



**HAL**  
open science

# Market Efficiency, Risk Neutral Pricing and Choice Among Representations: a "mini-model"

Christian Walter

► **To cite this version:**

Christian Walter. Market Efficiency, Risk Neutral Pricing and Choice Among Representations: a "mini-model". 2024. hal-04578320

**HAL Id: hal-04578320**

**<https://hal.science/hal-04578320>**

Preprint submitted on 16 May 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0  
International License

# Market Efficiency, Risk Neutral Pricing and Choice Among Representations: a “mini-model”

Christian WALTER<sup>1</sup>

January 12, 2024

## Abstract

The so-called “risk-neutral probability” is a technical tool that has received considerable attention in financial practices over the past thirty years, to the point that risk-neutral pricing techniques are now a routine part of the day-to-day business in the finance industry. I present an simplified “mini-model” of risk-neutral pricing to allow the black box to be opened to philosophical investigation by displaying without any mathematics the fair value pricing mechanism in a complete arbitrated free market. I argue that the difficulties of the puzzling methods used to value financial assets using risk-neutral valuation techniques are more conceptual than mathematical, and discuss the epistemological issues enlightened by this mini-model, particularly in terms of narratives. The mini-model reveals how the Efficient Markets Hypothesis (EMH) is a narrative imposed on the real finance, and how the EMH narratives correspond to each other as quantification conventions which draws a “representation format” of EMH. It raises the ethical question of choosing a narrative for financial purposes, particularly when ethical, climatic and environmental issues are at stake.

Keywords: Risk-Neutral pricing, Efficient Market Hypothesis, Fair value, Quantification convention, Representation format

JEL codes: A2, B16, B26, B41, C6, G12, G14

---

<sup>1</sup>ISJPS, universit  Paris 1 Panth on-Sorbonne, Af2i (Association fran aise des investisseurs institutionnels). E-mail: christian.walter@live.fr

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>The mini-model</b>	<b>6</b>
2.1	Extremely simple models . . . . .	6
2.2	The economy . . . . .	7
<b>3</b>	<b>The mini-model at work</b>	<b>10</b>
3.1	Trading strategies and financial arbitrage . . . . .	10
3.2	The change of probability . . . . .	13
<b>4</b>	<b>The alternative representations of EMH</b>	<b>17</b>
4.1	The three discounting conventions . . . . .	18
4.2	Dialogue with the general equilibrium theory . . . . .	23
4.3	The representation formats of EMH . . . . .	25
<b>5</b>	<b>Conclusion</b>	<b>27</b>

# 1 Introduction

The so-called “risk-neutral probability” is a technical tool that has received considerable attention in financial practices over the past thirty years, to the point that risk-neutral pricing techniques are now a routine part of the day-to-day business in the finance industry. The risk-neutral probability is a probability measure in a dual world, in general written down as  $Q$  in the financial literature, in distinction to the “real” (or “physical”) probability of the phenomenon under consideration, denoted  $P$ . The nature of probability  $Q$  is well understood in the mathematical theory of finance. There are numerous textbooks which provide the detailed technical background to this (Cochrane 2001, Dothan 1990, Föllmer and Schied 2002, Karatzas and Shreve 1998, Pliska 1997). The shift from  $P$  to  $Q$  has been extensively described in mathematical finance with the key role of the Radon-Nikodym operator  $L = Q/P$  known in financial literature as the “state price density” (SPD). Mathematical finance has made extensive use of this technique, via the Girsanov theorem. In the case of the standard model of price fluctuations using the Brownian representation for price dynamics (Walter 1996, 2001, 2019), since Brownian motion is a continuous stochastic process, Girsanov’s theorem enables the probability to be changed from the “physical” world ( $P$ ) to the “dual” world built by the risk-neutral probability ( $Q$ ). As shown by Harrison and Kreps (1979), the SPD exists if the market is arbitrage-free (absence of arbitrage opportunity, AOA), and it is unique if the market is complete. This uniqueness is the core condition for obtaining the “fair value” of any asset or liability and, reciprocally, a value is assumed “fair” if and only if the SPD is unique. Hence there is a “*va-et-vient*” between the change of probability and the notion of fair value. The risk-neutral probability  $Q$  became a convenient way of determining the “fair value” of given assets or liabilities.

In financial economics, fair value is the financial consequence of the notion of the so-called “Efficient Market Hypothesis” (EMH) alternatively known as the Efficient Market Theory. If EMH holds, it is possible to state that the market price of a given asset is “fair”. With EMH, one can hypothesize that the market is a “good” mechanism for pricing any asset or liability. The history of financial thought has deeply examined the formation and the development of EMH (for a recent philosophical approach, see Walter 2017). EMH has been extensively discussed since its introduction by Eugene Fama in 1970 (Fama 1970). In particular, the joint hypothesis problem is now well identified (Jarrow and Larson 2012), which raises issues for Hempel’s criteria for the logic of test of a hypothesis in the case of auxiliary hypotheses (Hempel 1966, p. 22). To be falsified (Popper), EMH must be tested jointly with a given model for expected returns (Fama 1991) and another given model for

expected risk (Walter 1996).

A large number of mathematical models exist to portray the situation of an informationally efficient market, as alternative representations of EMH. All these “EMH-models” have been widely documented. There is a lot of discussion and controversy about these models. In general, one works on rather complex models, and controversies do not ignore this complexity. This complexity is necessary for the accuracy of the models. It is useful in order to better apply a selected EMH-model to a given business context (e.g. among others Boyle 2001, Habart-Corlosquet et al. 2013, 2015, McNeil et al. 2015, Ottaviani 1995, Reghai 2015) in a specific “culture of models” (Pickering 1992) with a practice-oriented approach to modelling for decision-making (Svetlova and Dirksen 2014). The representation of a given mathematical framework plays a key role in testing EMH and in decision making (asset valuation, risk metrics, portfolio management).

Two main mathematical frameworks exist for mathematising EMH, I name the  $P$ -world and the  $Q$ -world. These two frameworks have been characterised in Chiapello and Walter (2016) as two “quantification conventions” (Desrosières 2003) of the real finance, in the sense that a quantification convention for financiers is a metrology that allows them to work easily on a day-to-day business. These two quantifications are respectively the “mean-variance convention” in the  $P$ -world and the “market consistent convention” in the  $Q$ -world. The periodisation introduced in Chiapello and Walter (2016) started with the actuarial convention, i.e. before the development of modern financial theory and EMH. Here I consider only the period starting after the birth of the financial theory and EMH. This theory evolved through two quantifications, from the  $P$ -world to the  $Q$ -world. Hence, these two quantifications conventions act as the first and second quantification of the EMH. In this paper, the  $P$ -world is named the first quantification convention of EMH and the  $Q$ -world is named the second quantification convention of EMH. The paper expands and elaborates on Chiapello and Walter (2016) by adding to it a simplified integrated approach to these two conventions, exhibiting the correspondence between these two worlds.

The fact that the probability  $Q$  is widely used in financial markets for pricing derivatives is well known. It is also well known that  $Q$  is the cornerstone of all the techniques of contemporary finance. For example, Accounting Standard Codification ASC paragraph 718-10-55-16: “Established principles of financial economic theory represent fundamental propositions that form the basis of modern corporate finance for example, the time value of money and risk-neutral valuation”. It is not as well known that it is also used in the framework of financial regulation. In fact, a very interesting aspect of the practice of using probability  $Q$  is that this mathe-

mathematical concept is fully incorporated into a prudential regulatory framework. The calculation technique under  $Q$  is the backbone of financial regulations implementing the MCV such as the European Solvency II Directive. Hence, the use of  $Q$  has also become a reality in areas seemingly distant from the derivatives markets. For example, a “market consistent value” (MCV) of an asset or liability is its market value, “if its is readily traded on a market at the point in time that the valuation is struck and (...) a reasoned best estimate of what market value would have been had it been readily traded at the relevant valuation point” (Kemp 2009). Given the importance of  $Q$  in the regulatory framework of finance, it seems that  $Q$  is a key part of the machinery of finance if we look at the “view from inside” of finance (Ippoliti 2017, p. 121) namely rules, laws, institutions, and regulators. The understanding of how the SPD works is important to the accurate understanding of the financial system.

Given the role that  $Q$  played in the transition of finance from the 1960s to the 1990s and in the regulatory framework, as well as in the possible financialisation of the economy, it would be important to be able to explain the stakes of the changes of worlds without needing to use sophisticated mathematics. But in this literature, there are many mathematical barriers to a simple approach to understanding the change of probability. In most cases, asset pricing models are expressed in the language of continuous-time stochastic differential equations using a martingale approach. The richness of these models allows for a general formula but the complexity of these models is high. It could be useful to simplify these complex models by constructing a simplified artificial world. Hence, my aim here is to use a elementary static “mini-model” for risk-neutral pricing allowing to illuminate with a extremely simple approach the correspondence between the  $P$ -world and the  $Q$ -world, as well as the causes of the uniqueness of  $Q$  with AOA condition. This minimalist view allows to cognitively grasp the relations between the two frameworks  $P$  and  $Q$  and their mathematical representations, the  $P$ -world and the  $Q$ -world, without any mathematical complexity. The paper is organised as follows.

Section 2 introduces the mini-model and the economy. Section 3 uses the mini-model and describes the arbitrage mechanism. I present the arbitrage effect on prices, the change of probability and why the probability  $Q$  becomes unique after arbitrage. Section 4 presents the different representation formats of the EMH with their discounting convention, their mathematical correspondence and the link between CAPM, risk-neutral probability  $Q$ , market pricing kernel and Arrow-Debreu securities in the mini-model. Section 5 concludes and opens the debate by presenting the epistemic contributions of the mini-model for non-specialists and financial practitioners.

## 2 The mini-model

The static mini-model introduced here is a mini “EMH-model”. It allows to deeply understand the idea of EMH in a extremely simple artificial case.

### 2.1 Extremely simple models

From an epistemological standpoint, an highly idealized and extremely simple model is sometimes called a “toy model” (Reutlinger et al., 2018). According to these authors, there are two kinds of toy models, named “embedded toy model” and “autonomous toy model” (Reutlinger et al., 2018). An autonomous toy model is designed without any background theory, as for instance the canonical Schelling’s checkerboard model for segregation (Schelling 1971). An embedded toy model is a that of “empirically well-confirmed framework theory”. According to Reutlinger et al. (2018), the mini-model I present here is an embedded toy model. The theories I use are that of the framework of mean-variance optimisation introduced by Harry Markowitz (1952) for portfolio management, leading to the Capital Asset Pricing Model (CAPM) of William Sharpe (1964), and the framework of risk-neutral pricing introduced by Harrison and Kreps (1979), Harrison and Pliska (1981).

From a mathematical standpoint, I use a well-known framework in financial modelling since the Cox, Ross and Rubinstein (1979) model, the binomial one, including two risky assets, one money market and two scenarios for the future. The relevance of the binomial framework is very known today. This framework is proving very useful in explaining the functioning of the pricing methods. With respect to the existing literature on the binomial framework in finance and the very numerous examples that can be found on the field, my aim here is to carry this framework as far as possible by using it to the point of highlighting the values of the Radon-Nikodym operator  $L$  in a static model with a binomial framework. The static mini-model allows to grasp the signification of the Radon-Nikodym function without mathematical expertise, enlightening the interplay between these two mathematical representations that are used in day-to-day financial practices. The novelty in the paper lies in the way I demonstrate the connection between the representations of the two worlds (“real” and “dual”) achieved by presenting numerical values and linking them with the CAPM numbers

From an educational standpoint, the proposed mini-model is a piece which provides access to a theoretical overview of the mathematical scene of neoclassical finance to non-experts in mathematical finance and stochastic calculus. It provides

an easy epistemic access to understanding the key role of the risk-neutral probability  $Q$  in work related to asset pricing models. Hence my ambitiously stated proposal is that the epistemic goal of my mini-model is to enable non-specialists of stochastic calculus (“non-quant” finance practitioners or researchers in the humanities and social sciences) to understand some of the epistemic difficulties of the risk-neutral pricing, namely the intertwin between the  $P$ -world and the  $Q$ -world and the uniqueness of  $Q$  with AOA in complete markets.

A possible objection to the relevance or usefulness of this mini-model is that an extremely simplified model is too simple or too abstract to be explanatory (Cartwright 2009). But many scientists consider that an abstract and simplified model sheds light on important aspects that are often obscured by the current use of complex models, even if the simplifications make the model highly unrealistic (Ylikoski and Aydinonat 2014). Abstract and simplified idealized models have an important role in many sciences and creating an stylized world can contribute to the scientific understanding of complex concepts. Here, the mini-model allows to simply but deeply understand the strange mechanism of uniqueness of  $Q$  after arbitrage in complete market, it allows to debunk the mathematical trick which is at the basis of any pricing in modern finance.

I now present the mini-model and its use to exhibit extremely simply the uniqueness of risk-neutral probability  $Q$  with AOA. The uniqueness of  $Q$  emerges from a rule-of-three calculation, without mathematical puzzles. The two values of the Radon-Nikodym operator  $L$  (state price density) are calculated in this very simple framework. The epistemic value of the quotes has changed after arbitrage.

## 2.2 The economy

Let us begin with the information set. Fama (1970)’s celebrated review about EMH divided the notion of market efficiency into three categories: (1) weak-form (tests to scrutinize if past returns predict future returns), (2) semi-strong-form (tests to check if security prices reflect public information announcements), and (3) strong-form (investigation if investors have private information that is not fully reflected in market prices). For the simplicity of my purpose, I now assume that all the available information is used to elaborate scenarios for future values of assets. Note that, instead of information about companies, it could be possible to use data sets as information, following the case-base decision theory (Gilboa and Schmeidler 2012).

It should be noted that there is no guarantee that the forecasted future values have any connection with the economic situation of the companies. It would be



equally possible to imagine future forecast values that are not based on financial analysis calculations, but on rumours about potential developments in the companies, without any business scenario calculations. To put it differently, the forecasts can be either “rational expectations” or any kind of rumours or “noise”. The agents can be rational investors, noise traders or chartists. For the purpose of the mini-model, at this stage, this distinction is not crucial.

Let us continue with the description of uncertainty. We assume that uncertainty of the outcome is measurable and, following Frank Knight (1921, for recent debates see Diebold, Doherty and Herring, 2010, Svetlova 2021 and Dimand 2021), we use the term “risk” to designate measurable uncertainty. That means that the probability distribution of the outcome is known, “either through calculation *a priori* or from statistics of past experience” (Knight 1921, p. 233). We use the Arrow-Debreu (1954) framework of a general equilibrium under uncertainty. Uncertainty about the outcomes of an action is described as choice over state-contingent outcomes by a finite set of well-defined and perfectly known states of the world representing the known “unknowns” denoted  $\Omega$  with  $\omega$  being a generic element. An observable subset of  $\Omega$  is called an “event”. It can be a single state or a collection of states. The scenarios for the contingent futures can be thought as possible states or “events” of the future. It is well known that this theory was extended by the subjective probability distribution introduced by Savage (1954) and criticised by Ellsberg (1961) and later Kahneman and Tversky (1979) to take account of the many puzzles raised by the Arrow-Debreu approach. Nevertheless, in order to address issues relating to EMH and the mini-model without too much complexity, we will remain here within the classic framework. We can, nonetheless, note that this extended framework makes it possible to interpret the risk-neutral probability  $Q$  as the objectification of an implicit and shared subjective measure of the market.

Let us imagine a company A for which financial analysts have elaborated at date  $t = 0$  scenarios for a given time horizon, say  $t = 1$ . It is possible to understand these scenarios as subjective beliefs of analysts, i.e. subjective predictions about the occurrence of the states the world. For the simplicity of the model, we suppose there are only two scenarios, growth or decline, corresponding to pessimistic forecasts and optimistic forecasts. The set  $\Omega$  of scenarios is (“down”, “up”) written as  $\Omega = \{\omega_d, \omega_u\}$ . On the date  $t = 0$  of the valuation, the share price of company A quoted on the stock exchange (or its estimated value if the company is not quoted on a stock exchange) is  $A_0 = 150$  euros. In the first scenario (an optimistic version of the future of the company), the value of the share at  $t = 1$  would be, for example,  $A_1(\omega_u) = 200$  euros. In the second scenario (more pessimistic about the business prospects), the share value would fall to  $A_1(\omega_d) = 140$  euros.

Real world probability $P$		0.50	0.50
Scenario		pessimistic	optimistic
Type of the asset	Asset price at $t = 0$	Forecast for $T = 1$ (epistemic values)	
A-company share	150	140	200
B-company share	66	50	120

Table 1: The price system of the mini-model of economy before arbitrage

Let  $P(\omega)$  be the probability of the scenario  $\omega$ . The probability  $P$  is the probability of the “phenomenon” related to the future of the company, the future state of “event” of the “physical world” or “real world”. The chosen framework guarantees that the occurrence of future events can be calculated with probabilities. In this conceptual framework, the future share value  $A_1(\omega)$  is a random variable depending of the event  $\omega$ . Suppose that neither scenario is preferred to the other: the two business prospects are assumed to be equally likely, and the probability of each scenario is therefore  $P(\omega_u) = P(\omega_d) = 0.50$ . Of course, this probability could be different (see Knight above) and any other values would be possible as long as the sum of the probabilities of each scenario remains equal to 1. The two probabilities of 0.50 applied to the two forecast values of 140 and 200 euros represent the probability distribution of the future value of the company. Since there are two possible outcomes for the future value of the shares, each weighted by a probability, it is termed a “risky prospect”. The word “risky” means that the uncertain future is calculated with probabilities.

Let us now enlarge our example by considering the “economy”. To keep the toy nature of the model and to preserve the mathematical simplicity of the example, we limit the economy to two companies. There is a second company denoted B which, like the first, is the subject of financial forecasts. Let us imagine that the share price at  $t = 0$  is  $B_0 = 66$  euros. In the first business scenario, at  $T = 1$  the share value would be, for example,  $B_1(\omega_d) = 50$  euros. In the second scenario, it would rise to  $B_1(\omega_u) = 120$  euros. The market is complete. The probability of the scenarios is the same as for company A. It exists a money market in the economy. This market provides a risk-free rate, i.e. a rate of return whose value does not depend on the scenarios, unlike the risky prospects. That means that, whatever the scenarios, the future value of the money market asset will be the same. The risk-free rate is equal to 2%.

Table 1 summarises the “mini-economy” with both risky prospects. Table 2 present the money market. In table 1, the matrix  $[140, 200 ; 50, 120]$  represents

Real world probability $P$			<b>1</b>
Type of the asset	Asset price at $t = 0$	Risk free rate	value for $T = 1$
Money market risk free rate	1	2%	1.02

Table 2: The money market of the mini-model of economy

the forecasts based on the information or rumours available to the participants of market, yielding the price system [150 ; 66] . In this sense, [140, 200 ; 50, 120] are the “epistemic values” of the quotes [150 ; 66]. In table 2, a risk-free asset of value 1 at  $t = 0$  produces 1.02 at  $T = 1$ .

### 3 The mini-model at work

We now use the mini-model to show in an extremely simple way how arbitrage in a complete market generates the uniqueness of the risk-neutral probability  $Q$ .

#### 3.1 Trading strategies and financial arbitrage

In the mini-model, a trading strategy is an asset allocation of funds between the two companies A and B. It is possible to define this portfolio with number of shares or weight of companies.

The notion of arbitrage is crucial in finance and financial theory. It is the cornerstone of the option pricing theory due to Black and Scholes (1973) but more generally for pricing any asset in practice. In finance, a financial arbitrage is a trading strategy that takes advantage of price inefficiencies, i.e. a trading strategy that allows obtaining a sure gain without bearing the slightest risk.

##### 3.1.1 Searching for arbitrage

Is there an arbitrage opportunity in this mini-economy? In general, a market is screened for arbitrage by exploring the possibilities of obtaining a sure (non-random) return using a mix of risky prospects bearing random returns. In the mini-model presented, a trading strategy will consist on an asset allocation between the two companies A and B that allows obtaining a sure return. If there are price inefficiencies in the price system (150, 66), there will be opportunities to obtain a sure

return greater than the risk-free rate (2%) with an appropriate portfolio.

Let us imagine that we now seek such a trading strategy. We aim to build a portfolio of A and B such that, whatever the future scenarios, the value of this portfolio is 100,000 euros at date 1 whatever the future state of the world, i.e. it will be “risk-free”. We have to find the holdings of companies A and B. Let  $\theta_A$  and  $\theta_B$  be the quantities (“holdings”) of the shares A and B at date 0. Given the price system (150, 66), the value of this portfolio at date 0 is:

$$V_0 = \theta_A \times 150 + \theta_B \times 66 \quad (1)$$

At date 1, the value of this portfolio is depending on the scenario  $\omega$ :

$$\theta_A \times A_1(\omega) + \theta_B \times B_1(\omega) = V_1(\omega) \quad (2)$$

We want  $V_1=100,000$  whatever the scenario  $\omega$  is. This leads to a system of two equations:

$$\begin{cases} \theta_A \times 140 + \theta_B \times 50 & = & 100,000 \\ \theta_A \times 200 + \theta_B \times 120 & = & 100,000 \end{cases} \quad (3)$$

whose solution is  $\theta_A = 1,029.41$  and  $\theta_B = -882.35$ . The negative sign of  $\theta_B$  means that we have to “short” the shares of company B. A short position is created when a trader sells a security first with the intention of repurchasing it at the end of the arbitrage. One can easily check that whatever the possible outcomes, the portfolio will be valued at 100,000 euros on date 1:

$$\begin{cases} 1,029.41 \times 140 - 882.35 \times 50 & = & 100,000 \\ 1,029.41 \times 200 - 882.35 \times 120 & = & 100,000 \end{cases}$$

At date 0, following (1) the value of this portfolio is:

$$V_0 = 1,029.41 \times 150 - 882.35 \times 66 = 96,176.47$$

The return of this portfolio is:  $100,000/96,176.47 - 1 = 3.98\%$  without any uncertainty about the result. For this reason, the return of 3.98% is said “risk-free”. Such a situation is very interesting: it would be enough to borrow from the money market (2%) the amount needed to buy this portfolio, to obtain the sure return (3.98%) without risk, and then to cash in the difference. It is therefore an anomaly in the market, since it would be possible to make very large gains without taking any risk: just buy very large quantities of A, borrow the money to finance the purchase, and short as much B as necessary. With  $\theta_A = 1,029.41$  and  $\theta_B = -882.45$ , the resulting gain is 1,900 euros. That means that you can make 1,900 euros with zero euros spent, a kind of “free lunch”.

Such an anomaly represents an inconsistency in the price formation according to the target prices forecasted in the case of EMH. Following Fama (1970), that means it exists some information which is not reflected in the price system (150,66) at date  $t = 0$ . This inefficient situation cannot last long because sooner or later someone (a trader, an investor, an arbitrageur) will detect it and intervene in the market to take advantage of it. By doing so, he will eliminate the inefficiency so that, at the end of the trading strategy, the market will be “cleared” of all possible arbitrage. To put it differently using the financial jargon, the condition of “absence of opportunity of arbitrage” (AOA) will be satisfied. Let us now move inside the mechanism of arbitrage, the principle that produces the market equilibrium, that is to say, what steps are required to produce the equilibrium.

### 3.1.2 Implementing arbitrage and effect on prices

The effect of the arbitrage is to move prices. Buying 1,029.41 shares of company A on the market will have the effect of putting upward pressure on the price, while selling 882.35 shares of company B will have the opposite effect. For example, let us imagine that the new price system is (151.25, 65.75). The implied returns resulting from  $\theta_A$  and  $\theta_B$  with this new price system would be  $2.37\% > 2\%$ . The market is not arbitrated since it is possible to find a positive risk-free rate that allows for a sure gain without an initial capital payment.

What would be the right equilibrium price level? By toying with the model, we realize that we have to do a specific hypothesis on the price changes. Suppose that only the price of company B changes. This may mean that the information on company B was not fully taken into account in the first valuation. In this case, company B was yielding too low a return relative to company A. The arbitrage will have the effect of increasing the return of company B. The price of company B will fall until its new value is low enough that any arbitrage no longer yields a definite return higher than that of existing risk-free investments.

The arbitrage mechanism is the following. Keeping in mind that:

$$1,029.41 A_1 - 882.35 B_1 = 100,000$$

whatever the state of the economy at date  $t = 1$ , we should therefore expect to have at  $t = 0$  a return of 2% corresponding to 98,039.22 euros :

$$1,029.41 A_0 - 882.35 B_0 = \frac{100,000}{1.02} = 98,039.22$$

$t = 0$			Forecast values for $t = 1$	
Before	After		pessimistic	optimistic
<b>Arbitrage</b>		Real-world probability $P$	0.5000	0.5000
150.00	150.00	Price of company A	140	200
66.00	<b>63.89</b>	Price of company B	50	120

Table 3: The price system of the mini-economy before and after arbitrage

Only the price of company B changes. In this case, the price adjustment mechanism would lead to:

$$B_0 = \frac{1}{882.35} (1,029 A_0 - 98,039.22) = 63.89$$

We find that the value of B after arbitrage is  $B_0 = 63.89$  euros. Through the intervention of arbitrageurs, prices have readjusted to a new price level, leading to the new price system of table 3. The valuation of B at the new price 63.89 is called “market consistent valuation” in the regulatory framework (Wuthrich et al. 2015).

We notice that arbitrage is always relative and does not indicate what the return of company B “should be” from an economic point of view. It is just a test of the consistency of investors’ expectations on the two companies simultaneously, whether these expectations are economically sound or not. If company A was different, another coherence would have arisen from the market with the couple (A,B).

## 3.2 The change of probability

Any asset pricing model stems from one simple concept: price equals expected discounted payoff. This leads to the present value equation.

### 3.2.1 The present value equation

The expected payoff of company A at date  $t = 1$  is  $140 \times 0.50 + 200 \times 0.50$ . It follows that the share value 150 at date  $t = 0$  is generated by the discounting of expected payoff at date  $t = 1$ . One can write the “present value equation” for company A:

$$150 = \frac{140 \times 0.50 + 200 \times 0.50}{1 + 0.1333} \quad (4)$$

The rate of 13.33% is the discount rate used to equate the expected future values at date  $t = 1$  and the quoted price at date  $t = 0$ .

In the case of company B, the same calculation would yield the following present value equation:

$$66 = \frac{50 \times 0.50 + 120 \times 0.50}{1 + 0.2879} \quad (5)$$

The rate of 28.79% is the discount rate used to equate the expected future values and the quoted price.

### 3.2.2 The present value under $P$ and under $Q$

Now imagine that the beliefs about the probabilities of future scenarios are not the same. Instead of having an equally weighted probability distribution (50, 50), imagine that the beliefs overweight the pessimistic scenario, for example (70, 30). Considering the present value equation, there are two options: change the share price at date 0 or change the discount rate. If the share price is fixed, the present value equation becomes:

$$150 = \frac{140 \times 0.70 + 200 \times 0.30}{1 + \text{new discount rate}} \quad (6)$$

It is straightforward to see that these constraints give the value of 5.33% for the new discount rate. We see that, given the initial share value and the expected scenarios, there is a relationship between the probability distribution and the expected rate of return.

But it can also be considered that, in this case, the expected rate of return of 5.33% for company A is not the “right” one, in the sense that it does not reflect what can be expected of a return in this economic area. The next question could be: if a rate of return is specified, what should be the probabilities of the scenarios? The answer is given with the following equation:

$$150 = \frac{140 \times p + 200 \times (1 - p)}{1 + \text{specified discount rate}} \quad (7)$$

We can see that, once the discount rate is fixed, the probability values follow.

Let us now imagine that we decide on the expected return of company A as the risk-free rate of return, 2%. This unusual situation (because company A is risky for the investor, it is a risky prospect) would mean that the investor does not demand a risk premium for buying company A. This would mean that the investor is not affected by risk-taking, that he is “risk neutral”. In this case, the present value equation becomes:

$$150 = \frac{140 \times p + 200 \times (1 - p)}{1 + 0.02} \quad (8)$$

$t = 0$		Forecast values for $t = 1$	
		pessimistic	optimistic
	Real world probability $P$	0.5000	0.5000
150	Price of company A	140	200
66	Price of company B	50	120
	Risk neutral probability $Q_A$	0.7833	0.2167
	Risk neutral probability $Q_B$	0.7526	0.2474

Table 4: The  $Q$  values in the mini-economy before arbitrage

It is straightforward to see that this constraint gives the value of 0.7833 for the new probability, so that:

$$150 = \frac{\mathbf{140} \times 0.7833 + \mathbf{200} \times 0.2167}{1 + 0.02} \quad (9)$$

This new probability distribution (0.7833, 0.2167) is called the “risk-neutral probability” for company A. This new probability depends on company A, as its values are extracted from the market calibrated to the quoted price (or value) of company A. For this reason, it is denoted  $Q_A$  with the subscript A. The price 150 is now a present value in a “ $Q_A$ -world”. The informational market equilibrium is written here in the “dual” world using the probability  $Q_A$  in the asset pricing equation.

But, it could be argued, risk aversion has not vanished from the psychology of investors. We can see that the risk aversion has been mathematically transferred from the risk premium to the new probability distribution, from the denominator of the present value to the numerator. In the  $Q_A$ -world, the risk aversion is present in the new probability distribution.

In the case of company B, the same calculation would yield the following present value equation:

$$66 = \frac{\mathbf{50} \times 0.7526 + \mathbf{120} \times 0.2474}{1 + 0.02} \quad (10)$$

The probability  $Q_B = (0.7526, 0.2474)$  resulting from risk neutral constraint on company B is not the same as the probability  $Q_A = (0.7833, 0.2167)$  resulting from risk neutral constraint on company A. Thus, the constraint of calculation with risk neutrality leads to as many  $Q$  distributions as there are companies. In the mini-model with two companies, we find two risk neutral probabilities  $Q_A$  and  $Q_B$ .



### 3.2.3 The uniqueness of $Q$ after arbitrage

Again, one might argue, what is the point of calculating a present value that gives the same result but with a different probability if, in addition, this probability depends, like risk premiums and betas, on each company? The answer is given by the transformation of prices after arbitrage. Once the market is arbitrated, the probability  $Q$  becomes unique.

With the arbitrated price of company B, the present value calculation yields the new present value equation:

$$63.89 = \frac{50 \times 0.50 + 120 \times 0.50}{1 + 0.3304} \quad (11)$$

The rate of 33.04% is the discount rate used to equate the expected future values and the quoted price in the arbitrated market.

If we now recalculate the probability  $Q_B$  from the equilibrium price of B after arbitrage, we find that  $Q_B$  (pessimistic scenario) = 0.7833 and  $Q_B$  (optimistic scenario) = 0.2167, hence we get the following:

$$63.89 = \frac{50 \times 0.7833 + 120 \times 0.2167}{1 + 0.02} \quad (12)$$

Let us recall that  $Q_A$  (pessimistic scenario) = 0.7833 and  $Q_A$  (optimistic scenario) = 0.2167. We observe that, after arbitrage,  $Q_A(\omega) = Q_B(\omega)$  for each scenario  $\omega$ . Let us denote  $Q$  the unique risk-neutral probability of the arbitrage-free economy. The uniqueness of the probability  $Q$  derived from the arbitrated market is due to the market is complete. This uniqueness allows both companies A and B to be valued with the same discount rate, the risk-free rate. This is the “risk neutral valuation” or “risk neutral pricing”. The uniqueness of the probability  $Q$  is closely linked to the absence of arbitrage, the “no arbitrage condition” (“absence of arbitrage opportunity”, AOA). The risk-neutral probability  $Q$  is the mathematical “natural child” of the EMH.

The change in probability measure is the cornerstone of rational pricing. This calculation technique is used to transform “real world” probability  $P$  into risk-neutral probability  $Q$ . Once the uniqueness of  $Q$  is obtained after arbitrage if the market is complete, all variables relevant to financial calculations (pricing, hedging, position fixing etc.) are requantified. How this requantification works is the role of the state price density  $L = Q/P$ , the mathematical operator which transforms the  $P$ -world into the  $Q$ -world. We can see that moving from  $P$  to  $Q$  is equivalent to overweighting

$t = 0$		Forecast values for $t = 1$	
		pessimistic	optimistic
	Real world probability $P$	0.5000	0.5000
150	Price of company A	140	200
63.89	Price of company B	50	120
	Risk neutral probability $Q_A = Q$	0.7833	0.2167
	Risk neutral probability $Q_B = Q$	0.7833	0.2167
	State Price Density $L$	1.5666	0.4334

Table 5: The uniqueness of  $Q$  in the mini-economy after arbitrage

the pessimistic scenario by  $0.7833/0.5000 = 1.5666$ . More generally, we can calculate the ratios of the shift from the  $P$ -world to the  $Q$ -world for the probabilities of each scenario:  $Q$  (pessimistic)/ $P$  (pessimistic) = 1.5666 and  $Q$  (optimistic)/ $P$  (optimistic) = 0.4334. As  $L(\omega) = Q(\omega)/P(\omega)$ , we have  $L$  (pessimistic) = 1.5666 and  $L$  (optimistic) = 0.4334. The mini-model allows to easily calculate the values of the state price density.

Hence, the mini-model confronts the two quantifications of EMH provided by the use of  $P$  and  $Q$  in the same very tangible and simple situation, with a stylized elementary market moving towards equilibrium.

## 4 The alternative representations of EMH

Now that we have described the mini-model and the impact of arbitrage on prices, we move on to the pricing of assets. A common definition of asset pricing is that of Cochrane (2001, p. xiii): “asset pricing theory tries to understand the prices of values of claims to uncertain payments” adding that “a low price implies a high rate of return, so one can also think of the theory as explaining why some assets pay high average returns than others”.

The alternative representations of EMH are alternative asset pricing methods using a given discounting convention. We now elaborate on this.

## 4.1 The three discounting conventions

At the most general level, a discount factor is just a variable that generates prices from payoffs. It has been shown in Chiapello and Walter (2016) that three “discounting conventions” have organized the history of financial practices. The discount factor has been transformed by the emergence of each discounting convention. The mini-model allows to grasp the nature of these changes.

### 4.1.1 The actuarial convention

The first discounting convention is the “actuarial convention”. With this convention, the present value is determined through a simple calculation: known cash flows were discounted to present value using a constant interest rate. The discount factor is deterministic.

As seen above in the present value equation (4), the share value 150 is generated by the beliefs and the discounting of possible future values:

$$150 = \frac{\mathbf{140} \times 0.50 + \mathbf{200} \times 0.50}{1 + 0.1333}$$

We now rewrite this equation by highlighting the discount factor:

$$150 = \mathbf{140} \times \frac{1}{1.1333} \times 0.50 + \mathbf{200} \times \frac{1}{1.1333} \times 0.50$$

The quantity  $1/1.1333 = 0.8824$  is called the discount factor of company A. We rewrite the previous relationship by making the discount factor visible:

$$150 = \mathbf{140} \times 0.8824 \times 0.50 + \mathbf{200} \times 0.8824 \times 0.50$$

The equilibrium price 150 is the mathematical conditional expectation of the payoff given the information set, “discounted” using the discount factor 0.8824. In Fama’s view (1970), this relation says that 150 “fully reflects” the available information on A.

Doing the same thing with company B, we find:

$$63.89 = \mathbf{50} \times 0.7517 \times 0.50 + \mathbf{120} \times 0.7517 \times 0.50$$

The quantity  $1/1.3304 = 0.7517$  is the discount factor of company B.

Now it is possible to write the price system with company-specific discount factors:

$$\begin{cases} 150.00 = \mathbf{140} \times 0.8824 \times 0.5000 + \mathbf{200} \times 0.8824 \times 0.5000 \\ 63.89 = \mathbf{50} \times 0.7517 \times 0.5000 + \mathbf{120} \times 0.7517 \times 0.5000 \end{cases} \quad (13)$$

This price system (13) reflects the EMH with company-specific risk-adjusted discount factors: 0.8824 for company A and 0.7517 for company B.

#### 4.1.2 The mean-variance convention

The price system (13) describes the EMH with company-specific risk-adjusted discount factors. The issue is the choice or the calculation of the company-specific discount factors. In most of cases, it is possible to choose a given discount factor resulting from financial analysis of the company. While some idea of risk is empirically taken into account by the choice of a higher or lower discount rate, this risk is not based on a statistical calculation in the first discounting convention.

The main interest of the second discounting convention is to allow to find a discount factor including a risk level with the CAPM, a risk premium. In the “mean-variance convention”, following Markowitz (1952), the risk is defined by the variance (or its square root, the standard deviation). The level of the risk premium is determined using the CAPM, which gives the level of the risk premium as a linear relation to the beta coefficient. Under the second discounting convention, the discount factor is no longer deterministic and becomes variable, since it depends on the beta coefficient. As discount factors are given under CAPM, we will speak of EMH-CAPM, a label which exemplifies the joint hypothesis problem.

Let us emphasise this point. The second discounting convention introduces a new and extremely important idea for the financial practices: the relevant discount rate for calculating a present value is related to the rate of return on a specific portfolio known as the “mean-variance-optimal tangent” portfolio. This portfolio has been considered equivalent to the “market” since Sharpe’s (1964) seminal paper. Apart from the technicity of this concept, the new development is that valuation is now associated with market equilibrium, paving the way for the “financialisation of valuation” (Chiapello, 2015). In the second discounting convention, valuation of any item requires a mean-variance optimal tangent portfolio, which in practice means actors must keep up with an index. And conversely, any mean-variance optimal tangent portfolio (or market index) becomes a possible instrument for asset valuation. Hence the “market” needs a proxy representation in order to apply the theoretical financial research to make practical real-life decisions. Serving as

proxies is precisely the function of market indexes such as the Dow Jones Euro Stoxx (DJEX).

However it is well known today that a lot of non-normality features (see Cont 2001, for example) invalidate the pure mean-variance (MV) framework. To take account for this, researchers consider augmenting the classic MV optimisation with higher order moments, namely skewness  $S$  and kurtosis  $K$  parameters for portfolio selection, where skewness and kurtosis define the asymmetry and peakedness of risk profile (see for example Jondeau and Rockinger 2006, Jurczenko and Maillet 2012). More precisely, kurtosis is the parameter which measures the tail thickness of return distribution. This approach gives birth to a multi-moments approach to risk, a generalised mean-variance-skewness-kurtosis (MVSK) framework. Including  $S$  and  $K$  in the higher dimensional portfolio selection problem is a convenient way to capture extreme risk, and it would be possible to sketch an EMH-MVSK model. But with extreme risk, the exhibition of a simple Radon-Nikodyn measure  $L$  is not easy (see Le Courtois and Xia Xu, 2023, for discussion and details). To not overcomplicate the mini-model, we decide to not take account of the issue of asset allocation with extreme risk and to stay in the MV framework of the mean-variance convention (see Le Courtois and Walter, 2016, for an example of treatment of the asset allocation problem with extreme risk).

Coming back to the mini-model in the mean-variance convention, to exhibit the EMH-CAPM representation, we need to build a “mini CAPM”. To be able to do this, we have to find an equivalent of the “market portfolio”, i.e. an index representative of the two companies A and B, for example, with reference to the DJ Euro Stoxx 50 (DJEX50), the “DJ Euro Stoxx 2” (DJEX2). The DJEX2 index consists of a portfolio of only companies A and B, whose weights in the portfolio are calculated using Markowitz mean-variance optimisation with the values for each scenario. Let us denote  $w_A^M$  and  $w_B^M$  the weights of company A and company B in the DJEX2. The weights  $w_A^M$  and  $w_B^M$  are the mean-variance-optimal weights of the companies in the DJEX2 index. Following Sharpe’s CAPM, they represent their “market capitalisation” in the mini-economy in the mini EMH-model. Using the mini-model, we find  $w_A^M = 0.7326$  and  $w_B^M = 0.2674$  (details of calculation available on request).

By applying these weights to the expected returns of A and B, it is easy to deduce the Markowitz’ risk-return representation of the DJEX2 index :

$$\begin{cases} \text{expected return} & = & 18.60\% \\ \text{expected risk} & = & 16.05\% \end{cases} \quad (14)$$

The table 6 presents the mean-variance convention and the mini CAPM resulting

	Portfolio weights		Market price of risk without higher moments			
	A	B	Expected return $\mathbb{E}[\cdot]$	Risk premium $\pi(\cdot)$	Expected volatility $\sigma$	Expected beta $\beta$
Money market			+2.00%	0	0.00%	0.00
Company A	1	0	+13.33%	+11.33%	20.00%	0.68
Company B	0	1	+33.04%	+31.04%	54.78%	1.87
<b>DJEX2</b>	0.7326	0.2574	<b>+18.60%</b>	<b>+16.60%</b>	<b>16.05%</b>	<b>1.00</b>

Table 6: The mean-variance convention and the mini CAPM

from the model, the risk level being characterised only by the standard deviation without any higher moment ( $S$  or  $K$ ). Let us assume that the risk-free rate is 2%. The risk premium of market index is  $18.60\% - 2\% = 16.60\%$ .

Now we have to find the betas of A and B in the mini-model, obtained from the prices of 150 and 63.89 euros. A calculation with the mini-model yields for company A the value of 0.68 and for B the value of 1.87, respectively. This means that if the return on the DJEX 2 index increases by 1%, the return on company A will increase by 0.68% and that on company B will increase by 1.87%. Company B amplifies market movements, while company A dampens them. These fluctuations are obviously related to the “MV-optimal market portfolio” of the CAPM. With another market portfolio or a “MVS $K$ -optimal market portfolio”, the values of the betas would be different.

We can therefore write the EMH-CAPM in the mini-model with the beta equation as follows:

$$\begin{cases} 13.33\% = 2\% + 0.68 \times (18.60\% - 2\%) \\ 33.04\% = 2\% + 1.87 \times (18.60\% - 2\%) \end{cases} \quad (15)$$

This system is the exact Sharpe’s (1964) CAPM. We have now written the CAPM in the mini-model, i.e. we have built a “mini CAPM”. There are only two equations since there are only two companies.

#### 4.1.3 The market consistent convention

As explained in Chiapello and Walter (2016), when a new convention arises, it does not mean the disappearance of the old one. In the case of risk-neutral probability,

the actuarial discounting convention is still used with the new probability  $Q$ . The share value 150 is generated by the beliefs and the discounting of possible future values with the new probability and the risk-free rate, so that the present value equation (4) becomes:

$$150 = \frac{\mathbf{140} \times 0.7833 + \mathbf{200} \times 0.2167}{1 + 0.02}$$

We now rewrite this equation by highlighting the discount factor:

$$150 = \mathbf{140} \times \frac{1}{1.02} \times 0.7833 + \mathbf{200} \times \frac{1}{1.02} \times 0.2167$$

The quantity  $1/1.02 = 0.9804$  is called the risk-free rate discount factor. We rewrite the previous relationship by making the discount factor visible:

$$150.00 = \mathbf{140} \times 0.9804 \times 0.7833 + \mathbf{200} \times 0.9804 \times 0.2167 \quad (16)$$

And the same calculation for company B.

Now it is possible to write the price system with risk-free rate discount factor, a single discount factor for the two companies:

$$\begin{cases} 150.00 = \mathbf{140} \times 0.9804 \times 0.7833 + \mathbf{200} \times 0.9804 \times 0.2167 \\ 63.89 = \mathbf{50} \times 0.9804 \times 0.7833 + \mathbf{120} \times 0.9804 \times 0.2167 \end{cases} \quad (17)$$

The price system (17) reflects the EMH with risk-neutral probability  $Q$  and risk-free rate for discounting, written EMH- $Q$ .

In the previous cases, the discount factor was constant or variable but not random. A natural generalisation of this framework is to view the discount factor as stochastic, i.e. contingent on scenarios  $\omega_1$  and  $\omega_2$  or “states of the world”. The discount factor, which in the second discounting convention only varied with the investments studied (i.e. the risk specific to each one, measured by the beta), has now become random. “Stochastic discounting” replaces traditional discounting, whether the discount rate used is given (under the first discounting convention) or results from an equilibrium model such as the CAPM (under the second discounting convention). This constitutes the third discounting convention named in Chiapello and Walter (2016) the “market-consistent convention”.

The notion of the “stochastic discount factor” (SDF) was introduced by Duffie (1996 [2016]). Within the general framework of stochastic discounting, the SDF method provides a unified general approach to the econometric analysis of asset

pricing models. The SDF expands the notions of variable but not random discount factor and it is used to represent the pricing operator in dynamic stochastic economies. The SDF is also termed the “deflator”, just as a traditional operation deflates nominal values to real values. The SDF is the another terminology of the “Market Pricing Kernel” (MPK) of Robert Lucas (1978). In financial literature, the terms MPK and SDF are considered interchangeable.

We now display the SDF in the mini-model. Let us denote  $\alpha(\omega)$  the SDF with the  $\omega$  state. The relation between the SDF and the state price density (SPD) is:

$$\alpha(\omega) = \frac{L(\omega)}{1+r} = \frac{1}{1+r} \frac{Q(\omega)}{P(\omega)} \quad (18)$$

This means that, for a given state of the economy, the SDF is equal to the SPD adjusted by the risk-free rate. It follows from (18) that:

$$\begin{cases} \alpha(\text{pessimistic}) & = 1.5359 \\ \alpha(\text{optimistic}) & = 0.4248 \end{cases} \quad (19)$$

Using the market pricing kernel, the price system at equilibrium is;

$$\begin{cases} 150.00 & = \mathbf{140} \times 1.5359 \times 0.5000 + \mathbf{200} \times 0.4248 \times 0.5000 \\ 63.89 & = \mathbf{50} \times 1.5359 \times 0.5000 + \mathbf{120} \times 0.4248 \times 0.5000 \end{cases} \quad (20)$$

The price system (20) reflects the EMH with MPK, written EMH-MPK.

## 4.2 Dialogue with the general equilibrium theory

We now engage in a dialogue with general equilibrium theory of Kenneth Arrow (1964) and Gérard Debreu (1959) by highlighting the relationship between the EMH and this theory. The idea of a complete market means that all financial products can be replicated by a linear combination of elementary assets, as “elementary particles” of all financial products, the so-called “Arrow-Debreu securities”. The value of the discount factor for each different state of the market is the price of the Arrow-Debreu security quantifying this particular state adjusted by the probability of the state. For this reason, the completeness of the market and the uniqueness of the risk-neutral probability  $Q$  are like two sides of the same coin, the game of fortune in an arbitrated market, shadowed by the Arrow-Debreu securities.

For this purpose, we consider the market states (the two scenarios) as “states of the world” in the sense of the general equilibrium model of Arrow and Debreu. We



$t = 0$		Forecast values for $t = 1$	
		pessimistic	optimistic
	Real world probability $P$	0.5000	0.5000
1	Money market	1.02	1.02
150	Price of company A	140	200
63.89	Price of company B	50	120
	Risk neutral probability $Q$	0.7833	0.2167
	State price density $L = Q/P$	1.5667	0.4333
	Market pricing kernel $\alpha$	1.5359	0.4248
0.7680	Arrow-Debreu security (pessimistic)	1	0
0.2124	Arrow-Debreu security (optimistic)	0	1

Table 7: The market consistent convention and the mini EMH-model

now introduce the notion of Arrow-Debreu security. An Arrow-Debreu security, also known as a pure security or a primitive asset, is a contract that pays one unit of cash (a currency or commodity) if a particular state of affairs occurs at a particular time in the future and pays zero cash in all other states. These fictional assets are the fundamental elements on the basis of which economic equilibrium in an uncertain environment is currently described.

In the static mini-model, let us note  $e_t^k$  for  $t = 0, 1$  the price at date  $t$  of the elementary asset giving to its holder an income of 1 at date 1 if the event  $k = \text{pessimistic}$  or  $\text{optimistic}$  occurs. It follows from the definition of an Arrow-Debreu security that the prices of the Arrow-Debreu securities in the mini-model are:

$$\begin{cases} e_0^{\text{pessimistic}} &= \frac{Q(\text{pessimistic})}{1.02} = \frac{0.7833}{1.02} = 0.7680 \\ e_0^{\text{optimistic}} &= \frac{Q(\text{optimistic})}{1.02} = \frac{0.2167}{1.02} = 0.2124 \end{cases} \quad (21)$$

The table 7 summarizes these results.

If Debreu (1959) thought of his general equilibrium model as a formal abstraction with no idea of being applicable in the “real world” and did not imagine that it could be otherwise, the mathematisation of finance in the 1980s, what we call the second quantification of finance, brought this abstraction into the daily professional practice of financiers. Every day, real markets buy and sell Arrow-Debreu prices in transactions based on risk-neutral pricing.

An interesting property of AOA in complete markets is that any income flow from a risky security can be interpreted as a linear combination (weighted aver-

age/portfolio) of the Arrow-Debreu securities. Hence, for the risky security  $A$  (resp.  $B$ ) at date  $t = 1$ , we have the pricing equation:

$$A_0 = A_1(\text{pessimistic}) \times e_0^{\text{pessimistic}} + A_1(\text{optimistic}) \times e_0^{\text{optimistic}}$$

It follows that:

$$\begin{cases} 150.00 = \mathbf{140} \times 0.7680 + \mathbf{200} \times 0.2124 \\ 63.89 = \mathbf{50} \times 0.7680 + \mathbf{120} \times 0.2124 \end{cases} \quad (22)$$

The price system (22) reflects the EMH with Arrow-Debreu securities, or EMH-AD.

Thus, the mini-model allows to illuminate the correspondence between the Arrow-Debreu securities, the risk-neutral probability  $Q$  and EMH.

### 4.3 The representation formats of EMH

In a philosophical paper, Marion Vorms (2011) introduces the notion of representation format as follows.

quotation

#### 4.3.1 The alternative representation formats

Using the mini-model, it is now possible to easily compare the representations of EMH for the same price system (150.00, 63.89):

EMH-CAPM: First quantification of financial theory - Mean-variance representation

$$\begin{cases} 150.00 = \mathbf{140} \times 0.8824 \times 0.5000 + \mathbf{200} \times 0.8824 \times 0.5000 \\ 63.89 = \mathbf{50} \times 0.7517 \times 0.5000 + \mathbf{120} \times 0.7517 \times 0.5000 \end{cases} \quad (23)$$

EMH- $Q$ : Second quantification of financial theory - Risk-Neutral pricing

$$\begin{cases} 150.00 = \mathbf{140} \times 0.9804 \times 0.7833 + \mathbf{200} \times 0.9804 \times 0.2167 \\ 63.89 = \mathbf{50} \times 0.9804 \times 0.7833 + \mathbf{120} \times 0.9804 \times 0.2167 \end{cases} \quad (24)$$

EMH-MPK: Market Pricing Kernel representation

$$\begin{cases} 150.00 = \mathbf{140} \times 1.5359 \times 0.5000 + \mathbf{200} \times 0.4248 \times 0.5000 \\ 63.89 = \mathbf{50} \times 1.5359 \times 0.5000 + \mathbf{120} \times 0.4248 \times 0.5000 \end{cases} \quad (25)$$

EMH-AD: General Equilibrium theory

$$\begin{cases} 150.00 = \mathbf{140} \times 0.7680 + \mathbf{200} \times 0.2124 \\ 63.89 = \mathbf{50} \times 0.7680 + \mathbf{120} \times 0.2124 \end{cases} \quad (26)$$

Each mathematical structure of the model designs a specific representation of EMH, which could be thought as a “representation format” (Vorms 2011).

### 4.3.2 Some reflections on the choice of representations

The use of the simplified mini-model allows one to easily understand how these formats act as fictional representations of the financial “reality”, representations that have had a significant importance for professional practices, because of the performativity effect revealed in the impact of financial theories on the business world, meaning that theories shape the real world (MacKenzie 2006). In this sense, representation formats of financial models can be thought as “metaphors” (McGoun, 2003) and as “narratives” (Shiller 2019). They are also “explanatory fictions” (Bokulich 2009), fictions that can explain in the sense that they are like helpful fictions that make it possible to design a “useful framework” for practices, namely the “*P*-world” for portfolio management and risk assessment, and the the “*Q*-world” for asset pricing.

The epistemic contribution of the mini-model is that thus makes it possible to better discover what is fictional in the EMH representations and how the fictions act. The frameworks of real-world probability *P* and risk-neutral probability *Q* are two equivalent fictions. To argue this is to support the idea that the probabilistic framework of finance can be considered as a “game with the world” (Shafer and Vovk 2001), a kind of Baudrillard’s hyperreality (Macintosh 2003) where “hyperreal” finance means a finance which refers to nothing but itself (McGoun 1997). This financial hyperreality raises the ethical issue of the financialisation of the economy.

Another issue important with the risk-neutral pricing is the performativity of financial models (Callon 1998, MacKenzie 2006). The pricing machinery of SPD “speaks” and what it “says” occurs, like a “speech act” (Austin 1962). Performativity raises the question: what kind of world do we want to see “spoken”? This echoes the ethical issue of financialisation. The financialisation of the economy is a phenomenon that has been both observed and decried for the last thirty years (Epstein 2005). The process of financialisation of the economy has been described in many ways: the growing influence of financial markets in the economic and financial regulation of investments, banking disintermediation, the uncontrolled inventiveness of financial engineering, the growing weight of financial activities in the GDP

of developed countries, etc. This process of financialisation brings with it methods of problem analysis, calculation techniques and decision-making principles which, originally created for a limited number of specific cases, have been generalised to all human issues and activities (Chiapello 2015). Among calculation techniques are the risk-neutral pricing methods. Chiapello and Walter (2016) show how the financialisation grew with the conceptual transition from the  $P$ -world of the 1960s to the  $Q$ -world of the 1980s. A consequence of the shift to the second quantification is that the mathematical machinery makes it possible to price any asset even in the absence of a market for that asset, extending the concept of AOA-“fair value” beyond its original scope. This pricing machinery of financialisation has been named the “financial *Logos*” (Walter, 2016). The financial *Logos* is a vector of financialisation.

Let us now consider the topic of green finance under the influence of financialisation. Another key problem for today is the greening of post-crisis finance. Green finance has become a key strategy for the financial industry in the wake of the 2008 financial crisis and understanding green finance becomes increasingly crucial (Jager and Dziwok, 2024). Revelli and Walter (2024) have suggested that the risk-neutral pricing could be at the origin of limitations of private green finance strategies, because of the financialisation of these strategies due to risk-neutral paradigm. The Radon-Nikodym operator could be thought as a “green-destroying operator” that produces natural disasters. In this case, the switch from  $P$  to  $Q$  with the notion of uniqueness of  $Q$  by AOA on complete markets could be seen as a trigger mechanism for financialisation and an obstacle to the greening of finance.

## 5 Conclusion

The narrative of  $Q$  risk-neutral pricing has become pervasive in finance. The mini-model can be thought as an conceptual help to understand the presence of this narrative and the narrative itself. It sheds light on the issues related to the valuation of financial assets with the risk-neutral pricing techniques, issues that are actually more conceptual than mathematical. It raises the issue of what does it mean to move from the real world to the risk-neutral world.

The three epistemic gains of the mini-model are the following. It enables “non-quant” financial practitioners and non-specialists of stochastic calculus to understand some of the epistemic difficulties of risk-neutral pricing and, in that sense, it is epistemically valuable and pedagogically useful. It captures one of the most puzzling conceptual aspects of risk-neutral pricing, the uniqueness of the risk-neutral probability  $Q$  in a complete arbitrage-free market. It allows one to grasp the signification

of the Radon-Nikodym function  $L$  without mathematical expertise, enlightening the interplay between the two mathematical representations ( $P$ -world and  $Q$ -world) that are used in day-to-day financial practices for financial valuation. At the end, the mini-model paves the way for considering fethical issues related to the pervasive use of the risk-neutral pricing methods.

## Acknowledgments

This is a new version of a paper prepared for disentangling the notions of fair value, efficient market hypothesis and risk-neutral pricing. Former versions of this paper have been presented in several seminars and conferences in particular the Phinance Biennial Conference, Roma, 8-9 September 2022. I would like to thank Emiliano Ippoliti and participants for their comments. For helpful conversations over the past several years on topics covered in this paper, I want to thank Jean-Philippe Bouchaud, Ernst Eberlein, Monique Jeanblanc, Olivier Le Courtois and Dilip Madan. I am grateful to Robert Dimand for helpful comments on the history of uncertainty in economics and to Ewa Dziwok for many constructive remarks and for checking the mathematics of this paper. I thank two anonymous referees for helpful comments on its structure and content. I also benefited of conversations with practitioners as Hubert Rodarie who provided relevant insights on european financial regulation. All errors remain mine.

## References

- Austin, John L. 1962. *How to do things with words*. Oxford: Oxford University Press.
- Arrow, Kenneth J., and Gérard Debreu. 1954. “Existence of an Equilibrium for a Competitive Economy”. *Econometrica* 22(3): 265-290.
- Boyle, Phelim, and Feidhlim Boyle. 2001. *Derivarives. The tools that changed finance*. London: Risk Books.
- Bokulich, Alisa. 2009. “Explanatory fictions”. In *Fictions in Science: Philosophical Essays on Modeling and Idealization*, edited by Mauricio Suarez. London: Routledge.
- Callon, Michel. 1998. “What does it mean to say that economics is performative?”. In *Do Economists make Markets?* edited by Donald MacKenzie, Fabian Muniesa, Lucia Siu: 310-357. Princeton: Princeton University Press.
- Campbell, John, Andrew Lo and Craig MacKinlay. 1997. *The Econometrics of Financial Markets*. Princeton: Princeton University Press.
- Cartwright, Nancy. 2009. “If no capacities then no credible worlds”. *Erkenntnis*, 70: 45-58.
- Chiapello, Eve. 2015. “Financialisation of valuation”. *Human studies* 38(1): 13-35.
- Chiapello, Eve, and Christian Walter. 2016. “The three ages of financial quantification: a conventionalist approach to the financier’s metrology”. *Historical Social Research* 41(2): 155-177.
- Cochrane, John. 2001. *Asset Pricing*. Princeton: Princeton University Press.
- Cont, Rama. 2001. “Empirical properties of asset returns: Stylized facts and statistical issues”. *Quantitative Finance* 1(2): 223-236.
- Cox, John, Stephen A. Ross, and Mark Rubinstein. 1979. “Option pricing: A simplified approach”. *Journal of Financial Economics* 7(3): 229-263.
- Desrosières, Alain. 2003. “Managing the economy: The State, the market and statistics”. In *The Cambridge history of science Vol. 7: Modern social and behavioral sciences*, edited by Theodore M. Porter and Dorothy Ross, 553-64. Cambridge: Cambridge University Press.

- Dimand, Robert. 2021. “Keynes, Knight, and fundamental uncertainty: a double centenary 1921-2021”. *Review of Political Economy* 33(4): 570-584.
- Diebold, Francis X., Neil A. Doherty, Richard J. Herring. 2010. *The Known, the Unknown, and the Unknowable in Financial Risk Management*. Princeton: Princeton University Press.
- Dothan, Michael. 1990. *Prices in Financial Markets*. Oxford: Oxford University Press.
- Duffie, Darrell. 1996. *Dynamic Asset Pricing Theory*. 3rd edition 2016. Princeton: Princeton University Press.
- Ellsberg, Daniel. 1961. “Risk, Ambiguity, and the Savage Axioms”. *The Quarterly Journal of Economics* 75(4): 643-669.
- Epstein, Gerald. 2005. *Financialization and the World Economy*. Edward Elgar Publishing.
- Fama, Eugene. 1970. “Efficient capital markets: a review of theory and empirical work”. *Journal of Finance*, 25(2): 383-417 and discussion p. 418-423.
- Föllmer Hans and Alexander Schied. 2002. *Stochastic Finance. An Introduction in Discrete Time*. Berlin; New York: de Gruyter.
- Gilboa, Itzhak and David Schmeidler. 2012. *Case-Based Predictions: An Axiomatic Approach to Predictions, Classification and Statistical Learning*, Singapore, World Scientific Publishing.
- Habart-Corlosquet Marine, William Gehin, Jacques Janssen and Raimondo Manca. 2015. *Asset and Liability Management for Banks and Insurance Companies*. London: ISTE Wiley.
- Habart-Corlosquet, Marine, Jacques Janssen and Raimondo Manca. 2013. *VaR Methodology for Non-Gaussian Finance*. London: ISTE Wiley.
- J. Michael Harrison and David M. Kreps. 1979. “Martingales and Arbitrage in Multiperiod Securities Markets”. *Journal of Economic Theory* 20(3): 381-408.
- J. Michel Harrison and Stanley R. Pliska. 1981. “Martingales and Stochastic Integrals in the Theory of Continuous Trading”. *Stochastic Processes and Applications* 11(3): 215-260.

- Hempel, Carl. 1966. *Philosophy of Natural Science*. Princeton: Prentice-Hall.
- Ippoliti, Emiliano. 2017. “Methods and Finance. A View from Inside”. In *Methods and Finance. A Unifying View on Finance, Mathematics and Philosophy*, edited by Emiliano Ippoliti and Ping Chen. Springer, p. 121-128.
- Jager, Johannes and Ewa Dziwok (ed.). 2024. *Understanding Green Finance. A Critical Assessment and Alternative Perspectives*. Cheltenham: Edward Elgar Publishing.
- Jarrow, Robert and Martin Larson. 2012. “The meaning of market efficiency”. *Mathematical Finance* 22(1): 1-30.
- Jondeau, Eric, Michael Rockinger. 2006. “Optimal portfolio allocation under higher moments”. *European Financial Management*. 12(1): 29-55.
- Jurczenko, Emmanuel and Bertrand Maillet. 2012. “The four-moment capital asset pricing model. Between asset pricing and asset allocation”. In *Multi-moment Asset Allocation and Pricing Models*, edited by Emmanuel Jurczenko and Bertrand Maillet. Chichester: John Wiley, 113-163.
- Kahneman, Daniel and Amos Tversky. 1979. “Prospect Theory: An Analysis of Decision under Risk”. *Econometrica* 47(2): 263-291.
- Ioannis Karatzas and Steven E. Shreve. 1998. *Methods of Mathematical Finance*. Berlin: Springer.
- Malcolm Kemp. 2009. *Market Consistency. Model Calibration in Imperfect Markets*. John Wiley.
- Knight, Frank H. 1921. *Risk, Uncertainty and Profit*. Boston and New York: Houghton Mifflin Company.
- Le Courtois, Olivier and Xu Xia. 2023. “Efficient portfolios and extreme risks: a Pareto-Dirichlet approach”. *Annals of Operations Research*. <https://doi.org/10.1007/s10479-023-05507-y>
- Le Courtois, Olivier and Christian Walter. 2016. *Extreme financial risk and asset allocation*. London: Imperial College Press.
- Lo, Andrew (ed.). 1997. *Market Efficiency: Stock Market Theory Behaviour in Theory and Practice*, Volumes I and II. Cheltenham: Edward Elgar Publishing.



- Lucas, Robert E. 1978. "Asset Prices in an Exchange Economy". *Econometrica* 46(6): 1429-1445.
- Macintosh, Norman. 2003. "From rationality to hyperreality: Paradigm poker". *International Review of Financial Analysis* 12: 453-465.
- MacKenzie, Donald. 2006. *An Engine, Not a Camera. How Financial Models Shape Markets*, Cambridge (MA): The MIT Press.
- Markowitz, Harry. 1952. "Portfolio Selection". *The Journal of Finance* 7(1): 77-91.
- McGoun, Elton G. 1997. "Hyperreal Finance". *Critical Perspectives on Accounting*. 8(1-2): 97-122.
- McGoun, Elton G. 2003. "Finance models as metaphors". *International Review of Financial Analysis* 12(4): 421-433.
- McNeil, Alexander, Frey Rudiger and Paul Embrechts. 2015. *Quantitative Risk Management. Concept, Techniques and Tools* (revised version. Princeton: Princeton University Press.
- Ottaviani, Giuseppe (ed.). 1995. *Financial Risk in Insurance*. New York: Springer.
- Pickering, Andrew. 1992. *Science as Practice and Culture*. Chicago: The University of Chicago Press.
- Pliska, Stanley. 1997. *Introduction to Mathematical Finance. Discrete Time Models*. Oxford: Blackwell.
- Quine, Willard Van Orman. 1953. "Two dogmas of empiricism". Ch. II in *From a logical point of view*. Cambridge (MA): Harvard University Press.
- Reghai, Adil. 2015. *Quantitative Finance. Back to Basic Principles*. Palgrave Macmillan.
- Reutlinger, Alexander, Dominik Hangleiter and Stephan Hartmann. 2018. "Understanding (with) Toy Models", *The British Journal for the Philosophy of Science* 69(4): 1069-1099.
- Revelli, Christophe and Christian Walter. 2024. "Limitations of conventional private green finance industry and strategies". In *Understanding Green Finance. A Critical Assessment and Alternative Perspectives*, edited by Johannes Jager and Ewa Dziwok. Edward Elgar Publishing.

- Savage, John. 1954. *The Foundations of Statistics*. New York: John Wiley & Sons.
- Schelling, Thomas. 1971. “Dynamic models of segregation”. *The Journal of Mathematical Sociology* 1(2): 143-186.
- Shafer Glenn and Vladimir Vovk. 2001. *Probability and Finance. It’s Only a Game!*. London: Wiley & Sons.
- Sharpe, William. 1964. “Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk”. *The Journal of Finance* 19(3): 425-442.
- Shiller, Robert. 2019. *Narrative Economics: How Stories Go Viral and Drive Major Economic Events*. Princeton: Princeton University Press.
- Svetlova, Ekaterina. 2012. “On the Performative Power of Financial Models”. *Economy and Society* 41(3): 418-434.
- Svetlova, Ekaterina. 2021. “On the relevance of Knight, Keynes and Shackle for unawareness research”. *Cambridge Journal of Economics* 45: 989-1007.
- Svetlova, Ekaterina and Vanessa Dirksen. 2014. “Models at Work – Models in Decision Making”. *Science in Context* 27(4): 561-577.
- Vorms, Marion. 2011. “Representing with imaginary models: Formats matter”. *Studies in History and Philosophy of Science Part A* 42(2): 287-295.
- Walter, Christian. 1996. “Une histoire du concept d’efficience sur les marchés financiers”. *Annales. Histoire Sciences Sociales* 51(4): 873-905.
- Walter, Christian. 2001. “The Efficient Market Hypothesis, the Gaussian Assumption, and the Investment Management Industry”. EFMA 2001 Lugano Meetings, SSRN: <https://ssrn.com/abstract=267443>
- Walter, Christian. 2016. “The financial Logos: The framing of financial decision-making by mathematical modelling”. *Research in International Business and Finance* 37: 597-604.
- Walter, Christian. 2017. “Philosophie de la finance : l’exemple de l’efficacité informationnelle d’un marché”. In *Philosophie économique. Un état des lieux*, edited by Gilles Campagnolo and Jean-Sébastien Gharbi, Paris: Éditions matériologiques, p. 579-626.
- Walter, Christian. 2019. “The Brownian motion in finance: an epistemological puzzle”. *Topoi* 40: 1-17.

Wütrich Mario, Hans Bühlman and Hansjorg Furrer. 2015. *Market-Consistent Actuarial Valuation*. Springer.

Ylikoski, Petri and Emrah Aydinonat. 2014. Understanding with Theoretical Models. *Journal of Economic Methodology*. 21(1): 19-36.