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RAPID STABILIZATION AND FINITE TIME STABILIZATION OF THE BILINEAR SCHRÖDINGER EQUATION

HOAI-MINH NGUYEN

ABSTRACT. We propose a method to establish the rapid stabilization of the bilinear Schrödinger control system and its linearized system, and the finite time stabilization of the linearized system using the Grammian operators. The analysis of the rapid stabilization involves a new quantity (variable) which is inspired by the adjoint state in the optimal control theory and is proposed in our recent work on control systems associated with strongly continuous group. The analysis of the finite time stabilization follows the strategy introduced by Coron and Nguyen in the study of the finite time stabilization of the heat equation and incorporate a new ingredient involving the estimate of the cost of controls of the linearized system in small time derived in this paper.

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1. INTRODUCTION

1.1. Statement of the main results. We consider the following bilinear control Schrödinger system, with $I = (0, 1)$,

$$(1.1) \quad \begin{cases} i\Psi_t = -\Delta\Psi - u(t)\mu(x)\Psi(t, x) & \text{in } \mathbb{R}_+ \times I, \\ \Psi(t, 0) = \Psi(t, 1) = 0 & \text{in } \mathbb{R}_+, \\ \Psi(0) = \Psi_0 & \text{in } I, \end{cases}$$

where Ψ_0 is the initial data,

the control u is *real*,

and μ is a given *real* function, around the fundamental state. Here Ψ is the complex-valued wave function of a particle confined in a $1d$ infinite square potential well. The particle is subjected

to an electric field inside the domain with the amplitude u , and μ is the dipolar moment of the particle. For detailed approximations leading to this first-order interaction Hamiltonian we refer for example to [25, Chapter 2].

Let $\lambda_1 < \lambda_2 < \dots, \lambda_k < \dots$ be the set of eigenvalues of the Laplace equation in I with the zero Dirichlet boundary condition and let (φ_k) be the standard orthogonal basis in $L^2(I)$ formed by the corresponding eigenfunctions. Thus

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k & \text{in } I, \\ \varphi_k = 0 & \text{on } \partial I. \end{cases}$$

Explicitly, for $k \geq 1$,

$$(1.2) \quad \lambda_k = \pi^2 k^2 \quad \text{and} \quad \varphi_k(x) = \sqrt{2} \sin(\pi k x) \text{ in } I.$$

It is clear that

$$(1.3) \quad e^{-i\lambda_1 t} \varphi_1 \text{ is a solution of (1.1) with } u = 0 \text{ and } \Psi_0 = \varphi_1.$$

We are interested in the stabilization of the system (1.1) around the state $e^{-i\lambda_1 t} \varphi_1$. To this end, it is convenient to introduce

$$(1.4) \quad \tilde{\Psi}(t, x) = e^{i\lambda_1 t} \Psi(t, x) \text{ in } \mathbb{R}_+ \times I \quad \text{and} \quad \tilde{\Psi}_0(x) = \Psi_0(x) \text{ in } I.$$

We then have, by (1.1),

$$(1.5) \quad \begin{cases} i\tilde{\Psi}_t = -\Delta\tilde{\Psi} - \lambda_1\tilde{\Psi} - u(t)\mu(x)\tilde{\Psi}(t, x) & \text{in } \mathbb{R}_+ \times I, \\ \tilde{\Psi}(t, 0) = \tilde{\Psi}(t, 1) = 0 & \text{in } \mathbb{R}_+. \end{cases}$$

The linearized system of (1.5) when $\tilde{\Psi}$ is closed to φ_1 , i.e., $\Psi(t, x)$ is closed to $e^{-i\lambda_1 t} \varphi_1$, is

$$(1.6) \quad \begin{cases} i\tilde{\Psi}_t = -\Delta\tilde{\Psi} - \lambda_1\tilde{\Psi} - u(t)\mu(x)\varphi_1(x) & \text{in } \mathbb{R}_+ \times I, \\ \tilde{\Psi}(t, 0) = \tilde{\Psi}(t, 1) = 0 & \text{in } \mathbb{R}_+. \end{cases}$$

In what follows, we *always* assume that

$$(1.7) \quad \mu \in H^3(I, \mathbb{R}).$$

The following condition on μ is repeated used later:

$$(1.8) \quad |\langle \mu \varphi_1, \varphi_k \rangle_{L^2(I)}| \geq \frac{c}{k^3} \text{ for } k \in \mathbb{N}_+,$$

for some positive constant c unless stated differently ¹ We are interested in the solutions of the above Schrödinger systems with controls u in $L^2_{loc}([0, +\infty); \mathbb{R})$ (we insist again that we are interested in the controls which are real).

The condition (1.8) is a sufficient condition to have the exact controllability of the linearized systems in small time and this implies the local exact controllability of the nonlinear systems, as shown by Beauchard and Laurent [6]. This condition is also a necessary condition to ensure that the nonlinear systems are locally exactly controllable in small time, see the work of Beauchard and Morancey [8]. The condition (1.8) is generic, see [6, Appendix A].

As in previous works, see, e.g., [6, 8, 18], we are interested in the solutions in the space \mathbf{H} (for each time t) defined by

$$(1.9) \quad \mathbf{H} = \left\{ \Psi \in H^1_0(I; \mathbb{C}); \sum_{k \geq 1} |k^3 \langle \Psi, \varphi_k \rangle_{L^2(I)}|^2 < +\infty \right\},$$

¹Hereafter, given a Hilbert space \mathcal{H} , we denote $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ its scalar product.

and

$$(1.10) \quad \mathbf{H}_{1,\#} = \left\{ \Psi \in \mathbf{H} \text{ such that } \Re \langle \Psi, \varphi_1 \rangle_{L^2(I)} = 0 \right\}.$$

Here and in what follows, for a complex number z , we denote its real part, its complex part, and its complex conjugate by $\Re z$, $\Im z$, and \bar{z} , respectively. We equip the following scalar product for the spaces \mathbf{H} and $\mathbf{H}_{1,\#}$:

$$(1.11) \quad \langle \Psi, \tilde{\Psi} \rangle_{\mathbf{H}} = \langle \Psi, \tilde{\Psi} \rangle_{H^3(I)} := \int_I \left(\Psi \tilde{\Psi} + \Psi' \tilde{\Psi}' + \Psi'' \tilde{\Psi}'' + \Psi''' \tilde{\Psi}''' \right) ds \text{ for } y, \tilde{y} \in \mathbf{H},$$

and

$$(1.12) \quad \langle \Psi, \tilde{\Psi} \rangle_{\mathbf{H}_{1,\#}} = \langle y, \tilde{y} \rangle_{\mathbf{H}} \text{ for } \Psi, \tilde{\Psi} \in \mathbf{H}_{1,\#}.$$

One can show that, for the linearized system (1.6),

$$(1.13) \quad \Psi(t) \in \mathbf{H}_{1,\#} \text{ for } t \geq 0 \text{ if } \Psi_0 \in \mathbf{H}_{1,\#}.$$

This property does not hold for the nonlinear system. Note that the exact controllability has been established for solutions in $C([0, T]; \mathbf{H})$, which requires roughly three derivatives in the space variable of the solutions. It is known from a general result of Ball, Marsden, and Slemrod [2] that the Schrödinger system (1.5) is not exactly controllable for solutions in $C([0, T]; H_0^1(I))$ or in $C([0, T]; H_0^1(I) \cap H^2(I))$ when μ is smooth since the control operator is bounded in this case.

It is convenient to consider the real part and the imaginary part of $\tilde{\Psi}$ separately. Assume that

$$\tilde{\Psi} = \tilde{\Psi}_1 + i\tilde{\Psi}_2 \text{ in } \mathbb{R}_+ \times I,$$

where $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$ are the real and the imaginary parts of $\tilde{\Psi}$. System (1.5) can be written under the form

$$(1.14) \quad \begin{cases} \tilde{\Psi}_{1,t} = -\Delta \tilde{\Psi}_2 - \lambda_1 \tilde{\Psi}_2 - u(t)\mu(x)\tilde{\Psi}_2 & \text{in } \mathbb{R}_+ \times I, \\ \tilde{\Psi}_{2,t} = \Delta \tilde{\Psi}_1 + \lambda_1 \tilde{\Psi}_1 + u(t)\mu(x)\tilde{\Psi}_1 & \text{in } \mathbb{R}_+ \times I, \end{cases}$$

and system (1.6) can be written under the form

$$(1.15) \quad \begin{cases} \tilde{\Psi}_{1,t} = -\Delta \tilde{\Psi}_2 - \lambda_1 \tilde{\Psi}_2 & \text{in } \mathbb{R}_+ \times I, \\ \tilde{\Psi}_{2,t} = \Delta \tilde{\Psi}_1 + \lambda_1 \tilde{\Psi}_1 + u(t)\mu(x)\varphi_1 & \text{in } \mathbb{R}_+ \times I. \end{cases}$$

Denote

$$(1.16) \quad \mathbb{H} = \left\{ y = (y_1, y_2)^\top \in H_0^1(I; \mathbb{R}^2); \sum_{\ell=1}^2 \sum_{k \geq 1} |k^3 \langle y_\ell, \varphi_k \rangle_{L^2(I)}|^2 < +\infty \right\},$$

and

$$(1.17) \quad \mathbb{H}_{1,\#} = \left\{ y = (y_1, y_2)^\top \in \mathbb{H} \text{ such that } \langle y_1, \varphi_1 \rangle_{L^2(I)} = 0 \right\},$$

and we equip with the following scalar products for the spaces \mathbb{H} and $\mathbb{H}_{1,\#}$:

$$(1.18) \quad \langle y, \tilde{y} \rangle_{\mathbb{H}} := \langle y, \tilde{y} \rangle_{H^3(I)} = \int_I \sum_{\ell=1}^2 \left(y_\ell \tilde{y}_\ell + y'_\ell \tilde{y}'_\ell + y''_\ell \tilde{y}''_\ell + y'''_\ell \tilde{y}'''_\ell \right) ds \text{ for } y = (y_1, y_2)^\top, \tilde{y} = (\tilde{y}_1, \tilde{y}_2)^\top \in \mathbb{H},$$

and

$$(1.19) \quad \langle y, \tilde{y} \rangle_{\mathbb{H}_{1,\#}} = \langle y, \tilde{y} \rangle_{\mathbb{H}} \text{ for } y, \tilde{y} \in \mathbb{H}_{1,\#}.$$

It is clear that

$$\Psi \in \mathbf{H} \text{ if and only if } (\Psi_1, \Psi_2)^\top \in \mathbb{H}$$

and

$$\Psi \in \mathbf{H}_{1,\#} \text{ if and only if } (\Psi_1, \Psi_2)^\top \in \mathbb{H}_{1,\#},$$

where Ψ_1 and Ψ_2 are the real part and the imaginary part of Ψ , respectively.

One can check, see e.g., [18], that

$$(1.20) \quad \mathbb{H} = \left\{ y = (y_1, y_2)^\top \in H^3(I; \mathbb{R}^2); y_1(x) = y_2(x) = y_1''(x) = y_2''(x) = 0 \text{ on } \partial I \right\}$$

and

$$(1.21) \quad \mathbb{H}_{1,\#} = \left\{ y = (y_1, y_2)^\top \in H^3(I; \mathbb{R}^2); \right. \\ \left. y_1(x) = y_2(x) = y_1''(x) = y_2''(x) = 0 \text{ on } \partial I \text{ and } \langle y_1, \varphi_1 \rangle_{L^2(I)} = 0 \right\}.$$

Note that $\mathbf{H}_{1,\#}$ is not a subspace of \mathbf{H} (with respect to the scalar field \mathbb{C}) whilst $\mathbb{H}_{1,\#}$ is a subspace of \mathbb{H} (with respect to the scalar field \mathbb{R}).

Consider $A : \mathcal{D}(A) \subset \mathbb{H} \rightarrow \mathbb{H}$ defined by

$$(1.22) \quad Ay = \begin{pmatrix} -\Delta y_2 - \lambda_1 y_2 \\ \Delta y_1 + \lambda_1 y_1 \end{pmatrix} \quad \text{and} \quad \mathcal{D}(A) = \left\{ y \in \mathbb{H}; Ay \in \mathbb{H} \right\}.$$

Then $\mathcal{D}(A)$ is dense in \mathbb{H} and A is skew-adjoint (see Lemmas 2.1 and 2.2). We equip $\mathcal{D}(A)$ with the standard scalar product for the graph-norm and denote $\mathcal{D}(A)'$ the dual space of $\mathcal{D}(A)$. It is worth noting that our definition of A and the domain $\mathcal{D}(A)$ are different from [6, 18]. Our definitions are motivated by the theory of stabilization developed for control systems associated with a strongly continuous group [48] and will be clear later when the feedback operator is introduced (see, e.g., (1.34), see also (1.31)).

Let $(A^*, \mathcal{D}(A^*))$ denote the adjoint of $(A, \mathcal{D}(A))$ and let

$$B : \mathbb{R} \rightarrow H^3(I; \mathbb{R}^2) \cap H_0^1(I; \mathbb{R}^2) \subset \mathcal{D}(A^*)'$$

be defined by, with $y = (y_1, y_2)^\top \in \mathcal{D}(A^*)$,

$$(1.23) \quad \langle Bu, y \rangle_{\mathcal{D}(A^*)', \mathcal{D}(A^*)} = u \left(\langle \mu \varphi_1, y_2 \rangle_{H^3(I)} - (\mu \varphi_1)_{xx}(1) y_{2,xxx}(1) + (\mu \varphi_1)_{xx}(0) y_{2,xxx}(0) \right).$$

The linear system (1.15) can be written under the form

$$(1.24) \quad y' = Ay + Bu \text{ in } \mathbb{R}_+$$

and the nonlinear system (1.14) can be written under the form

$$(1.25) \quad y' = Ay + Bu + uF(y - \Phi_1) \text{ in } \mathbb{R}_+,$$

where

$$(1.26) \quad \Phi_1 = (\varphi_1, 0)^\top \quad \text{and} \quad F(y) = (-\mu y_2, \mu y_1)^\top,$$

and, for all $\varphi \in \mathcal{D}(A^*)$,

$$(1.27) \quad \langle uF(y), \varphi \rangle_{\mathcal{D}(A^*)', \mathcal{D}(A^*)} \\ = u \left(\langle F(y), \varphi \rangle_{H^3(I)} - \langle (F(y))_{xx}(1), \varphi_{xxx}(1) \rangle_{\mathbb{R}^2} + \langle (F(y))_{xx}(0), \varphi_{xxx}(0) \rangle_{\mathbb{R}^2} \right)$$

(see Lemmas 3.3 and 3.7).

One cannot extend B as a bounded operator from \mathbb{R} into \mathbb{H} . This is the main source of the difficulties in the study of the stabilization using feedback of the linearized system (1.24) and more critical in the study of the nonlinear system (1.25) since $uF(y) \notin L^1((0, T); \mathbb{H})$. Nevertheless, B

is an *admissible* control operator with respect to the semi-group $(e^{tA})_{t \geq 0}$ generated by A in the sense that, for all $u \in L^2([0, T]; \mathbb{R})$, it holds that

$$(1.28) \quad y \in C([0, T]; \mathbb{H}) \text{ where } y(t) := \int_0^t e^{(t-s)A} B u(s) ds$$

(see Lemmas 3.2 and 3.3). As a consequence of the closed graph theorem, see e.g., [13], one has

$$(1.29) \quad \|y\|_{C([0, T]; \mathbb{H})} \leq C_T \|u\|_{L^2((0, T); \mathbb{R})}.$$

Thus, see e.g., [16, 54], that, for $T > 0$, there exists $C_T > 0$ such that

$$(1.30) \quad \int_0^T |B^* e^{-sA^*} z|^2 \leq C_T \|z\|_{\mathbb{H}}^2 \text{ for all } z \in \mathbb{H},$$

where B^* is the adjoint of B , and $(e^{tA^*})_{t \in \mathbb{R}}$ is the group generated by A^* (recall that A is skew-adjoint).

Note that

$$B^* : \mathcal{D}(A^*) \rightarrow \mathbb{R}.$$

and, with $v = (v_1, v_2)^\top \in \mathcal{D}(A^*)$, which is also $\mathcal{D}(A)$ since A is skew-adjoint,

$$(1.31) \quad B^* v = \langle \mu \varphi_1, v_2 \rangle_{H^3(I)} - (\mu \varphi_1)_{xx}(1) v_{2,xxx}(1) + (\mu \varphi_1)_{xx}(0), v_{2,xxx}(0)$$

since

$$(1.32) \quad \langle B u, v \rangle_{\mathbb{H}} = \langle u, B^* v \rangle_{\mathbb{R}}.$$

This paper is devoted to the stabilization of the nonlinear system (1.5) and its linearized system (1.6). The rapid stabilization of the linearized control system (1.6) was established by Coron, Gagnon, and Morancey [18] using techniques related to backstepping methods. The idea is to transform the original system into a damping one for which the stabilization is an easier task. Their transformations are of Fredholm type and different from the standard Volterra ones in the backstepping method. The existence of these transformations is ensured by the controllability of the linearized system, which follows from (1.8). The main technical difficulty in the work of Coron, Gagnon, and Morancey [18] is to deal with a control operator that is only admissible but *not* bounded. It is worth noting that the backstepping technique and its extended versions are useful tools to stabilize various equations in one-dimensional space such as heat equations [38], Schrödinger equations [31], KdV equations [14, 20], hyperbolic systems [24, 22, 23] and the reference therein. The backstepping can be also used to get finite-time stabilization for heat equations, see [21]. A concise introduction to the backstepping technique can be found in [32]. At this stage, to our knowledge, [18] is the only work dealing with the rapid stabilization of the linearized Schrödinger system using bilinear controls, and the analysis in [18] has not been successfully extended to the nonlinear system.

The goal of this paper is to present another method to obtain the rapid stabilization of the linearized control system (1.6) and of the bilinear control system (1.5), and the finite time stabilization of the linearized control system (1.6). Our approach is inspired by our recent work [48] in which we study the stabilization of systems associated with a strongly continuous group for unbounded control operators using Gramian operators. For control systems associated with a strongly continuous group, under the assumption that the systems are exactly controllable, it is shown in [48] that one can obtain rapid stabilization using static feedback in a trajectory sense or using dynamic feedback. The static trajectory feedback has its roots in the linear quadratic optimal control theory, as developed in Flandoli, Lasiecka, and Triggiani [27] (see also [33, 57, 59, 51, 53]). It is known from the optimal control theory that there exists static feedback in a *weak* sense to rapidly stabilize the system. Such feedback is understood in a weak sense since it is defined only on a dense set of the space state depending on the feedback operator (see [48, Proposition 4.1]).

The use of Gramian operators to rapidly stabilize exactly controllable systems associated with a strongly continuous group has been previously considered in [28, 55, 56] via the optimal control theory, and the feedback is thus understood in the *weak* sense. One cannot use Gramian operators to stabilize nonlinear settings using the theories developed in [28, 55, 56] as discussed in [48] (see also [18]). In this paper, we show that, for the considered bilinear control Schrödinger systems, even if the control operator is unbounded one can still obtain static feedback in the usual sense, the feedback is defined for all elements in the state space, to achieve the rapid stabilization. Moreover, we construct piecewise constant feedback to reach the stabilization in finite time for the linearized system.

Before introducing the feedback, we state the observability inequality for the exact controllability of the linearized system (1.6) as a consequence of the exact controllability result of Beauchard and Laurent [6] and Lemmas 3.2, 3.3 and 3.7 (see also Proposition 5.4 for a more quantitative version for small T).

Lemma 1.1. *Let $\mu \in H^3(I; \mathbb{R})$ verify (1.8) and let $T > 0$. We have*

$$(1.33) \quad \int_0^T |B^* e^{-sA^*} z|^2 \geq C_T \|z\|_{\mathbb{H}}^2 \text{ for all } z \in \mathbb{H}_{1, \sharp},$$

for some positive constant C_T independent of z .

We are ready to introduce the Gramian operator to stabilize the linearized system (1.6) and the nonlinear system (1.5). Let $\mu \in H^3(I; \mathbb{R})$ and $\lambda > 0$. Define $Q = Q(\lambda) : \mathbb{H} \rightarrow \mathbb{H}$ by

$$(1.34) \quad \langle Qz, \tilde{z} \rangle_{\mathbb{H}} = \int_0^\infty e^{-2\lambda s} \langle B^* e^{-sA^*} z, B^* e^{-sA^*} \tilde{z} \rangle_{\mathbb{R}} ds \text{ for } z, \tilde{z} \in \mathbb{H}.$$

Since A is skew-adjoint by Lemma 2.2 in Section 2, it follows from (1.30) that Q is well-defined and is symmetric in \mathbb{H} . We also have, by [48, Proposition 5.1] (see also [28, 55]),

$$(1.35) \quad AQ + QA^* - BB^* + 2\lambda Q = 0$$

in the following sense

$$(1.36) \quad \langle Qz, A^* \tilde{z} \rangle_{\mathbb{H}} + \langle A^* z, Q\tilde{z} \rangle_{\mathbb{H}} - \langle B^* z, B^* \tilde{z} \rangle_{\mathbb{H}} + 2\lambda \langle Qz, \tilde{z} \rangle_{\mathbb{H}} \text{ for } z, \tilde{z} \in \mathcal{D}(A^*).$$

Moreover, if the condition (1.8) on μ holds then, by Lemma 1.1,

$$(1.37) \quad \mathcal{Q} := \text{proj}_{\mathbb{H}_{1, \sharp}} \circ Q : \mathbb{H}_{1, \sharp} \rightarrow \mathbb{H}_{1, \sharp} \text{ is positive, i.e., } \langle \mathcal{Q}z, z \rangle_{\mathbb{H}_{1, \sharp}} \geq C \|z\|_{\mathbb{H}_{1, \sharp}}^2 \text{ for all } z \in \mathbb{H}_{1, \sharp}.$$

Concerning the rapid stabilization of (1.15), we prove the following result.

Theorem 1.1. *Let $\mu \in H^3(I, \mathbb{R})$ be such that (1.8) holds and let $\lambda > 0$. Given $y_0 \in \mathbb{H}_{1, \sharp}$, let $y \in C([0, T]; \mathbb{H})$ be the unique weak solution of the system*

$$(1.38) \quad \begin{cases} y' = Ay + Bu \text{ in } \mathbb{R}_+, \\ y(0) = y_0, \end{cases}$$

with

$$(1.39) \quad u = -B^* \mathcal{Q}^{-1} \text{proj}_{\mathbb{H}_{1, \sharp}} y.$$

Then $y(t) \in \mathbb{H}_{1, \sharp}$ for $t \geq 0$, and

$$\|\mathcal{Q}^{-1}y(t)\|_{\mathbb{H}} = e^{-2\lambda t} \|\mathcal{Q}^{-1}y_0\|_{\mathbb{H}} \text{ for } t \geq 0.$$

Consequently, there exist two positive constants C_1, C_2 independent of y_0 such that

$$(1.40) \quad C_1 e^{-2\lambda t} \|y_0\|_{\mathbb{H}} \leq \|y(t)\|_{\mathbb{H}} \leq C_2 e^{-2\lambda t} \|y_0\|_{\mathbb{H}} \text{ for } t \geq 0.$$

Remark 1.1. The meaning of the weak solutions are given in Definition A.1 in Appendix A for which one considers Bu as a source term. The well-posedness of (1.38) is a part of the conclusion of Theorem 1.1. Note that B^*z is also well-defined for $z \in \mathbb{H}$ by (1.31).

As a consequence of Theorem 1.1, the linearized system (1.15) is rapidly stabilizable by feedback controls. Equivalently, the linearized system (1.6) is rapidly stabilizable by feedback controls.

Remark 1.2. Note that $y(t) \in \mathbb{H}_{1,\sharp}$ for $t \geq 0$. One can hence replace the term $\text{proj}_{\mathbb{H}_{1,\sharp}} y$ by y in Theorem 1.1.

Concerning the non-linear system (1.14), which is equivalent to system (1.5), we have the following result.

Theorem 1.2. *Let $\mu \in H^3(I, \mathbb{R})$ be such that (1.8) holds and let $\lambda > 0$. For $0 < \hat{\lambda} < \lambda$, there exist two positive constants $\varepsilon_0 > 0$ and $C > 0$ such that*

$$(1.41) \quad \|y(t, \cdot) - \Phi_1\|_{\mathbb{H}} \leq Ce^{-2\hat{\lambda}t} \|y(0, \cdot) - \Phi_1\|_{\mathbb{H}} \quad \text{for } t \geq 0,$$

for all $y_0 \in \mathbb{H}_{1,\sharp}$ with $\|y_0\|_{L^2(I)} = 1$ and $\|y_0 - \Phi_1\|_{\mathbb{H}} \leq \varepsilon_0$, where $y(t, \cdot) \in C([0, T]; \mathbb{H})$ is the unique weak solution of the system

$$(1.42) \quad \begin{cases} y' = Ay + Bu + uF(y - \Phi_1) \text{ in } \mathbb{R}_+, \\ y(0) = y_0, \end{cases}$$

with

$$u = -B^*Q^{-1}\text{proj}_{\mathbb{H}_{1,\sharp}}(y - \Phi_1).$$

Remark 1.3. The meaning of the weak solutions are given in Definition A.1 in Appendix A for which one considers $Bu + uF(y - \Phi_1)$ as the source term.

As a consequence of Theorem 1.2, the nonlinear bilinear control system (1.14) is locally rapidly stabilizable by feedback controls. Equivalently, the nonlinear bilinear control system (1.6) is locally rapidly stabilizable by feedback controls.

Concerning the finite time stabilization, we have the following result on the linearized bilinear control system (1.24).

Theorem 1.3. *Let $\mu \in H^3(I, \mathbb{R})$ be such that (1.8) holds and let $T > 0$. There exists $\mathcal{K} : [0, T) \times \mathbb{H} \rightarrow \mathbb{R}$ such that \mathcal{K} is piecewise constant with respect to the first variable and linear continuous with respect to the second variable, and for every $y_0 \in \mathbb{H}_{1,\sharp}$, there exists a unique solution $y \in C([0, T); \mathbb{H})$ of the system*

$$(1.43) \quad \begin{cases} y' = Ay + Bu \text{ in } [0, T) \\ y(0) = y_0, \end{cases}$$

with

$$(1.44) \quad u(t) = \mathcal{K}(t, y(t, \cdot)).$$

Moreover,

$$(1.45) \quad y(t, \cdot) \rightarrow 0 \text{ in } \mathbb{H} \text{ as } t \rightarrow T_-$$

and

$$(1.46) \quad u(t, \cdot) \rightarrow 0 \text{ as } t \rightarrow T_-.$$

1.2. Ideas of the proof. The approach used in this paper is inspired by our recent work [48]. We first discuss the analysis of Theorem 1.1 and Theorem 1.2. Concerning the linearized system (1.6) (whose results are given in Theorem 1.1), one of the main parts of the analysis is to develop the theory in [48] to take into account the intrinsic constraint (1.13). Concerning the rapid stabilization of the nonlinear system (1.5) (whose results are given in Theorem 1.2), in addition to the ingredients used for the linearized system, we essentially use the fact that the solutions of the Schrödinger system conserve the L^2 -norm. This fact is used to control the component of the solution which is orthogonal to $\mathbb{H}_{1,\sharp}$ with respect to the $L^2(I)$ -scalar product (or the \mathbb{H} -scalar product). Additional technical ingredients for all the stabilization results are the well-posedness of the nonlinear feedback control systems, which are of nonlinear, nonlocal, and non-bounded nature (see Section 3), and the way to translate the results between the original systems (1.5) and (1.6) and the corresponding systems written under in the semi-group language (1.24) and (1.25) (see Lemma 3.7 and Appendix A).

To take one step further from the rapid stabilization to obtain the finite time stabilization (Theorem 1.3) for the linearized system, we follow the strategy of Coron and Nguyen [21]. The idea is to stabilize the system more and more as the time t goes to T_- . More precisely, we use $Q = Q_n := Q(\lambda_n)$ in the time interval $[t_n, t_{n+1})$ for a suitable positive sequence $(\lambda_n) \rightarrow +\infty$ and for a suitable increasing sequence $(t_n) \rightarrow T$. To be able to apply the strategy in [21], one needs to understand the size of $\|Q_n^{-1}\|_{\mathcal{L}(\mathbb{H}_{1,\sharp})}^2$ as a function of λ_n (a good bound for the size of $\|Q_n\|_{\mathcal{L}(\mathbb{H})}$ follows from the admissibility of the control operator B , see (1.30)). This is given in Lemma 5.1 after establishing the cost of the control for small time (see Proposition 5.4). This result is interesting in itself and its proof uses similar techniques as in [52]. The way to gain suitable information in each time interval $[t_n, t_{n+1})$ here is different from the one in [21] for which precise estimates of kernels of transformations from the backstepping technique are derived using the information of the kernels. Our new way to get appropriate information to be able to apply the strategy in [21] is quite robust and can be used in different contexts where the size of the control cost is understood for small time. An application of this approach will be given in [49] to study the finite-time stabilization of a KdV control system.

1.3. Previous related results. The controllability properties for the Schrödinger equation were mostly studied in the usual linear setting (in contrast to the bilinear control problems considered here). For the control of the linear Schrödinger equation with internal control (localized on a subdomain), we refer to [35, 39], the survey [34], and the references therein. In this setting, we mention [40] for the stabilization. The first local controllability results on the bilinear Schrödinger equation appear in [3, 4]. These local controllability results have been extended under weaker assumptions in [6, 8], in a more general setting in infinite time [45], and also in the case of simultaneous controllability of a finite number of particles [42], and the references therein. Note that, despite the infinite speed of propagation, it was proved that a minimal amount of time is required for the controllability of some bilinear Schrödinger equations, see [16, 5, 8] (see also [42, 12]) and the references therein. In addition to the exact controllability and the stabilization, various aspects of the controllability of the bilinear Schrödinger systems have been investigated. Concerning the approximative controllability, this has been studied by the geometric control techniques via appropriate Galerkin approximations, see e.g., [15, 10, 11] and the references therein. The Lyapunov technique has been used to obtain the global controllability results, see, e.g., [41, 7, 43, 44] though no indication of the convergence rate is given.

1.4. The organization of the paper. The paper is organized as follows. In Section 2, we establish several results on A and $\mathcal{D}(A)$. In particular, we prove that A is skew-adjoint in Lemma 2.2.

²Hereafter, given a Hilbert space \mathcal{H} , we denote $\mathcal{L}(\mathcal{H})$ the space of all continuous linear applications from \mathcal{H} to \mathcal{H} equipped with the standard norm.

In Section 3, we establish the well-posedness and the stability of various linear and nonlinear Schrödinger systems. These results will be used in the proof of the main theorems mentioned above. Section 4 is devoted to the rapid stabilization, in particular, we prove Theorems 1.1 and 1.2 there. In Section 5, we study the finite time stabilization. We prove Theorem 1.3 using estimates on the cost of controls for the linearized system established there (see Propositions 5.1 and 5.3). The analysis of the upper bound (Proposition 5.1) is based on the moment method. The analysis of the lower bound (Proposition 5.3) is based on a lower bound of the cost of a singular perturbation control problem (Proposition 5.2). In Appendix A, we discuss a well-posedness result on control systems associated with operator semi-groups, which is used throughout the paper.

2. PRELIMINARIES

In this section, we will prove some properties related to A defined in (1.22) and Q defined in (1.34). We begin with

Lemma 2.1. *Let $\gamma_1, \gamma_2 \in \mathbb{R}$ and let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathbb{H} \rightarrow \mathbb{H}$ be defined by*

$$(2.1) \quad \mathcal{A}y = \begin{pmatrix} -\Delta y_2 + \gamma_1 y_2 \\ \Delta y_1 + \gamma_2 y_1 \end{pmatrix} \quad \text{with} \quad \mathcal{D}(\mathcal{A}) = \left\{ y \in \mathbb{H}; \mathcal{A}y \in \mathbb{H} \right\}.$$

We have

- i) *The set $\mathcal{D}(\mathcal{A}^\infty)$ is dense in \mathbb{H} .*
- ii) *The set $\mathcal{D}(\mathcal{A}^\infty) \cap \mathbb{H}_{1,\sharp}$ is dense in $\mathbb{H}_{1,\sharp}$.*
- iii) *The set $\mathcal{D}(\mathcal{A}^\infty) \cap \mathbb{H}_{1,\sharp}$ is dense in $\mathcal{D}(\mathcal{A}) \cap \mathbb{H}_{1,\sharp}$ equipped the graph-norm of $\mathcal{D}(\mathcal{A})$.*

Recall that

$$\mathcal{D}(\mathcal{A}^\infty) = \bigcap_{k \geq 1} \mathcal{D}(\mathcal{A}^k).$$

Proof. We first prove *i*). Let $y = (y_1, y_2)^\top \in \mathbb{H}$. Then

$$(2.2) \quad y_1 = \sum_{k=1}^{\infty} a_k \varphi_k \quad \text{and} \quad y_2 = \sum_{k=1}^{\infty} b_k \varphi_k,$$

for some $(a_k), (b_k) \subset \mathbb{R}$ such that $\sum_{k \geq 1} \lambda_k^3 (|a_k|^2 + |b_k|^2) < +\infty$. Denote

$$(2.3) \quad y_{1,n} = \sum_{k=1}^n a_k \varphi_k \quad \text{and} \quad y_{2,n} = \sum_{k=1}^n b_k \varphi_k,$$

and set

$$y_n = (y_{1,n}, y_{2,n})^\top.$$

Since

$$(2.4) \quad \varphi_k'' = -\lambda_k \varphi_k \text{ in } I \quad \text{and} \quad \varphi_k = 0 \text{ on } \partial I,$$

it follows from (1.20) that

$$(2.5) \quad y_n \in \mathcal{D}(\mathcal{A}^\infty).$$

It is clear that

$$(2.6) \quad y_n = (y_{1,n}, y_{2,n})^\top \rightarrow y \text{ in } \mathbb{H}.$$

Assertion *i*) now follows from (2.5) and (2.6).

We next deal with *ii*). We first note that

$$\mathcal{D}(\mathcal{A}^\infty) \cap \mathbb{H}_{1,\sharp} = \left\{ y = (y_1, y_2)^\top \in \mathcal{D}(\mathcal{A}^\infty); \int_I y_1 \varphi_1 = 0 \right\}$$

Let $y = (y_1, y_2)^\top \in \mathbb{H}_{1,\#}$ and define y_n by (2.3) using (2.2). Then

$$(2.7) \quad y_n \in \mathcal{D}(\mathcal{A}^\infty) \quad \text{and} \quad y_n \rightarrow y \text{ in } \mathbb{H}.$$

Define $\hat{y}_n = (\hat{y}_{1,n}, \hat{y}_{2,n})$ by

$$(2.8) \quad \hat{y}_{1,n} = y_{1,n} - \langle y_{1,n}, \varphi_1 \rangle_{L^2(I)} \varphi_1 \quad \text{and} \quad \hat{y}_{2,n} = y_{2,n},$$

and denote

$$\hat{y}_n = (\hat{y}_{1,n}, \hat{y}_{2,n})^\top.$$

It follows from (2.4) that

$$(2.9) \quad \hat{y}_n \in \mathcal{D}(\mathcal{A}^\infty) \cap \mathbb{H}_{1,\#}.$$

Using the fact

$$(2.10) \quad \langle y_{1,n}, \varphi_1 \rangle_{L^2(I)} \rightarrow \langle y_1, \varphi_1 \rangle_{L^2(I)} \stackrel{y \in \mathbb{H}_{1,\#}}{=} 0,$$

we derive from (2.7) and (2.8) that

$$\hat{y}_n \rightarrow y \text{ in } \mathbb{H}.$$

Assertion *ii*) is proved.

We finally establish *iii*). Let $y = (y_1, y_2)^\top \in \mathcal{D}(\mathcal{A}) \cap \mathbb{H}_{1,\#}$ and define y_n by (2.3) using (2.2). Then

$$(2.11) \quad y_n \in \mathcal{D}(\mathcal{A}^\infty) \quad \text{and} \quad y_n \rightarrow y \text{ in } \mathcal{D}(\mathcal{A}).$$

Define $\hat{y}_n = (\hat{y}_{1,n}, \hat{y}_{2,n})^\top$ by (2.8). Then, by (2.9),

$$\hat{y}_n \in \mathcal{D}(\mathcal{A}^\infty) \cap \mathbb{H}_{1,\#}.$$

Using (2.10), we derive from (2.11) that

$$\hat{y}_n \rightarrow y \text{ in } \mathcal{D}(\mathcal{A}).$$

Assertion *iii*) is established.

The proof is complete. □

We next establish a result which implies that A is skew-adjoint.

Lemma 2.2. *Let $\gamma \in \mathbb{R}$ and let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathbb{H} \rightarrow \mathbb{H}$ be defined by*

$$(2.12) \quad \mathcal{A}y = \begin{pmatrix} -\Delta y_2 - \gamma y_2 \\ \Delta y_1 + \gamma y_1 \end{pmatrix} \quad \text{and} \quad \mathcal{D}(\mathcal{A}) = \left\{ y \in \mathbb{H}; \mathcal{A}y \in \mathbb{H} \right\}.$$

Then \mathcal{A} is skew-adjoint, i.e., $\mathcal{D}(\mathcal{A}^) = \mathcal{D}(\mathcal{A})$ and $\mathcal{A}^* = -\mathcal{A}$ in $\mathcal{D}(\mathcal{A})$.*

We recall, by Lemma 2.1, that $\mathcal{D}(\mathcal{A})$ is dense in \mathbb{H} .

Proof. Since

$$\varphi_k'' = -\lambda_k \varphi_k \text{ in } I,$$

we derive from the definition of \mathbb{H} that $y = (y_1, y_2) \in \mathcal{D}(\mathcal{A})$ if and only if

$$(2.13) \quad y \in [H^5(I)]^2,$$

$$(2.14) \quad y_\ell(x) = y_\ell''(x) = y_\ell''''(x) = 0 \text{ for } x \in \partial I, \ell = 1, 2,$$

Using this fact, we derive, by integration by parts, for $y = (y_1, y_2)^\top \in \mathcal{D}(\mathcal{A})$ and $z = (z_1, z_2) \in \mathcal{D}(\mathcal{A})$, that

$$(2.15) \quad - \int_I \Delta y_2 z_1 = - \int_I y_2 \Delta z_1, \quad - \int_I \Delta y_2' z_1' = - \int_I y_2' \Delta z_1',$$

$$(2.16) \quad - \int_I \Delta y_2'' z_1'' = - \int_I y_2'' \Delta z_1'', \quad - \int_I \Delta y_2''' z_1''' = - \int_I y_2''' \Delta z_1''',$$

$$(2.17) \quad \int_I \Delta y_1 z_2 = \int_I y_1 \Delta z_2, \quad \int_I \Delta y_1' z_2' = \int_I y_1' \Delta z_2',$$

$$(2.18) \quad \int_I \Delta y_1'' z_2'' = \int_I y_1'' \Delta z_2'', \quad \int_I \Delta y_1''' z_2''' = \int_I y_1''' \Delta z_2''.$$

It follows that, for $y \in \mathcal{D}(\mathcal{A})$ and $z \in \mathcal{D}(\mathcal{A})$,

$$\langle \mathcal{A}y, z \rangle_{\mathbb{H}} = \langle y, -\mathcal{A}z \rangle_{\mathbb{H}}.$$

It remains to show that $\mathcal{D}(\mathcal{A}^*) \subset \mathcal{D}(\mathcal{A})$. This is equivalent to establish that if $z \in \mathbb{H}$ is such that

$$(2.19) \quad |\langle \mathcal{A}y, z \rangle_{\mathbb{H}}| \leq C \|y\|_{\mathbb{H}} \text{ for all } y \in \mathcal{D}(\mathcal{A})$$

for some positive constant $C = C(z)$ independent of y , then $z \in \mathcal{D}(\mathcal{A})$.

Indeed, fix such a z . From (2.19), we deduce from (2.15) and (2.16) that, for $y \in \mathcal{D}(\mathcal{A})$,

$$(2.20) \quad \left| - \int_I \Delta y_2''' z_1''' + \int_I \Delta y_1''' z_2''' \right| \leq C \|y\|_{\mathbb{H}}.$$

By taking $y_1 = 0$ in (2.20), we obtain that, for $y_2 \in H^5(I)$ with $y_2 = y_2'' = y_2''' = 0$ on ∂I , it holds

$$(2.21) \quad \left| - \int_I \Delta y_2''' z_1''' \right| \leq C \|y_2\|_{H^3(I)}.$$

Given $\varphi \in C^\infty(\bar{I}; \mathbb{R})$, define, for $x \in \bar{I}$,

$$(2.22) \quad \begin{aligned} \xi_3(x) &= \varphi(x) - \frac{1}{2} \varphi'(1) x^2 + \frac{1}{2} \varphi'(0) (1-x)^2, \\ \xi_2(x) &= \int_0^x \xi_3(s) ds - x \int_0^1 \xi_3(s) ds, \quad \xi_1(x) = \int_0^x \xi_2(s) ds, \end{aligned}$$

and

$$(2.23) \quad y_2(x) = \int_0^x \xi_1(s) ds - x \int_0^1 \xi_1(s) ds.$$

Simple computations give, for $x \in \bar{I}$,

$$y_2' = \xi_1 - \int_0^1 \xi_1(s) ds, \quad y_2'' = \xi_2, \quad y_2''' = \xi_3 - \int_0^1 \xi_3(s) ds, \quad y_2'''' = \varphi' - \varphi'(1)x - \varphi'(0)(1-x).$$

One can then check that $y_2 \in H^5(I)$ with $y_2 = y_2'' = y_2''' = 0$ on ∂I . It follows from (2.21) applied to y_2 given by (2.23) that

$$\left| - \int_I (\Delta \varphi - \varphi'(1) + \varphi'(0)) z_1''' \right| \leq C \|\xi_3\|_{L^2(I)} \text{ for } \varphi \in C^\infty(\bar{I}),$$

where ξ_3 is defined by (2.22). Since $z_1'' = 0$ on ∂I , we deduce that

$$\left| - \int_I \Delta \varphi z_1''' \right| \leq C \|\xi_3\|_{L^2(I)} \text{ for } \varphi \in C^\infty(\bar{I}),$$

where ξ_3 is defined by (2.22). By first considering $\varphi \in C_c^\infty(I)$ and then using $\varphi \in C^\infty(\bar{I})$ with $\varphi' = 0$ on ∂I , we obtain

$$(2.24) \quad z_1''' \in H^2(I) \quad \text{and} \quad z_1'''' = 0 \text{ on } \partial I.$$

Similarly, by taking $y_2 = 0$ in (2.20), we derive that

$$(2.25) \quad z_2''' \in H^2(I) \quad \text{and} \quad z_2'''' = 0 \text{ on } \partial I.$$

Combining (2.24) and (2.25) yields

$$z \in \mathcal{D}(\mathbf{A}).$$

The proof is complete. \square

3. WELL-POSEDNESS AND STABILITY OF SCHRÖDINGER SYSTEMS

In this section, we establish the well-posedness and the stability of various systems related to the linear system (1.6) and the nonlinear system (1.5). The main goal is to formulate and establish results which are compatible with the theory of control systems associated with semi-group. Without the language of semi-group, some related results can be found in [6].

We first introduce $\mathbf{A} : \mathcal{D}(\mathbf{A}) \subset \mathbf{H} \rightarrow \mathbf{H}$ defined by

$$(3.1) \quad \mathbf{A}\Psi = i\Delta\Psi \quad \text{and} \quad \mathcal{D}(\mathbf{A}) = \left\{ \Psi \in \mathbf{H}; \mathbf{A}\Psi \in \mathbf{H} \right\}.$$

We have

Lemma 3.1. *We have*

$\mathcal{D}(\mathbf{A})$ is dense in \mathbf{H} and \mathbf{A} is skew-adjoint.

Proof. The conclusion is a consequence of Lemmas 2.1 and 2.2 with $\gamma = 0$ after considering the real part and the imaginary part of Ψ and $\mathbf{A}\Psi$. \square

We next introduce a useful operator related to the definitions of B in (1.23) and $uF(y)$ in (1.27).

Definition 3.1. *Given $T > 0$. Define*

$$\mathbf{T} : L^2((0, T); H^3(I; \mathbb{C}) \cap H_0^1(I; \mathbb{C})) \rightarrow L^1((0, T); \mathcal{D}(\mathbf{A}^*)')$$

by, for all $\varphi \in \mathcal{D}(\mathbf{A}^*)$,

$$(3.2) \quad \langle \mathbf{T}(f)(t, \cdot), \varphi \rangle_{\mathcal{D}(\mathbf{A}^*)', \mathcal{D}(\mathbf{A}^*)} = \langle f(t, \cdot), \varphi \rangle_{H^3(I)} - f_{xx}(t, 1)\varphi_{xxx}(1) + f_{xx}(t, 0)\varphi_{xxx}(0),$$

for $f \in L^2((0, T); H^3(I; \mathbb{C}) \cap H_0^1(I; \mathbb{C}))$.

We next discuss the well-posedness and the stability of linear systems.

Lemma 3.2. *Let $0 < T < T_0$ and $\lambda \in \mathbb{R}$. Let $\Phi_0 \in \mathbf{H}$ and $\mathbf{f} \in L^1((0, T); \mathcal{D}(\mathbf{A}^*)')$. There exists a unique weak solution $\Phi \in C([0, T]; \mathcal{D}(\mathbf{A}^*))$ to the system*

$$(3.3) \quad \begin{cases} i\Phi_t = -\Delta\Phi - \lambda\Phi + \mathbf{f} & \text{in } (0, T) \times I, \\ \Phi(t, 0) = \Phi(t, 1) = 0 & \text{in } (0, T), \\ \Phi(0, \cdot) = \Phi_0 & \text{in } I, \end{cases}$$

i.e.

$$(3.4) \quad i \frac{d}{dt} \langle \Phi, \Psi \rangle_{\mathbf{H}} = -\langle \Phi, \Delta\Psi \rangle_{\mathbf{H}} - \lambda \langle \Phi, \Psi \rangle_{\mathbf{H}} + \langle \mathbf{f}, \Psi \rangle_{\mathcal{D}(\mathbf{A}^*)', \mathcal{D}(\mathbf{A}^*)} \quad \text{in } (0, T)$$

in the distributional sense for all $\Psi \in \mathcal{D}(\mathbf{A}^*)$. Let $f \in L^2((0, T); H^3(I) \cap H_0^1(I))$. Define $\mathbf{f} = \mathbf{T}(f) \in L^1((0, T); \mathcal{D}(\mathbf{A}^*)')$. Then the weak solution Ψ of (3.3) satisfies $\Psi \in C([0, T]; \mathbf{H})$ and

$$(3.5) \quad \|\Phi(t, \cdot)\|_{H^3(I)} \leq C \left(\|\Phi_0\|_{H^3(I)} + \|f\|_{L^2((0, t); H^3(I))} \right) \quad \text{in } [0, T],$$

for some positive constant C depending only on T_0 .

Proof of Lemma 3.2. The existence and uniqueness of solutions in $C([0, T]; \mathcal{D}(\mathbf{A}^*))'$ follows from Proposition A.1 in the appendix. It then suffices to show the existence of a solution $\Psi \in C([0, T]; \mathbf{H})$ satisfying (3.5).

We first deal with the system

$$(3.6) \quad \begin{cases} i\Phi_t = -\Delta\Phi + \mathbf{f} & \text{in } (0, T) \times I, \\ \Phi(t, 0) = \Phi(t, 1) = 0 & \text{in } (0, T), \\ \Phi(0, \cdot) = \Phi_0 & \text{in } I \end{cases}$$

instead of (3.3), i.e., we consider (3.3) with $\lambda = 0$. We search $\Phi \in C([0, T]; \mathbf{H})$ under the form

$$\Phi(t, x) = \sum_{k \geq 1} a_k(t) \varphi_k(x) \text{ in } (0, T) \times I.$$

Using (3.16) with $\lambda = 0$ and $\varphi = \varphi_k$, we obtain

$$(3.7) \quad i(1 + \lambda_k + \lambda_k^2 + \lambda_k^3) a'_k = \lambda_k(1 + \lambda_k + \lambda_k^2 + \lambda_k^3) a_k + c_k \text{ in } (0, T),$$

where

$$(3.8) \quad c_k(t) = \langle f(t, \cdot), \varphi_k \rangle_{H^3(I)} - f_{xx}(t, 1) \varphi_{k,xxx}(1) + f_{xx}(t, 0) \varphi_{k,xxx}(0) \text{ in } (0, T).$$

We derive from (3.7) that

$$a'_k = -i\lambda_k a_k - i b_k \text{ in } (0, T) \text{ where } b_k = \frac{c_k}{1 + \lambda_k + \lambda_k^2 + \lambda_k^3}.$$

We then get

$$(3.9) \quad a_k(t) = e^{-i\lambda_k t} a_k(0) - i \int_0^t e^{-i\lambda_k(t-s)} b_k(s) \text{ in } (0, T).$$

Combining (3.8) and (3.9) yields

$$(3.10) \quad \begin{aligned} \sum_{k \geq 1} k^6 |a_k(t)|^2 &\leq C \|\Phi_0\|_{H^3(I)}^2 + C \sum_{k \geq 1} \frac{1}{\lambda_k^3} \int_0^t |\langle f, \varphi_k \rangle_{H^3(I)}|^2 dt \\ &\quad + C \sum_{k \geq 1} \left| \int_0^t e^{i\lambda_k s} f_{xx}(s, 1) ds \right|^2 + C \sum_{k \geq 1} \left| \int_0^t e^{i\lambda_k s} f_{xx}(s, 0) ds \right|^2. \end{aligned}$$

Here and in what follows in this proof, C denotes a positive constant depending only on T_0 . We have

$$(3.11) \quad \sum_{k \geq 1} \frac{1}{\lambda_k^3} \int_0^t |\langle f, \varphi_k \rangle_{H^3(I)}|^2 dt \leq C \int_0^t \|f(s, \cdot)\|_{L^2((0, T); H^3(I))}^2 ds.$$

Applying Ingham's inequality (see, e.g., [29, Theorem 4.3 on page 59]) and using the properties of Riesz basis, see, e.g., [58, Theorem 9 on page 32], we obtain

$$(3.12) \quad \sum_{k \geq 1} \left| \int_0^t e^{i\lambda_k s} f_{xx}(s, 1) ds \right|^2 \leq C \int_0^t |f_{xx}(s, 1)|^2 ds$$

and

$$(3.13) \quad \sum_{k \geq 1} \left| \int_0^t e^{i\lambda_k s} f_{xx}(s, 0) ds \right|^2 \leq C \int_0^t |f_{xx}(s, 0)|^2 ds.$$

Combining (3.10), (3.11), (3.12), and (3.13) yields

$$\|\Phi(t)\|_{H^3(I)}^2 \leq C \left(\|\Phi_0\|_{H^3(I)}^2 + \int_0^t \|f(s, \cdot)\|_{H^3(I)}^2 ds \right) \text{ in } [0, T].$$

One can also check that $\Phi \in C([0, T]; \mathbf{H})$ is also a weak solution of (3.3). The conclusion in the case $\lambda = 0$ follows.

To obtain the conclusion for (3.3) for a general λ , one first notes that if Φ is a solution of (3.6) then $\Phi(t, x)e^{-i\lambda t}$ is a solution of (3.3) with the same initial condition and with the source $e^{-i\lambda t}f(t, x)$ and then apply the result in the case $\lambda = 0$ to reach the conclusion.

The proof is complete. \square

The following simple result is useful to compare with previous results and motivates the definition of the operator \mathbf{T} in (3.2).

Lemma 3.3. *Let $0 < T < T_0$ and $\lambda \in \mathbb{R}$. Let $\Phi_0 \in \mathbf{H}$ and let $f \in L^2((0, T); H^3(I) \cap H_0^1(I))$. Define $\mathbf{f} = \mathbf{T}(f) \in L^1((0, T); \mathcal{D}(\mathbf{A}^*))$. Then $\Phi \in C([0, T]; \mathbf{H})$ is a unique weak solution to the system*

$$(3.14) \quad \begin{cases} i\Phi_t = -\Delta\Phi - \lambda\Phi + \mathbf{f} & \text{in } (0, T) \times I, \\ \Phi(t, 0) = \Phi(t, 1) = 0 & \text{in } (0, T), \\ \Phi(0, \cdot) = \Phi_0 & \text{in } I, \end{cases}$$

if and only if $\Phi \in C([0, T]; \mathbf{H})$ is a (weak) solution of the system

$$(3.15) \quad \begin{cases} i\Phi_t = -\Delta\Phi - \lambda\Phi + f & \text{in } (0, T) \times I, \\ \Phi(t, 0) = \Phi(t, 1) = 0 & \text{in } (0, T), \\ \Phi(0, \cdot) = \Phi_0 & \text{in } I, \end{cases}$$

in the sense that

$$(3.16) \quad i \frac{d}{dt} \langle \Phi, \varphi_k \rangle_{L^2(I)} = -\langle \Delta\Phi, \varphi_k \rangle_{L^2(I)} - \lambda \langle \Phi, \varphi_k \rangle_{L^2(I)} + \langle f, \varphi_k \rangle_{L^2(I)} \text{ in } (0, T)$$

in the distributional sense for all $k \geq 1$.

Proof. Let $\varphi \in H^3(I; \mathbb{C}) \cap H_0^1(I; \mathbb{C})$. We have

$$(1 + \lambda_k + \lambda_k^2 + \lambda_k^3) \langle \varphi, \varphi_k \rangle_{L^2(I)} = \langle \varphi, \varphi_k \rangle_{H^3(I)} - \varphi_{xx}(1)\varphi_{k,xxx}(1) + \varphi_{xx}(0)\varphi_{k,xxx}(0).$$

One can thus rewrite (3.9) under the form

$$(3.17) \quad i \frac{d}{dt} \langle \Phi, \varphi_k \rangle_{L^2(I)} = -\langle \Phi, \Delta\varphi_k \rangle_{L^2(I)} + \langle f, \varphi_k \rangle_{L^2(I)} \text{ in } (0, T).$$

The conclusion follows in the case $\lambda = 0$. The general case follows similarly. \square

We next make a connection with the definition of weak solutions used in [6]. Let $e^{i\Delta t} : L^2(I; \mathbb{C}) \rightarrow L^2(I; \mathbb{C})$ be defined by, for $\varphi \in L^2(I; \mathbb{C})$,

$$(3.18) \quad e^{it\Delta}\varphi = \sum_{k=1}^{\infty} \langle \varphi, \varphi_k \rangle_{L^2(I)} e^{-i\lambda_k t} \varphi_k,$$

and, for $\gamma \in \mathbb{R}$, let $e^{i(\Delta+\gamma)t} : L^2(I; \mathbb{C}) \rightarrow L^2(I; \mathbb{C})$ be defined by, for $\varphi \in L^2(I; \mathbb{C})$,

$$(3.19) \quad e^{it(\Delta+\gamma)}\varphi = \sum_{k=1}^{\infty} \langle \varphi, \varphi_k \rangle_{L^2(I)} e^{i(-\lambda_k+\gamma)t} \varphi_k.$$

We have

Lemma 3.4. *Let $0 < T < T_0$ and $\lambda \in \mathbb{R}$. Let $\Phi_0 \in \mathbf{H}$ and let $f \in L^2((0, T); H^3(I; \mathbb{C}) \cap H_0^1(I; \mathbb{C}))$. Define $\mathbf{f} = \mathbf{T}(f) \in L^1((0, T); \mathcal{D}(\mathbf{A}^*)')$. Then $\Phi \in C([0, T]; \mathbf{H})$ is a weak solution of (3.3) if and only if either*

$$(3.20) \quad \Phi(t, \cdot) = e^{it(\Delta+\gamma)}\Phi_0 - i \int_0^t e^{i(t-s)(\Delta+\gamma)}(f(s, \cdot) + (\gamma - \lambda)\Phi(s, \cdot)) ds \text{ for } t \in [0, T].$$

Proof. Assume that (3.20) holds for $\Phi \in C([0, T]; \mathbf{H})$. By taking the scalar product in $L^2(I)$ of the corresponding identity with φ_k , one derives that Φ is a weak solution of (3.3) by Lemma 3.3.

We next assume that $\Phi \in C([0, T]; \mathbf{H})$ is a weak solution of (3.3). We will prove (3.20). Since $\Phi \in C([0, T]; \mathbf{H})$ is a weak solution of (3.3), we deduce from Lemma 3.3 that

$$(3.21) \quad \langle \Phi(t, \cdot), \varphi_k \rangle_{L^2(I)} = \langle e^{it(\Delta+\gamma)}\Phi_0, \varphi_k \rangle_{L^2(I)} - \langle i \int_0^t e^{i(t-s)(\Delta+\lambda)}(f(s, \cdot) + (\gamma - \lambda)\Phi(s, \cdot)) ds, \varphi_k \rangle_{L^2(I)} \text{ for } t \in [0, T].$$

Set

$$\Psi(t, \cdot) = \int_0^t e^{i(t-s)(\Delta+\lambda)}(f(s, \cdot) + (\gamma - \lambda)\Phi(s, \cdot)) ds.$$

Since the space spanned by set of $(\varphi_k)_{k \geq 1}$ is dense in $L^2(I)$ and $\Psi \in C([0, T]; \mathbf{H})$, we obtain (3.20). \square

Remark 3.1. In [6, Proposition 2], the definition of the weak solutions in the sense of (3.20) is considered with $\gamma = 0$.

We next establish the well-posedness and stability of linear feedback systems.

Lemma 3.5. *Let $0 < T \leq T_0$ and $\Psi \in C([0, T]; H^3(I; \mathbb{C}) \cap H_0^1(I; \mathbb{C}))$, and let $\mathcal{L} \in \mathcal{L}(\mathbf{H}; \mathbb{C})$. Let $\Phi_0 \in \mathbf{H}$ and $f \in L^2((0, T); H^3(I; \mathbb{C}) \cap H_0^1(I; \mathbb{C}))$. There exists a unique weak solution $\Phi \in C([0, T]; \mathbf{H})$ to the system³*

$$(3.22) \quad \begin{cases} i\Phi_t = -\Delta\Phi - \lambda_1\Phi + \mathbf{g} & \text{in } (0, T) \times I, \\ \Phi(t, 0) = \Phi(t, 1) = 0 & \text{in } (0, T), \\ \Phi(0, \cdot) = \Phi_0 & \text{in } I, \end{cases}$$

where $\mathbf{g} \in L^1((0, T); \mathcal{D}(A^*)')$ is defined by $\mathbf{g} = \mathbf{T}(g)$ with

$$g(t, \cdot) = \mathcal{L}(\Phi(t, \cdot))\Psi(t, \cdot) + f(t, \cdot).$$

Moreover, there exist a positive constant C depending only on T_0 such that

$$(3.23) \quad \|\Phi(t, \cdot)\|_{H^3(I)} \leq e^{C(\|\mathcal{L}\|_{\mathcal{L}(\mathbf{H}; \mathbb{C})}^2 \|\Psi\|_{L^2((0, T); H^3(I))}^2)^{+1}} \left(\|\Phi_0\|_{H^3(I)} + \|f\|_{L^2((0, t); H^3(I))} \right) \text{ in } [0, T].$$

It is convenient to denote

$$(3.24) \quad \mathcal{X} := C([0, T]; \mathbf{H})$$

and to equip this space with the following standard norm

$$(3.25) \quad \|\Phi\|_{\mathcal{X}} := \sup_{[0, T]} \|\Phi(t, \cdot)\|_{\mathbf{H}}.$$

Then \mathcal{X} is a Banach space.

³The weak solution is understood in the sense given in Lemma 3.2.

Proof. Define $\mathcal{F} : \mathcal{X} \rightarrow \mathcal{X}$ by $\mathcal{F}(\tilde{\Phi}) = \Phi$, where Φ is the unique weak solution of the system

$$(3.26) \quad \begin{cases} i\Phi_t = -\Delta\Phi - \lambda_1\Phi + \mathbf{T}(\mathcal{L}(\tilde{\Phi}(t, \cdot)\Psi + f(t, \cdot))) & \text{in } (0, T) \times I, \\ \Phi(t, 0) = \Phi(t, 1) = 0 & \text{in } (0, T), \\ \Phi(0, \cdot) = \Phi_0 & \text{in } I. \end{cases}$$

Let $\tilde{\Phi}_1, \tilde{\Phi}_2 \in \mathcal{X}$ and denote $\Phi_1 = \mathcal{F}(\tilde{\Phi}_1)$ and $\Phi_2 = \mathcal{F}(\tilde{\Phi}_2)$. Set, in $[0, T] \times I$,

$$\delta\tilde{\Phi} = \tilde{\Phi}_2 - \tilde{\Phi}_1 \quad \text{and} \quad \delta\Phi = \Phi_2 - \Phi_1.$$

Define $\mathbf{h} \in L^1((0, T); \mathcal{D}(A^*)')$ by $\mathbf{h} = \mathbf{T}(h)$ where

$$h(t, \cdot) = \mathcal{L}(\delta\tilde{\Phi}(t, \cdot))\Psi(t, \cdot).$$

Then $\delta\Phi \in C([0, T]; \mathbf{H})$ is the unique weak solution of the system

$$(3.27) \quad \begin{cases} i\delta\Phi_t = -\Delta\delta\Phi - \lambda_1\delta\Phi + \mathbf{h} & \text{in } (0, T) \times I, \\ \Phi(t, 0) = \Phi(t, 1) = 0 & \text{in } (0, T), \\ \delta\Phi(0) = 0 & \text{in } I. \end{cases}$$

Applying Lemma 3.2 with $f = h$, we obtain

$$\|\delta\Phi\|_{\mathcal{X}} \leq C\|\mathcal{L}\|_{\mathcal{L}(\mathbf{H}, \mathbf{C})}\|\Psi\|_{L^2((0, T); H^3(I))}\|\delta\tilde{\Phi}\|_{\mathcal{X}}.$$

Here and in what follows in this proof, C denotes a positive constant depending only on T_0 .

Thus if $C\|\mathcal{L}\|_{\mathcal{L}(\mathbf{H}, \mathbf{C})}\|\Psi\|_{C([0, T]; H^3(I))} \leq 1/2$, then \mathcal{F} is a contracting map from \mathcal{X} into itself. Therefore, there exists a unique weak solution $\Phi \in C([0, T]; \mathbf{H})$ of the equation

$$\mathcal{F}(\Phi) = \Phi,$$

which is also a unique weak solution of (3.22). Moreover, we have, by Lemma 3.2,

$$\|\Phi\|_{\mathcal{X}} \leq C\|\Phi_0\|_{\mathbf{H}} + C\|f\|_{L^2((0, T); H^3(I))} + \frac{1}{2}\|\Phi\|_{\mathcal{X}},$$

which yields

$$\|\Phi\|_{\mathcal{X}} \leq C\left(\|\Phi_0\|_{\mathbf{H}} + \|f\|_{L^2((0, T); H^3(I))}\right).$$

The general case can be then proved as follows. Devide the interval $[0, T]$ into subintervals $[T_0, T_1], [T_1, T_2], \dots, [T_{n-1}, T_n]$ (with $T_0 = 0$ and $T_n = T$) such that $C\|\mathcal{L}\|_{\mathcal{L}(\mathbf{H}, \mathbf{C})}\|\Psi\|_{L^2((T_{i-1}, T_i); H^3(I))} \leq 1/2$ and note that n can be bounded above by $C(\|\mathcal{L}\|_{\mathcal{L}(\mathbf{H}, \mathbf{C})}^2\|\Psi\|_{L^2((0, T); H^3(I))}^2 + 1)$. We then have

$$(3.28) \quad \begin{aligned} \|\Phi\|_{\mathcal{X}} &\leq C^n\left(\|\Phi_0\|_{\mathbf{H}} + \|f\|_{L^2((0, T); H^3(I))}\right) \\ &\leq e^{C(\|\mathcal{L}\|_{\mathcal{L}(\mathbf{H}, \mathbf{C})}^2\|\Psi\|_{L^2((0, T); H^3(I))}^2 + 1)}\left(\|\Phi_0\|_{\mathbf{H}} + \|f\|_{L^2((0, T); H^3(I))}\right), \end{aligned}$$

which is the conclusion. \square

Remark 3.2. Let $u \in L^2((0, T); \mathbb{R})$. A related result corresponding to the case $\mathcal{L}(\Phi) = \Phi$ and $\Psi(t, x) = u(t)\mu(x)$ is considered in [6, Proposition 2].

We next study the local well-posedness and the stability of nonlinear feedback systems.

Lemma 3.6. *Let $0 < T < T_0$, $\Psi \in C([0, T]; H^3(I; \mathbb{C}) \cap H_0^1(I; \mathbb{C}))$, and let $\mathcal{L} \in \mathcal{L}(\mathbf{H}; \mathbb{C})$. There exists a positive constant C depending only on T_0 such that for $\Phi_0 \in \mathbf{H}$ and for $f \in L^2((0, T); H^3(I; \mathbb{C}) \cap H_0^1(I; \mathbb{C}))$ satisfying*

$$(3.29) \quad \|\Phi_0\|_{H^3(I)} \leq \varepsilon, \quad \|f\|_{L^2((0, T); H^3(I))} \leq \varepsilon,$$

and

$$(3.30) \quad \|\mathcal{L}\|_{\mathcal{L}(\mathbf{H};\mathbf{C})} e^{C(\|\mathcal{L}\|_{\mathcal{L}(\mathbf{H};\mathbf{C})}^2 \|\Psi\|_{L^2((0,T);H^3(I))}^2)^{+1}} \varepsilon \leq 1/2,$$

there exists a unique weak solution to the system

$$(3.31) \quad \begin{cases} i\Phi_t = -\Delta\Phi - \lambda_1\Phi + \mathbf{g} & \text{in } (0, T) \times I, \\ \Phi(t, 0) = \Phi(t, 1) = 0 & \text{in } (0, T), \\ \Phi(0, \cdot) = \Phi_0 & \text{in } I, \end{cases}$$

where $\mathbf{g} \in L^1((0, T); \mathcal{D}(A^*))$ is defined by $\mathbf{g} = \mathbf{T}(g)$ with

$$g = \mathcal{L}(\Phi(t, \cdot))\Psi + \mathcal{L}(\Phi(t, \cdot))\Phi + f.$$

Moreover,

$$(3.32) \quad \|\Phi(t, \cdot)\|_{\mathbf{H}} \leq C \left(\|\Phi_0\|_{\mathbb{H}} + \|f\|_{L^2((0,T);H^3(I))} \right) \text{ in } [0, T].$$

Proof. Let \mathcal{X} be defined in (3.24) with the norm given in (3.25). Let $\hat{\Phi} \in C([0, T]; \mathbf{H})$ be the unique weak solution of the linear system

$$(3.33) \quad \begin{cases} i\hat{\Phi}_t = -\Delta\hat{\Phi} - \lambda_1\hat{\Phi} + \mathbf{T}(\mathcal{L}(\hat{\Phi}(t, \cdot))\Psi + f) & \text{in } (0, T) \times I, \\ \hat{\Phi}(t, 0) = \hat{\Phi}(t, 1) = 0 & \text{in } (0, T), \\ \hat{\Phi}(0, \cdot) = \Phi_0 & \text{in } I. \end{cases}$$

For $\varepsilon > 0$, let $B_{\mathcal{X}}(\hat{\Phi}, \varepsilon)$ denote the open ball of radius ε centered at $\hat{\Phi}$ in \mathcal{X} and let $\overline{B_{\mathcal{X}}(\hat{\Phi}, \varepsilon)}$ be its closure in \mathcal{X} . Assume (3.29) with ε small.

Set

$$\alpha = \|\mathcal{L}\|_{\mathcal{L}(\mathbf{H};\mathbf{C})}^2 \|\Psi\|_{L^2((0,T);H^3(I))}^2 + 1$$

and

$$Data = \|\Phi_0\|_{\mathbb{H}} + \|f\|_{L^2((0,T);H^3(I))}.$$

Define

$$\mathcal{F} : \overline{B_{\mathcal{X}}(\hat{\Phi}, \varepsilon)} \rightarrow \mathcal{X},$$

where, for $\tilde{\Phi} \in \overline{B_{\mathcal{X}}(\hat{\Phi}, \varepsilon)}$, $\mathcal{F}(\tilde{\Phi}) = \Phi \in C([0, T]; \mathbf{H})$ is the unique weak solution of the system

$$(3.34) \quad \begin{cases} i\Phi_t = -\Delta\Phi - \lambda_1\Phi + \mathbf{T}(\mathcal{L}(\Phi(t, \cdot))\Psi + \mathcal{L}(\Phi(t, \cdot))\tilde{\Phi}(t, x) + f) & \text{in } (0, T) \times I, \\ \Phi(t, 0) = \Phi(t, 1) = 0 & \text{in } (0, T), \\ \Phi(0, \cdot) = \Phi_0 & \text{in } I. \end{cases}$$

Applying Lemma 3.5 to $\hat{\Phi}$ and Φ , we have

$$(3.35) \quad \|\hat{\Phi}\|_{\mathcal{X}} \leq e^{C\alpha} Data.$$

and

$$(3.36) \quad \begin{aligned} \|\Phi\|_{\mathcal{X}} &\leq e^{C\alpha} \left(Data + \|\mathcal{L}(\Phi(t, \cdot))\tilde{\Phi}(t, x)\|_{L^2((0,T);H^3(I))} \right) \\ &\leq e^{C\alpha} \left(Data + \|\mathcal{L}\|_{\mathcal{L}(\mathbf{H};\mathbf{C})} \|\Phi\|_{\mathcal{X}} \|\tilde{\Phi}\|_{L^2((0,T);H^3(I))} \right) \\ &\leq e^{C\alpha} \left(Data + \|\mathcal{L}\|_{\mathcal{L}(\mathbf{H};\mathbf{C})} \|\Phi\|_{\mathcal{X}} e^{C\alpha} \varepsilon \right). \end{aligned}$$

Here and in what follows C denotes a positive constant depending only on T_0 and can change from one place to another.

In what follows, we assume that

$$(3.37) \quad \|\mathcal{L}\|_{\mathcal{L}(\mathbf{H};\mathbb{C})} e^{C\alpha} \varepsilon \leq 1/2.$$

We derive from (3.36) that

$$(3.38) \quad \|\Phi\|_{\mathcal{X}} \leq e^{C\alpha} Data.$$

Combining (3.33) and (3.34) yields

$$i(\Phi - \hat{\Phi})_t = -\Delta(\Phi - \hat{\Phi}) - \lambda_1(\Phi - \hat{\Phi}) + \mathbf{T}(\mathcal{L}(\Phi(t, \cdot) - \hat{\Phi}(t, \cdot))\Psi + \mathcal{L}(\Phi(t, \cdot))\tilde{\Phi}(t, \cdot)) \text{ in } (0, T) \times I,$$

and

$$(3.39) \quad \begin{cases} (\Phi - \hat{\Phi})(t, 0) = (\Phi - \hat{\Phi})(t, 1) = 0 & \text{in } (0, T), \\ (\Phi - \hat{\Phi})(0, \cdot) = 0. \end{cases}$$

Applying Lemma 3.5 to $\Phi - \hat{\Phi}$, we derive that

$$(3.40) \quad \begin{aligned} \|\Phi - \hat{\Phi}\|_{\mathcal{X}} &\leq e^{C\alpha} \|\mathcal{L}(\Phi(t, \cdot))\tilde{\Phi}(t, x)\|_{L^2((0, T); H^3(I))} \\ &\leq e^{C\alpha} \|\mathcal{L}\|_{\mathcal{L}(\mathbf{H};\mathbb{C})} \|\Phi\|_{\mathcal{X}} \|\tilde{\Phi}\|_{\mathcal{X}} \stackrel{(3.35), (3.38)}{\leq} e^{C\alpha} \|\mathcal{L}\|_{\mathcal{L}(\mathbf{H};\mathbb{C})} \varepsilon^2. \end{aligned}$$

Thus \mathcal{F} maps $\overline{B_{\mathcal{X}}(\Psi, \varepsilon)}$ into itself provided that

$$e^{C\alpha} \|\mathcal{L}\|_{\mathcal{L}(\mathbf{H};\mathbb{C})} \varepsilon \leq 1/2.$$

With $\Phi_1 = \mathcal{F}(\tilde{\Phi}_1)$ and $\Phi_2 = \mathcal{F}(\tilde{\Phi}_2)$, one has

$$(3.41) \quad \begin{aligned} i(\Phi_1 - \Phi_2)_t &= -\Delta(\Phi_1 - \Phi_2) - \lambda_1(\Phi_1 - \Phi_2) + \mathbf{T}(\mathcal{L}(\Phi_1(t, \cdot) - \Phi_2(t, \cdot))\Psi \\ &\quad + \mathcal{L}(\Phi_1(t, \cdot))\tilde{\Phi}_1 - \mathcal{L}(\Phi_2(t, \cdot))\tilde{\Phi}_2) \text{ in } (0, T) \times I. \end{aligned}$$

Since

$$\mathcal{L}(\Phi_1(t, \cdot))\tilde{\Phi}_1 - \mathcal{L}(\Phi_2(t, \cdot))\tilde{\Phi}_2 = \mathcal{L}(\Phi_1(t, \cdot) - \Phi_2(t, \cdot))\tilde{\Phi}_1 + \mathcal{L}(\Phi_2(t, \cdot))(\tilde{\Phi}_1 - \tilde{\Phi}_2),$$

it follows from (3.35) and (3.38) that

$$(3.42) \quad \begin{aligned} \|\mathcal{L}(\Phi_1(t, \cdot))\tilde{\Phi}_1 - \mathcal{L}(\Phi_2(t, \cdot))\tilde{\Phi}_2\|_{L^2((0, T); H^3(I))} \\ \leq e^{C\alpha} \|\mathcal{L}\|_{\mathcal{L}(\mathbf{H};\mathbb{C})} \varepsilon \|\Phi_1 - \Phi_2\|_{\mathcal{X}} + e^{C\alpha} \|\mathcal{L}\|_{\mathcal{L}(\mathbf{H};\mathbb{C})} \varepsilon \|\tilde{\Phi}_1 - \tilde{\Phi}_2\|_{\mathcal{X}} \end{aligned}$$

Combining (3.41) and (3.42) and applying Lemma 3.5 to $\Phi_1 - \Phi_2$, we derive that

$$(3.43) \quad \|\Phi_1 - \Phi_2\|_{\mathcal{X}} \leq e^{C\alpha} \left(\|\mathcal{L}\|_{\mathcal{L}(\mathbf{H};\mathbb{C})} \varepsilon \|\Phi_1 - \Phi_2\|_{\mathcal{X}} + \|\mathcal{L}\|_{\mathcal{L}(\mathbf{H};\mathbb{C})} \varepsilon \|\tilde{\Phi}_1 - \tilde{\Phi}_2\|_{\mathcal{X}} \right).$$

Thus \mathcal{F} is contracting provided that

$$\|\mathcal{L}\|_{\mathcal{L}(\mathbf{H};\mathbb{C})} e^{C\alpha} \varepsilon \leq 1/2.$$

The conclusion follows from a standard fixed point theorem and (3.36). \square

We next translate the previous well-posedness result to the semi-group related to A defined in (1.22), which involves the definition of Q and the feedback. We only do it for Lemma 3.2. The statement and the proof of the corresponding variants of Lemma 3.5 and Lemma 3.6 are omitted to avoid repetition. Concerning Lemma 3.2, we have

Lemma 3.7. *Let $T > 0$, $\lambda \in \mathbb{R}$, and let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathbb{H} \rightarrow \mathbb{H}$ be defined by*

$$(3.44) \quad \mathcal{A}y = \begin{pmatrix} -\Delta y_2 - \lambda y_2 \\ \Delta y_1 + \lambda y_1 \end{pmatrix} \quad \text{and} \quad \mathcal{D}(\mathcal{A}) = \left\{ y \in \mathbb{H}; \mathcal{A}y \in \mathbb{H} \right\}.$$

Let $y_0 \in \mathbb{H}$ and $\mathbf{g} \in L^1((0, T); \mathcal{D}(\mathcal{A}^)')$. There exists a unique weak solution $y \in C([0, T]; \mathcal{D}(\mathcal{A}^*)')$ to the system*

$$(3.45) \quad \begin{cases} y_t = \mathcal{A}y + \mathbf{g} & \text{in } (0, T) \times I, \\ y(0, \cdot) = y_0 & \text{in } I. \end{cases}$$

Let $g \in L^2((0, T); H^3(I) \cap H_0^1(I))$. Define $\mathbf{g} = \mathcal{T}(g) \in L^1((0, T); \mathcal{D}(\mathcal{A}^)')$ (see Definition 3.2 below). Then $y \in C([0, T]; \mathbb{H})$. Moreover, $y = (y_1, y_2)^\top$ is a weak solution of (3.3) if and only if $\Phi := y_1 + iy_2$ is a weak solution of (3.2) with*

$$\Phi_0 = y_1(0, \cdot) + iy_2(0, \cdot) \quad \text{and} \quad f = -g_2 + ig_1 \quad \text{where } g = (g_1, g_2),$$

and $\mathbf{f} = \mathbf{T}(f)$. We also have

$$\|y(t)\|_{\mathbb{H}} \leq C \left(\|y_0\|_{\mathbb{H}} + \|g\|_{L^2((0, T); H^3(I))} \right) \quad \text{in } [0, T],$$

where C is a positive constant depending only on T_0 .

In Lemma 3.7, we used the following definition.

Definition 3.2. *Let $T > 0$ and $\lambda \in \mathbb{R}$, and let $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ be defined by (3.44). Define*

$$\mathcal{T} : L^2((0, T); H^3(I; \mathbb{R}^2) \cap H_0^1(I; \mathbb{R}^2)) \rightarrow L^1((0, T); \mathcal{D}(\mathcal{A}^*)')$$

by

$$(3.46) \quad \langle \mathcal{T}(g)(t, \cdot), \varphi \rangle_{\mathcal{D}(\mathcal{A}^*)', \mathcal{D}(\mathcal{A}^*)} = \langle g(t, \cdot), \varphi \rangle_{H^3(I)} - \langle g(t, 1), \varphi_{xxx}(1) \rangle_{\mathbb{R}^2} + \langle g(t, 0), \varphi_{xxx}(0) \rangle_{\mathbb{R}^2}.$$

Proof of Lemma 3.7. By Proposition A.1 in the appendix, there exists a unique weak solution $y \in C([0, T]; \mathcal{D}(\mathcal{A}^*)')$. Let $\Phi \in C([0, T]; \mathbf{H})$ be the unique weak solution of (3.3) with

$$\Phi_0 = y_1(0, \cdot) + iy_2(0, \cdot) \quad \text{and} \quad f = -g_2 + ig_1.$$

Let y_1 and y_2 be the real part and the imaginary part of Φ , respectively, and denote $y = (y_1, y_2)^\top$. Then

$$y \in C([0, T]; \mathbb{H}).$$

By Lemma 3.2, it suffices to prove that y is a weak solution of (3.45). This follows from the definition of weak solutions associated with \mathbf{A} . \square

4. RAPID STABILIZATION - PROOF OF THEOREM 1.1 AND THEOREM 1.2

This section containing two subsections is devoted to the proof of Theorem 1.1 and Theorem 1.2. The proof of Theorem 1.1 is given in the first subsection and the proof of Theorem 1.2 is given in the second one.

4.1. **Proof of Theorem 1.1.** Denote

$$A_\lambda = A + \lambda I.$$

Let $(y_\lambda, \tilde{y}_\lambda)^\top \in C([0, +\infty); \mathbb{H})$ be the unique weak solution of the system

$$(4.1) \quad \begin{cases} y'_\lambda = A_\lambda y_\lambda - BB^* \tilde{y}_\lambda & \text{in } (0, T), \\ \tilde{y}'_\lambda = -A_\lambda^* \tilde{y}_\lambda & \text{in } (0, T), \\ \hat{y}_\lambda(0) = y_0, \quad \tilde{y}_\lambda(0) = \mathcal{Q}^{-1}y_0. \end{cases}$$

Let $\tau \in (0, T]$ and $\varphi_\tau \in \mathbb{H}$, and let $\varphi \in C([0, \tau]; \mathbb{H})$ be the unique weak solution of

$$(4.2) \quad \begin{cases} \varphi' = -A_\lambda^* \varphi & \text{in } (0, \tau), \\ \varphi(\tau) = \varphi_\tau. \end{cases}$$

Applying [48, Lemma 2.1] for A_λ with $t = \tau$, we derive from (4.1) and (4.2) that

$$(4.3) \quad \langle y_\lambda(\tau), \varphi(\tau) \rangle_{\mathbb{H}} - \langle y_\lambda(0), \varphi(0) \rangle_{\mathbb{H}} = - \int_0^\tau \langle B^* \tilde{y}_\lambda(s), B^* \varphi(s) \rangle_{\mathbb{R}} ds.$$

Applying [48, Lemma 3.1] to $\tilde{y}_\lambda(\tau - \cdot)$ and $\varphi(\tau - \cdot)$, we obtain

$$(4.4) \quad \langle Q\tilde{y}_\lambda(0), \varphi(0) \rangle_{\mathbb{H}} - \langle Q\tilde{y}_\lambda(\tau), \varphi(\tau) \rangle_{\mathbb{H}} = \int_0^\tau \langle B^* \tilde{y}_\lambda(s), B^* \varphi(s) \rangle_{\mathbb{R}} ds.$$

Summing (4.3) and (4.4), after using the fact that $Q\tilde{y}_\lambda(0) = y_\lambda(0)$, we deduce that

$$\langle y_\lambda(\tau) - Q\tilde{y}_\lambda(\tau), \varphi(\tau) \rangle_{\mathbb{H}} = 0.$$

Since $\varphi(\tau) \in \mathbb{H}$ is arbitrary, we derive that

$$(4.5) \quad y_\lambda(\tau) - Q\tilde{y}_\lambda(\tau) = 0.$$

Set

$$\hat{y}(t) = e^{-\lambda t} y_\lambda(t), \quad \text{and} \quad \tilde{y}(t) = e^{-\lambda t} \tilde{y}_\lambda(t).$$

Then, from (4.1) and (4.5), we have

$$(4.6) \quad \begin{cases} \hat{y}' = A\hat{y} - BB^*\tilde{y} & \text{in } (0, T), \\ \tilde{y}' = -A^*\tilde{y} - 2\lambda\tilde{y} & \text{in } (0, T), \\ \hat{y}(0) = y_0, \quad \tilde{y}(0) = \mathcal{Q}^{-1}y_0, \end{cases}$$

and

$$(4.7) \quad \hat{y} - Q\tilde{y} = 0 \text{ in } [0, T].$$

Since $\hat{y}(t) \in \mathbb{H}_{1, \#}$ in $[0, T]$, it follows that

$$y = \hat{y} \text{ in } [0, T].$$

Since

$$\tilde{y}' = A\tilde{y} - 2\lambda\tilde{y},$$

and A is skew-adjoint by Lemma 2.2, it follows that

$$(4.8) \quad \|\tilde{y}(t)\|_{\mathbb{H}} = e^{-2\lambda t} \|\tilde{y}_0\|_{\mathbb{H}}.$$

The conclusion now follows from (4.8) and the fact that $y(t) = Q\tilde{y}(t)$ for $t \geq 0$. □

4.2. Proof of Theorem 1.2. Let $y_P(t)$ be the projection of $y(t)$ into $\mathbb{H}_{1,\sharp}$ using the \mathbb{H} -scalar product and set $z(t) = y(t) - y_P(t)$. Let $b : [0, +\infty) \rightarrow \mathbb{R}$ be the real function such that

$$(4.9) \quad z = b\Phi_1$$

(recall that $\Phi_1 = (\varphi_1, 0)^\top$). Note that

$$(4.10) \quad u(t) = -B^* \mathcal{Q}^{-1} y_P(t).$$

Since $Az = 0$, we derive from (1.42) that

$$(4.11) \quad y' = y'_P + z' = Ay_P + Bu + uF(y - \Phi_1) \text{ for } t > 0.$$

Taking the scalar product in $[L^2(I)]^2$ of this equation with Φ_1 and integrating by parts, we obtain, by Lemma 3.3,

$$(4.12) \quad b' = u \langle F(y - \Phi_1), \Phi_1 \rangle_{L^2(I)}.$$

We derive from (4.11) and (4.12) that

$$(4.13) \quad y'_P = Ay_P + Bu + uF(y - \Phi_1) - z' = Ay_P + Bu + uF(y - \Phi_1) - b'\Phi_1.$$

Fix $T_0 > 0$ and let $0 < T < T_0$. Applying Lemma 3.5 and Theorem 1.1, we have

$$(4.14) \quad \|y_P(t)\|_{\mathbb{H}} \leq \|\mathcal{Q}^{-1}\|_{\mathcal{L}(\mathbb{H}_{1,\sharp})} e^{-2\lambda t} \|y_P(0)\|_{\mathbb{H}} + C_{\lambda, T_0} \|uF(y - \Phi_1) - b'\Phi_1\|_{L^2((0, T); H^3(I))} \text{ in } [0, T].$$

Since, for $t \in [0, T]$,

$$\|u\|_{L^2(I)} \stackrel{(1.31)}{\leq} C_{T_0} \|y_P\|_{L^2((0, T); \mathbb{H})},$$

$$\|F(y - \Phi_1)\|_{L^\infty((0, T); H^3(I))} \stackrel{(1.26)}{\leq} C_{T_0} \|y - \Phi_1\|_{L^\infty((0, T); \mathbb{H})},$$

and

$$\|b'\|_{L^2(0, T)} \stackrel{(4.12)}{\leq} C_{T_0} \|u\|_{L^2(0, T)} \|y - \Phi_1\|_{L^\infty(0, T)} \leq C_{T_0} \|y_P\|_{L^2((0, T); \mathbb{H})} \|y - \Phi_1\|_{L^\infty((0, T); \mathbb{H})},$$

it follows from (4.14) that

$$(4.15) \quad \|y_P(t)\|_{\mathbb{H}} \leq \|\mathcal{Q}^{-1}\|_{\mathcal{L}(\mathbb{H}_{1,\sharp})} e^{-2\lambda t} \|y_P(0)\|_{\mathbb{H}} + C_{\lambda, T_0} \|y_P\|_{L^2((0, T); \mathbb{H})} \|y - \Phi_1\|_{L^\infty((0, T); \mathbb{H})} \text{ in } [0, T].$$

By the conservation of L^2 -norm of y , we derive from the fact that z and y_P are orthogonal in $L^2(I; \mathbb{R}^2)$,

$$b(t)^2 + \|y_P(t)\|_{L^2(I)}^2 = 1.$$

It follows that if $\|y(t) - \Phi_1\|_{L^2(I)} \leq 1/4$ in $[0, T]$, then

$$(4.16) \quad \|b(t) - 1\|_{L^2(I)} \leq C \|y_P(t)\|_{L^2(I)}^2.$$

By taking ε_0 sufficiently small and $\|y_0\|_{\mathbb{H}} \leq \varepsilon_0$, we derive from (4.15), and (4.16) that

$$(4.17) \quad \|y(t) - \Phi_1\|_{\mathbb{H}} \leq 2 \|\mathcal{Q}^{-1}\|_{\mathcal{L}(\mathbb{H}_{1,\sharp})} e^{-2\lambda t} \|y_P(0)\|_{\mathbb{H}} \text{ in } [0, T_0].$$

Taking T_0 large enough such that $2 \|\mathcal{Q}^{-1}\|_{\mathcal{L}(\mathbb{H}_{1,\sharp})} e^{-2\lambda T_0} \leq e^{-2\lambda T_0}$, we then can repeat the argument for the interval $[T_0, 2T_0]$, \dots , and obtain the conclusion.

5. FINITE TIME STABILIZATION - PROOF OF THEOREM 1.3

This section containing two subsections is devoted to the proof of Theorem 1.3. In the first subsection, we establish the cost of controls for the linearized system in small time. The proof of Theorem 1.3 using the results in the first section is given in the second one.

5.1. Cost of control of the linearized system for small time. We begin this section by establishing an upper bound of the cost of control of the linearized system for small time.

Proposition 5.1. *Let $\mu \in H^3(I, \mathbb{R})$ be such that (1.8) holds. For all $\Psi_0 \in \mathbf{H}_{1,\sharp}$, there exists $u \in L^2(0, T; \mathbb{R})$ such that*

$$\Psi(T, \cdot) = \Psi_0$$

and

$$\|u\|_{L^2(0, T)} \leq e^{\frac{C}{T}} \|\Psi_0\|_{\mathbf{H}},$$

where $\Psi \in C([0, T]; \mathbf{H})$ is the unique weak solution of the system

$$(5.1) \quad \begin{cases} i\Psi_t = -\Delta\Psi - \lambda_1\Psi - \mathbf{T}(u(t)\mu\varphi_1) & \text{in } (0, T) \times I, \\ \Psi(t, 0) = \Psi(t, 1) = 0 & \text{in } (0, T), \\ \Psi(0, \cdot) = 0 & \text{in } I. \end{cases}$$

Recall that \mathbf{T} is defined in Definition 3.1.

Proof. By a translation of time, it suffices to prove the following result. For all $\Psi_0 \in \mathbf{H}_{1,\sharp}$, there exists $u \in L^2((-T/2, T/2); \mathbb{R})$ such that

$$\Psi(T/2, \cdot) = \Psi_0$$

and

$$\|u\|_{L^2(-T/2, T/2)} \leq e^{\frac{C}{T}} \|\Psi_0\|_{\mathbf{H}},$$

where $\Psi \in C([-T/2, T/2]; \mathbf{H})$ is the unique weak solution of the system

$$(5.2) \quad \begin{cases} i\Psi_t = -\Delta\Psi - \lambda_1\Psi - \mathbf{T}(u(t)\mu\varphi_1) & \text{in } (-T/2, T/2) \times I, \\ \Psi(t, 0) = \Psi(t, 1) = 0 & \text{in } (-T/2, T/2), \\ \Psi(-T/2, \cdot) = 0 & \text{in } I. \end{cases}$$

The proof of this fact is based on the moment method, see, e.g., [52]. We represent Ψ under the form

$$\Psi(t, x) = \sum_{k \geq 1} a_k(t) \varphi_k(x) \text{ in } (-T/2, T/2) \times I.$$

We then have, see the proof of Lemma 3.3,

$$ia'_k = (\lambda_k - \lambda_1)a_k - c_k u(t) \text{ in } (-T/2, T/2),$$

where

$$(5.3) \quad c_k = \langle \mu\varphi_1, \varphi_k \rangle_{L^2(I)} \text{ in } (-T/2, T/2).$$

Thus

$$a'_k = -i(\lambda_k - \lambda_1)a_k + ic_k u(t) \text{ in } (-T/2, T/2).$$

Since $a_k(-T/2) = 0$ for $k \geq 1$, we then have

$$a_k(T/2) = ic_k \int_{-T/2}^{T/2} e^{-i(\lambda_k - \lambda_1)(T/2 - s)} u(s) ds.$$

Set

$$(5.4) \quad v(t) = u(T/2 - t), \quad \omega_k = \lambda_k - \lambda_1 \text{ for } k \geq 1,$$

and

$$(5.5) \quad d_k = a_k(T/2)/(ic_k) \text{ for } k \geq 1.$$

Since $a_1(T/2) = \Im a_1(T/2)$ for $\Phi_0 \in \mathbf{H}_{1,\#}$, it follows from (5.3) that

$$d_1 \in \mathbb{R}.$$

We then have

$$\int_{-T/2}^{T/2} e^{-i\omega_k s} v(s) ds = d_k \text{ in } (-T/2, T/2) \text{ for } k \geq 1,$$

By (1.8) and (5.5), we have

$$(5.6) \quad \sum_{k \geq 1} |d_k|^2 \leq C \|\Psi_0\|_{\mathbf{H}}^2.$$

By [52, Lemma 4.1] there exists $\beta > 0$ such that, for all $k \geq 1$,⁴

$$(5.7) \quad \prod_{n \geq 1, n \neq k} \left| 1 - \frac{z}{\omega_n - \omega_k} \right| + \prod_{n \geq 1, n \neq k} \left| 1 + \frac{z}{\omega_n + \omega_k} \right| \leq \beta e^{\beta|z|^{1/2}} \text{ for } z \in \mathbb{C}.$$

By [52, Lemma 4.2], for all $\gamma > 1$, there exist $C = C(\gamma)$ (independent of $T \in (0, T_0)$) and an analytic function H such that

$$(5.8) \quad H(0) = 1 \quad \text{and} \quad |H(z)| \leq e^{\frac{C}{T}} e^{-\gamma|z|^{1/2}} \text{ for } z \in \mathbb{R}, \quad |H(z)| \leq C_T e^{T|\Im z|/4}.$$

Fix $\gamma > 2\beta$ and a corresponding analytic function H . For $N \geq 2$, we define the function $\xi_N : \mathbb{C} \rightarrow \mathbb{C}$ as follows, for $z \in \mathbb{C}$,

$$(5.9) \quad \begin{aligned} \xi_N(z) = & \sum_{k=2}^N d_k H(z - \omega_k) \prod_{n \geq 1, n \neq k} \left(1 - \frac{z - \omega_k}{\omega_n - \omega_k} \right) \prod_{l \geq 1} \left(1 - \frac{z - \omega_k}{-\omega_l - \omega_k} \right) \\ & + \sum_{k=2}^N \bar{d}_k H(-z - \omega_k) \prod_{n \geq 1, n \neq k} \left(1 - \frac{z + \omega_k}{-\omega_n + \omega_k} \right) \prod_{l \geq 1} \left(1 - \frac{z + \omega_k}{\omega_l + \omega_k} \right) \\ & + d_1 H(z) \prod_{n \geq 2} \left(1 - \frac{z}{\omega_n} \right) \prod_{l \geq 2} \left(1 - \frac{z}{-\omega_l} \right). \end{aligned}$$

It follows from (5.6), (5.7), and (5.8) that the function ξ_N is well-defined and is analytic on \mathbb{C} . From the definition of ξ , we have

$$(5.10) \quad \xi_N(\omega_k) = d_k, \quad \xi_N(-\omega_k) = \bar{d}_k \text{ for } 2 \leq k \leq N,$$

and

$$(5.11) \quad \xi_N(0) = \xi_N(\omega_1) = d_1.$$

For all $c > 0$, there exists $c_1 > 0$ such that it holds

$$\begin{aligned} \int_{\mathbb{R}} e^{-c|z - \omega_m|^{1/2}} e^{-c|z - \omega_n|^{1/2}} dz + \int_{\mathbb{R}} e^{-c|z + \omega_m|^{1/2}} e^{-c|z + \omega_n|^{1/2}} dz \\ + \int_{\mathbb{R}} e^{-c|z - \omega_m|^{1/2}} e^{-c|z + \omega_n|^{1/2}} dz \leq c_1 e^{-\frac{c}{2}|\omega_n - \omega_m|^{1/2}}. \end{aligned}$$

We derive from (5.6), (5.7), and (5.8) that

$$(5.12) \quad \|\xi_N\|_{L^2(\mathbb{R})} \leq e^{\frac{C}{T}} \|\Psi_0\|_{\mathbf{H}}.$$

⁴[52, Lemma 4.1] only gives the estimate for the first term; nevertheless, the estimate for the second term can be done in the same manner.

and

the restriction of ξ_N on \mathbb{R} is a Cauchy sequence in $L^2(\mathbb{R})$.

By Paley-Wiener's theorem, see, e.g., [50, Theorem 19.3], there thus exists $v_N \in L^2(-T/2, T/2; \mathbb{C})$ such that $\hat{v}_N = \xi_N$ and ξ_N is a Cauchy sequence in $L^2(-T/2, T/2)$. Let v be the limit of the sequence (v_N) in $L^2(-T/2, T/2)$. Set

$$u = \frac{1}{2}(v + \bar{v}).$$

Then, for $k \geq 1$,

$$\hat{u}(\omega_k) = \frac{1}{2} \lim_{N \rightarrow +\infty} \left(\xi_N(\omega_k) + \overline{\xi_N(-\omega_k)} \right) \stackrel{(5.10), (5.11)}{=} d_k.$$

Here we used the fact that d_1 is real. The conclusion follows since

$$\|u\|_{L^2(\mathbb{R})} \leq C \|v\|_{L^2(\mathbb{R})} \leq C \limsup_{N \rightarrow +\infty} \|\xi_N\|_{L^2(\mathbb{R})} \stackrel{(5.12)}{\leq} e^{\frac{C}{T}} \|\Psi_0\|_{\mathbf{H}}$$

The proof is complete. \square

For the completeness, we next establish a lower bound of the cost of control of the linearized system for small time. To this end, we first prove the following result.

Proposition 5.2. *Let $0 < \varepsilon < 1/2$, $\varepsilon^3 < T < 3/2$. Let $\mu \in L^1((0, T); \mathbb{R})$ be such that $\int_I \mu \varphi_1^2 dx \neq 0$. If $u \in L^1((0, T); \mathbb{C})$ is a control which steers the control system*

$$(5.13) \quad \begin{cases} iv_t = -\varepsilon v_{xx} - \varepsilon \lambda_1 v + \frac{i}{4\varepsilon} v - \varepsilon u(t) \mu(x) \varphi_1 & \text{in } (0, T) \times (0, 1), \\ v(t, 0) = v(t, 1) & \text{in } (0, T) \end{cases}$$

from φ_1 at time 0 to 0 at the time T in the sense that there exists $v \in L^2((0, T); H_0^1(I)) \cap C([0, T]; L^2(I))$ such that

$$(5.14) \quad i \frac{d}{dt} \langle v, \varphi_k \rangle_{L^2(I)} = -\varepsilon \langle v, \Delta \varphi_k \rangle_{L^2(I)} - \varepsilon \lambda_1 \langle v, \varphi_k \rangle_{L^2(I)} + \frac{i}{4\varepsilon} \langle v, \varphi_k \rangle_{L^2(I)} - \varepsilon u(t) \langle \mu \varphi_1, \varphi_k \rangle_{L^2(I)} \quad \text{in } (0, T)$$

in the distributional sense for all $k \geq 1$, and $v(0, \cdot) = \varphi_1$ in I and $v(T, \cdot) = 0$ in I , then

$$\ln \|u\|_{L^1(0, T)} \geq \frac{1}{\varepsilon} \left(\frac{1}{2} - \frac{T}{4} \right) - C \ln \varepsilon^{-1},$$

for some positive constant C independent of ε and T .

Proof. The proof uses tools from complex analysis, see, e.g., [30]. Define

$$\Lambda_k := \varepsilon \lambda_k - \varepsilon \lambda_1 + \frac{i}{4\varepsilon} \quad \text{and} \quad \Phi_k(t, x) = \varphi_k(x) e^{-i\Lambda_k t} \quad \text{in } \mathbb{R}_+ \times (0, 1).$$

One can check that

$$(5.15) \quad \begin{cases} i\Phi_{k,t} = -\varepsilon \Phi_{k,xx} - \varepsilon \lambda_1 \Phi_k + \frac{i}{4\varepsilon} \Phi_k = 0 & \text{in } \mathbb{R}_+ \times [0, 1], \\ \Phi_k(t, 0) = \Phi_k(t, 1) = 0 & \text{for } t \in \mathbb{R}_+. \end{cases}$$

Multiplying the equation of v by Φ_k , integrating by parts, and using the fact $v(T, \cdot) = 0$, we have

$$(5.16) \quad i \int_0^1 v(0, x) \Phi_k(0, x) dx = \varepsilon \int_0^T u(t) e^{-i\Lambda_k t} dt \int_I \mu \varphi_1 \varphi_k dx.$$

Define, for $z \in \mathbb{C}$,

$$(5.17) \quad \mathcal{F}(z) := \int_{-T/2}^{T/2} u(t + T/2) e^{-izt} dt = e^{iTz/2} \int_0^T u(t) e^{-izt} dt.$$

It follows from (5.16) that, with $c_1 = \int_I \mu \varphi_1^2 dx$,

$$(5.18) \quad \mathcal{F}(\Lambda_1) = \frac{i}{c_1 \varepsilon} e^{i\Lambda_1 T/2} \quad \text{and} \quad \mathcal{F}(\Lambda_k) = 0 \text{ for } k \geq 2.$$

Applying the representation of entire functions of exponential type for \mathcal{F} , see e.g., [30, page 56], we derive from (5.18) that, for $z \in \mathbb{C}$ with $\Im z > 0$,

$$(5.19) \quad \ln |\mathcal{F}(z)| \leq I_0(z) + I_1(z) + \sigma \Im(z),$$

where

$$(5.20) \quad I_0(z) = \sum_{k \geq 2} \ln \frac{|\Lambda_k - z|}{|\bar{\Lambda}_k - z|}, \quad I_1(z) = \frac{\Im(z)}{\pi} \int_{-\infty}^{\infty} \frac{\ln |\mathcal{F}(\tau)|}{|\tau - z|^2} d\tau,$$

and

$$(5.21) \quad \sigma = \limsup_{y \rightarrow +\infty} \frac{\ln |\mathcal{F}(iy)|}{y}.$$

From the definition of \mathcal{F} in (5.17), we have, for $y \in \mathbb{R}_+$,

$$|\mathcal{F}(iy)| \leq \|u\|_{L^1(0,T)} e^{Ty/2}.$$

This implies

$$(5.22) \quad \sigma = \limsup_{y \rightarrow +\infty} \frac{\ln |\mathcal{F}(iy)|}{y} \leq T/2.$$

From (5.18), we derive that

$$(5.23) \quad \ln |\mathcal{F}(\Lambda_1)| \geq -\frac{T}{8\varepsilon} + C \ln \varepsilon^{-1}.$$

Here and in what follows in this proof, C denotes a positive constant independent of k and it can change from one place to another. Similar to [17, (2.597)] (see also [19]), we obtain

$$(5.24) \quad \begin{aligned} I_0(\Lambda_1) &= \sum_{k \geq 2} \ln \frac{\varepsilon(k^2 - 1)\pi^2}{([\varepsilon(k^2 - 1)\pi^2]^2 + [1/(2\varepsilon)]^2)^{1/2}} \leq \sum_{k \geq 2} \ln \frac{\varepsilon^2 k^2 \pi^2}{([\varepsilon^2 k^2 \pi^2]^2 + [1/2]^2)^{1/2}} \\ &\leq \int_1^{\infty} \ln \left(\frac{\varepsilon^2 \pi^2 x^2}{(\varepsilon^4 \pi^4 x^4 + 1/4)^{1/2}} \right) dx = \frac{1}{\varepsilon \pi \sqrt{2}} \int_{\varepsilon \pi \sqrt{2}}^{\infty} \ln \left(\frac{x^2}{\sqrt{x^4 + 1}} \right) dx. \end{aligned}$$

Since

$$\int_0^{\infty} \ln \left(\frac{x^2}{\sqrt{x^4 + 1}} \right) dx = -\frac{\pi}{\sqrt{2}},$$

it follows from (5.24) that

$$(5.25) \quad I_0(\Lambda_1) \leq -\frac{1}{2\varepsilon} + C \ln \varepsilon^{-1}.$$

We next estimate $I_1(\Lambda_1)$. From (5.17), we have, for $s \in \mathbb{R}$,

$$\ln |\mathcal{F}(s)| \leq \|u\|_{L^1(0,T)},$$

which yields

$$(5.26) \quad I_1(\Lambda_1) \leq \frac{1}{4\varepsilon\pi} \int_{-\infty}^{\infty} \frac{\ln \|u\|_{L^1(0,T)}}{s^2 + (1/4\varepsilon)^2} ds.$$

Since, for $a > 0$,

$$\frac{a}{\pi} \int_{-\infty}^{\infty} \frac{1}{s^2 + a^2} ds = 1,$$

it follows from (5.26) that

$$(5.27) \quad I_1(\Lambda_1) \leq \ln \|u\|_{L^1(0,T)}.$$

Combining (5.19), (5.23), (5.25), and (5.27) yields

$$-\frac{T}{8\varepsilon} \leq -\frac{1}{2\varepsilon} + \ln \|u\|_{L^1(0,T)} + \frac{T}{8\varepsilon} + C \ln \varepsilon^{-1}.$$

This implies

$$\ln \|u\|_{L^1(0,T)} \geq \frac{1}{\varepsilon} \left(\frac{1}{2} - \frac{T}{4} \right) - C \ln \varepsilon^{-1}.$$

The proof is complete. \square

We are ready to obtain a lower bound for the cost of control viewing Lemma 3.3.

Proposition 5.3. *Let $\mu \in L^1((0, T); \mathbb{C})$ be such that $\int_I \mu \varphi_1^2 dx \neq 0$, and let $0 < T < 1$. Then if $u \in L^1((0, T); \mathbb{C})$ is a control which steers the control system*

$$(5.28) \quad \begin{cases} iv_t = -v_{xx} - \lambda_1 v - u(t)\mu(x)\varphi_1 & \text{in } (0, T) \times (0, 1), \\ v(t, 0) = v(t, 1) & \text{in } (0, T) \end{cases}$$

from φ_1 at time 0 to 0 at the time T in the sense that there exists $v \in L^2((0, T); H_0^1(I)) \cap C([0, T]; L^2(I))$ such that

$$(5.29) \quad i \frac{d}{dt} \langle v, \varphi_k \rangle_{L^2(I)} = -\langle v, \Delta \varphi_k \rangle_{L^2(I)} - \lambda_1 \langle v, \varphi_k \rangle_{L^2(I)} - u(t) \langle \mu \varphi_1, \varphi_k \rangle_{L^2(I)} \text{ in } (0, T)$$

in the distributional sense for all $k \geq 1$, and $v(0, \cdot) = \varphi_1$ in I and $v(T, \cdot) = 0$ in I , then

$$\ln \|u\|_{L^1(0,T)} \geq \frac{1}{4T} - C \ln T^{-1},$$

for some positive constant C independent of T .

Proof. Define

$$\tilde{v}(t, x) := v(\varepsilon t, x) e^{-\frac{t}{4\varepsilon}} \text{ for } (t, x) \in (0, T/\varepsilon) \times (0, 1)$$

and set $\tilde{u}(t) = u(\varepsilon t) e^{-\frac{t}{4\varepsilon}}$ for $t \in (0, T/\varepsilon)$. Then

$$i\tilde{v}_t = -\varepsilon\tilde{v}_{xx} - \varepsilon\lambda_1\tilde{v} + \frac{i}{4\varepsilon}\tilde{v} - \varepsilon\tilde{u}(t)\mu(x)\varphi_1 \text{ in } (0, T/\varepsilon) \times (0, 1).$$

Applying Proposition 5.2 to \tilde{v} with $(\varepsilon, T) = (T, T)$, we have

$$\ln \|u\|_{L^1(0,T)} \geq \ln \|\tilde{u}\|_{L^1(0,1)} \geq \frac{1}{T} \left(\frac{1}{2} - \frac{1}{4} \right) - C \ln T^{-1} \geq \frac{1}{4T} - C \ln T^{-1},$$

which is the conclusion. \square

Remark 5.1. Similar arguments as in the proof of Proposition 5.3 can be found in [37]. For the boundary controls, the cost of controls for small time is also of the order $e^{C/T}$, see, e.g., [52].

Remark 5.2. It is shown in [47] that the cost of controls of the heat equation depends on the support of the data and the controlled region. This is based on the strategy of Lebeau and Robbiano [36] and the three-sphere inequalities with partial data established by Nguyen [46]. It would be interesting to study whether or not the cost of controls depends on the support of the initial data for the KdV system.

5.2. Proof of Theorem 1.3. We first give an estimate for $Q = Q(\lambda)$. The following result is a direct consequence of Proposition 5.1 and Hilbert uniqueness method.

Proposition 5.4. *Let $\mu \in H^3(I; \mathbb{R})$ be such that (1.8) holds and let $0 < T < T_0$. We have, for some positive constant C independent of T ,*

$$\int_0^T |B^* e^{-sA^*} z|^2 \geq e^{-\frac{C}{T}} \|z\|_{\mathbb{H}}^2 \text{ for all } z \in \mathbb{H}_{1,\sharp}.$$

Using Proposition 5.4, we can prove the following result.

Lemma 5.1. *Let $\lambda \geq \lambda_0$ and let $Q = Q(\lambda)$ be defined by (1.34). There exists a positive constant C independent of λ such that*

$$(5.30) \quad \langle Qz, z \rangle_{\mathbb{H}} \geq e^{-C\sqrt{\lambda}} \|z\|_{\mathbb{H}}^2 \text{ for all } z \in \mathbb{H}_{1,\sharp}.$$

Proof. We have

$$(5.31) \quad \begin{aligned} \langle Qz, z \rangle_{\mathbb{H}} &= \int_0^\infty e^{-2\lambda s} |B^* e^{-sA^*} z|^2 ds \geq \int_{1/\sqrt{\lambda}}^{2/\sqrt{\lambda}} e^{-2\lambda s} |B^* e^{-sA^*} z|^2 ds \\ &\geq \frac{1}{\sqrt{\lambda}} e^{-4\sqrt{\lambda}} \int_{1/\sqrt{\lambda}}^{2/\sqrt{\lambda}} |B^* e^{-sA^*} z|^2 ds \stackrel{\text{Proposition 5.4}}{\geq} \frac{C_1}{\sqrt{\lambda}} e^{-4\sqrt{\lambda}} e^{-C_2\sqrt{\lambda}} \|e^{-\frac{1}{\sqrt{\lambda}}A^*} z\|_{\mathbb{H}}, \end{aligned}$$

which yields

$$(5.32) \quad \langle Qz, z \rangle_{\mathbb{H}} \geq \frac{C_1}{\sqrt{\lambda}} e^{-4\sqrt{\lambda}} e^{-C_2\sqrt{\lambda}} \|z\|_{\mathbb{H}}.$$

The conclusion follows. \square

Theorem 1.3 is now a consequence of the following result.

Proposition 5.5. *Let $\mu \in H^3(I, \mathbb{R})$ be such that (1.8) holds and let $T > 0$. Let (t_n) be an increasing sequence that converges to T with $t_0 = 0$ and let $(\lambda_n) \subset \mathbb{R}_+$ be an increasing sequence. Define, for $t_n \leq t < t_{n+1}$ and $n \geq 0$,*

$$\mathcal{K}(t, z) = -B^* Q_n^{-1} \text{proj}_{\mathbb{H}_{1,\sharp}} z \text{ for } z \in \mathbb{H},$$

where $Q_n = Q(\lambda_n)$ defined by (1.34) with $\lambda = \lambda_n$. Set $s_0 = 0$ and $s_n = \sum_{k=0}^{n-1} \lambda_k (t_{k+1} - t_k)$ for $n \geq 1$. Let $y \in C([0, T]; \mathbb{H})$ of system (1.24) with

$$u(t) = \mathcal{K}(t, y(t, \cdot)) \text{ for } t \in [0, T].$$

There exists a positive constant γ such that, if for large n ,

$$(t_{n+1} - t_n) \lambda_n \geq \gamma \sqrt{\lambda_n},$$

then it holds, for $t_{n-1} \leq t \leq t_n$ and for $n \geq 1$,

$$\|y(t, \cdot)\|_{\mathbb{H}} \leq e^{-s_{n-1} + Cn} \|y_0\|_{\mathbb{H}}$$

and

$$|u(t)| \leq C e^{-s_{n-1}/4 + Cn} \|y_0\|_{\mathbb{H}},$$

for some positive constant C independent of n . In particular, if, in addition, we have that

$$\lim_{n \rightarrow +\infty} \frac{s_n}{n + \sqrt{\lambda_{n+1}}} = +\infty,$$

then

$$y(t, \cdot) \rightarrow 0 \text{ in } \mathbb{H} \text{ as } t \rightarrow T_-$$

and

$$u(t, \cdot) \rightarrow 0 \text{ as } t \rightarrow T_-.$$

Remark 5.3. There are sequences (t_n) and (λ_n) which satisfy the conditions given in the above proposition, for example, $t_n = T - T/n^2$ and $\lambda_n = n^8$ for large n .

Proof. Applying Theorem 1.1 and Lemma 5.1, we have

$$(5.33) \quad \|y(t_n)\|_{\mathbb{H}} \leq e^{-2\lambda_{n-1}(t_n - t_{n-1}) + C(1 + \sqrt{\lambda_{n-1}})} \|y(t_{n-1})\|_{\mathbb{H}} \text{ for } n \geq 1.$$

It follows that

$$(5.34) \quad \|y(t_n)\|_{\mathbb{H}} \leq e^{-s_{n-1} + Cn} \|y_0\|_{\mathbb{H}} \text{ for } n \geq 1.$$

We have, by (1.31), for $t_{n-1} \leq t \leq t_n$ and for $n \geq 1$,

$$|u(t)| = |B^* Q_{n-1}^{-1} y(t, \cdot)| \leq C e^{C\sqrt{\lambda_{n-1}}} \|y(t, \cdot)\|_{\mathbb{H}}.$$

The conclusion now follows from Theorem 1.1 and Lemma 5.1. \square

Remark 5.4. We are not able to extend the finite time stabilization to the nonlinear setting. This is due to the fact we cannot ensure that the well-posedness for the time interval $[t_n, t_{n+1})$ for large n .

APPENDIX A. CONTROL SYSTEMS ASSOCIATED WITH OPERATOR SEMI-GROUPS

In this section, we recall and establish some facts on the control systems associated with a strongly continuous semigroup. The standard references are [59, 26, 16, 9, 54].

Let \mathcal{H} and \mathcal{U} be two Hilbert spaces which denote the state space and the control space, respectively. The corresponding scalar products are $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{U}}$, and the corresponding norms are $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{U}}$. Let $(S(t))_{t \geq 0} \subset \mathcal{L}(\mathcal{H})$ be a strongly continuous semi-group on \mathcal{H} . Let $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ be the infinitesimal generator of $(S(t))_{t \geq 0}$ and denote $S(t)^*$ the adjoint of $S(t)$ for $t \geq 0$. Then $(S(t)^*)_{t \geq 0}$ is also a strongly continuous semigroup of continuous linear operators and its infinitesimal generator is $(\mathcal{A}^*, \mathcal{D}(\mathcal{A}^*))$, which is the adjoint of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$. As usual, we equip the domain $\mathcal{D}(\mathcal{A}^*)$ with the scalar product

$$\langle z_1, z_2 \rangle_{\mathcal{D}(\mathcal{A}^*)} = \langle z_1, z_2 \rangle_{\mathcal{H}} + \langle \mathcal{A}^* z_1, \mathcal{A}^* z_2 \rangle_{\mathcal{H}} \text{ for } z_1, z_2 \in \mathcal{D}(\mathcal{A}^*).$$

Then $\mathcal{D}(\mathcal{A}^*)$ is a Hilbert space. Denote $\mathcal{D}(\mathcal{A}^*)'$ the dual space of $\mathcal{D}(\mathcal{A}^*)$ with respect to \mathcal{H} . Then

$$\mathcal{D}(\mathcal{A}^*) \subset \mathcal{H} \subset \mathcal{D}(\mathcal{A}^*)'.$$

Let

$$\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{D}(\mathcal{A}^*)').$$

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Let

$$\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{D}(\mathcal{A}^*)').$$

Consider the control system

$$(A.1) \quad \begin{cases} y' = \mathcal{A}y + f + \mathcal{B}u \text{ in } t \in (0, T), \\ y(0) = y_0, \end{cases}$$

with $y_0 \in \mathcal{D}(\mathcal{A}^*)'$, and $f \in L^1((0, T); (\mathcal{D}(\mathcal{A}^*))')$ and $u \in L^1((0, T); \mathcal{U})$. We are interested in weak solutions of (A.1).

Definition A.1. A weak solution y of (A.1) is understood as an element $y \in C([0, T]; (\mathcal{D}(\mathcal{A}^*))')$ such that

$$(A.2) \quad \begin{cases} \frac{d}{dt} \langle y, \varphi \rangle_{\mathcal{H}} = \langle \mathcal{A}y + f + \mathcal{B}u, \varphi \rangle_{\mathcal{H}} \text{ in } (0, T) \\ y(0) = y_0 \end{cases} \quad \text{for all } \varphi \in \mathcal{D}(\mathcal{A}^{*\infty})$$

for which

- i) the differential equation in (A.2) is understood in the distributional sense,
- ii) the term $\langle \mathcal{A}y + f + \mathcal{B}u + \mathcal{M}y, \varphi \rangle_{\mathcal{H}}$ is understood as $\langle y, \mathcal{A}^* \varphi \rangle_{\mathcal{H}} + \langle f, \varphi \rangle_{\mathcal{D}(\mathcal{A}^*)', \mathcal{D}(\mathcal{A}^*)} + \langle u, \mathcal{B}^* \varphi \rangle_{\mathcal{U}}$.

The convention in ii) will be used from later on. Recall that $\mathcal{D}(\mathcal{A}^{*\infty})$ is dense in $\mathcal{D}(\mathcal{A}^*)$, see e.g., [26, Proposition 1.7]. The following result is on the well-posedness of weak solutions of (A.1).

Proposition A.1. Let $T > 0$, $y_0 \in \mathcal{D}(\mathcal{A}^*)'$, $u \in L^1((0, T); \mathcal{U})$, and $f \in L^1((0, T); \mathcal{D}(\mathcal{A}^*)')$. Then $y \in C([0, T], \mathcal{D}(\mathcal{A}^*)')$ is a weak solution of (A.1) if and only if, with $\tilde{f} := f + \mathcal{B}u$, it holds⁵

$$(A.3) \quad y(t) = S(t)y_0 + \int_0^t S(t-s)\tilde{f}(s) ds \text{ for } t \in [0, T].$$

Proof. We first prove that $y \in C([0, T]; \mathcal{D}(\mathcal{A}^*)')$ is a weak solution of (A.1) if and only if $y \in C([0, T]; \mathcal{D}(\mathcal{A}^*)')$ and (A.3) holds.

Assume first that $y \in C([0, T]; \mathcal{D}(\mathcal{A}^*)')$ and (A.3) holds. We will prove that y is a weak solution of (A.1). Here and in what follows, for notational ease, we denote $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ by $\langle \cdot, \cdot \rangle$. From (A.3), we obtain, with $\varphi \in \mathcal{D}(\mathcal{A}^{*\infty})$,

$$\langle y(t), \varphi \rangle = \langle S(t)y_0 + \int_0^t S(t-s)\tilde{f}(s) ds - y_0, \varphi \rangle \text{ for } t \in [0, T].$$

Set

$$(A.4) \quad \psi(t) = \langle S(t)y_0 + \int_0^t S(t-s)\tilde{f}(s) ds - y_0, \varphi \rangle \text{ for } t \in [0, T].$$

Then, for $t \in [0, T]$,

$$(A.5) \quad \psi(t) = \langle y_0, S(t)^* \varphi \rangle + \int_0^t \langle \tilde{f}(s), S(t-s)^* \varphi \rangle ds - \langle y_0, \varphi \rangle.$$

Since $\varphi \in \mathcal{D}(\mathcal{A}^{*\infty})$, we derive from (A.5) that, for $t \in [0, T]$,

$$\begin{aligned} \psi'(t) &= \langle y_0, \mathcal{A}^* S(t)^* \varphi \rangle + \int_0^t \langle \tilde{f}(s), \mathcal{A}^* S(t-s)^* \varphi \rangle ds + \langle \tilde{f}(t), \varphi \rangle \\ &= \langle y_0, S(t)^* \mathcal{A}^* \varphi \rangle + \int_0^t \langle \tilde{f}(s), S(t-s)^* \mathcal{A}^* \varphi \rangle ds + \langle \tilde{f}(t), \varphi \rangle, \end{aligned}$$

⁵This identity is understood in $\mathcal{D}(\mathcal{A}^*)'$, i.e., $\langle y(t), \varphi \rangle_{\mathbb{H}} = \langle S(t)y_0, \varphi \rangle_{\mathbb{H}} + \int_0^t \langle S(t-s)\tilde{f}(s), \varphi \rangle_{\mathbb{H}} ds$ in $[0, T]$ for all $\varphi \in \mathcal{D}(\mathcal{A}^{*\infty})$. The solutions defined by (A.3) are called mild solutions.

which yields, by (A.3),

$$(A.6) \quad \psi'(t) = \langle y, \mathcal{A}^* \varphi \rangle + \langle \tilde{f}(t), \varphi \rangle.$$

Integrating (A.6) and using (A.5) and (A.3), we obtain, in $[0, T]$,

$$\langle y(t) - y_0, \varphi \rangle = \int_0^t \langle y, \mathcal{A}^* \varphi \rangle + \langle \tilde{f}(t), \varphi \rangle,$$

which in turn implies (A.2).

We now prove that if y is a weak solution of (A.1), then y satisfies (A.3). We first assume that $f \in C([0, T]; \mathcal{H})$ and $y_0 \in \mathcal{D}(\mathcal{A})$. For $t > 0$, set, with $\varphi \in \mathcal{D}(A^{*\infty})$ and $s \in [0, t]$,

$$\chi(s) = \langle S(t-s)y(s), \varphi \rangle = \langle y(s), S(t-s)^* \varphi \rangle.$$

Then

$$\chi'(s) = -\langle y(s), S(t-s)^* A^* \varphi \rangle + \langle Ay(s) + \tilde{f}(s), S(t-s)^* \varphi \rangle = \langle S(t-s)\tilde{f}(s), \varphi \rangle.$$

It follows that

$$\chi(t) - \chi(0) = \int_0^t \langle S(t-s)\tilde{f}(s), \varphi \rangle ds,$$

which yields the identity. The proof in the general case follows by density.

The proof is complete. \square

Remark A.1. The equivalence between weak solutions and mild solutions was first proved in the case B is bounded and $f \in C([0, T]; \mathcal{H})$ by Ball [1], see also [9, Chapter 1 of Part II] for related results when \mathcal{B} is bounded.

REFERENCES

- [1] John M. Ball, *Strongly continuous semigroups, weak solutions, and the variation of constants formula*, Proc. Amer. Math. Soc. **63** (1977), no. 2, 370–373. MR 442748
- [2] John M. Ball, Jerrold E. Marsden, and Marshall Slemrod, *Controllability for distributed bilinear systems*, SIAM J. Control Optim. **20** (1982), no. 4, 575–597. MR 661034
- [3] Karine Beauchard, *Local controllability of a 1-D Schrödinger equation*, J. Math. Pures Appl. (9) **84** (2005), no. 7, 851–956. MR 2144647
- [4] Karine Beauchard and Jean-Michel Coron, *Controllability of a quantum particle in a moving potential well*, J. Funct. Anal. **232** (2006), no. 2, 328–389. MR 2200740
- [5] Karine Beauchard, Jean-Michel Coron, and Holger Teismann, *Minimal time for the bilinear control of Schrödinger equations*, Systems Control Lett. **71** (2014), 1–6. MR 3250374
- [6] Karine Beauchard and Camille Laurent, *Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control*, J. Math. Pures Appl. (9) **94** (2010), no. 5, 520–554. MR 2732927
- [7] Karine Beauchard and Mazyar Mirrahimi, *Practical stabilization of a quantum particle in a one-dimensional infinite square potential well*, SIAM J. Control Optim. **48** (2009), no. 2, 1179–1205. MR 2491595
- [8] Karine Beauchard and Morgan Morancey, *Local controllability of 1D Schrödinger equations with bilinear control and minimal time*, Math. Control Relat. Fields **4** (2014), no. 2, 125–160. MR 3167929
- [9] Alain Bensoussan, Giuseppe Da Prato, Michel C. Delfour, and Sanjoy K. Mitter, *Representation and control of infinite dimensional systems*, second ed., Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA, 2007. MR 2273323
- [10] Ugo Boscain, Marco Caponigro, Thomas Chambrion, and Mario Sigalotti, *A weak spectral condition for the controllability of the bilinear Schrödinger equation with application to the control of a rotating planar molecule*, Comm. Math. Phys. **311** (2012), no. 2, 423–455. MR 2902195
- [11] Ugo Boscain, Marco Caponigro, and Mario Sigalotti, *Multi-input Schrödinger equation: controllability, tracking, and application to the quantum angular momentum*, J. Differential Equations **256** (2014), no. 11, 3524–3551. MR 3186837
- [12] Mégane Bournoissou, *Quadratic behaviors of the 1D linear Schrödinger equation with bilinear control*, J. Differential Equations **351** (2023), 324–360. MR 4542546
- [13] Haim Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011. MR 2759829

- [14] Eduardo Cerpa and Jean-Michel Coron, *Rapid stabilization for a Korteweg-de Vries equation from the left Dirichlet boundary condition*, IEEE Trans. Automat. Control **58** (2013), no. 7, 1688–1695. MR 3072853
- [15] Thomas Chambrion, Paolo Mason, Mario Sigalotti, and Ugo Boscain, *Controllability of the discrete-spectrum Schrödinger equation driven by an external field*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **26** (2009), no. 1, 329–349. MR 2483824
- [16] Jean-Michel Coron, *On the small-time local controllability of a quantum particle in a moving one-dimensional infinite square potential well*, C. R. Math. Acad. Sci. Paris **342** (2006), no. 2, 103–108. MR 2193655
- [17] Jean-Michel Coron, *Control and nonlinearity*, Mathematical Surveys and Monographs, vol. 136, American Mathematical Society, Providence, RI, 2007. MR 2302744
- [18] Jean-Michel Coron, Ludovick Gagnon, and Morgan Morancey, *Rapid stabilization of a linearized bilinear 1-D Schrödinger equation*, J. Math. Pures Appl. (9) **115** (2018), 24–73. MR 3808341
- [19] Jean-Michel Coron and Sergio Guerrero, *Singular optimal control: a linear 1-D parabolic-hyperbolic example*, Asymptot. Anal. **44** (2005), no. 3-4, 237–257. MR 2176274
- [20] Jean-Michel Coron and Qi Lü, *Local rapid stabilization for a Korteweg-de Vries equation with a Neumann boundary control on the right*, J. Math. Pures Appl. (9) **102** (2014), no. 6, 1080–1120. MR 3277436
- [21] Jean-Michel Coron and Hoai-Minh Nguyen, *Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach*, Arch. Rational Mech. Anal. **225** (2017), 993–1023.
- [22] Jean-Michel Coron and Hoai-Minh Nguyen, *Optimal time for the controllability of linear hyperbolic systems in one-dimensional space*, SIAM J. Control Optim. **57** (2019), no. 2, 1127–1156. MR 3932617
- [23] Jean-Michel Coron and Hoai-Minh Nguyen, *Null-controllability of linear hyperbolic systems in one dimensional space*, Systems Control Lett. **148** (2021), 104851.
- [24] Jean-Michel Coron, Rafael Vazquez, Miroslav Krstic, and Georges Bastin, *Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping*, SIAM J. Control Optim. **51** (2013), no. 3, 2005–2035. MR 3049647
- [25] Domenico D’Alessandro, *Introduction to quantum control and dynamics*, Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series, Chapman & Hall/CRC, Boca Raton, FL, 2008. MR 2357229
- [26] Klaus-Jochen Engel and Rainer Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000, With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. MR 1721989
- [27] Franco Flandoli, Irena Lasiecka, and Roberto Triggiani, *Algebraic Riccati equations with nonsmoothing observation arising in hyperbolic and Euler-Bernoulli boundary control problems*, Ann. Mat. Pura Appl. (4) **153** (1988), 307–382. MR 1008349
- [28] Vilmos Komornik, *Rapid boundary stabilization of linear distributed systems*, SIAM J. Control Optim. **35** (1997), no. 5, 1591–1613. MR 1466918
- [29] Vilmos Komornik and Paola Loreti, *Fourier series in control theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2005. MR 2114325
- [30] P. Koosis, *The logarithmic integral*, vol. 1, Cambridge university press, 1998.
- [31] Miroslav Krstic, Bao-Zhu Guo, and Andrey Smyshlyaev, *Boundary controllers and observers for the linearized Schrödinger equation*, SIAM J. Control Optim. **49** (2011), no. 4, 1479–1497. MR 2817487
- [32] Miroslav Krstic and Andrey Smyshlyaev, *Boundary control of PDEs*, Advances in Design and Control, vol. 16, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008, A course on backstepping designs. MR 2412038
- [33] Irena Lasiecka and Roberto Triggiani, *Differential and algebraic Riccati equations with application to boundary-/point control problems: continuous theory and approximation theory*, Lecture Notes in Control and Information Sciences, vol. 164, Springer-Verlag, Berlin, 1991. MR 1132440
- [34] Camille Laurent, *Internal control of the Schrödinger equation*, Math. Control Relat. Fields **4** (2014), no. 2, 161–186. MR 3167930
- [35] Gilles Lebeau, *Contrôle de l’équation de Schrödinger*, J. Math. Pures Appl. (9) **71** (1992), no. 3, 267–291. MR 1172452
- [36] Gilles Lebeau and Luc Robbiano, *Contrôle exact de l’équation de la chaleur*, Comm. Partial Differential Equations **20** (1995), no. 1-2, 335–356. MR 1312710
- [37] Pierre Lissy, *A link between the cost of fast controls for the 1-D heat equation and the uniform controllability of a 1-D transport-diffusion equation*, C. R. Math. Acad. Sci. Paris **350** (2012), no. 11-12, 591–595. MR 2956149
- [38] Wei-Jiu Liu and Miroslav Krstić, *Backstepping boundary control of Burgers’ equation with actuator dynamics*, Systems Control Lett. **41** (2000), no. 4, 291–303. MR 1830371

- [39] Elaine Machtyngier, *Exact controllability for the Schrödinger equation*, SIAM J. Control Optim. **32** (1994), no. 1, 24–34. MR 1255957
- [40] Elaine Machtyngier and Enrique Zuazua, *Stabilization of the Schrödinger equation*, Portugal. Math. **51** (1994), no. 2, 243–256. MR 1277994
- [41] Mazyar Mirrahimi, *Lyapunov control of a quantum particle in a decaying potential*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **26** (2009), no. 5, 1743–1765. MR 2566708
- [42] Morgan Morancey, *Simultaneous local exact controllability of 1D bilinear Schrödinger equations*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **31** (2014), no. 3, 501–529. MR 3208452
- [43] Vahagn Nersisyan, *Growth of Sobolev norms and controllability of the Schrödinger equation*, Comm. Math. Phys. **290** (2009), no. 1, 371–387. MR 2520519
- [44] Vahagn Nersisyan, *Global approximate controllability for Schrödinger equation in higher Sobolev norms and applications*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **27** (2010), no. 3, 901–915. MR 2629885
- [45] Vahagn Nersisyan and Hayk Nersisyan, *Global exact controllability in infinite time of Schrödinger equation*, J. Math. Pures Appl. (9) **97** (2012), no. 4, 295–317. MR 2899810
- [46] Hoai-Minh Nguyen, *Cloaking property of a plasmonic structure in doubly complementary media and three-sphere inequalities with partial data*, SIAM J. Math. Anal. **54** (2022), no. 2, 2040–2096. MR 4405196
- [47] Hoai-Minh Nguyen, *A dependence of the cost of fast controls for the heat equation on the support of initial datum*, SIAM J. Control Optim. **60** (2022), no. 1, 530–544. MR 4384038
- [48] Hoai-Minh Nguyen, *Stabilization of control systems associated with a strongly continuous group*, (2024), <https://arxiv.org/abs/2402.07560>.
- [49] Hoai-Minh Nguyen, *Finite time stabilization of a KDV system*, (2024), preprint.
- [50] Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987. MR 924157
- [51] Olof Staffans, *Well-posed linear systems*, Encyclopedia of Mathematics and its Applications, vol. 103, Cambridge University Press, Cambridge, 2005. MR 2154892
- [52] Gerald Tenenbaum and Marius Tucsnak, *New blow-up rates for fast controls of Schrödinger and heat equations*, J. Differential Equations **243** (2007), no. 1, 70–100. MR 2363470
- [53] Emmanuel Trélat, Gengsheng Wang, and Yashan Xu, *Characterization by observability inequalities of controllability and stabilization properties*, Pure Appl. Anal. **2** (2020), no. 1, 93–122. MR 4041279
- [54] Marius Tucsnak and George Weiss, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2009. MR 2502023
- [55] Jose Manuel Urquiza, *Rapid exponential feedback stabilization with unbounded control operators*, SIAM J. Control Optim. **43** (2005), no. 6, 2233–2244. MR 2179485
- [56] Ambroise Vest, *Rapid stabilization in a semigroup framework*, SIAM J. Control Optim. **51** (2013), no. 5, 4169–4188. MR 3120757
- [57] George Weiss and Richard Rebarber, *Optimizability and estimatability for infinite-dimensional linear systems*, SIAM J. Control Optim. **39** (2000), no. 4, 1204–1232. MR 1814273
- [58] Robert M. Young, *An introduction to nonharmonic Fourier series*, Pure and Applied Mathematics, vol. 93, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980. MR 591684
- [59] Hans J. Zwart, *Linear quadratic optimal control for abstract linear systems*, Modelling and optimization of distributed parameter systems (Warsaw, 1995), Chapman & Hall, New York, 1996, pp. 175–182. MR 1388531

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