \in I}^{\mathfrak{S}} \sigma_i} = \bigwedge_{i \in I} \mathfrak{a}_{\sigma_i}, \quad (74)$$

Then, we have

$$\exists ! l \in \mathfrak{E}_{\mathfrak{S}} \mid \forall \sigma \in \mathfrak{S}, \varepsilon_l(\sigma) = \mathfrak{a}_\sigma. \quad (75)$$

■

Proof. Straightforward. If $\{\sigma \mid \mathfrak{a}_\sigma = \mathbf{Y}\}$ and $\{\sigma \mid \mathfrak{a}_\sigma = \mathbf{N}\}$ are not empty, it suffices to define $\Sigma_A := \prod^{\mathfrak{S}} \{\sigma \mid \mathfrak{a}_\sigma = \mathbf{Y}\}$, $\Sigma'_A := \prod^{\mathfrak{S}} \{\sigma \mid \mathfrak{a}_\sigma = \mathbf{N}\}$ and $l := l_{(\Sigma_A, \Sigma'_A)}$ (the case where some or all of these subsets are empty is treated immediately). □

Remark 4. In order to obtain a strengthened existence result of the type $l \in \overline{\mathfrak{E}_{\mathfrak{S}}}$ it suffices to add a constraint on the map A defined by $\forall \sigma \in \mathfrak{S}, \mathfrak{a}_{\sigma^*} = \overline{\mathfrak{a}_\sigma}$.

The space of channels from the space of states \mathfrak{S}_A to the space of states \mathfrak{S}_B will be denoted $\mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_B)$. It is the space of maps from \mathfrak{S}_A to \mathfrak{S}_B that is order-preserving and satisfies (67).

The composition of a channel (f, f^*) from $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$ to $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$ by another channel (g, g^*) defined from $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$ to $(\mathfrak{S}_C, \mathfrak{E}_C, \varepsilon^{\mathfrak{S}_C})$ is given by the pair of maps $(g \circ f, f^* \circ g^*)$ defining a valid channel (i.e. a Chu morphism) from $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$ to $(\mathfrak{S}_C, \mathfrak{E}_C, \varepsilon^{\mathfrak{S}_C})$.

The infimum of a channel (f, f^*) from $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$ to $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$ with another channel (g, g^*) defined from $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$ to $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$ is given by the pair of maps $(g \sqcap f, f^* \sqcap g^*)$ defining a valid channel (i.e. a Chu morphism) from $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$ to $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$.

We note that the identity map $(id_{\mathfrak{S}}, id_{\mathfrak{E}})$ is a channel from the States/Effects Chu space $(\mathfrak{S}, \mathfrak{E}, \varepsilon)$ to itself.

We have to note the following useful theorem.

Theorem 4. Let us consider a map $(B : \mathfrak{E} \longrightarrow \mathfrak{B}, l \mapsto b_l)$ satisfying

$$\forall l, l' \in \mathfrak{E}, \quad (l \sqsubseteq_{\mathfrak{E}} l') \Rightarrow (b_l \leq b_{l'}), \quad (76)$$

$$\forall \{l_i \mid i \in I\} \subseteq \mathfrak{E}, \quad b_{\prod_{i \in I} l_i} = \bigwedge_{i \in I} b_{l_i}, \quad (77)$$

$$\forall l \in \mathfrak{E}, \quad b_{\overline{l}} = \overline{b_l}, \quad (78)$$

$$b_{\mathfrak{Y}_{\mathfrak{E}}} = \mathbf{Y}. \quad (79)$$

Then, we have

$$\exists! \sigma \in \mathfrak{S} \mid \forall l \in \mathfrak{E}, \varepsilon_l(\sigma) = b_l. \quad (80)$$

■

Proof. Let us consider $l_B := \prod^{\mathfrak{E}} \{l \in \mathfrak{E} \mid b_l = \mathbf{Y}\}$. Note that l_B exists because \mathfrak{E} is a down-complete Inf semi-lattice and the subset $\{l \in \mathfrak{E} \mid b_l = \mathbf{Y}\}$ contains at least the element $\mathfrak{Y}_{\mathfrak{E}}$. Moreover, $b_{l_B} = \mathbf{Y}$ because of the relation (77). Note also that $l \sqsupseteq_{\mathfrak{E}} l_B$ implies $b_l = \mathbf{Y}$ because of the relation (76), and conversely $b_l = \mathbf{Y}$ implies $l \sqsupseteq_{\mathfrak{E}} l_B$ due to the definition of l_B . Let us now introduce $\Sigma_{l_B} = \prod^{\mathfrak{S}} \varepsilon_{l_B}^{-1}(\mathbf{Y})$. For any $l \in \mathfrak{E}$ such that $l \sqsupseteq_{\mathfrak{E}} l_B$, we have $\varepsilon_l(\Sigma_{l_B}) \geq \varepsilon_{l_B}(\Sigma_{l_B}) = \mathbf{Y}$, i.e. $\varepsilon_l(\Sigma_{l_B}) = \mathbf{Y}$. We could suppose that $l_B = l_{(\Sigma_B, \Sigma'_B)}$ for a certain $\Sigma'_B \in \mathfrak{S}$. However, we note that, because of (77) and (79), we have $b_{l_{(\Sigma_B, \Sigma'_B)}} = b_{l_{(\Sigma_B, \Sigma'_B)} \sqcap_{\mathfrak{E}} \mathfrak{Y}_{\mathfrak{E}}} = b_{l_{(\Sigma_B, \Sigma'_B)}} \wedge b_{\mathfrak{Y}_{\mathfrak{E}}} = \mathbf{Y}$. Hence, we have to accept that $l_B = l_{(\Sigma_B, \cdot)}$. Thus, we note that, for any $l_{(\Sigma, \Sigma')} \in \mathfrak{E}$, $l_{(\Sigma, \Sigma')} \not\sqsupseteq_{\mathfrak{E}} l_B$ is then equivalent to $\Sigma \not\sqsupseteq_{\mathfrak{S}} \Sigma_B$. Then, if $l_{(\Sigma, \Sigma')} \not\sqsupseteq_{\mathfrak{E}} l_B$ we cannot have $\varepsilon_{l_{(\Sigma, \Sigma')}}(\Sigma_{l_B}) = \mathbf{Y}$. We then conclude that $\varepsilon_l(\Sigma_{l_B}) = \mathbf{Y}$ is equivalent to $l \sqsupseteq_{\mathfrak{E}} l_B$, or in other words $\varepsilon_l(\Sigma_{l_B}) = \mathbf{Y}$ is equivalent to $b_l = \mathbf{Y}$. Using (78) and (20), we deduce that $(\varepsilon_l(\Sigma_{l_B}) = \mathbf{N}) \Leftrightarrow (\varepsilon_{\overline{l}}(\Sigma_{l_B}) = \mathbf{Y}) \Leftrightarrow (b_{\overline{l}} = \mathbf{Y}) \Leftrightarrow (b_l = \mathbf{N})$. As a result, we have for any $l \in \mathfrak{E}$ the equality $\varepsilon_l(\Sigma_B) = b_l$. As a conclusion, there exists $\sigma \in \mathfrak{S}$ such that $\forall l \in \mathfrak{E}, \varepsilon_l(\sigma) = b_l$.

Let us finally suppose that there exists $\sigma_1, \sigma_2 \in \mathfrak{S}$ such that $\forall l \in \mathfrak{E}, \varepsilon_l(\sigma_1) = b_l = \varepsilon_l(\sigma_2)$. Due to property (19) we then obtain $\sigma_1 = \sigma_2$. The state σ is then unique. \square

It appears clearly that "measurements" (realized by an observer on the given system) are channels which send states to statements (i.e. to elements of \mathfrak{B}).

Definition 6. A *measurement* is a channel m from the state space \mathfrak{S} to the space of statements \mathfrak{B} . The space of measurements associated to \mathfrak{S} will be denoted $\mathfrak{M}_{\mathfrak{S}}$.

Theorem 5. The space of measurements $\mathfrak{M}_{\mathfrak{S}}$ is isomorphic to the space of effects $\mathfrak{E}_{\mathfrak{S}}$. ■

Proof. An effect $l \in \mathfrak{E}_{\mathfrak{S}}$ being given, the map m_l defined by

$$m_l(\sigma) := \varepsilon_l^{\mathfrak{S}}(\sigma), \quad \forall \sigma \in \mathfrak{S}. \quad (81)$$

is a well defined homomorphism from \mathfrak{S} to \mathfrak{B} . Hence, the map

$$\begin{aligned} \varphi : \mathfrak{E}_{\mathfrak{S}} &\longrightarrow \mathfrak{M}_{\mathfrak{S}} \\ l &\longmapsto m_l \end{aligned} \quad (82)$$

is an homomorphism. It is moreover bijective. Indeed, let us consider any m in $\mathfrak{M}_{\mathfrak{S}}$. If $\{\sigma \mid m(\sigma) = \mathbf{Y}\}$ and $\{\sigma \mid m(\sigma) = \mathbf{N}\}$ are not empty, they are principal filters, and it suffices to associate to m the states $\sigma_A := \prod^{\mathfrak{S}} \{\sigma \mid m(\sigma) = \mathbf{Y}\}$, $\sigma'_A := \prod^{\mathfrak{S}} \{\sigma \mid m(\sigma) = \mathbf{N}\}$ and the effect $l := l_{(\sigma_A, \sigma'_A)}$ (the case where some or all of these subsets are empty is treated immediately).

As usual, the data m_l suffices to define a Chu morphism (m_l, m_l^*) . Indeed, the map $m_l^* : \mathfrak{E}_{\mathfrak{B}} \longrightarrow \mathfrak{E}_{\mathfrak{S}}$ is defined by the duality relation (66), i.e.

$$\forall \sigma \in \mathfrak{S}, \forall u \in \mathfrak{E}_{\mathfrak{B}}, \quad \varepsilon_{m_l^*(u)}^{\mathfrak{S}}(\sigma) = \varepsilon_u^{\mathfrak{B}}(m_l(\sigma)) = \varepsilon_u^{\mathfrak{B}}(\varepsilon_l^{\mathfrak{S}}(\sigma)) \quad (83)$$

The expression of $m_l^*(u)$ for any u in $\mathfrak{E}_{\mathfrak{B}}$ is reduced by the application of the three relations (69)(71)(72) to the property

$$m_l^*(l_{(\mathbf{Y}, \mathbf{N})}) := l \quad (84)$$

which validity can be checked directly. □

2.7 Bipartite experiments

In this subsection, we recall some elements, presented in [8], about the way to describe experiments implying two parties : Alice and Bob. The bipartite state space will be formed from two given spaces of states \mathfrak{S}_A and \mathfrak{S}_B . It appeared clearly in [8] that this notion of bipartite space of states is ambiguous and different elementary constructions were proposed.

We will denote by $\mathfrak{S}_{AB} = \mathfrak{S}_A \boxtimes \mathfrak{S}_B$ the bipartite space of states built from two given spaces of states \mathfrak{S}_A and \mathfrak{S}_B .^[3] We will also denote by $\mathfrak{E}_{AB} = \mathfrak{E}_A \boxtimes \mathfrak{E}_B$ the bipartite effect space build from two given effect spaces \mathfrak{E}_A and \mathfrak{E}_B . We will denote $\varepsilon^{\mathfrak{S}_{AB}}$ the corresponding bipartite evaluation map from \mathfrak{E}_{AB} to $\mathfrak{B}^{\mathfrak{S}_{AB}}$. We will assume the following requirements about these elements.

First of all, we have to ensure that $(\mathfrak{S}_{AB}, \mathfrak{E}_{AB}, \varepsilon^{\mathfrak{S}_{AB}})$ is a valid Spaces/Effects Chu space.

Secondly, for every effects l_A and l_B realized independently by each party respectively, we will assume that there must exist a unique associated bipartite effect in \mathfrak{E}_{AB} . As a consequence, we will assume that there are maps $\iota^{\mathfrak{E}_{AB}} : \mathfrak{E}_A \times \mathfrak{E}_B \hookrightarrow \mathfrak{E}_{AB}$ which describe the inclusion of 'pure tensors' in \mathfrak{E}_{AB} (for readability, we shall write $l_A \boxtimes l_B$ rather than $\iota^{\mathfrak{E}_{AB}}(l_A, l_B)$). Moreover, if

³Throughout this short axiomatic introduction, we adopt the notation \boxtimes for the tensor product in order to allow for different candidates for this tensor product. These different candidates will be denoted $\otimes, \tilde{\otimes}, \dots$

one of the parties chooses a mixture of effects, then this results in a mixture of the respective bipartite effects. In other words,

$$\left(\prod_{i \in I} \mathfrak{E}_A^{i,A}\right) \boxtimes \mathfrak{E}_B = \prod_{i \in I} \mathfrak{E}_B^{i,A} (\mathfrak{E}_A \boxtimes \mathfrak{E}_B), \quad (85)$$

$$\mathfrak{E}_A \boxtimes \left(\prod_{i \in I} \mathfrak{E}_B^{i,B}\right) = \prod_{i \in I} \mathfrak{E}_A^{i,B} (\mathfrak{E}_A \boxtimes \mathfrak{E}_B). \quad (86)$$

In the same logic, for every states $\sigma_A \in \mathfrak{S}_A$ and $\sigma_B \in \mathfrak{S}_B$, prepared independently by each parties, we will assume that there must exist a unique associated bipartite state in \mathfrak{S}_{AB} . As a consequence, we will assume that there are maps $\iota^{\mathfrak{S}_{AB}} : \mathfrak{S}_A \times \mathfrak{S}_B \hookrightarrow \mathfrak{S}_{AB}$ which describe the inclusion of 'pure tensors' in \mathfrak{S}_{AB} (for readability, we shall write $\sigma_A \boxtimes \sigma_B$ rather than $\iota^{\mathfrak{S}_{AB}}(\sigma_A, \sigma_B)$). Moreover, if any of the parties prepares a mixture of states, then this results in a mixture of the respective bipartite states. In other words,

$$\left(\prod_{i \in I} \mathfrak{S}_A^{i,A}\right) \boxtimes \sigma_B = \prod_{i \in I} \mathfrak{S}_B^{i,A} (\sigma_{i,A} \boxtimes \sigma_B), \quad (87)$$

$$\sigma_A \boxtimes \left(\prod_{i \in I} \mathfrak{S}_B^{i,B}\right) = \prod_{i \in I} \mathfrak{S}_A^{i,B} (\sigma_A \boxtimes \sigma_{i,B}). \quad (88)$$

Thirdly, for every $\sigma_{AB}, \sigma'_{AB} \in \mathfrak{S}_{AB}$ such that $\sigma_{AB} \neq \sigma'_{AB}$, we will assume that there must exist effects $\mathfrak{E}_A \in \mathfrak{E}_A$ and $\mathfrak{E}_B \in \mathfrak{E}_B$ such that when the two parties prepare σ_{AB} and apply \mathfrak{E}_A and \mathfrak{E}_B respectively, the resulting determination is different from the experiment where the parties prepare σ'_{AB} and apply \mathfrak{E}_A and \mathfrak{E}_B respectively. As a summary, applying effects locally is sufficient to distinguish all of the states in \mathfrak{S}_{AB} (this principle is called "tomographic locality"), i.e.

$$\forall \sigma_{AB}, \sigma'_{AB} \in \mathfrak{S}_{AB}, \quad (\forall \mathfrak{E}_A \in \mathfrak{E}_A, \mathfrak{E}_B \in \mathfrak{E}_B, \quad \varepsilon_{\mathfrak{E}_A \boxtimes \mathfrak{E}_B}^{AB}(\sigma_{AB}) = \varepsilon_{\mathfrak{E}_A \boxtimes \mathfrak{E}_B}^{AB}(\sigma'_{AB})) \Leftrightarrow (\sigma_{AB} = \sigma'_{AB}). \quad (89)$$

Endly, we will require the existence of homomorphisms $\theta_L^{\mathfrak{S}_A \mathfrak{S}_B}$ and $\theta_R^{\mathfrak{S}_A \mathfrak{S}_B}$ called partial traces

$$\theta_L^{\mathfrak{S}_A \mathfrak{S}_B} : \mathfrak{S}_A \boxtimes \mathfrak{S}_B \longrightarrow \mathfrak{S}_A \quad \theta_R^{\mathfrak{S}_A \mathfrak{S}_B} : \mathfrak{S}_A \boxtimes \mathfrak{S}_B \longrightarrow \mathfrak{S}_B \quad (90)$$

In [8] we proposed two complementary solutions of the previous axiomatic.

We will denote by $\check{\mathfrak{S}}_{AB}$ (or equivalently by $\mathfrak{S}_A \check{\otimes} \mathfrak{S}_B$) the set of maps Σ from $\mathfrak{E}_A \times \mathfrak{E}_B$ to $\mathfrak{E}_\perp \cong \mathfrak{B}$ satisfying

$$\forall \{\mathfrak{E}_{i,A} \mid i \in I\} \subseteq \mathfrak{E}_A, \forall \mathfrak{E}_B \in \mathfrak{E}_B, \quad \Sigma\left(\prod_{i \in I} \mathfrak{E}_{i,A}, \mathfrak{E}_B\right) = \bigwedge_{i \in I} \Sigma(\mathfrak{E}_{i,A}, \mathfrak{E}_B) \quad (91)$$

$$\forall \{\mathfrak{E}_{j,B} \mid j \in J\} \subseteq \mathfrak{E}_B, \forall \mathfrak{E}_A \in \mathfrak{E}_A, \quad \Sigma\left(\mathfrak{E}_A, \prod_{j \in J} \mathfrak{E}_{j,B}\right) = \bigwedge_{j \in J} \Sigma(\mathfrak{E}_A, \mathfrak{E}_{j,B}), \quad (92)$$

and

$$\forall \mathfrak{E}_A \in \mathfrak{E}_A, \quad \Sigma(\overline{\mathfrak{E}_A}, \mathfrak{E}_B) = \overline{\Sigma(\mathfrak{E}_A, \mathfrak{E}_B)}, \quad (93)$$

$$\forall \mathfrak{E}_B \in \mathfrak{E}_B, \quad \Sigma(\mathfrak{E}_A, \overline{\mathfrak{E}_B}) = \overline{\Sigma(\mathfrak{E}_A, \mathfrak{E}_B)}, \quad (94)$$

$$\Sigma(\mathfrak{E}_A, \mathfrak{E}_B) = \mathbf{Y}. \quad (95)$$

$\check{\mathfrak{S}}_{AB}$ is called *the maximal tensor product* of \mathfrak{S}_A and \mathfrak{S}_B .

Remark 5. Note that $\check{\mathfrak{S}}_{AB}$ depends crucially on the choice of the spaces of effects \mathfrak{E}_A and \mathfrak{E}_B within respectively $\mathfrak{E}_{\mathfrak{S}_A}$ and $\mathfrak{E}_{\mathfrak{S}_B}$, and not only on the structure of \mathfrak{S}_A and \mathfrak{S}_B .

The inclusion of pure tensors in \check{S}_{AB} is realized as follows :

$$\begin{aligned} \iota^{\check{S}_{AB}} : \mathfrak{S}_A \times \mathfrak{S}_B &\hookrightarrow \check{S}_{AB} \\ (\sigma_A, \sigma_B) &\mapsto \iota^{\check{S}_{AB}}(\sigma_A, \sigma_B) \mid \forall (l_A, l_B) \in \mathfrak{E}_A \times \mathfrak{E}_B, \\ &\iota^{\check{S}_{AB}}(\sigma_A, \sigma_B)(l_A, l_B) := \varepsilon_{l_A}^{\mathfrak{S}_A}(\sigma_A) \bullet \varepsilon_{l_B}^{\mathfrak{S}_B}(\sigma_B) \in \mathfrak{B}. \end{aligned} \quad (96)$$

\check{S}_{AB} is equipped with the pointwise partial order. It is a down-complete Inf semi-lattice with

$$\forall \{\Sigma_i \mid i \in I\} \subseteq \check{S}_{AB}, \forall (l_A, l_B) \in \mathfrak{E}_A \times \mathfrak{E}_B, \quad \left(\prod_{i \in I}^{\check{S}_{AB}} \Sigma_i\right)(l_A, l_B) := \bigwedge_{i \in I} \Sigma_i(l_A, l_B). \quad (97)$$

Let us introduce the following homomorphisms

$$\begin{aligned} \eta : \check{S}_{AB} &\longrightarrow \mathfrak{E}_A^* & \lambda : \check{S}_{AB} &\longrightarrow \mathfrak{E}_B^* \\ \Sigma &\mapsto \Sigma(\cdot, \mathfrak{Y}_{\mathfrak{E}_B}) & \Sigma &\mapsto \Sigma(\mathfrak{Y}_{\mathfrak{E}_A}, \cdot). \end{aligned} \quad (98)$$

As noted in Theorem 4, we have the following isomorphism

$$\begin{aligned} \rho_{\mathfrak{S}} : \mathfrak{S} &\longrightarrow \mathfrak{E}^* := \{ \psi \in \mathfrak{C}(\mathfrak{E}, \mathfrak{B}) \mid \forall l \in \mathfrak{E}, \psi(\bar{l}) = \overline{\psi(l)} \text{ and } \psi(\mathfrak{Y}_{\mathfrak{E}}) = \mathbf{Y} \} \\ \sigma &\mapsto \rho_{\mathfrak{S}}(\sigma) \mid \rho_{\mathfrak{S}}(\sigma)(l) := \varepsilon_l^{\mathfrak{S}}(\sigma), \forall l \in \mathfrak{E}. \end{aligned} \quad (99)$$

We can then define

$$\theta_{(1)}^{\mathfrak{S}_A \mathfrak{S}_B} := \rho_{\mathfrak{S}_A}^{-1} \circ \eta \quad \text{and} \quad \theta_{(2)}^{\mathfrak{S}_A \mathfrak{S}_B} := \rho_{\mathfrak{S}_B}^{-1} \circ \lambda. \quad (100)$$

We define \tilde{S}_{AB} to be the sub poset of \check{S}_{AB} generated by the elements $\iota^{\check{S}_{AB}}(\sigma_A, \sigma_B)$ for any $(\sigma_A, \sigma_B) \in \mathfrak{S}_A \times \mathfrak{S}_B$. \tilde{S}_{AB} will be equivalently denoted $\mathfrak{S}_A \tilde{\otimes} \mathfrak{S}_B$ and called *the minimal tensor product* of \mathfrak{S}_A and \mathfrak{S}_B .

In [8], it is proved that \tilde{S}_{AB} is in fact a down-complete Inf semi-lattice.

The partial traces are given by the following homomorphisms

$$\begin{aligned} \theta_{(1)}^{\mathfrak{S}_A \mathfrak{S}_B} : \tilde{S}_{AB} &\longrightarrow \mathfrak{S}_A & \theta_{(2)}^{\mathfrak{S}_A \mathfrak{S}_B} : \tilde{S}_{AB} &\longrightarrow \mathfrak{S}_B \\ \prod_{i \in I}^{\tilde{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B} &\mapsto \prod_{i \in I}^{\mathfrak{S}_A} \sigma_{i,A} & \prod_{i \in I}^{\tilde{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B} &\mapsto \prod_{i \in I}^{\mathfrak{S}_B} \sigma_{i,B} \end{aligned} \quad (101)$$

Moreover, in [8, Lemma 20], some insights are given on the structure of \tilde{S}_{AB} .

Let us consider $\prod_{i \in I}^{\tilde{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B}$ an element of \tilde{S}_{AB} . We have explicitly, for any $\sigma_A \in \mathfrak{S}_A$ and $\sigma_B \in \mathfrak{S}_B$, the following equivalence

$$\begin{aligned} \prod_{i \in I}^{\tilde{S}_{AB}} \sigma_{i,A} \tilde{\otimes} \sigma_{i,B} \sqsubseteq_{\tilde{S}_{AB}} \sigma_A \tilde{\otimes} \sigma_B &\Leftrightarrow \left(\left(\prod_{k \in I}^{\mathfrak{S}_A} \sigma_{k,A} \right) \sqsubseteq_{\mathfrak{S}_A} \sigma_A \text{ and } \left(\prod_{m \in I}^{\mathfrak{S}_B} \sigma_{m,B} \right) \sqsubseteq_{\mathfrak{S}_B} \sigma_B \text{ and} \right. \\ &\left. \left(\forall \emptyset \subsetneq K \subsetneq I, \left(\prod_{k \in K}^{\mathfrak{S}_A} \sigma_{k,A} \right) \sqsubseteq_{\mathfrak{S}_A} \sigma_A \text{ or } \left(\prod_{m \in I-K}^{\mathfrak{S}_B} \sigma_{m,B} \right) \sqsubseteq_{\mathfrak{S}_B} \sigma_B \right) \right). \end{aligned} \quad (102)$$

In [8] and [9], it has then been directly deduced from (102) that, if \mathfrak{S}_A and \mathfrak{S}_B admit a description in terms of pure states, then \tilde{S}_{AB} admit also a description in terms of pure states. More explicitly, we have

$$\forall \sigma \in \tilde{S}_{AB}, \quad \sigma = \prod_{\sigma_{\tilde{S}_{AB}}}^{\tilde{S}_{AB}} \sigma_{\tilde{S}_{AB}}, \quad \text{where } \sigma_{\tilde{S}_{AB}} = (\tilde{S}_{AB}^{\text{pure}} \cap (\uparrow^{\tilde{S}_{AB}} \sigma)), \quad (103)$$

$$\text{with } \tilde{S}_{AB}^{\text{pure}} = \{ \sigma_A \tilde{\otimes} \sigma_B \mid \sigma_A \in \mathfrak{S}_A^{\text{pure}}, \sigma_B \in \mathfrak{S}_B^{\text{pure}} \} = \text{Max}(\tilde{S}_{AB}) \quad (104)$$

In [9] it has been shown that a certain reduction of $\mathfrak{S}_A \check{\otimes} \mathfrak{S}_B$ can be defined in order to eliminate spurious states. It will play the fundamental role of tensor product in the following. We define *the regular tensor product* $\mathfrak{S}_A \hat{\otimes} \mathfrak{S}_B$ as follows

$$\mathfrak{S}_A \hat{\otimes} \mathfrak{S}_B \subseteq \mathfrak{S}_A \check{\otimes} \mathfrak{S}_B \quad (105)$$

$\forall \Phi \in \mathfrak{S}_A \hat{\otimes} \mathfrak{S}_B$ we require

$$\forall l_A \in \mathfrak{E}_A, \forall l_B \in \mathfrak{E}_B, \quad \Phi(l_A, \overline{\mathfrak{Y}_{\mathfrak{E}_B}}) = \Phi(\overline{\mathfrak{Y}_{\mathfrak{E}_A}}, l_B) = \mathbf{N}, \quad (106)$$

$$\begin{aligned} \forall l_A \in \mathfrak{E}_A, \quad \Phi(l_A, \mathfrak{Y}_{\mathfrak{E}_B}) = \mathbf{Y} &\Rightarrow \forall l_B \in \mathfrak{E}_B, \\ &(\Phi(l_A, l_B), \Phi(l_A, \overline{l_B})) \in \{(\mathbf{Y}, \mathbf{N}), (\mathbf{N}, \mathbf{Y}), (\perp, \perp)\}, \end{aligned} \quad (107)$$

$$\begin{aligned} \forall l_B \in \mathfrak{E}_B, \quad \Phi(\mathfrak{Y}_{\mathfrak{E}_A}, l_B) = \mathbf{Y} &\Rightarrow \forall l_A \in \mathfrak{E}_A, \\ &(\Phi(l_A, l_B), \Phi(\overline{l_A}, l_B)) \in \{(\mathbf{Y}, \mathbf{N}), (\mathbf{N}, \mathbf{Y}), (\perp, \perp)\}, \end{aligned} \quad (108)$$

$$\begin{aligned} \forall l_A \in \mathfrak{E}_A, \quad \Phi(l_A, \mathfrak{Y}_{\mathfrak{E}_B}) = \perp &\Rightarrow \forall l_B, l'_B \in \mathfrak{E}_B \mid l_B \sqcap_{\mathfrak{E}_B} l'_B = \perp_{\mathfrak{E}_B}, \\ &(\Phi(l_A, l_B), \Phi(l_A, l'_B)) \in \{(\perp, \mathbf{N}), (\mathbf{N}, \perp), (\perp, \perp)\}, \end{aligned} \quad (109)$$

$$\begin{aligned} \forall l_B \in \mathfrak{E}_B, \quad \Phi(\mathfrak{Y}_{\mathfrak{E}_A}, l_B) = \perp &\Rightarrow \forall l_A, l'_A \in \mathfrak{E}_A \mid l_A \sqcap_{\mathfrak{E}_A} l'_A = \perp_{\mathfrak{E}_A}, \\ &(\Phi(l_A, l_B), \Phi(l'_A, l_B)) \in \{(\perp, \mathbf{N}), (\mathbf{N}, \perp), (\perp, \perp)\}. \end{aligned} \quad (110)$$

$\mathfrak{S}_A \hat{\otimes} \mathfrak{S}_B$ is a sub Inf semi-lattice of the maximal tensor product $\mathfrak{S}_A \check{\otimes} \mathfrak{S}_B$ satisfying

$$\mathfrak{S}_A \check{\otimes} \mathfrak{S}_B \supseteq \mathfrak{S}_A \hat{\otimes} \mathfrak{S}_B. \quad (111)$$

We have the following fundamental result shown in [9] :

$$(\mathfrak{S}_A \text{ or } \mathfrak{S}_B \text{ simplex}) \Rightarrow \mathfrak{S}_A \hat{\otimes} \mathfrak{S}_B = \mathfrak{S}_A \check{\otimes} \mathfrak{S}_B = \mathfrak{S}_A \otimes \mathfrak{S}_B. \quad (112)$$

The following result has been shown in [9] and will be recalled in subsection 3.1.

$$(\mathfrak{S}_A \text{ and } \mathfrak{S}_B \text{ non-simplex orthocomplemented}) \Rightarrow \mathfrak{S}_A \check{\otimes} \mathfrak{S}_B \subsetneq \mathfrak{S}_A \hat{\otimes} \mathfrak{S}_B. \quad (113)$$

These two fundamental facts justify the adoption of the regular tensor product as our basic tensor product.

It is important to note that the laws $\check{\otimes}$, $\hat{\otimes}$ and \otimes are all associative.

2.8 Channels of the bipartite experiments

Let us consider a channel (f, f^*) from a States/Effects Chu space $(\mathfrak{S}_{A_1}, \mathfrak{E}_{A_1}, \mathfrak{E}^{\mathfrak{S}_{A_1}})$ to another States/Effects Chu space $(\mathfrak{S}_{A_2}, \mathfrak{E}_{A_2}, \mathfrak{E}^{\mathfrak{S}_{A_2}})$. Let us also consider a channel (g, g^*) from the Chu space $(\mathfrak{S}_{B_1}, \mathfrak{E}_{B_1}, \mathfrak{E}^{\mathfrak{S}_{B_1}})$ to the Chu space $(\mathfrak{S}_{B_2}, \mathfrak{E}_{B_2}, \mathfrak{E}^{\mathfrak{S}_{B_2}})$.

We define the channel $((f \hat{\otimes} g), (f \hat{\otimes} g)^*)$ from the States/Effects Chu space $(\widehat{\mathfrak{S}}_{A_1 B_1}, \widehat{\mathfrak{E}}_{A_1 B_1}, \mathfrak{E}^{\widehat{\mathfrak{S}}_{A_1 B_1}})$ to the States/Effects Chu space $(\widehat{\mathfrak{S}}_{A_2 B_2}, \widehat{\mathfrak{E}}_{A_2 B_2}, \mathfrak{E}^{\widehat{\mathfrak{S}}_{A_2 B_2}})$ by

$$((f \hat{\otimes} g)(\Sigma))(l_A, l_B) := \Sigma(f^*(l_A), g^*(l_B)) \quad (114)$$

We define the channel $((f \check{\otimes} g), (f \check{\otimes} g)^*)$ from the States/Effects Chu space $(\widetilde{\mathfrak{S}}_{A_1 B_1}, \widetilde{\mathfrak{E}}_{A_1 B_1}, \mathfrak{E}^{\widetilde{\mathfrak{S}}_{A_1 B_1}})$ to the States/Effects Chu space $(\widetilde{\mathfrak{S}}_{A_2 B_2}, \widetilde{\mathfrak{E}}_{A_2 B_2}, \mathfrak{E}^{\widetilde{\mathfrak{S}}_{A_2 B_2}})$ by

$$(f \check{\otimes} g)(\prod_{i \in I} \sigma_{i, A_1} \check{\otimes} \sigma_{i, B_1}) := \prod_{i \in I} f(\sigma_{i, A_1}) \check{\otimes} g(\sigma_{i, B_1}). \quad (115)$$

2.9 Incompatibility of measurements

Definition 7. N channels $\phi_i \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_{B_i})$ for $i = 1, \dots, N$ are said to be *jointly-compatible* iff there exists a *joint channel* $\Psi_{(\phi_i)_{i=1, \dots, N}} \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_{B_1} \boxtimes \dots \boxtimes \mathfrak{S}_{B_N})$ such that

$$\phi_i = \theta_{(i)}^{\mathfrak{S}_{B_1} \dots \mathfrak{S}_{B_N}} \circ \Psi_{(\phi_i)_{i=1, \dots, N}} \quad (116)$$

where $\theta_{(i)}^{\mathfrak{S}_{B_1} \dots \mathfrak{S}_{B_N}}$ is the partial trace projecting the tensor product $\mathfrak{S}_{B_1} \boxtimes \dots \boxtimes \mathfrak{S}_{B_N}$ onto its i -th component \mathfrak{S}_{B_i} .

It is generally easy to prove that two channels are compatible : it suffices to provide the corresponding joint channel. It is however not so easy to prove that two given channels are incompatible. We intent to propose a test to check the incompatibility of channels in a quite general setting.

Definition 8. Let be given two channels $\phi_1 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_B)$ and $\phi_2 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_C)$.

We will say that a triple $\sigma_{1,A}, \sigma_{2,A}, \sigma_{3,A} \in \mathfrak{S}_A$ satisfying

$$\sigma_{1,A} \sqsupseteq_{\mathfrak{S}_A} (\sigma_{2,A} \sqcap_{\mathfrak{S}_A} \sigma_{3,A}) \quad \text{and} \quad \neg \widehat{\sigma_{1,A} \sigma_{2,A}}^{\mathfrak{S}_A} \quad \text{and} \quad \neg \widehat{\sigma_{1,A} \sigma_{3,A}}^{\mathfrak{S}_A} \quad \text{and} \quad \sigma_{2,A} \parallel_{\mathfrak{S}_A} \sigma_{3,A} \quad (117)$$

certifies the incompatibility of ϕ_1 and ϕ_2 iff there does not exist any triple $\rho_{1,BC}, \rho_{2,BC}, \rho_{3,BC} \in \mathfrak{S}_B \widetilde{\otimes} \mathfrak{S}_C$ satisfying

$$\rho_{1,BC} \sqsupseteq_{\mathfrak{S}_{BC}} (\rho_{2,BC} \sqcap_{\mathfrak{S}_{BC}} \rho_{3,BC}) \quad (118)$$

and

$$\forall i = 1, 2, 3, \quad \theta_{(1)}^{\mathfrak{S}_B \mathfrak{S}_C}(\rho_{i,BC}) = \phi_1(\sigma_{i,A}) \quad \text{and} \quad \theta_{(2)}^{\mathfrak{S}_B \mathfrak{S}_C}(\rho_{i,BC}) = \phi_2(\sigma_{i,A}). \quad (119)$$

Theorem 6. Let be given two channels $\phi_1 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_B)$ and $\phi_2 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_C)$.

If there exists a triple $\sigma_{1,A}, \sigma_{2,A}, \sigma_{3,A} \in \mathfrak{S}$ that certifies incompatibility of ϕ_1 and ϕ_2 , then ϕ_1 and ϕ_2 are incompatible. \blacksquare

Proof. Let us suppose that ϕ_1 and ϕ_2 are compatible and let us introduce the joint channel $\Psi_{(\phi_1, \phi_2)}$. We note that the triple $\rho_{1,BC}, \rho_{2,BC}, \rho_{3,BC} \in \mathfrak{S}_B \widetilde{\otimes} \mathfrak{S}_C$ defined by

$$\rho_{i,BC} := \Psi_{(\phi_1, \phi_2)}(\sigma_{i,A}) \quad (120)$$

satisfies (118) because $\Psi_{(\phi_1, \phi_2)}$ satisfies the defining properties of a channel, and satisfies (119) because $\Psi_{(\phi_1, \phi_2)}$ is the joint channel ensuring the compatibility of ϕ_1 and ϕ_2 . We have then obtained a contradiction. ϕ_1 and ϕ_2 are then incompatible. \square

We momentarily assume that \mathfrak{S} can be described in terms of pure states.

Theorem 7. Let \mathfrak{S} be a simplex. Let us consider any two measurement channels $m_{(\sigma_1, \sigma'_1)}$ and $m_{(\sigma_2, \sigma'_2)}$. These channels are necessarily compatible. \blacksquare

Proof. We define the map Ψ as a map from \mathfrak{S} to $\mathfrak{B} \tilde{\otimes} \mathfrak{B}$ which satisfies, for any $\sigma \in \mathfrak{S}$,

$$\Psi(\sigma) := \prod_{\sigma' \in \underline{\mathfrak{S}}}^{\mathfrak{B} \tilde{\otimes} \mathfrak{B}} \Psi(\sigma') \quad (121)$$

and for any $\sigma \in \mathfrak{S}^{pure} = Max(\mathfrak{S})$

$$\left\{ \begin{array}{ll} \Psi(\sigma) = \mathbf{Y} \tilde{\otimes} \mathbf{Y} & \text{if } \varepsilon_{l(\sigma_1, \sigma'_1)}^{\mathfrak{S}}(\sigma) = \mathbf{Y} \text{ and } \varepsilon_{l(\sigma_2, \sigma'_2)}^{\mathfrak{S}}(\sigma) = \mathbf{Y}, \\ \Psi(\sigma) = \mathbf{Y} \tilde{\otimes} \mathbf{Y} \prod_{\mathfrak{B} \tilde{\otimes} \mathfrak{B}} \mathbf{Y} \tilde{\otimes} \mathbf{N} & \text{if } \varepsilon_{l(\sigma_1, \sigma'_1)}^{\mathfrak{S}}(\sigma) = \mathbf{Y} \text{ and } \varepsilon_{l(\sigma_2, \sigma'_2)}^{\mathfrak{S}}(\sigma) = \perp, \\ \Psi(\sigma) = \mathbf{Y} \tilde{\otimes} \mathbf{Y} \prod_{\mathfrak{B} \tilde{\otimes} \mathfrak{B}} \mathbf{N} \tilde{\otimes} \mathbf{Y} & \text{if } \varepsilon_{l(\sigma_1, \sigma'_1)}^{\mathfrak{S}}(\sigma) = \perp \text{ and } \varepsilon_{l(\sigma_2, \sigma'_2)}^{\mathfrak{S}}(\sigma) = \mathbf{Y}, \\ \Psi(\sigma) = \mathbf{N} \tilde{\otimes} \mathbf{N} & \text{if } \varepsilon_{l(\sigma_1, \sigma'_1)}^{\mathfrak{S}}(\sigma) = \mathbf{N} \text{ and } \varepsilon_{l(\sigma_2, \sigma'_2)}^{\mathfrak{S}}(\sigma) = \mathbf{N}, \\ \Psi(\sigma) = \mathbf{N} \tilde{\otimes} \mathbf{N} \prod_{\mathfrak{B} \tilde{\otimes} \mathfrak{B}} \mathbf{N} \tilde{\otimes} \mathbf{Y} & \text{if } \varepsilon_{l(\sigma_1, \sigma'_1)}^{\mathfrak{S}}(\sigma) = \mathbf{N} \text{ and } \varepsilon_{l(\sigma_2, \sigma'_2)}^{\mathfrak{S}}(\sigma) = \perp, \\ \Psi(\sigma) = \mathbf{N} \tilde{\otimes} \mathbf{N} \prod_{\mathfrak{B} \tilde{\otimes} \mathfrak{B}} \mathbf{Y} \tilde{\otimes} \mathbf{N} & \text{if } \varepsilon_{l(\sigma_1, \sigma'_1)}^{\mathfrak{S}}(\sigma) = \perp \text{ and } \varepsilon_{l(\sigma_2, \sigma'_2)}^{\mathfrak{S}}(\sigma) = \mathbf{N}, \\ \Psi(\sigma) = \mathbf{Y} \tilde{\otimes} \mathbf{N} & \text{if } \varepsilon_{l(\sigma_1, \sigma'_1)}^{\mathfrak{S}}(\sigma) = \mathbf{Y} \text{ and } \varepsilon_{l(\sigma_2, \sigma'_2)}^{\mathfrak{S}}(\sigma) = \mathbf{N}, \\ \Psi(\sigma) = \mathbf{N} \tilde{\otimes} \mathbf{Y} & \text{if } \varepsilon_{l(\sigma_1, \sigma'_1)}^{\mathfrak{S}}(\sigma) = \mathbf{N} \text{ and } \varepsilon_{l(\sigma_2, \sigma'_2)}^{\mathfrak{S}}(\sigma) = \mathbf{Y}, \\ \Psi(\sigma) = \perp \tilde{\otimes} \perp & \text{if } \varepsilon_{l(\sigma_1, \sigma'_1)}^{\mathfrak{S}}(\sigma) = \perp \text{ and } \varepsilon_{l(\sigma_2, \sigma'_2)}^{\mathfrak{S}}(\sigma) = \perp. \end{array} \right. \quad (122)$$

We observe that $\Psi_{(m_{l(\sigma_1, \sigma'_1)}, m_{l(\sigma_2, \sigma'_2)})}$ is unambiguously defined by (121) because \mathfrak{S} is a simplex.

Secondly, $\Psi_{(m_{l(\sigma_1, \sigma'_1)}, m_{l(\sigma_2, \sigma'_2)})}$ is a channel by construction.

Endly, we can check easily for any $\sigma \in \mathfrak{S}$ (the proof begins on each expressions of (122) by a simple computation and is extended to the whole set of elements of \mathfrak{S} using the expression (121) and the channel properties of $m_{l(\sigma_1, \sigma'_1)}$ and $m_{l(\sigma_2, \sigma'_2)}$)

$$(\theta_{(1)}^{\mathfrak{B} \tilde{\otimes} \mathfrak{B}} \circ \Psi_{(m_{l(\sigma_1, \sigma'_1)}, m_{l(\sigma_2, \sigma'_2)})})(\sigma) = m_{l(\sigma_1, \sigma'_1)}(\sigma) \quad (123)$$

$$(\theta_{(2)}^{\mathfrak{B} \tilde{\otimes} \mathfrak{B}} \circ \Psi_{(m_{l(\sigma_1, \sigma'_1)}, m_{l(\sigma_2, \sigma'_2)})})(\sigma) = m_{l(\sigma_2, \sigma'_2)}(\sigma). \quad (124)$$

Ψ is then the joint channel $\Psi_{(m_{l(\sigma_1, \sigma'_1)}, m_{l(\sigma_2, \sigma'_2)})}$. □

More generally, we can show the following result.

Theorem 8. If \mathfrak{S} is a simplex then the whole set of elements of $\{m_l \mid l \in \mathfrak{E}_{\mathfrak{S}}\}$ are jointly-compatible measurement channels. ■

Proof. Straightforward using the generalization of the construction used in previous Theorem for the joint-channel. □

3 Entanglement, steering and Bell non-locality

Let us begin with our basic definition. In this definition, $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$ and $(\mathfrak{S}_B, \mathfrak{E}_B, \varepsilon^{\mathfrak{S}_B})$ are generic states-effects Chu spaces. The tensor products $\mathfrak{S}_A \hat{\otimes} \mathfrak{S}_B$ and $\mathfrak{S}_A \tilde{\otimes} \mathfrak{S}_B$ are defined according to subsection 2.7.

Definition 9. An element of $\mathfrak{S}_A \widehat{\otimes} \mathfrak{S}_B$ which is not an element of $\mathfrak{S}_A \widetilde{\otimes} \mathfrak{S}_B$ is called an *entangled state*. The elements of $\mathfrak{S}_A \widetilde{\otimes} \mathfrak{S}_B$ are said to be *separable states*.

It is recalled in [17] that the very definition of entangled states in the specific case of quantum theory corresponds to this definition.

3.1 Existence of entangled states

Our aim is now to construct generically some entangled states.

Theorem 9. $(\mathfrak{S}_A, \mathfrak{E}_A, \mathfrak{E}^{\mathfrak{S}_A})$ and $(\mathfrak{S}_B, \mathfrak{E}_B, \mathfrak{E}^{\mathfrak{S}_B})$ are states-effects Chu spaces where \mathfrak{S}_A and \mathfrak{S}_B are chosen to be non-simplex orthocomplemented spaces of states and where \mathfrak{E}_A and \mathfrak{E}_B are chosen respectively to be the reduced effects spaces $\overline{\mathfrak{E}}_{\mathfrak{S}_A}$ and $\overline{\mathfrak{E}}_{\mathfrak{S}_B}$ (see subsection 2.5). The tensor product $\mathfrak{S}_A \widehat{\otimes} \mathfrak{S}_B$ and $\mathfrak{S}_A \widetilde{\otimes} \mathfrak{S}_B$ are defined according to subsection 2.7 with respect to these choices of reduced effects spaces.

As a result, entangled states do exist. In other words,

$$\mathfrak{S}_A \widetilde{\otimes} \mathfrak{S}_B \subsetneq \mathfrak{S}_A \widehat{\otimes} \mathfrak{S}_B. \quad (125)$$

■

Proof. \mathfrak{S}_A being a non-simplex orthocomplemented space of states, there exists a pair of states α_1, α_2 in \mathfrak{S}_A such that $\alpha_1^* \not\sqsubseteq_{\mathfrak{S}_A} \alpha_2$ and $\alpha_1 \not\sqsubseteq_{\mathfrak{S}_A} \alpha_2$. Indeed, if it was wrong that such a pair exist, we could, for any α , define a channel $\psi_\alpha := m_{(\alpha, \alpha^*)}$ such that $\psi_\alpha(\alpha) = \mathbf{Y}$ and $\psi_\alpha(\mathfrak{S}_A^{pure}) \subseteq \{\mathbf{Y}, \mathbf{N}\}$, which would impose \mathfrak{S}_A to be a simplex (see Lemmas 1 and ??). Let us fix such a pair (α_1, α_2) . Let us consider the subset I_{α_1} of $\underline{\alpha_1}_{\mathfrak{S}_A}$ defined by $I_{\alpha_1} := \{\alpha \in \underline{\alpha_1}_{\mathfrak{S}_A} \mid \alpha^* \sqsubseteq_{\mathfrak{S}_A} \alpha_2\}$.

We have necessarily $I_{\alpha_1} \subsetneq \underline{\alpha_1}_{\mathfrak{S}_A}$, because $I_{\alpha_1} = \underline{\alpha_1}_{\mathfrak{S}_A}$ would imply $\alpha_1^* = (\prod^{\mathfrak{S}_A} I_{\alpha_1})^* \sqsubseteq_{\mathfrak{S}_A} \alpha_2$ which is false by assumption. Then, we can always choose $\sigma_1 \in (\underline{\alpha_1}_{\mathfrak{S}_A} \setminus I_{\alpha_1}) \subseteq \mathfrak{S}_A^{pure}$ and $\sigma_2 \in \underline{\alpha_2}_{\mathfrak{S}_A} \subseteq \mathfrak{S}_A^{pure}$ such that $\sigma_1^* \not\sqsubseteq_{\mathfrak{S}_A} \sigma_2$ and $\sigma_1 \neq \sigma_2$.

Analogously, we can choose a pair of states τ_1, τ_2 in \mathfrak{S}_B^{pure} such that $\tau_1^* \not\sqsubseteq_{\mathfrak{S}_B} \tau_2$ and $\tau_1 \neq \tau_2$.

It is now rather easy to check that the following supremum exists in $\check{\mathfrak{S}}_{AB}$

$$\Sigma_{AB} := (\sigma_1 \widetilde{\otimes} \tau_1 \sqcap_{\check{\mathfrak{S}}_{AB}} \sigma_2 \widetilde{\otimes} \tau_2) \sqcup_{\check{\mathfrak{S}}_{AB}} (\sigma_1^* \widetilde{\otimes} \perp_{\mathfrak{S}_B} \sqcap_{\check{\mathfrak{S}}_{AB}} \perp_{\mathfrak{S}_A} \widetilde{\otimes} \tau_1^*) \quad (126)$$

the explicit expression is indeed simply given by a decomposition in terms of pure effects (see Theorem 2)

$$\Sigma_{AB}(l_A, l_B) := \bigwedge_{l'_A \in \underline{l_A}_{\mathfrak{E}_A}} \bigwedge_{l'_B \in \underline{l_B}_{\mathfrak{E}_B}} \Sigma_{AB}(l'_A, l'_B) \quad (127)$$

with

$$\left\{ \begin{array}{l} \Sigma_{AB}(\mathfrak{Y}_{\mathfrak{E}_A}, \mathfrak{Y}_{\mathfrak{E}_B}) = \mathbf{Y} \\ \Sigma_{AB}(\mathfrak{Y}_{\mathfrak{E}_A}, l_B) = \Sigma_{AB}(l_A, \overline{\mathfrak{Y}_{\mathfrak{E}_B}}) = \mathbf{N}, \quad \forall l_A \in \mathfrak{E}_A^{pure}, \forall l_B \in \mathfrak{E}_B^{pure}, \\ \Sigma_{AB}(\mathfrak{Y}_{\mathfrak{E}_A}, l_B) = \Sigma_{AB}(l_A, \mathfrak{Y}_{\mathfrak{E}_B}) = \perp, \quad \forall l_A \in \mathfrak{E}_A^{pure}, \forall l_B \in \mathfrak{E}_B^{pure}, \\ \Sigma_{AB}(l(\sigma_1, \sigma_1^*), l(\tau_1, \tau_1^*)) = \mathbf{N} \\ \Sigma_{AB}(l(\sigma_1^*, \sigma_1), l(\tau_2^*, \tau_2)) = \mathbf{N} \\ \Sigma_{AB}(l(\sigma_2^*, \sigma_2), l(\tau_1^*, \tau_1)) = \mathbf{N} \\ \Sigma_{AB}(l_A, l_B) = \perp \end{array} \right. \quad (128)$$

for any other pair $(l_A, l_B) \in \mathfrak{E}_A^{pure} \times \mathfrak{E}_B^{pure}$.

Using (102), $\sigma_1^* \not\sqsubseteq_{\mathfrak{S}_A} \sigma_1$, $\sigma_1^* \not\sqsubseteq_{\mathfrak{S}_A} \sigma_2$, $\sigma_1 \neq \sigma_2$, $\tau_1^* \not\sqsubseteq_{\mathfrak{S}_B} \tau_1$, $\tau_1^* \not\sqsubseteq_{\mathfrak{S}_B} \tau_2$ and $\tau_1 \neq \tau_2$, we deduce

$$(\sigma_1^* \tilde{\perp}_{\mathfrak{S}_B} \sqcap_{\tilde{\mathfrak{S}}_{AB}} \perp_{\mathfrak{S}_A} \tilde{\otimes} \tau_1^*) \not\sqsubseteq_{\tilde{\mathfrak{S}}_{AB}} \sigma_1 \tilde{\otimes} \tau_1, \quad (129)$$

$$(\sigma_1^* \tilde{\perp}_{\mathfrak{S}_B} \sqcap_{\tilde{\mathfrak{S}}_{AB}} \perp_{\mathfrak{S}_A} \tilde{\otimes} \tau_1^*) \not\sqsubseteq_{\tilde{\mathfrak{S}}_{AB}} \sigma_2 \tilde{\otimes} \tau_2, \quad (130)$$

$$\frac{(\sigma_1 \tilde{\otimes} \tau_1 \sqcap_{\tilde{\mathfrak{S}}_{AB}} \sigma_2 \tilde{\otimes} \tau_2)}{\tilde{\mathfrak{S}}_{AB}} = \{ \sigma_1 \tilde{\otimes} \tau_1, \sigma_2 \tilde{\otimes} \tau_2 \}. \quad (131)$$

As a consequence, we conclude that Σ_{AB} is an entangled state :

$$\Sigma_{AB} \in \hat{\mathfrak{S}}_{AB} \setminus \tilde{\mathfrak{S}}_{AB}. \quad (132)$$

□

3.2 Steering and entanglement-assisted incompatibility tests

We now intent to exploit entangled states to build entanglement-assisted incompatibility tests and to introduce the notion of steering in Generalized possibilistic Theories. Several results of this subsection are close from that presented in [16][17, subsection 7.3]. To clarify the notion of steering in quantum theory, we suggest to read the report [19].

Except if explicitly mentioned, $(\mathfrak{S}_A, \mathfrak{E}_A, \varepsilon^{\mathfrak{S}_A})$ and $(\mathfrak{S}_D, \mathfrak{E}_D, \varepsilon^{\mathfrak{S}_D})$ are generic states-effects Chu spaces. The tensor products $\mathfrak{S}_A \hat{\otimes} \mathfrak{S}_D$ and $\mathfrak{S}_A \tilde{\otimes} \mathfrak{S}_D$ are defined according to subsection 2.7.

Lemma 4. Let Σ_{AD} be an element of $\mathfrak{S}_A \hat{\otimes} \mathfrak{S}_D$, and let $\phi_1 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_B)$ and $\phi_2 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_C)$ be compatible channels. There is an element $\Lambda_{BCD} \in \mathfrak{S}_B \hat{\otimes} \mathfrak{S}_C \hat{\otimes} \mathfrak{S}_D$ such that

$$(\theta_{(1)}^{\mathfrak{S}_B \mathfrak{S}_C} \hat{\otimes} id)(\Lambda_{BCD}) = (\phi_1 \hat{\otimes} id)(\Sigma_{AD}) \quad (133)$$

$$(\theta_{(2)}^{\mathfrak{S}_B \mathfrak{S}_C} \hat{\otimes} id)(\Lambda_{BCD}) = (\phi_2 \hat{\otimes} id)(\Sigma_{AD}) \quad (134)$$

■

Proof. It suffices to define

$$\Lambda_{BCD} := (\Psi_{(\phi_1, \phi_2)} \hat{\otimes} id)(\Sigma_{AD}), \quad (135)$$

where $\Psi_{(\phi_1, \phi_2)} \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_B \hat{\otimes} \mathfrak{S}_C)$ is the joint channel associated to ϕ_1 and ϕ_2 . □

The previous result suggests a new way to certify the incompatibility of the channels ϕ_1 and ϕ_2 . We must note however that, due to the following result, Σ_{AD} must be an entangled state. For this reason, this new type of test will be called *entanglement-assisted incompatibility test*.

Lemma 5. Let Σ_{AD} be in fact an element of $\mathfrak{S}_A \tilde{\otimes} \mathfrak{S}_D$, i.e. Σ_{AD} is a separable state. Then for any $\phi_1 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_B)$ and $\phi_2 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_C)$ there exists $\Lambda_{BCD} \in \mathfrak{S}_B \tilde{\otimes} \mathfrak{S}_C \tilde{\otimes} \mathfrak{S}_D$ such that (133) and (134). ■

Proof. If $\Sigma_{AD} = \prod_{i \in I}^{\mathfrak{S}_A \tilde{\otimes} \mathfrak{S}_D} \sigma_{i,A} \tilde{\otimes} \sigma_{i,D}$, it suffices to define

$$\Lambda_{BCD} := \prod_{i \in I}^{\mathfrak{S}_B \tilde{\otimes} \mathfrak{S}_C \tilde{\otimes} \mathfrak{S}_D} \phi_1(\sigma_{i,A}) \tilde{\otimes} \phi_2(\sigma_{i,A}) \tilde{\otimes} \sigma_{i,D} \quad (136)$$

□

It is now natural to introduce the notion of steering, which has a very old history in quantum theory [19].

Definition 10. For $\phi_1 \in \mathcal{C}(\mathfrak{S}_A, \mathfrak{S}_B)$ and $\phi_2 \in \mathcal{C}(\mathfrak{S}_A, \mathfrak{S}_C)$, we will say that (ϕ_1, ϕ_2) *steer* the state Σ_{AD} iff there is no $\Lambda_{BCD} \in \mathfrak{S}_B \widehat{\otimes} \mathfrak{S}_C \widehat{\otimes} \mathfrak{S}_D$ such that (133) and (134).

The entanglement-assisted incompatibility test is now firmly established.

Theorem 10. As long as Σ_{AD} is an entangled state, if (ϕ_1, ϕ_2) *steer* the state Σ_{AD} , then ϕ_1 and ϕ_2 are incompatible channels. ■

Definition 11. We will say that the state Σ_{AD} is *steerable by channels* iff there exist some channels $\phi_1 \in \mathcal{C}(\mathfrak{S}_A, \mathfrak{S}_B)$ and $\phi_2 \in \mathcal{C}(\mathfrak{S}_A, \mathfrak{S}_C)$ steering the state Σ_{AD} .

Definition 12. We will say that the state Σ_{AD} is *steerable by measurements* iff there exist some measurements $\phi_1 \in \mathcal{C}(\mathfrak{S}_A, \mathfrak{B})$ and $\phi_2 \in \mathcal{C}(\mathfrak{S}_A, \mathfrak{B})$ steering the state Σ_{AD} . Obviously, if Σ_{AD} is steerable by measurements, then it is steerable by channels.

From Lemma 5, we can obviously view the steering by a pair of incompatible channels as a test for a state to be entangled :

Theorem 11. If a state Σ_{AD} is steerable by channels, then Σ_{AD} is necessarily an entangled state. ■

We have already shown that entangled states do exist. However, the fact that a state is steerable by channels is a more restrictive notion. A natural question is then to prove that GpT can admit steering by channels.

From now, \mathfrak{S}_A and \mathfrak{S}_D are chosen to be non-simplex orthocomplemented spaces of states, \mathfrak{E}_A and \mathfrak{E}_D are chosen respectively to be the reduced effects spaces $\overline{\mathfrak{E}}_{\mathfrak{S}_A}$ and $\overline{\mathfrak{E}}_{\mathfrak{S}_D}$ (see subsection 2.5). The tensor products $\mathfrak{S}_A \widehat{\otimes} \mathfrak{S}_D$ and $\mathfrak{S}_A \widetilde{\otimes} \mathfrak{S}_D$ are defined according to subsection 2.7 with these choices.

Theorem 12. States steerable by measurements (and then also states steerable by channels) do exist in $\mathfrak{S}_A \widehat{\otimes} \mathfrak{S}_D$. ■

Proof. Here again, we will choose σ_1 and σ_2 two pure states in \mathfrak{S}_A such that $\sigma_1 \neq \sigma_2$ and $\sigma_1^* \not\perp_{\mathfrak{S}_A} \sigma_2$, and τ_1 and τ_2 two pure states in \mathfrak{S}_D such that $\tau_1 \neq \tau_2$ and $\tau_1^* \not\perp_{\mathfrak{S}_A} \tau_2$. and we will choose the regular entangled state Σ to be defined formally as in (126), i.e. explicitly given by (127)(128).

For simplicity we will denote $\mathfrak{w}_A := \iota_{(\sigma_1, \sigma_1^*)}$, $\mathfrak{v}_A := \iota_{(\sigma_2, \sigma_2^*)}$ and $\mathfrak{w}_D := \iota_{(\tau_1, \tau_1^*)}$, $\mathfrak{v}_D := \iota_{(\tau_2, \tau_2^*)}$. In the following, we will also use the already adopted notations $\mathfrak{u} := \iota_{(\mathbf{Y}, \mathbf{N})} \in \mathfrak{E}_{\mathfrak{B}}$.

Obviously, we have chosen \mathfrak{S}_A and \mathfrak{S}_D to be non-simplex spaces of states, in order for pairs of incompatible measurements to exist. Now we choose $\phi_1 := \mathfrak{m}_{\mathfrak{w}_A}$ and $\phi_2 := \mathfrak{m}_{\mathfrak{v}_A}$ (we recall the expression of \mathfrak{m}_1^* in (84)).

We now intent to prove that the entangled state Σ_{AD} (defined formally as in (126), i.e. explicitly given by (127)(128)) is steered by the incompatible measurements ϕ_1 and ϕ_2 .

Let us then consider the bimorphic map defined from $\mathfrak{E}_{\mathfrak{B}} \times \mathfrak{E}_D$ to \mathfrak{B} by $\Phi_1(l, l') := \Sigma_{AD}(m_{\mathfrak{w}_A}^*(l), l')$. We have

$$\begin{aligned} \Phi_1(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}, \mathfrak{Y}_{\mathfrak{E}_D}) &= \mathbf{Y}, & \Phi_1(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}}, l_D) &= \mathbf{N}, \quad \forall l_D \in \mathfrak{E}_D^{pure} & \Phi_1(u, \overline{\mathfrak{Y}_{\mathfrak{E}_B}}) &= \Phi_1(\bar{u}, \overline{\mathfrak{Y}_{\mathfrak{E}_B}}) = \mathbf{N}, \\ \Phi_1(u, \mathfrak{w}_D) &= \mathbf{N}, & \Phi_1(\bar{u}, \overline{\mathfrak{v}_D}) &= \mathbf{N}, & \Phi_1(l, l_D) &= \perp, \quad \text{for any other pair } (l, l_D) \in \mathfrak{E}_{\mathfrak{B}}^{pure} \times \mathfrak{E}_D^{pure}. \end{aligned} \quad (137)$$

Let us also consider the bimorphic map defined from $\mathfrak{E}_{\mathfrak{B}} \times \mathfrak{E}_D$ to \mathfrak{B} by $\Phi_2(l, l') := \Sigma_{AD}(m_{\mathfrak{v}_D}^*(l), l')$. We have

$$\begin{aligned} \Phi_2(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}, \mathfrak{Y}_{\mathfrak{E}_D}) &= \mathbf{Y}, & \Phi_2(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}}, l_D) &= \mathbf{N}, \quad \forall l_D \in \mathfrak{E}_D^{pure} & \Phi_2(u, \overline{\mathfrak{Y}_{\mathfrak{E}_B}}) &= \Phi_2(\bar{u}, \overline{\mathfrak{Y}_{\mathfrak{E}_B}}) = \mathbf{N}, \\ \Phi_2(u, \mathfrak{w}_D) &= \mathbf{N}, & \Phi_2(\bar{u}, \overline{\mathfrak{v}_D}) &= \mathbf{N}, & \Phi_2(l, l_D) &= \perp, \quad \text{for any other pair } (l, l_D) \in \mathfrak{E}_{\mathfrak{B}}^{pure} \times \mathfrak{E}_D^{pure}. \end{aligned} \quad (138)$$

Let us now assume that an element $\Lambda \in \mathfrak{B} \hat{\otimes} \mathfrak{B} \hat{\otimes} \mathfrak{S}_D$ exists such that (133) and (134) are satisfied, and let us obtain a contradiction. The property $\Phi_1(u, \mathfrak{w}_D) = \mathbf{N}$ implies $\Lambda(u, \bar{u}, \mathfrak{w}_D) = \mathbf{N}$. The property $\Phi_2(\bar{u}, \overline{\mathfrak{v}_D}) = \mathbf{N}$ implies $\Lambda(u, \bar{u}, \overline{\mathfrak{v}_D}) = \mathbf{N}$. From these two relations, using the homomorphic property satisfied by Λ , we obtain $\Lambda(u, \bar{u}, \perp_{\mathfrak{E}_D}) = \mathbf{N}$. As a consequence, we have necessarily $\Lambda(u, \bar{u}, \mathfrak{v}_D) = \mathbf{N}$ and $\Lambda(u, \bar{u}, \overline{\mathfrak{v}_D}) = \mathbf{N}$. On another part, $\Phi_1(\bar{u}, \overline{\mathfrak{v}_D}) = \mathbf{N}$ implies $\Lambda(\bar{u}, \bar{u}, \overline{\mathfrak{v}_D}) = \mathbf{N}$. We now collect the relations $\Lambda(u, \bar{u}, \overline{\mathfrak{v}_D}) = \mathbf{N}$ and $\Lambda(\bar{u}, \bar{u}, \overline{\mathfrak{v}_D}) = \mathbf{N}$ to deduce $\Lambda(\perp_{\mathfrak{E}_{\mathfrak{B}}}, \bar{u}, \overline{\mathfrak{v}_D}) = \mathbf{N}$, and then $\Lambda(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}, \bar{u}, \overline{\mathfrak{v}_D}) = \mathbf{N}$, i.e. $\Phi_2(\bar{u}, \overline{\mathfrak{v}_D}) = \mathbf{N}$. This result contradicts the fact that $\Phi_2(\bar{u}, \overline{\mathfrak{v}_D}) = \perp$ by definition. We have then obtained the announced contradiction and Λ cannot exist as an element of $\mathfrak{B} \hat{\otimes} \mathfrak{B} \hat{\otimes} \mathfrak{S}_D$. As a conclusion, Σ is steered by the incompatible measurements ϕ_1 and ϕ_2 . \square

Remark 6. As a consequence of previous result and of Theorem 11, we have implicitly confirmed the fact that the state Σ_{AD} is an entangled state.

Remark 7. Moreover, as a consequence of previous result and of Theorem 10, we have also implicitly proved that there always exist some pairs of incompatible measurements in any non-simplex orthocomplemented space of states. Such a pair of incompatible measurements is given by : $\phi_1 := m_{l_{(\sigma_1, \sigma_1^*)}}$ and $\phi_2 := m_{l_{(\sigma_2, \sigma_2^*)}}$ with $\sigma_1, \sigma_2 \in \mathfrak{S}_A^{pure}$ chosen such that $\sigma_1 \neq \sigma_2$ and $\sigma_1^* \not\sqsubseteq_{\mathfrak{S}_A} \sigma_2$ ^[4].

3.3 Bell non-locality

We now intent to clarify some results about Bell non-locality (see [6] for a complete report on this standard notion) in the context of Generalized possibilistic Theories. Several results of this subsection are close from that presented in [16][17, subsection 7.3].

Except if explicitly mentioned, $(\mathfrak{S}_A, \mathfrak{E}_A, \mathfrak{E}^{\mathfrak{S}_A})$ and $(\mathfrak{S}_D, \mathfrak{E}_D, \mathfrak{E}^{\mathfrak{S}_D})$ are generic states-effects Chu spaces. The tensor products $\mathfrak{S}_A \hat{\otimes} \mathfrak{S}_D$ and $\mathfrak{S}_A \tilde{\otimes} \mathfrak{S}_D$ are defined according to subsection 2.7 with these choices.

Lemma 6. Let Σ_{AD} be an element of $\mathfrak{S}_A \hat{\otimes} \mathfrak{S}_D$. Let $\phi_1 \in \mathcal{C}(\mathfrak{S}_A, \mathfrak{S}_B)$ and $\phi_2 \in \mathcal{C}(\mathfrak{S}_A, \mathfrak{S}_C)$ be a pair of compatible channels, and $\rho_1 \in \mathcal{C}(\mathfrak{S}_D, \mathfrak{S}_E)$ and $\rho_2 \in \mathcal{C}(\mathfrak{S}_D, \mathfrak{S}_F)$ be another

⁴such a pair of pure states always exists in a non-simplex orthocomplemented space of states, as shown at the beginning of the proof of Theorem 9

pair of compatible channels. There is an element $\Lambda_{BCEF} \in \mathfrak{S}_B \widehat{\otimes} \mathfrak{S}_C \widehat{\otimes} \mathfrak{S}_E \widehat{\otimes} \mathfrak{S}_F$ such that

$$\theta_{(1)(3)}^{\mathfrak{S}_B \mathfrak{S}_C \mathfrak{S}_E \mathfrak{S}_F}(\Lambda_{BCEF}) = (\phi_1 \widehat{\otimes} \rho_1)(\Sigma_{AD}) \quad (139)$$

$$\theta_{(1)(4)}^{\mathfrak{S}_B \mathfrak{S}_C \mathfrak{S}_E \mathfrak{S}_F}(\Lambda_{BCEF}) = (\phi_1 \widehat{\otimes} \rho_2)(\Sigma_{AD}) \quad (140)$$

$$\theta_{(2)(3)}^{\mathfrak{S}_B \mathfrak{S}_C \mathfrak{S}_E \mathfrak{S}_F}(\Lambda_{BCEF}) = (\phi_2 \widehat{\otimes} \rho_1)(\Sigma_{AD}) \quad (141)$$

$$\theta_{(2)(4)}^{\mathfrak{S}_B \mathfrak{S}_C \mathfrak{S}_E \mathfrak{S}_F}(\Lambda_{BCEF}) = (\phi_2 \widehat{\otimes} \rho_2)(\Sigma_{AD}) \quad (142)$$

■

Proof. It suffices to define

$$\Lambda_{BCEF} := (\Psi_{(\phi_1, \phi_2)} \widehat{\otimes} \Psi_{(\rho_1, \rho_2)})(\Sigma_{AD}), \quad (143)$$

where $\Psi_{(\phi_1, \phi_2)} \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_B \widehat{\otimes} \mathfrak{S}_C)$ is the joint channel associated to ϕ_1 and ϕ_2 and $\Psi_{(\rho_1, \rho_2)} \in \mathfrak{C}(\mathfrak{S}_D, \mathfrak{S}_E \widehat{\otimes} \mathfrak{S}_F)$ is the joint channel associated to ρ_1 and ρ_2 . □

Lemma 7. If the state Σ_{AD} is a separable state, then there exists an element $\Lambda_{BCEF} \in \mathfrak{S}_B \widehat{\otimes} \mathfrak{S}_C \widehat{\otimes} \mathfrak{S}_E \widehat{\otimes} \mathfrak{S}_F$ such that (139)(140)(141)(142) are satisfied. ■

Proof. If $\Sigma_{AD} = \prod_{i \in I}^{\mathfrak{S}_A \widehat{\otimes} \mathfrak{S}_D} \sigma_{i,A} \widehat{\otimes} \sigma_{i,D}$, it suffices to define

$$\Lambda_{BCEF} = \prod_{i \in I}^{\mathfrak{S}_B \widehat{\otimes} \mathfrak{S}_C \widehat{\otimes} \mathfrak{S}_E \widehat{\otimes} \mathfrak{S}_F} \phi_1(\sigma_{i,A}) \widehat{\otimes} \phi_2(\sigma_{i,A}) \widehat{\otimes} \rho_1(\sigma_{i,D}) \widehat{\otimes} \rho_2(\sigma_{i,D}). \quad (144)$$

□

Definition 13. The state Σ_{AD} is said to be *Bell non-local with respect to the pairs of measurements* $\phi_1 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{B})$ and $\phi_2 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{B})$, $\rho_1 \in \mathfrak{C}(\mathfrak{S}_D, \mathfrak{B})$ and $\rho_2 \in \mathfrak{C}(\mathfrak{S}_D, \mathfrak{B})$ iff there is no state $\Lambda \in \mathfrak{B}^{\widehat{\otimes} 4}$ such that (139)(140)(141)(142) are satisfied.

Definition 14. The state Σ_{AD} is said to be *a Bell non-local state* if there exist pairs of measurements $\phi_1 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{B})$ and $\phi_2 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{B})$, $\rho_1 \in \mathfrak{C}(\mathfrak{S}_D, \mathfrak{B})$ and $\rho_2 \in \mathfrak{C}(\mathfrak{S}_D, \mathfrak{B})$ such that Σ_{AD} is Bell non-local with respect to the pairs of measurements (ϕ_1, ϕ_2) and (ρ_1, ρ_2) .

Definition 15. The state Σ_{AD} is said to be *Bell non-local with respect to the pairs of channels* $\phi_1 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_B)$ and $\phi_2 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_C)$, $\rho_1 \in \mathfrak{C}(\mathfrak{S}_D, \mathfrak{S}_E)$ and $\rho_2 \in \mathfrak{C}(\mathfrak{S}_D, \mathfrak{S}_F)$ iff there is no state $\Lambda_{BCEF} \in \mathfrak{S}_B \widehat{\otimes} \mathfrak{S}_C \widehat{\otimes} \mathfrak{S}_E \widehat{\otimes} \mathfrak{S}_F$ such that (139)(140)(141)(142) are satisfied.

Definition 16. The state Σ_{AD} is said to be *a generalized Bell non-local state* iff there exist pairs of channels $\phi_1 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_B)$ and $\phi_2 \in \mathfrak{C}(\mathfrak{S}_A, \mathfrak{S}_C)$, $\rho_1 \in \mathfrak{C}(\mathfrak{S}_D, \mathfrak{S}_E)$ and $\rho_2 \in \mathfrak{C}(\mathfrak{S}_D, \mathfrak{S}_F)$ such that Σ_{AD} is Bell non-local with respect to the pairs of channels (ϕ_1, ϕ_2) and (ρ_1, ρ_2) . Obviously, if Σ_{AD} is a Bell non-local state, it is also a generalized Bell non-local state.

The notion of Bell non-local state is a priori more restrictive than the notion of entangled state. It is then a fundamental skill to prove that Generalized possibilistic Theories do admit such objects.

From now, \mathfrak{S}_A and \mathfrak{S}_B are chosen to be non-simplex orthocomplemented spaces of states, \mathfrak{E}_A and \mathfrak{E}_B are chosen respectively to be the reduced effects spaces $\overline{\mathfrak{E}}_{\mathfrak{S}_A}$ and $\overline{\mathfrak{E}}_{\mathfrak{S}_B}$ (see subsection 2.5). The tensor products $\mathfrak{S}_A \widehat{\otimes} \mathfrak{S}_B$ and $\mathfrak{S}_A \widetilde{\otimes} \mathfrak{S}_B$ are defined according to subsection 2.7.

Theorem 13. Bell non-local states do exist in $\mathfrak{S}_A \widehat{\otimes} \mathfrak{S}_D$. ■

Proof. Here again, we will choose σ_1 and σ_2 two pure states in \mathfrak{S}_A such that $\sigma_1 \neq \sigma_2$ and $\sigma_1^* \not\sqsubseteq_{\mathfrak{S}_A} \sigma_2$, and τ_1 and τ_2 two pure states in \mathfrak{S}_D such that $\tau_1 \neq \tau_2$ and $\tau_1^* \not\sqsubseteq_{\mathfrak{S}_A} \tau_2$. and we will choose the regular entangled state Σ_{AD} to be defined formally as in (126), i.e. explicitly given by (127)(128).

For simplicity we will denote $\mathfrak{w}_A := l_{(\sigma_1, \sigma_1^*)}$, $\mathfrak{v}_A := l_{(\sigma_2, \sigma_2^*)}$ and $\mathfrak{w}_D := l_{(\tau_1, \tau_1^*)}$, $\mathfrak{v}_D := l_{(\tau_2, \tau_2^*)}$. In the following, we will also use the already adopted notations $\mathfrak{u} := l_{(\mathbf{Y}, \mathbf{N})} \in \mathfrak{E}_{\mathfrak{B}}$.

Now we choose $\phi_1 := \mathfrak{m}_{\mathfrak{w}_A}$, $\phi_2 := \mathfrak{m}_{\mathfrak{v}_A}$, $\rho_1 := \mathfrak{m}_{\mathfrak{w}_D}$, $\rho_2 := \mathfrak{m}_{\mathfrak{v}_D}$ (we recall the expression of \mathfrak{m}_l^* in (84)). Moreover, we consider the state $\Sigma_{AD} \in \mathfrak{S}_A \widehat{\otimes} \mathfrak{S}_D$ defined formally as in (126), i.e. explicitly given by (127)(128). We intent to prove that Σ_{AD} is a Bell non-local state with respect to the two pairs of incompatible measurements (ϕ_1, ϕ_2) and (ρ_1, ρ_2) .

Let us then consider the bimorphic map defined from $\mathfrak{E}_{\mathfrak{B}} \times \mathfrak{E}_{\mathfrak{B}}$ to \mathfrak{B} by $\Phi_{13}(l, l') := \Sigma_{AD}(\mathfrak{m}_{\mathfrak{w}_A}^*(l), \mathfrak{m}_{\mathfrak{w}_D}^*(l'))$. We have

$$\begin{aligned} \Phi_{13}(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}) &= \mathbf{Y}, & \Phi_{13}(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}}, l) &= \Phi_{13}(l, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}}) = \mathbf{N}, \quad \forall l \in \mathfrak{E}_{\mathfrak{B}} \\ \Phi_{13}(\mathfrak{u}, \mathfrak{u}) &= \mathbf{N}, & \Phi_{13}(l, l') &= \perp, \quad \text{for any other pair } (l, l') \in (\mathfrak{E}_{\mathfrak{B}}^{\text{pure}})^{\times 2}. \end{aligned} \quad (145)$$

Let us then consider the bimorphic map defined from $\mathfrak{E}_{\mathfrak{B}} \times \mathfrak{E}_{\mathfrak{B}}$ to \mathfrak{B} by $\Phi_{14}(l, l') := \Sigma_{AD}(\mathfrak{m}_{\mathfrak{v}_A}^*(l), \mathfrak{m}_{\mathfrak{v}_D}^*(l'))$. We have

$$\begin{aligned} \Phi_{14}(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}) &= \mathbf{Y}, & \Phi_{14}(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}}, l) &= \Phi_{14}(l, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}}) = \mathbf{N}, \quad \forall l \in \mathfrak{E}_{\mathfrak{B}} \\ \Phi_{14}(\overline{\mathfrak{u}}, \overline{\mathfrak{u}}) &= \mathbf{N}, & \Phi_{14}(l, l') &= \perp, \quad \text{for any other pair } (l, l') \in (\mathfrak{E}_{\mathfrak{B}}^{\text{pure}})^{\times 2}. \end{aligned} \quad (146)$$

Let us now consider the bimorphic map defined from $\mathfrak{E}_{\mathfrak{B}} \times \mathfrak{E}_{\mathfrak{B}}$ to \mathfrak{B} by $\Phi_{23}(l, l') := \Sigma_{AD}(\mathfrak{m}_{\mathfrak{w}_A}^*(l), \mathfrak{m}_{\mathfrak{v}_D}^*(l'))$. We have (the equations coming from axioms (95) and (106) are not mentioned)

$$\begin{aligned} \Phi_{23}(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}) &= \mathbf{Y}, & \Phi_{23}(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}}, l) &= \Phi_{23}(l, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}}) = \mathbf{N}, \quad \forall l \in \mathfrak{E}_{\mathfrak{B}} \\ \Phi_{23}(\overline{\mathfrak{u}}, \overline{\mathfrak{u}}) &= \mathbf{N}, & \Phi_{23}(l, l') &= \perp, \quad \text{for any other pair } (l, l') \in (\mathfrak{E}_{\mathfrak{B}}^{\text{pure}})^{\times 2}. \end{aligned} \quad (147)$$

Let us endly consider the bimorphic map defined from $\mathfrak{E}_{\mathfrak{B}} \times \mathfrak{E}_{\mathfrak{B}}$ to \mathfrak{B} by $\Phi_{24}(l, l') := \Sigma_{AD}(\mathfrak{m}_{\mathfrak{v}_A}^*(l), \mathfrak{m}_{\mathfrak{w}_D}^*(l'))$. We have (the equations coming from axioms (95) and (106) are not mentioned)

$$\begin{aligned} \Phi_{24}(\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}) &= \mathbf{Y}, & \Phi_{24}(\overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}}, l) &= \Phi_{24}(l, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}}) = \mathbf{N}, \quad \forall l \in \mathfrak{E}_{\mathfrak{B}} \\ \Phi_{24}(l, l') &= \perp, & & \text{for any other pair } (l, l') \in (\mathfrak{E}_{\mathfrak{B}}^{\text{pure}})^{\times 2}. \end{aligned} \quad (148)$$

We now suppose that there exists a state Λ in $\mathfrak{B}^{\widehat{\otimes} 4}$ such that (139)(140)(141)(142) are satisfied and we intent to exhibit a contradiction.

From $\theta_{(1)(3)}^{\mathfrak{B}\mathfrak{B}\mathfrak{B}\mathfrak{B}}(\Lambda) = \Phi_{13}$ and $\Phi_{13}(\mathfrak{u}, \mathfrak{u}) = \mathbf{N}$, we have $\Lambda(\mathfrak{u}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}, \mathfrak{u}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}) = \mathbf{N}$. Using the axiom $\Lambda(\mathfrak{u}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}}, \mathfrak{u}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}) = \mathbf{N}$ and the homomorphic property satisfied by Λ , we then deduce $\Lambda(\mathfrak{u}, \perp_{\mathfrak{E}_{\mathfrak{B}}}, \mathfrak{u}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}) = \mathbf{N}$. We must then necessarily have $\Lambda(\mathfrak{u}, \mathfrak{u}, \mathfrak{u}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}) = \mathbf{N}$ and $\Lambda(\mathfrak{u}, \overline{\mathfrak{u}}, \mathfrak{u}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}) = \mathbf{N}$. We now use $\Lambda(\mathfrak{u}, \overline{\mathfrak{u}}, \mathfrak{u}, \mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}) = \mathbf{N}$, the axiom $\Lambda(\mathfrak{u}, \overline{\mathfrak{u}}, \mathfrak{u}, \overline{\mathfrak{Y}_{\mathfrak{E}_{\mathfrak{B}}}}) = \mathbf{N}$ and the homomorphic property satisfied by Λ , to deduce $\Lambda(\mathfrak{u}, \overline{\mathfrak{u}}, \mathfrak{u}, \perp_{\mathfrak{E}_{\mathfrak{B}}}) = \mathbf{N}$ and then necessarily $\Lambda(\mathfrak{u}, \overline{\mathfrak{u}}, \mathfrak{u}, \mathfrak{u}) = \mathbf{N}$ and $\Lambda(\mathfrak{u}, \overline{\mathfrak{u}}, \mathfrak{u}, \overline{\mathfrak{u}}) = \mathbf{N}$.

In a completely symmetric path, we deduce in particular from $\theta_{(1)(4)}^{\mathfrak{B}\mathfrak{B}\mathfrak{B}\mathfrak{B}}(\Lambda) = \Phi_{14}$ and $\Phi_{14}(\overline{\mathfrak{u}}, \overline{\mathfrak{u}}) =$

\mathbf{N} the property $\Lambda(\bar{u}, \bar{u}, u, \bar{u}) = \mathbf{N}$.

As a consequence of previous partial results (i.e. from $\Lambda(u, \bar{u}, u, \bar{u}) = \mathbf{N}$ and $\Lambda(\bar{u}, \bar{u}, u, \bar{u}) = \mathbf{N}$), we deduce $\Lambda(\perp_{\mathcal{E}_{\mathfrak{B}}}, \bar{u}, u, \bar{u}) = \mathbf{N}$ and then necessarily $\Lambda(\mathfrak{Y}_{\mathcal{E}_{\mathfrak{B}}}, \bar{u}, u, \bar{u}) = \mathbf{N}$.

From $\theta_{(2)(3)}^{\mathfrak{B}\mathfrak{B}\mathfrak{B}}(\Lambda) = \Phi_{23}$ and $\Phi_{23}(\bar{u}, \bar{u}) = \mathbf{N}$, we have $\Lambda(\mathfrak{Y}_{\mathcal{E}_{\mathfrak{B}}}, \bar{u}, \bar{u}, \mathfrak{Y}_{\mathcal{E}_{\mathfrak{B}}}) = \mathbf{N}$. Using the axiom $\Lambda(\mathfrak{Y}_{\mathcal{E}_{\mathfrak{B}}}, \bar{u}, \bar{u}, \overline{\mathfrak{Y}_{\mathcal{E}_{\mathfrak{B}}}}) = \mathbf{N}$ and the homomorphic property satisfied by Λ , we then deduce $\Lambda(\mathfrak{Y}_{\mathcal{E}_{\mathfrak{B}}}, \bar{u}, \bar{u}, \perp_{\mathcal{E}_{\mathfrak{B}}}) = \mathbf{N}$. We then have necessarily $\Lambda(\mathfrak{Y}_{\mathcal{E}_{\mathfrak{B}}}, \bar{u}, \bar{u}, u) = \mathbf{N}$ and $\Lambda(\mathfrak{Y}_{\mathcal{E}_{\mathfrak{B}}}, \bar{u}, \bar{u}, \bar{u}) = \mathbf{N}$.

We now collect the two obtained properties $\Lambda(\mathfrak{Y}_{\mathcal{E}_{\mathfrak{B}}}, \bar{u}, u, \bar{u}) = \mathbf{N}$ and $\Lambda(\mathfrak{Y}_{\mathcal{E}_{\mathfrak{B}}}, \bar{u}, \bar{u}, \bar{u}) = \mathbf{N}$. As a consequence we obtain, using the homomorphic property satisfied by Λ , the property $\Lambda(\mathfrak{Y}_{\mathcal{E}_{\mathfrak{B}}}, \bar{u}, \perp_{\mathcal{E}_{\mathfrak{B}}}, \bar{u}) = \mathbf{N}$ and then necessarily $\Lambda(\mathfrak{Y}_{\mathcal{E}_{\mathfrak{B}}}, \bar{u}, \mathfrak{Y}_{\mathcal{E}_{\mathfrak{B}}}, \bar{u}) = \mathbf{N}$, i.e. $\Phi_{24}(\bar{u}, \bar{u}) = \mathbf{N}$. However, this result contradicts the property $\Phi_{24}(\bar{u}, \bar{u}) = \perp$ assumed by definition. We have then obtained the announced contradiction : the state Λ does not exist as an element of $\mathfrak{B}^{\otimes 4}$.

As a conclusion, Σ_{AD} is a Bell non-local state with respect to the two pairs of incompatible measurements (ϕ_1, ϕ_2) and (ρ_1, ρ_2) . \square

4 Conclusion

Inspired by the *operational quantum logic program*, we have the contention that probabilities can be viewed as a derived concept, even in a reconstruction program of Quantum Mechanics. The already cited remark of S. Abramsky [2, Theorem 4.4] can be viewed as another justification of this perspective on quantum mechanics. These two perspectives have stimulated our desire to build an operational description based on a possibilistic semantic (in a sense, the 'probabilities' are replaced by statements associated to a semantic domain made of three values 'indeterminate', 'definitely YES', 'definitely NO'). The present paper intends to develop one step further such an operational formalism. It will be called Generalized possibilistic Theory (GpT) as it is partly inspired by the formalism of Generalized Probabilistic Theory (GPT).

In [8] and [9], we clarified the notion of tensor product of our spaces of states allowing to describe compound systems. This result is recalled in a preliminary subsection 2.7. These results are exploited to show the generic existence of entangled states in the tensor product of non-simplex spaces of states build around reduced effects spaces (subsection 3.1). The entangled states exhibited in this subsection are exploited to propose entanglement-assisted incompatibility tests and to show that some states are steered by pairs of incompatible measurements (subsection 3.2). In the subsection 3.3, we go a little step further and we show that the phenomenon of Bell non-locality exists in the description of compound systems based on the tensor product of non-simplex spaces of states build around reduced effects spaces.

Statements and Declarations

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References

- [1] Abramsky, S., Brandenburger, A. The sheaf-theoretic structure of non-locality and contextuality. *New J. Phys.* , 13:113036, Nov 2011.

- [2] Abramsky, S. Big toy models. *Synthese*, 186(3):697–718, Jun 2012.
- [3] Abramsky, S., Coecke, B. Categorical Quantum Mechanics. In Engesser, K., Gabbay, D.M., Lehmann, D., editor, *Handbook of Quantum Logic and Quantum Structures*, pages 261 – 323. Elsevier, Amsterdam, 2009.
- [4] Bell, J.S. On the problem of hidden variables in quantum mechanics. *Rev. Mod. Phys.* , 38:447–452, 1966.
- [5] Bohr, N. The quantum postulate and the recent development of atomic theory. *Nature*, 121:580–590, 1928.
- [6] Brunner, N., Cavalcanti, D., Pironio, S., Scarani, V., Wehner, S. Bell nonlocality. *Rev. Mod. Phys.*, 86 (2) 419—478, 2014.
- [7] Buffenoir, E. Reconstructing quantum theory from its possibilistic operational formalism. *Quantum Stud.: Math. Found.*, 10:115-159, 2023. <https://doi.org/10.1007/s40509-022-00286-w>
- [8] Buffenoir, E. Generalized probabilistic theories : multipartite experiments. *Quantum Stud.: Math. Found.*, 10:443–482, 2023. <https://doi.org/10.1007/s40509-023-00306-3>
- [9] Buffenoir, E. A new perspective on the tensor product of semi-lattices. <https://arxiv.org/abs/2306.16420>
- [10] Fine, A.. Hidden variables, joint probability, and the bell inequalities. *Phys. Rev. Lett.*, 48:291–295, 1982.
- [11] Grätzer, G. Lattice Theory: First Concepts and Distributive Lattices. 1971.
- [12] Heinosaari, T., Miyadera, T., Ziman, M. An invitation to quantum incompatibility. *J. Phys. A*, 49 (12):123001, 2016.
- [13] Heisenberg, W. Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Z. Phys.*, 43:172–198, 1927.
- [14] Kochen, S., Specker, E. The problem of hidden variables in quantum mechanics. *J. Math. Mech.* , 17 (1):59–87, 1967.
- [15] Masanes, L., Acin, A., Gisin, N. General properties of nonsignaling theories. *Phys. Rev. A*, 73:012112, 2006.
- [16] Plavala, M. Conditions for the compatibility of channels in general probabilistic theory and their connection to steering and Bell nonlocality. *Phys. Rev. A*, 96 (5) 052127, 2017.
- [17] Plavala, M. General probabilistic theories: An introduction. *Phys. Rep.*, 1033:1–64, 2023.
- [18] Pratt, V.R. Chu Spaces, 1999.
- [19] Uola, R., Costa, A.C.S., Nguyen, H.C., Gühne, O. Quantum steering. *Rev. Mod. Phys.*, 92 (1) 015001, 2020.
- [20] Wolf, M.M., Perez-Garcia, D. , Fernandez, C. Measurements incompatible in quantum theory cannot be measured jointly in any other no-signaling theory. *Phys. Rev. Lett.*, 103:230402, 2009.