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STABILIZATION OF CONTROL SYSTEMS ASSOCIATED WITH A STRONGLY CONTINUOUS GROUP

HOAI-MINH NGUYEN

ABSTRACT. This paper is devoted to the stabilization of a linear control system $y' = Ay + Bu$ and its suitable non-linear variants where $(A, \mathcal{D}(A))$ is an infinitesimal generator of a strongly continuous group in a Hilbert space \mathbb{H} , and B defined in a Hilbert space \mathbb{U} is an admissible control operator with respect to the semigroup generated by A . Let $\lambda \in \mathbb{R}$ and assume that, for some *positive* symmetric, invertible $Q = Q(\lambda) \in \mathcal{L}(\mathbb{H})$, for some *non-negative*, symmetric $R = R(\lambda) \in \mathcal{L}(\mathbb{H})$, and for some *non-negative*, symmetric $W = W(\lambda) \in \mathcal{L}(\mathbb{U})$, it holds

$$AQ + QA^* - BWB^* + QRQ + 2\lambda Q = 0$$

in the sense that

$$\langle Qx, A^*y \rangle_{\mathbb{H}} + \langle A^*x, Qy \rangle_{\mathbb{H}} - \langle WB^*x, B^*y \rangle_{\mathbb{U}} + \langle RQx, Qy \rangle_{\mathbb{H}} + 2\lambda \langle Qx, y \rangle_{\mathbb{H}} = 0 \quad \forall x, y \in \mathcal{D}(A^*),$$

where A^* is the adjoint of A and $\mathcal{D}(A^*)$ is its domain. We present a new method to study the stabilization of such a system and its suitable nonlinear variants. Both the stabilization using dynamic feedback controls and the stabilization using static feedback controls in a weak sense are investigated. To our knowledge, the stabilization by dynamic feedback controls is new even in the linear setting. The nonlinear case is out of reach previously when B is unbounded for both types of stabilization. Consequently, we derive that if the control system is exactly controllable in some positive time, then it is rapidly stabilizable.

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1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

In this paper, we study the stabilization of a linear control system associated with a strongly continuous group and its related nonlinear systems. Let \mathbb{H} and \mathbb{U} be two Hilbert spaces which denote the state space and the control space, respectively. The corresponding scalar products are $\langle \cdot, \cdot \rangle_{\mathbb{H}}$

and $\langle \cdot, \cdot \rangle_{\mathbb{U}}$, and the corresponding norms are $\| \cdot \|_{\mathbb{H}}$ and $\| \cdot \|_{\mathbb{U}}$. Let $(S(t))_{t \in \mathbb{R}} \subset \mathcal{L}(\mathbb{H})$ be a strongly continuous *group* on \mathbb{H} , i.e.,

$$\begin{aligned} S(0) &= Id \text{ (the identity),} \\ S(t_1 + t_2) &= S(t_1) \circ S(t_2) \quad \forall t_1, t_2 \in \mathbb{R}, \end{aligned}$$

and

$$\lim_{t \rightarrow 0} S(t)x = x \quad \forall x \in \mathbb{H}.$$

Here and in what follows, for two Hilbert spaces \mathbb{X}_1 and \mathbb{X}_2 , we denote $\mathcal{L}(\mathbb{X}_1, \mathbb{X}_2)$ the Banach space of all bounded linear applications from \mathbb{X}_1 to \mathbb{X}_2 with the usual norm, and we simply denote $\mathcal{L}(\mathbb{X}_1, \mathbb{X}_1)$ by $\mathcal{L}(\mathbb{X}_1)$.

Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of $(S(t))_{t \in \mathbb{R}}$ and denote $S(t)^*$ the adjoint of $S(t)$ for $t \in \mathbb{R}$. Then $(S(t)^*)_{t \in \mathbb{R}}$ is also a strongly continuous *group* of continuous linear operators and its infinitesimal generator is $(A^*, \mathcal{D}(A^*))$, which is the adjoint of $(A, \mathcal{D}(A))$. As usual, we equip the domain $\mathcal{D}(A^*)$ with the scalar product

$$\langle z_1, z_2 \rangle_{\mathcal{D}(A^*)} = \langle z_1, z_2 \rangle_{\mathbb{H}} + \langle A^* z_1, A^* z_2 \rangle_{\mathbb{H}} \text{ for } z_1, z_2 \in \mathcal{D}(A^*).$$

Then $\mathcal{D}(A^*)$ is a Hilbert space. Denote $\mathcal{D}(A^*)'$ the dual space of $\mathcal{D}(A^*)$ with respect to \mathbb{H} . Then

$$\mathcal{D}(A^*) \subset \mathbb{H} \subset \mathcal{D}(A^*)'.$$

Let

$$B \in \mathcal{L}(\mathbb{U}, \mathcal{D}(A^*)').$$

In this paper, we consider the following control system, for $T > 0$,

$$(1.1) \quad \begin{cases} y' = Ay + Bu \text{ for } t \in (0, T), \\ y(0) = y_0, \end{cases}$$

where, at time t , the control is $u(t) \in \mathbb{U}$ and the state is $y(t) \in \mathbb{H}$, and $y_0 \in \mathbb{H}$ is an initial datum. This control setting is standard and used to model many control systems, see, e.g., [5, 28]. Interesting aspects of the controllability and the stability of (1.1) can be found in [18, 23, 9, 10, 5, 33, 28, 26] and the references therein.

As usual, see, e.g., [5, 28], we assume that B is an *admissible* control operator with respect to the semi-group $(S(t))_{t \geq 0}$ in the sense that, for all $u \in L^2([0, T]; \mathbb{U})$, it holds that

$$(1.2) \quad \varphi \in C([0, T]; \mathbb{H}) \text{ where } \varphi(t) := \int_0^t S(t-s)Bu(s) ds.$$

As a consequence of the closed graph theorem, see e.g., [4], one has

$$(1.3) \quad \|\varphi\|_{C([0, T]; \mathbb{H})} \leq C_T \|u\|_{L^2((0, T); \mathbb{U})}.$$

Let $\lambda \in \mathbb{R}$ and assume that, for some *positive*, symmetric, invertible $Q = Q(\lambda) \in \mathcal{L}(\mathbb{H})$, for some *non-negative*, symmetric $R = R(\lambda) \in \mathcal{L}(\mathbb{H})$, and for some *non-negative*, symmetric $W = W(\lambda) \in \mathcal{L}(\mathbb{U})$, it holds

$$(1.4) \quad AQ + QA^* - BWB^* + QRQ + 2\lambda Q = 0,$$

where (1.4) is understood in the following sense

$$(1.5) \quad \langle Qx, A^*y \rangle + \langle A^*x, Qy \rangle - \langle WB^*x, B^*y \rangle + \langle RQx, Qy \rangle + 2\lambda \langle Qx, y \rangle = 0 \quad \forall x, y \in \mathcal{D}(A^*).$$

In this paper, given a Hilbert space $\tilde{\mathbb{H}}$ and an operator $\tilde{R} \in \mathcal{L}(\tilde{\mathbb{H}})$ being symmetric, one says that \tilde{R} is non-negative if

$$\langle \tilde{R}x, x \rangle_{\tilde{\mathbb{H}}} \geq 0 \text{ for all } x \in \tilde{\mathbb{H}},$$

and one says that \tilde{R} is positive if, for some positive constant C , it holds ¹

$$\langle \tilde{R}x, x \rangle_{\tilde{\mathbb{H}}} \geq C \|x\|_{\tilde{\mathbb{H}}}^2 \text{ for all } x \in \tilde{\mathbb{H}}.$$

¹Thus positivity here means coercivity.

Recall that system (1.1) is called to be exactly controllable in some positive time T if for all $y_0, y_T \in \mathbb{H}$, there exists $u \in L^2((0, T); \mathbb{U})$ such that

$$y(T) = y_T,$$

where y is the unique weak solution of (1.1) (the definition of the weak solutions is recalled in Section 2). In this case, we also call that the pair (A, B) is exactly controllable in some positive time T . It is known that (1.1) is exactly controllable in time $T > 0$ if and only if the following observability inequality holds, see e.g., [5, 28],

$$(1.6) \quad \int_0^T \|B^* e^{sA^*} x\|_{\mathbb{U}}^2 ds \geq C \|x\|_{\mathbb{H}}^2 \text{ for all } x \in \mathbb{H},$$

where C is a positive constant independent of x . Here and in what follows, if \tilde{A} is the infinitesimal generator of the semigroup $(\tilde{S}(t))_{t \geq 0}$ in a Hilbert space $\tilde{\mathbb{H}}$, we also denote $\tilde{S}(t)$ by $e^{t\tilde{A}}$ for $t \geq 0$.

Identity (1.4) also appears in the linear quadratic optimal control theory under assumptions that are discussed now. Given a *non-negative*, symmetric $R = R(\lambda) \in \mathcal{L}(\mathbb{H})$, consider the cost function

$$(1.7) \quad J_T(u, y) = \int_0^T \langle Ry, y \rangle_{\mathbb{H}}(s) + \langle u, u \rangle_{\mathbb{U}}(s) ds \text{ for } T \in (0, +\infty].$$

For $T \in (0, +\infty]$, one is interested in minimizing $J_T(u, y)$ overall $u \in L^2((0, T); \mathbb{U})$ where y is the weak solution of (1.1). For $0 < T < +\infty$, let $P_T \in \mathcal{L}(\mathbb{H})$ be symmetric and satisfy

$$\langle P_T y_0, y_0 \rangle_{\mathbb{H}} = \inf_{u \in L^2((0, T), \mathbb{U})} J_T(u, y),$$

where y is the weak solution of (1.1) corresponding to u . The finite cost assumption is given by the condition

$$\inf_{u \in L^2((0, +\infty), \mathbb{U})} J_{\infty}(u, y) < +\infty,$$

for all $y_0 \in \mathbb{H}$.

The following results are known from the linear quadratic optimal control theory, as developed in Flandoli, Lasiecka, and Triggiani [12] (see also [17, 31, 34, 25]). Assume that the finite cost condition holds. Let u_{opt} and y_{opt} be the unique solution corresponding to the minimizing problem $\inf_{u \in L^2((0, +\infty), \mathbb{U})} J_{\infty}(u, y)$, i.e.,

$$(1.8) \quad J_{\infty}(u_{opt}, y_{opt}) = \inf_{u \in L^2((0, +\infty), \mathbb{U})} J_{\infty}(u, y),$$

where y is the weak solution of (1.1). Define

$$(1.9) \quad S_{opt}(t)y_0 = y_{opt}(t).$$

Then

$$(1.10) \quad S_{opt}(t)y_0 = S(t)(y_0) + \int_0^t S(t-s)Bu_{opt}(s) ds \text{ for } t \geq 0.$$

Let $(A_{opt}, \mathcal{D}(A_{opt}))$ be the infinitesimal generator of $(S_{opt}(t))_{t \geq 0}$. Then the pointwise limit of P_T as $T \rightarrow +\infty$ exists. Denote this limit by P_{∞} . It follows that $P_{\infty} : \mathcal{D}(A_{opt}) \rightarrow \mathcal{D}(A^*)$ and

$$(1.11) \quad u_{opt}(t) = -B^*P_{\infty}y_{opt}(t) \quad \text{if } y_0 \in \mathcal{D}(A_{opt}).$$

Assume that the system is optimizable, i.e., the finite cost condition holds with R being invertible. Then

$$(1.12) \quad (S_{opt}(t))_{t \geq 0} \text{ is exponentially stable.}$$

Assertions (1.9)-(1.12) thus give the stabilization of (1.1) by static feedback controls in a weak sense since $-B^*P_{\infty}$ is not defined for every element in \mathbb{H} when B is not bounded or equivalently when B^* is not bounded.

More information on P_∞ is known from the work of Flandoli, Lasieka, Triggiani [12] (see also [20, 17]). First, $P_\infty \in \mathcal{L}(\mathbb{H})$ is non-negative and symmetric; moreover, $A^*P_\infty \in \mathcal{L}(\mathcal{D}(A_{opt}), \mathbb{H})$, $B^*P_\infty \in \mathcal{L}(\mathcal{D}(A_{opt}), \mathbb{U})$, and P_∞ satisfies the algebraic Riccati equation

$$(1.13) \quad A^*P_\infty + P_\infty A + R - P_\infty B B^* P_\infty = 0,$$

in the following sense

$$(1.14) \quad \langle A^*P_\infty x, z \rangle_{\mathbb{H}} + \langle x, A^*P_\infty z \rangle_{\mathbb{H}} + \langle R x, z \rangle_{\mathbb{H}} = \langle B^*P_\infty x, B^*P_\infty z \rangle_{\mathbb{U}} \quad \text{for all } x, z \in \mathcal{D}(A_{opt}).$$

Second, assume that $(A^*, R^{1/2})$ is exactly controllable in some positive time, then P_∞ is the unique solution of (1.13) in the sense of (1.14); moreover, it is invertible. Denote

$$(1.15) \quad Q_\infty = P_\infty^{-1}.$$

When B is bounded, the algebraic Riccati equation (1.13) holds in the classic sense, i.e.,

$$(1.16) \quad \langle P_\infty x, A z \rangle_{\mathbb{H}} + \langle A x, P_\infty z \rangle_{\mathbb{H}} + \langle R x, z \rangle_{\mathbb{H}} = \langle B^* P_\infty x, B^* P_\infty z \rangle_{\mathbb{U}} \quad \text{for all } x, z \in \mathcal{D}(A),$$

and the stabilization by static feedback controls in the classic sense is fully understood via the linear quadratic optimal control theory, see, e.g., [21, 33] and the references therein. When B is unbounded, it might happen that P_∞ satisfies different Riccati equations on $\mathcal{D}(A_{opt})$ and $\mathcal{D}(A)$, see [32] (see also [31, 29]). These above facts partly reveal the subtleties of the theory of the Riccati equation.

What has been mentioned so far does not require the group property. When $(S(t))_{t \in \mathbb{R}}$ is a group, more information can be derived from the theory of linear quadratic optimal control theory. Consider the dual system

$$(1.17) \quad \begin{cases} z' = -A^*z + R^{1/2}v \text{ for } t \in (0, T), \\ z(0) = z_0, \end{cases}$$

and consider the dual cost

$$(1.18) \quad \hat{J}_T(v, z) = \int_0^T \langle B^*z, B^*z \rangle_{\mathbb{U}}(s) + \langle v, v \rangle_{\mathbb{H}}(s) ds \text{ for } T \in (0, +\infty].$$

Here and in what follows, for a Hilbert space \mathbb{X}_1 and an operator $R_1 \in \mathcal{L}(\mathbb{X}_1)$ being symmetric and nonnegative, one denotes $R_1^{1/2}$ its non-negative square root. One also uses $R_1^{-1/2}$ to denote the non-negative square root of R_1^{-1} if R_1 is additionally positive.

For $0 < T < +\infty$, let $\hat{Q}_T \in \mathcal{L}(\mathbb{H})$ be symmetric and satisfy

$$\langle \hat{Q}_T z_0, z_0 \rangle_{\mathbb{H}} = \inf_{u \in L^2((0, T), \mathbb{U})} J_T(u, y),$$

where z is the solution of (1.17), for all $z_0 \in \mathbb{H}$. The finite cost assumption of the dual system is given by the condition

$$\inf_{v \in L^2((0, +\infty), \mathbb{U})} \hat{J}_\infty(v, z) < +\infty,$$

for all $z_0 \in \mathbb{H}$, where z is the solution of (1.17). Under the finite cost condition of the dual system, the following results are known from the work of Flandoli, Lasieka, and Triggiani [12] (see also [17]). First, the pointwise limit of \hat{Q}_T as $T \rightarrow +\infty$ exists. Denote this limit by \hat{Q}_∞ . Then \hat{Q}_∞ is non-negative, symmetric, and satisfies the dual algebraic Riccati equation

$$(1.19) \quad A\hat{Q}_\infty + \hat{Q}_\infty A^* + \hat{Q}_\infty R \hat{Q}_\infty - B B^* = 0,$$

in the following sense

$$(1.20) \quad \langle \hat{Q}_\infty x, A^*z \rangle_{\mathbb{H}} + \langle A^*x, \hat{Q}_\infty z \rangle_{\mathbb{H}} + \langle R \hat{Q}_\infty x, \hat{Q}_\infty z \rangle_{\mathbb{H}} = \langle B^*x, B^*z \rangle_{\mathbb{U}} \quad \text{for all } x, z \in \mathcal{D}(A^*).$$

Second, assume in addition that $(-A, B)$ is exactly controllable in some positive time, then \hat{Q}_∞ is the unique solution of (1.19) in the sense of (1.20) and is invertible. Moreover, the following result holds

$$(1.21) \quad Q_\infty = \hat{Q}^\infty \text{ when } (A^*, R^{1/2}) \text{ and } (-A, B) \text{ are exactly controllable in some positive time.}$$

Identity (1.19) (in the sense of (1.20)) is a special case of (1.4) (in the sense of (1.5)) for which $W = I$ and $\lambda = 0$.

We next discuss known results related to (1.4) that come from Gramian operators. Let $\tilde{\lambda} > 0$ and assume that system (1.1) is exactly controllable in time $T > 0$. Thus (1.6) holds. Set, with $T_* = T + \frac{1}{2\tilde{\lambda}}$,

$$(1.22) \quad e(s) = \begin{cases} e^{-2\tilde{\lambda}s} & \text{in } [0, T], \\ 2\tilde{\lambda}e^{-2\tilde{\lambda}T}(T_* - s) & \text{in } (T, T_*]. \end{cases}$$

It is showed in [16] that (1.4) holds for W being the identity, and for $Q \in \mathcal{L}(\mathbb{H})$ being defined by

$$(1.23) \quad \langle Qx_1, x_2 \rangle_{\mathbb{H}} = \int_0^{T_*} e(s) \langle B^* e^{-sA^*} x_1, B^* e^{-sA^*} x_2 \rangle_{\mathbb{U}} ds,$$

for $\lambda = 0$, and for $R \in \mathcal{L}(\mathbb{H})$ being symmetric and defined by

$$\langle RQx, Qx \rangle_{\mathbb{H}} = - \int_0^{T_*} e'(s) \|B^* e^{-sA^*}\|_{\mathbb{U}}^2 ds.$$

Previous results when B is bounded were due to Slemrod [23]. These works are inspired by the ones of Lukes [19] and Kleinman [15] where the Gramian operators were introduced in the finite-dimensional setting. In [29], Urquiza observed in the case A is skew-adjoint and $\tilde{\lambda} > 0$ that (1.4) holds for W being identity, for Q being defined by

$$(1.24) \quad \langle Qx_1, x_2 \rangle_{\mathbb{H}} = \int_0^\infty e^{-2\tilde{\lambda}s} \langle B^* e^{-sA^*} x_1, B^* e^{-sA^*} x_2 \rangle_{\mathbb{U}} ds,$$

for $\lambda = 0$, and for $R = 2\tilde{\lambda}Q^{-1}$. The result of Urquiza was inspired by the Bass method previously discussed by Russell [22, page 114-115] following [5, Section 10.3].

In the settings of Komornik and Urquiza mentioned above, one can check that

$$Q \text{ is invertible by (1.6) using the fact } A \text{ is an infinitesimal of a group,}$$

and

$$(A^*, R^{1/2}) \text{ is exactly controllable,}$$

since A is an infinitesimal of a group and R is invertible and positive by (1.6) since A is an infinitesimal of a group and $\tilde{\lambda} > 0$. Thus, under the additional assumption that $(-A, B)$ is exactly controllable in some positive time, one can apply the linear quadratic optimal control theory mentioned above. It then follows that system (1.1) is stabilizable by static feedback controls in the weak sense (1.11). Urquiza [29] established that (1.1) is not only stabilizable but also stabilizable with the rate $2\tilde{\lambda}$ in the weak sense when A is skew-adjoint. Komornik also gave in [16] that (1.1) is not only stabilizable but also stabilizable with the rate $\tilde{\lambda}$ in the weak sense without assuming that $(-A, B)$ is exactly controllable in some positive time. Further information on the linear quadratic optimal control theory was used in the analysis in [16, 29] (see also [30]).

The goal of this paper is to present a new method to study the stabilization of (1.1) and its suitable nonlinear variants under condition (1.4). We study the stabilization of (1.1) by dynamic feedback controls and by static feedback controls in a weak sense, which we call a trajectory sense. A system is called dynamically stabilizable if it can be embedded as a subsystem of a larger, exponentially stable well-posed system. This definition has been used for finite dimensions, see e.g., [5, chapter 11], and for linear systems in infinite dimension, see e.g., [31].

Our approach is essentially based on the construction of new auxiliary dynamics for both types of stabilization (see Theorem 1.3 and Theorem 1.4) and “integration by parts arguments” (see Section 2 and Lemma 3.1). The new adding variable is inspired by the adjoint state in the linear quadratic optimal control theory and the way to choose controls in the Hilbert Uniqueness Method (HUM) principle. The advantage of our approach is at least twofold. First, the method works well in both linear and *nonlinear* settings. Second, a Lyapunov function is also provided for static feedback controls. To our knowledge, the stabilization of such systems by dynamic feedback controls is new even in the linear setting. The nonlinear case is out of reach previously when B is unbounded for both types of stabilization. Concerning the static feedback controls, as far as we know, a Lyapunov function is not known even in the case where B is bounded and A is not; a Lyapunov function was previously given in the finite-dimensional case [5, 15]. Consequently, we derive that if the system is exactly controllable in some positive time, then the system is rapidly stabilizable.

Adding a new variable is very natural and has been used a long time ago in the control theory even in finite dimensions, see e.g., [5, Section 11.3] and [24, Chapter 7], even for linear control systems. Coron and Pradly [7] showed that there exists a nonlinear system in finite dimensions for which the system cannot be stabilized by static feedback controls but can be stabilized by dynamic feedback ones. Dynamic feedback controls of finite dimensional nature, i.e., the complement system is a system of differential equations, have been previously implemented in the infinite dimensions, see e.g., [8, 6]. It is interesting to know whether or not adding a new variable is necessary in the setting in this paper.

The rest of this section is organized as follows. In the first subsection, we discuss the stabilization (1.1) by dynamic feedback controls. In the second subsection, we discuss the stabilization of (1.1) by static feedback controls (in the trajectory sense). In the last subsection, we give the organization of the paper.

1.1. Stabilization by dynamic feedback controls. In this section, we present the results on the dynamic stabilization of (1.1) and its suitable nonlinear variants. Given an infinitesimal generator \tilde{A} of a semigroup in a Hilbert space $\tilde{\mathbb{H}}$, set

$$\omega_0(\tilde{A}) = \inf_{t>0} \log \|e^{t\tilde{A}}\|_{\mathcal{L}(\tilde{\mathbb{H}})},$$

which denotes the growth of the $e^{t\tilde{A}}$ for $t \geq 0$. It is known, see e.g., [10], that

$$-\infty \leq \omega_0(\tilde{A}) < +\infty.$$

Concerning the dynamic stabilization of (1.1), we have the following result.

Theorem 1.1. *Let $\lambda \in \mathbb{R}$ and assume (1.4) with $R = 0$, and let $\lambda_1 \in \mathbb{R}$. Let $\hat{\omega}_0(A) \geq \omega_0(A)$ and $\hat{\omega}_0(-A^*) \geq \omega_0(-A^*)$ be two real constants such that, for some positive constant c ,*

$$\|e^{tA}\|_{\mathcal{L}(\mathbb{H})} \leq ce^{t\hat{\omega}_0(A)} \text{ for } t \geq 0 \quad \text{and} \quad \|e^{-tA^*}\|_{\mathcal{L}(\mathbb{H})} \leq ce^{t\hat{\omega}_0(-A^*)} \text{ for } t \geq 0,$$

and assume that

$$(1.25) \quad \lambda_1 - 2\lambda > \hat{\omega}_0(A) - \hat{\omega}_0(-A^*).$$

Given $y_0, \tilde{y}_0 \in \mathbb{H}$ arbitrary, let $(y, \tilde{y}) \in (C^0([0, T]; \mathbb{H}))^2$ be the unique weak solution of the system

$$(1.26) \quad \begin{cases} y' = Ay - BWB^*\tilde{y} & \text{in } (0, +\infty), \\ \tilde{y}' = -A^*\tilde{y} - 2\lambda\tilde{y} + \lambda_1 Q^{-1}(y - Q\tilde{y}) & \text{in } (0, +\infty), \\ y(0) = y_0, \quad \tilde{y}(0) = \tilde{y}_0. \end{cases}$$

Then

$$(1.27) \quad \|y(t)\|_{\mathbb{H}} + \|\tilde{y}(t)\|_{\mathbb{H}} \leq Ce^{(\hat{\omega}_0(-A^*) - 2\lambda)t} (\|y(0)\|_{\mathbb{H}} + \|\tilde{y}(0)\|_{\mathbb{H}}) \text{ for } t \geq 0,$$

where C is a positive constant independent of t and (y_0, \tilde{y}_0) . Consequently, if A is skew-adjoint and $\lambda_1 > 2\lambda$, then

$$(1.28) \quad \|y(t)\|_{\mathbb{H}} + \|\tilde{y}(t)\|_{\mathbb{H}} \leq C e^{-2\lambda t} (\|y(0)\|_{\mathbb{H}} + \|\tilde{y}(0)\|_{\mathbb{H}}) \text{ for } t \geq 0.$$

Remark 1.1. The definition of weak solutions is recalled in Section 2. The well-posedness of the weak solutions in Theorem 1.1 is established in Lemma 3.2.

We next illustrate how this result can be extended to a nonlinear setting. We just present here a quite standard nonlinear setting to highlight the robustness of the method. Let $f : \mathbb{H} \rightarrow \mathbb{H}$ be continuous such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(1.29) \quad \|f(x)\|_{\mathbb{H}} \leq \varepsilon \|x\|_{\mathbb{H}} \text{ for } x \in \mathbb{H} \text{ with } \|x\|_{\mathbb{H}} < \delta,$$

and f is Lipschitz in a neighborhood of 0 in \mathbb{H} , i.e., there exist $r > 0$ and $\Lambda > 0$ such that

$$(1.30) \quad \|f(x) - f(y)\|_{\mathbb{H}} \leq \Lambda \|x - y\|_{\mathbb{H}} \text{ for } x, y \in \mathbb{H} \text{ with } \|x\|_{\mathbb{H}}, \|y\|_{\mathbb{H}} < r.$$

We consider the following control problem

$$(1.31) \quad \begin{cases} y' = Ay + f(y) + Bu \text{ for } t \in (0, T), \\ y(0) = y_0 \in \mathbb{H}. \end{cases}$$

Concerning the local stabilization of (1.31), we have the following stabilization results by dynamic feedback controls.

Theorem 1.2. *Let $\lambda \in \mathbb{R}$ and assume that (1.4) holds with $R = 0$, and let $\lambda_1, \gamma \in \mathbb{R}$ be such that $\gamma < \lambda$. Let $\hat{\omega}_0(A) \geq \omega_0(A)$ and $\hat{\omega}_0(-A^*) \geq \omega_0(-A^*)$ be two real constants such that, for some positive constant c ,*

$$\|e^{tA}\|_{\mathcal{L}(\mathbb{H})} \leq c e^{t\hat{\omega}_0(A)} \text{ for } t \geq 0 \quad \text{and} \quad \|e^{-tA^*}\|_{\mathcal{L}(\mathbb{H})} \leq c e^{t\hat{\omega}_0(-A^*)} \text{ for } t \geq 0,$$

and assume that

$$(1.32) \quad \lambda_1 - 2\lambda > \hat{\omega}_0(A) - \hat{\omega}_0(-A^*), \quad 2\gamma - \hat{\omega}_0(-A^*) > 0,$$

and (1.29) and (1.30) hold. There exists $\varepsilon > 0$ (small) such that for $y_0, \tilde{y}_0 \in \mathbb{H}$ with $\|(y_0, \tilde{y}_0)\|_{\mathbb{H}} \leq \varepsilon$, there exists a unique solution $(y, \tilde{y}) \in (C^0([0, T]; \mathbb{H}))^2$ of the system

$$(1.33) \quad \begin{cases} y' = Ay + f(y) - BWB^*\tilde{y} & \text{in } (0, +\infty), \\ \tilde{y}' = -A^*\tilde{y} - 2\lambda\tilde{y} + Q^{-1}f(Q\tilde{y}) + \lambda_1 Q^{-1}(y - Q\tilde{y}) & \text{in } (0, +\infty), \\ y(0) = y_0, \quad \tilde{y}(0) = \tilde{y}_0. \end{cases}$$

Moreover, we have

$$(1.34) \quad \|y(t)\|_{\mathbb{H}} + \|\tilde{y}(t)\|_{\mathbb{H}} \leq C e^{(\hat{\omega}_0(-A^*) - 2\gamma)t} (\|y(0)\|_{\mathbb{H}} + \|\tilde{y}(0)\|_{\mathbb{H}}) \text{ for } t \geq 0,$$

where C is a positive constant independent of t and (y_0, \tilde{y}_0) . Consequently, if A is skew-adjoint and $\lambda_1 > 2\lambda > 2\gamma > 0$ then

$$(1.35) \quad \|y(t)\|_{\mathbb{H}} + \|\tilde{y}(t)\|_{\mathbb{H}} \leq C e^{-2\gamma t} (\|y(0)\|_{\mathbb{H}} + \|\tilde{y}(0)\|_{\mathbb{H}}) \text{ for } t \geq 0.$$

Remark 1.2. The weak solutions given in Theorem 1.2 are understood in the sense of the weak solutions given in Section 2 where the nonlinear terms play as a part of the source term.

Remark 1.3. The well-posedness of the weak solutions in Theorem 1.2 is a part of the proof. In comparison with Theorem 1.1, λ is supposed to satisfy the condition $2\lambda - \hat{\omega}_0(-A^*) > 0$ in Theorem 1.2 to make sure that the solution remains small for large time to ensure the well-posedness.

As a consequence of Theorem 1.1 and Theorem 1.2 (see also Proposition 5.2), we obtain the following results.

Proposition 1.1. *Assume that system (1.1) is exactly controllable in some positive time. System (1.1) is rapidly dynamically stabilizable.*

Proposition 1.2. *Assume that system (1.1) is exactly controllable in some positive time, and (1.29) and (1.30) hold. System (1.31) is locally rapidly dynamically stabilizable.*

Recall that system (1.1) is called rapidly dynamically stabilizable if it can be dynamically exponentially stabilizable with an arbitrary decay rate. A similar meaning with suitable modifications is used for system (1.31).

1.2. Stabilization by static feedback controls. In this section, we present the results on the stabilization of (1.1) and its suitable nonlinear variants using static feedback controls in a weak sense, which will be precise. We begin with (1.1).

Theorem 1.3. *Let $\lambda \in \mathbb{R}$ and assume (1.4). Given $y_0 \in \mathbb{H}$, let $(y, \tilde{y}) \in (C^0([0, T]; \mathbb{H}))^2$ be the unique weak solution of the system*

$$(1.36) \quad \begin{cases} y' = Ay - BWB^*\tilde{y} & \text{in } (0, +\infty), \\ \tilde{y}' = -A^*\tilde{y} - 2\lambda\tilde{y} - RQ\tilde{y} & \text{in } (0, +\infty), \\ y(0) = y_0, \quad \tilde{y}(0) = \tilde{y}_0 := Q^{-1}y_0. \end{cases}$$

Then

$$(1.37) \quad \tilde{y}(t) = Q^{-1}y(t) \text{ for } t \geq 0,$$

and

$$(1.38) \quad \begin{aligned} & \|Q^{-1/2}y(t)\|_{\mathbb{H}}^2 - \|Q^{-1/2}y(\tau)\|_{\mathbb{H}}^2 \\ &= -2\lambda \int_{\tau}^t \|Q^{-1/2}y(s)\|_{\mathbb{H}}^2 ds - \int_{\tau}^t \left(\|W^{1/2}B^*\tilde{y}(s)\|_{\mathbb{U}}^2 + \|R^{1/2}y(s)\|_{\mathbb{H}}^2 \right) ds \text{ for } t \geq \tau \geq 0. \end{aligned}$$

Consequently,

$$(1.39) \quad \|Q^{-1/2}y(t)\|_{\mathbb{H}} \leq e^{-\lambda t} \|Q^{-1/2}y(0)\|_{\mathbb{H}} \text{ for } t \geq 0.$$

Some comments on Theorem 1.3 are in order. Since

$$\tilde{y}' = -A^*\tilde{y} - 2\lambda\tilde{y} - RQ\tilde{y} \text{ in } (0, +\infty)$$

and $\tilde{y}(0) \in \mathbb{H}$, it follows from Lemma 2.1 given in Section 2 that $\tilde{y} \in C([0, T]; \mathbb{H})$ is well-defined for all $T > 0$ and moreover,

$$B^*\tilde{y} \in L^2((0, T), \mathbb{H}) \text{ for all } T > 0.$$

We thus derive that system (1.36) is well-posed and (1.38) makes sense. Combing (1.37) and the equation of y

$$y' = Ay - BWB^*\tilde{y},$$

we have thus shown that the control system $y' = Ay + Bu$ with the *static feedback* control

$$(1.40) \quad "u = -WB^*Q^{-1}y" \text{ for } t \geq 0,$$

is well-posed in the sense given in Theorem 1.3. We only consider (1.40) as static feedback controls in a weak sense, which we call a trajectory sense, since for $y \in \mathbb{H}$, it is not clear how to give the sense to the action $-WB^*Q^{-1}y$. In comparison with the static feedback controls in the sense given by (1.11), the static feedback controls given (1.40) are well-defined in the sense of Theorem 1.3 for all initial data $y_0 \in \mathbb{H}$. Theorem 1.3 can be considered as a new way to view the feedback controls given in (1.11).

It is important in Theorem 1.3 that $\tilde{y}_0 = Q^{-1}y_0$ in (1.36). Due to this fact, one cannot derive from Theorem 1.3 that system (1.1) is dynamically stabilizable via the system

$$(1.41) \quad \begin{cases} y' = Ay - BWB^*\tilde{y} & \text{in } (0, +\infty), \\ \tilde{y}' = -A^*\tilde{y} - 2\lambda\tilde{y} - RQ\tilde{y} & \text{in } (0, +\infty). \end{cases}$$

This is the reason to introduce the term $\lambda_1 Q^{-1}(y - Q\tilde{y})$ in Theorem 1.1.

Remark 1.4. Assertion (1.37) was known in the case where $\lambda = 0$, $W = I$, and under the additional assumptions that $(A^*, R^{1/2})$ and $(-A, B)$ are exactly controllable in some positive time, see [12, Theorems 2.4, 2.6, and 2.7].

Remark 1.5. From (1.38), the quantity $\|Q^{-1/2}y(t)\|_{\mathbb{H}}^2$ can be viewed as the Lyapunov function of the system. This fact seems new to us even in the case where B is bounded and A is not.

We next present a consequence of Theorem 1.3 in the case where A is also a skew-adjoint operator and $R = 0$.

Corollary 1.1. *Let $\lambda \in \mathbb{R}$, and assume that (1.4) holds with $R = 0$ and A is skew-adjoint. Given $y_0 \in \mathbb{H}$, let $(y, z) \in (C^0([0, T]; \mathbb{H}))^2$ be the unique weak solution of the system*

$$(1.42) \quad \begin{cases} y' = Ay - BWB^*\tilde{y} & \text{in } (0, +\infty), \\ \tilde{y}' = -A^*\tilde{y} - 2\lambda\tilde{y} & \text{in } (0, +\infty), \\ y(0) = y_0, \quad \tilde{y}(0) = \tilde{y}_0 := Q^{-1}y_0. \end{cases}$$

Then

$$(1.43) \quad \tilde{y}(t) = Q^{-1}y(t) \text{ for } t \geq 0$$

and, for some positive constants C_1, C_2 , independent of y_0 ,

$$(1.44) \quad C_1 e^{-2\lambda t} \leq \|y(t)\|_{\mathbb{H}} \leq C_2 e^{-2\lambda t} \|y_0\|_{\mathbb{H}} \text{ for } t \geq 0.$$

Corollary 1.1 is a direct consequence of Theorem 1.3. Indeed, (1.43) is a consequence of Theorem 1.3. Since A is skew-adjoint, it follows from the equation of \tilde{y} that

$$(1.45) \quad \|\tilde{y}(t)\| = e^{-2\lambda t} \|\tilde{y}(0)\| \text{ for } t \geq 0.$$

Assertion (1.44) is now a consequence of (1.43) and (1.45).

We next deal with the local stabilization of (1.31) by static feedback controls in the trajectory sense.

Theorem 1.4. *Let $\lambda > 0$ and assume (1.4), (1.29), and (1.30). There exists $\varepsilon > 0$ (small) such that for $y_0 \in \mathbb{H}$ with $\|y_0\|_{\mathbb{H}} \leq \varepsilon$, there exists a unique weak solution $(y, \tilde{y}) \in (C^0([0, T]; \mathbb{H}))^2$ of the system*

$$(1.46) \quad \begin{cases} y' = Ay + f(y) - BWB^*\tilde{y} & \text{in } (0, +\infty), \\ \tilde{y}' = -A^*\tilde{y} - 2\lambda\tilde{y} - RQ\tilde{y} + Q^{-1}f(Q\tilde{y}) & \text{in } (0, +\infty), \\ y(0) = y_0, \quad \tilde{y}(0) = \tilde{y}_0 := Q^{-1}y_0. \end{cases}$$

Moreover, we have

$$(1.47) \quad \tilde{y} = Q^{-1}y \text{ for } t \geq 0,$$

and

$$(1.48) \quad \|Q^{-1/2}y(t)\|_{\mathbb{H}}^2 - \|Q^{-1/2}y(\tau)\|_{\mathbb{H}}^2 = -2\lambda \int_{\tau}^t \|Q^{-1/2}y(s)\|_{\mathbb{H}}^2 ds \\ - \int_{\tau}^t \left(\|W^{1/2}B^*\tilde{y}(s)\|_{\mathbb{U}}^2 + \|R^{1/2}y(s)\|_{\mathbb{H}}^2 \right) ds + 2 \int_{\tau}^t \langle f(y(s)), Q^{-1}y(s) \rangle ds \text{ for } t \geq \tau \geq 0.$$

Consequently, for all $0 < \gamma < \lambda$, there exists ε_{γ} such that for $y_0 \in \mathbb{H}$ with $\|y_0\|_{\mathbb{H}} \leq \varepsilon_{\gamma}$, it holds

$$(1.49) \quad \|Q^{-1/2}y(t)\|_{\mathbb{H}} \leq e^{-\gamma t} \|Q^{-1/2}y(0)\|_{\mathbb{H}} \text{ for } t \geq 0.$$

Remark 1.6. The weak solutions given in Theorem 1.4 are understood in the sense of the weak solutions given in Section 2 where the nonlinear term plays as a part of the source term.

Remark 1.7. In comparison with Theorem 1.1, λ is supposed to be positive in Theorem 1.2 to make sure that the solution remains small for large time.

Here is a variant of Corollary 1.2 in the nonlinear setting, which is a direct consequence of Theorem 1.4, and the proof is omitted.

Corollary 1.2. *Let $\lambda > 0$, and assume that (1.4) holds with $R = 0$ and A is skew-adjoint. Assume (1.29) and (1.30). There exists $\varepsilon > 0$ (small) such that for $y_0 \in \mathbb{H}$ with $\|y_0\|_{\mathbb{H}} \leq \varepsilon$, there exists a unique solution $(y, \tilde{y}) \in (C^0([0, T]; \mathbb{H}))^2$ of the system*

$$(1.50) \quad \begin{cases} y' = Ay + f(y) - BWB^*\tilde{y} & \text{in } (0, +\infty), \\ \tilde{y}' = -A^*\tilde{y} - 2\lambda\tilde{y} + Q^{-1}f(Q\tilde{y}) & \text{in } (0, +\infty), \\ y(0) = y_0, \quad \tilde{y}(0) = \tilde{y}_0 := Q^{-1}y_0. \end{cases}$$

Moreover, we have

$$(1.51) \quad \tilde{y} = Q^{-1}y \text{ for } t \geq 0,$$

and, for all $0 < \gamma < \lambda$, there exists ε_γ such that for $y_0 \in \mathbb{H}$ with $\|y_0\|_{\mathbb{H}} \leq \varepsilon_\gamma$, it holds, for some positive constants C , independent of y_0 ,

$$(1.52) \quad \|y(t)\|_{\mathbb{H}} \leq Ce^{-2\gamma t}\|y_0\|_{\mathbb{H}} \text{ for } t \geq 0.$$

As a consequence of Theorem 1.3 and Theorem 1.4 (see also Proposition 5.1), we obtain the following results.

Proposition 1.3. *Assume that system (1.1) is exactly controllable in some positive time. System (1.1) is rapidly statically stabilizable in the trajectory sense.*

Proposition 1.4. *Assume that system (1.1) is exactly controllable in some positive time, and (1.29) and (1.30) hold. System (1.31) is locally rapidly statically stabilizable in the trajectory sense.*

1.3. Organisation of the paper. The paper is organized as follows. Section 2 is devoted to the well-posedness and some properties of various linear systems considered in this paper. Weak solutions are mainly considered, however, their connections with mild solutions and transposition solutions are also discussed. These are used later in establishing Theorems 1.1 to 1.4. Section 3 is devoted to the proofs of Theorem 1.1 and Theorem 1.2. Section 4 is mainly devoted to the proofs of Theorem 1.3 and Theorem 1.4, which seems new to us. In the last subsection of Section 4 (section Section 4.3), we also discuss the infinitesimal generator of the semigroup associated with the static feedback controls given in Theorem 1.3. In the last section, Section 5, we discuss choices of Q (and also R and W) when the system is exactly controllable.

2. PRELIMINARIES

In this section, we state and prove the well-posedness and some properties of various linear control systems considered in this paper. It is more convenient to consider a slightly more general system

$$(2.1) \quad \begin{cases} y' = Ay + f + Bu + My \text{ in } t \in (0, T), \\ y(0) = y_0, \end{cases}$$

with $y_0 \in \mathbb{H}$, and $f \in L^1((0, T); \mathbb{H})$, and $M \in \mathcal{L}(\mathbb{H})$. Recall that B is assumed to be an admissible control operator with respect to the semigroup $(S(t))_{t \geq 0} \subset \mathcal{L}(\mathbb{H})$ generated by the operator A throughout the paper. In this section, we only assume that $(S(t))_{t \geq 0} \subset \mathcal{L}(\mathbb{H})$ is a strongly continuous semigroup. Thus the condition that $(S(t))_{t \in \mathbb{R}} \subset \mathcal{L}(\mathbb{H})$ is a strongly continuous group on \mathbb{H} is not required in this section. In this paper, we mainly consider the concept of weak solutions. We also

discuss very briefly its connection with the notions of mild solutions and transposition solutions. A weak solution y of (2.1) is understood as an element $y \in C([0, T]; \mathbb{H})$ such that

$$(2.2) \quad \begin{cases} \frac{d}{dt} \langle y, \varphi \rangle_{\mathbb{H}} = \langle Ay + f + Bu + My, \varphi \rangle_{\mathbb{H}} & \text{in } (0, T) \\ y(0) = y_0 \end{cases} \quad \text{for all } \varphi \in \mathcal{D}(A^*)$$

for which

- i)* the differential equation in (2.2) is understood in the distributional sense,
- ii)* the term $\langle Ay + f + Bu + My, \varphi \rangle_{\mathbb{H}}$ is understood as $\langle y, A^* \varphi \rangle_{\mathbb{H}} + \langle f + My, \varphi \rangle_{\mathbb{H}} + \langle u, B^* \varphi \rangle_{\mathbb{U}}$.

The convention in *ii)* will be used throughout this section.

We begin by recalling the well-posedness of (2.1), see [28, Sections 4.1 and 4.2] (in particular, [28, Remark 4.1.2 and Proposition 4.2.5])².

Proposition 2.1. *Let $T > 0$, $y_0 \in \mathbb{H}$, $f \in L^1((0, T); \mathbb{H})$, and $M \in \mathcal{L}(\mathbb{H})$. Then*

- i)* $y \in C([0, T], \mathbb{H})$ is a weak solution of (2.1) if and only if, with $\tilde{f} := f + Bu + My$, it holds³

$$(2.3) \quad y(t) = S(t)y_0 + \int_0^t S(t-s)\tilde{f}(s) ds \quad \text{for } t \in [0, T].$$

- ii)* there exists a unique weak solution $y \in C([0, T], \mathbb{H})$ of (2.1).

Remark 2.1. The equivalence between weak solutions and mild solutions was first proved in the case B is bounded and $f \in C([0, T]; \mathbb{H})$ by Ball [1], see also [3, Chapter 1 of Part II] for related results when B is bounded.

The unique weak solution given in Proposition 2.1 also satisfies the transposition meaning as established in the following result.

Lemma 2.1. *Let $T > 0$, $y_0 \in \mathbb{H}$, $f \in L^1((0, T); \mathbb{H})$, and $M \in \mathcal{L}(\mathbb{H})$, and let $y \in C([0, T]; \mathbb{H})$ be the unique weak solution of (2.1). We have, for $t \in (0, T]$, for $z_t \in \mathcal{D}(A^*)$, and for $g \in C([0, t]; \mathcal{D}(A^*))$,*

$$(2.4) \quad \begin{aligned} \langle y(t), z_t \rangle_{\mathbb{H}} - \langle y_0, z(0) \rangle_{\mathbb{H}} &= \int_0^t \langle u(s), B^* z(s) \rangle_{\mathbb{U}} ds \\ &\quad - \int_0^t \langle g(s), y(s) \rangle_{\mathbb{H}} ds + \int_0^t \langle f(s), z(s) \rangle_{\mathbb{H}} ds + \int_0^t \langle My(s), z(s) \rangle_{\mathbb{H}} ds, \end{aligned}$$

where $z \in C([0, t]; \mathbb{H})$ is the unique weak solution of the backward system

$$(2.5) \quad \begin{cases} z' = -A^* z - g & \text{in } (0, t), \\ z(t) = z_t. \end{cases}$$

Consequently, for $z_T \in \mathbb{H}$ and $g \in L^1((0, T); \mathbb{H})$, the unique weak solution $z \in C([0, T]; \mathbb{H})$ of (2.5) with $t = T$ satisfies

$$(2.6) \quad \|B^* z\|_{L^2((0, T); \mathbb{U})} \leq C_T \left(\|g\|_{L^1((0, T); \mathbb{H})} + \|z_T\|_{\mathbb{H}} \right),$$

and (2.4) holds for $z_t \in \mathbb{H}$ and $g \in L^1((0, t); \mathbb{H})$. Here C_T denotes a position constant independent of g , f , and z_T .

²There is no f in the statement of [28, Proposition 4.2.5] but the result also holds with $f \in L^1((0, T); \mathbb{H})$ and the analysis is the same.

³This identity is understood in $\mathcal{D}(A^*)'$, i.e., $\langle y(t), \varphi \rangle_{\mathbb{H}} = \langle S(t)y_0, \varphi \rangle_{\mathbb{H}} + \int_0^t \langle S(t-s)\tilde{f}(s), \varphi \rangle_{\mathbb{H}} ds$ in $[0, T]$ for all $\varphi \in \mathcal{D}(A^*)$.

Remark 2.2. For $0 < T \leq T_0$, the constant C_T in (2.6) can be chosen independent of T . In fact, extend g by 0 for $t < 0$ and denote this extension by \tilde{g} . Consider the weak solution \tilde{z} of the system

$$(2.7) \quad \begin{cases} \tilde{z}' = -A^* \tilde{z} - g \text{ in } (T - T_0, T), \\ \tilde{z}(T) = z_T. \end{cases}$$

By (2.6), we have

$$\|B^* \tilde{z}\|_{L^2((T-T_0, T); \mathbb{U})} \leq C_{T_0} \left(\|\tilde{g}\|_{L^1((T-T_0, T); \mathbb{H})} + \|z_T\|_{\mathbb{H}} \right).$$

The desired assertion follows by noting that $\tilde{z} = z$ in $(0, T)$ and using the definition of g .

Before giving the proof of Lemma 2.1, let us give/recall the meaning of transposition solutions of system (2.1).

Definition 2.1. Let $T > 0$, $y_0 \in \mathbb{H}$, $f \in L^1((0, T); \mathbb{H})$, and $M \in \mathcal{L}(\mathbb{H})$. A function $y \in C([0, T]; \mathbb{H})$ is called a *transposition solution* of (2.1) if for all $t \in (0, T]$, $z_t \in \mathbb{H}$, and $g \in L^1((0, t); \mathbb{H})$, identity (2.4) holds where $z \in C([0, t]; \mathbb{H})$ is the unique solution of (2.5).

In what follows, for notational ease, we use $\langle \cdot, \cdot, \cdot \rangle$ to denote $\langle \cdot, \cdot, \cdot \rangle_{\mathbb{H}}$ or $\langle \cdot, \cdot, \cdot \rangle_{\mathbb{U}}$ in a clear context. We now give the proof of Lemma 2.1.

Proof of Lemma 2.1. Let $z_t \in \mathcal{D}(A^*)$ and $g \in C([0, t]; \mathcal{D}(A^*))$, and let $z \in C([0, t]; \mathbb{H})$ be the unique weak solution of (2.5). We have, for $n \geq 2$,

$$\langle y(t), z(t) \rangle - \langle y(0), z(0) \rangle = \sum_{i=1}^n \left(\langle y(t_i), z(t_i) \rangle - \langle y(t_{i-1}), z(t_{i-1}) \rangle \right),$$

where $t_0 = 0$ and $t_i = t_{i-1} + t/n$ for $1 \leq i \leq n$.

Since $z_t \in \mathcal{D}(A^*)$ and $g \in C([0, t]; \mathcal{D}(A^*))$, it follows that $z \in C([0, t]; \mathcal{D}(A^*))$. We thus obtain

$$(2.8) \quad \langle y(t_i), z(t_i) \rangle - \langle y(t_{i-1}), z(t_{i-1}) \rangle = \langle y(t_i), z(t_i) - z(t_{i-1}) \rangle + \langle y(t_i) - y(t_{i-1}), z(t_{i-1}) \rangle \\ \stackrel{(2.5)}{=} \langle y(t_i), \int_{t_{i-1}}^{t_i} (-A^* z(s) - g(s)) ds \rangle + \int_{t_{i-1}}^{t_i} \langle Ay(s) + \tilde{f}(s), z(t_{i-1}) \rangle ds.$$

where $\tilde{f} = f + Bu + My$. Recall that the convention *ii*) in the definition of the weak solutions of (2.2) is used here. Using the fact $z \in C([0, t]; \mathcal{D}(A^*))$ and $y \in C([0, t]; \mathbb{H})$, we derive that

$$(2.9) \quad \langle y(t_i), \int_{t_{i-1}}^{t_i} (-A^* z(s) - g(s)) ds \rangle + \int_{t_{i-1}}^{t_i} \langle Ay(s) + \tilde{f}(s), z(t_{i-1}) \rangle ds \\ = \int_{t_{i-1}}^{t_i} \langle y(s), (-A^* z(s) - g(s)) ds \rangle + \int_{t_{i-1}}^{t_i} \langle Ay(s) + \tilde{f}(s), z(s) \rangle ds + o(t_i - t_{i-1}).$$

Here the standard notation of $o(\cdot)$ is used: $o(s)/|s| \rightarrow 0$ as $s \rightarrow 0$. Combining (2.8) and (2.9) yields

$$\langle y(t_i), z(t_i) \rangle - \langle y(t_{i-1}), z(t_{i-1}) \rangle = - \int_{t_{i-1}}^{t_i} \langle y(s), g(s) ds \rangle + \int_{t_{i-1}}^{t_i} \langle \tilde{f}(s), z(s) \rangle ds + o(t_i - t_{i-1}).$$

Using the definition of \tilde{f} , we derive that

$$\langle y(t_i), z(t_i) \rangle - \langle y(t_{i-1}), z(t_{i-1}) \rangle = \int_{t_{i-1}}^{t_i} \langle u(s), B^* z(s) \rangle ds - \int_{t_{i-1}}^{t_i} \langle g(s), y(s) \rangle ds \\ + \int_{t_{i-1}}^{t_i} \langle f(s), z(s) \rangle ds + \int_{t_{i-1}}^{t_i} \langle My(s), z(s) \rangle ds + o(t_i - t_{i-1}).$$

Summing with respect to n and letting $n \rightarrow +\infty$, we reach (2.4) for $z_t \in \mathcal{D}(A^*)$ and $g \in C([0, t]; \mathcal{D}(A^*))$.

We next deal with (2.6). Fix $z_T \in \mathcal{D}(A^*)$ and $g \in C([0, T]; \mathcal{D}(A^*))$. Let $u \in L^2((0, T); \mathbb{U})$ and let $y \in C([0, T]; \mathbb{H})$ be the unique weak solution of (2.1) with $f = 0$, $y_0 = 0$, and $M = 0$. Applying (2.4) with $t = T$, we have

$$(2.10) \quad \int_0^T \langle u(s), B^* z(s) \rangle ds = \langle y(T), z_T \rangle + \int_0^T \langle g(s), y(s) \rangle ds.$$

Since

$$(2.11) \quad |\langle y(T), z_T \rangle| + \int_0^T |\langle g(s), y(s) \rangle| ds \leq \|y(T)\| \|z_T\| + \|g\|_{L^1((0, T); \mathbb{H})} \|y\|_{L^\infty((0, T); \mathbb{H})} \\ \stackrel{(1.3), \text{Proposition 2.1}}{\leq} C \|u\|_{L^2((0, T); \mathbb{U})} \left(\|z_T\| + \|g\|_{L^1((0, T); \mathbb{H})} \right).$$

Combining (2.10) and (2.11) yields

$$\|B^* z\|_{L^2((0, T); \mathbb{U})} \leq C \left(\|z_T\| + \|g\|_{L^1((0, T); \mathbb{H})} \right).$$

Assertion of (2.6) in the case $z_T \in \mathbb{H}$ and $g \in L^1((0, T); \mathbb{H})$ follows from this case by density.

Finally, (2.4) with $z_t \in \mathbb{H}$ and $g \in L^1((0, t); \mathbb{H})$ also follows from the case $z_t \in \mathcal{D}(A^*)$ and $g \in C([0, t]; \mathcal{D}(A^*))$ by density. \square

We next prove that the solutions in the transposition sense (2.4) are also unique. Their existence is a direct consequence of Proposition 2.1 and Lemma 2.1. More precisely, we have the following result.

Lemma 2.2. *Let $T > 0$, $y_0 \in \mathbb{H}$, and $u \in L^2((0, T); \mathbb{U})$. There exists a unique transposition solution $y \in C^0([0, T]; \mathbb{H})$ of (2.1). Moreover,*

$$(2.12) \quad \|y(\tau)\|_{\mathbb{H}} \leq C_T \left(\|y_0\| + \|u\|_{L^2((0, T); \mathbb{U})} \right),$$

for some positive constant C_T , independent of y_0 and u .

Remark 2.3. Let $0 < T \leq T_0$. By the arguments as in Remark 2.2, one can chose the constant C_T in (2.12) independent of T .

Proof. By Proposition 2.1 and Lemma 2.1, it suffices to prove the uniqueness.

Let $\mu > 0$ be large. We equip $C^0([0, T]; \mathbb{H})$ with the following norm

$$\|y\| = \sup_{t \in [0, T]} e^{-\mu t} \|y(t)\|_{\mathbb{H}}.$$

Recall that y is a transposition solution if, for $t \geq 0$,

$$(2.13) \quad \langle y(t), z_t \rangle - \langle y_0, z(0) \rangle = \int_0^t \langle u(s), B^* z(s) \rangle ds + \int_0^t \langle f(s), z(s) \rangle ds + \int_0^t \langle y(s), M^* z(s) \rangle ds,$$

where $z_t \in \mathbb{H}$ and z is a solution of the backward system

$$(2.14) \quad \begin{cases} z' = -A^* z \text{ in } (0, t), \\ z(t) = z_t. \end{cases}$$

Thus if y and \hat{y} are two transposition solutions, then

$$(2.15) \quad \langle y(t) - \hat{y}(t), z_t \rangle = \int_0^t \langle y(s) - \hat{y}(s), M^* z(s) \rangle ds.$$

This implies

$$e^{-\mu t} \|y(t) - \hat{y}(t)\| \leq C e^{-\mu t} \int_0^t \|y(s) - \hat{y}(s)\| ds \leq \frac{C}{\mu} \|y - \hat{y}\|.$$

Here and in what follows in this proof, C denotes a positive constant independent of y , \hat{y} , and μ . Thus

$$\|y - \hat{y}\| \leq \frac{C}{\mu} \|y - \hat{y}\|.$$

The uniqueness follows and the proof is complete. \square

Remark 2.4. Similar results in the case $M = 0$, $f = 0$, and $g = 0$ can be found in [5, Section 2.3 of Chapter 2].

3. DYNAMIC FEEDBACK CONTROLS

This section is devoted to the proof of Theorem 1.1 and Theorem 1.2. It contains three subsections and is organized as follows. In the first subsection, we state and prove two useful lemmas, which will be used in the proofs of Theorem 1.1. The proofs of Theorem 1.1 and Theorem 1.2 are given in the last two subsections, respectively.

3.1. Two useful lemmas. Note that (1.4) can be written under an equivalent form as follows

$$(3.1) \quad A_\lambda Q + QA_\lambda^* - BWB^* + QRQ = 0,$$

where

$$(3.2) \quad A_\lambda = A + \lambda I.$$

The meaning of (1.5) can be rewritten as follows

$$(3.3) \quad \langle Qx, A_\lambda^* y \rangle + \langle A_\lambda^* x, Qy \rangle - \langle WB^* x, B^* y \rangle + \langle RQx, Qy \rangle = 0 \quad \forall x, y \in \mathcal{D}(A^*).$$

We have the following result concerning (1.4).

Lemma 3.1. *Assume (1.4), i.e., (1.5). Given $x_0, y_0 \in \mathbb{H}$ and $f, g \in L^1((0, T); \mathbb{H})$, let $x, y \in C([0, T]; \mathbb{H})$ be the unique weak solution of the systems*

$$\begin{cases} x' = A_\lambda^* x + f \text{ in } (0, T), \\ x(0) = x_0, \end{cases} \quad \text{and} \quad \begin{cases} y' = A_\lambda^* y + g \text{ in } (0, T), \\ y(0) = y_0. \end{cases}$$

We have, for $t \in [0, T]$,

$$(3.4) \quad \langle Qx(t), y(t) \rangle - \langle Qx_0, y_0 \rangle = \int_0^t \left(\langle WB^* x(s), B^* y(s) \rangle - \langle RQx(s), Qy(s) \rangle \right) ds + \int_0^t \left(\langle Qf(s), y(s) \rangle + \langle Qg(s), x(s) \rangle \right) ds.$$

Proof. We first assume that $x_0, y_0 \in \mathcal{D}(A^*)$ and $f, g \in C([0, T]; \mathcal{D}(A^*))$. Then $x, y \in C([0, T]; \mathcal{D}(A^*))$ and $x', y' \in C([0, T]; \mathbb{H})$. We have

$$\frac{d}{dt} \langle Qx, y \rangle = \langle x', Qy \rangle + \langle Qx, y' \rangle = \langle A_\lambda^* x, Qy \rangle + \langle Qx, A_\lambda^* y \rangle + \langle f, Qy \rangle + \langle Qx, g \rangle.$$

Using (1.5), since Q is symmetric, it follows that

$$\frac{d}{dt} \langle Qx, y \rangle = \langle WB^* x, B^* y \rangle - \langle RQx, Qy \rangle + \langle Qf, y \rangle + \langle Qg, x \rangle.$$

We thus obtain (3.4).

The proof in the general case is based on the previous case, a density argument, and Lemma 2.1. \square

We next deal with the well-posedness of (1.36) in Theorem 1.1. It might be more convenient to consider a slightly more general system

$$(3.5) \quad \begin{cases} y' = Ay + f - BWB^*\tilde{y} + M_1y + M_2\tilde{y} \text{ for } t \in (0, T), \\ \tilde{y}' = -A^*\tilde{y} + \tilde{f} + \widetilde{M}_1\tilde{y} + \widetilde{M}_2y \text{ for } t \in (0, T), \\ y(0) = y_0, \quad \tilde{y}(0) = \tilde{y}_0, \end{cases}$$

with $y_0, \tilde{y}_0 \in \mathbb{H}$, $f, \tilde{f} \in L^1((0, T); \mathbb{H})$, $M_1, M_2, \widetilde{M}_1, \widetilde{M}_2 \in \mathcal{L}(\mathbb{H})$, and $W \in \mathcal{L}(\mathbb{U})$. A weak solution y of (3.5) is understood as an element $(y, \tilde{y}) \in (C([0, T]; \mathbb{H}))^2$ such that

$$(3.6) \quad \begin{cases} \frac{d}{dt} \langle y, \varphi \rangle_{\mathbb{H}} = \langle Ay + f - BWB^*\tilde{y} + M_1y + M_2\tilde{y}, \varphi \rangle_{\mathbb{H}} \text{ in } [0, T] \\ \frac{d}{dt} \langle \tilde{y}, \tilde{\varphi} \rangle_{\mathbb{H}} = \langle -A^*\tilde{y} + \tilde{f} + \widetilde{M}_1\tilde{y} + \widetilde{M}_2y, \tilde{\varphi} \rangle_{\mathbb{H}} \text{ in } [0, T] \\ y(0) = y_0, \quad \tilde{y}(0) = \tilde{y}_0, \end{cases} \quad \text{for all } \varphi, \tilde{\varphi} \in \mathcal{D}(A^*),$$

for which

- i) the differential equations are understood in the distributional sense,
- ii) the term $\langle Ay + f - BWB^*\tilde{y} + M_1y + M_2\tilde{y}, \varphi \rangle_{\mathbb{H}}$ is understood as $\langle y, A^*\varphi \rangle_{\mathbb{H}} + \langle f + M_1y + M_2\tilde{y}, \varphi \rangle_{\mathbb{H}} - \langle WB^*\tilde{y}, B^*\varphi \rangle_{\mathbb{U}}$.

Note that $B^*\tilde{y} \in L^2(0, T; \mathbb{U})$ since B is an admissible control operator.

We have the following result on the well-posedness of (3.5).

Lemma 3.2. *Let A be an infinitesimal generator of a group, and let $M_1, M_2, \widetilde{M}_1, \widetilde{M}_2 \in \mathcal{L}(\mathbb{H})$ and $W \in \mathcal{L}(\mathbb{U})$. Let $T > 0$, $y_0, \tilde{y}_0 \in \mathbb{H}$, $f, \tilde{f} \in L^1((0, T); \mathbb{H})$. There exists a unique weak solution $(y, \tilde{y}) \in (C([0, T], \mathbb{H}))^2$ of (3.5). Moreover, with $g := f - BWB^* + My$ and $\tilde{g} := \tilde{f} + \widetilde{M}_1\tilde{y} + \widetilde{M}_2y$, we have ⁴*

$$(3.7) \quad y(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A}g(s) ds \text{ for } t \in [0, T],$$

and

$$(3.8) \quad \tilde{y}(t) = e^{-tA^*}\tilde{y}_0 + \int_0^t e^{-(t-s)A^*}\tilde{g}(s) ds \text{ for } t \in [0, T].$$

Moreover, we have

$$\|y(t), \tilde{y}(t)\|_{\mathbb{H}} \leq C \left(\|y_0, \tilde{y}_0\|_{\mathbb{H}} + \|(f, \tilde{f})\|_{L^1((0, T); \mathbb{H})} \right) \text{ in } [0, T],$$

for some positive constant C , independent of y_0, \tilde{y}_0, f , and \tilde{f} .

Remark 3.1. In Lemma 3.2, we does not require that W is symmetric (or non-negative).

Proof. We first note that $(y, \tilde{y}) \in (C([0, T]; \mathbb{H}))^2$ is a weak solution of (3.5) if and only if $(y, \tilde{y}) \in (C([0, T]; \mathbb{H}))^2$ and (2.3) holds. This is a consequence of Proposition 2.1.

We now establish the existence and uniqueness. Let $\mu > 0$ be large. We equip $C([0, T]; \mathbb{H})$ the following norm

$$\|y\| = \sup_{t \in [0, T]} e^{-\mu t} \|y(t)\|_{\mathbb{H}}.$$

Define $\mathcal{F} : (C([0, T]; \mathbb{H}))^2 \rightarrow (C([0, T]; \mathbb{H}))^2$ as follows

$$\mathcal{F} \begin{pmatrix} y(t) \\ \tilde{y}(t) \end{pmatrix} = \begin{pmatrix} e^{tA}y_0 + \int_0^t e^{(t-s)A}g(s) ds, \\ e^{-tA^*}\tilde{y}_0 + \int_0^t e^{-(t-s)A^*}\tilde{g}(s) ds \end{pmatrix} \text{ for } t \in [0, T].$$

⁴These identities below are understood in $\mathcal{D}(A^*)'$ and $\mathcal{D}(-A^*)'$, respectively.

We have

$$\begin{aligned} \mathcal{F} \begin{pmatrix} y_2(t) \\ \tilde{y}_2(t) \end{pmatrix} - \mathcal{F} \begin{pmatrix} y_1(t) \\ \tilde{y}_1(t) \end{pmatrix} &= \begin{pmatrix} \int_0^t e^{(t-s)A} \left(-BWB^*(\tilde{y}_2 - \tilde{y}_1) + M_1(y_2 - y_1) + M_2(\tilde{y}_2 - \tilde{y}_1) \right) ds, \\ \int_0^t e^{-(t-s)A^*} \left(\tilde{M}_1(\tilde{y}_2 - \tilde{y}_1) + \tilde{M}_2(y_2 - y_1) \right) ds \end{pmatrix}. \end{aligned}$$

It follows that

$$\begin{aligned} \left\| \mathcal{F} \begin{pmatrix} y_2(t) \\ \tilde{y}_2(t) \end{pmatrix} - \mathcal{F} \begin{pmatrix} y_1(t) \\ \tilde{y}_1(t) \end{pmatrix} \right\| &\leq C \left(\int_0^t \|(y_2, \tilde{y}_2)(s) - (y_1, \tilde{y}_1)(s)\| ds + \|B^*(\tilde{y}_2 - \tilde{y}_1)\|_{L^2((0,t);U)} \right). \end{aligned}$$

Here and in what follows in this proof, C denotes a positive constant independent of solutions and μ .

This implies, by (2.6) of Lemma 2.1,

$$e^{-\mu\tau} \left\| \mathcal{F} \begin{pmatrix} y_2(t) \\ \tilde{y}_2(t) \end{pmatrix} - \mathcal{F} \begin{pmatrix} y_1(t) \\ \tilde{y}_1(t) \end{pmatrix} \right\| \leq C e^{-\mu t} \int_0^t \|(y_2, \tilde{y}_2)(s) - (y_1, \tilde{y}_1)(s)\| ds.$$

This yields

$$e^{-\mu\tau} \left\| \mathcal{F} \begin{pmatrix} y_2(t) \\ \tilde{y}_2(t) \end{pmatrix} - \mathcal{F} \begin{pmatrix} y_1(t) \\ \tilde{y}_1(t) \end{pmatrix} (t) \right\| \leq \frac{C}{\mu} \|(y_2, \tilde{y}_2) - (y_1, \tilde{y}_1)\|.$$

We derive that

$$\left\| \mathcal{F} \begin{pmatrix} y_2(t) \\ \tilde{y}_2(t) \end{pmatrix} - \mathcal{F} \begin{pmatrix} y_1(t) \\ \tilde{y}_1(t) \end{pmatrix} \right\| \leq \frac{C}{\mu} \left\| \begin{pmatrix} y_2(t) \\ \tilde{y}_2(t) \end{pmatrix} - \begin{pmatrix} y_1(t) \\ \tilde{y}_1(t) \end{pmatrix} \right\|.$$

By considering μ large enough, the existence and uniqueness of the weak solutions follow from a standard fixed point theorem. \square

3.2. Dynamic feedback controls in the linear case - Proof of Theorem 1.1. Set, for $t \geq 0$,

$$(3.9) \quad y_\lambda(t) = e^{\lambda t} y(t) \quad \text{and} \quad \tilde{y}_\lambda(t) = e^{\lambda t} \tilde{y}(t),$$

and denote

$$A_\lambda = A + \lambda I.$$

We have

$$(3.10) \quad \begin{cases} y'_\lambda = A_\lambda y_\lambda - BWB^* \tilde{y}_\lambda & \text{in } (0, +\infty), \\ \tilde{y}'_\lambda = -A_\lambda^* \tilde{y}_\lambda + \lambda_1 Q^{-1}(y_\lambda - Q \tilde{y}_\lambda) & \text{in } (0, +\infty), \\ y_\lambda(0) = y(0), \quad \tilde{y}_\lambda(0) = \tilde{y}(0). \end{cases}$$

Set, for $t \geq 0$,

$$Z_\lambda(t) = y_\lambda(t) - Q \tilde{y}_\lambda(t).$$

We formally have

$$\begin{aligned} \frac{d}{dt} Z_\lambda &= A_\lambda y_\lambda - BWB^* \tilde{y}_\lambda + Q A_\lambda^* \tilde{y}_\lambda - \lambda_1 Z_\lambda \\ &= A_\lambda (y_\lambda - Q \tilde{y}_\lambda) + A_\lambda Q \tilde{y}_\lambda - BWB^* \tilde{y}_\lambda + Q A_\lambda^* \tilde{y}_\lambda - \lambda_1 Z_\lambda, \end{aligned}$$

which yields, since (1.4) holds with $R = 0$, that

$$(3.11) \quad \frac{d}{dt}Z_\lambda = A_\lambda Z_\lambda - \lambda_1 Z_\lambda.$$

We now give the rigor proof of (3.11). Let $\tau > 0$, $\varphi_\tau \in \mathbb{H}$ and let $\varphi \in C([0, \tau]; \mathbb{H})$ be the unique weak solution of the system

$$(3.12) \quad \begin{cases} \varphi' = -A_\lambda^* \varphi \text{ in } (0, \tau), \\ \varphi(\tau) = \varphi_\tau. \end{cases}$$

Applying Lemma 2.1 for A_λ with $t = \tau$, we derive from (3.10) and (3.12) that

$$(3.13) \quad \langle y_\lambda(\tau), \varphi(\tau) \rangle - \langle y_\lambda(0), \varphi(0) \rangle = - \int_0^\tau \langle WB^* \tilde{y}_\lambda(s), B^* \varphi(s) \rangle ds.$$

Applying Lemma 3.1 for A_λ , $\tilde{y}_\lambda(\tau - \cdot)$ and $\varphi(\tau - \cdot)$ ($R = 0$), we obtain

$$(3.14) \quad \begin{aligned} & \langle Q \tilde{y}_\lambda(0), \varphi(0) \rangle - \langle Q \tilde{y}_\lambda(\tau), \varphi(\tau) \rangle \\ &= \int_0^\tau \langle WB^* \tilde{y}_\lambda(\tau - s), B^* \varphi(\tau - s) \rangle ds - \lambda_1 \int_0^\tau \langle Z_\lambda(\tau - s), \varphi(\tau - s) \rangle ds. \end{aligned}$$

Summing (3.13) and (3.14), we deduce from (3.10) and (3.12) that

$$\langle Z_\lambda(\tau), \varphi(\tau) \rangle - \langle Z_\lambda(0), \varphi(0) \rangle = -\lambda_1 \int_0^\tau \langle Z_\lambda(\tau - s), \varphi(\tau - s) \rangle ds.$$

This yields

$$\langle Z_\lambda(\tau), \varphi(\tau) \rangle - \langle Z_\lambda(0), e^{\tau A^*} \varphi(\tau) \rangle = -\lambda_1 \int_0^\tau \langle Z_\lambda(\tau - s), e^{s A^*} \varphi(\tau) \rangle ds.$$

Since $\varphi(\tau) \in \mathbb{H}$ is arbitrary, we obtain

$$Z_\lambda(\tau) = e^{\tau A} Z_\lambda(0) - \lambda_1 \int_0^\tau e^{(\tau-s)A} Z_\lambda(s) ds,$$

which implies (3.11).

We derive from (3.11) that

$$(3.15) \quad \|Z_\lambda(t)\|_{\mathbb{H}} \leq C e^{(-\lambda_1 + \lambda + \hat{\omega}_0(A))t} \|Z_\lambda(0)\|_{\mathbb{H}},$$

which yields

$$(3.16) \quad \|y(t) - Q \tilde{y}(t)\|_{\mathbb{H}} \leq C e^{(-\lambda_1 + \hat{\omega}_0(A))t} \|y(0) - Q \tilde{y}(0)\|_{\mathbb{H}}.$$

Here and in what follows in this proof, C is a positive constant independent of t and (y_0, \tilde{y}_0) .

Since

$$\tilde{y}' = -A^* \tilde{y} - 2\lambda \tilde{y} + \lambda_1 Q^{-1}(y - Q \tilde{y}) \quad \text{in } (0, +\infty),$$

it follows that

$$\tilde{y}'_{2\lambda} = -A^* \tilde{y}_{2\lambda} + f(t) \text{ in } (0, +\infty),$$

where, in $(0, +\infty)$

$$\tilde{y}_{2\lambda} = e^{2\lambda t} \tilde{y}(t) \quad \text{and} \quad f(t) = \lambda_1 e^{2\lambda t} Q^{-1}(y(t) - Q \tilde{y}(t)).$$

We obtain

$$(3.17) \quad \tilde{y}_{2\lambda}(t) = e^{-t A^*} \tilde{y}_{2\lambda}(0) + \int_0^t e^{-(t-s)A^*} f(s) ds.$$

From the definition of f and (3.16), we have

$$\left\| \int_0^t e^{-(t-s)A^*} f(s) ds \right\|_{\mathbb{H}} \leq C \int_0^t e^{\hat{\omega}_0(-A^*)(t-s)} e^{(-\lambda_1 + \hat{\omega}_0(A) + 2\lambda)s} \|y(0) - Q \tilde{y}(0)\|_{\mathbb{H}} ds.$$

Since

$$-\hat{\omega}_0(-A^*) + \hat{\omega}_0(A) + 2\lambda - \lambda_1 \stackrel{(1.25)}{<} 0,$$

it follows that

$$(3.18) \quad \left\| \int_0^t e^{-(t-s)A^*} f(s) ds \right\|_{\mathbb{H}} \leq C e^{\hat{\omega}_0(-A^*)t} \|y(0) - Q\tilde{y}(0)\|_{\mathbb{H}}.$$

Combining (3.17) and (3.18) yields

$$\|\tilde{y}_{2\lambda}(t)\|_{\mathbb{H}} \leq C e^{\hat{\omega}_0(-A^*)t} (\|\tilde{y}(0)\|_{\mathbb{H}} + \|y(0) - Q\tilde{y}(0)\|_{\mathbb{H}}),$$

which yields

$$(3.19) \quad \|\tilde{y}(t)\|_{\mathbb{H}} \leq C e^{(\hat{\omega}_0(-A^*)-2\lambda)t} (\|y(0)\|_{\mathbb{H}} + \|\tilde{y}(0)\|_{\mathbb{H}}).$$

Combining (3.16) and (3.19), we obtain

$$(3.20) \quad \|y(t)\|_{\mathbb{H}} + \|\tilde{y}(t)\|_{\mathbb{H}} \leq C \left(e^{(-\lambda_1 + \hat{\omega}_0(A))t} + e^{(\hat{\omega}_0(-A^*)-2\lambda)t} \right) (\|y(0)\|_{\mathbb{H}} + \|\tilde{y}(0)\|_{\mathbb{H}}).$$

Since

$$\lambda_1 - \hat{\omega}_0(A) \stackrel{(1.25)}{>} 2\lambda - \hat{\omega}_0(-A^*),$$

it follows from (3.20) that

$$(3.21) \quad \|y(t)\|_{\mathbb{H}} + \|\tilde{y}(t)\|_{\mathbb{H}} \leq C e^{(\hat{\omega}_0(-A^*)-2\lambda)t} (\|y(0)\|_{\mathbb{H}} + \|\tilde{y}(0)\|_{\mathbb{H}}),$$

which is (1.27).

It is clear that (1.28) is a direct consequence of (1.27).

The proof is complete. \square

3.3. Dynamic feedback controls in the nonlinear case - Proof of Theorem 1.2. For each $T > 0$ there exists $\varepsilon = \varepsilon_T > 0$ such that (1.33) is well-posed in the time interval $[0, T]$. The global existence and uniqueness follow for small ε provided that (1.34) is established for each fixed time interval $[0, T]$ with ε_T sufficiently small.

As in the proof of Theorem 1.1, set, for $t \geq 0$,

$$(3.22) \quad y_\lambda(t) = e^{\lambda t} y(t) \quad \text{and} \quad \tilde{y}_\lambda(t) = e^{\lambda t} \tilde{y}(t),$$

and denote

$$A_\lambda = A + \lambda I.$$

We have

$$(3.23) \quad \begin{cases} y'_\lambda = A_\lambda y_\lambda + e^{\lambda \cdot} f(e^{-\lambda \cdot} y_\lambda) - BWB^* \tilde{y}_\lambda & \text{in } (0, +\infty), \\ \tilde{y}'_\lambda = -A_\lambda^* \tilde{y}_\lambda + Q^{-1} e^{\lambda \cdot} f(e^{-\lambda \cdot} Q \tilde{y}_\lambda) + \lambda_1 Q^{-1} (y_\lambda - Q \tilde{y}_\lambda) & \text{in } (0, +\infty), \\ y_\lambda(0) = y(0), \quad \tilde{y}_\lambda(0) = \tilde{y}(0). \end{cases}$$

Set, for $t \geq 0$,

$$Z_\lambda(t) = y_\lambda(t) - Q \tilde{y}_\lambda(t).$$

As in the proof of Theorem 1.1, we formally have

$$(3.24) \quad \begin{aligned} \frac{d}{dt} Z_\lambda &= A_\lambda y_\lambda + e^{\lambda \cdot} f(e^{-\lambda \cdot} y_\lambda) - BWB^* \tilde{y}_\lambda + QA_\lambda^* \tilde{y}_\lambda - e^{\lambda \cdot} f(e^{-\lambda \cdot} Q \tilde{y}_\lambda) - \lambda_1 Z_\lambda \\ &= A_\lambda (y_\lambda - Q \tilde{y}_\lambda) + A_\lambda Q \tilde{y}_\lambda + e^{\lambda \cdot} f(e^{-\lambda \cdot} y_\lambda) - BWB^* \tilde{y}_\lambda + QA_\lambda^* \tilde{y}_\lambda - e^{\lambda \cdot} f(e^{-\lambda \cdot} Q \tilde{y}_\lambda) - \lambda_1 Z_\lambda. \end{aligned}$$

Using (1.4) with $R = 0$, as in the proof of Theorem 1.1, we can rigorously derive that

$$(3.25) \quad \frac{d}{dt} Z_\lambda = A_\lambda Z_\lambda - \lambda_1 Z_\lambda + g_1,$$

where

$$g_1(t) = e^{\lambda t} \left(f(e^{-\lambda t} y_\lambda(t)) - f(e^{-\lambda t} Q \tilde{y}_\lambda(t)) \right).$$

It follows from (3.25) that

$$(3.26) \quad \|Z_\lambda(t)\|_{\mathbb{H}} \leq C e^{(-\lambda_1 + \lambda + \hat{\omega}_0(A))t} \|Z_\lambda(0)\|_{\mathbb{H}} + C \int_0^t e^{(-\lambda_1 + \lambda + \hat{\omega}_0(A))(t-s)} \|g_1(s)\|_{\mathbb{H}} ds.$$

Here and in what follows in this proof, C is a positive constant independent of t and (y_0, \tilde{y}_0) .

From (3.26), we obtain

$$(3.27) \quad \|y(t) - Q \tilde{y}(t)\|_{\mathbb{H}} \leq C e^{(-\lambda_1 + \hat{\omega}_0(A))t} \|y(0) - Q \tilde{y}(0)\|_{\mathbb{H}} + C e^{(-\lambda_1 + \hat{\omega}_0(A))t} \int_0^t e^{(-\lambda_1 + \lambda + \hat{\omega}_0(A))s} \|g_1(s)\|_{\mathbb{H}} ds.$$

Since

$$\tilde{y}' = -A^* \tilde{y} - 2\lambda \tilde{y} + e^{\lambda \cdot} Q^{-1} f(e^{-\lambda \cdot} Q \tilde{y}_\lambda) + \lambda_1 Q^{-1} (y - Q \tilde{y}) \quad \text{in } (0, +\infty),$$

it follows that

$$\tilde{y}'_{2\lambda} = -A^* \tilde{y}_{2\lambda} + f_1(t) + f(t) \quad \text{in } (0, +\infty),$$

where, in $(0, +\infty)$,

$$\tilde{y}_{2\lambda} = e^{2\lambda} \tilde{y}(t), \quad f(t) = \lambda_1 e^{2\lambda t} Q^{-1} (y(t) - Q \tilde{y}(t)), \quad \text{and} \quad f_1(t) = e^{3\lambda t} Q^{-1} f(e^{-\lambda t} Q \tilde{y}_\lambda(t)).$$

We obtain

$$(3.28) \quad \tilde{y}_{2\lambda}(t) = e^{-tA^*} \tilde{y}_{2\lambda}(0) + \int_0^t e^{-(t-s)A^*} (f(s) + f_1(s)) ds.$$

As in the proof of Theorem 1.1, we have

$$(3.29) \quad \left\| \int_0^t e^{-(t-s)A^*} f(s) ds \right\|_{\mathbb{H}} \leq C e^{t\hat{\omega}_0(-A^*)} \left(\|y(0) - Q \tilde{y}(0)\|_{\mathbb{H}} + \int_0^t e^{(-\lambda_1 + \lambda + \hat{\omega}_0(A))s} \|g_1(s)\|_{\mathbb{H}} ds \right) \text{ for } t \geq 0.$$

Using (1.29), we derive from (3.16), (3.28), and (3.29) that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|(y(t), \tilde{y}(t))\|_{\mathbb{H}} \leq \delta$ in $[0, T]$ for some $T > 0$, then

$$\begin{aligned} \|(y(t), \tilde{y}(t))\|_{\mathbb{H}} &\leq C e^{(\hat{\omega}_0(-A^*) - 2\lambda)t} \|(y_0, \tilde{y}_0)\|_{\mathbb{H}} \\ &\quad + C \varepsilon e^{(\hat{\omega}_0(-A^*) - 2\lambda)t} \int_0^t e^{(-\lambda_1 + \lambda + \hat{\omega}_0(A))s} \|(y(s), \tilde{y}(s))\|_{\mathbb{H}} ds \\ &\quad + C \varepsilon e^{(\hat{\omega}_0(-A^*) - 2\lambda)t} \int_0^t e^{-s\hat{\omega}_0(-A^*)} \|(y(s), \tilde{y}(s))\|_{\mathbb{H}} ds \text{ for } t \in [0, T]. \end{aligned}$$

Here C is a positive constant independent of T , ε and δ . Thus, for all $T > 0$, there exists $\delta > 0$ such that if $\|(y_0, \tilde{y}_0)\|_{\mathbb{H}} \leq \delta$ then

$$(3.30) \quad \|(y(t), \tilde{y}(t))\|_{\mathbb{H}} \leq C e^{(\hat{\omega}_0(-A^*) - 2\lambda)t} \|(y_0, \tilde{y}_0)\|_{\mathbb{H}} \text{ in } [0, T].$$

In particular, we derive that if T is chosen sufficiently large,

$$(3.31) \quad \|(y(T), \tilde{y}(T))\|_{\mathbb{H}} \leq e^{(\omega_0(-A^*) - 2\gamma)T} \|(y_0, \tilde{y}_0)\|_{\mathbb{H}}.$$

The conclusion follows from (3.30) and (3.31) by considering the time $nT \leq t \leq n(T+1)$ for $n \in \mathbb{N}$. \square

4. STATIC FEEDBACK CONTROLS IN THE TRAJECTORY SENSE

This section containing three subsections is mainly devoted to the proofs of Theorem 1.3 and Theorem 1.4. The proofs of Theorem 1.3 and Theorem 1.4 are given respectively in the first two subsections. The last subsection is devoted to the study of the infinitesimal generator of the semigroup associated with the static feedback controls given in Theorem 1.3.

4.1. Static feedback controls in the linear case - Proof of Theorem 1.3. Set, for $t \geq 0$,

$$(4.1) \quad y_\lambda(t) = e^{\lambda t}y(t) \quad \text{and} \quad \tilde{y}_\lambda(t) = e^{\lambda t}\tilde{y}(t).$$

We then have, with $A_\lambda = A + \lambda I$,

$$(4.2) \quad \begin{cases} y'_\lambda = A_\lambda y_\lambda - BWB^*\tilde{y}_\lambda & \text{in } (0, +\infty), \\ \tilde{y}'_\lambda = -A_\lambda^*\tilde{y}_\lambda - RQ\tilde{y}_\lambda & \text{in } (0, +\infty), \\ y_\lambda(0) = y(0), \quad \tilde{y}_\lambda(0) = \tilde{y}(0) (= Q^{-1}y(0)). \end{cases}$$

Set, for $t \geq 0$,

$$Z_\lambda(t) = y_\lambda(t) - \tilde{z}_\lambda(t).$$

Formally, we have

$$\begin{aligned} \frac{d}{dt}Z_\lambda &= A_\lambda y_\lambda - BWB^*\tilde{y}_\lambda + QA_\lambda^*\tilde{y}_\lambda + QRQ\tilde{y}_\lambda \\ &= A_\lambda(y_\lambda - Q\tilde{y}_\lambda) + A_\lambda Q\tilde{y}_\lambda - BWB^*\tilde{y}_\lambda + QA_\lambda^*\tilde{y}_\lambda + QRQ\tilde{y}_\lambda, \end{aligned}$$

which yields, by (1.4),

$$(4.3) \quad \frac{d}{dt}Z_\lambda = A_\lambda Z_\lambda.$$

This can be proved rigorously as in the proof of Theorem 1.1. Since

$$Z_\lambda(0) = 0,$$

it follows that

$$(4.4) \quad Z_\lambda(t) = 0 \text{ for } t \geq 0.$$

In other words, (1.37) holds.

We next deal with (1.38). Formally, we have

$$\begin{aligned} (4.5) \quad \frac{d}{dt}\langle y, \tilde{y} \rangle &= \langle Ay - BWB^*\tilde{y}, \tilde{y} \rangle_{\mathbb{H}} + \langle y, -A^*\tilde{y} - RQ\tilde{y} - 2\lambda\tilde{y} \rangle \\ &\stackrel{(1.37)}{=} \langle Ay - BWB^*\tilde{y}, \tilde{y} \rangle_{\mathbb{H}} + \langle y, -A^*\tilde{y} - Ry - 2\lambda Q^{-1}y \rangle \\ &= -\|W^{1/2}B^*\tilde{y}\|^2 - \|R^{1/2}y\|^2 - 2\lambda\langle Q^{-1}y, y \rangle, \end{aligned}$$

which yields (1.38). The rigor proof of (1.38) can be done by applying Lemma 2.1 for y and \tilde{y} .

To derive (1.39) from (1.38), one just needs to set

$$\rho(t) = \langle Q^{-1}y(t), y(t) \rangle \text{ for } t \geq 0,$$

and note that, by (1.38),

$$\rho \in W^{1,1}(0, T) \text{ for all } T > 0 \quad \text{and} \quad \rho'(t) \leq -2\lambda\rho(t) \text{ for } t \geq 0.$$

The proof is complete. \square

4.2. Static feedback controls in the nonlinear case - Proof of Theorem 1.4. For each $T > 0$ there exists $\varepsilon = \varepsilon_T > 0$ such that (4.7) is well-posed in the time interval $[0, T]$. The global existence and uniqueness follow for small ε provided that (1.47), (1.48), and (1.49) are established for each fixed time interval $[0, T]$ with ε_T sufficiently small.

We now establish (1.47), (1.48), (1.49) in $[0, T]$ for $\varepsilon < \varepsilon_T$ (small). Set, in $[0, T]$,

$$(4.6) \quad y_\lambda(t) = e^{\lambda t}y(t) \quad \text{and} \quad \tilde{y}_\lambda(t) = e^{\lambda t}\tilde{y}(t).$$

We then have, with $A_\lambda = A + \lambda I$,

$$(4.7) \quad \begin{cases} y'_\lambda = A_\lambda y_\lambda + e^{\lambda t}f(e^{-\lambda t}y_\lambda(t)) - BWB^*\tilde{y}_\lambda & \text{in } (0, T), \\ \tilde{y}'_\lambda = -A_\lambda^*\tilde{y}_\lambda - RQ\tilde{y}_\lambda + Q^{-1}e^{\lambda t}f(e^{-\lambda t}Q\tilde{y}_\lambda) & \text{in } (0, T), \\ y_\lambda(0) = y(0), \quad \tilde{y}_\lambda(0) = \tilde{y}(0) (= Q^{-1}y(0)). \end{cases}$$

Set, for $t \in [0, T]$,

$$Z_\lambda(t) = y_\lambda(t) - Q\tilde{y}_\lambda(t).$$

As in the proof of (4.3), we obtain

$$(4.8) \quad \frac{d}{dt}Z_\lambda = A_\lambda Z_\lambda + e^{\lambda \cdot} \left(f(e^{-\lambda \cdot}y_\lambda) - f(e^{-\lambda \cdot}Q\tilde{y}_\lambda(\cdot)) \right).$$

Since $Z_\lambda(0) = 0$, we obtain

$$y_\lambda(t) - Q\tilde{y}_\lambda(t) = \int_0^t e^{(t-s)A_\lambda} e^{\lambda s} \left(f(e^{-\lambda s}y_\lambda(s)) - f(e^{-\lambda s}Q\tilde{y}_\lambda(s)) \right) ds.$$

Using (1.30), we deduce that

$$y_\lambda(t) = Q\tilde{y}_\lambda(t) \text{ for } t \geq 0,$$

which implies (1.47).

We next deal with (1.48). The proof of (1.48) is similar to the one of (1.38) by applying Lemma 2.1 for y and \tilde{y} .

What have been done so far does not require $\lambda > 0$. The fact $\lambda > 0$ is used to derive (1.49) from (1.48). Set

$$\rho(t) = \langle Q^{-1}y(t), y(t) \rangle \text{ for } t \geq 0,$$

Note that, by (1.48), as in the proof of (3.30) for all $T > 0$, there exists $\delta > 0$ such that if $\|y_0\|_{\mathbb{H}} \leq \delta$ in $[0, T]$, then

$$(4.9) \quad \rho(t) \leq Ce^{-2\lambda t}\rho(0) \text{ in } [0, T].$$

In particular, we have, if T is chosen sufficiently large,

$$(4.10) \quad \|(y(T), \tilde{y}(T))\|_{\mathbb{H}} \leq e^{-2\gamma T}\|(y_0, \tilde{y}_0)\|_{\mathbb{H}}.$$

The conclusion follows from (4.9) and (4.10) by considering the time $nT \leq t \leq n(T+1)$ for $n \in \mathbb{N}$.

The proof is complete. \square

4.3. The infinitesimal generator of the semigroup associated with the static feedback controls. In this section, we derive the information of the infinitesimal generator of the semigroup associated with the static feedback controls from Theorem 1.3.

Here is the main result of this section.

Proposition 4.1. *Let $\lambda \in \mathbb{R}$ and assume (1.4). Let $y_0 \in \mathbb{H}$, set*

$$(4.11) \quad S^Q(t)(y_0) = y(t),$$

where (y, \tilde{y}) is the solution of (1.36). Then

$$(4.12) \quad (S^Q(t))_{t \geq 0} \text{ is a strongly continuous semigroup on } \mathbb{H}.$$

Moreover, the semigroup $(S^Q(t))_{t \geq 0}$ decays exponentially with the rate λ , i.e., there exists $C > 0$ such that

$$(4.13) \quad \|S^Q(t)\| \leq Ce^{-\lambda t} \text{ for } t \geq 0.$$

Let $(A^Q, \mathcal{D}(A^Q))$ be its infinitesimal generator. We have

$$(4.14) \quad \mathcal{D}(A^Q) = Q\mathcal{D}(A^*) := \left\{ Qx; x \in \mathcal{D}(A^*) \right\}$$

and

$$(4.15) \quad A^Q z = -QA^*Q^{-1}z - 2\lambda z - QRz \text{ for } z \in \mathcal{D}(A^Q).$$

We also have

i) if BWB^ is bounded, i.e., $BWB^* \in \mathcal{L}(\mathbb{H})$, then*

$$(4.16) \quad \mathcal{D}(A^Q) = \mathcal{D}(A) \quad \text{and} \quad A^Q x = Ax - BWB^*Q^{-1}x \text{ for } x \in \mathcal{D}(A) = \mathcal{D}(A^Q).$$

*ii) if $\mathcal{D}(A^Q) = \mathcal{D}(A)$, then $BWB^*x \in \mathbb{H}$ for $x \in \mathcal{D}(A^*)$, and*

$$(4.17) \quad \|BWB^*x\|_{\mathbb{H}} \leq \|AQx\|_{\mathbb{H}} + C(\|A^*x\|_{\mathbb{H}} + \|x\|_{\mathbb{H}}) \text{ for } x \in \mathcal{D}(A^*)$$

for some positive constant C independent of x .

Remark 4.1. Related results from the linear quadratic optimal control theory can be found in [11, 9, 2, 31, 27]. Known results established in the case $\lambda = 0$ and W being identity are connections between $\mathcal{D}(A^{Q^*})$ and $\mathcal{D}(A)$, see [27, Theorem 2.1]. This is different from (4.14) where a connection between $\mathcal{D}(A^Q)$ and $\mathcal{D}(A^*)$ is given. Assertion *i)* is equivalent to the fact that B is bounded, i.e., $B \in \mathcal{L}(\mathbb{U}, \mathbb{H})$ when W is positive; this case is well-known. Under additional appropriate assumptions related to the original and the dual system, one might also derive (4.15) using the results in [12] (see also [27]). The proof of Proposition 4.1 given below is quite simple.

Proof of Proposition 4.1. It is clear that (4.12) and (4.13) are the consequences of Theorem 1.3.

We now prove (4.14) and (4.15). Fix $y_0 \in Q\mathcal{D}(A^*)$ (arbitrary). Let (y, \tilde{y}) be the unique weak solution of (1.36). Since $\tilde{y}(0) = Q^{-1}y_0 \in \mathcal{D}(A^*)$, it follows that $\tilde{y} \in C^1([0, +\infty); \mathbb{H}) \cap C^0([0, +\infty); \mathcal{D}(A^*))$ and

$$(4.18) \quad \tilde{y}'(0) = A^*\tilde{y}(0) - 2\lambda\tilde{y}(0) - RQ\tilde{y}(0).$$

Since $y(t) = Q\tilde{y}(t)$ for $t \geq 0$ by Theorem 1.3, we derive that $y'(0)$ is well-defined and

$$y'(0) = Q\tilde{y}'(0) \stackrel{(4.18)}{=} -QA^*Q^{-1}y_0 - 2\lambda y_0 - QRQy_0.$$

Hence $y_0 \in \mathcal{D}(A^Q)$ and

$$A^Q y_0 = -QA^*Q^{-1}y_0 - 2\lambda y_0 - QRQy_0.$$

To complete the proof of (4.14) and (4.15), we now show that if $y_0 \in \mathcal{D}(A^Q)$ then $y_0 \in Q\mathcal{D}(A^*)$. Fix $y_0 \in \mathcal{D}(A^Q)$ (arbitrary) and let (y, \tilde{y}) be the unique solution of (1.36). Since $y_0 \in \mathcal{D}(A^Q)$ and $S^Q(t)(y_0) = y(t)$, it follows that $y \in C^1([0, +\infty); \mathbb{H}) \cap C^0([0, +\infty); \mathcal{D}(A^Q))$. In particular $y'(0)$ is well-defined. Since $y(t) = Q\tilde{y}(t)$ for $t \geq 0$ by Theorem 1.3, it follows from the equation of \tilde{y} in (1.36) that $\tilde{y}'(0)$ is well-defined and thus $\tilde{y}(0) \in \mathcal{D}(A^*)$. Since $\tilde{y}(0) = Q^{-1}y_0$, we derive that

$$Q^{-1}y_0 \in \mathcal{D}(A^*).$$

In other words, $y_0 \in Q\mathcal{D}(A^*)$.

We next establish (4.16). We first assume that $BWB^* \in \mathcal{L}(\mathbb{H})$. It follows that the generator of the semigroup $(S^Q(t))_{t \geq 0}$ is $A - BWB^*Q^{-1}$ with the domain $\mathcal{D}(A)$.

We finally derive (4.17). Assume that $\mathcal{D}(A^Q) = \mathcal{D}(A)$. From (1.5), we have, for $x, y \in \mathcal{D}(A^*)$,

$$\begin{aligned} |\langle WB^*x, B^*y \rangle_{\mathbb{U}}| &\leq |\langle Qx, A^*y \rangle_{\mathbb{H}}| + |\langle Qy, A^*x \rangle_{\mathbb{H}}| + |\langle RQx, Qy \rangle_{\mathbb{H}}| + 2|\lambda| |\langle Qx, y \rangle_{\mathbb{H}}| \\ &\leq (\|AQx\|_{\mathbb{H}} + C\|A^*x\|_{\mathbb{H}} + C\|x\|_{\mathbb{H}})\|y\|_{\mathbb{H}}. \end{aligned}$$

It follows that

$$\|BWB^*x\|_{\mathbb{H}} \leq \|AQx\|_{\mathbb{H}} + C(\|A^*x\|_{\mathbb{H}} + \|x\|_{\mathbb{H}}) \text{ for } x \in \mathcal{D}(A^*),$$

which is (4.17).

The proof is complete. \square

5. CHOICES OF Q FOR EXACTLY CONTROLLABLE SYSTEMS

In this section, we discuss how to choose Q for exactly controllable systems. Assume that the system is exactly controllable at time T . This is equivalent to the fact that (1.6) holds. Fix $\lambda \in \mathbb{R}$ and $T_* > T$ and let $\rho : [0, T_*] \rightarrow \mathbb{R}$ be such that

$$(5.1) \quad \rho \text{ is Lipschitz, } \rho \text{ is decreasing, } \rho(0) = 1, \rho(T) > 0, \text{ and } \rho(T_*) = 0.$$

Let $W \in \mathcal{L}(\mathbb{U})$ be symmetric and positive. Define $Q : \mathbb{H} \rightarrow \mathbb{H}$ as follows

$$(5.2) \quad \langle Qz_1, z_2 \rangle = \int_0^{T_*} \rho(s) e^{-2\lambda s} \langle WB^* e^{-sA^*} z_1, B^* e^{-sA^*} z_2 \rangle ds \text{ for } z_1, z_2 \in \mathbb{H}.$$

Then Q is linear, continuous, and symmetric. Moreover, since ρ is decreasing and $\rho(T) > 0$, A is an infinitesimal of a group, it follows from (1.6) that

$$(5.3) \quad Q \text{ is invertible.}$$

Formally, it follows from (5.2) that

$$(A + \lambda I)Q + Q(A + \lambda I)^* = \int_0^{T_*} \rho(s) \frac{d}{ds} \left(e^{-s(A+\lambda I)} BWB^* e^{-s(A+\lambda I)^*} \right) ds.$$

Since $\rho(0) = 1$ and $\rho(T_*) = 0$, an integration by parts yields,

$$(A + \lambda I)Q + Q(A + \lambda I)^* = -BWB^* - \int_0^{T_*} \rho'(s) e^{-s(A+\lambda I)} BWB^* e^{-s(A+\lambda I)^*} ds.$$

We then obtain

$$(5.4) \quad AQ + QA^* - BWB^* + QRQ + 2\lambda Q = 0$$

with $R : \mathbb{H} \rightarrow \mathbb{H}$ being defined by

$$(5.5) \quad \langle RQz_1, Qz_2 \rangle = \int_0^{T_*} \rho'(s) \langle WB^* e^{-s(A+\lambda I)^*} z_1, B^* e^{-s(A+\lambda I)^*} z_2 \rangle ds.$$

We thus reach (1.4).

Let us make these arguments rigorously, i.e., we verify (1.5):

$$(5.6) \quad \langle Qz_1, A^*z_2 \rangle + \langle A^*z_1, Qz_2 \rangle - \langle WB^*z_1, B^*z_2 \rangle \\ + \langle RQz_1, Qz_2 \rangle + 2\lambda \langle Qz_1, z_2 \rangle = 0 \quad \forall z_1, z_2 \in \mathcal{D}(A^*).$$

Indeed, for $z_1, z_2 \in \mathcal{D}(A^{*2})$, we have, from (5.2),

$$(5.7) \quad \langle Qz_1, (A + \lambda I)^*z_2 \rangle + \langle (A + \lambda I)^*z_1, Qz_2 \rangle \\ = \int_0^{T_*} \rho(s) e^{-2\lambda s} \langle WB^* e^{-sA^*} z_1, B^* e^{-sA^*} (A + \lambda I)^*z_2 \rangle ds \\ + \int_0^{T_*} \rho(s) e^{-2\lambda s} \langle WB^* e^{-sA^*} (A + \lambda I)^*z_1, B^* e^{-sA^*} z_2 \rangle ds.$$

This implies

$$(5.8) \quad \langle Qz_1, (A + \lambda I)^* z_2 \rangle + \langle (A + \lambda I)^* z_1, Qz_2 \rangle \\ = \int_0^{T_*} \rho(s) \langle WB^* e^{-s(A+\lambda I)^*} z_1, B^* e^{-s(A+\lambda I)^*} (A + \lambda I)^* z_2 \rangle ds \\ + \int_0^{T_*} \rho(s) \langle WB^* e^{-s(A+\lambda I)^*} (A + \lambda I)^* z_1, B^* e^{-s(A+\lambda I)^*} z_2 \rangle ds.$$

Since

$$\int_0^{T_*} \rho(s) \langle WB^* e^{-s(A+\lambda I)^*} z_1, B^* e^{-s(A+\lambda I)^*} (A + \lambda I)^* z_2 \rangle ds \\ = \int_0^{T_*} \rho(s) \langle WB^* e^{-s(A+\lambda I)^*} z_1, B^* \frac{d}{ds} (e^{-s(A+\lambda I)^*} z_2) \rangle ds$$

and

$$\int_0^{T_*} \rho(s) \langle WB^* e^{-s(A+\lambda I)^*} (A + \lambda I)^* z_1, B^* e^{-s(A+\lambda I)^*} z_2 \rangle ds \\ = \int_0^{T_*} \rho(s) \langle WB^* \frac{d}{ds} (e^{-s(A+\lambda I)^*} z_1), B^* e^{-s(A+\lambda I)^*} z_2 \rangle ds,$$

it follows from (5.8) that

$$(5.9) \quad \langle Qz_1, (A + \lambda I)^* z_2 \rangle + \langle (A + \lambda I)^* Qz_1, z_2 \rangle \\ = -\langle WB^* z_1, B^* z_2 \rangle - \int_0^{T_*} \rho'(s) \langle WB^* e^{-s(A+\lambda I)^*} z_1, B^* e^{-s(A+\lambda I)^*} z_2 \rangle ds,$$

which is (5.6) for $z_1, z_2 \in \mathcal{D}(A^{*2})$. The general case follows by density.

We have just proven the following result which is on the choice of Q in the Gramian spirit when the system is exactly controllable.

Proposition 5.1. *Assume that $(S(t))_{t \in \mathbb{R}} \subset \mathcal{L}(\mathbb{H})$ is a strongly continuous group in \mathbb{H} , B is an admissible control operator, and system (1.1) is exactly controllable in time T for some $T > 0$. Let $\lambda \in \mathbb{R}$, $T_* > T$, and $\rho : [0, T_*] \rightarrow \mathbb{R}$ be a function satisfying (5.1), and let $W \in \mathcal{L}(\mathbb{U})$ be symmetric and positive. Define $Q : \mathbb{H} \rightarrow \mathbb{H}$ by*

$$(5.10) \quad \langle Qz_1, z_2 \rangle = \int_0^{T_*} \rho(s) e^{-2\lambda s} \langle WB^* e^{-sA^*} z_1, B^* e^{-sA^*} z_2 \rangle ds \text{ for } z_1, z_2 \in \mathbb{H}.$$

Then Q is linear, continuous, symmetric, and invertible and (1.4) holds with R defined by (5.5), i.e., (1.5) is valid.

Remark 5.1. Proposition 5.1 covers the setting considered by Komornik. Indeed, set, with $T_* = T + \frac{1}{2\lambda}$

$$(5.11) \quad \rho(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq T, \\ 2\lambda e^{-2\lambda(T-t)}(T_* - t) & \text{for } T < t \leq T_*. \end{cases}$$

Then

$$e_{\lambda}(t) = e^{\lambda t} \rho(t) \text{ in } [0, T_*].$$

Since, for $T \leq t \leq T_* = T + \frac{1}{2\lambda}$,

$$\rho(t) = e\tau e^{-\tau} \text{ with } \tau = 2\lambda(T_* - t),$$

and the function $\tau e^{-\tau}$ is increasing in $[0, 1]$, it follows that ρ defined in (5.11) verifies (5.1).

When A is skew-adjoint and $R = 0$, one has the following result. This result was previously obtained by Urquiza [29] by a different approach using results of Grabowski in [13] (see also [14]), and can be derived in the same manner as the one of Proposition 5.1.

Proposition 5.2. *Assume that $(S(t))_{t \in \mathbb{R}} \subset \mathcal{L}(\mathbb{H})$ is a strongly continuous group, B is an admissible control operator, and system (1.1) is exactly controllable in time T for some $T > 0$. Let $\lambda \in \mathbb{R}$ and let $W \in \mathcal{L}(\mathbb{U})$ be symmetric and non-negative, and assume that $\lambda > \omega_0(-A^*)$. Define $Q : \mathbb{H} \rightarrow \mathbb{H}$ by*

$$(5.12) \quad \langle Qz_1, z_2 \rangle = \int_0^\infty e^{-2\lambda s} \langle WB^* e^{-sA^*} z_1, B^* e^{-sA^*} z_2 \rangle ds \text{ for } z_1, z_2 \in \mathbb{H}.$$

Then Q is linear, continuous, symmetric, and invertible, and (1.4) holds with $R = 0$, i.e., (1.5) is valid with $R = 0$.

Proof. The proof of (5.1) is almost the same as the one of Proposition 5.1. One just needs to note that the RHS of (5.12) is well-defined for $\lambda > \omega_0(-A^*)$. The details are omitted. \square

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