The random walk model in finance: a new taxonomy

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Abstract

The backbone of financial risk modeling in finance over a long time period of more than a century, the random walk hypothesis has shown substantial variations in its structure throughout its history. In this article, I revisit the history of the random walk model in finance by introducing a new way of describing what a random walk is, based on the Lévy measure in the Fourier space, a tool that has not yet been used in the history of financial thought. With this lens, we are able to understand the overview of the life of this model in finance over the entire 20th century, including the precursors of the 19th century.

1. Introduction

The backbone of financial risk modeling in finance over a long time period of more than a century, the random walk hypothesis has shown substantial variations in its structure throughout its history. In this paper, I wish to revisit the history of the random walk model in finance. I introduce a new way of conceptually describing what a random walk is, based on a mathematical shift. With this shift, we are able to understand the overview of the life of this model in finance over the entire 20th century, including the precursors of the 19th century. This way of conceptually describing what a random walk is has not yet been used in the history of financial thought. It is based on a very useful mathematical function for the description of random walks, called the Lévy measure, a function defined in the Fourier space. The shift in Fourier space reveals certain similarities that were previously considered to

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be unconnected, and allows us to group together families of models that were previously considered to belong to different modeling traditions.

More precisely, the new use of the Lévy measure as a technical key in the history of financial thought allows us to better understand some of the issues of financial modeling by random walks. It allows us to revisit many scientific debates or controversies by linking what was perceived as opposed, such as neoclassical modeling vs. fractals, or econophysics vs. mathematical finance. Very often, we believe that we are dealing with oppositions where there really are none and we create "false debates" because of a conceptual illusion. The use of the Lévy measure allows bypassing these spurious debates. Thus, my proposal allows us to remove some of the puzzles in understanding the historical evolution of financial modeling, by isolating key components of this modeling and by clearly identifying the respective positions of the different schools of thought. It allows us to group together the random walk models used in the long history of financial modeling in the same epistemological perspective. It allows to identify and compare the different research programs undertaken on the modeling of stock market dynamics throughout the 20th century until today. Using the Lévy measure, I introduce a new taxonomy in the history of financial thought, presented in a summarized table that spans the 20th century. The taxonomy is based on a "triplet" I introduce to describe the random walk model, I name the "risk triplet", combining three characteristics of any random walk, its activity, its variation and its variance. The different alternatives of this triplet are displayed in a table that synthesizes them, namely the "Table T". Table T helps to bring about a dialogue between theories, models and different ways of modeling financial risk in the case of random walk.

The outline of the article is as follows. Section 2 presents the taxonomy in the table T. Section 3 aims to distinguish the random walk hypothesis from the idea of unpredictability in general and to disentangle these notions that are sometimes confounded in the work on the history of the random walk model in finance. The IID hypothesis is introduced in this sense in order to highlight what is a random walk and to separate random walk from Brownian motion or Gaussian processes. Section 4 deals with scale invariance. I revisit the work of Jules Regnault, who has been described as a "precursor" of the random walk model in finance because of the so-called "square-root-of-time" rule. I identify this rule as a particular case of scale invariance in general, and to this purpose I introduce the fractal geometry of Benoît Mandelbrot. I show how and why neoclassical financial modeling is fractal in the sense of Mandelbrot, which allows us to avoid opposing too quickly neoclassical finance and fractal finance. I introduce

the stability-under-addition property to separate the scale invariance from the random walk model. Section 5 addresses the issue of heavy tails in random walk models. It connects the issue of extreme values with the scale of price changes, and it shows with two examples of Lévy measure the mathematical indistinguishability of the financial mathematicians' approach and the physicists' approach in the case of random walk models with heavy tails.

2. The table T

The taxonomy is built on the Lévy measure and more precisely the two main characteristics given by the Lévy measure, namely "activity" and "variation" of a Lévy process. I supplement these two characteristics with the second moment (if it exists), the "variance" or "diffusion component" of a Lévy process, to compose what I consider and name as the "characteristic triplet" of any random walk model for finance: activity, variation, variance. As we will see below, this triplet will be very useful for understanding some issues involved in financial modeling with random walks. My taxonomy gives four possible combinations with the components of the triplet.

The risk triplet is built on the Lévy measure, and it is in this regard that one of the main contributions of this paper is to introduce the Lévy measure in the field of history of financial thought in order to refresh the historical perspective on the random walk model in finance.

Preliminary definitions

I begin with some preliminary definitions which are necessary to give a clear understanding of the use of this taxonomy. I said previously that the taxonomy is built on the two main characteristics given by the Lévy measure, namely "activity" and "variation" of a Lévy process. To clarify this point without exaggerating the mathematical treatment and to avoid the use of complex formulas, we now move on to Fourier space, which allows us to describe "in words" certain mathematical subtleties. In this space, we pass from the density function to the characteristic function, which is the Fourier transform of the density function. To be a little more accurate, if X is a random variable, the Fourier transform of X is $\Phi_X(u) = \mathbf{E}[\exp iuX]$ where exp is the exponential function, the letter i is the imaginary unit of complex numbers and **E** the symbol of the mathematical expectation. For any random variable X, its characteristic function $\Phi_X(u)$ exhibits a quantity called "characteristic exponent" of X which is denoted $\Psi_X(u)$ such that $\Phi_X(u) = \exp(\Psi_X(u))$.

The notion of characteristic exponent of a random variable plays an essential role in the study of random walks. The characteristic exponent shapes precisely the morphology of uncertainty, these words denoting the irregularity: for example in finance, the stock market paths, the "roughness" of their appearance. Random walks can have varying degrees of roughness. Activity and variation quantify these degrees. Activity and variation shape the roughness of the "pattern", while variance indicates the "size" of this pattern, its scale. In this sense, the "pattern" of the risk morphology can change with scale (risk at one week, risk at one month) and its "shape" can be different or not depending on the scale. Risk morphology is not like Russian dolls, except in the case of stability-under-addition-property of Lévy processes (see section 4 below).

The characteristic exponent is proportional to time. If $X(t)$ is a Lévy process, then $\Psi_{X(t)}(u) = t \times \Psi_{X(1)}(u)$, meaning (in words) that the characteristic exponent at a given time t is easily obtained from the characteristic exponent at time 1:

Characteristic exponent at time t = t × Characteristic exponent at time 1

This is the reason why the characteristic exponent of a Lévy process is equal to t times that of its underlying infinitely divisible distribution, which is in fact the distribution X_1 This property expresses the infinite divisibility of random variables. This is one of the main attractions of Lévy processes, making them preferable to other types of model. Roughly speaking, the property of infinite divisibility states that it is possible to consider a given random variable as a sum of identical independent random variables, for example the random variable of one-year returns as the sum of the 52 random variables of one-week returns. But the distribution of one-year returns has not to have the same shape (scale-free property) as the distribution of one-week returns. The distribution can be infinitely divisible but not scale invariant. It should not be confounded with the "Russian dolls property" of alpha-stable processes (see section 4 below).

The Lévy measure has been explicitly introduced in the models of the 1990s, whereas it was only implicit in the models of the 1970s, with the exception of the models of Mandelbrot (1962) in which it appeared in a closed form of the characteristic exponent[2]. The explicit form

[2] When the literature on the history of the random walk model addresses Mandelbrot models, this closed form is generally used. In this paper we introduce the explicit form because it allows us to unify different modeling approaches that were previously perceived as

of the characteristic exponent of a random walk was obtained in the most general case by Paul Lévy in 1934 from his theory of processes with IID increments, the so-called Lévy-Khintchine formula. In words, the Lévy-Khintchine formula is decomposed in three parts:

$$\text{Characteristic exponent} = \underbrace{\text{expectation} + \text{diffusion coefficient}}_{\text{Brownian motion continuous component}} + \underbrace{\text{Lévy measure}}_{\text{discontinuous component}}$$

A Lévy process is fully determined by these three quantities. In financial words, the three parts of the Lévy-Khintchine formula mean:

$$\text{Market dynamics} = \underbrace{\text{trend of returns} + \text{scale of risk}}_{\text{continuous component}} + \underbrace{\text{morphology of risk}}_{\text{discontinuous component}}$$

For financial purposes, the Lévy-Khintchine formula may be analyzed as follows (see technical details in Le Courtois and Walter 2014, pp. 60 sq.). The first two terms represent the continuous component of the dynamics, which is a Brownian motion with mean and standard deviation. Financially speaking the standard deviation represents market volatility. The second term is the Lévy measure. This component of the Lévy process completely defines the structure of jumps. Heuristically speaking, the Lévy measure provides the average number of jumps per time unit as a function of their amplitude. It is thus the mathematical object that can quantify the occurrence and size of jumps and creates discontinuity in the path of the stochastic process.

Now we turn to the intuitive presentation of the activity and variation of a Lévy process. Intuitively, the greater the number of jumps per time unit, the more the path of the stochastic process will have a high degree of erraticness and the more discontinuous the random walk will be. A random walk will be highly erratic if the average number of jumps occurring per unit of time is very large. The average number of jumps per unit of time defines the so-called "intensity" of a Lévy process – also known as "activity" by analogy with turbulence. Consider for example a very simple Lévy process, the Poisson process with parameter λ: λ is the average number of jumps per unit of time. In this very simple Lévy process, the activity of the process is λ. In this extremely rudimentary case, the average number of jumps per unit of time is finite and is given by the Poisson parameter.

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contradictory. The transition from one form to another is precisely explained in Le Courtois and Walter (2014, p. 82-88).

Let us continue with this simple example to get an intuitive idea of what the Lévy measure does. Whenever a jump in the Poisson process occurs, the magnitude of the jump must be specified. Suppose that this magnitude is random and is pulled into a given probability distribution with known density. In this case, one faces a so-called "compound Poisson process". We see that the product activity-density captures both the occurrence rate of discontinuities and their magnitude. To say it differently, this product fully characterizes the jump structure of the process. This product is precisely the Lévy measure. If for example the distribution of jumps is Gaussian with a mean being the average size of jumps and a standard deviation being the "volatility" of the size of jumps, it will in this case be a compound Poisson process with a normal (Gaussian) distribution so-called "compound Poisson-normal" (CPN). The activity of this Lévy process (CPN process) is the Poisson parameter and the density of the distribution of jumps is the Gaussian distribution. In all cases in which one constructs a compound Poisson process with another distribution, the number of jumps per unit of time (the occurrence rate of discontinuities) is finite and the resulting Lévy process is of finite activity. In this situation one can clearly separate the activity from the density.

But there is no reason why the average number of small jumps per unit of time should stay finite. The advantage of generalizing in this way is that the very many small movements in the market can be taken into account and that this makes it possible to disentangle the notion of discontinuity from the occurrence of large moves only. In the case of infinite activity, it is no longer possible to separate the activity from the density. Both are "mixed" in the Lévy measure, which entirely shapes the morphology of the irregularity (the discontinuity) of the financial dynamics. In this situation, it becomes less necessary to add a continuous Brownian component. It is only when the average number of jumps is finite that it is necessary to add this continuous Brownian component for market movements occurring between the jumps. To sum up, the activity can be finite or infinite.

Consider now the average distance between two points of the process. The average distance too can be finite or infinite (the mean may or may not exist). This idea of average distance corresponds to what is called the variation of a Lévy process. The variation is another feature of the morphology of financial uncertainty. We see that the variation may be finite or infinite.

The taxonomy

I start by introducing my taxonomy. The table "T" below presents this taxonomy with some examples of financial models.

Lévy measure		Variance	Examples of random walk models in financial modeling
Activity	**Variation**		
0	N.S.	Finite	Samuelson (1965)
Infinite	Infinite	Infinite	Mandelbrot (1962)
Finite	Finite	Finite	Press (1967), Praetz (1972)
Infinite	Finite	Finite	Madan and Seneta (1990)
Infinite	Infinite	Finite	*Financial mathematicians*: Eberlein and Keller (1995), Barndorff-Nielsen (1998), Carr, Geman, Madan and Yor (2002)
			Physicists: Mantegna (1991), Mantegna and Stanley (1994), Koponen (1995), Bouchaud and Potters (1997)

Table "T". The "risk triplet" and the financial models of random walks

The reading of the table T shows the kinds of modeling a random walk, corresponding to the different periods of the history of the random walk model in finance. It allows to understand the manner in which my taxonomy allows to organize the history of the random walk model in finance and provide a better understanding of some parts of this history. Each line reflects an episode in the history of the random walk model in finance. The first line presents the continuous random walk models of the mainstream financial economics before the crisis opened by Mandelbrot in 1962, with the main canonical model which is that of Paul Samuelson (1965). The four lines that follow describe four stages in the history of the accounting of discontinuities by random walk models, according to the values of the triplet.

Now let us turn to the details of each line. The triplet "infinite / infinite / infinite" corresponds to Mandelbrot's radical data-driven approach, which clashed with the dominant view of financial modeling in the 1960s, namely the continuous random walk model first introduced by Louis Bachelier in his 1900 thesis (see section 3 below). To solve the "leptokurtic

phenomenon", characterized by a discontinuous price change dynamics creating fat tails on empirical price change distributions (high level of *kurtosis*), Mandelbrot adopted a very radical view on risk modeling by maintaining one of the fractal properties of neoclassical finance, the scale invariance of time-scaling of risk (see section 4 below) and extended this scale invariance to a subclass of Lévy processes, the alpha-stable processes, a family of stochastic processes which keeps the self-similar (fractal) property of risk morphology. In this very special case of scale invariance, the three components of the triplet are infinite. For this reason, and especially because of the infinite variance, it has been widely rejected by the mainstream financial economics academic community since its introduction, explained as producing "blood, sweat, toil and tears" (Cootner 1964, p. 337), as being "indigestible" (Mirowski 1995, p. 582), and a "monster" (MacKenzie 2006 p. 108). For example, the tenet concept of portfolio risk diversification, which has been central to neoclassical portfolio theory since the nobelised work of Harry Markowitz (1952), is cancelled out by an infinite variance (Le Courtois and Walter 2014, p. 339).

The triplet "finite / finite / finite" corresponds to the modeling answer of the mainstream financial economics in the 1970s, adding Poisson jumps to Brownian diffusion component to exhibits discontinuities inside a continuous Brownian framework. Poisson type jump components in jump diffusion models address this issue. The CPN process is a particular case of Lévy processes, because its increments are IID. The three quantities of the risk triplet are finite and this finiteness seemed very convenient and tractable for academics. It is the development of mathematical models that is at the origin of financial models, even if we try to partially take into account the problem of jumps. The aim is to maintain the diffusion framework by including extra-jumps. This modeling period of discontinuous random walks is characterized by the extensive use of jump-diffusion processes, losing the scale invariance property of Brownian motion (the Russian doll property of the risk pattern). As it appears, it is column by column the opposite of the triplet characterizing Mandelbrot's proposal. It is thus clear why it was impossible for the financial economics to originally accept Mandelbrot's hypotheses. At this time, it was impossible for the financial industry driven by the mainstream continuous approach to include this view. This direct confrontation triggered the "leptokurtic crisis" (Walter 2019).

The following triplet "infinite / finite / finite" releases one of the finite constraints and admits infinite activity. We find in this triplet the first attempts to take into account real market data in a fine tuning way, an approach that can be described as rediscovering the data-driven view:

the modelers work empirically on the basis of statistical data, and not by first looking for a development of mathematical models. Dilip Madan and Eugene Seneta (1990), followed by Madan and Milne (1991) introduce the (symmetric) Variance-Gamma process (symmetric VG process) to represent the price change dynamics and to price European options. The last triplet "infinite / infinite / finite" admits infinite activity and infinite variation. It corresponds to a second generation of financial models based on pure jump processes, in which the diffusion component has been removed as it is considered useless for modeling.

As we notice, the last two triplets bring infinity into financial risk modeling. Despite its novelty, and although it breaks with the continuity paradigm of neoclassical finance because it clashes abruptly with the continuous mainstream financial economics of the 1970s, this approach remains in the community of financial mathematics. The financial mathematicians of this approach are not heterodox mathematicians and they don't have to adopt a new economic paradigm. Borrowing a concept from David Colander (2007), I argue that, for this reason, these approaches can be considered as "inside-the-mainstream heterodoxy". On the other hand, as well as Mandelbrot and the physicists who are beginning to be interested in finance are situated from the outset on the periphery of financial mathematicians. For this reason the physicists would be more considered as "outside-the-mainstream" heterodoxy. But it is worth noting that the introduced category "infinite / infinite / finite" combines the two different communities of academics, the financial mathematicians and the physicists[3]: these two heterodoxies (two communities) share the same triplet in modeling discontinuities.

It is interesting here to see how the taxonomy based on Lévy measure allows the inclusion of physicists because the models of physicists that are IID match those of financial mathematicians with the considered triplet. The choice of the Fourier space gives access to a

[3] According to the canonical book of Mantegna and Stanley (2000), "econophysics" describes the work of physicists working in finance and the way the physicists can contribute to the science of economics. It is worth noting that physicists worked in finance before the name "econophysics" (Weatherall 2014). It has emerged in 1991 in France with Jean-Philippe Bouchaud and in Italy with Rosario Mantegna (see below). Two of the first "econophysics" events in 1998 were the "Les Houches winter school of physics" held at Les Houches, France, 23 February-6 Marsh and the "International Workshop on Econophysics and Statistical Finance" held at University of Palermo, Italy 28-30 September 1998. This workshop is the first workshop with Proceedings entirely dedicated to an econophysics conference.

"lens" of view of the history of the random walk model in finance which allows us not to be blurred by the different formats of the density function space. Mandelbrot's initial approach is at the origin of a twofold inspiration, which fertilized two streams of modeling: one that coming from physicists, the other that coming from financial mathematicians. The intellectual source of the random walk models of these two communities is the same. All these models address the previously unresolved issue of discontinuities in price changes. Hence another result of this taxonomy is the highlighting of an important phenomenon over the long period of financial modeling I name the "discontinuous turn" in financial modeling (Walter 2019).

The next section presents in the simplest and most intuitive possible way the main characteristics of random walks and Lévy processes[4] with embodying these characteristics with elements of the history of financial thought. For more mathematical details, one can refer to one of the many books that provide a comprehensive view of this topic, as for example Bertoin (1998), Sato (1999) or Applebaum (2009).

3. The IID hypothesis

There already exists a considerable literature on the random walk model in finance[5]. A great deal of work has been carried out on this theme. It is not the place in this article even to try to review the abundant existing work on this topic. So many works have already been published on the history of this idea in the academic literature that it is impossible here to give even a brief review. I just recall some technical issues for clarifying the notions which serve as a basis for the topic. When I think it is useful, I will engage some of this literature, although often in footnotes rather than in the body of the text.

[4] A random walk can be unidimensional (random walk in one dimension: a line) or multidimensional (random walk in higher dimensions: a walk in a space of dimension greater than one). In the paper, we consider only unidimensional random walks.

[5] According to Eric Brian and Christine Théré (1997, p. vii), the emergence of the topic of the history of the random walk model in academic audience was inaugurated by Christian Walter (1996). Walter (1996) cites Peter Bernstein (1992) book which mentions random walk model, but Bernstein's book was "written for a broad, non-academic audience" (Dimand 2009, p. 84).

The unpredictability of future prices

In order to clarify what we are talking about, let's write down the quantities we will discuss in this article. If $S(t)$ is the price of any given asset S (for Stock, Security, Share) at time t, the continuous compound return of asset S between dates 0 and t is:

$$X(t) = \ln S(t) - \ln S(0) \quad (1)$$

This means that prices evolve according to the equation:

$$S(t) = S(0) \exp X(t) \quad (2)$$

where "exp" is the exponential function. The random walk hypothesis in financial modeling assumes that $X(t)$ is a random walk. Nothing else. Nothing more.

The concept of random walk has been applied to price changes in financial markets since the beginning of the 20th century[6]. In his pioneering doctoral thesis[7], Louis Bachelier (1900) did not consider log-prices but what he called "true prices" (Bachelier 1900, p. 25). The market investigated by Bachelier was the Government bond market, not the stock market. The "true price" corresponds to what we would currently label on the bond market a clean price expressed as a percentage, i.e. excluding accrued interest, or "par value net of accrued interest". If equivalence had to be found on the stock market, one would consider the stock price of a given share net of its accrued dividend. Let us note $S^*(t)$ the true price of asset S at time t. Knowing that the true price is obtained from the price by removing the fraction of the accrued interest during the period, one can write the relationship between the true price and the price at date t as:

$$S^*(t) = S(t) - \mu t \quad (3)$$

where μ represents the accrued interest in continuous time. Bachelier's famous assertion is then what he names the "fundamental principle" of true prices: "the mathematical expectation of the speculator is zero" (Bachelier 1900, p. 34). For Bachelier, the speculator is the one who

[6] Strictly speaking, Jules Regnault's (1863) analyses (see Jovanovic 2004), although they may prefigure the idea of random walk, do not contain the idea of a stochastic process on prices. About the Regnault's "square-root-of-time-rule", see below.

[7] Translated in English by James Boness in Paul Cootner (1964), and by Mark David and Alison Etheridge (2006). See also Jean-Michel Courtault and Youri Kabanov (2002).

makes a bet on the variation of true prices, instead of looking for the yield of the annuity. Bachelier's assertion is therefore mathematically written as follows (Walter 2013, p. 73):

$$\mathbf{E}_0[S^*(t) - S^*(0)] = 0 \qquad (4)$$

where $\mathbf{E}_0$ [.] is the symbol of mathematical expectation at date t=0. This equation can be written:

$$\mathbf{E}_0[S^*(t)] = S^*(0) \qquad (5)$$

This relationship describes a stochastic process named a "martingale" on the true price. To say it differently, the true prices are unforecastable, but this model is not a random walk. This relation shows that the property of unpredictability of future prices is not identically associated with random walk. Martingales are sufficient for this[8].

The unpredictability of the future "true price" says nothing about the shape of the distribution of these future "true prices". This is the second aspect of Bachelier's thesis. In fact, using a law of large numbers argument, Bachelier introduces an additional hypothesis and obtains the normal distribution of returns (Walter 1996, p. 879). Bachelier proposes to use the Gaussian distribution because of the influence of Quetelet (Walter 1996, p. 904; Jovanovic 2004, p. 219) which leads him to think in terms of average and Gaussian law. This in fact amounts to only imagining square-integrable martingales. But it could be otherwise with the only initial hypothesis on the "true price".

Hence the random walk model appears in Bachelier's work, but indirectly. To put it simply but precisely, if $W(t)$ is a standard Brownian motion, then Bachelier states that:

$$S^*(t) = S^*(0) + \sigma\, W(t) \qquad (6)$$

This relation (6) is equivalent to:

$$S(t) = S(0) + \mu t + \sigma\, W(t) \qquad (7)$$

[8] Jovanovic and Le Gall (2001) argue that, for Jules Regnault, if "the expected profit is zero for each operation" hence "in a very pioneering way, price behaviour took the shape of a random walk model" (p. 340) Technically this is not correct: if "the expected profit is zero for each operation", price behavior follows a martingale, with or without square-integrability.

This is tantamount to saying that the price process is a Brownian motion. This new relationship will be used by Bachelier to price options. The parameter σ of relation (7) is the diffusion coefficient of Brownian motion. In the financial jargon, this is the "volatility" of markets. Harry Markowitz (1952) has proposed to summarize the risk by the variance of the returns. The distribution of $W(t)$ is a Gaussian distribution centered on zero with variance equaling t: the square root of t is the standard deviation of the distribution. This property of Brownian motion is at the origin of the so-called "square-root-of-time-rule" (Danielsson and Zigrand 2006) in the standard model of fluctuations.

The Osborne's (1959) modification is to replace $S(t)$ by $X(t)$. This amounts to considering logarithms of prices instead of prices, leading to (Samuelson, 1965):

$$S(t) = S(0) \exp(\mu t + \sigma W(t)) \qquad (8)$$

This above simple equation is the standard model of price fluctuations since the 1960s, the exponential of Brownian motion.

We can therefore see how Bachelier's thesis embeds both the idea of random walk with the use of Brownian motion for the pricing of options, and the idea of market informational efficiency with the use of martingales on true prices: the best possible forecast of the future true price is the present true price.

The independence property

We now turn to the difference between unpredictability and random walk. The association between random walk and unpredictability has sometimes been a source of some confusion. Let us start with an example taken from the financial literature. In his very celebrated and extremely renowned book on *A random walk down Wall Street* (1973), Burton Malkiel stated that "A random walk is one in which future steps of directions cannot be predicted on the basis of past actions" (Malkiel 1973, p. 24). But, strictly speaking, this heuristic definition is not the definition of a random walk but of a stochastic process in which the future is independent of the past, given the present. Such a process is not a random walk. It can be, but it might not be either. It could be a martingale, or a Markov process. We see in this example as a canonical pattern the confusions that have sometimes occurred in the financial literature and in the history of financial thought, about the notion of random walk.

What exactly does mean that prices are unforecastable? And what exactly are we referring to when we claim such a statement? Secondly, are we talking about unforecastability of prices or

price changes? How to define the notion of unforecastability? Is unforecastability associated with random walk? If we want to mean that prices are unpredictable, why use a random walk model? And if we use a random walk model, what does it mean about price changes? One of the important points that we will try to emphasize in this section is the following: it is a misunderstanding to assimilate unpredictability and random walk. Let us just illustrate this with an example. If returns are unpredictable but volatility presents a short-term memory, then we can say that, indeed, we have unpredictability on prices changes but not a random walk since future volatility, and hence the square of returns, is predictable. This simple example introduces the issue of what is a random walk and what is not.

In the simplest possible general terms by avoiding too technical definitions, what is called a "random walk" is a stochastic process whose increments are independently and identically distributed (hereafter IID): each increment is a random variable which has the same probability distribution as the others and all are mutually independent. This IID property characterizes Lévy processes, a family of stochastic processes named after their inventor, the French mathematician Paul Lévy[9]. Any Brownian motion is a Lévy process, but the opposite is not true. There are an infinite number of random walks that are not Brownian motions. The Brownian motion type of the random walk is only one of many, although it was the first to appear in the work of applied mathematics in finance with Bachelier (1900). We now elaborate on the issue of independence.

With the independence property, we deal with three notions, the time, the scale and the lag. To be clear, let us recall that we are talking about the process $X(t)$ and that we wonder if its increments are independent. To be able to define the increments of $X(t)$, it is necessary to choose a characteristic "size" of the increments (one day, one week, one month etc.), denoted by the Greek letter τ. The variable τ is the characteristic scale of the increments (daily scale, weekly scale etc.). Once this scale is chosen, it is possible to consider the increments of $X(t)$, which are periodical returns (one-day return, one-week return etc.) of the following form:

$$X(t) - X(t-\tau) \stackrel{\text{def}}{=} Z(t,\tau) \text{ or } Z_t^\tau \qquad (9)$$

[9] A Lévy process has not to be confused with the so-called "Lévy distributions" (see below). It is worth to note that this point is sometimes a source of confusion in the literature.

The random variable $Z(t, \tau)$ is the law of the increments of $X(t)$ for the scale τ. This notation is straightforward, but it uses a notation convention that deserves to be emphasized. We keep visible τ to highlight the fact that the structure of price changes and the tests for unpredictability are strongly dependent on the scale τ. The random walk model implies strong properties on the increments at various scales, as we will see below. Times lags will be denoted par the letter h which will be a multiple of τ. Again the notation convention allows to precisely grasping what we want to check. For example, if τ = 1 day and h = 10, then the quantity denoted $\mathrm{corr}(Z_t^{1\text{ day}}, Z_{t+10}^{1\text{ day}})$ defines the correlation between the daily return at date t and the daily return at date t+10 (10 days later). If, for that specific scale, the daily returns are independent, hence the autocorrelation is null. But the opposite is not true as we see now.

The independence of increments means not only uncorrelated increments but also any uncorrelated nonlinear functions of increments. Since this difference is subtle but very important for a precise understanding of the financial controversies about the random walk model in finance, it is elaborated a little more explicitly. The dependence between two random variables (say Z_1 and Z_2) is measured by the quantity $C_{f,g}(Z_1,Z_2)$ defined as:

$$C_{f,g}(Z_1, Z_2) = \mathbf{E}[f(Z_1) \times g(Z_2)] - \mathbf{E}[f(Z_1)] \times \mathbf{E}[g(Z_2)] \qquad (10)$$

where **E** is the symbol of mathematical expectation. Z_1 and Z_2 are independent if and only if $C_{f,g}(Z_1,Z_2) = 0$ for all functions f and g.

For financial modeling, we can think of Z_1 and Z_2 as two random variable representing increments of $X(t)$ according to a given scale τ. Therefore, the independence of price changes at given scale τ for a given lag h is verified if and only if $C^{\tau}_{f,g}(h) = 0$ in (note the superscript τ):

$$C^{\tau}_{f,g}(Z^{\tau}_t, Z^{\tau}_{t+h}) = \mathbf{E}[f(Z^{\tau}_t) \times g(Z^{\tau}_{t+h})] - \mathbf{E}[f(Z^{\tau}_t)] \times \mathbf{E}[g(Z^{\tau}_{t+h})] = C^{\tau}_{f,g}(h) \qquad (11)$$

which is heuristically interpretable as a relationship that depends only on the scale τ and the lag h between the two observed price changes. Once again, we keep visible τ in the writing of the formula to emphasize the fact that the structure of price changes is strongly dependent on τ. For this very reason, given functions f and g, the quantity C depends only on the time scale τ and the lag h, hence the notation convention $C^{\tau}_{f,g}(h)$.

From the point of view of the history of financial thought, the history of the controversies on the unpredictability of price changes coincides with the chronology of the different selections made for the functions *f* and *g*. The role of the scale τ is not emphasized in the beginning of financial econometrics. In the first work of the years 1930-1970, $f(x) = g(x) = x$. In this case, the function $C_{f,g}(h)$ is simply the autocovariance function $\gamma(h)$:

$$\gamma(h) = \text{cov}(Z_t, Z_{t+h}) = \mathbf{E}[Z_t \times Z_{t+h}] - \mathbf{E}[Z_t] \times \mathbf{E}[Z_{t+h}] \quad (12)$$

The independence of the increments is identical to the nullity of the correlation coefficient. The correlation between two price changes is null. This kind of function *C* with evidence of absence of serial linear autocorrelation gave birth to the idea of efficient market hypothesis (LeRoy 1989, Walter 1996, Campbell et al. 1997, Jovanovic 2009, Delcey 2019). It is the reason why, due to the first definition of the quantity *C*, the work on random walk model in financial economics overlaps with research on the EMH and the random walk hypothesis was taken to be equivalent to the EMH.

There are as many functions *C* as there are possible kinds of functions *f* and *g*. If for example $f(x) = g(x) = x^2$ i.e. a simple kind of nonlinear function of increments, independence means $C=0$ also for squared increments. But it is clear that it is possible to have $C=0$ for $f(x) = g(x) = x$ (simple increments) and $C \neq 0$ for $f(x) = g(x) = x^2$ (squared increments), meaning that price change are uncorrelated but non independent. For example, in the famous Nobel prized financial model using an autoregressive conditional heteroscedasticity process (ARCH) introduced by Robert Engle (1982), increments are uncorrelated but the squared increments are correlated: there is a "memory" of fluctuations compatible with uncorrelated increments ("volatility clusters"). That means that price changes are unforecastable but do not follow random walk because the volatility exhibits a short memory.

4. Scale invariance in financial prices

In the previous section, we have mentioned the so-called "square-root-of-time-rule" of Brownian motion. In fact, if $X(t)$ is a Brownian motion, the square-root-of-time-rule is a mathematical consequence of the scale invariance of Brownian motion. If risk is measured by the standard deviation, hence the square-root-of-time-rule means that risk at scale τ is scaled to risk at scale $a \times \tau$ by the multiplication of square root of *a*. This characteristic is heuristically described in Regnault (1863, p. 50) as follows: "The deviation of the prices increases with the square root of the time" (translation in Jovanovic and Le Gall 2001, p. 13). Regnault does not

speak in terms of stochastic processes, but if the Regnault's "deviation" can be thought as the standard deviation i.e. the "volatility" in financial jargon, the Regnault's sentence can denote in contemporary terminology that volatility scales with the square root of time.

A simple way to understand what is at stake with this scale invariance for financial modeling is to consider the following practical problem. Let us consider a given financial asset and its distribution of returns (i.e. its risk) over a given time period (one week, one month, etc.). Following our notational convention, the distribution of returns between time 0 and time t is denoted $X(t)$. Let us now consider the distribution of returns of the same asset over two time periods (two weeks, two months, etc.) knowing the distribution over both time periods and assuming that the continuous rate of return on assets follows a random walk. We are interested in the distribution of returns between time 0 and time $2t$, i.e. $X(2t)$. How do you get from a one-period risk to a cumulated two-period risk? If $X(t)$ is a random walk, then moving to the Fourier space:

$$\text{Characteristic exponent at time } 2t = 2 \times \text{Characteristic exponent at time } t$$

If the distribution of the two-week returns has the same shape as the distribution of the one-week returns, scaled by the time, the distribution is scale invariant. In this case, the random walk exhibits scale invariance, which is not the case for the other possible distributions eligible for random walks. We now turn back to the Regnault "square-root-of-time-rule" by explaining more in details its link with the random walk model.

The Regnault "square-root-of-time-rule"

The random walk model in finance is often associated with stylized fact or "patterns" observed on stock market fluctuations, "scaling laws" or "power laws"[10]. A power law is a relation of the type $Y = k X^{\alpha}$ where α is called the power law exponent, and k is a constant. A power law describes the fact that one quantity varies as a power of another. Power law exponents characterize the kind of fluctuations of the quantity changes. One of the features of the power laws is their scale invariance. This property is the origin of the second name "scaling law": scaling by a constant c simply multiplies the original power-law relationship by the constant $c^{-\alpha}$. Power laws are found in finance in distributions and correlations in financial time series. It follows from the previous section that the existence of power laws on the

[10] See for example Bouchaud 2001; Cont 2001; Mandelbrot 2001; Farmer and Lillo 2004; Lux and Alfarano 2016.

correlations pushes the modeling out of the family of random walks, since the IID hypothesis is not satisfied. We will therefore only be interested in power laws under the IID hypothesis.

The Regnault's statement describes a specific case of power law on moments. To understand in what the Regnault's rule is linked to the scale invariance of price changes, let us mathematically rewrite Regnault's sentence. The issue is to pass from the distribution of X (t) to the distribution of X (at), a multiple of time. Let λ (X, t) be a parameter of the distribution of X (t). In the case of Regnault's work, λ (X, t) is the standard deviation $\sigma(X, t)$. I write the "square-root-of-time-rule" of Regnault's sentence as following:

$$\sigma(X, a \times t) = \sqrt{a} \times \sigma(X, t) = a^{0,5} \times \sigma(X, t) \quad (13)$$

From which comes the scaling property on volatility with exponent 0.5 for square root. For a financial practitioner, this scaling property is the reason why:

$$\text{volatility}(X, 12 \text{ months}) = \sqrt{12} \times \text{volatility}(X, 1 \text{ month}) \quad (14)$$

The practical "square-root-of-time-rule" states that the annual volatility is obtained by the monthly volatility multiplied by the square root of the duration measured in months, i.e., 12.

The standard deviation is the square root of the variance, the central moment of order 2. In fact, among the parameters useful to qualify the risk of a distribution are the central moments of order k for k=2, 3, 4. These moments give an idea of the shape of the distribution (size, flat, peaked, symmetric, asymmetric, etc.). Let $m_k(X, t)$ be a central moment of order k of the random variable X (t): the Regnault relationship dealing with m_2. Now I write the Regnault sentence as follows:

$$m_2(X, a \times t) = a \times m_2(X, t) \quad (15)$$

The variance is linear in time. Hence the idea of a scaling law on stock market variations can be found in an incipient manner in the Regnault's book. More generally, we could look to obtain the moments $m_k(X, at)$ at a scale at knowing the moments $m_k(X, t)$ at scale t.

The relation (15) is at the origin of variance-ratio tests in the financial literature (Lo and MacKinlay 1988). Given the time scaling of variance in the case of Brownian motion, the variance of (say) two-period returns is twice the variance of one-period return if the random walk hypothesis holds, hence the variance ratio to be checked is:

$$\frac{m_2(X, a \times t)}{a \times m_2(X, t)} \tag{16}$$

If the variance ratio is equal to 1, $X(t)$ follows the Regnault rule. Many studies have exploited this scaling property of moments in the random walk hypothesis (e. g. Campbell et al. 1997, Charles and Darné 2009).

The square-root-of-time-rule is a mathematical consequence of the scale invariance of Brownian motion. In fact Brownian motion is a self-similar (fractal) process $B(t)$ such as:

$$B(a \times t) \equiv \sqrt{a} \times B(t) \tag{17}$$

where the symbol ≡ indicates an equality in distribution. The scaling exponent of Brownian motion is 0,5 from which it follows the Regnault law. The above equation means that the shape of the distribution of returns is invariant when the time scale is changed by the square root of time. This scale invariance of Brownian motion is also a core concept of the Bachelier (1900) model and, therefore, a core concept of neoclassical financial modeling. The Brownian motion type of the random walk allows annualizing the volatility from monthly measures of empirical volatility. The equation (17) describes a scale invariance property of the Brownian motion used in neoclassical risk modelling: in neoclassical finance, the risk is scaled by the square root of the time.

The time-scaling of risk in neoclassical finance represents the scale invariance of neoclassical finance. In the standard model of stock market variations, this feature of the Brownian motion describes one aspect of the fractal property of volatility. This is another way to say that the fractal property of financial risk is pervasive in neoclassical finance. Hence, the neoclassical finance is somewhere embedded in a fractal framework. The mostly used square-root-of-time rule of financial practitioners reveals the underlying fractal framework of neoclassical finance. The main model of pricing options, the Black-Scholes (1973) model, as using as a mathematical ingredient a Brownian motion, is embedded in an underlying fractal framework. Hence it is not possible to easily contrast neoclassical finance with “fractal finance”.

This square-root-of-time-rule has often been considered as a representation of the random walk model in finance, but one can have another scaling invariance under the IID hypothesis and outside the IID hypothesis. Let us give just an example. Instead of (17), on could have the following relation:

$$X(a \times t) \equiv a^H \times X(t) \quad (18)$$

where the scaling exponent H is the self-similarity index of the stochastic process $X(t)$. When H=1/2, X (t) is a Brownian motion. The Brownian motion is self-similar (fractal) with IID increments. When $H \neq 1/2$, the stochastic process X (t) is the fractional Brownian motion (FBM) denoted B_H (t) introduced by Mandelbrot (1965) and widely used in hydrology and climatology.

$$B_H(a \times t) \equiv a^H \times B_H(t) \quad (18)$$

In the case of FBM, the increments are identically distributed but not independent; hence in this case we depart from the random walk hypothesis. In the case of H=1/α, with stationary and independent increments, the stochastic process X (t) is an alpha-stable motion, self-similar with IID increments, and belongs to the random walk family. We can see how the fractality property can refer either to the random walk family of stochastic processes or to stochastic processes which are outside this family. To disentangle this puzzle, we now turn to the stability-under-addition property of the random walk model in finance.

The stability-under-addition property

The concept of scaling has been heuristically introduced by Augustin Cauchy in 1853 for expressing that in some cases the distribution of a random variable is invariant under non-random weighting, a form of sum of random variables. This scale invariance is satisfied by the Gaussian distribution (Regnault and Bachelier cases). The annual risk has the same shape as the monthly risk: a Gaussian distribution, scaled by the square root of time. Using financial words, this assumption means that the shape of financial risk remains Gaussian at every scale. More generally, every distribution which exhibits this invariance belongs to a family of distributions named "alpha-stable distributions" because the scaling property is defined and quantified by a characteristic exponent denoted α which varies between 0 and 2. For example, the Gaussian distribution is stable under addition: the sum of two Gaussian distributions is a Gaussian distribution. The Cauchy distribution is stable under addition: the sum of two Cauchy distributions is a Cauchy distribution. In the case of Brownian motion, α = 2 and (Regnault and Bachelier cases) in this sense the Brownian motion is an IID-2-stable process. In the case of Cauchy, α = 1 and in this sense the Cauchy motion is an IID-1-stable process[11].

[11] For detailed exposition on alpha-stable processes, see Janicki and Weron (1993), Samorodnitsky and Taqqu (1994) and Lévy-Véhel and Walter (2002).

At the beginning of the use of Lévy's processes in financial modeling, it should be noted that the vocabulary was not yet standardized. For example, in their 1991 papers, both the physicists Jean-Philippe Bouchaud and Rosario Mantegna used the term "Lévy flight" with the meaning introduced by Mandelbrot (1977, p. 113), a stochastic process with IID increments described by an alpha-stable distribution (Bouchaud et al., 1991, p. 1466; Mantegna 1991, p. 233). As they explain, the term "Lévy walk" refers to a stochastic model introduced in Physics in 1987 by Shlesinger, West and Klafter (reference [12] of Mantegna 1991 and [13] of Bouchaud et al. 1991)[12]. For Bouchaud, a "Lévy flight" is a "superdiffusion" process, a kind of anomalous diffusion in which the sum of IID increments is dominated by its largest term (Bouchaud et al., 1991, p. 1466). For Mantegna, superdiffusion is a random walk in which the variance of the increments grows faster than time. He concludes his paper by considering plausible such a stochastic process because he detected superdiffusion for the l-day variance of the index changes (Fig. 4 of Mantegna 1991). Thus the term "Lévy flight" was used into physics in 1991.

The word "fractal" is a neologism created by Mandelbrot from the Latin adjective "*fractus*", which has the same root as fraction and fragment and means "irregular or fragmented" (Mandelbrot 1977, p. 4). A fractal object is created from an initial object that is fragmented into small objects by following deterministic or stochastic rules involving internal homothety. Infinite repetition (iteration) is one of the essential aspects of fractals which are objects whose structure is invariant by certain translations (Mandelbrot 1977 p. 17). That means that the whole looks like the part, which itself looks like a smaller part. The pattern repeats itself at different scales. Some random walks exhibit fractal properties, some others not. $X(t)$ exhibits scaling characteristics if $X(t)$ is statistically identical to its transform by contraction in time followed by a corresponding change in intensity (Mandelbrot 1977, p. 254). Hence, "fractality" is another way to describe scale invariance.

5. A tale of fat tails: random walks with heavy tails

Let us now consider the random variable representing the law of IID increments at scale τ, namely $Z(t, \tau)$. If there is any topic of scientific discussion that runs through almost the entire 20th century in financial modeling, it is certainly the shape of the distribution of $Z(t, \tau)$. It has been well documented in the academic literature that, in general, the empirical distribution of

[12] Mantegna, personal communication to C. Walter, January 21, 2021.

$Z(t, \tau)$ has thick tails, too thick to be Gaussian[13]. This story is so well known today that it has been termed a "tale of fat tails" (Mandelbrot, 2001). The fat tails of empirical distributions of price changes have become an established stylized fact in the literature. I would just like to clarify two important technical points for the history of the random walk model in finance, enlightened by the choice of the Lévy measure.

The issue of extreme values

Firstly, it is important to keep in mind that the shape of the distribution has fat tails at a given time scale τ. Heavy tails are not necessarily the same on all scales. There in nothing like scale invariance in heavy tails, except in the case of fractal behavior of prices. Time scales in financial markets should be regarded as an area of study in their proper perspective (Walter 2001, 2002). But it has been shown in a large number of studies that the tails of the random variable $Z(t, \tau)$ follows a scaling law. With scale invariance, the characteristic length scale of markets fluctuations is infinite. This infinity leads to self-similar and scale-free fluctuations of prices. But in a lot of cases, scale matters and scale invariance is a too strong property of random walks. Place has to be made for other kinds of random walks, which keep the infinitely divisible property but without the scale invariance property. In fact, the so-called "scaling anomalies" seem to contradict the Mandelbrot's scale-invariant fractal model. The problem that appears for our purpose is that of the relationship between random walks, distribution tails and scaling laws.

But there are a huge number of ways to make fat tails occur on random walks. This is the second element of our comments, which refers to the modeling of extreme values[14]. The technical relationship between Lévy processes and extreme values is described in Le Courtois and Walter (2017).

The dialogue between theories

Addressing the history of the random walk model in financial thought through the consideration of heavy tails of distribution by highlighting them through the Lévy measure allows for a dialogue between the different competing theories that seek to describe these heavy tails. We will now show how this makes it possible to consider the models of financial

[13] A good introduction to heavy tailed distributions in finance is the handbook edited by Svetlozar Rachev (2003).

[14] A good introduction to extreme values in finance is the handbook edited by François Longin (2017).

mathematicians and the models of physicists as a single conceptual grid for the analysis of stock market variations. That is to say, thanks to the use of the Lévy measure, we are going to put forward the idea that the financial mathematicians' models and the physicists' models only differ in reality by the sociological origin of their scientists, but not by a different approach to the modeling of stock market dynamics. Hence, if we consider the Lévy measure of the models chosen by the physicists, we can observe that this measure is identical to that of the models implemented by the second generation of financial mathematicians. For example, the Lévy measure of the Koponen (1995) model is the same as that of the CGMY Carr et al. (2002) model[15]. The Lévy measure of the symmetric VG model of Madan et al (1998) is a special case of the Lévy measure of the Koponen (1995) model.

6. Conclusion

The history of the random walk model in finance is marked by a certain number of confusions or ambiguities in the different periods of its evolution during the 20th century. We have introduced a new historical "lens" to shed new light on this long history, with the use of a mathematical tool crucial for the description of random walks but never before used in the history of the random walk model, the Lévy measure.

With this new technical tool, we were able to establish the relationships between the different variants of the random walk model in finance. We introduced a triplet characteristic of random walks, the "triplet T". This triplet allowed us to contextualize Mandelbrot's work in relation to fractals and the scientific controversies over fat tails in empirical distributions of financial data.

Such a study of the random walk model in finance allows access to an epistemological perspective that is quite different from those that have been used to examine it up to now. We hope that this paper will be an encouragement to continue in this direction with the use of the Lévy measure, in order to pave the way for future historical and epistemological work on the history of the random walk model in finance, and more generally on the history of financial thought itself.

[15] See the precise mathematical expressions in Le Courtois and Walter (2014, p. 102).

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