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ESTIMATION OF THE INVARIANT MEASURE OF A MULTIDIMENSIONAL DIFFUSION FROM NOISY OBSERVATIONS

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ABSTRACT. We introduce a new approach for estimating the invariant density of a multidimensional diffusion when dealing with high-frequency observations blurred by independent noises. We consider the intermediate regime, where observations occur at discrete time instances $k\Delta_n$ for $k = 0, \dots, n$, under the conditions $\Delta_n \rightarrow 0$ and $n\Delta_n \rightarrow \infty$. Our methodology involves the construction of a kernel density estimator that uses a pre-averaging technique to proficiently remove noise from the data while preserving the analytical characteristics of the underlying signal and its asymptotic properties. The rate of convergence of our estimator depends on both the anisotropic regularity of the density and the intensity of the noise. We establish conditions on the intensity of the noise that ensure the recovery of convergence rates similar to those achievable without any noise. Furthermore, we prove a Bernstein concentration inequality for our estimator, from which we derive an adaptive procedure for the kernel bandwidth selection.

Mathematics Subject Classification (2010): 62G05, 62G07, 62G20, 60J60.

Keywords: Non-parametric estimation, high-frequency asymptotics, noisy observation, pre-averaging, ergodic diffusion, anisotropic density estimation, concentration inequality.

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1. INTRODUCTION

1.1. **Setting.** In this paper, we revisit the classical problem of estimating the distribution of a signal blurred by additive noise. We focus on a d -dimensional deconvolution model

$$(1) \quad Y_{i,n} = X_{i,n} + \tau_n \xi_{i,n}, \quad i = 0, \dots, n,$$

where the variables $(X_{i,n})_i$ are stationary with common distribution μ and $(\xi_{i,n})_{i,n}$ is an i.i.d sequence independent of X . The sequence (τ_n) represents the noise intensity and is assumed known. When the $(X_{i,n})_i$ are independent, the noise is Gaussian and the target density is α -Hölder regular, the best achievable pointwise quadratic rate of estimation is $\log(n)^{-\alpha/2} \tau_n^\alpha$, see [Fan91, CL13] for more details. When τ_n is of order 1, the resulting convergence rate becomes logarithmic, and the estimators cannot be used in practice. In this work, we show the situation improves when the process $(X_{i,n})_i$ exhibits Markovian properties. Such structure can significantly enhance the efficiency of denoising the observations.

Throughout this paper, we consider a d -dimensional stochastic process X , defined on an appropriate probability space $(\Omega, \mathcal{A}, \mathbb{P})$, governed by the following dynamics

$$(2) \quad dX_t = b(X_t) dt + \sigma(X_t) dW_t.$$

Here, W represents a d -dimensional Brownian motion, and the functions $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ are the transport and diffusion coefficients of X , respectively. We specifically consider the case where σ is the identity matrix. Under mild conditions on b and σ , the process X is ergodic and admits a unique stationary distribution, denoted by $\bar{\mu}^b$. Furthermore, $\bar{\mu}^b$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^d and has a density also denoted by $\bar{\mu}^b$.

We aim at estimating $\bar{\mu}^b$ from the $(Y_i)_{i=1, \dots, n}$ defined in Equation (1), with

$$(3) \quad X_{i,n} = X_{i\Delta_n}, \quad i = 0, \dots, n,$$

for some positive sequence $(\Delta_n)_n$. If Δ_n is of order 1 or larger, the variables $(X_{i,n})$ are exponentially β -mixing and we would retrieve results similar to the i.i.d case. When $\Delta_n \rightarrow 0$ and $n\Delta_n \rightarrow \infty$, we can use the Markovian and ergodic structure of (2) to build an estimator of $\bar{\mu}^b$ with polynomial rate even when $\tau_n = 1$.

1.2. Motivation. Historically, diffusion models were first introduced as approximations of discrete Markov chains. Over time, their relevance has significantly expanded across various domains of applied mathematics [Pap95, Ber93, Hul03, Bai57, Ric13]. Statistical inference for diffusion processes has attracted extensive study due to the model's significance in many applied fields. This research first included both parametric and non-parametric estimation of the parameters b and σ . The estimation of the invariant measure $\bar{\mu}^b$ is also studied [NZ79, Del80, Bos12], pushed by its association with various numerical methods, such as Markov Chain Monte Carlo [LP02, Pan08]. Note also that the non-parametric estimation of $\bar{\mu}^b$ and the estimation of the transport b are intertwined [Sch13].

Initiated by [Ros56, Par62], non-parametric density estimation has been extensively studied in the context of i.i.d observations [Tsy09]. A natural estimator of the common density $\bar{\mu}$ of i.i.d. observations $(X_{i,n})_i$ is given by

$$\hat{\mu}_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{K}_h(x - X_k)$$

where we write $\mathbf{K}_h(y) = \prod_i h_i^{-1} K(y_i h_i^{-1})$ for any $\mathbf{h} = (h_1, \dots, h_d) \in (0, +\infty)^d$ and $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, and where $K : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded kernel. Oracle inequalities show that this estimator can achieve the convergence rate $n^{-\alpha/(2\alpha+d)}$ where α is the Hölder regularity of $\bar{\mu}$. This rate is optimal, see e.g. [Tsy09] for details. The tuning parameter \mathbf{h} needs to be chosen in an adaptive way to achieve this rate, see [GL08, GL09, GL11]. The extension to ergodic processes is not trivial and earlier statistical studies focus on two different asymptotic regimes: continuous observations of the process $(X_t)_{t \leq T}$ with $T \rightarrow \infty$, or low frequency observations given by $(X_{i\Delta})_i$ with $0 \leq i \leq n$. Since X is exponentially β -mixing under mild assumptions on b , low frequency observations naturally relate to the i.i.d. case and present similar convergence rate. On the other hand, continuous observations can be studied using fine probabilistic tools for ergodic continuous-time Markovian dynamics [BCG08, CG08, Lez01, Pau15] and precise estimates on the transition densities [CW97, QRZ03, QZ04]. The seminal works of [DR06, DR07] first established convergence rates of kernel estimators under continuous observations of X over a time interval $[0, T]$. In these works, the invariant density is still estimated through a kernel based estimator as in the i.i.d. case. [DR06, DR07] also obtain the convergence rate

$$T^{-\alpha/(2\alpha+d-2)}$$

when $\bar{\mu}^b$ is α -Hölder and $d \geq 3$. When considering an anisotropic framework, the convergence rate depends on the effective average smoothness [Str18] and minimax rates are derived in [AG21].

Two significant limitations still need to be addressed in this setup: First, what happens if we access discrete high-frequency observations of X , i.e. we observe $(X_{i\Delta_n})_i$ for $0 \leq i \leq n$ when $\Delta_n \rightarrow 0$ and $n\Delta_n \rightarrow \infty$. Secondly, can these methods be applied if these observations are polluted with a noise.

The question of discrete observations naturally arise with the advent of high-frequency data collection, and in particular in finance [ASJ14]. Hence, understanding the estimation rates under different asymptotic conditions becomes crucial. This includes specifying conditions on Δ_n that determine when continuous or low-frequency observations are more analytically pertinent. While this topic has only recently received substantial attention, pioneering works like [GHR04] and [CT16], which explore random sampling times, stand out. However, this question has recently been addressed in full generalities in [AG23] where the breakeven point between the high frequency observations similar to continuous observations and low frequency observations similar to the i.i.d. case is identified and studied. The continuous rate is known to be optimal [AG21] in all dimensions, but the question of the optimality of the low frequency rate is still an open question, with the exception of the one dimensional case, solved in [AG22].

The second limitation in previous studies is to effectively incorporate noise into these analyses. Noise is an often unavoidable element in practical applications, such as financial modeling, biological experiments, and sensor data analysis due to measurement errors or external noise. Underestimating noise can lead to biased and unreliable estimations. This question was extensively studied in the context of noisy observations of i.i.d random variables. When noise is assumed to have an additive structure, existing literature uses Fourier inversion and kernel-based methods to recover the distribution of interest [Dev89, LT89, SC90]. Later works [CH88, Fan91, Fan93] establish minimax optimality of this procedure under the assumptions that the noise distribution is known and has a non-vanishing Fourier transform. It is important to note that in this setup, the convergence rates are slow. For instance, when the distribution of interest is α -Hölder regular and the noises are independent standard Gaussian variables, the optimal rate of convergence for any estimator is only $\log(n)^{-\alpha/2}$. The rates of convergences in such framework has been studied under different set of assumptions depending on both regularities of the noise and the density of interest in [CL13]. However, literature on noisy ergodic setups remains sparse, with a few notable contributions including [Sch11, Sch12]. Unlike the i.i.d. case, the structure of (2) and (3) allows better extraction of the information hidden by the noise, leading to improved estimation rates. Indeed, using the Hölder regularity of X , we can denoise high-frequency data while preserving the analytical properties of the signal. We achieve this through a pre-averaging technique, as in [JLM⁺09]. This approach has been widely studies and could be generalized to other noises, as demonstrated in [JPV10, HP13, JM15] for high-frequency statistics and [Sch11, Sch12] within an ergodic framework. In this paper, we show that when the noise is relatively small (see Section 4.1 for details) we can estimate the invariant density rates to the non-noisy case studied in [AG23]. When the noise τ_n is relatively large, we can still estimate the invariant density with convergence rate given by

$$(\tau_n^2 \Delta_n)^{\frac{\alpha}{2+2\alpha}}.$$

This rate is polynomial in Δ_n even for τ_n constant, which is a great improvement compared to the i.i.d. case. Note also that this rate does not depend on the dimension, see Section 4.1 for insights.

1.3. Organization of the paper. In Section 2, we present the statistical model, including the underlying assumptions and the probabilistic framework of our analysis. We provide clear definitions for our observation scheme and outline the requirements for the noise source. In Section 3, we systematically construct the kernel estimator for the invariant density $\hat{\mu}_{n,h,p}$. This estimator differs from the standard kernel estimator due to the need to preprocess the data, which arises from the presence of noise. To reduce the impact of noise, we use a preaveraging strategy. Specifically, for some integer $p \geq 1$, we divide our number of observations by p by averaging them over a range of size p . Section 4 presents the upper bounds for the quadratic risk $\mathbb{E}[|\hat{\mu}_{n,h,p}(x) - \bar{\mu}^b(x)|^2]$, providing insights into the optimal hyperparameters selection. Although the obtained rates may not achieve minimax optimality, they align with the expected non-parametric estimation rates in similar contexts. Furthermore, it enables us to precisely understand the noise intensity threshold above which averaging is required to achieve a better convergence rate, given by $\Delta_n^{1/\alpha}$, whenever $\bar{\mu}^b$ is assumed to be α -Hölder. Section 5 includes the derivation of a Bernstein-type concentration inequality allowing adaptive selection of the bandwidth h . We then proceed to a numerical analysis section (Section 6), which encompasses experiments and discussions on the estimation procedure. This is followed by a section consolidating essential probabilistic results instrumental in variance control of our estimator, see Section 7. Lastly, all proofs are gathered in the Appendix.

2. STATISTICAL AND PROBABILISTIC FRAMEWORK

2.1. Notation. For all $x \in \mathbb{R}^d$, we denote by $|x|^2 := x \cdot x$ the Euclidian norm. Throughout the paper, we denote by $\mathcal{P}(\mathbb{R}^d)$ the space of probability measures on \mathbb{R}^d . Moreover, for any differentiable function, $f : \mathbb{R}^d \rightarrow \mathbb{R}$, ∇f stands for the gradient of f . Similarly, if f admits k derivative in the i -th component, we denote this derivative by $\partial_i^k f$. Finally, for any $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\nabla \cdot g$ denotes the divergence of g .

For any σ -finite measure ν on \mathbb{R}^d , for any $q \geq 1$, we say that $f : \mathbb{R}^d \rightarrow \mathbb{R}$ belongs to $L^q(\nu)$ whenever

$$\|f\|_{L^q(\nu)}^q := \int_{\mathbb{R}^d} |f(x)|^q \nu(dx) < +\infty.$$

When ν is the Lebesgue measure on \mathbb{R}^d , we only denote L^q and the associated norm $\|\cdot\|_q$. Finally, when $q = +\infty$ and ν is the Lebesgue measure, we define

$$\|f\|_\infty := \sup_{x \in \mathbb{R}^d} |f(x)|.$$

2.2. The statistical diffusion model. We consider a stochastic process X defined on a rich enough probability space $(\Omega, \mathcal{A}, \mathbb{P})$. For a given Lipschitz continuous function b and probability measure μ on \mathbb{R}^d , we can define a probability \mathbb{P}_μ^b under which X is solution of the following stochastic differential equation

$$(4) \quad dX_t = b(X_t) dt + dW_t, \quad \mathcal{L}(X_0) = \mu,$$

where W is a d -dimensional \mathbb{P}_μ^b -Brownian motion. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the filtration generated by W . We write \mathbb{E}_μ^b for the expectation with respect to \mathbb{P}_μ^b . We also use \mathbb{P}_x^b and \mathbb{E}_x^b instead of $\mathbb{P}_{\delta_x}^b$ and $\mathbb{E}_{\delta_x}^b$. In the paper, for any probability measure $\nu \in \mathcal{P}(\mathbb{R}^d)$, for any $q > 1$, we say that $f : \mathbb{R}^d \rightarrow \mathbb{R}$ belongs to $L^q(\nu)$ whenever

$$\|f\|_{L^q(\nu)}^q := \int_{\mathbb{R}^d} |f(x)|^q \nu(dx) < +\infty.$$

When ν is the Lebesgue measure on \mathbb{R}^d , we only denote L^q and the associated norm $\|\cdot\|_q$. In the following, we always assume the following conditions on b .

Assumption 1. *The function b is differentiable and satisfies*

$$|b(0)| \leq b_0 \quad \text{and} \quad \forall i \in \{1, \dots, d\}, \quad \|\partial_i b\|_\infty \leq b_1/d,$$

where b_0 and b_1 are positive constants. This ensures in particular that $\|\nabla \cdot b\|_\infty \leq b_1$.

Assumption 2. *There exists a function $V : \mathbb{R}^d \rightarrow \mathbb{R}$ differentiable and bounded below by a constant V_0 such that $b = -\nabla V$ and $V(0) = 0$.*

Assumption 3. *There exists $\tilde{C}_b > 0$ and $\tilde{\rho}_b > 0$ such that $\langle x, b(x) \rangle \leq -\tilde{C}_b|x|$, $\forall x : |x| \geq \tilde{\rho}_b$.*

By definition, X solution of (2) is a Markov process. Assumption 1 shows that the drift force exhibits at most linear growth, which implies that there exists a constant $C_0 > 0$ such that the transition density $p_t^b(x, y)$ for all $t > 0$ and for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ with $|x - y|^2 < t$ satisfies

$$(5) \quad p_t^b(x, y) \leq C_0(t^{-d/2} + t^{3d/2}),$$

see for example [CW97]. This inequality is crucial to obtain robust upper bounds when estimating the invariant measure of a stochastic process, as demonstrated, for instance in [DR07, Str18]. Assumption 2 is also usual when studying the asymptotic behaviour of a diffusion process. However, it is not a necessary condition [Bha78, DR06], since the existence and uniqueness of the invariant measure primarily depend on the asymptotic behaviour of the drift force b and its growth at infinity. In the present paper, Assumption 2 is needed at a later stage in order to derive upper bounds similar to (5) for the transition density of the pre-averaged process, see Section G.1.

Under Assumptions 2 and 3, the process X defined by (4) admits a unique stationary distribution denoted by $\bar{\mu}^b$, absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^d whose density, also denoted by $\bar{\mu}^b$, is explicitly given for all $x \in \mathbb{R}^{2d}$ by

$$\bar{\mu}^b(x) = Z_V^{-1} \exp(-2V(x)) \quad \text{where} \quad Z_V = \int_{\mathbb{R}^d} \exp(-2V(y)) dy.$$

Assumption 3 is common to guarantee an exponentially fast convergence towards equilibrium. In the case where the drift force is derived from a potential V , a direct link exists between the classical Poincaré inequality and Equation 6, and Assumption 3 implies that X satisfies a Poincaré inequality as shown in [BGL⁺14]. More precisely, there exists $C_{PI} > 0$, depending only on \tilde{C}_b , such that for

any $f \in L^2(\bar{\mu}^b)$ satisfying $\mathbb{E}_{\bar{\mu}^b}^b[f(X_0)] = 0$ and any $t \geq 0$, we have

$$(6) \quad \text{Var}_{\bar{\mu}^b}^b[P_t^b f(X_0)] \leq e^{-2tC_{PI}^{-1}} \mathbb{E}_{\bar{\mu}^b}^b[f(X_0)^2],$$

where $(P_t^b)_{t \geq 0}$ is the semi-group associated to the process X , acting on any measurable functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$(7) \quad \forall t \geq 0, \forall x \in \mathbb{R}^d, P_t^b f(x) = \mathbb{E}_x^b[f(X_t)].$$

It is well known that the accuracy of the estimation of the invariant density $\bar{\mu}^b$ strongly depends on the regularity of $\bar{\mu}^b$ [DR07, Str18, AG21, AG23]. Therefore, we assume that $\bar{\mu}^b$ belongs to the anisotropic Hölder class $\mathcal{H}_d(\alpha, \mathcal{L})$, which is defined below.

Definition 1. Let $\alpha = (\alpha_1, \dots, \alpha_d) \in (0, \infty)^d$ and $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_d) \in (0, \infty)^d$. A function $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to belong to the anisotropic Hölder class $\mathcal{H}_d(\alpha, \mathcal{L})$ of functions if, for all $1 \leq i \leq d$, g is $[\alpha_i]$ -differentiable in the i -th variable and the partial derivatives satisfy for $0 \leq k \leq [\alpha_i]$

$$\|\partial_i^k g\|_\infty \leq \mathcal{L}_i \quad \text{and} \quad \forall t \in \mathbb{R}, \|\partial_i^{[\alpha_i]} g(\cdot + te_i) - \partial_i^{[\alpha_i]} g(\cdot)\|_\infty \leq \mathcal{L}_i |t|^{\alpha_i - [\alpha_i]}$$

where (e_1, \dots, e_d) is the canonical basis of \mathbb{R}^d .

Definition 2. Let $\alpha = (\alpha_1, \dots, \alpha_d) \in (0, \infty)^d$, $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_d) \in (0, \infty)^d$ and $b_0, b_1 > 0$. We write $\mathbf{b} = (\alpha, \mathcal{L}, b_0, b_1, V_0)$ and $\Sigma(\mathbf{b})$ the set of functions $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying Assumption 1 with constants b_0 and b_1 , Assumption 2 with constant V_0 , Assumption 3 and such that $\bar{\mu}^b$ belongs to the anisotropic Hölder class $\mathcal{H}_d(\alpha, \mathcal{L})$.

In this paper, we always assume that b belongs to $\Sigma(\mathbf{b})$, for some $\mathbf{b} = (\alpha, \mathcal{L}, b_0, b_1, V_0)$, with $\alpha_1 \leq \dots \leq \alpha_d$. Moreover, X is assumed to be observed at discrete times $i\Delta_n$, for $0 \leq i \leq n$, and blurred by a noise composed of independent standard Gaussian variables. We observe

$$Y_{i,n} = X_{i\Delta_n} + \tau_n \xi_{i,n},$$

where $\xi_{i,n}$ are i.i.d Gaussian variables, $\Delta_n \rightarrow 0$ and $n\Delta_n \rightarrow \infty$.

3. ESTIMATION PROCEDURE

In this section, we plan to use a kernel type estimation procedure. We consider a bounded kernel with compact support $K : \mathbb{R} \rightarrow \mathbb{R}$, that is a measurable function such that $\int_{\mathbb{R}} K(y) dy = 1$. We assume that it is of order $l \geq 1$, i.e. that for all $1 \leq k \leq l-1$, we have

$$(8) \quad \int_{\mathbb{R}} y^k K(y) dy = 0.$$

Now for any $\mathbf{h} = (h_1, \dots, h_d) \in (0, +\infty)^d$ and $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, we define

$$(9) \quad \mathbf{K}_{\mathbf{h}}(y) = \prod_{i=1}^d h_i^{-1} K(y_i h_i^{-1}).$$

In the case where the density of interest $\bar{\mu}^b$ belongs to the anisotropic Hölder class $\mathcal{H}_d(\alpha, \mathcal{L})$, we always assume that $l \geq \lceil \alpha_d \rceil$.

First, note that the natural kernel based estimator of the invariant density in absence of noise does not work in our context. Indeed, since $(Y_{k,n})$ is a stationary process, the estimator

$$(10) \quad \hat{\mu}_{KB,n,\mathbf{h},p}(x) = \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{K}_{\mathbf{h}}(x - Y_{k,n})$$

estimates the density of $Y_{1,n}$, which is given by

$$\bar{\mu}^b * \varphi_{\tau_n}(x) = \int_{\mathbb{R}^d} \bar{\mu}^b(x-y) \varphi_{\tau_n}(y) dy = \int_{\mathbb{R}^d} \bar{\mu}^b(x - \tau_n y) \varphi_1(\tau_n y) dy$$

where $\varphi_\tau(x) = (2\pi\tau^2)^{-d/2} \exp(-|x|^2/(2\tau^2))$. Therefore, the presence of noise in (10) creates an additional bias satisfying

$$(11) \quad |\bar{\mu}^b * \varphi_{\tau_n}(x) - \mu(x)| \leq \int_{\mathbb{R}^d} |\bar{\mu}^b(x - \tau_n y) - \bar{\mu}^b(x)| \varphi_1(\tau_n y) dy$$

which is of order τ_n . This bias is dominating the usual bias of kernel based estimators when τ_n is large. In particular, when τ_n is of order 1, the estimator is not consistent. Therefore, we need to reduce the influence of the noise in the observation.

To that extent, we implement in this paper a preaveraging approach and we compute local average the observations over batches of size p . This yields to the following modified observations $(p^{-1} \sum_{\ell=0}^{p-1} Y_{kp+\ell, n})_{1 \leq k \leq \lfloor n/p \rfloor}$. Intuitively, this approach should work because of the regularity of X . Indeed, we have

$$(12) \quad \frac{1}{p} \sum_{\ell=0}^{p-1} Y_{kp+\ell, n} = X_{kp\Delta_n} + \frac{1}{p} \sum_{\ell=0}^{p-1} (X_{(kp+\ell)\Delta_n} - X_{kp\Delta_n}) + \frac{\tau_n}{p} \sum_{\ell=0}^{p-1} \xi_{kp+\ell, n}.$$

This expression can be seen as $X_{kp\Delta_n} + \text{noise}$ and the effective sampling frequency becomes $(p\Delta_n)^{-1}$. Moreover, the noise now comes from two sources: in addition to the noise $(\xi_{i, n})_{i, n}$, we now have a preaveraging error $p^{-1} \sum_{\ell=0}^{p-1} (X_{(kp+\ell)\Delta_n} - X_{kp\Delta_n})$. Using the pathwise regularity of X , this error should remain small when p is not too large. Moreover, since the random variable $(\xi_{i, n})_{i, n}$ are independent standard Gaussian variables, we can rewrite $p^{-1} \sum_{\ell=0}^{p-1} \xi_{kp+\ell, n} = \tau_n p^{-1/2} \tilde{\xi}_{k, n}$ where $\tilde{\xi}_{k, n}$ is also a standard Gaussian variable. We next define a kernel estimator based on the preaveraged observations. For any non negative integer p and any $x \in \mathbb{R}^d$, we define

$$(13) \quad \hat{\nu}_{n, h, p}(x) = \frac{1}{\lfloor n/p \rfloor} \sum_{k=0}^{\lfloor n/p \rfloor - 1} \mathbf{K}_h \left(x - p^{-1} \sum_{\ell=0}^{p-1} Y_{kp+\ell, n} \right).$$

Remark that when $p = 1$, this estimator is the usual estimator used for estimating the invariant density from discrete observations, see [Str18, AG23]. From what precedes, it first seems natural that $\hat{\nu}_{n, h, p}$ estimates $\bar{\mu}^b * \varphi_{\tau_n p^{-1/2}}$ which is the invariant density of $X_{kp\Delta_n} + \tau_n p^{-1/2} \tilde{\xi}_{k, n}$. However, the preaveraging error $p^{-1} \sum_{\ell=0}^{p-1} (X_{(kp+\ell)\Delta_n} - X_{kp\Delta_n})$ induces an additional term. Indeed, we have

$$(14) \quad \frac{1}{p} \sum_{\ell=0}^{p-1} (X_{(kp+\ell)\Delta_n} - X_{kp\Delta_n}) = \frac{1}{p} \sum_{\ell=0}^{p-1} \left(\int_{kp\Delta_n}^{(kp+\ell)\Delta_n} b(X_s) ds \right) + \frac{1}{p} \sum_{\ell=0}^{p-1} (W_{(kp+\ell)\Delta_n} - W_{kp\Delta_n}).$$

The first sum is negligible compared to the second one. Moreover, $p^{-1} \sum_{\ell=0}^{p-1} (W_{(kp+\ell)\Delta_n} - W_{kp\Delta_n})$ is a centered Gaussian variable with variance $(12p)^{-1} (p-1)(2p-1)\Delta_n$, independent of $X_{kp\Delta_n}$. Therefore, this term has an effect on the estimation comparable to that of $\tau_n p^{-1/2} \tilde{\xi}_{k, n}$. Combining these two terms, we deduce that $\hat{\nu}_{n, h, p}$ estimates $\bar{\mu}^b * \varphi_{\tilde{\tau}_{n, p}}$ where

$$(15) \quad \tilde{\tau}_{n, p} = \left(\frac{\tau_n^2}{p} + \frac{(p-1)(2p-1)\Delta_n}{12p} \right)^{1/2}.$$

This analysis is formalised in the following proposition.

Proposition 1. *Under Assumptions 1, 2 and 3, there exists $C_B > 0$ such that for any $p \in \{1, \dots, \lfloor \Delta_n^{-1/2} \rfloor\}$ and any $x \in \mathbb{R}^d$, we have*

$$(16) \quad \left| \mathbb{E}_{\bar{\mu}^b} [\hat{\nu}_{n, h, p}(x)] - \bar{\mu}^b * \varphi_{\tilde{\tau}_{n, p}}(x) \right| \leq C_B \begin{cases} \sum_i h_i^{\alpha_i} & \text{if } p = 1, \\ \sum_i h_i^{\alpha_i} + \sqrt{p\Delta_n} & \text{if } p \geq 2. \end{cases}$$

Although bounding the bias of kernel based estimators is usually an easy task, this is not the case here. The proof of Proposition 1 is indeed quite delicate and can be found in Section B. Indeed, we cannot use Itô's formula because, from Equation 9, we see that for any $i \in \{1, \dots, d\}$, $\partial_i \mathbf{K}_h(x) = h_i^{-1} \prod_j h_j^{-1} K(h_j^{-1} x_j)$ which would interfere and create a contribution of order $\prod_i h_i^{-1}$. Instead, using crucially Assumption 2, we use Girsanov's theorem to remove the contribution of the drift term in (14). We then control the likelihood introduced via the change of measure which

introduce the additional term $\sqrt{p\Delta_n}$ in (16) when $p \geq 2$.

Proceeding as in (11), we see that (16) implies

$$(17) \quad \left| \mathbb{E}_{\bar{\mu}^b}^b[\hat{\nu}_{n,\mathbf{h},p}(x)] - \bar{\mu}^b(x) \right| \leq \tilde{C}_B \left(\sum_i h_i^{\alpha_i} + \tilde{\tau}_{n,p} \right)$$

for some $\tilde{C}_B \geq C_B$. This can be improved by a deconvolution procedure. To do so, note that

$$\bar{\mu}^b * \varphi_{\tilde{\tau}_{n,p}}(x) = \mathbb{E}[\bar{\mu}^b(x - \tilde{\tau}_{n,p}\zeta)]$$

where ζ is a standard d -dimensional Gaussian variable. Using the regularity of $\bar{\mu}^b$, we can proceed to a Taylor expansion of $\bar{\mu}^b(x + \tilde{\tau}_{n,p}(\gamma - \zeta))$ around x for all $\gamma = (\gamma_1, \dots, \gamma_d) \in \{0, \dots, l\}^d$. Computing explicitly the moments of $(\gamma - \zeta)$ appearing in this expression, we get an explicit expansion of $\bar{\mu}^b * \varphi_{\tilde{\tau}_{n,p}}(x + \tilde{\tau}_{n,p}\gamma)$ around $\bar{\mu}^b(x)$. We can isolate $\bar{\mu}^b(x)$ from this expression. Specifically, we introduce the matrix $A = (a_{k,i})_{0 \leq k, i \leq l}$, with coefficients given for any $k, i \in \{0, \dots, l\}$ by

$$(18) \quad a_{k,i} = \sum_{j=0}^k \binom{k}{j} (-1)^j m_j i^{k-j}$$

where m_j stands for the j -th moment of a standard Gaussian variable. The matrix A is invertible, as shown in Appendix A. We denote its inverse by A^{-1} , and $\mathbf{u} = (u_0, \dots, u_l)$ stands for its first column. Then, for all $k \in \{0, \dots, l\}$,

$$(19) \quad \sum_{i=0}^l u_i \left(\sum_{j=0}^k \frac{(-1)^j m_j i^{k-j}}{j!(k-j)!} \right) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For any multi-index $\gamma = (\gamma_1, \dots, \gamma_d) \in \{0, \dots, l\}^d$, we define $\mathbf{u}_\gamma = \prod_{i=1}^d u_{\gamma_i}$, and the following point-wise estimator

$$(20) \quad \hat{\mu}_{n,\mathbf{h},p}(x) = \sum \mathbf{u}_\gamma \hat{\nu}_{n,\mathbf{h},p}(x + \gamma \tilde{\tau}_{n,p})$$

where the sum holds over all $\gamma = (\gamma_1, \dots, \gamma_d) \in \{0, \dots, l\}^d$. The bias of $\hat{\mu}_{n,\mathbf{h},p}(x)$ is precised in the following proposition, proved in Section C

Proposition 2. *Suppose that $b \in \Sigma(\mathbf{b})$, with α such that $\alpha_1 \leq \dots \leq \alpha_d$. Then there exists a constant \tilde{C}_B depending only in \mathbf{b} so that for any $x \in \mathbb{R}^d$, any $p \in \{1, \dots, \lceil \Delta_n^{-1/2} \rceil\}$ and any $\mathbf{h} \in \mathbb{R}^d$, we have*

$$(21) \quad \left| \mathbb{E}_{\bar{\mu}^b}^b[\hat{\mu}_{n,\mathbf{h},p}(x)] - \bar{\mu}^b(x) \right| \leq C_B \begin{cases} \tau_n^{\alpha_1} + \sum_{i=1}^d h_i^{\alpha_i} & \text{if } p = 1, \\ \sqrt{p\Delta_n} + \frac{\tau_n^{\alpha_1}}{p^{\alpha_1/2}} + \sum_{i=1}^d h_i^{\alpha_i} & \text{if } p \geq 2. \end{cases}$$

We now turn to the variance of $\hat{\mu}_{n,\mathbf{h},p}(x)$. Before stating the upper bound of the variance let us now define $k_0 := k_0(\alpha)$ such that $\alpha_1 = \alpha_2 = \dots = \alpha_{k_0} < \alpha_{k_0+1} \leq \dots \leq \alpha_d$. Let us also define

$$(22) \quad \begin{aligned} D_1 &= \{(\alpha, k_0), k_0 = 1 \text{ or } k_0 = 2 \text{ and } \alpha_2 < \alpha_3\}; \\ D_2 &= \{(\alpha, k_0), k_0 \geq 3\}; \\ D_3 &= \{(\alpha, k_0), k_0 = 1 \text{ and } \alpha_2 = \alpha_3\}. \end{aligned}$$

The upper bound is stated in the following proposition.

Proposition 3. *Suppose that $b \in \Sigma(\mathbf{b})$. Suppose that $\alpha = (\alpha_1, \dots, \alpha_d)$ satisfies $\alpha_1 = \alpha_2 = \dots = \alpha_{k_0} < \alpha_{k_0+1} \leq \dots \leq \alpha_d$, for some $k_0 \in \{1, \dots, d\}$. If $\hat{\mu}_{n,\mathbf{h},p}$ is the estimator proposed in (20), then there exist $C_V > 0$ uniform over $\Sigma(\mathbf{b})$ and $n_0 > 0$ such that, for $n \geq n_0$, the following holds true for all $x \in \mathbb{R}^d$ and all $\mathbf{h} \in (0, 1]^d$.*

- If $d = 1$, then

$$(23) \quad \text{Var}_{\bar{\mu}^b}^b(\hat{\mu}_{n,\mathbf{h},p}(x)) \leq \frac{C_V}{T_n} \left(p\Delta_n h_1^{-1} + |\log(h_1)| \right).$$

- If $d = 2$, then

$$(24) \quad \text{Var}_{\bar{\mu}^b}^b(\hat{\mu}_{n,\mathbf{h},p}(x)) \leq \frac{C_V}{T_n} \left(p\Delta_n h_1^{-1} h_2^{-1} + |\log(p\Delta_n)| + |\log(h_1 h_2)| \right).$$

- If $d \geq 3$ and $(k_0, \alpha) \in D_1$, then

$$(25) \quad \text{Var}_{\bar{\mu}^b}^b(\hat{\mu}_{n,\mathbf{h},p}(x)) \leq \frac{C_V}{T_n} \left(p\Delta_n \prod_{i=1}^d h_i^{-1} + \sum_{i=1}^d |\log(h_i)| \prod_{i=3}^d h_i^{-1} \right).$$

- If $d \geq 3$ and $(k_0, \alpha) \in D_2$, then

$$(26) \quad \text{Var}_{\bar{\mu}^b}^b(\hat{\mu}_{n,\mathbf{h},p}(x)) \leq \frac{C_V}{T_n} \left(\left(\prod_{i=1}^{k_0} (h_i)^{(2-k_0)/k_0} \prod_{i=k_0+1}^d h_i^{-1} + p\Delta_n \prod_{i=1}^d h_i^{-1} + \sum_{i=1}^d |\log h_i| \right) \right).$$

- If $d \geq 3$ and $(k_0, \alpha) \in D_3$, then

$$(27) \quad \text{Var}_{\bar{\mu}^b}^b(\hat{\mu}_{n,\mathbf{h},p}(x)) \leq \frac{C_V}{T_n} \left(\sum_{i=1}^d |\log h_i| + p\Delta_n \prod_{i=1}^d h_i^{-1} + (h_2 h_3)^{-1/2} \prod_{i=4}^d h_i^{-1} \right).$$

The proof of Proposition 3 is delayed to Appendix D. It is important to point out that our results yield the same upper bounds for the variance term similar to those found in Proposition 1 of [AG23]. The only difference is the presence of a factor p in front of $\Delta_n (T_n \prod_{i=1}^d h_i)^{-1}$. This change is natural as this term directly comes from the discretisation of the process. Here, the preaveraging induces a sub-sampling of the data, grouping the observations on windows of length p . The discretisation step Δ_n therefore becomes $p\Delta_n$. As seen in [AG23], this term does not contribute, except when Δ_n is large. Here, the break-even point also depends on the noise intensity and is detailed in Section 4.1. Note that our proof and the proof Proposition 1 of [AG23] follows the same paths. However, the introduction of additive noise structure requires new bounds on transition densities of the preaveraged process, see Section 7. Note also that although Proposition 3 is stated on the final estimator $\hat{\mu}_{n,\mathbf{h},p}(x)$, the same result holds for $\hat{\nu}_{n,\mathbf{h},p}(x)$. Indeed, in Appendix D, we prove it for $\hat{\nu}_{n,\mathbf{h},p}(x)$ and the proof for $\hat{\mu}_{n,\mathbf{h},p}(x)$ follows from

$$(28) \quad \text{Var}_{\bar{\mu}^b}^b(\hat{\mu}_{n,\mathbf{h},p}(x)) \leq \left(\sum_{\gamma} |u_{\gamma}| \right)^{2d} \sup_{y \in \mathbb{R}^d} \text{Var}_{\bar{\mu}^b}^b(\hat{\nu}_{n,\mathbf{h},p}(y))$$

and the fact that $(\sum_{\gamma} |u_{\gamma}|)^{2d}$ is a constant independent of n .

4. UPPER BOUNDS AND HYPER-PARAMETER CHOICE

We quantify the quality of this estimation procedure by deriving an upper bound for the quadratic error

$$\mathcal{R}(\hat{\mu}, b; x) = \mathbb{E}_{\bar{\mu}^b}^b [|\hat{\mu}(x) - \bar{\mu}^b(x)|^2].$$

Using the classical Bias-Variance decomposition of the quadratic error

$$(29) \quad \mathcal{R}(\hat{\mu}_{n,\mathbf{h},p}, b; x) = B_{n,\mathbf{h},p}^b(x)^2 + V_{n,\mathbf{h},p}^b(x),$$

where for all $x \in \mathbb{R}^d$, we have

$$B_{n,\mathbf{h},p}^b(x) = |\mathbb{E}_{\bar{\mu}^b}^b[\hat{\mu}_{n,\mathbf{h},p}(x)] - \bar{\mu}^b(x)| \quad \text{and} \quad V_{n,\mathbf{h},p}^b(x) = \text{Var}_{\bar{\mu}^b}^b[\hat{\mu}_{n,\mathbf{h},p}(x)].$$

Upper bounds for both $B_{n,\mathbf{h},p}^b(x)$ and $V_{n,\mathbf{h},p}^b(x)$ are given by Propositions 2 and 3 respectively. The variance obtained in Proposition 3 depends on many case and the choice of the hyper-parameters p and \mathbf{h} naturally depends on these cases. In this section, we detail each case and the convergence rate that can be obtained using the best choices p^* and \mathbf{h}^* . For future use, we write

$$(30) \quad \bar{\alpha} = \left(\frac{1}{d} \sum_{i=1}^d \alpha_i^{-1} \right)^{-1} \quad \text{and} \quad \bar{\alpha}_3 = \left(\frac{1}{d-2} \sum_{i=3}^d \alpha_i^{-1} \right)^{-1}$$

4.1. General results. Before studying the quadratic risk in this model, we first recall the results from [AG23] where the estimation of $\bar{\mu}^b$ is studied from discrete non-noisy observations. In [AG23], the authors distinguish two regimes depending on Δ_n . The breaking point w_n^{HF} between these two

regimes is defined by

$$w_n^{HF} = \begin{cases} \log(T_n)T_n^{-1} & \text{if } d = 1, 2, \\ \log(T_n) \left(\frac{\log(T_n)}{T_n} \right)^{\frac{\bar{\alpha}_3}{(2\bar{\alpha}_3+d-2)} \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right)} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_1, \\ T_n^{\frac{-\bar{\alpha}_3}{(2\bar{\alpha}_3+d-2)} \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right)} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_2 \cup D_3. \end{cases}$$

They show that, when $\Delta_n \leq w_n^{HF}$, the estimator of the invariant measure behaves as in the continuous observation case and therefore exhibits the convergence rate $(v_n^{HF})^{1/2}$ defined as

$$v_n^{HF} = \begin{cases} \log(T_n)T_n^{-1} & \text{if } d = 1, 2, \\ \left(\frac{\log(T_n)}{T_n} \right)^{\frac{2\bar{\alpha}_3}{2\bar{\alpha}_3+d-2}} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_1, \\ T_n^{-\frac{2\bar{\alpha}_3}{2\bar{\alpha}_3+d-2}} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_2 \cup D_3. \end{cases}$$

In order to get this convergence rate, when $d \geq 3$ [AG23] identifies the optimal bandwidth choice given by

$$h_i^{*,HF} = \begin{cases} T_n^{-1/2} & \text{if } d = 1, 2, \\ \left(\frac{\log(T_n)}{T_n} \right)^{\frac{\bar{\alpha}_3}{\alpha_i(2\bar{\alpha}_3+d-2)}} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_1, \\ T_n^{-\frac{\bar{\alpha}_3}{\alpha_i(2\bar{\alpha}_3+d-2)}} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_2 \cup D_3. \end{cases}$$

When $\Delta_n \geq w_n^{HF}$, the estimator of the invariant measure behaves as in the i.i.d. case and exhibits a convergence rate $n^{\frac{\bar{\alpha}}{2\bar{\alpha}+d}}$. Here, we want to reproduce this behaviour while also incorporating noise. In order to get this convergence rate, [AG23] also identifies the optimal bandwidth choice $h^{*,LF}$ in that case, given by

$$h_i^{*,LF} = n^{\frac{-\bar{\alpha}}{\alpha_i(2\bar{\alpha}+d)}}.$$

More over, the convergence rate of the low frequency estimator is $(v_n^{LF})^{1/2}$ where

$$v_n^{LF} = n^{-\frac{2\bar{\alpha}}{2\bar{\alpha}+d}}.$$

In our case, the effective discretization step is $p\Delta_n$ and therefore, we expect that the switch between the two regimes of [AG23] appear when $p\Delta_n \approx w_n^{HF}$. However, p also needs to be chosen to minimize the quadratic error bound. Applying the same analysis as in [AG23], we then define $h^{*,p}$ for each p by

$$h_i^{*,p} = \begin{cases} h_i^{*,HF} & \text{if } p\Delta_n \leq w_n^{HF}, \\ \left(\frac{p}{n} \right)^{\frac{\bar{\alpha}}{\alpha_i(2\bar{\alpha}+d)}} & \text{if } p\Delta_n \geq w_n^{HF}. \end{cases}$$

Then we get for each $p \geq 1$,

$$(31) \quad \mathcal{R}(\hat{\mu}_{n,h^{*,p},p}, b; x) \lesssim \begin{cases} p\Delta_n \mathbb{1}_{p \geq 2} + \frac{\tau_n^{2\alpha_1}}{p^{\alpha_1}} + v_n^{HF} & \text{if } p\Delta_n \leq w_n^{HF}, \\ p\Delta_n \mathbb{1}_{p \geq 2} + \frac{\tau_n^{2\alpha_1}}{p^{\alpha_1}} + \left(\frac{p}{n} \right)^{\frac{2\bar{\alpha}}{\alpha_i(2\bar{\alpha}+d)}} & \text{if } p\Delta_n \geq w_n^{HF}. \end{cases}$$

We then want to simplify the case $p\Delta_n \geq w_n^{HF}$ by removing one of the dependence in p . To do so, we use that the condition $p\Delta_n \geq w_n^{HF}$ implies that $p\Delta_n \gtrsim \left(\frac{p}{n} \right)^{\frac{2\bar{\alpha}}{2\bar{\alpha}+d}}$. This statement is formalised in the following Lemma, proved in Section E.1.

Lemma 1. *For all $c_1 > 0$, there exists $c_2 > 0$ such that if $p\Delta_n \geq c_1 w_n^{HF}$, then $p\Delta_n \geq c_2 \left(\frac{p}{n} \right)^{\frac{2\bar{\alpha}}{2\bar{\alpha}+d}}$*

Using Lemma 1, we simplify (31). If $p = 1$, we have

$$(32) \quad \mathcal{R}(\hat{\mu}_{n,h^{*,1},1}, b; x) \lesssim \begin{cases} \tau_n^{2\alpha_1} + v_n^{HF} & \text{if } \Delta_n \leq w_n^{HF}, \\ \tau_n^{2\alpha_1} + n^{-\frac{2\bar{\alpha}}{2\bar{\alpha}+d}} & \text{if } \Delta_n \geq w_n^{HF}. \end{cases}$$

and for $p \geq 2$, we have

$$(33) \quad \mathcal{R}(\hat{\mu}_{n,h^{*,p},p}, b; x) \lesssim \begin{cases} p\Delta_n + \frac{\tau_n^{2\alpha_1}}{p^{\alpha_1}} + v_n^{HF} & \text{if } p\Delta_n \leq w_n^{HF}, \\ p\Delta_n + \frac{\tau_n^{2\alpha_1}}{p^{\alpha_1}} & \text{if } p\Delta_n \geq w_n^{HF}. \end{cases}$$

From these, we see that the optimal choice of p is given by

$$(34) \quad p^* = \left[(\tau_n^{2\alpha_1} \Delta_n^{-1})^{1/(1+\alpha_1)} \right] \vee 1$$

and from this choice, we take $\mathbf{h}^* = \mathbf{h}^{*,p^*}$. Note that this choice ensure that $p^* \Delta_n^{1/2}$ is bounded so that Proposition 2 applies. Four asymptotic regimes can be observed.

Proposition 4 (Small noise intensity, high sampling frequency). *Suppose that $p^* \Delta_n \leq w_n^{HF}$ and that $\tau_n^{2\alpha_1} \leq \Delta_n$. In that case, we have*

$$p^* = 1 \quad \text{and} \quad \mathbf{h}^* = \mathbf{h}^{*,HF}$$

and we get

$$\mathcal{R}(\hat{\mu}_{n,\mathbf{h}^*,p^*}, b; x) \lesssim v_n^{HF} \wedge \tau_n^{\alpha_1}.$$

Proposition 5 (Large noise intensity, high sampling frequency). *Suppose that $p^* \Delta_n \leq w_n^{HF}$ and that $\tau_n^{2\alpha_1} \geq \Delta_n$. In that case, we have*

$$p^* = \left[(\tau_n^{2\alpha_1} \Delta_n^{-1})^{1/(1+\alpha_1)} \right] \quad \text{and} \quad \mathbf{h}^* = \mathbf{h}^{*,HF}$$

and we get

$$\mathcal{R}(\hat{\mu}_{n,\mathbf{h}^*,p^*}, b; x) \lesssim v_n^{HF} \wedge (\tau_n^2 \Delta_n)^{\frac{\alpha_1}{1+\alpha_1}}.$$

Remark 1. *In the previous proposition, note that since $\tau_n^{2\alpha_1} \geq \Delta_n$, the condition $p^* \Delta_n \lesssim w_n^{HF}$ is equivalent to $\Delta_n \lesssim \tau_n^{-2} (w_n^{HF})^{\frac{1+\alpha_1}{\alpha_1}}$.*

Proposition 6 (Small noise intensity, Low sampling frequency). *Suppose that $p^* \Delta_n \geq w_n^{HF}$ and that $\tau_n^{2\alpha_1} \leq \Delta_n$. In that case, we have*

$$p^* = 1 \quad \text{and} \quad \mathbf{h}^* = \mathbf{h}^{*,1}$$

and we get

$$\mathcal{R}(\hat{\mu}_{n,\mathbf{h}^*,p^*}, b; x) \lesssim n^{-\frac{2\bar{\alpha}}{2\bar{\alpha}+d}} \wedge \tau_n^{2\alpha_1}.$$

Proposition 7 (Large noise intensity, Low sampling frequency). *Suppose that $p^* \Delta_n \geq w_n^{HF}$ and that $\tau_n^{2\alpha_1} \geq \Delta_n$. In that case, we have*

$$p^* = \left[(\tau_n^{2\alpha_1} \Delta_n^{-1})^{1/(1+\alpha_1)} \right] \quad \text{and} \quad \mathbf{h}^* = \mathbf{h}^{*,p^*}$$

and we get

$$\mathcal{R}(\hat{\mu}_{n,\mathbf{h}^*,p^*}, b; x) \lesssim (\tau_n^2 \Delta_n)^{\frac{\alpha_1}{1+\alpha_1}}.$$

4.2. Analysis of the convergence rate. In this section, we study the convergence rates found previously in the setup $\Delta_n = n^{-\theta}$ and $\tau_n = n^{-\kappa}$. Of course, since $\Delta_n \rightarrow 0$ and $T_n = n\Delta_n \rightarrow \infty$, we must impose $0 < \theta < 1$. We also require $\kappa \geq 0$ for convenience, although a deeper analysis shows that we can still estimate the invariant measure when $\kappa > -\theta/(2\alpha_1)$. For conciseness, we also ignore the logarithmic factors in the rates w_n^{HF} and v_n^{HF} . We also introduce the following notations

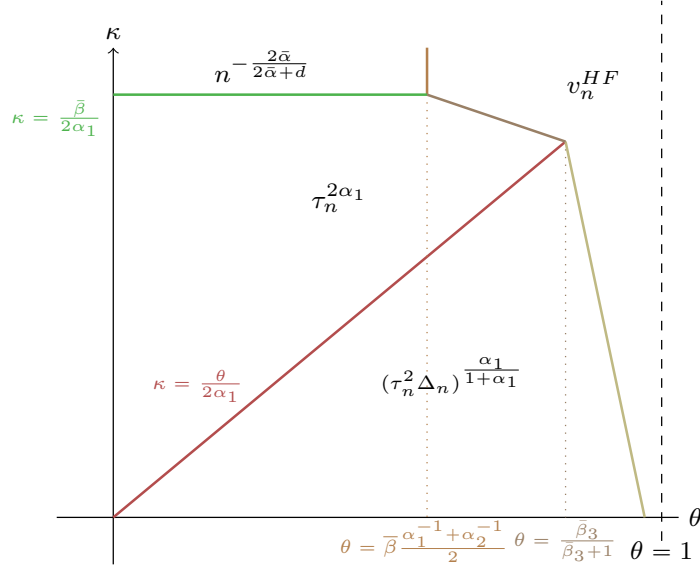
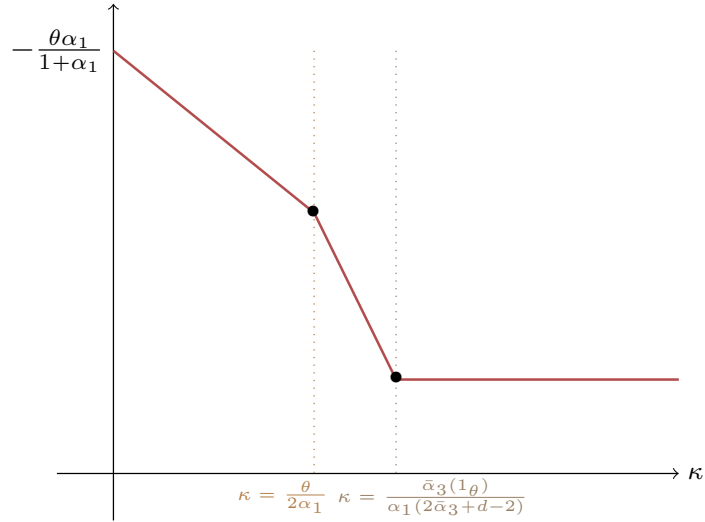
$$\bar{\beta} = \frac{2\bar{\alpha}}{2\bar{\alpha} + d} \quad \text{and} \quad \bar{\beta}_3 = \frac{2\bar{\alpha}_3}{2\bar{\alpha}_3 + d - 2}.$$

Rewriting the conditions for each proposition shows that

- Proposition 4 applies when $\kappa \geq \theta/(2\alpha_1)$ and $\theta \geq \bar{\beta}(\alpha_1^{-1} + \alpha_2^{-1})/2$,
- Proposition 5 applies when $\kappa \leq \theta/(2\alpha_1)$ and $\kappa \geq (1-\theta)\bar{\beta}_3(1+\alpha_1)(\alpha_1^{-1} + \alpha_2^{-1})/2 - \theta\alpha_1$,
- Proposition 6 applies when $\kappa \geq \theta/(2\alpha_1)$ and $\theta \leq \bar{\beta}(\alpha_1^{-1} + \alpha_2^{-1})/2$,
- Proposition 7 applies when $\kappa \leq \theta/(2\alpha_1)$ and $\kappa \leq (1-\theta)\bar{\beta}_3(1+\alpha_1)(\alpha_1^{-1} + \alpha_2^{-1})/2 - \theta\alpha_1$.

We can then identify the domains where each convergence rates operate and we refer to Figure 1 for an illustration of the different regimes

4.3. Comments. The rate w_n^{HF} is the same in the cases $d \geq 3, k_0 \geq 3$ and $d \geq 3, k_0 = 1$ and $\alpha_2 = \alpha_3$. The same remark holds for v_n^{HF} and in these two cases, the convergence rate of the estimator is therefore the same. We still distinguish these cases here because the proofs differ slightly. As noted in [AG23], it is not clear that the high-frequency and the low-frequency rates meet when $\Delta_n \approx w_n^{HF}$.

FIGURE 1. Rates of convergence of $\bar{\mu}_{n,\mathbf{h},p}(x)$.FIGURE 2. Evolution of the rate of convergence as a function of κ with fixed θ in a logarithmic scale.

5. BERNSTEIN INEQUALITY AND ADAPTIVE CHOICE OF THE HYPERPARAMETERS

In this section, we introduce a concentration inequality of Bernstein's type for the kernel estimator built in Section 3. This inequality provides a robust framework for analyzing the estimator's performance and variance. Leveraging this concentration inequality, we develop an adaptive approach for hyper-parameter selection in a data-driven way.

5.1. Bernstein inequality. Recalling that D_1, D_2 and D_3 are defined in Equation (22), we get the following result.

Theorem 1 (Bernstein inequality). *Suppose that $b \in \Sigma(\mathbf{b})$. Then there exists C, τ positive and uniformly bounded over $\Sigma(\mathbf{b})$ such that for all n, \mathbf{h}, p satisfying $p\Delta_n \leq 1$, we have*

$$\mathbb{P}_{\bar{\mu}^b} \left(\left| \hat{\mu}_{n,\mathbf{h},p}(x) - \mathbb{E}_{\bar{\mu}^b} [\hat{\mu}_{n,\mathbf{h},p}(x)] \right| > \varepsilon \right) \leq K \exp \left(- \frac{n_p^2 \varepsilon^2 \beta^2}{32n_p v^2(\alpha, n, \mathbf{h}, p) + \tau \beta \varepsilon \|\mathbf{K}_{\mathbf{h}}\|_{\infty} n_p \log n_p} \right)$$

where $n_p = \lfloor n/p \rfloor$, $\beta = 1/\sqrt{d\ell}\|u\|_2$, $\|u\|_2^2 = \sum_\gamma |u_\gamma|^2$ and

$$\begin{aligned} v^2(\alpha, n, \mathbf{h}, p) &= \text{Var}_{\mu^b} \left(\mathbf{K}_\mathbf{h} \left(x - p^{-1} \sum_{\ell=0}^{p-1} Y_{\ell, n} \right) \right) \\ &\quad + 2 \sum_{k=1}^{\infty} \text{Cov}_{\mu^b} \left(\mathbf{K}_\mathbf{h} \left(x - p^{-1} \sum_{\ell=0}^{p-1} Y_{\ell, n} \right), \mathbf{K}_\mathbf{h} \left(x - p^{-1} \sum_{\ell=0}^{p-1} Y_{k p + \ell, n} \right) \right). \end{aligned}$$

Moreover, we have the following estimates for $v^2(\alpha, n, \mathbf{h}, p)$:

$$v^2(\alpha, n, \mathbf{h}, p) \leq C \begin{cases} (h_1 h_2)^{-1} + \frac{|\log(p\Delta_n)|}{p\Delta_n} + \frac{|\log(h_1 h_2)|}{p\Delta_n} & \text{if } d = 1, 2, \\ \prod_{i=1}^d h_i^{-1} + \frac{\sum_{i=1}^d |\log(h_i)|}{p\Delta_n \prod_{i=3}^d h_i} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_1, \\ \prod_{i=1}^d h_i^{-1} + \frac{\prod_{i=1}^{k_0} (h_i)^{2/k_0}}{p\Delta_n \prod_{i=1}^d h_i} + \frac{\sum_{i=1}^d |\log h_i|}{p\Delta_n} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_2, \\ \prod_{i=1}^d h_i^{-1} + \frac{\sum_{i=1}^d |\log h_i|}{p\Delta_n} + \frac{1}{p\Delta_n (h_2 h_3)^{1/2} \prod_{i=4}^d h_i} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_3. \end{cases}$$

for some constant $C > 0$ uniform over $\Sigma(\mathbf{b})$ and independent of n, \mathbf{h} and p .

The Bernstein inequality presented in Theorem 1 is standard and based on [Lem21]. Our proof requires the derivation of an exponential rate of convergence toward the invariant measure for the Markov process embedding the preaveraged process $(p^{-1} \sum_{\ell=0}^{p-1} Y_{k p + \ell, n})_k$. We refer to Section F for details.

5.2. Choice of the hyperparameters. Bernstein inequality can be used to tune adaptively the parameter h , following the ideas of [GL08, GL09, GL11]. Fix $p \geq 1$. As previously discussed, for dimensions $d = 1$ and $d = 2$, the variance's upper bound does not rely on the smoothness. Consequently, there is no advantage to adopting a data-oriented adaptive strategy for $d < 3$.

For $d \geq 3$, the strategy is to consider a range of possible bandwidths and to select the one that minimizes the error. For this aim, we introduce a heuristic representation of the bias and a penalty term proportional to the variance bound of Proposition ... The optimal bandwidths are those minimizing the sum of these two latter. Let us begin by introducing the grid

$$\mathcal{H}_n^p \subset \left\{ \mathbf{h} \in (0, 1]^d : h_1 \leq \dots \leq h_d, \forall l = 1, \dots, d \quad \left(\frac{\log(n_p)^3}{n_p} \right)^{1/d} \leq h_l \leq 1 \right\},$$

where we recall that n_p stands for $\lfloor n/p \rfloor$ for the sake of clarity. We also assume that $\#\mathcal{H}_n^p \leq T_n$. According to the set of candidate bandwidths, we can introduce the set of candidate estimators:

$$\mathcal{F}(\mathcal{H}_n^p) := \{ \hat{\mu}_{n, \mathbf{h}, p}(x) : x \in \mathbb{R}^d, \mathbf{h} \in \mathcal{H}_n^p \}.$$

The goal of this section is to choose an estimator in the family $\mathcal{F}(\mathcal{H}_n^p)$, in a completely data-driven way. Following the idea in [GL08, GL09, GL11], our selection procedure relies on the introduction of auxiliary convolution estimators, $\hat{\mu}_{n, (\mathbf{h}, \boldsymbol{\eta}), p}$ for $(\mathbf{h}, \boldsymbol{\eta}) \in (\mathcal{H}_n^p)^2$ which is the same estimator as the one introduced in Section 3 but with kernel $\mathbf{K}_\mathbf{h} * \mathbf{K}_\boldsymbol{\eta}$. The first important remark is that when the regularity of the function of interest is unknown, the upper bound on the variance for p fixed can be rewritten for any $\mathbf{h} \in \mathcal{H}_n^p$,

$$\begin{aligned} v_n^p(\mathbf{h}) &= \frac{1}{T_n} \left(p\Delta_n \prod_{i=1}^d h_i^{-1} + \min \left(\sum_{i=1}^d |\log(h_i)| \prod_{i=3}^d h_i^{-1}, (h_2 h_3)^{-1/2} \prod_{i=4}^d h_i^{-1}, \right. \right. \\ &\quad \left. \left. \min_{k_0 \geq 3} \prod_{i=1}^{k_0} h_i^{\frac{2-k_0}{k_0}} \prod_{i=k_0+1}^d h_i^{-1} \right) \right). \end{aligned}$$

Moreover, one can write:

$$\begin{aligned} \sum_{i=1}^d |\log(h_i)| \prod_{i=3}^d h_i^{-1} &\leq \prod_{i=1}^3 h_i^{-\frac{1}{3}} \prod_{i=4}^d h_i^{-1} \\ \iff h_3^2 &\geq h_1 h_2 \sum_{i=1}^d |\log(h_i)|. \end{aligned}$$

Using the fact that $h_1 \leq \dots \leq h_d$, we obtain

$$v_n^p(\mathbf{h}) = \frac{1}{T_n} \left(p \Delta_n \prod_{i=1}^d h_i^{-1} + \min \left(\sum_{i=1}^d |\log(h_i)| \prod_{i=3}^d h_i^{-1}, (h_2 h_3)^{-1/2} \prod_{i=4}^d h_i^{-1} \right) \right).$$

With this purpose in mind, we introduce the following penalty function

$$V_n^p(\mathbf{h}) = \bar{\omega} \log(n_p) v_n^p(\mathbf{h}),$$

for some positive constant $\bar{\omega}$ which has to be taken large. Now, following once again the procedure of [GL08, GL09, GL11], we define for any $\mathbf{h} \in \mathcal{H}_n^p$,

$$(35) \quad A_n^p(\mathbf{h}) = \max_{\boldsymbol{\eta} \in \mathcal{H}_n^p} \left\{ \left| \hat{\mu}_{n,(\mathbf{h},\boldsymbol{\eta}),p}(x) - \hat{\mu}_{n,\boldsymbol{\eta},p}(x) \right|^2 - V_n^p(\boldsymbol{\eta}) \right\}_+.$$

Finally, we define the choice procedure

$$\mathbf{h}^* \in \operatorname{argmin}_{\mathbf{h} \in \mathcal{H}_n^p} \{ A_n^p(\mathbf{h}) + V_n^p(\mathbf{h}) \}.$$

Remark 2. The choice of the penalty $A_n^p(\mathbf{h})$ and the threshold $V_n^p(\mathbf{h})$ are standard in the Goldenshulger-Lepski methodology: $A_n^p(\mathbf{h})$ is a kind of proxy for the estimation of the squared bias of $\hat{\mu}_{n,\mathbf{h},p}(x)$ while $V_n^p(\mathbf{h})$ is the exact penalty needed to balance the size of the variance of the estimator in \mathbf{h} , inflated by a logarithmic term and tuned with $\bar{\omega} > 0$. This enables one to control all the stochastic deviation terms. See section F for details.

Setting this methodology up, we obtain for any pre-averaging level p , the following Oracle inequality

Proposition 8 (Oracle inequality). *Assume that Assumptions 1, 2 and 3 hold and that $d \geq 3$, then there exists $n_0 \in \mathbb{N}$, such that for any $n \geq n_0$,*

$$\mathbb{E} \left[\left| \hat{\mu}_{n,\mathbf{h}^*,p} - \bar{\mu}^b(x) \right|^2 \right] \leq c \inf_{\mathbf{h} \in \mathcal{H}_n^p} \{ B_{n,\mathbf{h},p}^b(x)^2 + V_n^p(\mathbf{h}) \} + c n_p^{-\gamma} + p \Delta_n \mathbf{1}_{p \geq 2} + \frac{\tau_n^{2\alpha_1}}{p_1^\alpha},$$

for some $c > 0$ and $\gamma > 1$, and where we recall that $B_{n,\mathbf{h},p}^b(x)$ is defined in Equation (29).

The proof of Proposition 8 is postponed to Section F.

6. NUMERICAL ANALYSIS

In this section, we study the estimators derived in Section 3. We first focus on the case $d = 1$. Following the results of Section 4 bandwidth h is given by $h = T_n^{-1}$. The main challenge is thus to choose the preaveraging parameter p . Theoretically, the best approach would be to consider the bias-corrected estimator $\hat{\mu}_{n,\mathbf{h},p}(x)$ defined in (20). However, this estimator is unstable, due to the constant $(\sum_\gamma |u_\gamma|)^{2d}$ appearing in (28). Indeed, consider for instance the case

$$V(x) = x^2/4 \quad \text{and} \quad b(x) = -x/2.$$

Taking $n = 2^{14}$, $\Delta_n = n^{-1/2} = 2^{-7}$ and $\tau_n = 1$, we can compute the bias and the variance of both $\hat{\nu}_{n,\mathbf{h},p}(x)$ and $\hat{\mu}_{n,\mathbf{h},p}(x)$. The results are presented in Table 1 with $p = \lfloor (\tau_n^2 \Delta_n^{-1})^{1/2} \rfloor \vee 1$. In this example, we can see clearly that the variance increases by a non-negligible factor when doing the bias-correction procedure and consequently the mean squared error is higher, even if the bias is substantially smaller.

	Without bias correction	With bias correction
Error	1.61e-3	8.2e-3
Bias	1.21e-3	3.48e-4
Variance	1.67 e-3	9.77e-3

TABLE 1. Impact of the bias correction.

Therefore, in most practical applications, the use of the initial estimator $\hat{\nu}_{n,h,p}(x)$ leads to better results compared to $\hat{\mu}_{n,h,p}(x)$. Therefore, in the following, we focus on $\hat{\nu}_{n,h,p}(x)$ in our simulations. In that case, we can choose p with easy asymptotics: indeed, the optimal choice for p minimises the bias in (17), and thus we take $p^* = \lfloor (\tau_n^2 \Delta_n^{-1})^{1/2} \rfloor \vee 1$.

We now illustrate the effectiveness of the pre-averaging strategy and stack it up against the straightforward method where $p = 1$. We still consider the case

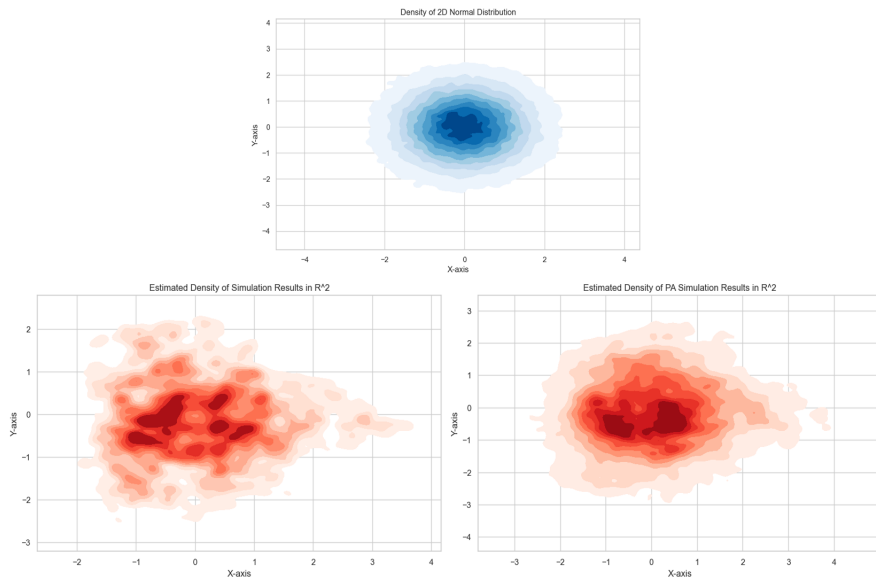
$$V(x) = x^2/4 \quad \text{and} \quad b(x) = -x/2.$$

with $n = 2^{14}$, $\Delta_n = n^{-1/2} = 2^{-7}$ and $\tau_n = 1$. The results are presented in Table 2 where the pointwise quadratic error is presented for different values of p and at different points x .

p	$x = 0$	$x = 0.25$	$x = 0.5$	$x = 1$	$x = 0.75$
1	1.29e-1	1.20e-1	9.75e-2	6.80e-2	4.12e-2
16	6.31e-2	5.62e-2	4.36e-2	2.57e-2	1.08e-2
p^*	1.04e-2	1.08e-2	6.49e-3	2.83e-3	1.70e-3
1024	7.07e-2	5.05e-2	4.06e-2	1.67e-2	2.02e-2
4096	7.49e-1	5.39e-1	3.25e-1	7.27e-2	5.22e-2

TABLE 2. Pointwise quadratic error.

We also illustrate the effectiveness of this method in dimension 2, with the potential $V(x) = |x|^2/4$. In that case $\bar{\mu}^b$ is the density of a standard Gaussian distribution. The following plots provides a visual comparison between the desired distribution and the outcomes of two estimation techniques in dimension $d = 2$, with the same parameters as in the unidimensional case. The first plot is the target density. The bottom left plot shows the estimated distribution using no preaveraging method while the bottom right plot shows the results with preaveraging. As expected, the second method exhibits better results, confirming that pre-averaging refines the estimation process when the noise intensity is constant.



Remark 3 (Non-Gaussian noise). *In this paper, we assume for simplicity that the noise is composed of independent d -dimensional standard Gaussian variables, also independent of the underlying process X . However, this assumption could be softened. Indeed, we only need that the noise is centered, independent from the diffusion X , identically distributed and that it has finite moment of order l . In the case that its distribution is that of a variable ξ , we should replace estimate (16) of Proposition 1 by*

$$\left| \mathbb{E}_{\bar{\mu}^b}^b[\hat{\nu}_{n,h,p}(x)] - \mathbb{E}[\bar{\mu}^b(x - p^{-1/2}\tau_n\tilde{\xi}_p - \tilde{\tau}_{n,p}\zeta)] \right| \leq C_B \begin{cases} \sum_i h_i^{\alpha_i} & \text{if } p = 1, \\ \sum_i h_i^{\alpha_i} + \sqrt{p\Delta_n} & \text{if } p \geq 2 \end{cases}$$

where ζ is a standard Gaussian variable, $\tilde{\tau}_{n,p}^2 = (p-1)(2p-1)\Delta_n/12p$ and $\tilde{\xi}_p = p^{-1/2}\sum_{k=1}^p \xi_k$ with ξ_1, \dots, ξ_n i.i.d with common distribution ξ . In that case, the bias correction procedure should be changed accordingly and should be split into two parts. First, we repeat the same procedure with $\tilde{\tau}_{n,p}^2 = (p-1)(2p-1)\Delta_n/12p$ to ensure the first bias corrected estimator centres around $\mathbb{E}[\bar{\mu}^b(x - p^{-1/2}\tau_n\tilde{\xi}_p)]$. Then we repeat the same procedure with $p^{-1/2}\tau_n$ instead of $\tilde{\tau}_{n,p}^2$ and where m_j stands for the moment of order j of $\tilde{\xi}_p$.

7. PROBABILISTIC TOOLS: THE MARKOVIAN STRUCTURE OF THE PRAVERAGED PROCESS

In this section, we gather some estimates about transition probabilities that will be very useful in the following proofs. Indeed, the proof of Proposition 3 is based on an in-depth study of the Markov chain

$$(36) \quad \left(p^{-1} \sum_{\ell=0}^{p-1} Y_{kp+\ell,n}, X_{(k+1)p\Delta_n} \right)_{k \in \mathbb{N}}.$$

First, note that the stationarity of X ensures that this Markov chain is also stationary under $\mathbb{P}_{\bar{\mu}^b}^b$. We write $\bar{\pi}^b$ for its stationary distribution on \mathbb{R}^{2d} . We now summarize the notations used in the following.

Notation 1. For any $t > 0$, $x, y \in \mathbb{R}^d$,

- $p_t^b(x; \cdot)$ stands for the transition density of X , that is density of X_t conditionally on $X_0 = x$;
- $\mathfrak{p}_{p,n,t}^b(x; \cdot)$ stands for the density of $p^{-1}\sum_{\ell=0}^{p-1} X_{\ell\Delta_n+t}$ conditionally on $X_0 = x$;
- $\mathfrak{p}_{p,n}^b(x; \cdot) = \mathfrak{p}_{p,n,0}^b(x; \cdot)$ stands for the density of $p^{-1}\sum_{\ell=0}^{p-1} X_{\ell\Delta_n}$ conditionally on $X_0 = x$;
- $\mathfrak{p}_{p,n}^b(x; \cdot, \cdot)$ stands for the joint density of $(p^{-1}\sum_{\ell=0}^{p-1} X_{\ell\Delta_n}, X_{p\Delta_n})$ conditionally on $X_0 = x$;
- $\bar{\pi}^b(\cdot, \cdot)$ is an invariant density of the Markov process $(p^{-1}\sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n}, X_{(k+1)p\Delta_n})_{k \in \mathbb{N}}$.

We now derive some bounds for these densities. The proofs of these results are relegated to Appendix G. We start with a short time control of the transition density for the pre-averaged process.

Lemma 2. *Suppose that Assumptions 1 and 2 hold. Then there exists positive constants κ_1 , λ_1 and η_1 , such that if $p\Delta_n \leq \eta_1$, we have for any $x, y \in \mathbb{R}^d$,*

$$(37) \quad \mathfrak{p}_{p,n}^b(x; y) \leq \frac{\kappa_1}{(p\Delta_n)^{d/2}} \exp\left(-\lambda_1 \frac{|y-x|^2}{p\Delta_n} + V(x)\right).$$

This lemma is arguably the most technical result in this section. Its proof, postponed to Appendix G.1, is strongly inspired by Theorem 4 in [GG08] where a similar result is shown for a unidimensional diffusion. As noted in [GG08], their approach cannot readily apply in a multidimensional setting because of a non-trivial time change. In our case the diffusion coefficient is assumed to be a real constant, and the outcomes can be extended. Our motivation for undertaking this proof also lies in the requirement to ensure uniformity of the bounds with respect to p, n . Moreover, here we want to get rid of the uniform boundedness of the drift term b . Its proof is based on the Girsanov theorem to remove the drift contribution. Then we condition at each time $\frac{i}{n}$ so that on each interval $[i/n, (i+1)/n]$, the Brownian Motion is a Brownian bridge. Then sharp estimates can be obtained

using the properties of the Brownian Bridge.

We now study the transition density of X in short time.

Lemma 3. *Under Assumptions 1, 2 and 3, there exists a positive constant κ_3 , such that for any $t \in (0, 1)$, $b \in \Sigma(\mathbf{b})$, and $z \in \mathbb{R}^d$:*

$$(38) \quad p_t^b(x; z) \leq \frac{\kappa_3}{t^{d/2}} \exp\left(-\frac{|x-z|^2}{2t} + V(x) - V(z)\right).$$

This Lemma is particularly useful when $t \ll 1$. In that case, it is usually stated in the weaker form (5). The sharper bound depending explicitly in the potential V allow for better estimates in Proposition 3. Combining Lemmas 2 and 3, we easily get the following corollary.

Corollary 1. *Under Assumption 1 and 2, let us assume that $p\Delta_n \leq \eta_1$, where $\eta_1 > 0$ is defined in Lemma 2. Then, for each $x, y \in \mathbb{R}^d$, and any $t \in (0, 1)$*

$$\bar{\pi}_{p,n,t}^b(x, y) \leq \kappa_1 (2\pi)^{-d/2} e^{b_1/2} (p\Delta_n + 2t\lambda_1)^{-d/2} \exp\left(-\frac{\lambda_1}{2t + p\Delta_n} |x - y|^2\right).$$

We finish this section with a result concerning the invariant density of (36).

Lemma 4. *For all $(x, y) \in \mathbb{R}^{2d}$, there exists a constant κ_2 independent of p, n and b such that*

$$\bar{\pi}^b(y, z) \leq \frac{\kappa_2}{(p\Delta_n)^{d/2}} \exp\left(-V(z) - \frac{\lambda_1}{p\Delta_n} |y - z|^2\right).$$

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In the following sections, we assume that Assumptions 1, 2 and 3 hold and that $p\Delta_n \leq 1$. Furthermore $a \lesssim b$ stands for $a \leq Cb$ where C can change from line to line and depending only on \mathfrak{b} .

APPENDIX A. INVERTIBILITY OF A

We show that the matrix A defined in Section 3 is invertible. To that extent, we compute explicitly $\det(A)$ and we show that it does not vanish. First, we write $a_{k,i,j} = \binom{k}{j}(-1)^j m_j i^{k-j}$ for $0 \leq j \leq k$ so that the (k, i) -th coefficient of A defined in Equation (18) is given by $a_{k,i} = \sum_{j=0}^k a_{k,i,j}$. By multilinearity of the determinant, we have then

$$\begin{aligned} \det(A) &= \sum_{j_0=0}^0 \sum_{j_1=0}^1 \cdots \sum_{j_\ell=0}^\ell \det((a_{k,i,j_k})_{k,i}) \\ &= \sum_{j_0=0}^0 \sum_{j_1=0}^1 \cdots \sum_{j_\ell=0}^\ell \det((i^{k-j_k})_{k,i}) \prod_{k=0}^\ell \left(\binom{k}{j_k} (-1)^{j_k} m_{j_k} \right). \end{aligned}$$

Note now that $\det((i^{k-j_k})_{k,i}) = 0$ when two lines are the same. This happens when $k - j_k = k' - j_{k'}$ for some $k \neq k'$. Thus $\det((i^{k-j_k})_{k,i}) = 0$ except when the $k - j_k$, $0 \leq k \leq \ell$ are all distinct. Since $0 \leq j_k \leq k$, we show by induction that the only possibility is $j_0 = j_1 = \cdots = j_\ell = 0$ so that

$$\det(A) = \det((i^k)_{k,i}) \prod_{k=0}^\ell \left(\binom{k}{0} (-1)^0 m_0 \right) = \prod_{0 \leq k < i \leq \ell} (i - k)$$

where the last equality is obtained using the expression of the Vandermonde determinant and using $m_0 = 1$ (by definition).

APPENDIX B. PROOF OF PROPOSITION 1

In this Section, we aim at studying the expectation $\mathbb{E}_{\bar{\mu}^b}^b[\hat{\nu}_{n,h,p}(x)]$. Let us assume in all the proof that there exists $L > 0$ such that

$$(39) \quad p^2 \Delta_n \leq L,$$

for all $n \geq 1$. By definition of $\hat{\nu}_{n,h,p}$ in Equation (13) and by stationarity of X under $\mathbb{P}_{\bar{\mu}^b}^b$, we have

$$\mathbb{E}_{\bar{\mu}^b}^b[\hat{\nu}_{n,h,p}(x)] = \int_{\mathbb{R}^d} \bar{\mu}^b(y) \mathbb{E}_y^b \left[\mathbf{K}_h \left(x - \frac{\tau_n}{p^{1/2}} \tilde{\xi}_{i,n} - \frac{1}{p} \sum_{\ell=0}^{p-1} X_{\ell \Delta_n} \right) \right] dy.$$

Recall that \mathbb{P}_y^b is the probability measure under which $X_0 = y$ almost surely. When $p = 1$, we observe that $p^{-1} \sum_{\ell=0}^{p-1} X_{\ell \Delta_n} = y$ holds almost surely under \mathbb{P}_y^b . This is not the case when $p > 1$. Instead, we have

$$\frac{1}{p} \sum_{\ell=0}^{p-1} X_{\ell \Delta_n} = y + \frac{1}{p} \sum_{\ell=0}^{p-1} (X_{(k+p\ell)\Delta_n} - X_{k p \Delta_n}).$$

The second part of the right hand side can be seen as a noise term that we decompose as

$$\frac{1}{p} \sum_{\ell=0}^{p-1} (X_{(kp+\ell)\Delta_n} - X_{kp\Delta_n}) = \frac{1}{p} \sum_{\ell=0}^{p-1} \left(\int_{kp\Delta_n}^{(kp+\ell)\Delta_n} b(X_s) ds \right) + \frac{1}{p} \sum_{\ell=0}^{p-1} (W_{(kp+\ell)\Delta_n} - W_{kp\Delta_n}).$$

We remove the effects of the drift b through the Girsanov Theorem. For all admissible drift b , we define

$$N_t^b = \int_0^t b(X_s) \cdot dW_s.$$

The following technical lemma is needed to apply the Girsanov Theorem.

Lemma 5. *There exists $\delta_0 > 0$ such that for any $T > 0$ and $x \in \mathbb{R}^d$,*

$$\sup_{t \in [0, T-\delta]} \mathbb{E}_x^b \left[\exp \left(\frac{1}{2} (\langle N^b \rangle_{t+\delta} - \langle N^b \rangle_t) \right) \right] \leq C_1,$$

for every $0 \leq \delta \leq \delta_0$ and some C_1 that depends on b .

Proof. Using Assumption 1, we get that for any $t \in [0, T]$,

$$|X_t| \leq |X_0| + b_0 T + |b|_{lip} \int_0^t |X_s| ds + \sup_{t \in [0, T]} |B_t|.$$

Therefore, from Gronwall Lemma, we have for all $p \geq 1$ and all $0 \leq t \leq T$

$$(40) \quad \mathbb{E}[|X_t|^{2p}] \leq 3^{2p-1} e^{2pt|b|_{lip}} (|x|^{2p} + T^p + (b_0 T)^{2p}).$$

Moreover, for any $\delta > 0$ and $t \in [0, T - \delta]$, using Jensen inequality,

$$\mathbb{E}_x^b \left[\exp \left(\frac{1}{2} \int_t^{t+\delta} |b(X_s)|^2 ds \right) \right] \leq \frac{1}{\delta} \int_t^{t+\delta} \mathbb{E}_x^b \left[\exp \left(\frac{\delta}{2} |b(X_s)|^2 \right) \right] ds.$$

Using Assumption 1, we get the following bound

$$\begin{aligned} \mathbb{E}_x^b \left[\exp \left(\frac{1}{2} \int_t^{t+\delta} |b(X_s)|^2 ds \right) \right] &\leq \frac{1}{\delta} \int_t^{t+\delta} \mathbb{E}_x^b \left[\exp \left(\frac{\delta}{2} (|b_0|^2 + |b|_{lip}^2 |X_s|^2) \right) \right] ds \\ &\leq \frac{e^{\delta|b_0|^2}}{\delta} \int_t^{t+\delta} \mathbb{E}_x^b \left[\exp \left(\frac{\delta}{2} |b|_{lip}^2 |X_s|^2 \right) \right] ds. \end{aligned}$$

Now, for $s \in [t, t + \delta]$, we have

$$\mathbb{E}_x^b \left[\exp \left(\frac{\delta}{2} |b|_{lip}^2 |X_s|^2 \right) \right] = 1 + \sum_{k \geq 1} \left(\frac{\delta |b|_{lip}}{2} \right)^k \frac{\mathbb{E}_x^b[|X_s|^{2k}]}{k!}.$$

From Equation (40), we get that $\mathbb{E}_x^b \left[\exp \left(\frac{\delta}{2} |b(X_s)|^2 \right) \right] < +\infty$, for all $\delta > 0$, which proves Lemma 5. \square

By Novikov's criterion – in its version developed in the classical textbook [KS91], Lemma 5.14, p.198 – Lemma 5 shows that the local martingale $\mathcal{E}_t^b(N) = \exp(-N_t^b - 1/2 \langle N^b \rangle_t)$ is indeed a (non-local) martingale under \mathbb{P}_x^b so we can apply Girsanov theorem. Let \mathbb{Q}_x^b be the probability defined by its restriction to \mathcal{F}_t by

$$\frac{d\mathbb{Q}_x^b}{d\mathbb{P}_x^b} \Big|_t = \mathcal{E}_t^b(N).$$

Note that under \mathbb{Q}_x^b , the process $W^{\mathbb{Q}^b}$ defined for $t \geq 0$ by

$$(41) \quad W_t^{\mathbb{Q}^b} = W_t + \int_0^t b(X_s) ds = X_t - X_0$$

is a d -dimensional Brownian motion. Then we have, using the Markov property

$$\begin{aligned} \mathbb{E}_{\bar{\mu}^b}^b[\hat{\nu}_{n,h,p}(x)] &= \int_{\mathbb{R}^d} \bar{\mu}^b(y) \mathbb{E}_y^b[\hat{\nu}_{n,h,p}(x)] dy \\ &= \int_{\mathbb{R}^d} \bar{\mu}^b(y) \mathbb{E}_y^{\mathbb{Q}^b} \left[\mathbf{K}_h \left(x - \frac{\tau_n}{p^{1/2}} \tilde{\xi}_{i,n} - \frac{1}{p} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n} \right) \frac{d\mathbb{P}_y^b}{d\mathbb{Q}_y^b} \Big|_{p\Delta_n} \right] dy. \end{aligned}$$

Moreover, we have $p^{-1} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n} = X_0 + p^{-1} \sum_{\ell=0}^{p-1} W_{\ell\Delta_n}^{\mathbb{Q}^b}$ and thus for all $t \geq 0$

$$\left. \frac{d\mathbb{P}_y^b}{d\mathbb{Q}_y^b} \right|_t = \exp \left(\int_0^t b(y + W_s^{\mathbb{Q}^b}) \cdot dW_s^{\mathbb{Q}^b} - \frac{1}{2} \int_0^t |b(y + W_s^{\mathbb{Q}^b})|^2 ds \right).$$

and

$$\mathbb{E}_{\bar{\mu}^b}^b[\widehat{\nu}_{n,h,p}(x)] = \int_{\mathbb{R}^d} \bar{\mu}^b(y) \mathbb{E}_y^{\mathbb{Q}^b} \left[\mathbf{K}_h \left(x - y - \frac{\tau_n}{p^{1/2}} \tilde{\xi}_{i,n} - \frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell\Delta_n}^{\mathbb{Q}^b} \right) \left. \frac{d\mathbb{P}_y^b}{d\mathbb{Q}_y^b} \right|_{p\Delta_n} \right] dy.$$

Note that in the expectation on the right hand side, $\left. \frac{d\mathbb{P}_y^b}{d\mathbb{Q}_y^b} \right|_{p\Delta_n}$ is entirely determined by $W^{\mathbb{Q}^b}$ which is a standard Brownian motion under \mathbb{Q}^b . Therefore, we have

$$\begin{aligned} & \mathbb{E}_y^{\mathbb{Q}^b} \left[\mathbf{K}_h \left(x - y - \frac{\tau_n}{p^{1/2}} \tilde{\xi}_{i,n} - \frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell\Delta_n}^{\mathbb{Q}^b} \right) \left. \frac{d\mathbb{P}_y^b}{d\mathbb{Q}_y^b} \right|_{p\Delta_n} \right] \\ &= \mathbb{E} \left[\mathbf{K}_h \left(x - y - \frac{\tau_n}{p^{1/2}} \tilde{\xi}_{0,n} - \frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell\Delta_n} \right) M_{p\Delta_n}^b(y) \right] \end{aligned}$$

where W is a standard Brownian motion, $\tilde{\xi}_{0,n}$ is a standard Gaussian variable and where $M_t^b(y)$ is defined by

$$M_t^b(y) = \exp \left(\int_0^t b(y + W_s) \cdot dW_s - \frac{1}{2} \int_0^t |b(y + W_s)|^2 ds \right).$$

Note that $(M_t^b(y))_t$ can be seen as the solution of the stochastic differential equation

$$M_t^b(y) = 1 + \int_0^t M_s^b(y) b(y + W_s) \cdot dW_s.$$

Using Assumption 2 and Ito's formula, we have

$$(42) \quad M_t^b(y) = \exp \left(V(y) - V(y + W_t) - \frac{1}{2} \int_0^t |b(y + W_s)|^2 + \nabla \cdot b(y + W_s) ds \right).$$

This expression ensures that $y \mapsto M_t^b(y)$ is continuous, and therefore $M_t^b(Y)$ is measurable for any random variable Y . Then we have

$$\begin{aligned} \mathbb{E}_{\bar{\mu}^b}^b[\widehat{\nu}_{n,h,p}(x)] &= \int_{\mathbb{R}^d} \bar{\mu}^b(y) \mathbb{E} \left[\mathbf{K}_h \left(x - y - \frac{\tau_n}{p^{1/2}} \tilde{\xi}_{0,n} - \frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell\Delta_n} \right) M_{p\Delta_n}^b(y) \right] dy \\ &= B_1(x) + B_2(x) \end{aligned}$$

where

$$(43) \quad B_1(x) := \int_{\mathbb{R}^d} \bar{\mu}^b(y) \mathbb{E} \left[\mathbf{K}_h \left(x - y - \frac{\tau_n}{p^{1/2}} \tilde{\xi}_{0,n} - \frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell\Delta_n} \right) \right] dy,$$

$$(44) \quad B_2(x) := \int_{\mathbb{R}^d} \bar{\mu}^b(y) \mathbb{E} \left[\mathbf{K}_h \left(x - y - \frac{\tau_n}{p^{1/2}} \tilde{\xi}_{0,n} - \frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell\Delta_n} \right) (M_{p\Delta_n}^b(y) - 1) \right] dy$$

and we study each term separately. More precisely, we will prove the bounds

$$|B_1(x) - \mathbb{E}[\bar{\mu}^b(x - \tilde{\tau}_{n,p}\zeta)]| \lesssim \sum_{i=1}^d h_i^{\alpha_i} \quad \text{and} \quad |B_2| \lesssim \sqrt{p\Delta_n}$$

which prove Proposition 1.

Control of $B_1(x)$. Note that $p^{-1} \sum_{\ell=0}^{p-1} W_{\ell\Delta_n}$ is a centred Gaussian variable with covariance matrix $(p-1)(2p-1)/(12p)\Delta_n I_d$, independent from $\tilde{\xi}_{0,n}$. Therefore, we get:

$$B_1(x) = \int_{\mathbb{R}^d} \bar{\mu}^b(y) \mathbb{E} \left[\mathbf{K}_h \left(x - y - \tilde{\tau}_{n,p}\zeta \right) \right] dy,$$

where $\tilde{\tau}_{n,p}^2 = \tau_n^2/p + (p-1)(2p-1)\Delta_n/12p$ is defined in Equation (15), and ζ is a standard Gaussian random variable on \mathbb{R}^d . Moreover, we have

$$|B_1(x) - \mathbb{E}[\bar{\mu}^b(x - \tilde{\tau}_{n,p}\zeta)]| = \left| \mathbb{E} \left[\int_{\mathbb{R}^d} K(z) [\bar{\mu}^b(x - h \odot z - \tilde{\tau}_{n,p}\zeta) - \bar{\mu}^b(x - \tilde{\tau}_{n,p}\zeta)] dz \right] \right|,$$

where $h \odot z = (h_1 z_1, \dots, h_d z_d)$. We then use a Taylor expansion and the Hölder regularity of the density $\bar{\mu}^b$ and the level of the kernel K , we get

$$\left| \mathbb{E} \left[\int_{\mathbb{R}^d} \mathbf{K}(z) [\bar{\mu}^b(x - h \odot z - \tilde{\tau}_{n,p}\zeta) - \bar{\mu}^b(x - \tilde{\tau}_{n,p}\zeta)] dz \right] \right| \lesssim \sum_{i=1}^d \frac{\int |z_j|^{\alpha_j} |\mathbf{K}(z)| dz}{[\alpha_i]!} h_i^{\alpha_i} \lesssim \sum_{i=1}^d h_i^{\alpha_i}.$$

Control of $B_2(x)$. First, when $p = 1$, we have

$$\int_{\mathbb{R}^d} \bar{\mu}^b(y) \mathbb{E} \left[\mathbf{K}_h \left(x - y - \frac{\tau_n}{p^{1/2}} \tilde{\xi}_{0,n} \right) (M_{p\Delta_n}^b(y) - 1) \right] dy = 0$$

since $(M_t^b(y))_t$ is a martingale independent of $\tilde{\xi}_{0,n}$. For $p \geq 2$, the situation becomes more intricate. We first state the following lemma.

Lemma 6. *Under Assumptions 1, 2 and 3, there exists a constant $M > 0$ which is uniform over $\Sigma(\mathbf{b})$, such that for all $x \in \mathbb{R}^d$*

$$|b(x)e^{-V(x)}| \leq M.$$

Proof. First, note that the function $x \mapsto b(x)e^{-V(x)}$ is continuous and hence bounded on any compact set. Precisely, this function is bounded on the ball centered to 0 and of radius $2\tilde{\rho}_b$, where $\tilde{\rho}_b$ is defined in Assumption 3. We now consider x , such that $|x| \geq 2\tilde{\rho}_b$. With a classical Taylor expansion argument, we get that $V(x) \geq V_0 + \tilde{C}_b|x|/2$.

Moreover, Assumption 1 ensures that

$$|b(x)| \leq C_1(1 + |x|),$$

for some constant $C_1 > 0$ so we get

$$|b(x)e^{-V(x)}| \leq C_1(1 + |x|)e^{-\tilde{C}_b|x|+V_0},$$

which is bounded, concluding the proof. \square

We can now come back to the control of $B_2(x)$, we write $\bar{W}_p^n = p^{-1} \sum_{k=0}^{p-1} W_{k\Delta_n}$ for conciseness. Note that since $\tilde{\xi}_{i,n}$ and \bar{W}_p^n are independent, we have

$$\begin{aligned} B_2(x) &= \int_{\mathbb{R}^d} \bar{\mu}^b(y) \mathbb{E} \left[\mathbf{K}_h \left(x - y - \frac{\tau_n}{p^{1/2}} \tilde{\xi}_{0,n} - \bar{W}_p^n \right) (M_{p\Delta_n}^b(y) - 1) \right] dy \\ &= \mathbb{E} \left[\int_{\mathbb{R}^d} \bar{\mu}^b(y) \mathbf{K}_h \left(x - y - \frac{\tau_n}{p^{1/2}} \tilde{\xi}_{0,n} - \bar{W}_p^n \right) (M_{p\Delta_n}^b(y) - 1) dy \right] \\ &= \mathbb{E} \left[\int_{\mathbb{R}^d} \mathbf{K}_h \left(x - y - \frac{\tau_n}{p^{1/2}} \tilde{\xi}_{0,n} \right) \bar{\mu}^b \left(y - \bar{W}_p^n \right) \left(M_{p\Delta_n}^b \left(y - \bar{W}_p^n \right) - 1 \right) dy \right] \\ &= \int_{\mathbb{R}^d} \mathbb{E} \left[\mathbf{K}_h \left(x - y - \frac{\tau_n}{p^{1/2}} \tilde{\xi}_{0,n} \right) \right] \mathbb{E} \left[\bar{\mu}^b \left(y - \bar{W}_p^n \right) \left(M_{p\Delta_n}^b \left(y - \bar{W}_p^n \right) - 1 \right) \right] dy. \end{aligned} \tag{45}$$

Since

$$\int_{\mathbb{R}^d} |\mathbf{K}_h(x)| dx = \int_{\mathbb{R}^d} |\mathbf{K}(x)| dx,$$

the proof of Proposition 1 is down to proving that

$$|\mathbb{E}[\bar{\mu}^b(y - \bar{W}_p^n)(M_{p\Delta_n}^b(y - \bar{W}_p^n) - 1)]| \lesssim \sqrt{p\Delta_n}$$

uniformly for $y \in \mathbb{R}^d$.

Step 1. The main idea of the proof is to perform a Taylor expansion of our quantity of interest around y . To do so, let us introduce for any $y \in \mathbb{R}^d$, $s \geq 0$ and $j \in \{1, \dots, d\}$,

$$Y_{y,s}^j : \mathbb{R} \rightarrow \mathbb{R}^d \\ u \mapsto (y_1 + W_s^1, \dots, y_{j-1} + W_s^{j-1}, u + W_s^j, y_{j+1} + W_s^{j+1}, \dots, y_d + W_s^d).$$

We also define for all $s \geq 0$, $j \in \{1, \dots, d\}$ and $y \in \mathbb{R}^d$ the function $\phi_{y,s}^j$ for $u \in \mathbb{R}$ by

$$\phi_{y,s}^j(u) = M_s^b(y_1, \dots, y_{j-1}, u, y_{j+1}, \dots, y_d).$$

The purpose of this first step is to show that for any $s \geq 0$, $j \in \{1, \dots, d\}$ and $y \in \mathbb{R}^d$, the function $\phi_{y,s}^j$ is continuously differentiable. Since $\bar{\mu}^b$ is continuously differentiable and for any $z \in \mathbb{R}^d$ $M_t^b(z)$ is positive, it is equivalent to prove that $\log(\phi_{y,s}^j)$ is continuously differentiable. In fact for any $y \in \mathbb{R}^d$, we write

$$\log(M_t^b(y)) = - \sum_{i=1}^d \int_0^t b^i(y + W_s) dW_s^i + \frac{1}{2} \int_0^t |b^i(y + W_s)|^2 ds,$$

where W^i and b^i stand for the i -th component of W and b respectively. Moreover, using the fact that b is continuously differentiable in any direction, we obtain

$$b^i(Y_{y,s}^j(y_j)) = \int_0^{y_j} \partial_j b^i(Y_{y,s}^j(u)) du + b^i(Y_{y,s}^j(0)).$$

Then we can rewrite the previous equation

$$\begin{aligned} \log(M_t^b(y)) &= - \sum_{i=1}^d \left\{ \int_0^t \left(\int_0^{y_j} \partial_j b^i(Y_{y,s}^j(u)) du + b^i(Y_{y,s}^j(0)) \right) dW_s^i \right\} \\ &\quad - \frac{1}{2} \sum_{i=1}^d \left\{ \int_0^t \left| \int_0^{y_j} \partial_j b^i(Y_{y,s}^j(u)) du + b^i(Y_{y,s}^j(0)) \right|^2 ds \right\}. \end{aligned}$$

It is clear that the second double integral is continuously differentiable with respect to y_j . The situation is less straightforward for the first one due to the stochastic integral. We plan to use the following Fubini's Theorem for stochastic integrals.

Theorem 2 (Theorem 2.2 in [Mar12]). *Let (X, Σ, μ) be a σ -finite measure space. Let $S = M + A$ be a continuous semimartingale. Let $\psi : X \times [0, T] \rightarrow \mathbb{R}$ be progressively measurable and such that almost surely, we have,*

$$\begin{aligned} \int_X \left(\int_0^T |\psi(x, t)|^2 d\langle M \rangle_t \right)^{\frac{1}{2}} d\mu(x) &< \infty, \\ \int_X \int_0^T |\psi(x, t)| dA_t d\mu(x) &< \infty. \end{aligned}$$

Then, for all $t \in [0, T]$, one has that almost surely

$$\int_X \int_0^t \psi(r, x) dS_r d\mu(x) = \int_0^t \int_X \psi(r, x) d\mu(x) dS_r.$$

Using the boundedness of the partial derivatives, we can easily see that this result applies in our case. Therefore, we get

$$\begin{aligned} \log(M_t^b(y)) &= - \sum_{i=1}^d \left\{ \int_0^{y_j} \left(\int_0^t \partial_j b^i(Y_{y,s}^j(u)) dW_s^i \right) du + \int_0^t b^i(Y_{y,s}^j(0)) dW_s^i \right\} \\ &\quad - \frac{1}{2} \sum_{i=1}^d \left\{ \int_0^t \left| \int_0^{y_j} \partial_j b^i(Y_{y,s}^j(u)) du + b^i(Y_{y,s}^j(0)) \right|^2 ds \right\}. \end{aligned}$$

Note also that $Y_{y,s}^j(u)$ and $Y_{y,s}^j(0)$ do not depend on y_j by definition. Therefore, for all $1 \leq j \leq d$, and for all $y \in \mathbb{R}^d$, the function $\phi_{y,s}^j$ is differentiable. Moreover, we have

$$(46) \quad \partial_j \log(M_t^b(y)) = - \sum_{i=1}^d \left\{ \int_0^t \partial_j b^i(y + W_s) dW_s^i + \int_0^t (b^i \partial_j b^i)(y + W_s) ds \right\},$$

as $y + W_s = Y_{y,s}^j(y_j)$. We now want to prove that these partial derivatives are continuous. To do so, we plan to apply component by component the following Kolmogorov–Chentsov theorem, due to [AL14]

Theorem 3 (Theorem 3.1. of [AL14]). *Let $D \subset \mathbb{R}^d$ be a bounded domain of cone type and let $X : \Omega \times D \rightarrow \mathbb{R}$ be a random field on D . Assume that there exist $m \geq 1$, $p > 1$, $\epsilon \in (0, p]$, and $C > 0$ such that the weak derivatives $\partial^\beta X$ are in $L^p(\Omega \times D)$ and*

$$\mathbb{E} \left(\left| \partial^\beta X(x) - \partial^\beta X(y) \right|^p \right) \leq C |x - y|^{d+\epsilon}$$

for all $x, y \in D$ and any multi-index $\beta \in \mathbb{N}^n$ with $|\beta| \leq d$. Then X has a modification that is locally of class \bar{C}^t for all $t < d + \min\{\epsilon/p, 1 - d/p\}$.

For all $u, v \in \mathbb{R}$, $p > 1$, and $t \geq 0$, we obtain

$$\begin{aligned} \left| \frac{d}{du} \log(\phi_{y,t}(u)) - \frac{d}{du} \log(\phi_{y,t}(v)) \right|^p &\lesssim \left| \int_0^t (\partial_j b(Y_{y,s}^j(u)) - \partial_j b(Y_{y,s}^j(v))) dW_s \right|^p \\ &+ \left| \int_0^t (b(Y_{y,s}^j(u)) - b(Y_{y,s}^j(v))) \partial_j b(Y_{y,s}^j(u)) ds \right|^p \\ &+ \left| \int_0^t (\partial_j b(Y_{y,s}^j(u)) - \partial_j b(Y_{y,s}^j(v))) b(Y_{y,s}^j(v)) ds \right|^p. \end{aligned}$$

Now, taking the expectation, we get

$$\mathbb{E} \left[\left| \frac{d}{du} \log(\phi_{y,t}^j(u)) - \frac{d}{du} \log(\phi_{y,t}^j(v)) \right|^p \right] \lesssim \text{I} + \text{II} + \text{III},$$

where

$$\begin{aligned} \text{I} &= \mathbb{E} \left[\left| \int_0^t (\partial_j b(Y_{y,s}^j(u)) - \partial_j b(Y_{y,s}^j(v))) dW_s \right|^p \right]; \\ \text{II} &= \mathbb{E} \left[\left| \int_0^t (b(Y_{y,s}^j(u)) - b(Y_{y,s}^j(v))) \partial_j b(Y_{y,s}^j(u)) ds \right|^p \right]; \\ \text{III} &= \mathbb{E} \left[\left| \int_0^t (\partial_j b(Y_{y,s}^j(u)) - \partial_j b(Y_{y,s}^j(v))) b(Y_{y,s}^j(v)) ds \right|^p \right]. \end{aligned}$$

We first control the term I using the Burkholder-Davis-Gundy's inequality and we get

$$\text{I} \leq \left| \int_0^t \mathbb{E} [|\partial_j b(Y_{y,s}^j(u)) - \partial_j b(Y_{y,s}^j(v))|^2] ds \right|^{p/2}.$$

Moreover, we know that $\partial_j b$ is $(\alpha_j - 1) \wedge 1$ -Hölder in the j -th variable. Using that $\alpha_j \geq 2$, we know that $\partial_j b$ is Lipschitz continuous. Then,

$$\text{I} \lesssim t^{p/2} |u - v|^p.$$

For the second term, we use the boundedness of the partial derivatives to get

$$\text{II} \lesssim t^p |u - v|^p.$$

Moreover, for the last term, using the fact that b is at most linear and the control of moments of order p of the Brownian motion, we get

$$\text{III} \lesssim |u - v|^p,$$

where the constant here may depend on $(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_d)$ and t . Finally, as $\alpha_j \geq 2$, we have that $p \geq 1/\epsilon$, for some $\epsilon \in (0, p]$. Applying Theorem 3, we get that $\phi_{y,t}^j$ is continuously differentiable on \mathbb{R} .

Step 2. For any $t \geq 0$, we define $\psi : \mathbb{R}^d \ni y \mapsto \bar{\mu}^b(y)(M_t^b(y) - 1)$. Moreover, recall that we write $\bar{W}_p^n = p^{-1} \sum_{\ell=0}^{p-1} W_{\ell \Delta_n}$. From the previous step, we write for any $y \in \mathbb{R}^d$,

$$\psi(y - \bar{W}_p^n) - \psi(y) = \sum_{j=1}^d E_j,$$

where

$$E_j = \psi((y - \bar{W}_p^n)_1, \dots, (y - \bar{W}_p^n)_j, y_{j+1}, \dots, y_d) - \psi((y - \bar{W}_p^n)_1, \dots, (y - \bar{W}_p^n)_{j-1}, y_j, \dots, y_d),$$

for any $j \in \{2, \dots, d-1\}$, and

$$E_1 = \psi((y - \bar{W}_p^n)_1, y_2, \dots, y_d) - \psi(y_1, \dots, y_d),$$

$$E_d = \psi((y - \bar{W}_p^n)_1, \dots, (y - \bar{W}_p^n)_d) - \psi((y - \bar{W}_p^n)_1, \dots, (y - \bar{W}_p^n)_{d-1}, y_d).$$

Then, using Step 1, we have for any $j \in \{1, \dots, d\}$

$$E_j = \int_0^1 (\partial_j \psi)((y - \bar{W}_p^n)_1, \dots, (y - \bar{W}_p^n)_{j-1}, (y - \lambda \bar{W}_p^n)_j, y_{j+1}, \dots, y_d) (\bar{W}_p^n)_j d\lambda.$$

Then, we have

$$\begin{aligned} & (p\Delta_n)^{-1/2} |\mathbb{E}[\psi(y - \bar{W}_p^n) - \psi(y)]| \\ & \leq \sum_{j=1}^d \mathbb{E} \left[\left| \int_0^1 (\partial_j \psi)((y - \bar{W}_p^n)_1, \dots, (y - \bar{W}_p^n)_{j-1}, (y - \lambda \bar{W}_p^n)_j, y_{j+1}, \dots, y_d) (\bar{W}_p^n)_j d\lambda \right|^2 \right]^{1/2}. \end{aligned}$$

We now introduce the following notation for any $\lambda \in (0, 1)$,

$$Y_{y,n,p}^j(\lambda) = ((y - \bar{W}_p^n)_1, \dots, (y - \bar{W}_p^n)_{j-1}, (y - \lambda \bar{W}_p^n)_j, y_{j+1}, \dots, y_d)$$

Moreover, we have the following decomposition

$$\mathbb{E} \left[\left| \int_0^1 \partial_j \psi(Y_{y,n,p}^j(\lambda)) d\lambda \right|^2 \right] \lesssim \mathbf{I} + \mathbf{II},$$

where

$$\mathbf{I} = \int_0^1 \mathbb{E} [|\partial_j \bar{\mu}^b(Y_{y,n,p}^j(\lambda))(M_{p\Delta_n}^b(Y_{y,n,p}^j(\lambda)) - 1)|^2] d\lambda$$

$$\mathbf{II} = \int_0^1 \mathbb{E} [|\mu^b(Y_{y,n,p}^j(\lambda)) \partial_j (M_{p\Delta_n}^b(Y_{y,n,p}^j(\lambda)) - 1)|^2] d\lambda.$$

In order to control the term \mathbf{I} , we write

$$\mathbf{I} = Z_V^{-1} \int_0^1 \mathbb{E} [|b^j(Y_{y,n,p}^j(\lambda)) \exp(-2Y_{y,n,p}^j(\lambda)) (M_{p\Delta_n}^b(Y_{y,n,p}^j(\lambda)) - 1)|^2] d\lambda.$$

Moreover, using Lemma 6, we get that

$$\mathbf{I} \lesssim 1 + \int_0^1 \mathbb{E} [|\mu^b(Y_{y,n,p}^j(\lambda))|^2] d\lambda.$$

Finally, using Equation (42) we obtain for any $\lambda \in (0, 1)$,

$$\begin{aligned} M_{p\Delta_n}^b(Y_{y,n,p}^j(\lambda)) &= \exp(V(Y_{y,n,p}^j(\lambda)) - V(Y_{y,n,p}^j(\lambda) + W_t)) \\ &\quad - \frac{1}{2} \int_0^t |b(Y_{y,n,p}^j(\lambda) + W_s)|^2 + \nabla \cdot b(Y_{y,n,p}^j(\lambda) + W_s) ds. \end{aligned}$$

Finally, leveraging on the fact that V is bounded from below, and both $\bar{\mu}^b$ and $\nabla \cdot b$ are bounded, we get

$$\mathbf{I} \lesssim 1.$$

Step 3. We now move on to the control of term \mathbf{II} . First, for any $j \in \{1, \dots, d\}$ we have

$$\frac{\partial_j M_{p\Delta_n}^b(Y_{y,n,p}^j(\lambda))}{M_{p\Delta_n}^b(Y_{y,n,p}^j(\lambda))} = - \sum_{i=1}^d \left(\int_0^{p\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_s) dW_s^i + \int_0^{p\Delta_n} b^i \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_s) ds \right)$$

Then we have $\mathbf{II} \leq \mathbf{II}_A + \mathbf{II}_B$, where

$$\mathbf{II}_A = 2 \int_0^1 \mathbb{E} \left[|\mu^b(Y_{y,n,p}^j(\lambda)) M_{p\Delta_n}^b(Y_{y,n,p}^j(\lambda)) \sum_{i=1}^d \int_0^{p\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_s) dW_s^i|^2 \right] d\lambda$$

$$\mathbb{I}_B = 2 \int_0^1 \mathbb{E} \left[|\bar{\mu}^b(Y_{y,n,p}^j(\lambda)) M_{p\Delta_n}^b(Y_{y,n,p}^j(\lambda)) \sum_{i=1}^d \int_0^{p\Delta_n} (b^i \partial_j b^i)(Y_{y,n,p}^j(\lambda) + W_s)|^2 \right] d\lambda$$

Moreover using consecutively Jensen and Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned} \mathbb{I}_A &\lesssim \sum_{i=1}^d \int_0^1 \mathbb{E} \left[\left| \bar{\mu}^b(Y_{y,n,p}^j(\lambda)) M_{p\Delta_n}^b(Y_{y,n,p}^j(\lambda)) \int_0^{p\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_u) dW_u^i \right|^2 \right] d\lambda \\ &\lesssim \sum_{i=1}^d \int_0^1 \mathbb{E} \left[|\bar{\mu}^b(Y_{y,n,p}^j(\lambda)) M_{p\Delta_n}^b(Y_{y,n,p}^j(\lambda))|^4 \right]^{1/2} \mathbb{E} \left[\left| \int_0^{p\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_u) dW_u^i \right|^4 \right]^{1/2} d\lambda. \end{aligned}$$

Moreover, from Equation (42) and the boundedness of $\bar{\mu}^b$, we get once again

$$\int_0^1 \mathbb{E} \left[|\bar{\mu}^b(Y_{y,n,p}^j(\lambda)) M_{p\Delta_n}^b(Y_{y,n,p}^j(\lambda))|^4 \right]^{1/2} \lesssim 1.$$

Then, we only need to deal for each $i \in \{1, \dots, d\}$ with

$$\mathbb{E} \left[\left| \int_0^{p\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_u) dW_u^i \right|^4 \right].$$

Let $\mathcal{G}_{p,\Delta_n} = \sigma(W_{k\Delta_n}, 0 \leq k \leq p)$, the σ -field generated by the discrete observation of W at times $k\Delta_n$ for $k \in \{0, \dots, p\}$, so that conditionally on \mathcal{G}_{p,Δ_n} ,

$$(47) \quad \begin{cases} dW_u = I_k du + d\mathcal{W}_u^{*,k}, \\ d\mathcal{W}_u^{*,k} = \frac{-\mathcal{W}_u^{*,k}}{(k+1)\Delta_n - u} du + dB_u^k, \end{cases}$$

in distribution, where for each $k \in \{0, \dots, p-1\}$

$$I_k := \frac{W_{(k+1)\Delta_n} - W_{k\Delta_n}}{\Delta_n},$$

$\mathcal{W}^{*,k}$ is a Brownian Bridge on $[k\Delta_n; (k+1)\Delta_n]$, and B^k is a d -dimensional Brownian motion independent of \mathcal{G}_{p,Δ_n} . Then, we write

$$\begin{aligned} &\mathbb{E}^b \left[\left| \int_0^{p\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_u) dW_u^i \right|^4 \right] = \mathbb{E}^b \left[\mathbb{E}^b \left[\left| \int_0^{p\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_u) dW_u^i \right|^4 \middle| \mathcal{G}_{p,\Delta} \right] \right] \\ &= \mathbb{E}^b \left[\mathbb{E}^b \left[\left| \sum_{\ell=0}^{p-1} \int_{\ell\Delta_n}^{(\ell+1)\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_u) dW_u^i \right|^4 \middle| \mathcal{G}_{p,\Delta} \right] \right] \\ &\lesssim \left(\mathbb{E}^b \left[\mathbb{E}^b \left[\left| \sum_{\ell=0}^{p-1} \int_{\ell\Delta_n}^{(\ell+1)\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_u) I_\ell^i du \right|^4 \middle| \mathcal{G}_{p,\Delta} \right] \right] \right. \\ &\quad + \mathbb{E}^b \left[\mathbb{E}^b \left[\left| \sum_{\ell=0}^{p-1} \int_{\ell\Delta_n}^{(\ell+1)\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_u) \frac{\mathcal{W}_u^{*,\ell,i}}{(\ell+1)\Delta_n - u} du \right|^4 \middle| \mathcal{G}_{p,\Delta} \right] \right] \\ &\quad \left. + \mathbb{E}^b \left[\mathbb{E}^b \left[\left| \sum_{\ell=0}^{p-1} \int_{\ell\Delta_n}^{(\ell+1)\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_u) dB_u^{\ell,i} \right|^4 \middle| \mathcal{G}_{p,\Delta} \right] \right] \right). \end{aligned}$$

Using both boundedness of the partial derivatives and Jensen inequality, we get

$$\mathbb{E}^b \left[\mathbb{E}^b \left[\left| \sum_{\ell=0}^{p-1} \int_{\ell\Delta_n}^{(\ell+1)\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_u) I_\ell^i du \right|^4 \middle| \mathcal{G}_{p,\Delta} \right] \right] \lesssim p^3 \Delta_n^4 \mathbb{E} \left[\sum_{\ell=0}^{p-1} I_\ell^4 \right] \lesssim p^4 \Delta_n^2.$$

For the second term, let us remark that by definition of the Brownian Bridge, for any $i \in \{1, \dots, d\}$, $\ell \in \{0, \dots, p-1\}$ and $u \in [\ell\Delta_n, (\ell+1)\Delta_n]$, we have $\mathcal{W}_u^{*,\ell,i}$ is gaussian variable with $\mathbb{E}[\mathcal{W}_u^{*,\ell,i}] = 0$ and $\text{Var}[\mathcal{W}_u^{*,\ell,i}] = u$, as $\mathcal{W}_{\ell\Delta_n}^{*,\ell,i} = \mathcal{W}_{(\ell+1)\Delta_n}^{*,\ell,i} = 0$. This allows us to obtain

$$\mathbb{E}[\mathcal{W}_u^{*,\ell,i}|^4] = \frac{((\ell+1)\Delta_n - u)^4 (u - \ell\Delta_n)^4}{\Delta_n^4}.$$

Using once again the boundedness of $\bar{\mu}^b$, we obtain

$$\mathbb{E}^b \left[\mathbb{E}^b \left[\left| \sum_{\ell=0}^{p-1} \int_{\ell\Delta_n}^{(\ell+1)\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_u) \frac{\mathcal{W}_u^{*,\ell,i}}{(\ell+1)\Delta_n - u} du \right|^4 \middle| \mathcal{G}_{p,\Delta} \right] \right] \lesssim p^4 \Delta_n^2.$$

Lastly, we use the Burkholder-Davis-Gundy Inequality to handle the final term. Every previous step has paved the way for this one. Conditionally to \mathcal{G}_{p,Δ_n} , we have that for each $i \in \{1, \dots, d\}$

$$\left(\int_0^t \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_u) dB_u^{\ell,i} \right)_{t \geq 0},$$

is a martingale. This would not have been true without the conditioning. Then, we get

$$\begin{aligned} \mathbb{E}^b \left[\mathbb{E}^b \left[\left| \sum_{\ell=0}^{p-1} \int_{\ell\Delta_n}^{(\ell+1)\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_u) dB_u^{\ell,i} \right|^4 \middle| \mathcal{G}_{p,\Delta} \right] \right] &\lesssim \left(\int_0^{p\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_u)^2 du \right)^2 \\ &\lesssim p^2 \Delta_n^2. \end{aligned}$$

We obtain

$$\mathbb{E}^b \left[\left| \int_0^{p\Delta_n} \partial_j b^i(Y_{y,n,p}^j(\lambda) + W_u) dW_u^i \right|^4 \right] \lesssim p^4 \Delta_n^2.$$

Combining this to (47) gives $\Pi_A \lesssim p^2 \Delta_n$ which is bounded, thanks to Equation (39).

Step 4. Control of Π_B . Using Cauchy-Schwarz inequality, we get

$$(48) \quad \Pi_B \lesssim \sum_{i=1}^d \int_0^1 \mathbb{E}^b \left[\left| \int_0^{p\Delta_n} \sqrt{\bar{\mu}^b(Y_{y,n,p}^j(\lambda))} (b^i \partial_j b^i)(Y_{y,n,p}^j(\lambda) + W_u) du \right|^4 \right]^{1/2} d\lambda.$$

Moreover for $i \in \{1, \dots, d\}$ and $\lambda \in (0, 1)$,

$$\begin{aligned} &\mathbb{E} \left[\left| \int_0^{p\Delta_n} \sqrt{\bar{\mu}^b(Y_{y,n,p}^j(\lambda))} (b^i \partial_j b^i)(Y_{y,n,p}^j(\lambda) + W_u) du \right|^4 \right] \\ &\lesssim (p\Delta_n)^3 \int_0^{p\Delta_n} \mathbb{E} [\bar{\mu}^b(Y_{y,n,p}^j(\lambda))^2 (b^i \partial_j b^i)(Y_{y,n,p}^j(\lambda) + W_u)^4] du \\ &\lesssim (p\Delta_n)^3 \int_0^{p\Delta_n} \mathbb{E} [\bar{\mu}^b(Y_{y,n,p}^j(\lambda))^2 b^i(Y_{y,n,p}^j(\lambda) + W_u)^4] du \\ &\lesssim (p\Delta_n)^4, \end{aligned}$$

where, at the last line we used a similar argument as the one of the proof of Lemma 6 to bound $\mathbb{E}^b [\bar{\mu}^b(Y_{y,n,p}^j(\lambda))^2 b^i(Y_{y,n,p}^j(\lambda) + W_u)^4]$ uniformly in n, p, y, u, i and λ . Therefore $\Pi_B \lesssim (p\Delta_n)^2$, and $\Pi \lesssim 1$.

Conclusion. Combining the previous four steps, we have

$$|\mathbb{E}[\bar{\mu}^b(y - \bar{W}_p^n)(M_{p\Delta_n}^b(y - \bar{W}_p^n) - 1)]| \lesssim \sqrt{p\Delta_n}$$

and therefore, for all $x \in \mathbb{R}^d$,

$$(49) \quad B_2(x) \lesssim \sqrt{p\Delta_n},$$

which concludes the proof.

APPENDIX C. PROOF OF PROPOSITION 2

We now plan to use Proposition 1 to prove the estimate (21). Recall that for all $x \in \mathbb{R}^d$, $\hat{\mu}_{n,h,p}(x)$ is defined in Equation (20) by

$$\hat{\mu}_{n,h,p}(x) = \sum \mathbf{u}_\gamma \hat{\nu}_{n,h,p}(x + \gamma \tilde{\tau}_{n,p})$$

where the sum holds over all $\gamma = (\gamma_1, \dots, \gamma_d) \in \{0, \dots, \ell\}^d$. Thus Proposition 1 ensures that (21) is proved once we control the following quantity

$$(50) \quad \left| \sum \mathbf{u}_\gamma \mathbb{E}[\bar{\mu}^b(x + \tilde{\tau}_{n,p}(\gamma - \zeta))] - \bar{\mu}^b(x) \right|$$

We first provide a technical Lemma in dimension $d = 1$.

Lemma 7. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a α -Hölder function and ζ be a standard real Gaussian variable. Then we have*

$$\left| \sum_{\gamma=0}^l u_\gamma \mathbb{E}[f(x + \tau(\gamma - \zeta))] - f(x) \right| \leq \tau^\alpha \frac{\mathcal{L}}{[\alpha]} \sum_{\gamma=0}^l \mathbb{E}[|\gamma - \zeta|^\alpha].$$

Proof. By Taylor formula and using the α -Hölder property of f , we get the existence of $\varepsilon \in [0, 1]$ such that

$$\begin{aligned} \sum_{\gamma=0}^l u_\gamma \mathbb{E}[f(x + \tau(\gamma - \zeta))] &= \sum_{\gamma=0}^l u_\gamma \sum_{k=0}^{[\alpha]-1} \frac{f^{(k)}(x)}{k!} \mathbb{E}[(\gamma - \zeta)^k] + R_{\alpha,l}(\varepsilon) \\ &= \sum_{k=0}^{[\alpha]-1} \frac{f^{(k)}(x)}{k!} \sum_{\gamma=0}^l u_\gamma \sum_{\beta=0}^k \frac{k!}{\beta!(k-\beta)!} (-1)^\beta m_\beta \gamma^{k-\beta} + R_{\alpha,l}(\varepsilon), \end{aligned}$$

where

$$R_{\alpha,l}(\varepsilon) = \mathbb{E} \left[\sum_{\gamma=0}^l \frac{f^{([\alpha])}(x + \varepsilon\tau(\gamma - \zeta))}{([\alpha])!} (\varepsilon\tau)^{[\alpha]} (\gamma - \zeta)^{[\alpha]} \right].$$

Then, from Equation (19), we get that $\sum_{\gamma=0}^l u_\gamma \sum_{k=0}^{[\alpha]-1} f^{(k)}(x) \mathbb{E}[(\gamma - \zeta)^k]/k! = f(x)$. Moreover as f is α -Hölder, we conclude that

$$R_{\alpha,l}(\varepsilon) \leq \tau^\alpha \frac{\mathcal{L}}{[\alpha]} \sum_{\gamma=0}^l \mathbb{E}[|\gamma - \zeta|^\alpha].$$

□

We now extend Lemma 7 to our setup using an appropriate induction. We first introduce $\theta = \theta(x, \gamma, n, p) = x + \gamma \tilde{\tau}_{n,p} \in \mathbb{R}^d$. We prove by induction on $0 \leq I \leq d$ that for any $0 \leq \gamma_{I+1}, \dots, \gamma_d \leq l$,

$$\left| \sum_{\gamma_{I+1}=0}^l \dots \sum_{\gamma_1=0}^l \prod_{i=1}^I u_{\gamma_i} \mathbb{E}[\bar{\mu}^b(\theta - \tilde{\tau}_{n,p}\zeta)] - \mathbb{E}[\bar{\mu}^b(x_1, \dots, x_I, \theta_{I+1} - \tilde{\tau}_{n,p}\zeta_{I+1}, \dots, \theta_d - \tilde{\tau}_{n,p}\zeta_d)] \right|,$$

is bounded up to some constant by $\sum_{i=1}^I \tilde{\tau}_{n,p}^{\alpha_i}$. For $I = 0$, this is trivial and for $I = d$, the second expectation does not contain any random variable any-more and thus equals $\bar{\mu}^b(x)$, which proves Equation (50). Let $I < d$ at which this property is proved. We fix $0 \leq \gamma_{I+2}, \dots, \gamma_d \leq l$. We have

$$\sum_{\gamma_{I+1}=0}^l \dots \sum_{\gamma_1=0}^l \prod_{i=1}^{I+1} u_{\gamma_i} \mathbb{E}[\bar{\mu}^b(\theta - \tilde{\tau}_{n,p}\zeta)] = \sum_{\gamma_{I+1}=0}^l u_{\gamma_{I+1}} \sum_{\gamma_I=0}^l \dots \sum_{\gamma_1=0}^l \prod_{i=1}^I u_{\gamma_i} \mathbb{E}[\bar{\mu}^b(\theta - \tilde{\tau}_{n,p}\zeta)]$$

Recall also that by Assumption, $\bar{\mu}^b$ is α_{I+1} -Hölder regular in the $(I+1)$ -th variable. Therefore, by induction and using Lemma 7 we get

$$\begin{aligned} &\left| \sum_{\gamma_{I+1}=0}^l u_{\gamma_{I+1}} \mathbb{E}[\bar{\mu}^b(x_1, \dots, x_I, \theta_{I+1} - \tilde{\tau}_{n,p}\zeta_{I+1}, \dots, \theta_d - \tilde{\tau}_{n,p}\zeta_d) | \zeta_{I+2}, \dots, \zeta_d] \right. \\ &\quad \left. - \bar{\mu}^b(x_1, \dots, x_{I+1}, \theta_{I+2} - \tilde{\tau}_{n,p}\zeta_{I+2}, \dots, \theta_d - \tilde{\tau}_{n,p}\zeta_d) \right| \lesssim \tilde{\tau}_{n,p}^{\alpha_{I+1}}. \end{aligned}$$

Note that Lemma 7 ensures that the last inequality holds up to a constant uniformly in n, p, b .

APPENDIX D. PROOF OF PROPOSITION 3

D.1. Structure and completion of the proof. First note that by (28), it enough to prove Proof of Proposition 3 with $\hat{v}_{n,h,p}(x)$ instead of $\hat{\mu}_{n,h,p}(x)$. We now introduce the main notations used in this proof. Let $n_p = \lfloor n/p \rfloor$. For each j , we write

$$\tilde{\xi}_j = \frac{1}{\sqrt{p}} \sum_{\ell=0}^{p-1} \xi_{j+\ell}$$

so that $(\tilde{\xi}_j)_j$ is a sequence of i.i.d standard d -dimensional Gaussian variables. Then we write

$$(51) \quad \mathfrak{C}_p^b(k; x) = \text{Cov}_{\bar{\mu}^b} \left(\mathbf{K}_h(x - p^{-1} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} + \frac{\tau_n}{\sqrt{p}} \tilde{\xi}_k), \mathbf{K}_h(x - p^{-1} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n} + \frac{\tau_n}{\sqrt{p}} \tilde{\xi}_0) \right).$$

for each k and x , and

$$(52) \quad \mathfrak{C}_p^b(k; x, u, v) = \text{Cov}_{\bar{\mu}^b} \left(\mathbf{K}_h(x - p^{-1} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} + u), \mathbf{K}_h(x - p^{-1} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n} + v) \right)$$

$$(53) \quad \mathfrak{D}_p^b(k; x, u, v) = \mathbb{E}_{\bar{\mu}^b} [\mathbf{K}_h(x - \frac{1}{p} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} + u) \mathbf{K}_h(x - \frac{1}{p} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n} + v)]$$

for each k, x, u and v . Note that for $k > 0$, the noises $\tilde{\xi}_0$ and $\tilde{\xi}_k$ are independent and therefore

$$(54) \quad \mathfrak{C}_p^b(k; x) \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathfrak{C}_p^b(k; x, u, v) \chi_{p,n}(du) \chi_{p,n}(dv)$$

where $\chi_{p,n}$ stands for the law of $\tau_n p^{-1/2} \tilde{\xi}_0$ so that $\chi_{p,n}(du) = \varphi_{\tau_n p^{-1/2}}(u) du$. Moreover, we also have

$$(55) \quad \mathfrak{C}_p^b(k; x) \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathfrak{D}_p^b(k; x, u, v) \chi_{p,n}(du) \chi_{p,n}(dv)$$

By definition of $\hat{\nu}_{n,h,p}(x)$, we have

$$\begin{aligned} \text{Var}_{\bar{\mu}^b}(\hat{\nu}_{n,h,p}(x)) &= \text{Var}_{\bar{\mu}^b} \left(\frac{1}{n_p} \sum_{k=0}^{n_p-1} \mathbf{K}_h(x - \frac{1}{p} \sum_{\ell=0}^{p-1} Y_{kp+\ell,n}) \right) \\ &= \frac{1}{n_p^2} \sum_{i,j=0}^{n_p-1} \text{Cov}_{\bar{\mu}^b} \left(\mathbf{K}_h(x - \frac{1}{p} \sum_{\ell=0}^{p-1} X_{(ip+\ell)\Delta_n} + \frac{\tau_n}{\sqrt{p}} \tilde{\xi}_i), \mathbf{K}_h(x - \frac{1}{p} \sum_{\ell=0}^{p-1} X_{(jp+\ell)\Delta_n} + \frac{\tau_n}{\sqrt{p}} \tilde{\xi}_j) \right). \end{aligned}$$

Since X and ξ are both ergodic and mutually independent, we deduce that

$$(56) \quad \text{Var}_{\bar{\mu}^b} \left(\frac{1}{n_p} \sum_{k=0}^{n_p-1} \mathbf{K}_h(x - \frac{1}{p} \sum_{\ell=0}^{p-1} Y_{kp+\ell,n}) \right) = \frac{2}{n_p^2} \sum_{k=0}^{n_p-1} (n_p - k) \mathfrak{C}_p^b(k; x).$$

We now present several bounds on the coefficients $\mathfrak{C}_p^b(k; x)$.

Lemma 8 (Instantaneous correlations). *For each $k \geq 0$, we have*

$$\mathfrak{C}_p^b(k; x) \lesssim \prod_{i=1}^d h_i^{-1}.$$

Lemma 9 (Short-term correlations). *For any $1 \leq k_1 \leq d$ and any $k \geq 2$, we have*

$$\mathfrak{C}_p^b(k; x) \lesssim (kp\Delta_n)^{-k_1/2} \prod_{i=k_1+1}^d h_i^{-1}.$$

Lemma 10 (Mid-term correlations). *For any $k \geq 1$ such that $(k-1)p\Delta_n \geq 1$, we have*

$$\mathfrak{C}_p^b(k; x) \lesssim 1.$$

Lemma 11 (Long-term correlations). *For any $k \geq 1$, we have*

$$\mathfrak{C}_p^b(k; x) \lesssim e^{-C_{PI}^{-1}(k-1)p\Delta_n} \left(\prod_{i=1}^d h_i^{-1} \right)^2.$$

Following [AG23], the rest of the proof consists in identifying the best way to combine these lemmas with (56), depending on d and k_0 . Recall that k_0 is such that $\alpha_1 = \dots = \alpha_{k_0} < \alpha_{k_0+1} \leq \dots \leq \alpha_d$. We now introduce $0 \leq j_1 \leq j_2 \leq j_3 \leq j_4$ such that $j_3 = \lceil (p\Delta_n)^{-1} \rceil$ and

$$j_4 = \max \left(\min \left(\left\lceil \frac{-2C_{PI} \log \prod_{i=1}^d h_i}{p\Delta_n} \right\rceil, n_p \right), j_3 \right).$$

Then we have

$$\frac{1}{n_p^2} \sum_{k=0}^{n_p-1} (n_p - k) \mathfrak{C}_p^b(k; x) = \frac{1}{n_p^2} \left(\sum_{k=0}^{j_1} + \sum_{k=j_1+1}^{j_2} + \sum_{k=j_2+1}^{j_3} + \sum_{k=j_3+1}^{j_4} + \sum_{k=j_4+1}^{n_p-1} \right) (n_p - k) \mathfrak{C}_p^b(k; x).$$

For the first sum, we use Lemma 8 so that we have

$$\frac{1}{n_p^2} \sum_{k=0}^{j_1} (n_p - k) \mathfrak{C}_p^b(k; x) \lesssim \frac{1}{n_p \prod_{i=1}^d h_i} (j_1 + 1).$$

For the second and third sums, we use Lemma 9 for some $1 \leq k_1 \leq d$ and $1 \leq k_2 \leq d$ to be chosen later. In that case, we get

$$\begin{cases} n_p^{-2} \sum_{k=j_1+1}^{j_2} (n_p - k) \mathfrak{C}_p^b(k; x) \lesssim n_p^{-1} \sum_{k=j_1+1}^{j_2} (kp\Delta_n)^{-k_1/2} \prod_{i=k_1+1}^d h_i^{-1}, \\ n_p^{-2} \sum_{k=j_2+1}^{j_3} (n_p - k) \mathfrak{C}_p^b(k; x) \lesssim n_p^{-1} \sum_{k=j_2+1}^{j_3} (kp\Delta_n)^{-k_2/2} \prod_{i=k_2+1}^d h_i^{-1}. \end{cases}$$

Since $j_3 k \Delta_n \geq 1$, we use Lemma 10 for the fourth sum and we get

$$\frac{1}{n_p^2} \sum_{k=j_3+1}^{j_4} (n_p - k) \mathfrak{C}_p^b(k; x) \lesssim \frac{1}{n_p} (j_4 - j_3).$$

Using the definition of j_4 , we have

$$\frac{1}{n_p^2} \sum_{k=j_3+1}^{j_4} (n_p - k) \mathfrak{C}_p^b(k; x) \lesssim \min \left(\frac{-2C_{PI} \log \prod_{i=1}^d h_i}{n_p p \Delta_n}, 1 \right).$$

Using also that $n_p p \Delta_n \gtrsim T_n$, we get

$$(57) \quad \frac{1}{n_p^2} \sum_{k=j_3+1}^{j_4} (n_p - k) \mathfrak{C}_p^b(k; x) \lesssim \min \left(\frac{\sum_{i=1}^d |\log h_i|}{T_n}, 1 \right).$$

For the last sum, we use Lemma 11 and we have

$$\begin{aligned} \frac{1}{n_p^2} \sum_{k=j_4+1}^{n_p-1} (n_p - k) \mathfrak{C}_p^b(k; x) &\lesssim \frac{1}{n_p} \sum_{k=j_4+1}^{n_p-1} \mathfrak{C}_p^b(k; x) \\ &\lesssim \frac{1}{n_p} \sum_{k=j_4+1}^{n_p-1} e^{-C_{PI}^{-1}(k-1)p\Delta_n} \left(\prod_{i=1}^d h_i^{-1} \right)^2 \\ &\lesssim \frac{1}{T_n} e^{-C_{PI}^{-1}j_4 p \Delta_n} \left(\prod_{i=1}^d h_i^{-1} \right)^2. \end{aligned}$$

Using the definition of j_4 , we have

$$(58) \quad \frac{1}{n_p^2} \sum_{k=j_4+1}^{n_p-1} (n_p - k) \mathfrak{C}_p^b(k; x) \lesssim \frac{1}{T_n} + \frac{1}{T_n} e^{-C_{PI}^{-1}T_n} \left(\prod_{i=1}^d h_i^{-1} \right)^2.$$

Case $d = 1, 2$. Taking $j_1 = 0$, $j_2 = j_3$ and $k_1 = d$, we obtain

$$\begin{aligned} \frac{2}{n_p^2} \sum_{k=j_1+1}^{j_2} \mathfrak{C}_p^b(k; x) &\lesssim \frac{2}{n_p^2} \sum_{k=1}^{j_2} (n_p - k) \frac{1}{kp\Delta_n} \\ &\lesssim \frac{\log(j_2)}{n_p p \Delta_n} = \frac{|\log(p\Delta_n)|}{T_n}. \end{aligned}$$

and therefore

$$\text{Var}_{\mu^b}^b(\hat{\nu}_{n, \mathbf{h}, p}(x)) \lesssim \frac{p\Delta_n}{T_n} \prod_{i=1}^d h_i^{-1} + \frac{|\log(p\Delta_n)|}{T_n} + \sum_{i=1}^d \frac{|\log(h_i)|}{T_n} + \frac{1}{T_n} e^{-CT_n} \left(\prod_{i=1}^d h_i^{-1} \right)^2.$$

Case $d \geq 3$, $k_0 = 1$ and $\alpha_2 < \alpha_3$ or $k_0 = 2$. We take $k_1 = 2$ and $k_2 = d$ and

$$j_1 = \left\lfloor \frac{h_1 h_2}{p \Delta_n} \right\rfloor \quad \text{and} \quad j_2 = \left\lfloor \frac{\left(\prod_{i \geq 3} h_i \right)^{\frac{d-2}{2}}}{p \Delta_n} \right\rfloor.$$

Therefore we have

$$\begin{aligned} \frac{1}{n_p^2} \sum_{k=0}^{j_1} (n_p - k) \mathfrak{E}_p^b(k; x) &\lesssim \frac{1}{n_p \prod_{i=1}^d h_i} \frac{h_1 h_2}{p \Delta_n} + \frac{1}{n_p} \prod_{i=1}^d h_i^{-1} \\ &\lesssim \frac{1}{T_n \prod_{i=3}^d h_i} + \frac{1}{n_p} \prod_{i=1}^d h_i^{-1} \end{aligned}$$

and

$$\begin{aligned} n_p^{-2} \sum_{k=j_1+1}^{j_2} (n_p - k) \mathfrak{E}_p^b(k; x) &\lesssim n_p^{-1} \sum_{k=j_1+1}^{j_2} (k p \Delta_n)^{-1} \prod_{i=3}^d h_i^{-1} \\ &\lesssim n_p^{-1} \log(j_2/j_1) (p \Delta_n)^{-1} \prod_{i=3}^d h_i^{-1} \\ &\lesssim T_n^{-1} \sum_{i=1}^d |\log(h_i)| \prod_{i=3}^d h_i^{-1} \end{aligned}$$

and

$$\begin{aligned} n_p^{-2} \sum_{k=j_2+1}^{j_3} (n_p - k) \mathfrak{E}_p^b(k; x) &\lesssim n_p^{-1} \sum_{k=j_2+1}^{j_3} (k p \Delta_n)^{-d/2} \\ &\lesssim n_p^{-1} j_2^{1-d/2} (p \Delta_n)^{-d/2} \\ &\lesssim T_n^{-1} \left(\prod_{i \geq 3} h_i \right)^{-1}. \end{aligned}$$

Combining these three bounds with (57) and (58), we get

$$\text{Var}_{\mu^b}(\hat{\nu}_{n, \mathbf{h}, p}(x)) \lesssim \frac{1}{T_n \prod_{i=3}^d h_i} + \frac{1}{n_p} \prod_{i=1}^d h_i^{-1} + \frac{1}{T_n} \sum_{i=1}^d |\log(h_i)| \prod_{i=3}^d h_i^{-1}$$

which proves (25).

Case $d \geq 3$, $k_0 \geq 3$. We take $k_1 = k_0$ and $k_2 = d$ and

$$j_1 = \left\lfloor \frac{\left(\prod_{i=1}^{k_0} h_i \right)^{2/k_0}}{p \Delta_n} \right\rfloor \quad \text{and} \quad j_2 = \left\lfloor \frac{1}{p \Delta_n} \right\rfloor.$$

Therefore, we have

$$\begin{aligned} \frac{1}{n_p^2} \sum_{k=0}^{j_1} (n_p - k) \mathfrak{E}_p^b(k; x) &\lesssim \frac{1}{n_p \prod_{i=1}^d h_i} \frac{\left(\prod_{i=1}^{k_0} h_i \right)^{2/k_0}}{p \Delta_n} + \frac{1}{n_p} \prod_{i=1}^d h_i^{-1} \\ &\lesssim \frac{1}{T_n} \prod_{i=1}^{k_0} (h_i)^{(2-k_0)/k_0} \prod_{i=k_0+1}^d h_i^{-1} + \frac{1}{n_p} \prod_{i=1}^d h_i^{-1}, \end{aligned}$$

and

$$\begin{aligned} n_p^{-2} \sum_{k=j_1+1}^{j_2} (n_p - k) \mathfrak{E}_p^b(k; x) &\lesssim n_p^{-1} \sum_{k=j_1+1}^{j_2} (k p \Delta_n)^{-k_0/2} \prod_{i=k_0+1}^d h_i^{-1} \\ &\lesssim n_p^{-1} j_1^{1-k_0/2} (p \Delta_n)^{-k_0/2} \prod_{i=k_0+1}^d h_i^{-1} \end{aligned}$$

$$\lesssim T_n^{-1} \prod_{i=1}^{k_0} (h_i)^{(2-k_0)/k_0} \prod_{i=k_0+1}^d h_i^{-1},$$

and using that $j_3 \leq j_2 + 1$

$$n_p^{-2} \sum_{k=j_2+1}^{j_3} (n_p - k) \mathfrak{E}_p^b(k; x) \lesssim n_p^{-1} (j_2 + 1)^{-d/2} (p\Delta_n)^{-d/2} \lesssim \frac{1}{n_p}.$$

Combining these three bounds with (57) and (58), we get

$$\text{Var}_{\bar{\mu}^b}(\hat{v}_{n, \mathbf{h}, p}(x)) \lesssim T_n^{-1} \prod_{i=1}^{k_0} (h_i)^{(2-k_0)/k_0} \prod_{i=k_0+1}^d h_i^{-1} + \frac{1}{n_p} \prod_{i=1}^d h_i^{-1} + \frac{\sum_{i=1}^d |\log h_i|}{T_n}$$

which proves (26).

Case $d \geq 3$, $k_0 = 1$ and $\alpha_2 = \alpha_3$ We take $k_1 = 1$ and $k_2 = 3$,

$$j_1 = 0 \quad \text{and} \quad j_2 = \left\lfloor \frac{h_2 h_3}{p\Delta_n} \right\rfloor.$$

Therefore, we have

$$\frac{1}{n_p^2} \sum_{k=0}^{j_1} (n_p - k) \mathfrak{E}_p^b(k; x) \lesssim \frac{1}{n_p \prod_{i=1}^d h_i}$$

and

$$\begin{aligned} n_p^{-2} \sum_{k=j_1+1}^{j_2} (n_p - k) \mathfrak{E}_p^b(k; x) &\lesssim n_p^{-1} \sum_{k=j_1+1}^{j_2} (kp\Delta_n)^{-1/2} \prod_2^d h_i^{-1} \\ &\lesssim n_p^{-1} j_2^{1/2} (p\Delta_n)^{-1/2} \prod_{i=2}^d h_i^{-1} \\ &\lesssim T_n^{-1} (h_2 h_3)^{-1/2} \prod_{i=4}^d h_i^{-1} \end{aligned}$$

and

$$\begin{aligned} n_p^{-2} \sum_{k=j_2+1}^{j_3} (n_p - k) \mathfrak{E}_p^b(k; x) &\lesssim n_p^{-1} \sum_{k=j_2+1}^{j_3} (kp\Delta_n)^{-3/2} \prod_{i=4}^d h_i^{-1} \\ &\lesssim n_p^{-1} j_2^{-1/2} (p\Delta_n)^{-3/2} \prod_{i=4}^d h_i^{-1} \\ &\lesssim T_n^{-1} (h_2 h_3)^{-1/2} \prod_{i=4}^d h_i^{-1} \end{aligned}$$

Combining these three bounds with (57) and (58), we get

$$\text{Var}_{\bar{\mu}^b}(\hat{v}_{n, \mathbf{h}, p}(x)) \lesssim \frac{\sum_{i=1}^d |\log h_i|}{T_n} + \frac{1}{n_p \prod_{i=1}^d h_i} + \frac{1}{T_n \sqrt{h_2 h_3}} \prod_{i=4}^d h_i^{-1}$$

which proves (27).

D.2. Proof of Lemma 8. Since X and ξ are both ergodic and mutually independent we have by Cauchy-Schwarz inequality

$$\begin{aligned} \mathfrak{E}_p^b(k; x) &\leq \text{Var}_{\bar{\mu}^b}(\mathbf{K}_{\mathbf{h}}(x - p^{-1} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} + \frac{\tau_n}{\sqrt{p}} \tilde{\xi}_k)) \\ &\leq \mathbb{E}_{\bar{\mu}^b} \left[\mathbf{K}_{\mathbf{h}}(x - p^{-1} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} + \frac{\tau_n}{\sqrt{p}} \tilde{\xi}_k)^2 \right]. \end{aligned}$$

Using the notations of Section 7, we know that $p^{-1} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} + \tau_n p^{-1/2} \tilde{\xi}_k$ has a density under $\mathbb{P}_{\bar{\mu}^b}^p$ given by

$$\bar{\mu}_{p,n}^b(z) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \bar{\mu}^b(y) \mathbf{p}_{p,n}^b(y; u) \varphi_{\tau_n p^{-1/2}}(z - u) dy du$$

Moreover, for all $z \in \mathbb{R}^d$,

$$\begin{aligned} |\bar{\mu}_{p,n}^b(z)| &\leq \|\bar{\mu}^b\|_\infty \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mathbf{p}_{p,n}^b(y; u) dy \right) \varphi_{\tau_n p^{-1/2}}(z - u) du \\ &\leq \|\bar{\mu}^b\|_\infty. \end{aligned}$$

Then, performing a change of variable gives

$$\mathfrak{C}_p^b(k; x) = \int_{\mathbb{R}^d} \bar{\mu}_{p,n}^b(z) \mathbf{K}_h(x - z)^2 dz = \left(\prod_{i=1}^d h_i^{-1} \right) \int_{\mathbb{R}^d} \bar{\mu}_{p,n}^b(x + \mathbf{h}z) \mathbf{K}(z)^2 dz \lesssim \prod_{i=1}^d h_i^{-1}$$

since $\mathbf{K} \in L^2$.

D.3. Proof of Lemma 9. First recall from Section 7 that the process

$$(p^{-1} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n}, X_{(k+1)p\Delta_n})_{k \in \mathbb{N}}$$

is stationary with law denoted $\bar{\pi}^b$ on \mathbb{R}^{2d} . Using also that X is a Markov process, we get that for all $(u, v) \in \mathbb{R}^d \times \mathbb{R}^d$, we have

$$\begin{aligned} \mathfrak{D}_p^b(k; x, u, v) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{E}^b[\mathbf{K}_h(x - p^{-1} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} + v) | X_{p\Delta_n} = z] \mathbf{K}_h(x - \tilde{\omega} + v) \bar{\pi}^b(\tilde{\omega}; z) dz d\tilde{\omega} \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{K}_h(x + v - \omega) \mathbf{K}_h(x + u - \tilde{\omega}) \mathbf{p}_{p,n,(k-1)p\Delta_n}^b(z; \omega) \bar{\pi}^b(\tilde{\omega}; z) d\tilde{\omega} d\omega dz. \end{aligned}$$

where $\mathbf{p}_{p,n,(k-1)p\Delta_n}^b$ is also defined in Section 7. Moreover, we know from Lemma 4 that there exists $\lambda_1 > 0$ such that

$$\bar{\pi}^b(\tilde{\omega}; z) \lesssim \frac{1}{(p\Delta_n)^{d/2}} e^{-V(z) - \frac{\lambda_1}{p\Delta_n} |\tilde{\omega} - z|^2} \lesssim \frac{1}{(p\Delta_n)^{d/2}} e^{-\frac{\lambda_1}{p\Delta_n} |\tilde{\omega} - z|^2}.$$

From Corollary 1, we also have

$$\mathbf{p}_{p,n,(k-1)p\Delta_n}^b(x, y) \lesssim \frac{1}{(2(k-1)p\lambda_1)^{d/2} (p\Delta_n)^{d/2}} \exp\left(-\frac{\lambda_1}{(1+2(k-1))\Delta_n} |x - y|^2\right).$$

Then, we combine these two estimates using Lemma 15. For all $\omega, \tilde{\omega} \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} \mathbf{p}_{p,n,(k-1)p\Delta_n}^b(z, \tilde{\omega}) \bar{\pi}^b(\omega, z) dz \lesssim \frac{1}{((1+(k-1)\lambda_1)p\Delta_n)^{d/2}} \exp\left(-\frac{|\omega - \tilde{\omega}|^2}{(1+(k-1)\lambda_1)p\Delta_n}\right).$$

Then,

$$\begin{aligned} \mathfrak{D}_p^b(k; x, u, v) &\lesssim \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathbf{K}_h(x + v - \omega) \mathbf{K}_h(x + u - \tilde{\omega})| s^{-d/2} \exp\left(-\frac{|\omega - \tilde{\omega}|^2}{s}\right) d\tilde{\omega} d\omega \\ &\leq \int_{\mathbb{R}^d} |\mathbf{K}_h(x + v + \omega)| \int_{\mathbb{R}^d} |\mathbf{K}_h(x + u - \tilde{\omega})| s^{-d/2} \exp\left(-\frac{|\omega - \tilde{\omega}|^2}{s}\right) d\tilde{\omega} d\omega. \end{aligned}$$

where $s = (1 + (k-1)\lambda_1)p\Delta_n$. We first focus on the inner integral. We recall that for any $z = (z_1, \dots, z_d) \in \mathbb{R}^d$,

$$\mathbf{K}_h(z) = \prod_{i=1}^d h_i^{-1} K(h_i^{-1} z_i),$$

and using that K is integrable, we obtain for any $i \in \{1, \dots, d\}$,

$$(59) \quad \begin{aligned} \int_{\mathbb{R}} h_i^{-1} \left| K\left(\frac{x_i + u_i + \tilde{\omega}_i}{h_i}\right) \right| \frac{1}{\sqrt{s}} \exp\left(-\frac{|\omega_i - \tilde{\omega}_i|^2}{s}\right) d\tilde{\omega}_i &\lesssim \int_{\mathbb{R}} h_i^{-1} \left| K\left(\frac{x_i + u_i + \tilde{\omega}_i}{h_i}\right) \right| \frac{1}{\sqrt{s}} d\tilde{\omega}_i \\ &\lesssim \frac{1}{\sqrt{s}} \int_{\mathbb{R}} \left| K(\tilde{\omega}_i) \right| d\tilde{\omega}_i \\ &\lesssim \frac{1}{\sqrt{s}}. \end{aligned}$$

Using that K is bounded, we also have

$$(60) \quad \begin{aligned} \int_{\mathbb{R}} h_i^{-1} \left| K\left(\frac{x_i + u_i + \tilde{\omega}_i}{h_i}\right) \right| \frac{1}{\sqrt{s}} \exp\left(-\frac{|\omega_i - \tilde{\omega}_i|^2}{s}\right) d\tilde{\omega}_i &\lesssim \int_{\mathbb{R}} \frac{1}{\sqrt{s} h_i} \exp\left(-\frac{|\omega_i - \tilde{\omega}_i|^2}{s}\right) d\tilde{\omega}_i \\ &\lesssim \frac{1}{h_i}. \end{aligned}$$

Combining the bounds given by Equations (59) and (60), we obtain that for any $1 \leq k_1 \leq d$,

$$\int_{\mathbb{R}^d} |\mathbf{K}_h(x + u - \tilde{\omega})| s^{-d/2} \exp\left(-\frac{|\omega - \tilde{\omega}|^2}{s}\right) d\tilde{\omega} \lesssim \frac{1}{s^{k_1/2} \prod_{i=k_1+1}^d h_i},$$

and therefore we have

$$\mathfrak{D}_p^b(k; x, u, v) \lesssim \frac{1}{s^{k_1/2} \prod_{i=k_1+1}^d h_i} \int_{\mathbb{R}^d} |\mathbf{K}_h(x + v + \omega)| d\omega \lesssim \frac{1}{s^{k_1/2} \prod_{i=k_1+1}^d h_i}.$$

By definition of s , we obtain

$$\mathfrak{D}_p^b(k; x, u, v) \lesssim \frac{1}{(kp\Delta_n)^{k_1/2} \prod_{i=k_1+1}^d h_i}.$$

The results on $\mathfrak{C}_p^b(k; x)$ follows by integrating (55).

D.4. Proof of Lemma 10. We begin by extending the bound on $\mathfrak{p}_{p,n,t}^b(\cdot, \cdot)$ given by Corollary 1 for $t > 1$. In fact, for $x, y \in \mathbb{R}^d$ and $t > 1$, we get using Markov property

$$\mathfrak{p}_{p,n,t}^b(x; y) = \int_{\mathbb{R}^d} p_t^b(x; z) \mathfrak{p}_{p,n}^b(z; y) dz.$$

Moreover, using the fact that X is a Markov process, we have that

$$\mathfrak{p}_{p,n,t}^b(x; y) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p_{1/2}^b(x, \omega) p_{t-1/2}^b(\omega, z) d\omega \right) \mathfrak{p}_{p,n}^b(z; y) dz$$

From Lemma 3, we know that

$$p_{1/2}^b(x, \omega) \lesssim \exp\left(-|x - \omega|^2 + V(x) - V(\omega)\right)$$

and therefore using that V is bounded below, we have

$$\mathfrak{p}_{p,n,t}^b(x; y) \lesssim e^{V(x)} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p_{t-1/2}^b(\omega; z) d\omega \right) \mathfrak{p}_{p,n}^b(z; y) dz \lesssim e^{V(x)}.$$

Moreover, we know from Lemma 4 that there exists $\lambda_1 > 0$ such that

$$\bar{\pi}^b(\tilde{\omega}; z) \lesssim \frac{1}{(p\Delta_n)^{d/2}} e^{-V(z) - \frac{\lambda_1}{p\Delta_n} |\tilde{\omega} - z|^2}.$$

Proceeding as for Lemma 9, we have

$$\begin{aligned} &\mathfrak{D}_p^b(k; x, u, v) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{K}_h(x + v - \omega) \mathbf{K}_h(x + u - \tilde{\omega}) \mathfrak{p}_{p,n,(k-1)p\Delta_n}^b(z; \omega) \bar{\pi}^b(\tilde{\omega}; z) d\tilde{\omega} d\omega dz \\ &\lesssim \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{K}_h(x + v - \omega) \mathbf{K}_h(x + u - \tilde{\omega}) \frac{1}{(p\Delta_n)^{d/2}} e^{-\frac{\lambda_1}{p\Delta_n} |\tilde{\omega} - z|^2} d\tilde{\omega} d\omega dz. \end{aligned}$$

Using that

$$\frac{1}{(p\Delta_n)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{\lambda_1}{p\Delta_n}|z-\omega|^2\right) dz = \left(\frac{4\pi}{\lambda_1}\right)^{d/2},$$

we obtain

$$\mathfrak{D}_p^b(k; x, u, v) \lesssim \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{K}_h(x+v-\omega) \mathbf{K}_h(x+u-\tilde{\omega}) d\tilde{\omega} d\omega$$

and using the usual change of variable and the fact that $|K|$ is integrable, we get

$$\mathfrak{D}_p^b(k; x, u, v) \lesssim 1.$$

The result on $\mathfrak{C}_p^b(k; x)$ follows by integrating (55).

D.5. Proof of Lemma 11. For $\mathbf{h} = (h_1, \dots, h_d) \in (0, 1)^d$ and $y \in \mathbb{R}^d$, we define

$$\mathfrak{K}_h(y) = \int_{\mathbb{R}^d} \mathbf{K}_h(x-y+u) \chi_{p,n}(du),$$

and

$$\mathfrak{K}_h^c(y) = \mathfrak{K}_h(y) - \mathbb{E}_{\bar{\mu}^b} \left[\mathfrak{K}_h \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n} \right) \right].$$

Then we have by definition

$$\begin{aligned} \mathfrak{C}_p^b(k; x) &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathfrak{C}_p^b(k; x, u, v) \chi_{p,n}(du) \chi_{p,n}(dv) \\ &\leq \mathbb{E}_{\bar{\mu}^b}^b \left[\mathfrak{K}_h \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} \right) \mathfrak{K}_h \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n} \right) \right] - \mathbb{E}_{\bar{\mu}^b}^b \left[\mathfrak{K}_h \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n} \right) \right]^2, \\ &\leq \mathbb{E}_{\bar{\mu}^b}^b \left[\mathfrak{K}_h^c \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} \right) \mathfrak{K}_h^c \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n} \right) \right]. \end{aligned}$$

Moreover, we write $\{\mathcal{A}_k, k \geq 0\}$ for the natural filtration induced by the lifted Markov chain $(p^{-1} \sum_{\ell=0}^{p-1} Y_{kp+\ell,n}, X_{(k+1)p\Delta_n})_{k \in \mathbb{N}}$. Then using Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\left| \mathbb{E}_{\bar{\mu}^b}^b \left[\mathfrak{K}_h^c \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} \right) \mathfrak{K}_h^c \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n} \right) \right] \right| \\ &= \left| \mathbb{E}_{\bar{\mu}^b}^b \left[\mathbb{E}_{\bar{\mu}^b}^b \left[\mathfrak{K}_h^c \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} \right) \middle| \mathcal{A}_0 \right] \mathfrak{K}_h^c \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n} \right) \right] \right| \\ &\leq \mathbb{E}_{\bar{\mu}^b}^b \left[\mathbb{E}_{\bar{\mu}^b}^b \left[\mathfrak{K}_h^c \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} \right) \middle| \mathcal{A}_0 \right]^2 \right]^{1/2} \mathbb{E}_{\bar{\mu}^b}^b \left[\mathfrak{K}_h^c \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n} \right)^2 \right]^{1/2}. \end{aligned}$$

The second expectation is treated as in Lemma 8 and we have

$$\mathbb{E}_{\bar{\mu}^b}^b \left[\mathfrak{K}_h^c \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n} \right)^2 \right]^{1/2} \lesssim \prod_{i=1}^d h_i^{-1}.$$

For the first one, we use that X is a stationary Markov process to get

$$\begin{aligned} &\mathbb{E}_{\bar{\mu}^b}^b \left[\mathbb{E}^b \left[\mathfrak{K}_h^c \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} \right) \middle| \mathcal{A}_0 \right]^2 \right] = \mathbb{E}_{\bar{\mu}^b}^b \left[\mathbb{E}^b \left[\mathfrak{K}_h^c \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} \right) \middle| \mathcal{F}_{p\Delta_n} \right]^2 \right] \\ &= \int_{\mathbb{R}^d} \mathbb{E}^b \left[\mathfrak{K}_h^c \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} \right) \middle| X_{p\Delta_n} = x \right]^2 \bar{\mu}^b(dx) \\ &= \int_{\mathbb{R}^d} \mathbb{E}^b \left[\mathbb{E}^b \left[\mathfrak{K}_h^c \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} \right) \middle| X_{kp\Delta_n} \right] \middle| X_{p\Delta_n} = x \right]^2 \bar{\mu}^b(dx) \end{aligned}$$

Defining $\varphi : y \mapsto \mathbb{E}^b \left[\mathcal{K}_h^c \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} \right) \middle| X_{kp\Delta_n} = y \right]$, we get that for any $x \in \mathbb{R}^d$,

$$\mathbb{E}^b [\varphi(X_{kp\Delta_n}) | X_{p\Delta_n} = x] = \mathbb{E}_x^b [\varphi(X_{(k-1)p\Delta_n})] = P_{(k-1)p\Delta_n}^b \varphi(x),$$

where (P_t^b) is defined in Equation (7). Moreover, from the definition of \mathcal{K}_c , we get $\mathbb{E}_{\mu^b}^b [\varphi(X_0)] = 0$. Then, using Equation (6), we have

$$\begin{aligned} \mathbb{E}_{\mu^b}^b \left[\mathbb{E}^b \left[\mathcal{K}_h^c \left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{(kp+\ell)\Delta_n} \right) \middle| \mathcal{F}_{p\Delta_n} \right]^2 \right] &= \text{Var}_{\mu^b}^b [P_{(k-1)p\Delta_n}^b \varphi(X_0)] \\ &\leq e^{-2C_{PI}^{-1}(k-1)p\Delta_n} \mathbb{E}_{\mu^b}^b [\varphi(X_0)^2]. \end{aligned}$$

Moreover, for all $x \in \mathbb{R}^d$, $|\varphi(x)| \leq 2\|\mathcal{K}_h^c\|_\infty \leq 2\|\mathbf{K}_h\|_\infty$. Then, we have $\|\varphi\|_\infty \lesssim \prod_{i=1}^d h_i^{-1}$. Combining all previous inequalities, we get

$$\mathfrak{C}_p^b(k; x) \lesssim e^{-C_{PI}^{-1}(k-1)p\Delta_n} \left(\prod_{i=1}^d h_i^{-1} \right)^2.$$

APPENDIX E. PROOF OF THE RESULTS OF SECTION 4.1

E.1. Preliminary results. In this section, we aim at proving Lemma 1 and other results used in the proof of the results of Section 4.1. We start by introducing a concise notation. We say that $A(\theta) \lesssim B(\theta)$ implies $C(\theta) \lesssim D(\theta)$, if for all $c_1 > 0$, there exists $c_2 > 0$ such that $A(\theta) \leq c_1 B(\theta)$ implies $C(\theta) \leq c_2 D(\theta)$. We define similarly the equivalence between $A(\theta) \lesssim B(\theta)$ and $C(\theta) \lesssim D(\theta)$. We can now state the main Lemma of this section.

Lemma 12. For $p \geq 1$, the condition $p\Delta_n \lesssim w_n^{HF}$ is equivalent to $p\Delta_n \lesssim \tilde{w}_{n,p}^{HF}$ with

$$\tilde{w}_{n,p}^{HF} = \begin{cases} \left(\frac{p}{n} \right)^{\frac{1}{2}} \log \left(\frac{n}{p} \right)^{\frac{1}{2}} & \text{if } d = 1, 2 \\ \left(\frac{p}{n} \right)^{\frac{\bar{\alpha}}{2\bar{\alpha}+d} \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right)} \log \left(\frac{n}{p} \right) & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_1, \\ \left(\frac{p}{n} \right)^{\frac{2\bar{\alpha}}{2\bar{\alpha}+d} \frac{1}{\alpha_1}} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_2, \\ \left(\frac{p}{n} \right)^{\frac{\bar{\alpha}}{(2\bar{\alpha}+d)} \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right)} & \text{if } d \geq 3 \text{ and } d \geq 3 \text{ and } (\alpha, k_0) \in D_3. \end{cases}$$

Proof. First note that since $T_n = \frac{np\Delta_n}{p}$, the condition

$$p\Delta_n \lesssim T_n^{-u} \log(T_n)^v$$

for some $u, v \geq 0$, is equivalent to

$$p\Delta_n \lesssim \left(\frac{p}{n} \right)^{\frac{u}{1+u}} \log \left(\frac{n}{p} \right)^{\frac{v}{1+u}}.$$

If $u = \frac{\bar{\alpha}_3}{2\bar{\alpha}_3+d-2} \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right)$, we have

$$u = \frac{1}{2 + \frac{d-2}{\bar{\alpha}_3}} \frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2} = \frac{1}{2 + \frac{d}{\bar{\alpha}} - \frac{1}{\alpha_1} - \frac{1}{\alpha_2}} \frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2} = \frac{\bar{\alpha}(\alpha_1 + \alpha_2)}{\alpha_1 \alpha_2 (2\bar{\alpha} + d) - \bar{\alpha}(\alpha_1 + \alpha_2)}$$

and therefore

$$\frac{u}{1+u} = \frac{\bar{\alpha}(\alpha_1 + \alpha_2)}{\alpha_1 \alpha_2 (2\bar{\alpha} + d)} = \frac{\bar{\alpha}}{2\bar{\alpha} + d} \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right),$$

which proves the case $d \geq 3$ and $k_0 = 1$ with $\alpha_2 < \alpha_3$ or $k_0 = 2$. The other cases are done similarly. \square

A particular case is the case where $p = 1$, restated as follows

Lemma 13. The condition $\Delta_n \lesssim w_n^{HF}$ is equivalent to $\Delta_n \lesssim \tilde{w}_n^{HF}$ with $\tilde{w}_n^{HF} = \tilde{w}_{n,1}^{HF}$

Proof of Lemma 1. From Lemma 12, we know that $p\Delta_n \lesssim w_n^{HF}$ is equivalent to $p\Delta_n \lesssim \tilde{w}_{n,p}^{HF}$. Using that $\alpha_1 > 2$, we check that

$$\tilde{w}_{n,p}^{HF} \geq \left(\frac{p}{n} \right)^{\frac{2\bar{\alpha}}{2\bar{\alpha}+d}}$$

and therefore we have Lemma 1. \square

E.2. Proof of Proposition 4. Suppose that $p^* \Delta_n \leq w_n^{HF}$ and that $\tau_n^{2\alpha_1} \leq \Delta_n$ and recall that in that case, we take

$$p^* = 1 \quad \text{and} \quad \mathbf{h}^* = \mathbf{h}^{*,HF}$$

From Proposition 2, we see by plugging these values that

$$B_{n,\mathbf{h},p}^b(x) \lesssim \tau_n^{2\alpha_1} + v_n^{HF}.$$

Note also that $|\log(h_i^*)|$ is of the order of $\log(T_n)$ and that by definition of \mathbf{h}^* , we have

$$v_n^{HF} \lesssim T_n^{-1} \begin{cases} \sum_{i=1}^d |\log(h_i^*)| & \text{if } d = 1, 2 \\ \sum_{i=1}^d |\log(h_i^*)| \prod_{i=3}^d (h_i^*)^{-1} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_1 \\ (\prod_{i=1}^{k_0} (h_i^*)^{(2-k_0)/k_0}) \prod_{i=k_0+1}^d (h_i^*)^{-1} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_2 \\ (h_2^* h_3^*)^{-1/2} \prod_{i=4}^d (h_i^*)^{-1} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_3. \end{cases}$$

Therefore, by factorizing by v_n^{HF} the variance bound from Proposition 3, we have

$$V_{n,\mathbf{h}^*,p^*}^b(x) \lesssim v_n^{HF} (1 + r_n^{HF}),$$

with

$$r_n^{HF} = \begin{cases} 0 & \text{if } d = 1, 2 \\ \Delta_n (h_1^* h_2^*)^{-1} \log(T_n)^{-1} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_1 \\ \Delta_n \prod_{i=1}^{k_0} (h_i^*)^{-2/k_0} + \log(T_n) \prod_{i=1}^{k_0} (h_i^*)^{-2/k_0} \prod_{i=1}^d h_i^* & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_2 \\ \log(T_n) (h_2^* h_3^*)^{1/2} \prod_{i=4}^d h_i^* + \Delta_n h_1^{-1} (h_2^* h_3^*)^{-1/2} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_3 \end{cases}$$

and it remains to prove that in each case, the remainder can be ignored so that $V_{n,\mathbf{h},p}^b(x) \lesssim v_n^{HF}$. First note that the condition $\Delta_n \lesssim w_n^{HF}$ and the definition of \mathbf{h}^* ensures that

$$\Delta_n \lesssim \begin{cases} h_1^* \log(T_n) & \text{if } d = 1 \\ h_1^* h_2^* \log(T_n) & \text{if } d = 2 \\ h_1^* h_2^* \log(T_n) & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_1 \\ \prod_{i=1}^{k_0} (h_i^*)^{2/k_0} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_2 \\ h_1^* (h_2^* h_3^*)^{1/2} & \text{if } dd \geq 3 \text{ and } (\alpha, k_0) \in D_3 \end{cases}$$

and therefore

$$r_n^{HF} \lesssim \begin{cases} 0 & \text{if } d = 1, 2 \text{ or } d \geq 3 \text{ and } (\alpha, k_0) \in D_1 \\ \log(T_n) \prod_{i=1}^{k_0} (h_i^*)^{-2/k_0} \prod_{i=1}^d h_i^* & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_2 \\ \log(T_n) (h_2^* h_3^*)^{1/2} \prod_{i=4}^d h_i^* & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_3 \end{cases}$$

Then we easily check that if $d \geq 3$ and $k_0 \geq 3$, we have

$$\log(T_n) \prod_{i=1}^{k_0} (h_i^*)^{-2/k_0} \prod_{i=1}^d h_i^* = \log(T_n) T_n^{-\frac{\bar{\alpha}_2}{\bar{\alpha}_3 + d - 2} \sum_{j=k_0+1}^d 1/\alpha_j} \rightarrow 0.$$

Analogously, if $d \geq 3$ and $k_0 = 1$ with $\alpha_2 = \alpha_3$, we have

$$\log(T_n) (h_2^* h_3^*)^{-1/2} \prod_{i=4}^d h_i^* = \log(T_n)^{c+1} T_n^{-c} \rightarrow 0,$$

with

$$c = \frac{\bar{\alpha}_3}{2\bar{\alpha}_3 + d - 2} \left(\frac{1}{2\alpha_2} + \frac{1}{2\alpha_3} + \sum_{i=4}^d \frac{1}{\alpha_i} \right) > 0$$

so that

$$V_{n,\mathbf{h}^*,p^*}^b(x) \lesssim v_n^{HF}$$

which concludes the proof of Proposition 4.

E.3. Proof of Proposition 5. Suppose that $p^* \Delta_n \leq w_n^{HF}$ and that $\tau_n^{2\alpha_1} \geq \Delta_n$ and recall that in that case, we take

$$p^* = \left[(\tau_n^{2\alpha_1} \Delta_n^{-1})^{1/(1+\alpha_1)} \right] \quad \text{and} \quad \mathbf{h}^* = \mathbf{h}^{*,HF}$$

From Proposition 2, we see by plugging these values that

$$B_{n,\mathbf{h}^*,p^*}^b(x) \lesssim \frac{\tau_n^{2\alpha_1}}{(p^*)^{\alpha_1}} + (p^* \Delta_n)^{1/2} + v_n^{HF} \lesssim (\tau_n^2 \Delta_n)^{\frac{\alpha_1}{1+\alpha_1}} + v_n^{HF}.$$

For the variance, we proceed as for Proposition 5: using that $p^* \Delta_n \lesssim w_n^{HF}$ and $\mathbf{h}^* = \mathbf{h}^{*,HF}$, we have from Proposition 3 that

$$V_{n,\mathbf{h}^*,p^*}^b(x) \lesssim v_n^{HF}.$$

so we can conclude.

E.4. Proof of Proposition 6. Suppose that $\Delta_n \geq w_n^{HF}$ and that $\tau_n^{2\alpha_1} \leq \Delta_n$ and recall that in that case, we take

$$p^* = 1 \quad \text{and} \quad \mathbf{h}^* = \mathbf{h}^{*,1} = \mathbf{h}^{*,LF}$$

From Proposition 2, we see by plugging these values that

$$B_{n,\mathbf{h},p}^b(x) \lesssim \tau_n^{2\alpha_1} + v_n^{HF}.$$

We now study the variance. First, note that

$$T_n^{-1} \Delta_n \prod_{i=1}^d (h_i^*)^{-1} = n^{\frac{-2\alpha}{2\alpha+d}} = v_n^{LF}.$$

Therefore, by factorizing by v_n^{LF} the variance bound from Proposition 3 and using that $|\log(h_i^*)|$ is of the order of $\log(n)$, we have

$$V_{n,\mathbf{h}^*,p^*}^b(x) \lesssim v_n^{LF} (1 + r_n^{LF}),$$

with

$$r_n^{LF} = \begin{cases} 0 & \text{if } d = 1, 2 \\ \Delta_n^{-1} \log(n) h_1^* h_2^* & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_1 \\ \Delta_n^{-1} \prod_{i=1}^{k_0} (h_i^*)^{2/k_0} + \log(n) \Delta_n^{-1} \prod_{i=1}^d h_i^* & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_2 \\ \log(n) \Delta_n^{-1} \prod_{i=1}^d h_i^* + \Delta_n^{-1} h_1 (h_2^* h_3^*)^{1/2} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_3 \end{cases}$$

Using Lemma 13 and the fact that $\Delta_n \gtrsim w_n^{HF}$, we know that $\Delta_n \gtrsim \tilde{w}_n^{HF}$. Using also the definition of \mathbf{h}^* , we check that

$$(61) \quad \Delta_n \gtrsim \begin{cases} |h_1^* \log(h_1^*)| & \text{if } d = 1 \\ h_1^* h_2^* |\log(h_1^* h_2^*)| & \text{if } d = 2 \\ h_1 h_2 \log(n) & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_1 \\ \prod_{i=1}^{k_0} (h_i^*)^{2/k_0} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_2 \\ h_1^* (h_2^* h_3^*)^{1/2} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_3. \end{cases}$$

and therefore

$$V_{n,\mathbf{h}^*,p^*}^b(x) \lesssim v_n^{LF} \begin{cases} 1 & \text{if } d = 1, 2 \text{ or } d \geq 3 \text{ and } (\alpha, k_0) \in D_1 \\ 1 + \log(n) \Delta_n^{-1} \prod_{i=1}^d h_i^* & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_2 \\ 1 + \log(n) \Delta_n^{-1} \prod_{i=1}^d h_i^* & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_3. \end{cases}$$

We conclude the proof of Proposition 6 using (61) to get

$$\log(n) \Delta_n^{-1} \prod_{i=1}^d h_i^* \lesssim \begin{cases} \log(n) \prod_{i=1}^{k_0} (h_i^*)^{(2-k_0)/k_0} \prod_{i=k_0+1}^d h_i^* & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_2 \\ (h_2^* h_3^*)^{1/2} \prod_{i=4}^d h_i^* & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_3. \end{cases}$$

so that

$$\log(n) \Delta_n^{-1} \prod_{i=1}^d h_i^* \rightarrow 0,$$

as $n \rightarrow \infty$.

E.5. Proof of Proposition 7. Suppose that $p^* \Delta_n \geq w_n^{HF}$ and that $\tau_n^{2\alpha_1} \geq \Delta_n$. and recall that in that case, we take

$$p^* = \left[(\tau_n^{2\alpha_1} \Delta_n^{-1})^{1/(1+\alpha_1)} \right] \quad \text{and} \quad \mathbf{h}^* = \mathbf{h}^{*,p^*}$$

From Proposition 2, we see by plugging these values that

$$B_{n,\mathbf{h}^*,p^*}^b(x) \lesssim (\tau_n^2 \Delta_n)^{\frac{\alpha_1}{1+\alpha_1}} + \left(\frac{p^*}{n} \right)^{\frac{2\bar{\alpha}}{2\bar{\alpha}+d}}$$

For the variance, we proceed as for Proposition 6: using that $p^* \Delta_n \gtrsim w_n^{HF}$ (and therefore that $p^* \Delta_n \gtrsim \tilde{w}_n^{HF}$ by Lemma 1) and $\mathbf{h}^* = \mathbf{h}^{*,p^*}$, we have from Proposition 3 that

$$V_{n,\mathbf{h}^*,p^*}^b(x) \lesssim \left(\frac{p^*}{n} \right)^{\frac{2\bar{\alpha}}{2\bar{\alpha}+d}}$$

and thus

$$\mathcal{R}(\hat{\mu}_{n,\mathbf{h}^*,p^*}, b; x) \lesssim (\tau_n^2 \Delta_n)^{\frac{\alpha_1}{1+\alpha_1}} + \left(\frac{p^*}{n} \right)^{\frac{2\bar{\alpha}}{2\bar{\alpha}+d}}.$$

It remains to prove that $(\tau_n^2 \Delta_n)^{\frac{\alpha_1}{1+\alpha_1}}$ is always dominating here. This term is indeed of the order of $p^* \Delta_n$. By Lemma 12, we have $p^* \Delta_n \gtrsim \tilde{w}_{n,p^*}^{HF}$ so it is enough to prove that

$$(62) \quad \left(\frac{p^*}{n} \right)^{\frac{2\bar{\alpha}}{2\bar{\alpha}+d}} \lesssim \tilde{w}_{n,p^*}^{HF}$$

Moreover, since $p^* \Delta_n \rightarrow 0$, it is clear that $p^*/n \rightarrow 0$. Using the definition of \tilde{w}_{n,p^*}^{HF} , we see that (62) is equivalent to

$$\begin{cases} 2\bar{\alpha} \geq d & \text{if } d = 1, 2 \\ \frac{\bar{\alpha}}{2\bar{\alpha}+d} \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right) \geq \frac{2\bar{\alpha}}{2\bar{\alpha}+d} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_1 \\ \frac{2\bar{\alpha}}{2\bar{\alpha}+d} \frac{1}{\alpha_1} \geq \frac{2\bar{\alpha}}{2\bar{\alpha}+d} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_2 \\ \frac{\bar{\alpha}}{(2\bar{\alpha}+d)} \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right) \geq \frac{2\bar{\alpha}}{2\bar{\alpha}+d} & \text{if } d \geq 3 \text{ and } (\alpha, k_0) \in D_3. \end{cases}$$

These inequalities always hold since $\alpha_1 > 1$ and $\alpha_2 > 1$ so we can conclude.

APPENDIX F. PROOF OF THE RESULTS OF SECTION 5

F.1. Proof of Theorem 1. The main idea of this proof is to use general results for Markov chains to the lifted Markov chain

$$(63) \quad \left(p^{-1} \sum_{\ell=0}^{p-1} Y_{kp+\ell,n}, X_{(k+1)p\Delta_n} \right)_{k \in \mathbb{N}}.$$

We first introduce few definitions for Markov chains and we refer to [KM12] for more details. Consider a Markov chain Υ on a given space (X, \mathcal{X}) . We say that Υ is ψ -irreducible and aperiodic if there exists a σ -finite measure ψ on (X, \mathcal{X}) such that for all $A \in \mathcal{X}$ such that $\psi(A) > 0$, any $x \in X$ and any n large enough, we have

$$P^n(x, A) > 0$$

where P denotes the transition semigroup of Υ . We say that Υ is geometrically ergodic if it admits an invariant measure π and functions $\rho : X \rightarrow (0, 1)$ and $\rho : X \rightarrow [1, \infty)$ such that for all $n \geq 0$ and for π -almost $x \in X$, we have

$$\|P^n(x, \cdot) - \pi\|_{TV} \leq C(x)\rho(x)^n.$$

Intuitively, geometrically ergodic converge fast to their invariant distribution starting from almost any point. Therefore, they must satisfy concentration properties such as Bernstein inequality. In this section, we plan to use the following result, from [Lem21].

Theorem 4 (Theorem 1.1 in [Lem21]). *Let Υ be a geometrically ergodic Markov chain with state space \mathcal{X} , and let π be its unique stationary probability measure. Moreover, let $f : \mathcal{X} \rightarrow \mathbb{R}$ be a bounded measurable function such that $\mathbb{E}_\pi f = 0$. Furthermore, let $x \in \mathcal{X}$. Then, we can find constants $K, \tau > 0$ depending only*

on x and the transition probability $P(\cdot, \cdot)$ such that for all $t > 0$,

$$\mathbb{P}_x \left(\frac{1}{N} \sum_{i=0}^{N-1} f(\Upsilon_i) > t \right) \leq K \exp \left(- \frac{N^2 t^2}{32N\sigma_{Mrv}^2 + \tau t \|f\|_\infty N \log N} \right),$$

where

$$\sigma_{Mrv}^2 = \text{Var}_\pi(f(\Upsilon_0)) + 2 \sum_{i=1}^{\infty} \text{Cov}_\pi(f(\Upsilon_0), f(\Upsilon_i))$$

denotes the asymptotic variance of the process $(f(\Upsilon_i))_i$.

We want to apply this Theorem to (63) and we need to check that this chain is geometrically ergodic. An easy way to check for geometric ergodicity is to use Chapter 15 from [MT12]: any ψ -irreducible and aperiodic Markov chain satisfying the drift condition V4 stated below is geometrically ergodic.

Assumption V4. *There exists a function $W : X \rightarrow [1, \infty)$, a set $C \in \mathcal{X}$ and constants $\delta > 0$ and $b < \infty$ such that*

$$(64) \quad PW \leq (1 - \delta)W + b\mathbb{1}_C$$

and such that there exists $n \geq 1$, $\varepsilon > 0$ and a probability measure ν on (X, \mathcal{X}) such that

$$(65) \quad P^n(x, A) \geq \varepsilon \mathbb{1}_C(x) \nu(A)$$

for all $x \in X$ and $A \in \mathcal{X}$. In that case, we say that W is a Lyapunov function and C is a small set.

Thus the plan is to prove that the lifted chain (63) is ψ -irreducible and aperiodic and satisfies Assumption V4.

First remark that the Markov chain $(X_{pk\Delta_n})_k$ is clearly aperiodic and ψ -irreducible. In fact for any $x \in \mathbb{R}^d$ and any Borelian $A \in \mathcal{B}_{\mathbb{R}^d}$ of positive Lebesgue measure, and for any $k \geq 1$, we have

$$(66) \quad \mathbb{P}^b(X_{pk\Delta_n} \in A \mid X_0 = x) > 0.$$

We deduce that the lifted Markov chain (63) is also aperiodic and ψ -irreducible by using that the density $\mathfrak{p}_{p,n}^b(x; \cdot, \cdot)$ is almost everywhere positive.

Moreover, using Assumption 3, we know that the chain is geometrically ergodic. Combined with aperiodicity and ψ -irreducibility, this ensures that $(X_{pk\Delta_n})_k$ satisfies Assumption V4 and we consider W and C as in Assumption V4. We want to prove that the same property hold for (63). We consider $\widetilde{W} : (x, y) \mapsto W(y)$ and $\widetilde{C} = \mathbb{R}^d \times C$. Then, for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, we have

$$\begin{aligned} \mathfrak{P}\widetilde{W}(x, y) &= \mathbb{E} \left[\widetilde{W} \left(p^{-1} \sum_{\ell=0}^{p-1} Y_{p+\ell, n}, X_{2p\Delta_n} \right) \middle| p^{-1} \sum_{\ell=0}^{p-1} Y_{p+\ell, n} = x, X_{p\Delta_n} = y \right] \\ &= \mathbb{E} \left[\widetilde{W} \left(p^{-1} \sum_{\ell=0}^{p-1} Y_{p+\ell, n}, X_{2p\Delta_n} \right) \middle| X_{p\Delta_n} = y \right]. \end{aligned}$$

and therefore

$$\mathfrak{P}\widetilde{W}(x, y) \leq \lambda \widetilde{W}(x, y) + b\mathbb{1}_C(y).$$

We only need to check that \widetilde{C} is indeed a *small* set. We already know that C is a *small* set and therefore there exists $\delta > 0$ and $\nu \in \mathcal{P}(\mathbb{R}^d)$ such that for some $k \geq 1$,

$$\forall x \in C, \quad \forall B \in \mathcal{B}_{\mathbb{R}^d}, \quad P^k(x, B) \geq \delta \nu(B).$$

We now define the following probability measure on \mathbb{R}^{2d}

$$\tilde{\nu}(dx, dy) = \int_{\mathbb{R}^d} \mathfrak{p}_{p,n}^b(y; x, z) dz \nu(dx) dy.$$

Then, for all $(\bar{x}, x) \in \tilde{C}$, we get for any Borel set of \mathbb{R}^d , $\bar{B}, B \in \mathcal{B}_{\mathbb{R}^d}$, conditioning with respect to $X_{pk\Delta_n}$ and using the Markov property:

$$\begin{aligned} \mathfrak{P}^{k+1}((\bar{x}, x), \bar{B} \times B) &= \int_{\mathbb{R}^d} p_{kp\Delta_n}(x, x') \int_{\bar{B} \times B} \mathfrak{P}_{p,n}^b(x'; y, z) dy dz dx' \\ &\geq \delta \int_{\bar{B} \times B} \int_{\mathbb{R}^d} \mathfrak{P}_{p,n}^b(x'; y, z) \nu(dx') dy dz \\ &\geq \delta \tilde{\nu}(\bar{B} \times B), \end{aligned}$$

where we used that $x \in C$ and C is a *small* set.

We are now ready to apply Theorem 4 to (63). We apply it with $f(x_1, x_2) = \mathbf{K}_{\mathbf{h}}(x - x_1)$ for all $x_1, x_2 \in \mathbb{R}^d$. Thus we get

$$\mathbb{P}_{\bar{\mu}^b}^b \left(|\hat{\nu}_{n,\mathbf{h},p}(x) - \mathbb{E}[\hat{\nu}_{n,\mathbf{h},p}(x)]| > \varepsilon \right) \leq K \exp \left(- \frac{N^2 \varepsilon^2}{32Nv^2(\alpha, n, \mathbf{h}, p) + \tau \varepsilon \|\mathbf{K}_{\mathbf{h}}\|_{\infty} N \log N} \right)$$

where $N = \lfloor n/p \rfloor$, and

$$\begin{aligned} v^2(\alpha, n, \mathbf{h}, p) &= \text{Var}_{\bar{\mu}^b}^b \left(\mathbf{K}_{\mathbf{h}}(x - p^{-1} \sum_{\ell=0}^{p-1} Y_{\ell,n}) \right) \\ &\quad + 2 \sum_{k=1}^{\infty} \text{Cov}_{\bar{\mu}^b}^b \left(\mathbf{K}_{\mathbf{h}}(x - p^{-1} \sum_{\ell=0}^{p-1} Y_{\ell,n}), \mathbf{K}_{\mathbf{h}}(x - p^{-1} \sum_{\ell=0}^{p-1} Y_{kp+\ell,n}) \right). \end{aligned}$$

In order to control the variance term, everything works exactly as in the proof of Proposition 3, except for the last term as the sum is infinite.

F.2. Proof of Proposition 8. In this section, we will repeatedly use estimates of the form

$$(67) \quad \int_{\nu}^{\infty} \exp(-z^r) dz \leq 2r^{-1} \nu^{1-r} \exp(-\nu^r), \quad \nu, r > 0, \quad \nu \geq (2/r)^{1/r}.$$

and

$$(68) \quad \exp \left(- \frac{az^p}{b + cz^{p/2}} \right) \leq \exp \left(- \frac{az^p}{2b} \right) + \exp \left(- \frac{az^{p/2}}{2c} \right), \quad a, b, c, z > 0.$$

Before delving into the proof of Proposition 8, let us state and prove the following Lemma

Lemma 14. Under Assumptions 1, 2, 3, there exists $\gamma > 1$ such that for any $\mathbf{h} \in \mathcal{H}_p^n$,

$$\mathbb{E}_{\bar{\mu}^b}^b [A_n^p(\mathbf{h})] \lesssim V_p^n(\mathbf{h}) + \mathcal{B}_{n,\mathbf{h},p}(x)^2 + n_p^{-\gamma} + \sqrt{p\Delta_n} + \frac{\tau_n^{2\alpha_1}}{p^{\alpha_1}}.$$

Proof of Lemma 14. For any $\boldsymbol{\eta} \in \mathcal{H}_n^p$, we have

$$\left\{ |\hat{\mu}_{n,(\mathbf{h},\boldsymbol{\eta}),p}(x) - \hat{\mu}_{n,\boldsymbol{\eta},p}(x)|^2 - V_n^p(\boldsymbol{\eta}) \right\}_+ \lesssim \mathbf{I}_{\mathbf{h},\boldsymbol{\eta}}^{n,p}(x) + \mathbf{II}_{\mathbf{h},\boldsymbol{\eta}}^{n,p}(x) + \mathbf{III}_{\mathbf{h},\boldsymbol{\eta}}^{n,p}(x). \quad (69)$$

where

$$\begin{aligned} \mathbf{I}_{\mathbf{h},\boldsymbol{\eta}}^{n,p}(x) &= \left\{ |\hat{\mu}_{n,\boldsymbol{\eta},p}(x) - \mathbb{E}[\hat{\mu}_{n,\boldsymbol{\eta},p}(x)]|^2 - V_n^p(\boldsymbol{\eta}) \right\}_+, \\ \mathbf{II}_{\mathbf{h},\boldsymbol{\eta}}^{n,p}(x) &= \left\{ |\hat{\mu}_{n,(\mathbf{h},\boldsymbol{\eta}),p}(x) - \mathbb{E}[\hat{\mu}_{n,(\mathbf{h},\boldsymbol{\eta}),p}(x)]|^2 - V_n^p(\boldsymbol{\eta}) \right\}_+, \\ \mathbf{III}_{\mathbf{h},\boldsymbol{\eta}}^{n,p}(x) &= \left| \mathbb{E}_{\bar{\mu}^b}^b [\hat{\mu}_{n,\boldsymbol{\eta},p}(x)] - \mathbb{E}_{\bar{\mu}^b}^b [\hat{\mu}_{n,(\mathbf{h},\boldsymbol{\eta}),p}(x)] \right|^2. \end{aligned}$$

Let us begin with the term $\mathbf{I}_{\mathbf{h},\boldsymbol{\eta}}^{n,p}(x)$. We write

$$\begin{aligned} &\mathbb{E}_{\bar{\mu}^b}^b \left[\left\{ |\hat{\mu}_{n,\boldsymbol{\eta},p}(x) - \mathbb{E}_{\bar{\mu}^b}^b [\hat{\mu}_{n,\boldsymbol{\eta},p}(x)]|^2 - V_n^p(\boldsymbol{\eta}) \right\}_+ \right] \\ &= \int_{V_n^p(\boldsymbol{\eta})}^{+\infty} \mathbb{P}_{\bar{\mu}^b}^b \left(\left| \hat{\mu}_{n,\boldsymbol{\eta},p}(x) - \mathbb{E}_{\bar{\mu}^b}^b [\hat{\mu}_{n,\boldsymbol{\eta},p}(x)] \right| \geq z^{1/2} \right) dz \\ &\leq K \int_{V_n^p(\boldsymbol{\eta})}^{+\infty} \exp \left(- \frac{n_p^2 \beta^2 z}{32n_p v^2(\alpha, n, \boldsymbol{\eta}, p) + \tau \beta \|\mathbb{K}_{\boldsymbol{\eta}}\| n_p \log(n_p) z^{1/2}} \right) dz, \end{aligned}$$

where we used at the last line the concentration inequality of Bernstein's type given by Theorem 1. Then, using Equation (68), we obtain

$$\mathbb{E}_{\bar{\mu}^b} \left[\left\{ \left| \hat{\mu}_{n,\eta,p}(x) - \mathbb{E}_{\bar{\mu}^b} [\hat{\mu}_{n,\eta,p}(x)] \right|^2 - V_n^p(\boldsymbol{\eta}) \right\}_+ \right] \lesssim \mathbf{I} + \mathbf{II},$$

where

$$\begin{aligned} \mathbf{I} &:= K \int_{V_n^p(\boldsymbol{\eta})}^{+\infty} \exp\left(-\frac{n_p \beta^2 z}{32v^2(\alpha, n, \boldsymbol{\eta}, p)}\right) dz; \\ \mathbf{II} &:= K \int_{V_n^p(\boldsymbol{\eta})}^{+\infty} \exp\left(-\frac{n_p \beta z^{1/2}}{\tau \log(n_p) \|\mathbb{K}_\boldsymbol{\eta}\|_\infty}\right) dz. \end{aligned}$$

The first term can be computed explicitly

$$\mathbf{I} = \frac{32Kv^2(\alpha, n, \boldsymbol{\eta}, p)}{n_p \beta^2} \exp\left(-\frac{n_p \beta^2 V_n^p(\boldsymbol{\eta})}{32v^2(\alpha, n, \boldsymbol{\eta}, p)}\right).$$

Moreover, by definition of $v^2(\alpha, n, \boldsymbol{\eta}, p)$ in Theorem 1, and considering the fact that $\eta_1 \leq \dots \leq \eta_d$, we obtain $v^2(\alpha, n, \boldsymbol{\eta}, p) = n_p v_n^p(\boldsymbol{\eta})$. Then,

$$\mathbf{I} \lesssim v_n^p(\boldsymbol{\eta}) \exp\left(-\frac{\bar{\omega} \log(n_p) \beta^2}{32}\right).$$

Moreover, we know that

$$v_n^p(\boldsymbol{\eta}) = \frac{1}{T_n} \max \left\{ p \Delta_n \prod_{i=1}^d \eta_i^{-1}, \min \left\{ \sum_{i=1}^d |\log(\eta_i)| \prod_{i=3}^d \eta_i^{-1}, (\eta_2 \eta_3)^{-1/2} \prod_{i=4}^d \eta_i^{-1} \right\} \right\}.$$

Using the particular structure of the grid \mathcal{H}_n^p , we obtain $v_n^p(\mathbf{h}) \lesssim 1 + n_p/T_n$, and

$$(70) \quad \mathbf{I} \lesssim n_p^{-\frac{\bar{\omega} \beta^2}{32}} \left(1 + \frac{n_p}{T_n}\right).$$

For the second term \mathbf{II} , using Equation (67) we obtain

$$\mathbf{II} \lesssim \frac{\log(n_p) \|\mathbb{K}_\boldsymbol{\eta}\|_\infty \sqrt{V_n^p(\boldsymbol{\eta})}}{n_p} \exp\left(-\frac{\beta n_p V_n^p(\boldsymbol{\eta})^{1/2}}{\tau \log(n_p) \|\mathbb{K}_\boldsymbol{\eta}\|_\infty}\right).$$

Moreover, we know that $\|\mathbb{K}_\boldsymbol{\eta}\|_\infty \lesssim \prod_{i=1}^d \eta_i^{-1}$. Then,

$$\frac{\log(n_p) \|\mathbb{K}_\boldsymbol{\eta}\|_\infty \sqrt{V_n^p(\boldsymbol{\eta})}}{n_p} \lesssim \frac{v_n^p(\mathbf{h})^{1/2} \log(n_p)^2 \bar{\omega}^{1/2}}{n_p \prod_{i=1}^d \eta_i}.$$

Using the definition of $v_n^p(\mathbf{h})$, we obtain

$$\begin{aligned} \prod_{i=1}^d \eta_i^{-1} v_n^p(\mathbf{h})^{1/2} &= \frac{1}{\sqrt{T_n}} \max \left\{ \sqrt{p \Delta_n} \prod_{i=1}^d h_i^{-3/2}, \right. \\ &\quad \left. \min \left\{ (h_1 h_2)^{-1} \sum_{i=1}^d |\log(h_i)| \prod_{i=3}^d h_i^{-3/2}, h_1^{-1} (h_2 h_3)^{-5/4} \prod_{i=4}^d h_i^{-3/2} \right\} \right\} \end{aligned}$$

Using once again the lower bound on the η_i for $i = 1, \dots, d$, we obtain

$$\prod_{i=1}^d \eta_i^{-1} v_n^p(\mathbf{h})^{1/2} \lesssim 1 + \frac{\sqrt{n_p}}{\sqrt{T_n}}.$$

Finally, we can write

$$(71) \quad \frac{n_p V_n^p(\boldsymbol{\eta})^{1/2}}{\log(n_p) \|\mathbb{K}_\boldsymbol{\eta}\|_\infty} \gtrsim \bar{\omega}^{1/2} \log(n_p)$$

In fact,

$$\frac{n_p V_n^p(\boldsymbol{\eta})^{1/2}}{\log(n_p) \|\mathbb{K}_\boldsymbol{\eta}\|_\infty} \gtrsim \frac{\bar{\omega}^{1/2} n_p \prod_{i=1}^d \eta_i v_n^p(\boldsymbol{\eta})^{1/2}}{\log(n_p)^{1/2}},$$

and

$$\prod_{i=1}^d \eta_i v_n^p(\boldsymbol{\eta})^{1/2} = \left(\prod_{i=1}^d \eta_i \right)^{1/2} \frac{1}{\sqrt{T_n}} \max \left\{ \sqrt{p\Delta_n}, \min \left\{ \left(\sum_{i=1}^d |\log(\eta_i)| \eta_1 \eta_2 \right)^{1/2}, \sqrt{\eta_1} (\eta_2 \eta_3)^{1/4} \right\} \right\}$$

This ensures that

$$\begin{aligned} \prod_{i=1}^d \eta_i v_n^p(\boldsymbol{\eta})^{1/2} &\geq \frac{1}{\sqrt{n_p}} \left(\prod_{i=1}^d \eta_i \right)^{1/2} \\ &\geq \frac{\log(n_p)^{3/2}}{n_p}, \end{aligned}$$

implying Equation (71). Finally, this ensures that for any $\mathbf{h}, \boldsymbol{\eta} \in \mathcal{H}_n^p$, we have

$$(72) \quad \mathbf{I}_{\mathbf{h}, \boldsymbol{\eta}}^{n,p}(x) \lesssim \frac{n_p^{1-\bar{\omega}\beta/32}}{T_n} + \frac{n_p^{1/2-\bar{\omega}^{1/2}\beta/\tau}}{\sqrt{T_n}}.$$

For the term $\mathbf{II}_{\mathbf{h}, \boldsymbol{\eta}}^{n,p}(x)$. Observe that such a term can be treated exactly as the previous one noting that

$$\|\mathbf{K}_{\mathbf{h}} * \mathbf{K}_{\boldsymbol{\eta}}\|_{\infty} \leq \|\mathbf{K}_{\boldsymbol{\eta}}\|_{\infty} \|\mathbf{K}_{\mathbf{h}}\|_1 \quad \text{and} \quad \|\mathbf{K}_{\mathbf{h}} * \mathbf{K}_{\boldsymbol{\eta}}\|_1 \leq \|\mathbf{K}_{\mathbf{h}}\|_1^2.$$

In fact, following the proofs of Lemmas 8, 9, 10 and 11, we get that for all $\varepsilon > 0$

$$\mathbb{P}_{\bar{\mu}^b}^b \left(\left| \hat{\mu}_{n,(\mathbf{h}, \boldsymbol{\eta}),p}(x) - \mathbb{E}_{\bar{\mu}^b}^b [\hat{\mu}_{n,(\mathbf{h}, \boldsymbol{\eta}),p}(x)] \right| > \varepsilon \right) \leq K \exp \left(- \frac{n_p \varepsilon^2 \beta^2}{32v^2(\alpha, n, \boldsymbol{\eta}, p) + \tau\beta\varepsilon \|\mathbf{K}_{\boldsymbol{\eta}}\|_{\infty} \log n_p} \right).$$

Finally, we obtain the same bound for $\mathbf{II}_{\mathbf{h}, \boldsymbol{\eta}}^{n,p}(x)$,

$$\mathbf{II}_{\mathbf{h}, \boldsymbol{\eta}}^{n,p}(x) \lesssim \frac{n_p^{1-\bar{\omega}\beta/32}}{T_n} + \frac{n_p^{1/2-\bar{\omega}^{1/2}\beta/\tau}}{\sqrt{T_n}}.$$

Finally, we consider the term $\mathbf{III}_{\mathbf{h}, \boldsymbol{\eta}}^{n,p}(x)$. We recall

$$\mathbf{III}_{\mathbf{h}, \boldsymbol{\eta}}^{n,p}(x) = \left| \mathbb{E}_{\bar{\mu}^b}^b [\hat{\mu}_{n,\boldsymbol{\eta},p}(x)] - \mathbb{E}_{\bar{\mu}^b}^b [\hat{\mu}_{n,(\mathbf{h}, \boldsymbol{\eta}),p}(x)] \right|^2.$$

In following, let us denote $\mu_{\boldsymbol{\eta}}(x) = \mathbb{E}_{\bar{\mu}^b}^b [\hat{\mu}_{n,\boldsymbol{\eta},p}(x)]$ and $\mu_{(\mathbf{h}, \boldsymbol{\eta})}(x) = \mathbb{E}_{\bar{\mu}^b}^b [\hat{\mu}_{n,(\mathbf{h}, \boldsymbol{\eta}),p}(x)]$. Moreover, recalling that $\tilde{\tau}_{n,p}$ is defined in Equation (15) and denoting by $\bar{\rho}_{n,p}$ the law of the preaveraged process defined in Equation (12) under $\mathbb{P}_{\bar{\mu}^b}^b$, we write

$$\begin{aligned} \mu_{(\mathbf{h}, \boldsymbol{\eta})}(x) &= \sum_{\gamma} \mathbf{u}_{\gamma} \mathbb{E}_{\bar{\mu}^b}^b [\hat{v}_{n,(\mathbf{h}, \boldsymbol{\eta}),p}(x + \gamma \tilde{\tau}_{n,p})] \\ &= \sum_{\gamma} \mathbf{u}_{\gamma} \int_{\mathbb{R}^d} \mathbf{K}_{\mathbf{h}} * \mathbf{K}_{\boldsymbol{\eta}}(x + \gamma \tilde{\tau}_{n,p} - y) \bar{\rho}_{n,p}(dy) \\ (73) \quad &= \sum_{\gamma} \mathbf{u}_{\gamma} ((\mathbf{K}_{\mathbf{h}} * \mathbf{K}_{\boldsymbol{\eta}}) * \bar{\rho}_{n,p})(x + \gamma \tilde{\tau}_{n,p}) \\ &= \mathbf{K}_{\boldsymbol{\eta}} * \left(\sum_{\gamma} \mathbf{u}_{\gamma} \mathbf{K}_{\mathbf{h}} * \bar{\rho}_{n,p}(\cdot + \gamma \tilde{\tau}_{n,p}) \right)(x) \\ &= \mathbf{K}_{\boldsymbol{\eta}} * \mu_{\mathbf{h}}(x). \end{aligned}$$

Now, let us perform the same decomposition as in Section B (see more precisely Equations (43) and (44)), and write

$$\begin{aligned} \mu_{\boldsymbol{\eta}}(x) &= \sum_{\gamma} \mathbf{u}_{\gamma} \mathbb{E}_{\bar{\mu}^b}^b [\hat{v}_{n,\boldsymbol{\eta},p}(x + \gamma \tilde{\tau}_{n,p})] \\ &= \sum_{\gamma} \mathbf{u}_{\gamma} B_1(x + \gamma \tilde{\tau}_{n,p}) + \sum_{\gamma} \mathbf{u}_{\gamma} B_2(x + \gamma \tilde{\tau}_{n,p}). \end{aligned}$$

For the second term on the right hand side, we obtain thanks to Equation (49) that

$$(74) \quad \sum_{\gamma} \mathbf{u}_{\gamma} B_2(x + \gamma \tilde{\tau}_{n,p}) \lesssim \sqrt{p\Delta_n}.$$

Moreover,

$$\begin{aligned} \sum_{\gamma} \mathbf{u}_{\gamma} B_1(x + \gamma \tilde{\tau}_{n,p}) &= \sum_{\gamma} \mathbf{u}_{\gamma} \mathbf{K}_{\eta} * (\bar{\mu}^b * \varphi_{\tilde{\tau}_{n,p}})(x + \gamma \tilde{\tau}_{n,p}) \\ &= \mathbf{K}_{\eta} * \left(\sum_{\gamma} \mathbf{u}_{\gamma} (\bar{\mu}^b * \varphi_{\tilde{\tau}_{n,p}})(\cdot + \gamma \tilde{\tau}_{n,p}) \right) (x). \end{aligned}$$

Moreover, it is easy to see using the definition of $(\mathbf{u}_{\gamma})_{\gamma}$, that for any $z \in \mathbb{R}^d$,

$$\left| \sum_{\gamma} \mathbf{u}_{\gamma} (\bar{\mu}^b * \varphi_{\tilde{\tau}_{n,p}})(z + \gamma \tilde{\tau}_{n,p}) - \bar{\mu}^b(z) \right| \lesssim \frac{\tau_n^{2\alpha_1}}{p^{\alpha_1}}.$$

Then, we can write

$$(75) \quad \mu_{\eta}(x) = \mathbf{K}_{\eta} * \bar{\mu}^b(x) + \varepsilon_{n,p}(x),$$

for some function $\varepsilon_{n,p}$ such that for all $x \in \mathbb{R}^d$, $|\varepsilon_{n,p}(x)| \lesssim \sqrt{p\Delta_n} + \tau_n^{2\alpha_1}/p^{\alpha_1}$. Then, combining Equations (73) and (75), we obtain that for all $x \in \mathbb{R}^d$,

$$\mathbb{I}_{\mathbf{h},\eta}^{n,p}(x) \lesssim B_{n,\mathbf{h},p}(x)^2 + p\Delta_n + \frac{\tau_n^{2\alpha_1}}{p^{\alpha_1}}.$$

Finally, using $\#\mathcal{H}_n^p \lesssim T_n$, we obtain that

$$\begin{aligned} \sup_{\eta \in \mathcal{H}_n^p} \left\{ |\hat{\mu}_{n,(\mathbf{h},\eta),p}(x) - \hat{\mu}_{n,\eta,p}(x)|^2 - V_n^p(\eta) \right\}_+ &\lesssim B_{n,\mathbf{h},p}(x)^2 + n_p^{1-\bar{\omega}\beta/32} \\ &\quad + \sqrt{T_n n_p^{1/2-\bar{\omega}^{1/2}\beta/\tau}} + p\Delta_n + \frac{\tau_n^{2\alpha_1}}{p^{\alpha_1}}. \end{aligned}$$

Finally, using the fact that $T_n \leq n_p$ and taking $\bar{\omega} > 2\tau^2/\beta^2 \vee 64/\beta$, we get the expected result. \square

We are now ready to move on to the proof of Proposition 8.

Proof of Proposition 8. Let us consider $\mathbf{h} \in \mathcal{H}_n^p$, then one can write

$$\begin{aligned} \mathbb{E}_{\bar{\mu}^b}^b \left[|\hat{\mu}_{n,\mathbf{h}^*,p}(x) - \bar{\mu}^b(x)|^2 \right] &\lesssim \mathbb{E}_{\bar{\mu}^b}^b \left[|\hat{\mu}_{n,\mathbf{h},p}(x) - \bar{\mu}^b(x)|^2 \right] \\ &\quad + \mathbb{E}_{\bar{\mu}^b}^b \left[|\hat{\mu}_{n,\mathbf{h}^*,p}(x) - \hat{\mu}_{n,\mathbf{h},p}(x)|^2 \right]. \end{aligned}$$

We know that the first term on the right-hand side can be controlled in the following way

$$\mathbb{E}_{\bar{\mu}^b}^b \left[|\hat{\mu}_{n,\mathbf{h},p}(x) - \bar{\mu}^b(x)|^2 \right] \leq B_{n,\mathbf{h},p}(x)^2 + V_n^p(\mathbf{h}).$$

For the second term, we obtain

$$\begin{aligned} \mathbb{E}_{\bar{\mu}^b}^b \left[|\hat{\mu}_{n,\mathbf{h}^*,p}(x) - \hat{\mu}_{n,\mathbf{h},p}(x)|^2 \right] &\lesssim \mathbb{E}_{\bar{\mu}^b}^b \left[|\hat{\mu}_{n,\mathbf{h}^*,p}(x) - \hat{\mu}_{n,(\mathbf{h}^*,\mathbf{h}),p}(x)|^2 \right] \\ &\quad + \mathbb{E}_{\bar{\mu}^b}^b \left[|\hat{\mu}_{n,\mathbf{h},p}(x) - \hat{\mu}_{n,(\mathbf{h}^*,\mathbf{h}),p}(x)|^2 \right]. \end{aligned}$$

This finally gives

$$\mathbb{E}_{\bar{\mu}^b}^b \left[|\hat{\mu}_{n,\mathbf{h}^*,p}(x) - \bar{\mu}^b(x)|^2 \right] \lesssim \mathbb{E}_{\bar{\mu}^b}^b [A_n^p(\mathbf{h})] + V_n^p(\mathbf{h}) + \mathbb{E}_{\bar{\mu}^b}^b \left[|\hat{\mu}_{n,\mathbf{h},p}(x) - \bar{\mu}^b(x)|^2 \right].$$

Moreover, using the result of Lemma 14, we obtain

$$\mathbb{E}_{\bar{\mu}^b}^b \left[|\hat{\mu}_{n,\mathbf{h}^*,p}(x) - \bar{\mu}^b(x)|^2 \right] \lesssim \inf_{\mathbf{h} \in \mathcal{H}_n^p} \{ B_{n,\mathbf{h},p}(x)^2 + V_n^p(\mathbf{h}) \} + n^{-\gamma} + p\Delta_n + \tau_n^{2\alpha_1}/p^{\alpha_1},$$

for some $\gamma > 1$. \square

APPENDIX G. PROOF OF THE RESULTS OF SECTION 7

First recall the following technical lemma which extensively used in the following.

Lemma 15. For $a_1, a_2 \in \mathbb{R}^d$, $\nu_1, \nu_2 \geq 0$, such that $\nu_1 + \nu_2 > 0$, we have

$$\int_{\mathbb{R}^d} e^{-\nu_1|x-a_1|^2 - \nu_2|x-a_2|^2} dx = \frac{\kappa_1}{(\nu_1 + \nu_2)^{d/2}} e^{-\frac{\nu_1\nu_2}{\nu_1+\nu_2}|a_1-a_2|^2},$$

for some constant $\kappa_1 > 0$ depending only on the dimension d .

G.1. Proof of Lemma 2. We first consider the joint density of $(p^{-1} \sum_{k=0}^{p-1} X_{k\Delta_n}, X_{p\Delta_n})$ conditional on $X_0 = x$, denoted $\mathbf{p}_{p,n}^b(x; \cdot, \cdot)$. We claim that under the assumptions of Lemma 2, there exists constants C_1, λ_1, c_1 and $\eta_1 > 0$ such that if $p\Delta_n \leq \eta_1$, we have for any x, y, z

$$(76) \quad \mathbf{p}_{p,n}^b(x; y, z) \leq \frac{C_1}{(p\Delta_n)^d} \exp\left(-\lambda_1 \frac{|y-x|^2 + |z-x|^2}{p\Delta_n} + V(x) - V(z)\right)$$

Using Lemma 15, (37) is a mere consequence of (76): since $V(z)$ is bounded below, it suffices to integrate (76) with respect to z to get (37). The rest of this proof is devoted to showing (76).

Note first that $\mathbf{p}_{p,n}^b(x; y, z) = (p\Delta_n)^{-d} \mathbf{q}_{p,n}^b(x; \sqrt{1/(p\Delta_n)}(y-x), \sqrt{1/(p\Delta_n)}(z-x))$ where $\mathbf{q}_{p,n}^b(x; \cdot, \cdot)$ is the joint density of $(p^{-1}(p\Delta_n)^{-1/2} \sum_{k=0}^{p-1} (X_{k\Delta_n} - x), (p\Delta_n)^{-1/2}(X_{p\Delta_n} - x))$ under the condition that $X_0 = x$. Therefore, it is enough to prove that

$$(77) \quad \mathbf{q}_{p,n}^b(x; y, z) \leq c_2^{-1} e^{-c_2(y^2+z^2)+V(x)-V(x+(p\Delta_n)^{1/2}z)},$$

for some $c_2 > 0$. The methodology we use to prove (77) closely resembles the one of Theorem 4 in [GG08]; we refer to the comments of Lemma 2 for more details. We introduce the process $\mathcal{X}_t^{p,n} = (p\Delta_n)^{1/2}(X_{tp\Delta_n} - x)$ which satisfies

$$d\mathcal{X}_t^{p,n} = b_{p,n}(\mathcal{X}_t^{p,n}) dt + dW_t^{p,n},$$

where $W^{p,n}$ is a d -dimensional Brownian motion and $b_{p,n}(w) = (p\Delta_n)^{1/2}b(x+(p\Delta_n)^{1/2}w)$. Moreover,

$$\frac{1}{p} \left(\frac{1}{p\Delta_n}\right)^{1/2} \sum_{k=0}^{p-1} (X_{k\Delta_n} - x) = \frac{1}{p} \sum_{\ell=0}^{p-1} \mathcal{X}_{\ell/p}^{p,n} \quad \text{and} \quad \left(\frac{1}{p\Delta_n}\right)^{1/2} (X_{p\Delta_n} - x) = \mathcal{X}_1^{p,n}.$$

Let us now define the stochastic process

$$(\mathcal{E}_t^{p,n})_{t \geq 0} = \left(\exp\left(-\int_0^t b_{p,n}(\mathcal{X}_s^{p,n}) dW_s^{p,n} - \frac{1}{2} \int_0^t |b_{p,n}(\mathcal{X}_s^{p,n})|^2 ds\right) \right)_{t \geq 0}.$$

Under Assumption 1, we get that Novikov criterion holds following the steps of the proof of Lemma 5. This ensures that $(\mathcal{E}_t^{p,n})_{t \geq 0}$ is a martingale with constant expectation equal to 1. Then it defines a change of measure and we can consider a probability measure \mathbb{Q}_x^b , under which we get rid of the influence of the drift for the dynamic of $\mathcal{X}^{p,n}$. More precisely, we define the probability measure \mathbb{Q}_x^b on $\sigma(\{W_t^{p,n}, t \leq 1\})$ by

$$\frac{d\mathbb{Q}_x^b}{d\mathbb{P}_x^b} = \exp\left(-\int_0^1 b_{p,n}(\mathcal{X}_t^{p,n}) dW_t^{p,n} - \frac{1}{2} \int_0^1 |b_{p,n}(\mathcal{X}_t^{p,n})|^2 dt\right).$$

Using the Itô formula, we obtain

$$\begin{aligned} \frac{d\mathbb{P}_x^b}{d\mathbb{Q}_x^b} &= \exp\left(\int_0^1 b_{p,n}(\mathcal{X}_t^{p,n}) d\mathcal{X}_t^{p,n} - \frac{1}{2} \int_0^1 |b_{p,n}(\mathcal{X}_t^{p,n})|^2 dt\right) \\ &= \exp\left(B_{p,n}(\mathcal{X}_1^{p,n}) - \frac{1}{2} \int_0^1 |b_{p,n}(\mathcal{X}_t^{p,n})|^2 + \nabla \cdot b_{p,n}(\mathcal{X}_t^{p,n}) dt\right), \end{aligned}$$

where for all $w \in \mathbb{R}^d$, $B_{p,n}(w) := V(x) - V(x + (p\Delta_n)^{1/2}w)$. Now let f and g be two bounded positive functions. Then we have

$$\begin{aligned} &\mathbb{E}_x \left[f\left(\frac{1}{p} \sum_{\ell=0}^{p-1} \mathcal{X}_{\ell/p}^{p,n}\right) g(\mathcal{X}_1^{p,n}) \right] \\ &= \mathbb{E}^{\mathbb{Q}_x^b} \left[f\left(\frac{1}{p} \sum_{\ell=0}^{p-1} \mathcal{X}_{\ell/p}^{p,n}\right) g(\mathcal{X}_1^{p,n}) \exp\left(B_{p,n}(\mathcal{X}_1^{p,n}) - \frac{1}{2} \int_0^1 |b_{p,n}(\mathcal{X}_t^{p,n})|^2 + \nabla \cdot b_{p,n}(\mathcal{X}_t^{p,n}) dt\right) \right]. \end{aligned}$$

Moreover, we have $\nabla \cdot b_{p,n}(w) \leq Cp\Delta_n$. Since Girsanov Theorem ensures that $\mathcal{X}^{p,n}$ is a Brownian motion under \mathbb{Q}_x^b , we get the bound

$$(78) \quad \mathbb{E}_x^b \left[f\left(\frac{1}{p} \sum_{\ell=0}^{p-1} \mathcal{X}_{\ell/p}^{p,n}\right) g(\mathcal{X}_1^{p,n}) \right] \leq C_1 \mathbb{E}^b \left[f\left(\frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell/p}\right) g(W_1) e^{B_{p,n}(W_1)} \right],$$

for some Brownian motion W and some constant C_1 , which does not depend on p nor n . Therefore, it is enough to prove that

$$(79) \quad \mathbb{E}^b \left[f \left(\frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell/p} \right) g(W_1) e^{B_{p,n}(W_1)} \right] \leq C'_1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u) g(v) e^{-C'_2(|u|^2 + |v|^2) + B_{p,n}(v)} du dv$$

holds for some positive constants C'_1 and C'_2 . Again, this is closely related to Lemma 4 of [GG08]. Recall that $W_t^* = W_t - tW_1$ defines a Brownian bridge on $[0, 1]$, independent of W_1 . Thus, if $h(v) = g(v)e^{B_{p,n}(v)}$, we get:

$$(80) \quad \begin{aligned} \mathbb{E}^b \left[f \left(\frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell/p} \right) h(W_1) \right] &= \mathbb{E}^b \left[f \left(\frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell/p}^* + \frac{\ell}{p} W_1 \right) h(W_1) \right] \\ &= \mathbb{E}^b \left[\mathbb{E}^b \left[f \left(\frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell/p}^* + \frac{\ell}{p} W_1 \right) h(W_1) \middle| W_1 \right] \right] \\ &= \mathbb{E}^b \left[\psi_{p,n}^b(W_1) \right], \end{aligned}$$

where for all $\omega \in \mathbb{R}^d$,

$$\psi_{p,n}^b(\omega) = \mathbb{E}^b \left[f \left(\frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell/p}^* + \frac{\ell}{p} \omega \right) h(\omega) \right].$$

We know that the Brownian Bridge W^* itself admits the following decomposition, see e.g [GG08]:

$$W_t^* = \xi \eta_t + W_t^{**},$$

where ξ is a standard random variable, η is the deterministic function

$$\eta_t = \begin{cases} t & \text{if } t \in [0, 1/2], \\ (1-t) & \text{if } t \in [1/2, 1], \end{cases}$$

and W^{**} is the process on $[0, 1]$ constructed as the concatenation of two independent Brownian bridges, on $[0, 1/2]$ and $[1/2, 1]$ respectively. Moreover in this decomposition the random variable η and the process W^{**} are independent. Then,

$$(81) \quad \psi_{p,n}^b(\omega) = h(\omega) \mathbb{E} \left[f \left(\frac{\xi}{p} \sum_{\ell=0}^{p-1} \eta_{\ell/p} + \frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell/p}^{**} + \frac{p-1}{2p} \omega \right) \right].$$

Let $c_p = p^{-1} \sum_{\ell=0}^{p-1} \eta_{\ell/p}$, which is bounded uniformly in p . Using the independence between ξ and W^{**} , we have:

$$\begin{aligned} \mathbb{E}^b \left[\left(\frac{\xi}{p} \sum_{\ell=0}^{p-1} \eta_{\ell/p} + \frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell/p}^{**} + \frac{p-1}{2p} \omega \right) \right] &= \mathbb{E}^b \left[\int_{\mathbb{R}^d} f \left(c_p v + \frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell/p}^{**} + \frac{p-1}{2p} \omega \right) \frac{1}{(2\pi)^{d/2}} e^{-|v|^2/2} dv \right] \\ &= \mathbb{E}^b \left[\int_{\mathbb{R}^d} f(u) (\sqrt{2\pi} c_p)^{-d} e^{-\frac{1}{2c_p^2} |u - \frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell/p}^{**} - \frac{p-1}{2p} \omega|^2} du \right]. \end{aligned}$$

We then use the fact that for $\varepsilon \in (0, 1)$, $|x - y|^2 \geq \varepsilon/(1 + \varepsilon)|x|^2 - \varepsilon|y|^2$, to get

$$(82) \quad \mathbb{E}^b \left[\psi_{p,n}^b(W_1) \right] \leq \int_{\mathbb{R}^d} \frac{1}{(\sqrt{2\pi} c_p)^d} f(u) e^{-\frac{|u|^2 \varepsilon}{2c_p^2(1+\varepsilon)} + \frac{c_p^2 \varepsilon (p-1)^2 |u|^2}{4c_p^2 p^2}} \mathbb{E} \left[e^{c_p^{-2} \varepsilon \sup_{t \in [0,1]} |W_t^{**}|^2} \right].$$

Using that W^{**} is a concatenation of two Brownian bridges, we know that there exists $\varepsilon_+ > 0$ and C_ε such that for all $\varepsilon \leq \varepsilon_+$,

$$\mathbb{E} \left[e^{c_p^2 \varepsilon \sup_{t \in [0,1]} |W_t^{**}|^2} \right] \leq C_\varepsilon.$$

Moreover, thanks to the boundedness of c_p with respect to p , we get that C_ε does not depend on p . Then, plugging (82) into Equation (80), we get that for $\varepsilon \leq \varepsilon_+$,

$$\mathbb{E}^b \left[f \left(\frac{1}{p} \sum_{\ell=0}^{p-1} W_{\ell/p} \right) h(W_1) \right] \leq \frac{C_\varepsilon}{(\sqrt{2\pi} c_p)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u) h(v) e^{-\frac{c_p^2 u^2 \varepsilon}{2(1+\varepsilon)} + \frac{c_p^2 \varepsilon (p-1)^2 |v|^2}{4p^2}} e^{-|v|^2/2} du dv$$

$$\leq \frac{C_\varepsilon}{(\sqrt{2\pi c_p})^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(u)g(v) e^{B_{p,n}(v)} e^{-\frac{u^2 \varepsilon}{2c_p^2(1+\varepsilon)} + \frac{\varepsilon(p-1)^2 |v|^2}{4c_p^2 p^2}} e^{-|v|^2/2} du dv.$$

Then, we get Equation (79) as soon as $\varepsilon < \min(2c_p p^2/(p-1)^2, \varepsilon_+)$, which concludes the proof.

G.2. Proof of Lemma 3. We consider $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ non-negative bounded with compact support. From Girsanov Theorem, we can show that for any $x \in \mathbb{R}^d$,

$$\mathbb{E}_x^b[\varphi(X_t)] \leq (2\pi)^{-d/2} e^{b_1 t/2} \frac{e^{V(x)}}{t^{d/2}} \int_{\mathbb{R}^d} \varphi(y) e^{-\frac{|x-y|^2}{2t} - V(y)} dy$$

and we conclude using $t \leq 1$, and the boundedness of $\bar{\mu}^b$.

G.3. Proof of Corollary 1. First note that since X is a Markov process, we have

$$\mathbf{p}_{p,n,t}^b(x; y) = \int_{\mathbb{R}^d} p_t^b(x; z) \mathbf{p}_{p,n}^b(z; y) dz.$$

Using the bounds given by Lemmas 2 and 3, combined with Lemma 15, we get

$$\mathbf{p}_{p,n,t}^b(x; y) \lesssim (p\Delta_n + 2t\lambda_1)^{-d/2} \exp\left(\frac{\lambda_1}{p\Delta_n + 2t}|x-y|^2 + V(x)\right).$$

G.4. Proof of Lemma 4. Note that since the distribution of X_0 is $\bar{\mu}^b$ which is the invariant measure of X , the distribution of $(p^{-1} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n}, X_{p\Delta_n})$ is $\bar{\pi}^b$. Thus for any non negative function $\varphi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$, we have

$$\begin{aligned} \mathbb{E}_{\bar{\mu}^b}^b\left[\varphi\left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n}, X_{p\Delta_n}\right)\right] &= \int_{\mathbb{R}^d} \mathbb{E}_x^b\left[\varphi\left(\frac{1}{p} \sum_{\ell=0}^{p-1} X_{\ell\Delta_n}, X_{p\Delta_n}\right)\right] \bar{\mu}^b(x) dx \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y, z) \mathbf{p}_{p,n}^b(x; y, z) \bar{\mu}^b(x) dx dz dy. \end{aligned}$$

From Equation (76), we get

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y, z) \mathbf{p}_{p,n}^b(x; y, z) \bar{\mu}^b(x) dx dz dy \\ &\leq \frac{C_1}{(p\Delta_n)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y, z) \exp\left(-\lambda_1 \frac{|y-z|^2 + |z-x|^2}{p\Delta_n}\right) e^{-V(z)+V(x)} \frac{e^{-2V(x)}}{Z_V} dx dz dy \\ &\leq \frac{C_1 \|\bar{\mu}^b\|_\infty^{1/2}}{Z_V^{1/2} (p\Delta_n)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y, z) e^{-V(z)} \left(\int_{\mathbb{R}^d} \exp\left(-\frac{\lambda_1}{p\Delta_n}(|y-x|^2 + |z-x|^2)\right) dx\right) dy dz. \end{aligned}$$

Use Lemma 15 to control the integral in the variable x , and get

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y, z) \mathbf{p}_{p,n}^b(x; y, z) \bar{\mu}^b(x) dx dz dy \\ &\leq \frac{C_1 \kappa_1 \|\bar{\mu}^b\|_\infty^{1/2}}{Z_V^{1/2} (p\Delta_n)^{d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y, z) e^{-V(z)} \exp\left(-\frac{\lambda_1}{2p\Delta_n}|y-z|^2\right) dy dz. \end{aligned}$$

We can conclude that for all $(y, z) \in \mathbb{R}^{2d}$, we have

$$\bar{\pi}^b(y, z) \leq \frac{C_1 \kappa_1 \|\bar{\mu}^b\|_\infty^{1/2}}{Z_V^{1/2} (p\Delta_n)^{d/2}} \exp\left(-\frac{\lambda_1}{2p\Delta_n}|y-z|^2 - V(z)\right).$$

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