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ABSTRACT

In this article, we examine a Networked Control System (NCS) in which the plant and the controller communicate over a network subject to a certain communication constraint. The plant is described by discrete-time *nonlinear dynamics* subject to bounded disturbances. Due to an overloaded communication network, we assume that the control signal and the information from the plant (the measured output signal) cannot be transmitted simultaneously and are subject to a multiplexing constraint. The goal is to design a switching strategy that allows us to sequentially communicate given these constraints while optimizing a quadratic cost over a finite horizon. Consequently, we proceed by emulation and assume that a controller that satisfies performance requirements is already provided. The resulting optimization problem is observed to be an integer programming problem that is generally NP-complete, i.e., the complexity is exponential in the time horizon considered. To overcome this issue, we provide a different perspective on this problem than what has been presented by the community before. Our main contribution is to reformulate the problem with all its constraints to a form that renders it amenable to apply the discrete-time Pontryagin Maximum Principle to get the necessary conditions for the optimality of the control action sequence. These necessary conditions are then solved numerically by a multiple-shooting method. To validate the approach, we present some illustrative numerical experiments on an inverted pendulum. Different setups are considered and numerically analyzed: usage of a predictor when the output is not transmitted and usage of the previous value of the output when the new value is not transmitted, with or without the choice of non-transmission.

1. Introduction

With recent advancements in communication technologies, Networked Control Systems (NCS) are finding applications in the Internet of Things (IoT) and cyber-physical systems; see [1], [2]. However, the increasing use of NCS in large-scale applications comes with challenges in communication and control. Major obstacles in communication are resource constraints, packet loss, time delays, etc. (see for instance [3, 4]). In this context, we study an NCS where both the sensor signals and the control commands are communicated over the same network with the constraint that communication may occur only in one way at any given time instant. Our goal is to determine the optimal sequence of communication decisions over a given finite time horizon such that a quadratic control and communication cost is minimized.

The literature on NCS is rich, going from stability analysis and optimal control to optimal scheduling while considering different constraints on communication and control. Early work on optimal scheduling of transmission of observation can be found in [5]. A general framework of an NCS with multiple sensors sending signals to a controller through a network was proposed in [6]. An optimum stochastic control method is employed to design the optimum policy. A resource-constrained linear system with noise is studied by imposing constraints on the transmission of observations in different ways [7, 8]. The objective of finding an optimal policy for observation transmission is achieved using a certainties equivalence control method. Event-based state estimation is examined in [9], where multiple sensors send sporadic information to the predictor. The event triggering is based on the variance of the observations coming from sensors. A joint decision-making task of optimal control and scheduling under the delay-dependent cost is considered in [10]. The authors find a value function-based solution to the optimal control and a mixed-integer programming-based solution to the optimal scheduling. Another work on a joint control and networking problem is studied in [11], with decisions on sampling, control, and scheduling to be taken. The problem is tackled by decomposing control and networking problems, which are solved using a certainty equivalence controller, and a dual optimization method, respectively. The aforementioned articles consider that only observations are transmitted through the network with different constraints. However, these articles do not consider the possibility of the control being transmitted through the network. A stochastic NCS with two different architectures based on the location of the controller with respect to the actuator and sensor is considered in [12]. The controller is triggered by a stopping criteria on the observation, which is then used for scheduling. In [12], only one way of communication is done through the network, i.e. control is transmitted to the actuator, or observation is transmitted from the sensor

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to the controller. Transmission in both ways through the network is not considered. An NCS with varying time intervals and transmission delays is considered in [13]. In this work, the authors have derived bounds on delays for a stable system, and the stability analysis is done using Lyapunov's method. This result focuses more on the stability analysis and does not discuss optimality. In a wireless communication network, energy is utilized in several layers [4]. This makes optimum utilization of the network capabilities while ensuring the quality of control important. One such work considers the packet loss probability as a function of allocated transmission power [14]. The joint objective of control and communication is achieved by separating standard LQR and Markov process minimization, respectively. In a similar problem with packet loss, [15] provides an energy-efficient approach for communication in an NCS. First, the authors give conditions for stabilizing time-triggered policy using stochastic allowable transmission intervals. Then a communication policy is designed that optimizes the energy while obeying the stability requirements.

In contrast to the aforementioned works, we consider a communication constraint arising out of a shared communication network. At any instant, the network has three choices: (i) transmit control, (ii) transmit observation or (iii) do not transmit any signal. Under these constraints, the objective is to optimize these choices to minimize a quadratic cost involving the control and communication costs. The contributions of this article are as follows:

1. We present a general framework of an NCS where the plant has uncertainties or bias, and the communication from the controller to the plant, and from the plant to the controller is over the same communication network. We use a predictor that calculates the state of the plant, in the absence of observations, using an approximate model. We consider a quadratic cost which has three terms corresponding to the plant state, the control effort, and the communication effort.

2. The resulting optimization is an integer programming problem, with the optimization variable being the communication decision vector over a given time horizon. Since this problem is NP-complete, we recast the problem as an optimal control problem by considering an augmented state and providing a solution using the discrete-time Pontryagin Maximum Principle (DPMP). This allows us to derive the necessary conditions of optimality. A numerical solution of the two-point boundary value problem arising out of the necessary conditions is obtained using the multiple-shooting method. In Section 4, we show that this method performs better than an exhaustive search method in terms of computation time.

The rest of the article is organized as follows. In Section 2, the framework is presented, and the problem formulation is stated. Section 3 is dedicated to providing the necessary conditions of optimality using the DPMP. The results are validated through numerical simulation in Section 4. Finally, concluding remarks and future work are stated in Section 5.

2. Problem Formulation

In this section, we set up a system with four key parts: a plant, a controller, a predictor, and a communication network that links the plant and controller. Our presentation will take the following sequence:

- First, we describe the plant and a feedback controller obtained by emulation, i.e., designed to stabilize the plant without considering the communication network.
- Then, we introduce the communication network and constraints in place and discuss the subsequent changes that this brings about in the controller design and the closed-loop dynamics. At this stage, we also introduce a local predictor for the plant dynamics that provides estimates when the network is not able to provide any information from the plant side.

2.1. Plant and Controller Model

Consider a discrete-time nonlinear plant described by

$$x_{k+1} = f(x_k, u_k) \quad \text{for all } k \in \{0, 1, 2, \dots\} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, and $u_k \in \mathbb{R}^p$ is the control at the actuator at the k^{th} instant respectively. $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a *smooth* map governing the dynamics of the system. Consider a closed-loop feedback controller given by:

$$u_k = g(x_k) \quad \text{for all } k \in \{0, 1, 2, \dots\} \tag{2}$$

where x_k is the state feedback received by the controller at the k^{th} instant. Note that the controller (2) is a controller for the plant (1) designed by emulation, i.e., it is obtained without considering any network.

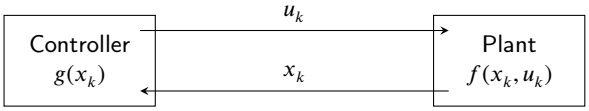

Figure 1: Schematic of the plant and the controller without the network

Assumption 1. The controller (2) stabilizes the system (1) asymptotically.

Please note that the design of the controller may not be related to stability but to other closed-loop performance criteria. We emphasize that our interest here lies in a networked system over a finite time horizon N, and a corresponding cost. Consequently, the controller can be designed to optimize the cost over the finite horizon without ensuring any stability guarantees. We denote $[N] := \{0, 1, 2, ..., N\}$ and $[N]^\star := \{1, 2, ..., N\}$. Next, we introduce the communication model and then give a complete model of the overall system.

2.2. Communication Constraints

Figure 2 is a schematic of our setup. As can be seen, there are four subsystems in place - the plant, the controller, the network and the predictor.The controller communicates to the plant via the network, and the plant sends the state information to the predictor via the network. We consider the following limitations on the network. The network has three choices at any instant:

i) it can transmit a control command from the controller to the plant or,

ii) it can transmit the observation from the plant to the predictor or,

iii) it does not communicate at all.

These three choices are encapsulated in the form of a switch variable q_k that takes values in a discrete set

$$\mathrm{S} := \{-1, 0, 1\} \tag{3}$$

where,

$$q_k := \begin{cases} 1 & \text{if the control is transmitted;} \\ -1 & \text{if the observation is transmitted;} \\ 0 & \text{if there is no communication.} \end{cases} \tag{4}$$

Remark 1. The admissible set of switch positions S is discrete, i.e., not convex, and it is challenging to solve an optimization problem over a non-convex set.

The multiplexing constraint described by (4) can be expressed algebraically as,

$$q_k(1 - q_k)\left(1 + q_k\right) = 0 \quad \text{for all } k \in [N]. \tag{5}$$

It is indeed straightforward to see that the equality $q_k(1 - q_k)(1 + q_k) = 0$ holds only when $q_k \in \{-1, 0, 1\}$. Alternatively, one of the three choices from $\{-1, 0, 1\}$ implies that the algebraic equality holds.

Remark 2. Note that the multiplexing constraint, in the form (4), cannot be incorporated as a constraint in the formulation of the discrete optimal control problem. The existing solution machinery is not able to handle this. To dovetail this constraint into a form that is amenable to the discrete optimal control machinery, we convert the constraint to an algebraic equality of the form (5).

Based on the switching, we can write the transmitted quantities as follows. The transmitted control information through the network can be written as,

$$\tilde{u}_k := \begin{cases} u_k & \text{if } q_k = 1; \\ \tilde{u}_{k-1} & \text{otherwise.} \end{cases} \tag{6}$$

2.3. State Update in the Controller Dynamics

In reality, there may be time intervals when the communication network does not convey any information from the plant. In such a scenario, it is advantageous to have a local, approximate model of the plant, which would help obtain a coarse estimate of the state based on a predicted estimate devoid of information update [16]. With this in mind, we introduce an element in the system called a predictor. In our framework, the predictor has an approximate model of the system. The function of the predictor is to update the state as per the approximate model of the system when the observation signal is not transmitted and transmit the actual state if available. The approximate model can be found using data-driven approaches or system identification methods [17, 18], and is of the form: $\hat{f} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ such that, in a neighborhood of interest $U \times V \subset \mathbb{R}^n \times \mathbb{R}^p$, there exists a finitely small $\epsilon > 0$, such that

$$\left\| \hat{f}(y, v) - f(y, v) \right\| \leq \epsilon \quad \text{for all } (y, v) \in U \times V. \tag{7}$$

This framework helps the controller to use the available information and send a prediction

$$\hat{x}_k := \begin{cases} x_k & \text{if } q_k = -1; \\ \hat{f}(\hat{x}_{k-1}, \tilde{u}_{k-1}) & \text{otherwise} \end{cases} \tag{8}$$

for all $k \in [N - 1]$. The algorithm we propose presents the necessary conditions for an optimal communication sequence for the N stages, accounts for all the constraints, and minimizes a quadratic performance measure (presented shortly ahead.) This measure of performance enforces a trade-off between the growth in the predictor error and sufficient transmission of information from the plant over the finite stages. With this framework of the NCS shown in the

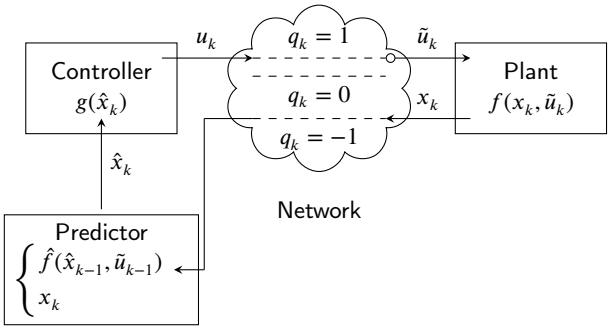

Figure 2: Schematic of a Networked Control System with a predictor in the loop

Fig 2 and described by (1)-(8), we derive overall dynamics of the system. Please note that (6) and (8) are written as cases based on certain conditions of the switch position. We incorporate the different cases of (6) and (8) in the following unified framework. Now the overall system dynamics can be

written as,

$$\begin{aligned} x_{k+1} &= f(x_k, \tilde{u}_k) \\ \tilde{u}_k &= \frac{1}{2}[q_k\left(1+q_k\right)u_k + \left(2+q_k\right)\left(1-q_k\right)\tilde{u}_{k-1}] \\ u_k &= g(\hat{x}_k) \\ \hat{x}_k &= -\frac{1}{2}q_k(1-q_k)x_k \\ &\quad + \left(1-\frac{q_k}{2}\right)\left(1+q_k\right)\left(\hat{f}(\hat{x}_{k-1}, \tilde{u}_{k-1})\right) \end{aligned} \tag{9}$$

for all $k \in [N-1]$. We provide a detailed derivation in Appendix A, which shows the second equation of (9) and (6) are equivalent. Similarly, the fourth equation of (9) and (8) are equivalent. For ease of notation, define a concatenated state $\xi_k := \begin{pmatrix} x_k & \tilde{u}_{k-1} & \hat{x}_{k-1} \end{pmatrix}^T$ and the switch q_k acts as the decision variable. Then, the dynamics for the concatenated state can be written as,

$$\xi_{k+1} = F\left(\xi_k, q_k\right) \tag{10}$$

where,

$$F\left(\xi_k, q_k\right) = \begin{pmatrix} f(x_k, \tilde{u}_k) \\ \frac{1}{2}\Big(q_k(1+q_k)g(\hat{x}_k) + (2+q_k)(1-q_k)\tilde{u}_{k-1}\Big) \\ \frac{-q_k}{2}(1-q_k)f(x_{k-1}, \tilde{u}_{k-1}) + \dots \\ \dots (1-\frac{q_k}{2})(1+q_k)\hat{f}(\hat{x}_{k-1}, \tilde{u}_{k-1}) \end{pmatrix}$$

for all $k \in [N-1]$, with initial conditions $\xi_0 = \begin{pmatrix} x_0 & 0 & x_0 \end{pmatrix}^T$, where x_0 is a user defined initial condition.

2.4. Problem Statement

Given the three dynamic blocks described in the previous section - the communication network, the predictor, and the plant - we formulate a measure of performance that weighs the usage of the network resource as well. The switching sequence is obtained by minimizing an objective function of the form:

$$\min_{\{q_k\}_{k=0}^{N-1}} \quad \sum_{k=0}^{N-1} J_k\left(\xi_k, q_k\right) + J_N\left(\xi_N\right)$$
$$\text{subject to} \quad \begin{cases} \text{dynamics (10)} \\ q_k \in \mathrm{S} \quad \text{for all } k \in [N-1] \\ \xi_0 = \xi_{\text{in}} \end{cases} \tag{P1}$$

where

i) $J_k : \mathbb{R}^{2n+p} \times \mathbb{R} \to \mathbb{R}$ denotes the cost incurred at each stage and is given by

$$J_k\left(\xi_k, q_k\right) := \xi_k^T A \xi_k + \lambda q_k^T q_k$$

for all $k \in [N-1]$ with $A = \operatorname{diag}\begin{pmatrix} Q & R & \mathbf{0}_{n\times n} \end{pmatrix}$.

ii) $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive semi-definite matrix.

iii) $R \in \mathbb{R}^{p\times p}$ is a symmetric positive definite matrix.

iv) $\lambda \in \mathbb{R}_{\geq 0}$ is a non-negative scalar weighting factor.

v) $J_N : \mathbb{R}^{2n+p} \times \mathbb{R} \to \mathbb{R}$ denotes the cost incurred at the last stage and is given by

$$J_N\left(\xi_N\right) := \xi_N^T B \xi_N$$

with $B = \operatorname{diag}\begin{pmatrix} Q & \mathbf{0}_{p\times p} & \mathbf{0}_{n\times n} \end{pmatrix}$.

vi) ξ_{in} is the user defined initial condition.

3. Main Result

The problem statement given by (P1), however, is not in a standard form of the optimal control problem as defined by Boltyanskii [19]. The main obstacle, as stated in Remark 1, is the admissible switching action set is discrete. To apply Boltyanskii's *method of tents* (tent - tangent convex cone) to establish the necessary conditions for optimality, we recast the problem (P1) into a form that is amenable to the solution proposed by Boltyanskii. To do so, we use an approach motivated by [20]. First, we define a set

$$\bar{\mathrm{S}} := [-a, a] \quad \text{for a fixed } a \in [1, \infty[\tag{11}$$

and a map $\Phi : \bar{\mathrm{S}} \to \mathbb{R}$ such that,

$$\Phi(q) := \|(q-1)q(q+1)\|^2 \tag{12}$$

where $q \in \bar{\mathrm{S}}$. We introduce an augmented state $\eta_k \in \mathbb{R}$ for all $k \in [N]$. Define the governing dynamics of the augmented state as

$$\eta_{k+1} = \eta_k + \Phi(q_k) \quad \text{for all } k \in [N-1]. \tag{13}$$

Lemma 1. *For the dynamics defined in* (13) *and an initial condition* $\eta_0 = 0$, $q_k \in \mathrm{S}$ *for all* $k \in [N-1]$ *if and only if* $\eta_N = 0$, *where* S *is given by* (3).

PROOF. ($\Longrightarrow$) Given the dynamics (13), $\eta_0 = 0$, and $\eta_N = 0$, then

$$\begin{aligned} & \eta_N = \eta_0 + \sum_{k=0}^{N-1} \Phi(q_k) \\ \Longrightarrow & \sum_{k=0}^{N-1} \Phi(q_k) = 0 \\ \Longrightarrow & \Phi(q_k) = 0 \quad \text{for all } k \in [N-1] \\ \Longrightarrow & q_k \in \mathrm{S} \quad \text{for all } k \in [N-1]. \end{aligned} \tag{14}$$

($\Longleftarrow$) Given the initial condition $\eta_0 = 0$, and $q_k \in \mathrm{S}$ for all $k \in [N-1]$, then using the dynamics (13) it is clear that $\eta_N = 0$.

The problem is recast using Lemma 1, and the equivalent optimal control problem is as follows.

$$\min_{\{q_k\}_{k=0}^{N-1}} \quad \sum_{k=0}^{N-1} J_k\left(\xi_k, q_k\right) + J_N\left(\xi_N\right)$$
$$\text{subject to} \quad \begin{cases} \text{dynamics (10)} \\ \text{dynamics (13)} \\ q_k \in \bar{\mathrm{S}} \quad \text{for all } k \in [N-1] \\ (\xi_0, \eta_0) = (\xi_{\text{in}}, 0) \\ \eta_N = 0 \end{cases} \tag{P2}$$

Here concatenated state ξ_k is augmented with another state variable η_k, and the adjoint p_k is augmented with χ_k.

3.1. Necessary conditions for optimality

Theorem 1. *Let $\left(\mathring{q}_k\right)_{k=0}^{N-1}$ be an optimal switching sequence for (P2) and $\left(\mathring{\xi}_k\right)_{k=0}^{N-1}$ be the corresponding state trajectory. For $\nu \in \{-1, 0\}$ and $\chi \in \mathbb{R}$, define the Hamiltonian*

$$[N-1] \times \left(\mathbb{R}^{2n+p}\right)^\star \times \mathbb{R}^{2n+p} \times \mathbb{R} \ni \left(t, p, \xi, q\right) \longmapsto \mathcal{H}^{\nu,\chi}(t, p, \xi, q) := \nu J_t(\xi, q) + \left\langle p,\, F\left(\xi, q\right)\right\rangle + \langle \chi,\, \Phi(q)\rangle \in \mathbb{R}. \tag{15}$$

Then there exists adjoint trajectories $\left(p_k\right)_{k=0}^{N-1} \in \left(\mathbb{R}^{2n+p}\right)^\star$ such that the following conditions hold:

(NC-1) the non-triviality condition:
the adjoint variables $\left(p_k\right)_{k=0}^{N-1}$ and the scalar ν do not vanish simultaneously;

(NC-2) the state and the adjoint dynamics:

$$\text{state:} \quad \mathring{\xi}_{k+1} = \mathcal{D}_p \mathcal{H}^{\nu,\chi}(k, p, \mathring{\xi}_k, \mathring{q}_k)$$
$$\text{adjoints:} \quad p_{k-1} = \mathcal{D}_\xi \mathcal{H}^{\nu,\chi}(k, p, \mathring{\xi}_k, \mathring{q}_k);$$

(NC-3) the transversality condition:

$$p_{N-1} = \nu \mathcal{D}_\xi J_N(\mathring{\xi}_N);$$

(NC-4) the Hamiltonian non-positive gradient condition:

$$\left\langle \mathcal{D}_q \mathcal{H}^{\nu,\chi}(k, p, \mathring{\xi}_k, \mathring{q}_k),\, q - \mathring{q}_k \right\rangle \le 0 \quad \text{for all } q \in \Omega_{\bar{\mathrm{S}}}\left(\mathring{q}_k\right)$$

where $\Omega_{\bar{\mathrm{S}}}\left(\mathring{q}_k\right)$ is the support cone of $\bar{\mathrm{S}}$ at with apex at $\mathring{q}_k$.

PROOF. We establish the necessary conditions for the optimal control problem (P2) where the multiplexing constraint has been modified as an augmented state with boundary conditions. Define the Hamiltonian,

$$[N-1] \times \left(\mathbb{R}^{2n+p}\right)^\star \times (\mathbb{R})^\star \times \mathbb{R}^{2n+p} \times \mathbb{R} \times \mathbb{R} \ni \left(t, p, \chi, \xi, \eta, q\right) \longmapsto \mathcal{H}^{\nu}(t, p, \chi, \xi, \eta, q) := \nu J_t(\xi, q) + \left\langle p,\, F\left(\xi, q\right)\right\rangle + \langle \chi,\, \eta + \Phi(q)\rangle \in \mathbb{R}. \tag{16}$$

Let $\left(\mathring{q}_k\right)_{k=0}^{N-1}$ be an optimal switching sequence for (P2) and $\left(\mathring{\xi}_k, \mathring{\eta}_k\right)_{k=0}^{N-1}$ be the corresponding state and augmented state trajectories. For $\nu \in \{-1, 0\}$ and the Hamiltonian (16), there exists adjoint trajectories $\left(p_k, \chi_k\right)_{k=0}^{N-1} \in \left(\mathbb{R}^{2n+p} \times \mathbb{R}\right)^\star$ such that the following conditions hold:

(nc-1) the non-triviality condition:
the adjoint variables $\left(p_k\right)_{k=0}^{N-1}$, χ_k and the scalar ν do not vanish simultaneously;

(nc-2) the state and the adjoint dynamics:

$$\text{state:} \quad \begin{cases} \mathring{\xi}_{k+1} = \mathcal{D}_p \mathcal{H}^{\nu}(k, p, \chi, \mathring{\xi}_k, \mathring{\eta}_k, \mathring{q}_k) \\ \mathring{\eta}_{k+1} = \mathcal{D}_\chi \mathcal{H}^{\nu}(k, p, \chi, \mathring{\xi}_k, \mathring{\eta}_k, \mathring{q}_k) \end{cases}$$
$$\text{adjoints:} \quad \begin{cases} p_{k-1} = \mathcal{D}_\xi \mathcal{H}^{\nu}(k, p, \chi, \mathring{\xi}_k, \mathring{\eta}_k, \mathring{q}_k) \\ \chi_{k-1} = \mathcal{D}_\eta \mathcal{H}^{\nu}(k, p, \chi, \mathring{\xi}_k, \mathring{\eta}_k, \mathring{q}_k) \end{cases}$$

(nc-3) the transversality condition:

$$p_{N-1} = \nu \mathcal{D}_\xi J_N(\mathring{\xi}_N);$$

(nc-4) the Hamiltonian non-positive gradient condition:

$$\left\langle \mathcal{D}_q \mathcal{H}^{\nu}(k, p, \chi, \mathring{\xi}_k, \mathring{\eta}_k, \mathring{q}_k),\, q - \mathring{q}_k \right\rangle \le 0 \quad \text{for all } q \in \Omega_{\bar{\mathrm{S}}}\left(\mathring{q}_k\right)$$

where $\Omega_{\bar{\mathrm{S}}}\left(\mathring{q}_k\right)$ is the support cone of $\bar{\mathrm{S}}$ at with apex at $\mathring{q}_k$.

Boltyanskii's method of tents requires the existence of a local approximation of the feasible set. The maps f, and J_k defined earlier are smooth, which ensures the existence of a local conical approximation of the feasible set. From (nc-2), it is evident that the χ_k remains constant since

$$\chi_{k-1} = \mathcal{D}_\eta \mathcal{H}^{\nu}(k, p, \chi, \mathring{\xi}_k, \mathring{\eta}_k, \mathring{q}_k) = \chi_k$$

for all $k \in [N]^\star$. This means the necessary conditions obtained using the Hamiltonian (15) do not depend on the η. The necessary conditions (NC-1) - (NC-4) obtained in Theorem 1, are equivalent of the necessary conditions (nc-1) -(nc-4).

To complete the proof, we prove that admissible trajectories in (P1) are admissible in (P2) and admissible trajectories in (P2) are admissible in (P1). Let $\gamma := \left(\xi_k, q_k\right)_{k=0}^{N-1}$ be an admissible trajectory in (P1). Augment the trajectory with $\eta_k = 0$ for all $k \in [N-1]$, then from Lemma 1 the trajectory $\tilde{\gamma} = \left(\xi_k, q_k, 0\right)_{k=0}^{N-1}$ is admissible in (P2). Let a trajectory be $\tilde{\beta} := \left(\xi_k, q_k, \eta_k\right)_{k=0}^{N-1}$, then from Lemma 1 the trajectory $\tilde{\beta}$ is admissible in (P2) only when $\eta_k = 0$ for all $k \in [N-1]$. Now for a trajectory $\tilde{\beta} = \left(\xi_k, q_k, 0\right)_{k=0}^{N-1}$ admissible in (P2), there exists a corresponding trajectory $\beta = \left(\xi_k, q_k\right)_{k=0}^{N-1}$ admissible in (P1). Hence, we establish a bijection.

3.2. Solution of the Necessary Conditions

We established the necessary conditions of optimality in the preceding subsection, which results in a two-point boundary value problem (TPBVP). Here, the aim is to find

a trajectory that satisfies the necessary conditions (nc-1) - (nc-4) with given initial and final conditions. We solve the TPBVP iteratively using the multiple shooting method [21].

We define a shooting variable,

$$s := \left(\left(\xi_k\right)_{k=1}^{N} \quad \left(\eta_k\right)_{k=1}^{N} \quad \left(p_k\right)_{k=1}^{N} \quad \left(\chi_k\right)_{k=1}^{N} \right)^T \tag{17}$$

and a shooting function

$$\phi(s) := \begin{pmatrix} \vdots \\ \xi_{k+1} - \mathcal{D}_p \mathcal{H}^v(k, p, \chi, \xi_k, \eta_k, q_k) \\ \eta_{k+1} - \mathcal{D}_\chi \mathcal{H}^v(k, p, \chi, \xi_k, \eta_k, q_k) \\ p_{k-1} - \mathcal{D}_\xi \mathcal{H}^v(k, p, \chi, \xi_k, \eta_k, q_k) \\ \chi_{k-1} - \mathcal{D}_\eta \mathcal{H}^v(k, p, \chi, \xi_k, \eta_k, q_k) \\ \vdots \\ p_{N-1} - \nu \mathcal{D}_\xi J_N(\xi_N) \end{pmatrix} \tag{18}$$

with boundary conditions $\xi_0 = \xi_{\text{in}}$, $\eta_0 = 0$, and $\eta_N = 0$. We obtain the q_k, required in (18) using the condition (nc-4). The solution of the nonlinear equation

$$\phi(s) = \mathbf{0} \tag{19}$$

is the optimal trajectory. The solution of a system of nonlinear equations (19) is obtained using `fsolve` function of the optimization toolbox from MATLAB [22].

From (18), it is clear that the number of nonlinear equations to be solved scales linearly with the number of time stages, given the dimension of the state space is constant. Similarly, the number of nonlinear equations to be solved scales linearly with the dimension of the state space, given the number of time stages is constant. In short, the computational complexity is $\mathcal{O}(nN)$, where n is the dimension of the state space and N is the number of time stages. Whereas an exhaustive search method has an exponential complexity $\mathcal{O}(3^N)$. The function `fsolve` requires an initial guess $s^{(0)}$ of the solution, and the solution is sensitive to the choice of the initial guess. The initial guess is chosen empirically, which poses a challenge.

4. Numerical Experiments

In this section, we validate our main result by a numerical experiment on an inverted pendulum. The state and system dynamics as per the notation defined in (9) are:

$$\begin{aligned} x_k &:= \left(\theta_k, \omega_k\right)^T \quad \text{and} \\ f(x_k, \tilde{u}_k) &:= \begin{pmatrix} \theta_k + h\omega_k \\ \omega_k + h\frac{ml}{J}\sin\theta_k - \frac{ml}{J}\tilde{u}_k \cos\theta_k + b \end{pmatrix}. \end{aligned} \tag{20}$$

Here, θ_k is the angle of suspension, ω_k is the angular velocity, and $\tilde{u}_k$ is the transmitted control at k^{th} instant. The mass of the pendulum bob is denoted by m, the length of the rod is l, J is the inertia, and b is a bias. The relation between control generated by the controller (u_k) and the transmitted control ($\tilde{u}_k$) is given in (9). Further, the control is given by,

$$u_k = K(E_k - E_d)\hat{\omega}_k \cos\hat{\theta}_k \tag{21}$$

Parameter	Value
step size (h)	0.1 s
no. of steps (N)	100
mass of the bob (m)	9 kg
length of the rod (l)	1 m
moment of inertia (J)	1 kg m^2
initial conditions (θ_0, ω_0)	$(\pi + 0.8, 0)$
controller gain (K)	0.3
weighting parameter (λ)	0.5
bias (b)	0.8

Table 1
Numerical experiment parameters

where K is the controller gain and $\hat{x}_k := \left(\hat{\theta}_k, \hat{\omega}_k\right)^T$ represent the predicted state at the k^{th} instant. The energy is given by $E_k = \frac{1}{2}J\hat{\omega}_k^2 + mgl(\cos\hat{\theta}_k - 1)$ and E_d is the energy in the desired state. Here, we use an established control law that stabilizes the inverted pendulum without bias to its upright position [23]. The predictor dynamics corresponding to (9) are given as,

$$\hat{f}(\hat{x}_k, \tilde{u}_k) := \begin{pmatrix} \hat{\theta}_k + h\hat{\omega}_k \\ \hat{\omega}_k + h\frac{ml}{J}\sin\hat{\theta}_k - \frac{ml}{J}\tilde{u}_k\cos\hat{\theta}_k \end{pmatrix}. \tag{22}$$

We choose the total cost as

$$J = J_{\text{sc}} + J_{\text{comm}} \tag{23}$$

where,

$$\begin{aligned} J_{\text{sc}} &:= \sum_{k=0}^{N-1} (\theta_k - \pi)^T Q(\theta_k - \pi) + \omega_k^T Q \omega_k + \tilde{u}_k^T R \tilde{u}_k \\ J_{\text{comm}} &:= \sum_{k=0}^{N-1} q_k^T q_k \end{aligned} \tag{24}$$

such that it penalizes the pendulum when it is farther from the upward equilibrium. Unless specified otherwise, the system parameters are chosen as given in the following Table 1. Within this section, we delve into the following key aspects:

(Obj1) A comparative performance analysis with and without the predictor integrated into the loop.

(Obj2) A study of available communication choices and their implications.

(Obj3) Examination of the varying impact of the weighting factor (λ).

(Obj4) An assessment of the time and computational complexities of our approach vis-à-vis an exhaustive search method.

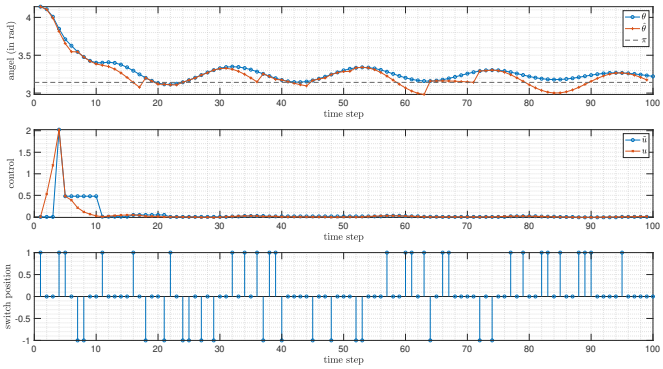

Figure 3: Setup 1 - top: actual and calculated angular position, middle: actual and transmitted control, bottom: optimal switching signal

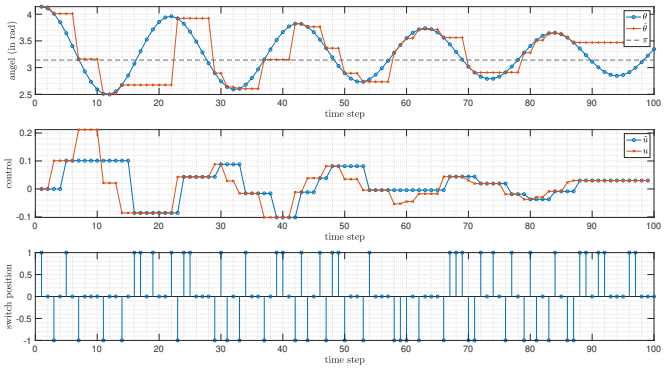

Figure 4: Setup 2 - top: actual and calculated angular position, middle: actual and transmitted control, bottom: optimal switching signal

4.1. Setup 1: Predictor in the Loop

First, we study a general setup discussed in the article. The predictor is integrated into the loop, and it updates the state information based on the latest observation available. The pendulum is perturbed from its upward equilibrium initially. For the initial 20 steps, we see in Fig. 3 that the difference between the actual state and the equilibrium position is significant. Thus for these steps, we observe that the control command is transmitted more than the observation. After 40 steps, the pendulum oscillates in a narrow range near the equilibrium due to the inherited nature of the control law (21), which is observed by [23].

4.2. Setup 2: Without the Predictor

In this setup, we discard the predictor from the loop and assume that when the observation is unavailable, we use the latest observation available. Then the observation is updated as,

$$\hat{x}_k := \begin{cases} x_k & \text{if } q_k = -1; \\ \hat{x}_{k-1} & \text{otherwise.} \end{cases} \tag{25}$$

The rest of the derivations were modified accordingly. In this case, consider the same three communication choices as in the previous case. Here, the controller does not have the latest information of the states at the plant, and the control command is a function of the delayed observation. Moreover, the *intelligence* provided by the predictor is also missing. This results in more significant oscillation, as observed in Fig. 4.

4.3. Setup 3: Two choices of Communication

We consider the system with the predictor in the loop in this setup. But the policy designer has two choices: i) send the control command or ii) receive the measured signal. The problem formulation and solution need some modification since the choice of q_k is restricted to -1 (receive the feedback) or 1 (send the control command). In Fig. 5, we find a similar result to the first setup, i.e., the control command is transmitted more in the first 40 steps. This indicates the optimal switching policy emphasizes regulating the pendulum position near the equilibrium.

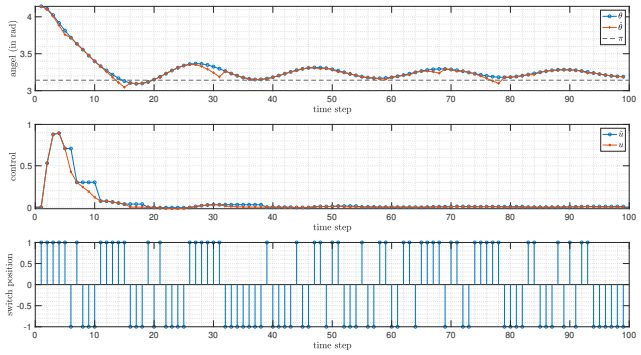

Figure 5: Setup 3 - top: actual and calculated angular position, middle: actual and transmitted control, bottom: optimal switching signal

	Setup 1	Setup 2	Setup 3
total cost	44.43	207.88	69.81
J_{sc}	22.93	178.88	19.81
J_{comm}	43	58	100

Table 2
Performance comparison of all setups with $\lambda = 0.5$

4.4. Discussion

With setups explained in the preceding subsections, we tabulate different costs for all setups in Table 2, and we present our observations (Obs1) - (Obs4).

(Obs1) Performace of setup 2 without a predictor in the loop is poor compared to setup 1 with a predictor, in terms of all costs. This indicates that the regulation is poorer and the communication requirement is higher.

(Obs2) In setup 3, the network is constantly communicating either way, still setup 1 performs better in terms of total

# of steps	Computation time (in sec)		Cost	
	exhaustive method	proposed method	exhaustive method	proposed method
5	6.45	2.57	9.37	9.41
6	24.69	2.7	11.82	12.01
7	87	2.84	13.40	13.52

Table 3
Comparison of time complexity and total cost between exhaustive search vs. proposed method

costs. Since the network is communicating at all times, the performance in terms of quadratic cost is better in setup 3. However, for that, the availability of the communication channel at all times is required.

(Obs3) We use setup 1 and vary the weighting factor (λ) in the total cost. Fig. 6 shows that when the weight of the

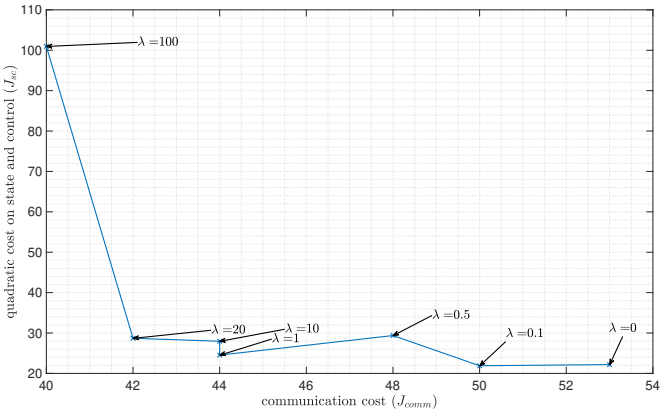

Figure 6: Trade-off between control cost and communication cost while varying λ

communication cost (J_{comm}) is high, there is less communication. As we decrease the weight of communication, we observe that the number of communication instances increases. This establishes a trade-off between better regulation and the number of communication instances. It is important to note the communication cost is discrete, therefore we do not observe a point, corresponding to $\lambda = 0.5$, that does not fit the curve exactly.

(Obs4) We employ an exhaustive search method to compare our results of setup 1 with it. In the exhaustive search method, we search among all possible combinations of switching sequences and note the outcomes. Since we have three choices of the switch position, the number of possibilities is 3^N, which is an exponential increase with the number of time steps. This is evident in the Table 3. Whereas the computation time in our proposed method does not scale as fast. Also, we tabulate the total cost in both methods, which is almost the same, and the minor variation can be attributed to the numerical approximation.

5. Conclusions

In this article, we proposed a general framework of an NCS with a communication channel shared for the transmission of the control and the observation. The problem of optimal scheduling of the transmission was tackled using DPMP by reformulating the problem using an augmented state. The empirical validation, conducted on a nonlinear system - the inverted pendulum, reaffirmed the theoretical conclusions. Notably, our proposed approach outperformed an exhaustive search method in terms of computational efficiency.

An extension of the problem can be considered where there are multiple channels in both ways of communication with different delays, and the goal is to find the optimal scheduling policy. Another avenue of future work is to design a joint policy for optimal control and optimal scheduling.

Appendix A. Derivation of Overall System Dynamics

The system description is given in (1)-(8) and is combined in (9) as follows. In (9), the first and the third equations are written as they appear in (1) and (2), respectively. From (6), the transmitted control can be expressed algebraically as,

$$\begin{aligned}\tilde{u}_k &= \frac{q_k(1+q_k)}{2}u_k + \frac{(1-q_k)}{2}\tilde{u}_{k-1} + \frac{(1-q_k)(1+q_k)}{2}\tilde{u}_{k-1} \\ &= \frac{q_k(1+q_k)}{2}u_k + \frac{(1-q_k)(2+q_k)}{2}\tilde{u}_{k-1} \\ &= \frac{1}{2}\left(q_k\left(1+q_k\right)u_k + \left(2+q_k\right)\left(1-q_k\right)\tilde{u}_{k-1}\right).\end{aligned} \tag{26}$$

As the switch position varies among three positions given by $q_k \in \{-1, 0, 1\}$, we verify that conditions given in (6) hold. If $q_k = 1$, then the first term in the first equality becomes u_k and the rest two go to zero. If $q = -1$, then the second term in the first equality becomes $\tilde{u}_{k-1}$, and the rest two terms go to zero. Similarly, if $q_k = 0$, then the first term goes to zero, and the rest two terms become $\frac{1}{2}\tilde{u}_{k-1}$. Hence (6) is verified. Similarly, from (8) the prediction can be expressed as follows,

$$\begin{aligned}\hat{x}_k &= \frac{-q_k(1-q_k)}{2}x_k + \frac{(1+q_k)}{2}\hat{f}(\hat{x}_{k-1}, \tilde{u}_{k-1}) \\ &\quad + \frac{(1-q_k)(1+q_k)}{2}\hat{f}(\hat{x}_{k-1}, \tilde{u}_{k-1}) \\ &= -\frac{1}{2}q_k(1-q_k)x_k \\ &\quad + \left(1-\frac{q_k}{2}\right)\left(1+q_k\right)\hat{f}(\hat{x}_{k-1}, \tilde{u}_{k-1}).\end{aligned} \tag{27}$$

If $q_k = -1$, then the first term in the first equality becomes x_k, and the rest two terms go to zero. If $q_k = 1$, then the second term in the first equality goes to $\hat{f}(\hat{x}_{k-1}, \tilde{u}_{k-1})$, and the rest two terms go to zero. If $q = 0$, then the first term goes

to zero and the rest two terms each go to $\frac{1}{2}\hat{f}(\hat{x}_{k-1}, \tilde{u}_{k-1})$. Thereby verifying the conditions of (8).

Equation (9) is re-written by replacing $\tilde{u}_k = g(\hat{x}_k)$ as

$$\begin{aligned} x_{k+1} &= f(x_k, \tilde{u}_k) \\ \tilde{u}_k &= \frac{1}{2}[q_k\left(1+q_k\right) g(\hat{x}_k) + \left(2+q_k\right)\left(1-q_k\right)\tilde{u}_{k-1}] \\ \hat{x}_k &= -\frac{1}{2}q_k(1-q_k)x_k \\ &\quad + \left(1-\frac{q_k}{2}\right)\left(1+q_k\right)\left(\hat{f}(\hat{x}_{k-1}, \tilde{u}_{k-1})\right) \end{aligned} \tag{28}$$

The above equation can be written in a compact form as (10) using the concatenated state ξ_k.

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