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Abstract. We are concerned with the construction of global-in-time strong solutions for the incompressible Vlasov-Navier-Stokes system in the whole three-dimensional space. One of our goals is to establish that small initial velocities with critical Sobolev regularity $H^{1/2}$ and sufficiently well localized initial kinetic distribution functions give rise to global and unique solutions. This constitutes an extension of the celebrated result for the incompressible Navier-Stokes equations (NS) that has been established in 1964 by Fujita and Kato [8]. If in addition the initial velocity is in L^1, we establish that the total energy of the system decays to 0 with the optimal rate $t^{-3/2}$, like for the weak solutions of (NS), see [15, 16].

Our results partly rely on the use of a higher order energy functional that controls the regularity H^1 of the velocity and seems to have been first introduced by Li, Shou and Zhang in [12] in the context of *nonhomogeneous* Vlasov-Navier-Stokes system. In the small data case, we show that this energy functional decays with the rate $t^{-5/2}$.

This paper is concerned with the proof of the existence of global-in-time strong solutions and the study of large time asymptotics for the following so-called Vlasov-Navier-Stokes system in $\mathbb{R}^3\times\mathbb{R}^3$:

$$(VNS)\qquad \begin{cases} \partial_t f + v\cdot\nabla_x f + \operatorname{div}_v\big(f(u-v)\big)=0,\\ \partial_t u + u\cdot\nabla_x u - \Delta_x u + \nabla_x P = \displaystyle\int_{\mathbb{R}^3} f(v-u)\,dv,\\ \operatorname{div} u = 0. \end{cases}$$

This system is a toy model for describing the dynamics of a spray, that is, a cloud of particles that are immersed in an incompressible viscous homogeneous fluid. The kinetic distribution function $f=f(t,x,v)$ then represents the density of particles with velocity $v\in\mathbb{R}^3$, that are located at $x\in\mathbb{R}^3$ at time $t\geq 0$, while $u=u(t,x)$ stands for the ambient velocity field obeying the classical incompressible (and homogeneous) Navier-Stokes equations, under the influence of a drag term, the so-called Brinkman force.

For $f\equiv 0$, System (VNS) reduces to the celebrated incompressible Navier-Stokes equations

$$(NS)\qquad \begin{cases} \partial_t u + u\cdot\nabla u - \Delta u + \nabla P = 0,\\ \operatorname{div} u = 0, \end{cases}$$

and the question arises as to which results for (NS) can be extended to (VNS).

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Like for (NS), there is an energy balance associated to (VNS), namely

$$\frac{d}{dt}E_0 + D_0 = 0 \quad \text{with} \quad E_0 := \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}^3_x)} + \frac{1}{2}\||v|^2 f\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \tag{0.1}$$
$$\text{and} \quad D_0 := \|\nabla u\|^2_{L^2(\mathbb{R}^3_x)} + \|f|u-v|^2\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}.$$

One can thus expect to have a global finite energy weak solutions theory similar to that established by J. Leray for (NS), in [11]. Such solutions have indeed been constructed by L. Boudin, L. Desvillettes, C. Grandmont and A. Moussa in [5] who leveraging a suitable approximation scheme and compactness arguments to prove an analogous of Leray's theorem, for (VNS). More precisely, in the case where the fluid domain is the three-dimensional torus $\mathbb{T}^3$, any initial data (f_0, u_0) such that

$$f_0 \in L^\infty(\mathbb{T}^3\times\mathbb{R}^3), \quad |v|^2 f_0 \in L^1(\mathbb{T}^3\times\mathbb{R}^3) \quad \text{and} \quad u_0 \in L^2(\mathbb{T}^3) \quad \text{with} \quad \operatorname{div} u_0 = 0$$

gives rise to a global-in-time distributional solution verifying

$$E_0(t) + \int_0^t D_0\,d\tau \leq E_{0,0} := \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^3_x)} + \frac{1}{2}\||v|^2 f_0\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)}, \quad \text{a.e. } t\in\mathbb{R}_+.$$

The result of [5] also holds in three-dimensional bounded domains [1], in the two-dimensional torus, and can be adapted to the $\mathbb{R}^2$ or $\mathbb{R}^3$ setting if f_0 is suitably well localized (see the appendix of [9]). Furthermore, as for (NS), in the 2D case, weak solutions are unique in the energy class (again, see [9]).

To continue the analogy with (NS), one can study whether smoother initial data generate smoother solutions. In particular, in the three dimensional case, if assuming that u_0 is small in the Sobolev space $\dot H^{1/2}(\mathbb{R}^3)$ and that f_0 satisfies appropriate conditions, do we have a global and unique solution ? A positive answer to this question would provide a result similar to that of H. Fujita and T. Kato [8] for (NS) in a three-dimensional bounded domain (and adapted to the $\mathbb{R}^3$ situation by J.-Y. Chemin in [6]).

In the present paper, we focus on the $\mathbb{R}^3$ case. We want to establish the existence and uniqueness of global solutions for (VNS) supplemented with an initial velocity satisfying Fujita-Kato type assumption, and to specify the rate of convergence to 0 of the energy E_0, and the long time behavior of f. To our knowledge, proving global existence and uniqueness for critical regularity is new. As for the proof of the long time behavior, it has been studied recently by D. Han Kwan in [7]. We here reach the optimal time decay for E_0, and also specify the rate of decay for the following higher order energy functional:

$$E_1 := \|\nabla u\|^2_{L^2} + \iint f|u-v|^2\,dv\,dx. \tag{0.2}$$

1. Main results

Although our final goal is to investigate (VNS) supplemented with initial velocity in the critical Sobolev space $H^{1/2}(\mathbb{R}^3)$, we shall first consider the smoother situation where u_0 belongs to the Sobolev space $H^1(\mathbb{R}^3)$. This is motivated by the fact that in the small solutions regime, the functional E_1 defined in (0.2) satisfies a quasi conservation law, in the spirit of (0.1) whenever the density ρ defined in (1.6) stays bounded and ∇u is in $L^1(\mathbb{R}_+;L^\infty)$. In this setting, one can achieve optimal time decay estimates (that is, with the rate of the heat equation) for both E_0 and E_1: we shall first establish the following result, which turns out to be the key to our second one, pertaining to initial velocities in the nonhomogeneous Sobolev space $H^{1/2}(\mathbb{R}^3)$.

Theorem 1.1. *Assume that the initial distribution function f_0 satisfies*

$$f_0 \in L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v) \cap L^\infty(\mathbb{R}^3_v \times \mathbb{R}^3_x) \quad \text{and} \quad |v|^2 f_0 \in L^1(\mathbb{R}^3_v; L^\infty(\mathbb{R}^3_x)) \tag{1.1}$$

and that u_0 is a divergence free vector-field in $L^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$.

There exists a small constant c_0 depending only on

$$\|u_0\|_{L^1(\mathbb{R}^3)}, \quad \|f_0\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}, \quad \|f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))} \quad \text{and} \quad \||v|^2 f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))}$$

such that if

$$\|u_0\|^2_{H^1(\mathbb{R}^3)} + \iint |v|^2 f_0 \, dv \, dx \le c_0, \tag{1.2}$$

then (VNS) admits a unique global solution (f, u, P) satisfying the energy balance:

$$E_0(t) + \int_0^t D_0 \, d\tau = E_{0,0} \quad \text{for all } t > 0, \tag{1.3}$$

and such that:

- $u \in \mathcal{C}_b(\mathbb{R}_+; H^1) \cap L^2(\mathbb{R}_+; L^\infty)$, $\nabla u \in L^1(\mathbb{R}_+; L^\infty)$, $\nabla^2 u \in L^2(\mathbb{R}_+ \times \mathbb{R}^3)$,
- $\nabla P \in L^2(\mathbb{R}_+ \times \mathbb{R}^3)$,
- $f \in L^\infty_{loc}(\mathbb{R}_+; L^\infty(\mathbb{R}^3_v \times \mathbb{R}^3_x))$, $|v|^2 f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^3_v; L^\infty(\mathbb{R}^3_x)))$ *and*

$$\|f(t)\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} = M_0 := \|f_0\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \quad \text{for all } \ t \in \mathbb{R}_+. \tag{1.4}$$

Furthermore, optimal time decay estimates hold for E_0 and E_1, namely

$$E_0(t) \le C_0 t^{-3/2} \quad \text{and} \quad E_1(t) \le C_1 t^{-5/2}, \tag{1.5}$$

for some positive constants C_0 and C_1 depending only on suitable norms of the initial data.

Remark 1.2. In contrast with the (NS) situation, we do not know how to prove global results without assuming more integrability on u_0 than that which is given by (1.2). In fact, in order to keep under control the following hydrodynamical quantities (density, momentum and distribution energy)[1]:

$$\rho(t,x) := \int f(t,x,v)\, dv, \quad j(t,x) := \int v f(t,x,v)\, dv, \quad m_2(t,x) := \int |v|^2 f(t,x,v)\, dv \tag{1.6}$$

we need ∇u to be small in $L^1(\mathbb{R}_+; L^\infty)$. This latter property requires sufficient decay for u.

Remark 1.3. As e.g. in the work [18] by J. Xu and S. Kawashima dedicated to partially dissipative systems, it is enough to have u_0 in the homogeneous Besov space $\dot{B}^{-3/2}_{2,\infty}(\mathbb{R}^3)$. The use of this space that corresponds to the critical embedding $L^1(\mathbb{R}^3) \hookrightarrow \dot{B}^{-3/2}_{2,\infty}(\mathbb{R}^3)$ is motivated by the fact that it lends itself much better to parabolic estimates than L^1.

Instead of $u_0 \in \dot{B}^{-3/2}_{2,\infty}(\mathbb{R}^3)$, one can just assume that $u_0 \in \dot{B}^{-\sigma}_{2,\infty}(\mathbb{R}^3)$ for some $\sigma \in (1, 3/2)$. Then, the decay exponents for E_0 and E_1 are σ and $\sigma+1$, respectively. In terms of Lebesgue spaces, the condition $\sigma > 1$ corresponds to $u_0 \in L^p$ for some $p < 6/5$.

As observed in e.g. [10], having decay estimates at hand for the velocity allows to get some insight on the long time asymptotics of f, ρ and j. To state it precisely, we need to introduce the Wasserstein distance $W_1(\mu,\nu)$ between two measures on $\mathbb{R}^3 \times \mathbb{R}^3$. In our context, it can be defined by:

$$W_1(\mu,\nu) := \sup\Big\{ \iint \phi \, d\mu(x,v) - \iint \phi \, d\nu(x,v), \ \phi \in C^{0,1}(\mathbb{R}^3 \times \mathbb{R}^3), \ \|\nabla_{x,v}\phi\|_{L^\infty} = 1 \Big\}.$$

[1] In all the paper, integration is performed on $\mathbb{R}^3$, unless otherwise specified.

Corollary 1.4. *Under the assumptions of Theorem 1.1, we have for some constant C_0 depending only on the norms of the data:*

$$\|(j-\rho u)(t)\|_{L^1} + W_1(f(t),\rho(t)\otimes\delta_{v=u(t)}) \leq C_0 t^{-5/4}. \tag{1.7}$$

Furthermore, there exists a vector-field j_∞ in $L^1(\mathbb{R}^3)$ such that

$$\|\rho(t)-\rho_0+\operatorname{div} j_\infty\|_{\dot W^{-1,1}} \leq C_0 t^{-1/4}. \tag{1.8}$$

Remark 1.5. This result has to be compared with that of [7] where it is stated that

$$W_1(f(t),\rho(t)\otimes\delta_{v=0}) \leq C_1\langle t\rangle^{-3/4+\varepsilon} \quad \text{for all} \quad \varepsilon>0.$$

Having the decay rate 5/4 might indicate that the monokinetic distribution $\rho(t)\otimes\delta_{v=u(t)}$ is a better approximation of f than just $\rho(t)\otimes\delta_{v=0}$. Having a decay information for E_1 and not only for E_0 is the key to this improvement.

Let us now come to the main aim of the paper: establishing a Fujita-Kato type result for (VNS). Again, in order to get a global result, we need to assume that u_0 has enough integrability. For technical reasons, we also need f_0 to have a better localization than in Theorem 1.1, namely that for some $q>5$, we have[2]

$$N_q(f_0) := \sup_{(x,v)\in\mathbb{R}^3\times\mathbb{R}^3} \langle v\rangle^q f_0(x,v) < \infty \quad \text{with} \quad \langle v\rangle := \sqrt{1+|v|^2}. \tag{1.9}$$

Theorem 1.6. *Let u_0 be a divergence free vector-field in $L^1(\mathbb{R}^3)\cap\dot H^{1/2}(\mathbb{R}^3)$, and assume that f_0 satisfies* (1.1) *and that $N_q(f_0)<\infty$ for some $q>5$. Then, there exists a constant c_0 depending only on $\|u_0\|_{L^1}$, $\|f_0\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}$ and $N_q(f_0)$ such that, if*

$$\|u_0\|^2_{H^{1/2}(\mathbb{R}^3)} + \iint |v|^2 f_0\,dv\,dx \leq c_0,$$

then (VNS) admits a unique global solution satisfying the energy equality (1.3)*, the mass conservation* (1.4)*, $u\in L^2(\mathbb{R}_+;L^\infty\cap\dot H^{3/2})\cap\mathcal{C}_b(\mathbb{R}_+;H^{1/2})$, and u satisfies the log Lipschitz property stated in* (5.10)*. Furthermore, we have optimal time decay estimates for E_0 and E_1, namely* (1.5)*, and the asymptotics* (1.7) *and* (1.8) *hold true.*

Remark 1.7. If we assume that u_0 is small in the critical Besov space $B^{1/2}_{2,1}$ (a 'large' subspace of $H^{1/2}$), then it suffices to make the hypothesis (1.1) on f_0. In fact, the constructed solution turns out to satisfy $\nabla u\in L^1(\mathbb{R}_+;L^\infty)$ so that one can bound ρ, j and m_2 as in Theorem 1.1 without requiring a stronger localization of f_0.

Let us give an overview of the main ideas leading to our results.

The first step in proving Theorem 1.1 is to establish global-in-time a priori estimates for E_0, E_1 and $t\|\partial_t u\|^2_{L^2}$. These estimates are *conditional*: they are valid on $[0,T]$ (for smooth enough solutions) provided that it is known that ρ is bounded and that ∇u is in $L^1(0,T;L^\infty)$. At this stage, no additional integrability is required for u_0.

The second step is dedicated to the proof of optimal time decay estimates for E_0 and E_1. In contrast with the recent work [7], we shall not use the Shonbeck Fourier splitting method of [15] (as it looks to entail a small loss in the decay rate) but rather an older approach that seems to originate from a work of J. Nash [14] on parabolic equations. Schematically,

[2]It should be noted that a similar condition was required in the previous works dedicated to (VNS) in the whole space and plane (see [7] and [9]). Whether it may be avoided is an open question.

the idea goes as follows. Assume that there exist nonnegative *Lyapunov functional* $\mathcal{L}$ and *dissipation rate* such that

$$\frac{d}{dt}\mathcal{L} + \mathcal{H} \leq 0 \quad \text{a.e. on } \mathbb{R}_+. \tag{1.10}$$

Suppose in addition that we have at hand the control of a 'lower order' functional $\mathcal{N}$:

$$\mathcal{N} \leq \mathcal{N}_0 \quad \text{a.e. on } \mathbb{R}_+. \tag{1.11}$$

Finally assume that $\mathcal{L}$ is an 'intermediate' functional between $\mathcal{N}$ and $\mathcal{H}$ in the sense that there exist $\theta \in (0,1)$ and $C > 0$ such that:

$$\mathcal{L} \leq C\mathcal{H}^\theta \mathcal{N}^{1-\theta}. \tag{1.12}$$

Then, inserting this inequality in (1.10) gives

$$\frac{d}{dt}\mathcal{L} + c_0\mathcal{L}^{1/\theta} \leq 0 \quad \text{with} \quad c_0 := C^{-1/\theta}\mathcal{N}_0^{1-1/\theta},$$

and thus, after time integration,

$$\mathcal{L}(t) \leq \mathcal{L}(0)\biggl(1 + \frac{1-\theta}{\theta}\, c_0 \mathcal{L}_0^{\frac{1-\theta}{\theta}} t\biggr)^{-\frac{\theta}{1-\theta}}. \tag{1.13}$$

In the present situation, $\mathcal{L}$ will be either E_0 or E_1, and $\mathcal{H}$ will be the corresponding dissipation rate, that is D_0 in the first case, and the functional D_1 defined in (2.5), in the second case. Finally, for $\mathcal{N}$ we use $\|u\|_{\dot{B}^{-3/2}_{2,\infty}}$ (rather than the L^1 norm). For some reason however, we do not manage to get the optimal rates first time. We first prove decay $t^{-3/2}$ for E_1, which will enable us to bound the Brinkman force in $L^1(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$. Then, in a second step, we obtain the optimal rates for both E_0 and E_1.

Let us also underline that establishing (1.12) might be not so straightforward since E_0, D_0, E_1 and D_1 contain two terms that are hardly comparable. Fortunately, being in the small solutions regime will spare us considering different cases, depending on whether one or the other term is dominant.

In the third step, we have to get rid of the a priori assumptions on the solution we made hitherto. Ensuring the smallness of ∇u in $L^1(\mathbb{R}_+; L^\infty)$ is the key. It will be obtained by combining the time decay results of the second step with the energy estimates of the first step and suitable Gagliardo-Nirenberg or Sobolev inequalities.

The next step is uniqueness. To this end, we follow the method of [9] that relies on the control of suitably weighted L^2 norms for the difference of velocities and of the characteristics associated to the transport equation for the distribution function (see the definition in (A.1)). The situation here is simpler since we have ∇u in $L^1(\mathbb{R}_+; L^\infty)$ and it is thus possible to conclude by means of the classical Gronwall lemma. Nevertheless, we give some details since this proof will be a model one to investigate the more complicated situation of uniqueness for Fujita-Kato type solutions, and will be also useful in the next step, dedicated to constructing global solutions.

The last step is dedicated to the proof of existence of a global solution. To do so, we use a Friedrichs regularization of the velocity equation and of the velocity field in the first equation of (VNS) (in order to keep the symmetric structure of the overall system). These 'regularized' solutions turn out to satisfy the same estimates as those of (VNS). Furthermore, by modifying suitably the proof of uniqueness, one can show the (strong) convergence to solutions of (VNS) in the energy space.

To get Theorem 1.6, the first step is to establish a short time existence result for u_0 in the Sobolev space $H^{1/2}$. As in [7], by looking at the velocity equation as a Navier-Stokes equation with a source term (namely the Brinkman force), we succeed in getting a solution on, say, the time interval $[0,1]$ provided the data are small enough. This solution is shown to have a log Lipschitz velocity, which enables us to adapt the method of [9] (based on Osgood lemma) to the 3D case. Next, since the constructed solution is H^1 for almost every positive time and as the negative Besov regularity is under control for small time, even if u is only a $H^{1/2}$ type solution, one can combine this result with the previous one to complete the proof of Theorem 1.6.

We end this section pointing out some interesting open questions related to our results.

- For (NS), assuming only that u_0 is small in $\dot{H}^{1/2}$ allows to get global-in-time result (see [8] or the adaptation to the $\mathbb{R}^3$ setting by J.-Y. Chemin in [6]). Can we do without any additional integrability integrability condition on u_0 in Theorems 1.1 and 1.6 ?
- The localization requirement for f_0 is rather strong and not natural inasmuch as it does not come into play (at least not directly) in the control of the energy functionals E_0 and E_1. Is it enough to make the same assumptions as in Theorem 1.1 ?
- Can we compare the solutions of (VNS) to those of the following pressureless Euler-Navier-Stokes system:

$$(ENS)\qquad \begin{cases} \partial_t\rho + \operatorname{div}(\rho w) = 0, \\ \partial_t(\rho w) + \operatorname{div}(\rho w\otimes w) + \rho(w-u) = 0, \\ \partial_t u + u\cdot\nabla u - \Delta u + \nabla_x P = \rho(w-u), \\ \operatorname{div} u = 0. \end{cases}$$

  Note that here, in the large time asymptotics, we rather proved that u behaves as the solution of the incompressible Navier-Stokes equations with no source term, and that the density ρ tends to be transported by its flow.

The rest of the paper unfolds as follows. The next section is devoted to the proof of (conditional) energy estimates for smooth enough solutions of (VNS). Then, in Section 3, we establish the optimal decay estimates for E_0 and E_1, and deduce the key Lipschitz control of the velocity field. Section 4 is devoted to completing the proof of Theorem 1.1 while Section 5 deals with the proof of Theorem 1.6 and Remark 1.7. Some technical results concerning the properties of the flow associated to the transport equation of (VNS) are postponed in Appendix.

Notation: For any normed space X, index $q\in[1,\infty]$ and time $T\in[0,\infty]$, we shall denote $\|z\|_{L^q_T(X)} := \big\|\|z(t)\|_X\big\|_{L^q(0,T)}$ and omit T if it is ∞. In the case where z has several components in X, we keep the same notation for the norm. It will be sometimes convenient to use the short notation u_t (resp. f_t) to designate the time derivative of u (resp. f).

Finally, the notation C stands for harmless positive real numbers, and we shall often write $A\lesssim B$ instead of $A\leq CB$. To emphasize the dependency with respect to parameters $a_1,\cdots,a_n$, we adopt the notation $C_{a_1,\cdots,a_n}$.

2. Energy estimates

Throughout this section, we assume that we are given a smooth solution (f, u, P) on the time interval $[0, T]$, with sufficient decay. Our goal is to derive a priori estimates for E_0, E_1 and $t\|u_t\|_{L^2}^2$ that are 'closable' for all time in the small data case.

As a first, let us recall that (0.1) can be obtained by taking the $L^2(\mathbb{R}^3;\mathbb{R}^3)$ scalar product of the velocity equation of (VNS) with u, integrating the equation of f on $\mathbb{R}^3 \times \mathbb{R}^3$ with respect to the measure $|v|^2\,dx\,dv$ and performing suitable integration by parts (see e.g. [5]).

Next, in order to bound the energy functional E_1 defined in (0.2), the starting point is to take the $L^2(\mathbb{R}^3;\mathbb{R}^3)$ scalar product of the velocity equation of (VNS) with u_t. This gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|u_t\|_{L^2}^2 = -\int (u\cdot\nabla u)\cdot u_t\,dx + \iint u_t\cdot((v-u)f)\,dv\,dx. \tag{2.1}$$

The convection term may be handled as follows for all $\varepsilon > 0$:

$$\begin{aligned}
-\int (u\cdot\nabla u)\cdot u_t\,dx &\le \|u\|_{L^6}\|\nabla u\|_{L^3}\|u_t\|_{L^2}\\
&\le C\|\nabla u\|_{L^2}^{3/2}\|\nabla^2 u\|_{L^2}^{1/2}\|u_t\|_{L^2}\\
&\le \frac{1}{4}\|u_t\|_{L^2}^2 + \frac{\varepsilon}{2}\|\nabla^2 u\|_{L^2}^2 + \frac{C}{\varepsilon}\|\nabla u\|_{L^2}^6.
\end{aligned}$$

To bound the last term of (2.1), we use the decomposition

$$\begin{aligned}
\iint u_t\cdot((v-u)f)\,dv\,dx &= -\frac{1}{2}\frac{d}{dt}\iint |v-u|^2 f\,dv\,dx + \frac{1}{2}\iint_{\mathbb{R}^3} |v-u|^2 f_t\,dv\,dx\\
&= -\frac{1}{2}\frac{d}{dt}\iint |v-u|^2 f\,dv\,dx - \frac{1}{2}\iint |v-u|^2 v\cdot\nabla_x f\,dv\,dx\\
&\qquad + \frac{1}{2}\iint |v-u|^2 \mathrm{div}_v\big(f(v-u)\big)\,dv\,dx\\
&= -\frac{1}{2}\frac{d}{dt}\iint |v-u|^2 f dv\,dx + \iint f(v-u)\cdot(v\cdot\nabla u)dv\,dx\\
&\qquad - \iint f|v-u|^2\,dv\,dx.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\iint u_t\cdot((v-u)f)\,dv\,dx &= -\frac{1}{2}\frac{d}{dt}\iint |v-u|^2 f\,dv\,dx + \iint f(v-u)\cdot(u\cdot\nabla u)\,dv\,dx\\
&\qquad + \iint f(v-u)\cdot((v-u)\cdot\nabla u)\,dv\,dx - \iint f|v-u|^2\,dv\,dx.
\end{aligned} \tag{2.2}$$

The first and last terms are parts of E_1 and of D_1 (defined in (0.2) and (2.5) below), respectively. For the third term, we have

$$\iint f(v-u)\cdot((v-u)\cdot\nabla u)\,dv\,dx \le \|\nabla u\|_{L^\infty} E_1.$$

Finally, observe that by Cauchy-Schwarz inequality with respect to the v variable, we have

$$\Big\|\int f(v-u)\,dv\Big\|_{L^2}^2 \le \|\rho\|_{L^\infty}\iint f|v-u|^2\,dx\,dv. \tag{2.3}$$

Hence

$$\begin{aligned}\iint f(v-u)\cdot(u\cdot\nabla u)\,dv\,dx &\leq \Big\|\int f(v-u)\,dv\Big\|_{L^2}\|u\|_{L^6}\|\nabla u\|_{L^3}\\ &\leq C\|\rho\|_{L^\infty}^{1/2}\biggl(\iint f|u-v|^2\,dv\,dx\biggr)^{1/2}\|\nabla u\|_{L^2}^{3/2}\|\nabla^2 u\|_{L^2}^{1/2}\\ &\leq \frac{\varepsilon}{4}\|\nabla^2 u\|_{L^2}^2+\frac12\iint f|u-v|^2dv\,dx+\frac{C}{\varepsilon}\|\rho\|_{L^\infty}^2\|\nabla u\|_{L^2}^6.\end{aligned}$$

Using the definition of E_1 in (0.2), we conclude that for all $\varepsilon>0$,

$$\begin{aligned}\frac{d}{dt}E_1+\|u_t\|_{L^2}^2+\iint f|u-v|^2\,dv\,dx &\leq \varepsilon\|\nabla^2u\|_{L^2}^2\\ &\quad+C\|\nabla u\|_{L^\infty}E_1+C\varepsilon^{-1}\max(1,\|\rho\|_{L^\infty}^2)\|\nabla u\|_{L^2}^6.\end{aligned}\tag{2.4}$$

To close the estimate, one has to bound $\|\nabla^2u\|_{L^2}$. This may be done by using the elliptic regularity provided by the following Stokes system:

$$-\Delta u+\nabla P=-u_t-u\cdot\nabla u+\int f(v-u)\,dv,\qquad \operatorname{div}u=0.$$

We have

$$\|\nabla^2u,\nabla P\|_{L^2}^2\leq 3\|u_t\|_{L^2}^2+3\|u\cdot\nabla u\|_{L^2}^2+3\Big\|\int f(v-u)\,dv\Big\|_{L^2}^2.$$

So, handling the convection term as above and using (2.3), we discover that

$$\|\nabla^2u,\nabla P\|_{L^2}^2\leq 3\|u_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^3\|\nabla^2u\|_{L^2}+3\|\rho\|_{L^\infty}\iint f|v-u|^2\,dx\,dv,$$

which, leveraging Young inequality, leads to

$$\|\nabla^2u,\nabla P\|_{L^2}^2\leq 6\|u_t\|_{L^2}^2+C\|\nabla u\|_{L^2}^6+6\|\rho\|_{L^\infty}\iint f|v-u|^2\,dx\,dv.$$

Let $R:=\max(1,\|\rho\|_{L^\infty})$ and

$$D_1:=\frac12\|u_t\|_{L^2}^2+\frac{1}{24R}\|\nabla^2u,\nabla P\|_{L^2}^2+\frac12\iint f|u-v|^2\,dx\,dv.\tag{2.5}$$

Using the above inequality and reverting to (2.4) with $\varepsilon=R^{-1}$, we end up with

$$\frac{d}{dt}E_1+D_1\leq C\|\nabla u\|_{L^\infty}E_1+CR^3\|\nabla u\|_{L^2}^6.\tag{2.6}$$

In order to control E_1 by means of the above inequality, we need both ρ to be bounded and ∇u to be in $L^1(0,T;L^\infty)$. According to (A.5) and provided f_0 belongs to $L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))$, the first property stems from the second one (from (A.3), more exactly). Owing to Sobolev embedding, achieving this key Lipschitz control requires our controlling the L^p norm of ∇_x^2u for some $p>3$, or in the Lorentz space $L^{3,1}$. Toward this, we shall use elliptic estimates for the following Stokes system:

$$-\Delta u+\nabla P=-u_t-u\cdot\nabla u-\int_{\mathbb{R}^3}f(v-u)\,dv,\qquad \operatorname{div}u=0.\tag{2.7}$$

Since our approach heavily relies on energy arguments, it is tempting to look for a control of ∇u_t in L^2, owing to the embedding $\dot H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$, which, following the approach of [12] suggest us to consider the higher order energy functional $E_2:=\|u_t\|_{L^2}^2$. However, since we only have u_0 in H^1, one cannot expect to get a uniform control of $E_2(t)$. Even for the heat

equation, this will require H^2 regularity. For scaling reasons, it is natural to compensate the missing space derivative by the weight $\sqrt{t}$ that is, to look for uniform estimates of $tE_2(t)$.

To proceed, let us differentiate the velocity equation with respect to time:

$$\begin{aligned} u_{tt} + u\cdot\nabla u_t + \nabla P_t - \Delta u_t + \rho u_t + u_t\cdot\nabla u &= -\int (u-v) f_t\,dv \\ &= \int (u-v)(v\cdot\nabla_x f)\,dv + \int (v-u) f\,dv. \end{aligned} \tag{2.8}$$

Taking the $L^2(\mathbb{R}^3;\mathbb{R}^3)$ scalar product with tu_t yields

$$\begin{aligned} &\frac12\frac{d}{dt}\|\sqrt{t}\,u_t\|_{L^2}^2 + \|\sqrt{t}\,\nabla u_t\|_{L^2}^2 + \|\sqrt{\rho t}\,u_t\|_{L^2}^2 - \frac12\|u_t\|_{L^2}^2 \\ &= -\int (tu_t)\cdot(u_t\cdot\nabla u)\,dx + \iint (v\cdot\nabla_x f)(u-v)\cdot(tu_t)\,dv\,dx + \iint (tu_t)\cdot(v-u) f\,dv\,dx. \end{aligned}$$

In light of (2.2), we have

$$\begin{aligned} \iint (tu_t)\cdot(v-u) f\,dv\,dx = -\frac12\frac{d}{dt}\iint t|v-u|^2 f\,dv\,dx + \iint tf(v-u)\cdot(u\cdot\nabla u)\,dv\,dx \\ + \iint tf(v-u)\cdot((v-u)\cdot\nabla u)\,dv\,dx - \frac12\iint f|v-u|^2\,dv\,dx. \end{aligned}$$

Therefore, setting

$$E_2 := \|u_t\|_{L^2}^2 + \iint f|v-u|^2\,dv\,dx \quad\text{and}\quad D_2 := \|\nabla u_t\|_{L^2}^2 + \|\sqrt{\rho}\,u_t\|_{L^2}^2,$$

we discover that

$$\begin{aligned} \frac12\frac{d}{dt}(tE_2) + tD_2 + \frac12\iint f|v-u|^2\,dv\,dx = \frac12\|u_t\|_{L^2}^2 - \int (tu_t)\cdot(u_t\cdot\nabla u)\,dx \\ + \iint (v\cdot\nabla_x f)(u-v)\cdot(tu_t)\,dv\,dx \\ + \iint tf(v-u)\cdot(u\cdot\nabla u)\,dv\,dx + \iint tf(v-u)\cdot((v-u)\cdot\nabla u)\,dv\,dx. \end{aligned} \tag{2.9}$$

For the second term of the right-hand side, combining Hölder, Sobolev, Gagliardo-Nirenberg (to bound the L^3 norm) and, eventually, Young inequalities gives

$$\begin{aligned} -\int (tu_t)\cdot(u_t\cdot\nabla u)\,dx &\le \|\sqrt{t}\,u_t\|_{L^6}\|\sqrt{t}\,u_t\|_{L^2}\|\nabla u\|_{L^3} \\ &\le \frac14\|\sqrt{t}\,\nabla u_t\|_{L^2}^2 + C\|\sqrt{t}\,u_t\|_{L^2}^2\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}. \end{aligned}$$

To handle the third term, we integrate by parts and use that $v = u + (v-u)$, getting :

$$\begin{aligned} \iint (v\cdot\nabla_x f)(u-v)\cdot(tu_t)\,dv\,dx = I_1 + I_2 + I_3 := \iint t(v-u)v\cdot\nabla u_t\, f\,dv\,dx \\ - \iint tu_t\cdot(u\cdot\nabla u)\, f\,dv\,dx + \iint tu_t\cdot((u-v)\cdot\nabla u)\, f\,dv\,dx. \end{aligned}$$

Combining Cauchy-Schwarz and Young inequalities, we easily get for all $\varepsilon > 0$,

$$I_1 \le \frac{\varepsilon}{4}\|\sqrt{tm_2}\,\nabla u_t\|_{L^2}^2 + \frac{1}{\varepsilon}\iint tf|v-u|^2\,dv\,dx.$$

Next, we have

$$\begin{aligned}I_2 &\leq \|\rho\|_{L^\infty}^{1/2}\|\sqrt{\rho t}\,u_t\|_{L^2}\|\nabla u\|_{L^3}\|\sqrt{t}u\|_{L^6}\\ &\leq \frac{1}{2}\|\sqrt{\rho t}\,u_t\|_{L^2}^2 + Ct\|\rho\|_{L^\infty}\|\nabla u\|_{L^2}^3\|\nabla^2 u\|_{L^2}\\ &\leq \frac{1}{2}\|\sqrt{\rho t}\,u_t\|_{L^2}^2 + \frac{1}{2}\|\nabla^2 u\|_{L^2}^2 + C\|\rho\|_{L^\infty}^2\|\nabla u\|_{L^2}^2\|\sqrt{t}\nabla u\|_{L^2}^4\end{aligned}$$

and, using (2.3) and Gagliardo-Nirenberg inequality to bound the L^3 norm,

$$\begin{aligned}I_3 &\leq \|\sqrt{t}u_t\|_{L^6}\|\nabla u\|_{L^3}\Big\|\sqrt{t}\int(v-u)f\Big\|_{L^2}\\ &\leq \frac{1}{4}\|\sqrt{t}\,\nabla u_t\|_{L^2}^2 + C\|\rho\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\iint tf|v-u|^2\,dv\,dx.\end{aligned}$$

Next, we write that

$$\begin{aligned}\iint tf(v-u)\cdot(u\cdot\nabla u)\,dv\,dx &\leq \|\rho\|_{L^\infty}^{1/2}\biggl(\iint tf|v-u|^2\,dv\,dx\biggr)^{1/2}\|\sqrt{t}\,u\|_{L^6}\|\nabla u\|_{L^3}\\ &\leq C\|\rho\|_{L^\infty}^{1/2}\biggl(\iint tf|v-u|^2\,dv\,dx\biggr)^{1/2}\|\sqrt{t}\,\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2}\\ &\leq \frac{1}{2}\iint tf|v-u|^2\,dv\,dx + C\|\rho\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\sqrt{t}\,\nabla u\|_{L^2}^2.\end{aligned}$$

Finally, it is obvious that

$$\iint tf(v-u)\cdot((v-u)\cdot\nabla u)\,dv\,dx \leq \|\nabla u\|_{L^\infty}\iint tf|u-v|^2\,dv\,dx.$$

Therefore, choosing $\varepsilon = \|m_2\|_{L^\infty}^{-1}$, reverting to (2.9) and using the definition of D_i and E_i for $i=0,1,2$, we end up with

$$\begin{aligned}&\frac{d}{dt}(tE_2) + tD_2 + \iint f|v-u|^2\,dv\,dx \leq 24RD_1 + 2\bigl(1+\|m_2\|_{L^\infty}\bigr)tD_1\\ &\quad + C\|\rho\|_{L^\infty}^2 D_0(tE_1)^2 + C\|\rho\|_{L^\infty}\sqrt{RD_0D_1}\,tE_1 + \bigl(2\|\nabla u\|_{L^\infty} + C\|\rho\|_{L^\infty}\sqrt{RD_0D_1}\bigr)tE_2.\end{aligned}\tag{2.10}$$

Let us set

$$\begin{aligned}&\mathcal{E}(t) := 2(2+\|m_2\|_{L^\infty})(tE_1(t)+2E_0(t)) + 25RE_1(t) + tE_2(t)\\ \text{and}\quad &\mathcal{D}(t) := 2(1+\|m_2\|_{L^\infty})D_0(t) + 2tD_1(t) + RD_1(t) + tD_2(t).\end{aligned}\tag{2.11}$$

Using a suitable combination of the above inequality with (0.1), (2.6) and the fact that

$$\frac{d}{dt}(tE_1) = t\frac{d}{dt}E_1 + D_0,$$

we discover that

$$\frac{d}{dt}\mathcal{E} + \mathcal{D} \leq C\bigl(U\mathcal{E} + \mathcal{D}\mathcal{E} + \|\rho\|_{L^\infty}^2 D_0\mathcal{E}^2\bigr)\quad\text{with}\quad U := \|\nabla u\|_{L^\infty}.\tag{2.12}$$

Consequently, as long as $CR^2\mathcal{E}\leq 1/2$, we just have

$$\frac{d}{dt}\mathcal{E} + \frac{1}{2}\mathcal{D} \leq C\bigl(U+D_0\bigr)\mathcal{E}.\tag{2.13}$$

The next step is to apply Gronwall lemma so as to get a control of $\mathcal{E}$ in terms of the initial data. The only issue is to control $t\|u_t\|_{L^2}^2$ near $t=0$ (note that for the heat equation, the

fact that $u_0 \in \dot{H}^1$ does not guarantee that $\|u_t\|_{L^2}$ is bounded for $t \to 0$). To do so, we observe that, according to (2.12), the function

$$\widetilde{\mathcal{E}}(t) := e^{-C\int_0^t (U+D_0)\,d\tau}\mathcal{E}(t)$$

is nonincreasing, hence has a limit in $\mathbb{R}^+ \cup \{+\infty\}$ at 0. Next, under our above smallness condition, Inequality (2.6) guarantees in particular that (say)

$$\int_0^t \|u_t\|_{L^2}^2 \leq 2E_{1,0}.$$

Hence, for all $t \in \mathbb{R}_+$, there exists $t_0 \in (0,t)$ such that $t_0\|u_t(t_0)\|_{L^2}^2 \leq 2E_{1,0}$. Remembering the definition of $\widetilde{\mathcal{E}}$ and of $\mathcal{E}$, one can thus conclude that one can find arbitrarily small positive t_0 for which

$$\mathcal{E}(t_0) \leq C\bigl(1 + \|\langle v\rangle^2 f_0\|_{L^1_v(\mathbb{R}^3;L^\infty_x(\mathbb{R}^3))}\bigr)\bigl(E_{0,0} + E_{1,0}\bigr)\cdot$$

Now, reverting to (2.13) and using Gronwall lemma and (0.1), we end up with

$$\mathcal{E}(t) + \int_0^t \mathcal{D}\,d\tau \leq C(1 + \|(\rho, m_2)\|_{L^\infty})\biggl(\|u_0\|_{H^1}^2 + \iint f_0|v|^2\,dv\,dx\biggr)e^{C\int_0^t \|\nabla u\|_{L^\infty}\,d\tau}e^{CE_{0,0}}.$$

Provided we have (A.3) the term $\|(\rho, m_2)\|_{L^\infty}$ may be bounded by means of (A.5) and of (A.6). In the end, we get an absolute constant c_0 such that, whenever

$$\bigl(1 + \|\langle v\rangle^2 f_0\|_{L^1_v(\mathbb{R}^3;L^\infty_x(\mathbb{R}^3))}\bigr)\bigl(\|u_0\|_{H^1}^2 + \iint f_0|v|^2\,dv\,dx\bigr) \leq c_0, \tag{2.14}$$

we have

$$\mathcal{E}(t) + \int_0^t \mathcal{D}\,d\tau \leq C\bigl(1 + \|\langle v\rangle^2 f_0\|_{L^1_v(\mathbb{R}^3;L^\infty_x(\mathbb{R}^3))}\bigr)\biggl(\|u_0\|_{H^1}^2 + \iint f_0|v|^2\,dv\,dx\biggr)e^{CE_{0,0}}. \tag{2.15}$$

3. Decay estimates and control of the Lipschitz norm of the velocity

The present section is devoted to proving optimal time decay for the energy functionals E_0 and E_1 associated to global solutions to (VNS) satisfying the smallness condition (A.3). Although our ultimate goal is to establish (1.5), we will first prove a non-optimal decay estimate for E_1 that is, nevertheless, sufficient to control the Brinkman force in $L^1(\mathbb{R}_+ \times \mathbb{R}^3)$. This latter control will enable us to bound the norm of u in $\dot{B}^{-3/2}_{2,\infty}$ uniformly in time, then to get (1.5) with the optimal exponents. In the last part of the section, we shall combine these estimates with a bootstrap argument so as to justify (A.3) in the case of small data.

Step 1: Propagation of negative regularity I. We plan to use the approach presented in Section 1 with the Lyapunov functional $\mathcal{L} = E_1$ and the dissipation rate $\mathcal{H} = D_1$. Having in mind the following classical interpolation inequality:

$$\|\nabla u\|_{L^2} \lesssim \|\nabla^2 u\|_{L^2}^{\theta}\|u\|_{\dot{B}^{-\sigma}_{2,\infty}}^{1-\theta}, \quad \sigma > -1, \quad \theta = \frac{\sigma+1}{\sigma+2}, \tag{3.1}$$

the 'low order' functional $\mathcal{N}$ with be a negative Besov norm of u that can be uniformly bounded in time.

According to (1.13), choosing $\sigma = 3/2$ would give the desired decay rate $5/2$. However, at this stage of the proof, we do not know whether the Brinkman force is sufficiently integrable with respect to time, so that we are not able to bound $\|u(t)\|_{\dot{B}^{-3/2}_{2,\infty}}$ uniformly in time.

To get round the difficulty, we consider the norm of u in $\dot B^{-1/2}_{2,\infty}$, which, in principle, should give the decay $t^{-3/2}$ for E_1. This turns out to be enough to bound the Brinkman force in $L^1(\mathbb{R}_+\times\mathbb{R}^3)$. Then, it will be possible to apply again the whole procedure with the optimal exponents.

This being said, we rewrite the velocity equation of (VNS) as

$$\partial_t u-\Delta u=-\mathcal{P}\mathrm{div}\,(u\cdot\nabla u)+\mathcal{P}\biggl(\int_{\mathbb{R}^3}f(v-u)\,dv\biggr)\cdot \tag{3.2}$$

Now, using the maximal regularity estimates stated in e.g. [4, (3.39)] and the fact that $\mathcal{P}$ is a continuous projector on any homogeneous Besov space, we get[3] for all $t\geq 0$,

$$\|u(t)\|_{\dot B^{-\frac12}_{2,\infty}}+\|u\|_{\widetilde L^2_t(\dot B^{\frac12}_{2,\infty})}\leq\|u_0\|_{\dot B^{-\frac12}_{2,\infty}}+\|u\cdot\nabla u\|_{\widetilde L^2_t(\dot B^{-\frac32}_{2,\infty})}+\biggl\|\int_{\mathbb{R}^3}f(v-u)\,dv\biggr\|_{\widetilde L^2_t(\dot B^{-\frac32}_{2,\infty})}\cdot$$

To bound the last term, we use that, according to Cauchy-Schwarz inequality, we have

$$\int\biggl|\int f(u-v)\,dv\biggr|dx\leq\biggl(\iint f\,dv\,dx\biggr)^{1/2}\biggl(\iint f|v-u|^2\,dv\,dx\biggr)^{1/2}\cdot \tag{3.3}$$

Hence, combining with the embedding

$$L^2(0,t;L^1(\mathbb{R}^3))\hookrightarrow L^2(0,t;\dot B^{-\frac32}_{2,\infty}(\mathbb{R}^3)), \tag{3.4}$$

we get

$$\begin{aligned}\biggl\|\int_{\mathbb{R}^3}f(v-u)\,dv\biggr\|_{\widetilde L^2_t(\dot B^{-\frac32}_{2,\infty})}&\lesssim\biggl\|\int_{\mathbb{R}^3}f(v-u)\,dv\biggr\|_{L^2_t(L^1)}\\&\lesssim\Bigl(\|f\|_{L^\infty(\mathbb{R}_+;L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v))}\|f|u-v|^2\|_{L^1(\mathbb{R}_+\times\mathbb{R}^3_x\times\mathbb{R}^3_v)}\Bigr)^{1/2}\cdot\end{aligned}$$

Using the relations (1.3) and (1.4), we end up with

$$\biggl\|\int_{\mathbb{R}^3}f(v-u)\,dv\biggr\|_{\widetilde L^2_t(\dot B^{-\frac32}_{2,\infty})}\leq C\sqrt{M_0E_{0,0}}.$$

Furthermore, combining (3.4), Cauchy-Schwarz inequality and (1.3) yields

$$\|u\cdot\nabla u\|_{\widetilde L^2_t(\dot B^{-\frac32}_{2,\infty})}\lesssim\|u\cdot\nabla u\|_{L^2_t(L^1)}\leq\|u\|_{L^\infty_t(L^2)}\|\nabla u\|_{L^2_t(L^2)}\leq\sqrt{2}\,E_{0,0}.$$

In conclusion, we have for all $t\geq 0$,

$$\|u\|_{L^\infty_t(\dot B^{-\frac12}_{2,\infty})}+\|u\|_{\widetilde L^2_t(\dot B^{\frac12}_{2,\infty})}\leq\|u_0\|_{\dot B^{-\frac12}_{2,\infty}}+C\sqrt{M_0E_{0,0}}+CE_{0,0}=:C_0. \tag{3.5}$$

[3]The reader is referred to [4, Def. 2.67] for the definition of norms $\|\cdot\|_{\widetilde L^r_t(\dot B^s_{2,\infty}(\mathbb{R}^3))}$. The only property that we need to know here is that $\|z\|_{\widetilde L^r_t(\dot B^s_{2,\infty}(\mathbb{R}^3))}\leq\|z\|_{L^r_t(\dot B^s_{2,\infty}(\mathbb{R}^3))}$ with equality if and only if $r=\infty$.

Step 2: A non optimal decay estimate for E_1. From (3.1) and (3.5), we infer that

$$\|\nabla^2 u\|_{L^2}^2 \gtrsim C_0^{-4/3}\big(\|\nabla u\|_{L^2}^2\big)^{5/3}.$$

Therefore, if we set

$$U := \|\nabla u\|_{L^\infty}, \quad R := \max(1, \|\rho\|_{L^\infty}), \quad X_1 := \|\nabla u\|_{L^2}^2 \quad \text{and} \quad Y_1 := \iint f|u-v|^2\,dx\,dv,$$

and use (2.6), we end up with

$$\frac{d}{dt}E_1 + 2c_0' X_1^{5/3} + Y_1 \leq CUE_1 + CR^3X_1^3 \quad \text{with} \quad c_0' := C_0^{-4/3}/8.$$

So, assuming the following a priori bound:

$$CR^3X_1^{4/3} \leq c_0', \tag{3.6}$$

we get

$$\frac{d}{dt}\widetilde{E}_1 + c_0'\widetilde{X}_1^{5/3} + \widetilde{Y}_1 \leq 0 \quad \text{with} \quad \widetilde{Z}(t) := e^{-C\int_0^t U(\tau)\,d\tau}Z(t) \quad \text{for} \quad Z \in \{E_1, X_1, Y_1\}. \tag{3.7}$$

Note that

$$\widetilde{Y}_1 \leq \widetilde{E}_1 \leq E_{1,0}.$$

Since we restricted our attention to the case of small data (meaning in particular that $E_{1,0}$ is small), one may assume with no loss of generality that $\widetilde{Y}_1 \geq c_0'\widetilde{Y}_1^{5/3}$, and thus Inequality (3.7) gives after a suitable harmless change of c_0':

$$\frac{d}{dt}\widetilde{E}_1 + c_0'\widetilde{E}_1^{5/3} \leq 0.$$

Hence, provided (A.3) holds, we have

$$E_1(t) \leq CE_{1,0}(1+a_0t)^{-3/2} \tag{3.8}$$

with a_0 depending only on the initial data, on M_0, $E_{0,0}$ and $\|u_0\|_{\dot{B}^{-1/2}_{2,\infty}}$.

From this inequality, one can point out more integrability of D_1. Indeed, assuming that

$$CR^3E_1 \leq 1, \tag{3.9}$$

Inequality (2.6) implies that

$$\frac{d}{dt}E_1 + D_1 \leq C(U+D_0)E_1. \tag{3.10}$$

Let us set for some $\beta \in (0, 3/2)$,

$$\check{E}_1(t) := (1+a_0t)^\beta e^{-C\int_0^t (U+D_0)\,d\tau}E_1(t) \quad \text{and} \quad \check{D}_1(t) := (1+a_0t)^\beta e^{-C\int_0^t (U+D_0)\,d\tau}D_1(t).$$

Then (3.10) implies that

$$\frac{d}{dt}\check{E}_1 + \check{D}_1 \leq \beta a_0(1+a_0t)^{\beta-1}\widetilde{E}_1.$$

Hence, using the bound (3.8) and observing that $t \mapsto (1+a_0t)^{\beta-5/2}$ is integrable on $\mathbb{R}_+$, we end up with[4]

$$\int_0^t (1+a_0\tau)^\beta \widetilde{D}_1(\tau)\,d\tau \leq \frac{2\beta}{3/2-\beta}\,E_{1,0}.$$

[4]Note that, if taking $\beta = 3/2$, then one gets $\displaystyle\int_0^t (1+a_0\tau)^{3/2}\widetilde{D}_1(\tau)\,d\tau \leq 2\big(1+\log(1+a_0t)\big)E_{1,0}$.

Therefore, assuming (A.3) and (3.9) gives

$$(3/2-\beta)\int_0^t (1+a_0\tau)^\beta D_1(\tau)\,d\tau \leq C\,e^{CE_{0,0}}\,E_{1,0}, \qquad \beta\in(0,3/2). \tag{3.11}$$

Step 3. Propagation of negative regularity II. Leveraging (3.11) with some $\beta\in(1,3/2)$ ensures that the Brinkman force belongs to $L^1(\mathbb{R}_+\times\mathbb{R}^3)$. Indeed, remembering (3.3) and using Cauchy-Schwarz inequality, one can write that

$$\int_{\mathbb{R}_+}\int\biggl|\int f(u-v)\,dv\biggr|dxdt \leq \sqrt{M_0}\int_{\mathbb{R}_+}\sqrt{(1+a_0t)^\beta\,D_1(t)}\,(1+a_0t)^{-\beta/2}\,dt \leq C_{0,\beta}. \tag{3.12}$$

This additional information will enable us to control $\|u(t)\|_{\dot B^{-\frac32}_{2,\infty}}$ since maximal regularity estimates for (3.2) also ensure that

$$\|u\|_{L^\infty_t(\dot B^{-\frac32}_{2,\infty})}+\|u\|_{\widetilde L^1_t(\dot B^{\frac12}_{2,\infty})}\leq\|u_0\|_{\dot B^{-\frac32}_{2,\infty}}+\|u\cdot\nabla u\|_{\widetilde L^1_t(\dot B^{-\frac32}_{2,\infty})}+\biggl\|\int_{\mathbb{R}^3}f(v-u)\,dv\biggr\|_{\widetilde L^1_t(\dot B^{-\frac32}_{2,\infty})}.$$

To bound the last term, we combine (3.12) with the embeddings

$$L^1(0,t\times\mathbb{R}^3)\hookrightarrow L^1(0,t;\dot B^{-\frac32}_{2,\infty}(\mathbb{R}^3))\hookrightarrow\widetilde L^1_t(\dot B^{-\frac32}_{2,\infty}(\mathbb{R}^3)).$$

For the convection term, we use

$$\|u\cdot\nabla u\|_{L^1_t(L^1)}\leq\|u\|_{L^2_t(L^2)}\|\nabla u\|_{L^2_t(L^2)}$$

and the interpolation inequality[5]

$$\|u\|_{L^2_t(L^2)}\lesssim\|u\|^{2/3}_{\widetilde L^2_t(\dot B^{-\frac12}_{2,\infty})}\|\nabla u\|^{1/3}_{L^2_t(L^2)}.$$

Combining with the inequality

$$\|u\|_{\widetilde L^2_t(\dot B^{-\frac12}_{2,\infty})}\leq\|u\|^{1/2}_{L^\infty_t(\dot B^{-\frac32}_{2,\infty})}\|u\|^{1/2}_{\widetilde L^1_t(\dot B^{\frac12}_{2,\infty})},$$

we discover that

$$\|u\cdot\nabla u\|_{L^1_t(L^1)}\lesssim\|\nabla u\|^{4/3}_{L^2_t(L^2)}\|u\|^{1/3}_{L^\infty_t(\dot B^{-\frac32}_{2,\infty})}\|u\|^{1/3}_{\widetilde L^1_t(\dot B^{\frac12}_{2,\infty})}.$$

Hence, owing to (0.1) and (3.12), we have

$$\|u\|_{L^\infty_t(\dot B^{-\frac32}_{2,\infty})\cap\widetilde L^1_t(\dot B^{\frac12}_{2,\infty})}\leq\|u_0\|_{\dot B^{-\frac32}_{2,\infty}}+CE^{2/3}_{0,0}\|u\|^{2/3}_{L^\infty_t(\dot B^{-\frac32}_{2,\infty})\cap\widetilde L^1_t(\dot B^{\frac12}_{2,\infty})}+C_0.$$

Consequently, using Young inequality and (0.1), one can conclude that there exists C_0' depending only on $E_{0,0}$, on $\|u_0\|_{\dot B^{-\frac32}_{2,\infty}}$ such that if $E_{1,0}$ is small enough and (A.3) is satisfied, then we have

$$\|u\|_{L^\infty_t(\dot B^{-\frac32}_{2,\infty})}+\|u\|_{\widetilde L^1_t(\dot B^{\frac12}_{2,\infty})}\leq C_0, \qquad t>0. \tag{3.13}$$

[5]The reader should not be intimidated by the use of tilde norms. The regularity exponent behaves as usual, and the Lebesgue exponent, according to Hölder inequality, see [4, Section 2.6.3].

Step 4. Optimal decay for the energy functionals. Since

$$\|u\|_{L^2} \lesssim \|\nabla u\|_{L^2}^{3/5} \|u\|_{\dot{B}^{-\frac{3}{2}}_{2,\infty}}^{2/5},$$

Inequality (3.13) ensures that

$$\|\nabla u\|_{L^2}^2 \geq C_0^{-4/3} \biggl(\frac{1}{2} \|u\|_{L^2}^2 \biggr)^{5/3}.$$

At this stage, one can mimic the proof of (3.8), (3.11) and eventually get for some positive numbers a_0 and K_0 depending only on suitable norms of the data that

$$\sup_{t\in\mathbb{R}_+} \bigl((1+a_0t)^{3/2} E_0(t)\bigr) + (3/2-\alpha) \int_{\mathbb{R}_+} (1+a_0t)^\alpha D_0(t)\,dt \leq K_0 E_{0,0}, \qquad \alpha \in (1,3/2). \tag{3.14}$$

One can also improve the decay of E_1 and D_1 by a factor 1 : it suffices to use (3.1) with $\sigma = 3/2$ which, owing to (3.13), gives

$$\|\nabla^2 u\|_{L^2}^2 \gtrsim C_0^{-4/5} \bigl(\|\nabla u\|_{L^2}^2 \bigr)^{7/5}.$$

Hence, we have for a new constant c_0 depending only on $\|u_0\|_{\dot{B}^{-3/2}_{2,\infty}}$, $E_{0,0}$ and $E_{1,0}$,

$$\frac{d}{dt} E_1 + 2c_0 X_1^{7/5} + Y_1 \leq CUE_1 + CR^3 X_1^3.$$

Defining $\widetilde{E}_1$, $\widetilde{X}_1$ and $\widetilde{Y}_1$ as before, we discover that whenever $CR^3 X_1^{8/5} \leq c_0$, we have

$$\frac{d}{dt} \widetilde{E}_1 + 2c_0 \widetilde{X}_1^{7/5} + \widetilde{Y}_1 \leq 0,$$

which leads for all $\beta \in (1,5/2)$ to

$$\sup_{t\in\mathbb{R}_+} \bigl((1+a_0t)^{5/2} E_1(t)\bigr) + (5/2-\beta) \int_{\mathbb{R}_+} (1+a_0t)^\beta D_1(t)\,dt \leq K_0(E_{0,0}+E_{1,0}). \tag{3.15}$$

Step 5: The Lipschitz bound. The Lipschitz bound will follow from the embedding:

$$\|z\|_{L^\infty} \lesssim \|\nabla z\|_{L^{3,1}}, \tag{3.16}$$

where the Lorentz space $L^{3,1}$ is defined by real interpolation as follows:

$$L^{3,1} := [L^2, L^\infty]_{1/3,1}. \tag{3.17}$$

Now, taking advantage of the regularity properties of the Stokes system (2.7) that are also valid in $L^{3,1}$ (just argue by interpolation from Lebesgue spaces), we can write that

$$\|\nabla^2 u\|_{L^{3,1}} \lesssim \|u_t\|_{L^{3,1}} + \|u\cdot\nabla u\|_{L^{3,1}} + \Bigl\| \int f(v-u)\,dv \Bigr\|_{L^{3,1}}.$$

Therefore we have

$$\int_0^\infty \|\nabla u\|_{L^\infty}\,dt \lesssim \int_0^\infty \biggl(\|u_t\|_{L^{3,1}} + \|u\cdot\nabla u\|_{L^{3,1}} + \Bigl\| \int f(v-u)\,dv \Bigr\|_{L^{3,1}} \biggr) dt. \tag{3.18}$$

On the one hand, since we also have $L^{3,1} = [L^2, L^6]_{1/2,1}$, one can write owing to $\dot{H}^1 \hookrightarrow L^6$ that

$$\|u_t\|_{L^{3,1}} \lesssim \|u_t\|_{L^2}^{1/2} \|\nabla u_t\|_{L^2}^{1/2}. \tag{3.19}$$

On the other hand, using Hölder inequality and (3.16), one can write that

$$\|u\cdot\nabla u\|_{L^{3,1}} \leq \|u\|_{L^\infty} \|\nabla u\|_{L^{3,1}} \lesssim \|\nabla u\|_{L^{3,1}}^2 \lesssim \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}.$$

Finally, we have by (3.17) and (2.3) that

$$\Big\|\int f(v-u)\,dv\Big\|_{L^{3,1}} \lesssim \Big\|\int f(v-u)\,dv\Big\|_{L^2}^{2/3}\Big\|\int f(v-u)\,dv\Big\|_{L^\infty}^{1/3}$$
$$\lesssim \|\rho\|_{L^\infty}^{1/3}\biggl(\iint f|v-u|^2\,dv\,dx\biggr)^{1/3}\bigl(\|j\|_{L^\infty}+\|\rho\|_{L^\infty}\|u\|_{L^\infty}\bigr)^{1/3}.$$

Hence, since

$$\|u\|_{L^\infty}\lesssim \|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2},$$

we obtain

$$\Big\|\int f(v-u)\,dv\Big\|_{L^{3,1}} \le \|\rho\|_{L^\infty}^{1/3}\biggl(\iint f|v-u|^2\,dv\,dx\biggr)^{1/3}\bigl(\|j\|_{L^\infty}^{1/3}+\|\rho\|_{L^\infty}^{1/3}\|\nabla u\|_{L^2}^{1/6}\|\nabla^2 u\|_{L^2}^{1/6}\bigr).$$

Therefore, reverting to (3.18) and remembering the definition of D_0, D_1 and D_2, we discover that we have for some constant C_R depending only on $\|\rho\|_{L^\infty}$ and $\|j\|_{L^\infty}$,

$$\int_0^\infty \|\nabla u\|_{L^\infty}\,dt \le C_R\int_0^\infty \bigl(D_1^{1/4}D_2^{1/4}+D_0^{1/2}D_1^{1/2}+D_1^{1/3}+D_0^{1/12}D_1^{5/12}\bigr)dt.$$

In order to show that the right-hand side is finite, we take α (resp. β) close enough to $3/2$ in (3.14) (resp. $5/2$ in (3.15)) and observe that the above inequality may be rewritten

$$\int_0^\infty \|\nabla u\|_{L^\infty}\,dt \le C_R\int_0^\infty \bigl(t^{-1/4}(1+a_0t)^{-\beta/4}((1+a_0t)^\beta D_1)^{1/4}(tD_2)^{1/4}+D_0^{1/2}D_1^{1/2}$$
$$+(1+a_0t)^{-\beta/3}((1+a_0t)^\beta D_1)^{1/3}+(1+a_0t)^{-\frac{\alpha+5\beta}{12}}((1+a_0t)^\alpha D_0)^{1/12}((1+a_0t)^\beta D_1)^{5/12}\bigr)dt.$$

Using Hölder inequality, Inequality (2.15) and the decay estimates (3.14) and (3.15), one can conclude that the right-hand side may be bounded in terms of the initial data whenever $\beta>2$ and $\alpha+5\beta>2$, which is indeed possible[6] for α and β close enough to $3/2$ and $5/2$, respectively. Hence

$$\int_0^\infty \|\nabla u\|_{L^\infty}\le C_0$$

with C_0 depending only on the norms of the initial data and tending to 0 when $E_{0,0}+E_{1,0}$ tends to 0. Let us underline that we do not need $\|u_0\|_{\dot B^{-3/2}_{2,\infty}}$ to be small.

4. Proving theorem 1.1

This section is devoted to the proof of our first main result. We shall first present a stability result that will entail the uniqueness in Theorem 1.1, and will be also used in the second part dedicated to the existence. The last part is devoted to the study of the large time behavior for the distribution function.

[6]In accordance with Remark 1.3, if assuming that $u_0\in\dot B^{-\sigma}_{2,\infty}$, the same approach would give $\alpha=\sigma$ and $\beta=\sigma+1$. This gives the restriction $\sigma>1$.

4.1. **A stability result.** Since the result we have in mind will be also useful to prove the strong convergence of the solutions to some approximate (VNS) system (see the next part), we shall here consider the following slightly more general system

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + \mathrm{div}_v\big(f(w-v)\big) = 0, \\ \partial_t u - \Delta u + w\cdot\nabla u + \displaystyle\int f(w-v)\,dv + \nabla P = S, \\ \mathrm{div}\, v = 0. \end{cases} \tag{4.1}$$

Both the transport field w and the source term S are given sufficiently smooth time-dependent vector-fields.

Our goal is to prove a stability estimate for (4.1), that implies a uniqueness result for (VNS) in the functional framework of Theorem 1.1. To do this, we shall adapt the approach of [9] . Let us underline that the proof here is simpler since the characteristics associated to the first equation of (4.1) are Lipschitz.

So, let us consider two solutions u_1 and u_2 of (4.1) associated with vector-fields w_1 and w_2, source terms S_1 and S_2, and initial velocities ($u_{0,1}$ and $u_{0,2}$). For simplicity, we shall assume that We shall allow for possibly different but will keep the initial distribution function f_0 is the same for the two solutions.

Let us set $\delta f := f_2 - f_1$, $\delta u := u_2 - u_1$, $\delta w := w_2 - w_1$ and $\delta S := S_2 - S_1$. We denote by ρ_i and $m_{2,i}$ the density (resp. energy) associated with f_i through (1.6). We also introduce the characteristics $Z_1 = (X_1, V_1)$ and $Z_2 = (X_2, V_2)$ associated with w_1 and w_2 respectively (instead of u), through System (A.1), and use the notation $Y_j(t,x,v) := Y_j(t;0,x,v)$ for $Y \in \{V, X, Z\}$. Finally, we set:

$$\delta X := X_2 - X_1, \quad \delta V := V_2 - V_1 \quad \text{and} \quad \delta Z := Z_2 - Z_1.$$

The system satisfied by δu reads:

$$\begin{cases} \partial_t \delta u + w_1\cdot\nabla\delta u - \Delta\delta u + \rho_2\delta w = \mathcal{P}\Big(\delta S - \delta w\cdot\nabla u_2 + \displaystyle\int \delta f(v-w_1)\,dv\Big), \\ \mathrm{div}\,\delta u = 0. \end{cases} \tag{4.2}$$

Taking the $L^2(\mathbb{R}^3;\mathbb{R}^3)$ scalar product with δu immediately gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\delta u\|_{L^2}^2 + \|\nabla\delta u\|_{L^2}^2 + \|\sqrt{\rho_2}\,\delta u\|_{L^2}^2 \\ = -\int (\delta w\cdot\nabla u_2)\cdot\delta u\,dx + \iint \delta f(v-w_1)\cdot\delta u\,dv\,dx + \int \delta u\cdot\delta S\,dx. \end{aligned} \tag{4.3}$$

For the first term of the right-hand side, combining Hölder, Sobolev and Young inequalities yields

$$-\int (\delta w\cdot\nabla u_2)\cdot\delta u\,dx \le \frac{1}{10}\|\nabla\delta u\|_{L^2}^2 + C\|\nabla u_2\|_{L^3}^2\|\delta w\|_{L^2}^2. \tag{4.4}$$

To handle the second term (let us call it δF), using Formula (A.2) for f_1 and f_2 gives

$$\delta F = e^{3t}\iint \big(f_0(Z_2^{-1}(t)) - f_0(Z_1^{-1}(t))\big)(v-w_1)\cdot\delta u\,dv\,dx. \tag{4.5}$$

Performing the changes of variables $(x',v') = Z_2^{-1}(t,x,v)$ and $(x',v') = Z_1^{-1}(t,x,v)$, respectively and using the properties of the flow that have been recalled in Appendix, we

obtain

$$\delta F = \iint \bigl(V_2(t) - w_1(t, X_2(t))\bigr) \cdot \delta u(t, X_2(t))\, f_0\, dv'\, dx' \\ - \iint \bigl(V_1(t) - w_1(t, X_1(t))\bigr) \cdot \delta u(t, X_1(t))\, f_0\, dv'\, dx'.$$

We further decompose δF into $A_1 + A_2 + A_3 + A_4$ with

$$A_1(t) := \iint f_0(x,v) \delta V(t,x,v) \cdot \delta u(t, X_2(t,x,v))\, dv\, dx$$
$$A_2(t) := \iint f_0(x,v) \bigl(w_1(t, X_1(t,x,v)) - w_1(t, X_2(t,x,v))\bigr) \cdot \delta u(t, X_1(t,x,v))\, dv\, dx$$
$$A_3(t) := \iint f_0(x,v) w_1(t, X_2(t,x,v)) \bigl(\delta u(t, X_1(t,x,v)) - \delta u(t, X_2(t,x,v))\bigr)\, dv\, dx$$
$$A_4(t) := \iint f_0(x,v) V_1(t,x,v) \cdot \bigl(\delta u(t, X_2(t,x,v)) - \delta u(t, X_1(t,x,v))\bigr)\, dv\, dx.$$

For bounding A_1, we use Young inequality:

$$A_1 \leq \frac{1}{2} \| \sqrt{f_0}\, \delta V \|_{L^2}^2 + \frac{1}{2} \int f_0(x,v) |\delta u(t, X_2(t,x,v))|^2\, dx\, dv.$$

From (A.2), we deduce that

$$\int f_0(x,v) |\delta u(t, X_2(t,x,v))|^2\, dx\, dv = \iint f_2(t,x,v) |\delta u(t,x)|^2\, dv\, dx = \| \sqrt{\rho_2} \delta u(t) \|_{L^2}^2.$$

Hence

$$A_1 \leq \frac{1}{2} \| \sqrt{f_0}\, \delta V \|_{L^2}^2 + \frac{1}{2} \| \sqrt{\rho_2} \delta u(t) \|_{L^2}^2. \tag{4.6}$$

Next, as ∇u_1 is bounded, using Cauchy-Schwarz inequality and a change of variable gives

$$\begin{aligned} A_2 &\leq \|\nabla w_1\|_{L^\infty} \iint f_0(x,v) |\delta X(t,x,v)| |\delta u(t, X_1(t,x,v))|\, dv dx \\ &\leq \|\nabla w_1\|_{L^\infty} \| \sqrt{f}_0 \delta X \|_{L^2} \biggl(\iint f_0(x,v) |\delta u(t, X_1(t,x,v))|^2\, dv dx \biggr)^{1/2} \\ &\leq \|\nabla w_1\|_{L^\infty} \| \sqrt{f}_0 \delta X \|_{L^2} \biggl(\iint f_1(t,x,v) |\delta u(t,x)|^2\, dv dx \biggr)^{1/2}. \end{aligned}$$

Hence we have

$$A_2 \leq \|\nabla w_1\|_{L^\infty} \| \sqrt{f}_0 \delta X \|_{L^2} \| \sqrt{\rho_1}\, \delta u \|_{L^2}. \tag{4.7}$$

In order to bound A_3 and A_4, we need to resort to the following inequality (see [17]) that involves Hardy's maximal function:

$$|z(y) - z(x)| \leq C |x-y| \bigl(M(\nabla z)(y) + M(\nabla z)(x) \bigr). \tag{4.8}$$

This enables us to write that

$$A_3 \lesssim \int f_0(x,v) |w_1(t, X_2(t,x,v))| \bigl(M(\nabla \delta u)(t, X_1(t,x,v)) \\ + M(\nabla \delta u)(t, X_2(t,x,v)) \bigr) \delta X(t,x,v)\, dx dv. \tag{4.9}$$

Now, from the usual change of variable, (A.2) and the continuity of the maximal function on L^2, we gather that for $j=1,2$,

$$\iint (M(\nabla\delta u)(X_j))^2\, f_0\, dvdx = \int (M(\nabla\delta u))^2 \Big(\int f_j\, dv\Big) dx \lesssim \|\rho_j\|_{L^\infty}\|\nabla\delta u\|_{L^2}^2.$$

Hence, reverting to (4.9) gives

$$A_3 \leq \frac{1}{10}\|\nabla\delta u\|_{L^2}^2 + C\|w_1\|_{L^\infty}^2\|(\rho_1,\rho_2)\|_{L^\infty}\|\sqrt{f_0}\delta X\|_{L^2}^2. \tag{4.10}$$

Term A_4 is the most involved. To handle it, we observe that $A_4 \leq A_{4,1}+A_{4,2}+A_{4,3}$ where

$$A_{4,3} = \iint f_0(x,v)\delta V(t,x,v)|\delta X(t,x,v)|M(\nabla\delta u)(t,X_2(t,x,v))\, dvdx$$

and, for $j=1,2$,

$$A_{4,j} = \iint f_0(x,v)|\delta X(t,x,v)||V_j(t,x,v)|M(\nabla\delta u)(t,X_j(t,x,v))\, dvdx.$$

By Cauchy-Schwarz inequality, change of variable, formula (A.2) and the continuity in L^2 of the maximal function, we have for $j=1,2$,

$$\begin{aligned} A_{4,j} &\leq \|\sqrt{f_0}\,\delta X\|_{L^2}\Big(\iint f_0|V_j|^2(M(\nabla\delta u)(X_j))^2\, dvdx\Big)^{1/2} \\ &\leq \|\sqrt{f_0}\,\delta X\|_{L^2}\|m_{2,j}\|_{L^\infty}^{1/2}\|M(\nabla\delta u)\|_{L^2} \\ &\leq \frac{1}{10}\|\nabla\delta u\|_{L^2}^2 + C\|m_{2,j}\|_{L^\infty}\|\sqrt{f_0}\,\delta X\|_{L^2}^2. \end{aligned} \tag{4.11}$$

Similarly,

$$A_{4,3} \leq \frac{1}{10}\|\nabla\delta u\|_{L^2}^2 + C\|\rho_2\|_{L^\infty}\|\delta V\|_{L^\infty}^2\|\sqrt{f_0}\,\delta X\|_{L^2}^2. \tag{4.12}$$

Putting together Inequalities (4.4), (4.6), (4.7), (4.10), (4.11) and (4.12) yields

$$\begin{aligned} \frac{d}{dt}\|\delta u\|_{L^2}^2 + \|\nabla\delta u\|_{L^2}^2 + \|\sqrt{\rho_2}\,\delta u\|_{L^2}^2 &\leq 2\int \delta S\cdot\delta u\, dx + \|\sqrt{f_0}\,\delta V\|_{L^2}^2 \\ &+ C\Big(\|\nabla u_2\|_{L^3}^2\|\delta w\|_{L^2}^2 + \|\rho_1\|_{L^\infty}^{1/2}\|\nabla w_1\|_{L^\infty}\|\sqrt{f_0}\,\delta X\|_{L^2}\|\delta u\|_{L^2} \\ &+ \big(\|(\rho_1,\rho_2)\|_{L^\infty}\|w_1\|_{L^\infty}^2 + \|(m_{2,1},m_{2,2})\|_{L^\infty} + \|\rho_2\|_{L^\infty}\|\delta V\|_{L^\infty}^2\big)\|\sqrt{f_0}\,\delta X\|_{L^2}^2\Big). \end{aligned} \tag{4.13}$$

We have to keep in mind that (A.8) implies that

$$\delta V(t,x,v) = \int_0^t e^{s-t}\big(w_2(s,X_2(s,x,v)) - w_1(s,X_1(s,x,v))\big)\, ds, \tag{4.14}$$

and thus

$$\|\delta V\|_{L^2(0,t;L^\infty)} \leq \|w_1\|_{L^2(0,t;L^\infty)} + \|w_2\|_{L^2(0,t;L^\infty)}. \tag{4.15}$$

To complete the proof, we have to look at the time evolution of $\|\sqrt{f_0}\,\delta Z(t)\|_{L^2(\mathbb{R}^3\times\mathbb{R}^3)}$. Using (A.1), we see that

$$\delta X_t = \delta V \quad\text{and}\quad \delta V_t = w_2(X_2) - w_1(X_1) - \delta V.$$

Hence

$$\frac12\frac{d}{dt}\|\sqrt{f_0}\,\delta Z\|_{L^2}^2+\|\sqrt{f_0}\,\delta V\|_{L^2}^2=\iint f_0\delta X\cdot\delta V\,dvdx+\iint f_0\big(w_2(X_2)-w_1(X_1)\big)\cdot\delta V\,dvdx$$
$$\leq\frac12\|\sqrt{f_0}\,\delta X\|_{L^2}^2+\frac12\|\sqrt{f_0}\,\delta V\|_{L^2}^2+B_1+B_2$$

with

$$B_1:=\iint f_0\big(w_1(X_2)-w_1(X_1)\big)\cdot\delta V\,dvdx\quad\text{and}\quad B_2:=\iint f_0\big(w_2(X_2)-w_1(X_2)\big)\cdot\delta V\,dvdx.$$

It is obvious that

$$B_1\leq\|\nabla w_1\|_{L^\infty}\iint f_0|\delta X||\delta V|\,dv\,dx, \tag{4.16}$$

$$B_2\leq\frac12\iint f_0|\delta V|^2+\frac12\iint f_0(x,v)|\delta w(t,X_2(t))|^2\,dvdx$$
$$\leq\frac12\iint f_0|\delta V|^2+\frac12\iint f_2(t,x,v)|\delta w(t)|^2\,dvdx. \tag{4.17}$$

Hence, we end up with

$$\frac{d}{dt}\|\sqrt{f_0}\,\delta Z\|_{L^2}^2\leq(1+\|\nabla w_1\|_{L^\infty})\|\sqrt{f_0}\,\delta Z\|_{L^2}^2+\|\sqrt{\rho_2}\,\delta w\|_{L^2}^2. \tag{4.18}$$

Putting this inequality together with (4.15) and (4.13) yields

$$\frac{d}{dt}\big(\|\delta u\|_{L^2}^2+\|\sqrt{f_0}\,\delta Z\|_{L^2}^2\big)+\|\nabla\delta u\|_{L^2}^2+\|\sqrt{\rho_2}\,\delta u\|_{L^2}^2\leq2\int\delta S\cdot\delta u\,dx+\|\sqrt{\rho_2}\,\delta w\|_{L^2}^2 \tag{4.19}$$
$$+C\bigg(\|\nabla u_2\|_{L^3}^2\|\delta u\|_{L^2}^2+\big((1+\|\rho_1\|_{L^\infty}^{1/2})\|\nabla w_1\|_{L^\infty}+1\big)\big(\|\delta u\|_{L^2}^2+\|\sqrt{f_0}\,\delta Z\|_{L^2}^2\big)$$
$$+\big(\|(\rho_1,\rho_2)\|_{L^\infty}\|w_1\|_{L^\infty}^2+\|(m_{2,1},m_{2,2})\|_{L^\infty}+\|\rho_2\|_{L^\infty}\|(w_1,w_2)\|_{L^2(0,t;L^\infty)}\big)\|\sqrt{f_0}\,\delta X\|_{L^2}^2\bigg).$$

By Gronwall lemma the above inequality readily implies the uniqueness part of Theorem 1.1 (take $S_1=S_2=0$, $w_1=u_1$ and $w_2=u_2$). Indeed, we then have ∇u_1 in $L^1_{loc}(\mathbb{R}_+;L^\infty)$, ∇u_2 in $L^2(\mathbb{R}_+;L^3)$, u_1,u_2 belong to $L^2_{loc}(\mathbb{R}_+;L^\infty)$ and $\rho_1,\rho_2,m_{2,1},m_{2,2}$ are in $L^\infty_{loc}(\mathbb{R}_+;L^\infty)$.

4.2. **The proof of existence.** We consider the following Friedrichs approximation of the Vlasov-Navier-Stokes system for all $n\in\mathbb{N}$:

$$(VNS_n)\qquad\begin{cases}\partial_t f+v\cdot\nabla_x f+\operatorname{div}_v\big(f(J_nu-v)\big)=0,\\ \partial_t u+J_n(u\cdot\nabla_x u)-\Delta_x u=J_n\bigg(\displaystyle\int_{\mathbb{R}^3}f(v-u)\,dv\bigg),\end{cases}$$

where the spectral orthogonal projector J_n of $L^2(\mathbb{R}^3;\mathbb{R}^3)$ is defined by $J_n=\mathcal{P}1_{B(0,n)}(D)$. Note that the range of J_n is the space

$$L^2_n:=\big\{z\in L^2(\mathbb{R}^3;\mathbb{R}^3),\ \operatorname{div}z=0\quad\text{and}\quad\operatorname{Supp}\widehat{z}\subset\bar B(0,n)\big\}.$$

Let us admit temporarily that System (VNS_n) supplemented with initial data (f_0,J_nu_0) has a unique solution (f^n,u^n) on some time interval $[0,T]$ such that f^n satisfies the properties stated in Theorem 1.1 and u^n belongs to the space

$$E^n_T:=\mathcal{C}([0,T];L^2_n).$$

Owing to the spectral localization, the vector-field u^n belongs to all Sobolev spaces and since $J_n u^n = u^n$ and J_n is a L^2 orthogonal projector, all the energy estimates of Sections 2 are valid for (f^n, u^n). In fact, from (2.15) and a bootstrap argument, we see that provided we have (A.3) on $[0,T]$ and the smallness condition (1.2) is satisfied, then

$$\mathcal{E}^n(t) + \int_0^t \mathcal{D}^n \, d\tau \leq Ce^{CE_{0,0}} \biggl(\bigl(1 + \|\langle v\rangle^2 f_0\|_{L^1_v(\mathbb{R}^3; L^\infty_x(\mathbb{R}^3))}\bigr) \biggl(\|J_n u_0\|_{H^1}^2 + \iint f_0 |v|^2 \, dv \, dx \biggr) \biggr) \tag{4.20}$$

where the functionals $\mathcal{D}^n$ and $\mathcal{E}^n$ corresponding to (f^n, u^n) are defined as in (2.11).

As J_n is also continuous on all Besov spaces $\dot{B}^{-\sigma}_{2,\infty}$, one can repeat the computations of Section 3 and eventually get the key Lipschitz control (A.3). This allows to continue the solution (f^n, u^n) for all positive time.

In short, we have constructed a sequence $(f^n, u^n)_{n\in\mathbb{N}}$ of global approximate solutions, that satisfy (4.20) and all the estimates of Sections 2 and 3, uniformly with respect to n (since the right-hand side may be bounded independently of n). Furthermore, the distribution f^n satisfies the estimates (A.2) to (A.5), in particular

$$\|f^n(t)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} \leq e^{3t} \|f_0\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} \quad \text{for all} \quad t \geq 0. \tag{4.21}$$

From Inequalities (4.20) and (4.21), we deduce that, up to subsequence,

$$f^n \rightharpoonup f \quad \text{in} \quad L^\infty \quad \text{and} \quad u^n \rightharpoonup u \quad \text{in} \quad L^\infty(\mathbb{R}_+; H^1) \cap L^2_{loc}(\mathbb{R}_+; H^2) \qquad \text{weak} \ * \, .$$

We claim that $(u^n)_{n\in\mathbb{N}}$ converges *strongly* to u in the energy space. In fact, we are going to show that it is a Cauchy sequence in $\mathcal{C}([0,T]; L^2) \cap L^2(0,T; H^1)$ for all $T > 0$. To prove it, we observe that for all $n \in \mathbb{N}$, the velocity field u^n satisfies the second equation of (VNS) with source term

$$S^n := (\mathcal{P} - J_n)(u^n \cdot \nabla u^n) + (\mathcal{P} - J_n) \int f^n (u^n - v) \, dv.$$

Now, owing to the spectral cut-off, Inequality (2.3), Hölder and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} \|S^n\|_{\dot{H}^{-1}} &\leq n^{-1} \biggl(\|u^n \cdot \nabla u^n\|_{L^2} + \biggl\| \int f^n (u^n - v) \, dv \biggr\|_{L^2} \biggr) \\ &\leq Cn^{-1} \biggl(\|\nabla u^n\|_{L^2}^{3/2} \|\nabla^2 u^n\|_{L^2}^{1/2} + \|\rho^n\|_{L^\infty}^{1/2} \biggl(\iint f^n |u^n - v|^2 \, dv \, dx \biggr)^{1/2} \biggr). \end{aligned}$$

Hence, introducing the functionals E_0^n, D_0^n, E_1^n and D_1^n like in (0.1), (0.2) and (2.5), we discover that for all $t \geq 0$,

$$\int_0^t \|S^n\|_{\dot{H}^{-1}}^2 \, d\tau \leq Cn^{-2} \biggl(\int_0^t \bigl(D_1^n + D_0^n (E_1^n)^2 + \|f_0\|_{L^1(\mathbb{R}^3_v; L^\infty(\mathbb{R}^3_x))} D_0^n \bigr) d\tau \biggr).$$

Note also that

$$\|J_n u_0 - J_m u_0\|_{L^2} \leq n^{-1} \|u_0\|_{\dot{H}^1} \quad \text{for } m > n > 0.$$

Therefore, taking advantage of Inequalities (4.19) and (4.20) and of Gronwall lemma, one can conclude that for all $n \in \mathbb{N}$ and $m > n$, we have for some increasing function C_0 depending

on the data, but *independent of* n,

$$\|(u^m-u^n)(t)\|_{L^2}^2+\int_0^t\|\nabla(u^m-u^n)\|_{L^2}^2\,d\tau\leq C_0(t)\,n^{-2}.$$

This completes the proof of our claim.

Since the convergence is strong in the energy space, it is now easy to conclude that (f,u) satisfies (VNS) in the sense of distributions. Furthermore, since for all $n\in\mathbb{N}$, the approximate solution (f^n,u^n) satisfies the energy balance

$$\frac12\biggl(\|u^n(t)\|_{L^2}^2+\iint f^n(t)|v|^2\,dv\,dx\biggr)+\int_0^t\biggl(\|\nabla u^n\|_{L^2}^2+\iint f|v-u^n|^2\,dv\,dx\biggr)d\tau \\ =\frac12\biggl(\|u_0^n\|_{L^2}^2+\int\!\!f_0|v|^2\,dv\,dx\biggr),$$

the strong convergence guarantees that (f,u) satisfies (1.3).

Finally, that $u\in\mathcal{C}(\mathbb{R}_+;H^1)$ stems from the fact that u satisfies a heat equation with initial data in H^1 and right-hand side in $L^2_{loc}(\mathbb{R}_+;L^2)$.

For completeness, let us explain how to solve (VNS_n) locally in time for fixed $n\in\mathbb{N}$ and small enough $T>0$. Toward this, we consider the map Φ defined on E_T^n by $\Phi(w)=u$, where u stands for the solution of the *linear* parabolic equation

$$\partial_t u-\Delta u=J_n\biggl(\int f(v-w)\,dv-w\cdot\nabla u\biggr),\qquad u|_{t=0}=J_nu_0, \tag{4.22}$$

and f stands for the unique bounded solution of the (linear) transport equation :

$$\partial_t f+v\cdot\nabla_x f+\mathrm{div}_v\bigl(f(w-v)\bigr)=0,\qquad f|_{t=0}=f_0. \tag{4.23}$$

To implement the fixed point theorem, it will be convenient to endow E_T^n with the following norm :

$$\|z\|_{E_T^n}:=\biggl(\sup_{t\in[0,T]}\|z(t)\|_{L^2}^2+\int_0^T\|\nabla z\|_{L^2}^2\,dt\biggr)^{1/2}.$$

Since $w\in E_T^n$, we actually have $w\in\mathcal{C}([0,T];C^{1,0})$ owing to the spectral localization, which guarantees the existence and uniqueness of a solution to (4.23), that can be computed by the method of characteristics (see (A.2)).

We claim that if T is small enough, then Φ has a fixed point in E_T^n. Indeed, by a method similar to that used for establishing (0.1), we readily get

$$\frac12\frac{d}{dt}\biggl(\|u\|_{L^2}^2+\iint f|v|^2\,dv\,dx\biggr)+\|\nabla u\|_{L^2}^2+\iint f|v-u|^2\,dv\,dx=\iint f(w-u)\cdot(v-u)\,dv\,dx.$$

Combining Hölder, Sobolev and Young inequalities gives

$$\iint f(w-u)\cdot(v-u)\,dv\,dx\leq\iint f|v-u|^2\,dv\,dx+\frac14\|\rho\|_{L^\infty}\|w-u\|_{L^2}^2$$

where ρ denotes the mass associated to f through (1.6).

To continue the proof, we assume that

$$\int_0^T\|\nabla w\|_{L^\infty}\,d\tau\leq\delta, \tag{4.24}$$

where δ is chosen as in (A.3). This ensures that

$$\|\rho\|_{L^\infty(0,T;L^\infty)} \leq 2\|f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))}.$$

Then, the above inequality combined with Gronwall's lemma gives for all $t \in [0,T]$,

$$\begin{aligned}(4.25)\quad &\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u\|_{L^2}^2\,d\tau \\ &\leq \biggl(\|u_0\|_{L^2}^2 + \iint f_0|v|^2\,dv\,dx + t\|f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))}\|w\|_{L^\infty(0,t:L^2)}^2\biggr) e^{t\|f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))}^2}.\end{aligned}$$

Let us assume that w belongs to the closed ball $\bar{B}_{E_T^n}(0,M)$ of E_T^n with M defined by

$$M^2 = 2\biggl(1 + \|u_0\|_{L^2}^2 + \iint f_0|v|^2\,dv\,dx\biggr).$$

The spectral localization of w guarantees that

$$\|\nabla w\|_{L^\infty} \leq C\|\nabla w\|_{\dot{H}^1}^{1/2}\|\nabla w\|_{\dot{H}^2}^{1/2} \leq Cn^{3/2}\|\nabla w\|_{L^2},$$

whence

$$\int_0^T \|\nabla w\|_{L^\infty} \leq Cn^{3/2}T^{1/2}\|w\|_{E_T^n}.$$

Since we want (4.24) to be satisfied, we conclude that if T is such that

$$(4.26)\quad Cn^{3/2}T^{1/2}M \leq \delta, \quad T\|f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))}^2 \leq \log 2 \quad\text{and}\quad \|f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))}TM^2 \leq 1,$$

then (4.25) ensures that u is in $\bar{B}_{E_T^n}(0,M)$, too.

In order to complete the proof of the existence of a fixed point in $\bar{B}_{E_T^n}(0,M)$, it suffices to show that Φ is a contracting mapping on this ball. To do this, we consider two elements w_1 and w_2 of $\bar{B}_{E_T^n}(0,M)$ with T satisfying (4.26). Then, both $u_1 := \Phi(w_1)$ and $u_2 := \Phi(w_2)$ are in $\bar{B}_{E_T^n}(0,M)$. Let us denote by f_i the distribution associated to w_i through the transport equation (4.23). In order to get the desired property of contraction, one can adapt the method leading to the stability estimate (4.19), to the following system verified by (f_i,u_i) with $i=1,2$:

$$\begin{cases}\partial_t u_i - \Delta u_i + J_n\biggl(w_i\cdot\nabla u_i + \displaystyle\int f_i(w_i-v)\,dv\biggr) = 0,\\ \partial_t f_i + v\cdot\nabla_x f_i + \mathrm{div}_v\bigl(f_i(w_i-v)\bigr) = 0.\end{cases}$$

Note that the only difference compared to System (4.1) is the orthogonal projector J_n in the first equation. However, J_n restricted to L^2_n is just the identity so that it has no effect in the energy estimates, and (4.19) still holds. Now, remembering that (4.24) and (4.26) are satisfied, and taking advantage of the control on ρ_i and $m_{2,i}$ provided by (A.5) and (A.6), we get

$$\frac{d}{dt}\delta\mathcal{U} + \delta H \leq C\bigl(\|\nabla u_2\|_{L^3}^2 + \|\rho_2\|_{L^\infty}\bigr)\delta W + C_{f_0}\bigl(1 + \|w_1\|_{L^\infty}^2 + \|\nabla w_1\|_{L^\infty}\bigr)\delta\mathcal{U},$$

with C_{f_0} depending only on $\|f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))}$,

$$\delta\mathcal{U}(t) := \|\delta u(t)\|_{L^2}^2 + \|\sqrt{f_0}\delta Z(t)\|_{L^2}, \quad \delta W(t) := \|\delta w(t)\|_{L^2}^2 \quad\text{and}\quad \delta H(t) := \|\nabla\delta u(t)\|_{L^2}^2.$$

Hence, using Gronwall lemma gives

$$\delta\mathcal{U}(t) + \int_0^t \delta H\,d\tau \leq C\int_0^t e^{C_{f_0}\int_\tau^t(1+\|w_1\|_{L^\infty}^2+\|\nabla w_1\|_{L^\infty})\,d\tau'}\bigl(C_{f_0} + \|\nabla u_2\|_{L^3}^2\bigr)\delta W\,d\tau.$$

Recall that w_1 satisfies (4.24). Furthermore, the spectral localization of w_1 and u_2 ensures that

$$\|w_1\|_{L^\infty} \leq Cn^{1/2}\|w_1\|_{L^2} \quad \text{and} \quad \|\nabla u_2\|_{L^3}^2 \leq Cn^3\|u_2\|_{L^2}^2.$$

Hence, for small enough t, we have

$$\|\delta u\|_{L^\infty(0,t;L^2)}^2 + \int_0^t \|\nabla \delta u\|_{L^2}^2\,d\tau \leq 2Ct(C_{f_0}+n^3M)\|\delta w\|_{L^\infty(0,t;L^2)}^2,$$

which implies that the map Φ is contractive in $\bar{B}_{E_T^n}(0,M)$, if T is small enough.

The Banach fixed point theorem allows to conclude that Φ admits a (unique) fixed point in $\bar{B}_{E_T^n}(0,M)$. □

4.3. **Convergence of the distribution of particles.** Here we prove Corollary 1.4. Having the time decay of E_1 at hand, establishing the convergence of the distribution function f is a tiny modification of [10]. Indeed, by definition of $\rho(t)$ and of the Wasserstein distance W_1 between $f(t)$ and $\rho(t)\otimes\delta_{u(t)}$ and of $\rho(t)$, we have

$$W_1(f(t),\rho(t)\otimes\delta_{u(t)}) = \sup_{\|\nabla_{x,v}\phi\|_{L^\infty}=1} \iint f(t,x,v)\big(\phi(x,v)-\phi(x,u(t,x))\big)dv\,dx.$$

Using the definition of the Lipschitz norm, then Cauchy-Schwarz inequality and (1.4), one can write for all function ϕ with Lipschitz norm equal to 1,

$$\begin{aligned}\iint f(t,x,v)\big(\phi(x,v)-\phi(x,u(t,x))\big)dx\,dv &\leq \iint f(t,x,v)|v-u(t,x)|\,dv\,dx\\ &\leq \left(\int\rho(t)\,dx\iint f(t)|v-u(t)|^2\,dv\,dx\right)^{1/2}\\ &\leq \sqrt{M_0\,E_1(t)}.\end{aligned}$$

Keeping (1.5) in mind completes the proof of the decay for the Wasserstein distance.

We also have (1.7) since, by definition of ρ and of j,

$$\begin{aligned}\|j(t)-(\rho u)(t)\|_{L^1} &\leq \iint f(t)|v-u(t)|\,dv\,dx\\ &\leq \sqrt{M_0\,E_1(t)}.\end{aligned}$$

Finally, integrating the first equation of (VNS) with respect to the v variable gives

$$\partial_t\rho + \operatorname{div} j = 0,$$

and thus

$$\rho(t) = \rho_0 - \operatorname{div}\left(\int_0^t j\,d\tau\right). \tag{4.27}$$

Now, we have $j=(j-\rho u)+\rho u$ with $j-\rho u$ satisfying (1.7). Hence $(j-\rho u)$ is in $L^1(\mathbb{R}_+;L^1)$. Furthermore, by Hölder and Sobolev inequality, and using (1.4), we have

$$\|(\rho u)(t)\|_{L^1} \leq \|\rho(t)\|_{L^{6/5}}\|u(t)\|_{L^6} \leq CM_0^{5/6}\|\rho(t)\|_{L^\infty}^{1/6}\|\nabla u(t)\|_{L^2}.$$

From Theorem 1.1, the density ρ is uniformly bounded on $\mathbb{R}_+$, and $\|\nabla u(t)\|_{L^2}$ decays according to (1.5). Hence ρu is also in $L^1(\mathbb{R}_+;L^1)$. One can thus set

$$j_\infty := \int_0^\infty j\,d\tau \quad \text{and} \quad \rho_\infty := \rho_0 - \operatorname{div} j_\infty$$

so that (4.27) implies that

$$\rho_\infty - \rho(t) = -\mathrm{div}\left(\int_t^\infty (j-\rho u)\,d\tau + \int_t^\infty \rho u\,d\tau\right).$$

Taking advantage of (1.5) and of the above bounds for $j-\rho u$ and ρu, one readily gets (1.8).

5. The Fujita-Kato theorem for the Vlasov-Navier-Stokes system

This section is devoted to the proof of Theorem 1.6, that is, to the extension of Theorem 1.1 to the case where u_0 has only critical Sobolev regularity $H^{1/2}$.

The key step is to establish that if $\|u_0\|_{\dot H^{1/2}}$ and E_0 are small enough, then $u(t)$ remains in $\dot H^{1/2}$ until time $t=1$ and possesses the log Lipschitz regularity that is needed for getting uniqueness *within the class of Fujita-Kato solutions.* All these properties will be achieved by a bootstrap argument, assuming beforehand that $\|u\|_{L^1(0,1;L^\infty)}$ is small enough. As a result, we shall obtain some $t_0\in(0,1)$ for which the smallness condition (1.2) is satisfied by $(f(t_0),u(t_0))$. As proving the propagation of the regularity $\dot B^{-3/2}_{2,\infty}$ is not an issue, all the hypotheses of Theorem 1.1 are satisfied at time t_0, which allows to construct a solution on the time interval $[t_0,\infty)$ with the properties stated therein (in particular the optimal decay rates for E_0 and E_1). The uniqueness result that will be proved below ensures that the two solutions coincide on $[t_0,1]$, which will yield the full statement of Theorem 1.6.

5.1. **The incompressible Navier-Stokes equations with source term.** In order to perform the key step, we consider the following Navier-Stokes equation with source term:

$$\left\{\begin{array}{l} \partial_t u + u\cdot\nabla u - \Delta u + \nabla P = S \in L^{4/3}(0,T;L^2(\mathbb{R}^3)),\\ \mathrm{div}\,u = 0,\\ u|_{t=0} = u_0 \in \dot H^{1/2}. \end{array}\right. \tag{5.1}$$

We denote by u_L the solution to the linearized equation, namely $u_L := u_L^1 + u_L^2$ with

$$u_L^1(t) := e^{t\Delta}u_0 \quad\text{and}\quad u_L^2(t) := \int_0^t e^{(t-\tau)\Delta}\mathcal{P}S(\tau)\,d\tau \tag{5.2}$$

where $(e^{t\Delta})_{t>0}$ stands for the heat semi-group and $\mathcal{P}$ for the orthogonal projector on divergence free vector-fields.

Proposition 5.1. *The solution u_L^1 to the free heat equation with initial data u_0 belongs to*

$$\mathcal{C}(\mathbb{R}_+;\dot H^{1/2})\cap L^2(0,T;\dot B^{3/2}_{2,1})\cap \widetilde{L}^1(\mathbb{R}_+;\dot H^{5/2})$$

where $\widetilde{L}^1(\mathbb{R}_+;\dot H^{5/2})$ stands for the superspace of $L^1(\mathbb{R}_+;\dot H^{5/2})$ that has been defined in [6, Def. 2.67]*, and we have*

$$\|u_L^1\|_{L^\infty(\mathbb{R}_+;\dot H^{1/2})} + \|u_L\|_{L^2(0,T;\dot B^{3/2}_{2,1})} + \|u_L^1\|_{\widetilde{L}^1(\mathbb{R}_+;\dot H^{5/2})} \leq C\|u_0\|_{\dot H^{1/2}}. \tag{5.3}$$

The solution u_L^2 to the heat equation with source term S belongs to

$$\mathcal{C}([0,T];\dot H^{1/2})\cap L^2(0,T;\dot B^{3/2}_{2,1})\cap L^{4/3}(0,T;\dot H^2),$$

and satisfies

$$\|u_L^2\|_{L^\infty(0,T;\dot H^{1/2})} + \|u_L^2\|_{L^2(0,T;\dot B^{3/2}_{2,1})} + \|u_L^2\|_{L^{4/3}(\mathbb{R}_+;\dot H^2)} \leq C\|S\|_{L^{4/3}(0,T;L^2)}. \tag{5.4}$$

Finally, there exists a universal constant c such that if

$$\|u_0\|_{\dot H^{1/2}} + \|S\|_{L^{4/3}(0,T;L^2)} \leq c, \tag{5.5}$$

then (5.1) *admits a unique solution u on $[0,T]$ with*

$$\widetilde u := u - u^1_L - u^2_L \quad in \quad \mathcal{C}([0,T];\dot B^{1/2}_{2,1}) \cap L^2(0,T;\dot B^{3/2}_{2,1}) \cap L^1(0,T;\dot B^{5/2}_{2,1}),$$

and we have

$$\|\widetilde u\|_{L^\infty(0,T;\dot B^{1/2}_{2,1})} + \|\widetilde u\|_{L^2(0,T;\dot B^{3/2}_{2,1})} + \|\widetilde u\|_{L^1(0,T;\dot B^{5/2}_{2,1})} \leq C\big(\|u_0\|^2_{\dot H^{1/2}} + \|S\|^2_{L^{4/3}(0,T;L^2)}\big)\cdot \tag{5.6}$$

Proof. The first two parts of the proposition follow from standard properties of the heat flow (see e.g. [4, Chap. 3]) as regards the fact that the linear solution belongs to the two extremal spaces, and the observation by D. Arsénio in [2] and D. Arsenio and H. Houamed [3] that the solution actually belongs to $L^2(0,T;\dot B^{3/2}_{2,1})$ instead of just $L^2(0,T;H^{3/2})$. To prove the last part of the statement, we observe that $\widetilde u$ satisfies

$$\partial_t \widetilde u - \Delta \widetilde u + \nabla P = -u\cdot\nabla u, \quad \operatorname{div}\widetilde u = 0 \quad \text{and} \quad \widetilde u|_{t=0} = 0.$$

Hence, using again parabolic maximal regularity in Besov spaces (see [4, Chap. 3]) yields for all $t\in[0,T]$,

$$\|\widetilde u\|_{L^\infty(0,t;\dot B^{1/2}_{2,1})} + \|\widetilde u\|_{L^2(0,t;\dot B^{3/2}_{2,1})} + \|\widetilde u\|_{L^1(0,t;\dot B^{5/2}_{2,1})} \leq C\|u\cdot\nabla u\|_{L^1(0,t;\dot B^{1/2}_{2,1})}.$$

Now, taking advantage of the product law $\dot B^{1/2}_{2,1}\times\dot B^{3/2}_{2,1}\to\dot B^{1/2}_{2,1}$ and of Cauchy-Schwarz inequality with respect to the time variable, we obtain

$$\begin{aligned}\|\widetilde u\|_{L^\infty(0,t;\dot B^{1/2}_{2,1})} + \|\widetilde u\|_{L^2(0,t;\dot B^{3/2}_{2,1})} + \|\widetilde u\|_{L^1(0,t;\dot B^{5/2}_{2,1})}&\\ \leq C\big(\|u^1_L\|^2_{L^2(0,t;\dot B^{3/2}_{2,1})} + \|u^2_L\|^2_{L^2(0,t;\dot B^{3/2}_{2,1})} + \|\widetilde u\|^2_{L^2(0,t;\dot B^{3/2}_{2,1})}\big)&\cdot\end{aligned}$$

Provided the smallness condition (5.5) is satisfied, it is easy to work out from this inequality a fixed point scheme so as to construct a solution u of (5.1) with the desired properties. □

5.2. **Controlling the solution on an interval of size** 1**.** We aim at applying Proposition 5.1 with $S = j - \rho u$ and $T = 1$. As a first, let us observe that owing to Cauchy-Schwarz inequality, we have

$$\|j-\rho u\|^2_{L^2(0,T\times\mathbb{R}^3)} \leq \|\rho\|_{L^\infty(0,T\times\mathbb{R}^3)}\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\rho|v-u|^2\,dv\,dx\,dt.$$

Hence, taking advantage of Identity (1.3) and of Inequality (A.9), we discover that, provided $q>3$, we have for some constant C_q depending only on q,

$$\|j-\rho u\|_{L^2(0,T\times\mathbb{R}^3)} \leq C_q e^{3T/2}\big(1+\|u\|^{q/2}_{L^1(0,T;L^\infty)}\big)N_q^{1/2}E_0^{1/2}. \tag{5.7}$$

Let us take $T=1$. Then, remembering (5.3) and (5.5), using the embedding $\dot B^{3/2}_{2,1}\hookrightarrow L^\infty$ and Hölder inequality (to bound S suitably), one can find an absolute positive constant c such that if

$$\|u_0\|_{\dot H^{1/2}} \leq c \quad \text{and} \quad \|j-\rho u\|_{L^2(0,1\times\mathbb{R}^3)} \leq c, \tag{5.8}$$

then we have in particular

$$\|u\|_{L^1(0,1;L^\infty)} \leq C\|u\|_{L^2(0,1;\dot B^{3/2}_{2,1})} \leq Cc. \tag{5.9}$$

Hence, assuming with no loss of generality that $Cc \leq 1$ and using (5.7) with $T = 1$, we gather that the second part of (5.8) is satisfied provided that

$$2e^{3/2} C_q N_q^{1/2} E_0^{1/2} \leq c.$$

Using a bootstrap argument we conclude that if

$$\|u_0\|_{\dot{H}^{1/2}} < c \quad \text{and} \quad N_q E_0 < c^2/(4C_q^2 e^3)$$

then (VNS) supplemented with initial data u_0 and f_0 admits a solution on the time interval $[0,1]$ that satisfies the basic identity balance (1.3) and all the properties stated in Proposition 5.1 with $T = 1$. Taking c smaller if needed, one can ensure the Lipschitz smallness condition (A.3) on $[0,1]$ so that we have for all $t \in [0,1]$,

$$\|f(t)\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))} + \||v|^2 f(t)\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))} \lesssim \|f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))} + \||v|^2 f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))}.$$

Note that Proposition 5.1 combined with interpolation also guarantees that

$$\|u\|_{L^4(0,1;\dot{H}^1)} \leq Cc$$

and since $\|u\|_{L^\infty(0,1;L^2)}$ is small (as $E_{0,0}$ is small), there exists[7] some $t_0 \in (0,1)$ such that $\|u(t_0)\|_{\dot{H}^1} \leq 2Cc$. Consequently, taking c smaller if needed, one can apply Theorem 1.1 with initial data $f(t_0)$ and $u(t_0)$. Combining with the uniqueness result that will be proved in the next part, we conclude that it is possible to continue our solution beyond $t = t_0$ into a global-in-time H^1 type solution, which satisfies the regularity and time decay properties stated in Theorem 1.1. This completes the global existence part of Theorem 1.6, with optimal time decay.

5.3. **Uniqueness.** Compared to Theorem 1.1, the difficulty is that the solution (f,u,P) of Theorem 1.6 does not need to satisfy $\nabla u \in L^1_{loc}(\mathbb{R}_+;L^\infty)$ in the case where u_0 is in $\dot{H}^{1/2}$ for the simple reason that $e^{t\Delta}u_0$ does not satisfy this property. We claim however that for all $T > 0$ and $\eta \in (0,1/2)$, we have

$$\int_0^T \|u\|_{C_{\widetilde{\omega}_\eta}}\, dt \quad \text{with} \quad \widetilde{\omega}_\eta := r(1-\log r)^{1-\eta}, \tag{5.10}$$

where, for all continuous increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = 0$, we have used the notation

$$\|f\|_{C_\omega} := \sup_{x \neq y} \frac{|f(y) - f(x)|}{\omega(|y-x|)}.$$

Following the proof of the 2D case that has been proposed in [9], we shall see that it ensures uniqueness in the framework of Fujita-Kato's solutions.

To prove (5.10), let us split u into $\widetilde{u} + u_L^1 + u_L^2$ where u_L^1 and u_L^2 have been defined in (5.2) with $S = j - \rho u$. The properties of (f,u) already ensure that $S \in L^2_{loc}(\mathbb{R}_+;L^2)$. Hence, Proposition 5.1 gives us that u_L^1 and $\widetilde{u}$ are in $\widetilde{L}^1(0,T;\dot{H}^{5/2})$ for all $T > 0$, so that these two functions are also in $L^1(0,T;C_{\widetilde{\omega}_\eta})$ for all $\eta \in (0,1/2)$ since $\widetilde{L}^1(0,T;\dot{H}^{5/2})$ is embedded in $L^1(0,T;C_{\widetilde{\omega}_\eta})$ (see [4, p. 237]).

That $S \in L^2_{loc}(\mathbb{R}_+;L^2)$ does not ensure that u_L^2 is in $L^1_{loc}(\mathbb{R}_+;C_{\widetilde{\omega}_\eta})$ (one can only guarantee that $u_L^2 \in L^2_{loc}(\mathbb{R}_+;\dot{H}^2)$). However, since in addition ρ and j are bounded then, remembering that $u \in L^2(\mathbb{R}_+;L^\infty)$, we have $S \in L^2_{loc}(\mathbb{R}_+;L^\infty)$ and thus, by real interpolation, that

[7] By using an energy method in the spirit of that of Section 2, it is actually possible to prove that $t^{1/4}\|\nabla u(t)\|_{L^2} \leq Cc$ for all $t \in [0,1]$.

$S \in L^2_{loc}(\mathbb{R}_+; L^{3,1})$ (where $L^{3,1}$ stands for the Lorentz space we defined in (3.17)). Then, parabolic maximal regularity gives $\nabla^2 u^2_L \in L^2_{loc}(\mathbb{R}_+; L^{3,1})$ and thus $\nabla u^2_L \in L^2_{loc}(\mathbb{R}_+; L^\infty)$ by embedding. This completes the proof of (5.10).

One can now tackle the proof of uniqueness. We assume that we are given two solutions (f_1, u_1, P_1) and (f_2, u_2, P_2) fulfilling the properties listed in Theorem 1.6 and (5.10). We assume in addition that ρ_i, j_i and $m_{2,i}$ are in $L^\infty(0,T; L^\infty)$ for $i = 1, 2$. According to (A.9), (A.10) and (A.11), this is ensured if $N_q < \infty$ for some $q > 5$.

Then, in order to prove uniqueness, we start again from (4.3), bounding the first term of the right-hand side according to (4.4) and still splitting δF (defined in (4.5)) into $A_1 + A_2 + A_3 + A_4$. Term A_2 has to be treated differently as before as we do not have any control on $\|\nabla u_1\|_{L^\infty}$. The second change is that when bounding δZ, one cannot use (4.16). Now, by using Cauchy-Schwarz inequality and (A.2) with $Z = Z_1$, we get

$$A_2 \leq \|\sqrt{\rho_1}\,\delta u\|_{L^2} \biggl(\iint f_0(x,v) |u_1(t, X_1(t,x,v)) - u_1(t, X_2(t,x,v))|^2 \,dv\,dx \biggr)^{1/2}$$

Let $\omega_\eta : r \mapsto r(2 - \log r)^{2-2\eta}$. Remembering (5.10), one can assert that there exists an integrable function α on $[0,T]$ such that

$$\iint |u_1(t, X_1(t)) - u_1(t, X_2(t))|^2 \, f_0 \,dv\,dx \leq \alpha^2(t) \iint \omega_\eta(|\delta X(t)|^2) \, f_0 \,dv\,dx.$$

Observe that the function ω_η is concave and increasing on some nontrivial interval $[0, r_0]$ and that we have $|\delta X(t,x,v)|^2 \leq r_0$ pointwise for all $t \in [0,T]$ and $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ provided $T > 0$ is small enough. Indeed, relation (A.7) implies that

$$\delta X(t,x,v) = \int_0^t \bigl(e^{\tau - t} - 1\bigr) \bigl(u_2(\tau, X_2(\tau,x,v)) - u_1(\tau, X_1(\tau,x,v))\bigr) \,d\tau,$$

and thus

$$|\delta X(t,x,v)| \leq \sqrt{t} \, \bigl(\|u_1\|_{L^2(0,t;L^\infty)} + \|u_2\|_{L^2(0,t;L^\infty)} \bigr) \cdot$$

Therefore, assuming with no loss of generality that the total mass M_0 defined in (1.4) is equal to 1, Jensen inequality allows us to get

$$A_2(t) \leq \alpha(t) \|\sqrt{\rho_1}\,\delta u\|_{L^2} \bigl(\omega_\eta(\|\sqrt{f_0}\,\delta X\|^2_{L^2}) \bigr)^{1/2}.$$

Similarly, we have

$$\begin{aligned} B_1(t) &\leq \|\sqrt{f_0}\,\delta V(t)\|_{L^2} \biggl(\iint |u_1(t, X_1(t)) - u_1(t, X_2(t))|^2 \, f_0 \,dv\,dx \biggr)^{1/2} \\ &\leq \alpha(t) \|\sqrt{f_0}\,\delta V(t)\|_{L^2} \end{aligned}$$

Let $Y := \|\delta u\|^2_{L^2} + \|\sqrt{f_0}\,\delta Z\|^2_{L^2}$. In the end we get the following inequality:

$$\begin{aligned} &\frac{d}{dt} Y + \|\nabla \delta u\|^2_{L^2} \lesssim (1 + \|\rho_1\|_{L^\infty})^{1/2} \alpha \, \omega_\eta(Y) \\ &+ \bigl(1 + \|\nabla u_2\|^2_{L^3} + \|(\rho_1, \rho_2)\|_{L^\infty} \|u_1\|^2_{L^\infty} + \|(m_{2,1}, m_{2,2})\|_{L^\infty} + \|\rho_2\|_{L^\infty} \|(u_1, u_2)\|_{L^2(0,t;L^\infty)} \bigr) Y. \end{aligned}$$

Note that $\rho_1, \rho_2, m_{2,1}, m_{2,2}$ are bounded, u_1, u_2 are in $L^2(0,T; L^\infty)$ and α is integrable. Since the modulus of continuity ω_η satisfies Osgood's condition, that is

$$\int_0^{r_0} \omega_\eta^{-1}(r) \,dr = \infty \quad \text{for some} \quad r_0 > 0,$$

one can conclude that $Y \equiv 0$ on $[0,T]$ (apply e.g. [4, Lem. 3.4]). This completes the proof of the uniqueness part of Theorem 1.6.

For the sake of completeness, let us briefly explain how to construct a global solution satisfying the properties of Theorem 1.6. One can for instance consider the regularized system (VNS_n) used to build a solution in Theorem 1.1. Again, the spectral cut-off has not effect on the estimates, which allows to bound the sequence of approximate solutions in the desired space. In order to prove the strong convergence and justify the energy balance (1.3), one can adapt the proof of uniqueness to the case of source terms, like we did for Theorem 1.1, but in the functional framework of Theorem 1.6. The details are left to the reader.

5.4. **The case of an initial velocity in $\dot{B}^{1/2}_{2,1}$.** Here we justify Remark 1.7: we explain how to handle the case where the initial data (f_0, u_0) satisfy (1.1) and

$$\|u_0\|_{B^{1/2}_{2,1}(\mathbb{R}^3)} + \iint |v|^2 f_0 \, dv \, dx \leq c \ll 1.$$

Compared to the case $u_0 \in \dot{H}^{1/2}$, the key difference lies in the fact that $u^1_L = e^{t\Delta}u_0$ satisfies

$$u^1_L \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}^{1/2}_{2,1} \cap L^1(\mathbb{R}_+; \dot{B}^{5/2}_{2,1}),$$

which implies that $\nabla u^1_L \in L^1(\mathbb{R}_+; L^\infty)$ with

$$\int_0^\infty \|\nabla u^1_L\|_{L^\infty} \leq Cc.$$

Consequently, one can bound ρ and m_2 in $L^\infty(0, T\times\mathbb{R}^3)$ (and thus j) according to (A.5) and (A.6) whenever (A.3) holds true. Now, these latter properties enable us to bound $S := j - \rho u$ in $L^2_{loc}(\mathbb{R}_+; L^\infty)$ and one can argue like in Subsection 5.2 to prove (A.3) on, say, the time interval $[0,1]$.

The rest of the proof goes as before. Compared to Theorem 1.6, the gain is that we do not need to assume any longer that $N_q(f_0) < \infty$ for some $q > 5$.

Appendix A. The control of the flow, and applications

Most of the regularity estimates that were used in this paper required boundedness of suitable Lebesgue norms of the density ρ (and, to a lesser extent, of the momentum j or of the distribution energy m_2).

In order to derive these bounds, one can take advantage of formulae for ρ, j and m_2 in terms of the characteristics $Z := (X, V)$ associated to the f equation. More precisely, looking at $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ as parameters, we consider the following system of ODEs:

$$\text{(A.1)} \qquad \begin{cases} \partial_s X(s;t,x,v) = V(s;t,x,v) \\ \partial_s V(s;t,x,v) = u(s;X(s;t,x,v)) - V(s;t,x,v) \\ X(t;t,x,v) = x \quad \text{and} \quad V(t;t,x,v) = v. \end{cases}$$

In other words, $Z = (X,V)$ is the flow of the time-dependent vector-field

$$F(t,x,v) = (v, u(t,x) - v), \qquad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3.$$

The Cauchy-Lipschitz theorem ensures that for (A.1) to be solvable, it suffices that $u \in L^1_{loc}(\mathbb{R}_+; W^{1,\infty})$. Furthermore, since $\mathrm{Div}_{x,v} F = -3$, we have

$$\det D_{x,v} Z(s;t,x,v) = e^{3(t-s)}.$$

The solution of the first equation of (VNS) then reads:

$$f(t,x,v) = e^{3t} f_0\big(X(0;t,x,v), V(0;t,x,v)\big), \tag{A.2}$$

and thus

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,dv = e^{3t}\int_{\mathbb{R}^3} f_0\big(X(0;t,x,v), V(0;t,x,v)\big)\,dv.$$

According to [10, Lemma 4.4], there exists an absolute constant δ such that if

$$\int_0^T \|\nabla u\|_{L^\infty}\,dt \leq \delta, \tag{A.3}$$

then for all $t \in [0,T]$ and $x \in \mathbb{R}^3$, the map $\Gamma_{t,x} : v \mapsto V(0;t,x,v)$ is a bilipschitz homeomorphism on $\mathbb{R}^3$ satisfying

$$\det D_v \Gamma_{t,x}(v) \geq e^{3t}/2. \tag{A.4}$$

So, performing the change of variable $w = \Gamma_{t,x}(v)$ in (A.2), we get

$$\rho(t,x) = e^{3t}\int_{\mathbb{R}^3} f_0(X(0;t,x,\Gamma_{t,x}^{-1}(w)),w)\det D_w\Gamma_{t,x}^{-1}(w)\,dw,$$

and thus, owing to (A.4),

$$|\rho(t,x)| \leq 2\|f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))}. \tag{A.5}$$

Similarly, we have

$$\begin{aligned} m_2(t,x) &= e^{3t}\int_{\mathbb{R}^3} |v|^2 f_0\big(X(0;t,x,v), V(0;t,x,v)\big)\,dv \\ &= e^{3t}\int_{\mathbb{R}^3} |\Gamma_{t,x}^{-1}(w)|^2 f_0(X(0;t,x,\Gamma_{t,x}^{-1}(w)),w)\det D_w\Gamma_{t,x}^{-1}(w)\,dw. \end{aligned}$$

In order to bound $|\Gamma_{t,x}^{-1}(w)|^2$, we integrate the second line of (A.1) on $[0,t]$, getting

$$\Gamma_{t,x}^{-1}(w) = e^{-t}\biggl(w + \int_0^t e^s u\big(s, X(s;t,x,\Gamma_{t,x}^{-1}(w))\big)ds\biggr),$$

whence

$$\big|\Gamma_{t,x}^{-1}(w)\big| \leq e^{-t}|w| + \int_0^t e^{s-t}\|u(s)\|_{L^\infty}\,ds.$$

Therefore, using again (A.4), it is easy to get

$$m_2(t,x) \leq 4e^{-2t}\||v|^2 f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))} + 4\biggl(\int_0^t e^{s-t}\|u(s)\|_{L^\infty}\,ds\biggr)^2 \|f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))}.$$

Now, we have, according to Gagliardo-Nirenberg and Cauchy-Schwarz inequalities,

$$\biggl(\int_0^t e^{s-t}\|u(s)\|_{L^\infty}\,ds\biggr)^2 \leq C\biggl(\int_0^t e^{2s-2t}\,ds\biggr)\int_0^t \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\,ds.$$

Using Cauchy-Schwarz inequality for bounding the last integral and remembering Inequalities (1.3) and (2.15), one can find C_0 depending only on the initial data such that

$$m_2(t,x) \leq 4e^{-2t}\||v|^2 f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))} + C_0(1-e^{-2t})\|f_0\|_{L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x))}. \tag{A.6}$$

Note that the above computations actually allow to obtain a control of f and of $|v|^2 f$ in $L^\infty(\mathbb{R}_+;L^1(\mathbb{R}^3_v;L^\infty(\mathbb{R}^3_x)))$.

We now look at the situation where we do not assume any longer that (A.3) holds, but where u satisfies the log Lipschitz property (5.10). Then, Osgood lemma (see e.g. [4, Chap. 3]) still guarantees that System (A.1) has a unique solution. Furthermore, direct manipulations reveal that

$$X(s;t,x,v) = x + (1-e^{t-s})v + \int_s^t \big(e^{\tau-s}-1\big)u(\tau, X(\tau;t,x,v))\,d\tau, \tag{A.7}$$

$$V(s;t,x,v) = e^{t-s}v - \int_s^t e^{\tau-s}u(\tau, X(\tau;t,x,v))\,d\tau. \tag{A.8}$$

This implies that for all $t \in [0,T]$, we have

$$|v| \leq e^{-t}|V(0;t,x,v)| + \int_0^t e^{\tau-t}\|u(\tau)\|_{L^\infty}\,d\tau.$$

Hence, since $f(t,x,v) = e^{3t}f_0(Z(0;t,x,v))$, we have for all $q > 3$,

$$\begin{aligned}
\rho(t,x) = \int f(t,x,v)\,dv &= e^{3t}\int \langle v\rangle^q f_0(Z(0;t,x,v))\,\langle v\rangle^{-q}\,dv \\
&\leq C_q e^{3t}\left(\sup_{v\in\mathbb{R}^3}\langle v\rangle^q f_0(Z(0;t,x,v))\right) \\
&\leq C_q e^{3t}\Bigg(\sup_{v\in\mathbb{R}^3}\langle V(0;t,x,v)\rangle^q f_0(Z(0;t,x,v)) \\
&\qquad + \left(\int_0^t e^{-\tau}\|u(\tau)\|_{L^\infty}\,d\tau\right)^q \sup_{v\in\mathbb{R}^3} f_0(Z(0;t,x,v))\Bigg).
\end{aligned}$$

Denoting $N_q = \sup_{(x,v)\in\mathbb{R}^3\times\mathbb{R}^3}\langle v\rangle^q f_0(x,v)$, one can conclude that

$$\|\rho(t)\|_{L^\infty} \leq C_q e^{3t}\big(1+\|u\|_{L^1(0,t;L^\infty)}\big)^q N_q. \tag{A.9}$$

Similarly, we have, if $q > 3$,

$$\begin{aligned}
j(t,x) = \int v f(t,x,v)\,dv &= e^{3t}\int v\langle v\rangle^q f_0(Z(0;t,x,v))\,\langle v\rangle^{-q}\,dv \\
&\leq C_{q+1}e^{3t}\left(\sup_{v\in\mathbb{R}^3}\langle v\rangle^{q+1} f_0(Z(0;t,x,v))\right).
\end{aligned}$$

Hence,

$$\|j(t)\|_{L^\infty} \leq C_{q+1}e^{3t}\big(1+\|u\|_{L^1(0,t;L^\infty)}\big)^{q+1} N_{q+1}. \tag{A.10}$$

The same argument leads to

$$\|m_2(t)\|_{L^\infty} \leq C_{q+2}e^{3t}\big(1+\|u\|_{L^1(0,t;L^\infty)}\big)^{q+2} N_{q+2}. \tag{A.11}$$

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