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CONUNDRUMS FOR CONTINUOUS LEBESGUE MEASURE-PRESERVING INTERVAL MAPS

JOZEF BOBOK, JERNEJ ČINČ, PIOTR OPROCHA, AND SERGE TROUBETZKOY

ABSTRACT. We consider continuous maps of the interval which preserve the Lebesgue measure. Except for the identity map or $1 - \text{id}$ all such maps have topological entropy at least $\log 2/2$ and generically they have infinite topological entropy. In this article we show that the generic map has zero measure-theoretic entropy. This implies that there are dramatic differences in the topological versus metric behavior both for injectivity as well as for the structure of the level sets of generic maps.

1. INTRODUCTION

Genericity of zero metric entropy has a long history starting with a result of Rokhlin. Let (X, μ) denote a Lebesgue probability space and $\text{Aut}(X, \mu)$ the group of measure preserving automorphisms of X . Consider the weak topology: a sequence $\{f_i\} \subset \text{Aut}(X, \mu)$ converges to f if for every measurable subset $A \subset X$ we have $\mu(f_i(A) \Delta f(A)) \rightarrow 0$ as $i \rightarrow \infty$. This makes $\text{Aut}(X, \mu)$ into a Polish space. In 1959 Rokhlin showed that the subset of all transformations $T \in \text{Aut}(X, \mu)$ that have zero entropy with respect to μ contains a dense G_δ subset [18]. Later Rudolf extended this result to countable amenable actions [12] and L. Bowen further extended this to the non-amenable case [8].

In the late 60s Katok and Stepin developed a quantitative approximation technique and were able to apply it not only for automorphisms but also for homeomorphisms of a compact manifold M of dimension at least 2 with respect to the uniform topology, i.e., induced by the metric

$$\rho(f, g) = \sup_{x \in M} d(f(x), g(x))$$

where d is the standard metric on M . In particular they showed that the generic Lebesgue measure-preserving homeomorphisms of M have zero metric entropy [15, 16]. These results have modern proofs established by Guihéneuf [14].

On the other hand Parry proved similar results in two non-invertible settings. First he considered the one-sided full 2-shift. He considered the weak-topology

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on the set of invariant ergodic probability measures, and showed that the set of zero entropy (ergodic, fully supported, and non-atomic) measures is a dense G_δ [17]. This result immediately translates to the doubling map on the circle. Then he considered the space S of strictly increasing degree 2 maps of the circle which preserve the Lebesgue measure with the uniform topology. In this setting he showed that the set of ergodic zero entropy maps is a dense G_δ .

It is clear that Parry's result also holds for full tent maps preserving the Lebesgue measure. We consider the generalization of this question when we drop the (quite strong) unimodal restriction. Let $I := [0, 1]$. Define C_λ to be the set of continuous maps of I which preserve the Lebesgue measure λ . We consider the uniform topology. The main result of this article is that there is a dense G_δ set of maps in C_λ with zero metric entropy, more precisely

Theorem 1. *The set of zero metric entropy maps is a dense G_δ set $G \subset C_\lambda$.*

Theorem 1 holds more generally for C_μ where μ is any non-atomic probability measure with full support.

By combining and adapting various known results we obtain several corollaries of Theorem 1 which contrast the measure theoretic and topological behaviors of generic maps in C_λ . Before turning to them we adapt the results of Bruckner and Garg [9] to show that generic maps in C_λ are nowhere differentiable, even stronger

Theorem 2. *Generic maps in C_λ are of nonmonotonic type, i.e., for each generic f we have $\overline{D}f(x) = +\infty$ and $\underline{D}f(x) = -\infty$ for every x .*

Clearly there are no invertible interval maps that are nowhere differentiable. A map $f \in C_\lambda$ is called *invertible a.e.* if there is an f -invariant set $X \subset I$ of full Lebesgue measure such that the restriction $f|_X$ is invertible. According to [21, Corollary 4.14.3] $h(f) = 0$ implies that f is invertible a.e.. Combining this with Theorem 2 yields the following surprising corollary.

Corollary 3. *Generic maps in C_λ are of nonmonotonic type and invertible a.e., thus for each generic f there exists a set $X_f \subset I$ of Lebesgue measure 1 such that f is a bijection on X_f and $\lambda(M) = \lambda(M \cap X_f) = \lambda(f^{-1}(M \cap X_f))$ for each Borel set $M \subset I$.*

Remark 4. *Corollary 3 is in a strong contrast to the topological behavior of typical maps in C_λ . Let*

$$B_0 := \{x \in I : \#f^{-1}(x) > 1\}.$$

It follows from [6, Corollary 3.2] that except for the two exception maps id and $1 - \text{id}$ the set B_0 contains an interval, and thus we always have a set of positive measure on which f is not one-to-one.¹

Remark 5. *The reverse implication, f is invertible a.e. implies that $h(f) = 0$, is not true in general. It is known that any compact manifold of dimension $d > 1$ which carries a minimal uniquely ergodic homeomorphism also carries a minimal uniquely ergodic homeomorphism with positive topological entropy [1] and the measure can be assumed to be the Lebesgue measure. It seems unknown if this reverse implication holds in C_λ .*

Our next result further contrasts the topological and metric behavior of generic maps. For every $c \in I$ the set $\{x : f(x) = c\}$ is called a *level set* of f .

Theorem 6. *For each generic map f in C_λ there exists a set $X_f \subset I$ of Lebesgue measure 1 such that for each $c \in (0, 1)$ the level set $\{x : f(x) = c\}$ contains a perfect set, while for each c the set $X_f \cap \{x : f(x) = c\}$ contains at most one point.*

For comparison we resume other generic properties of maps from C_λ ; in our context the item (6) in the following theorem is particularly interesting. For the definitions of the notions we will mention in the following theorem we refer the reader to the associated articles.

Theorem. *There is a dense G_δ subset of C_λ such that for each $f \in G$*

- (1) *f has infinite topological entropy [3],*
- (2) *f is weakly mixing with respect to λ [3],*
- (3) *f is leo and thus satisfies the specification property [3],*
- (4) *f has the shadowing property [4],*
- (5) *periodic points have Cantor set structure, see [4] for details.*
- (6) *f has λ -a.e. point a knot point [2],*
- (7) *the graph of f has Hausdorff dimension and lower box dimension 1 and upper box dimension 2 [3],[19].*
- (8) *f is δ -crooked [11].*

The leo and specification properties are actually satisfied not only generically but on an open dense subset of C_λ [7].

Two other related results were proven by Sigmund. In his first result he considered the full shift on $\sigma : \mathcal{R}^{\mathbb{Z}} \rightarrow \mathcal{R}^{\mathbb{Z}}$ and showed that the set of probability measures with zero entropy form a dense G_δ in the weak topology. He also consider the set of all subshifts with a fixed finite alphabet with respect to the

¹On the first page of the article [3] there is an incorrect statement that maps other than id and $1 - \text{id}$ have non-zero metric entropy. This need not be the case as exhibited by the results of this article.

Vietoris topology and showed that the generic subshift has zero topological entropy [20].

A generalization of Parry's result on genericity of zero entropy measures was obtained by Carvalho and Condori [10]. They considered uniformly continuous maps f defined over a Polish metric space and showed that the set of f -invariant measures with zero metric entropy is a G_δ set in the weak topology and that this set is dense and thus generic if the set of f -periodic measures is dense in the set of f -invariant measures.

2. PROOF OF THEOREM 1

Let $\mathcal{P} := \{P_1, \dots, P_k\}$ be a finite partition of I and let

$$H(\mathcal{P}) := - \sum_{i=1}^k \lambda(P_i) \log \lambda(P_i).$$

For $f \in C_\lambda$ we denote

$$h_n(f, \mathcal{P}) := \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} f^{-i}\mathcal{P}\right).$$

This function is decreasing in n thus the limit $h(f, \mathcal{P}) := \lim_{n \rightarrow \infty} h_n(f, \mathcal{P})$ exists. Finally we define the *entropy* of f by

$$h(f) := \sup\{h(f, \mathcal{P}) : \mathcal{P} \text{ is a finite partition of } I\}.$$

Let \mathcal{P}_i be the partition of I into dyadic intervals $[\frac{j}{2^i}, \frac{j+1}{2^i}]$ and \mathcal{A}_i the sub-algebra generated by \mathcal{P}_i . Then $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots$ and $\bigvee_{n=1}^{\infty} \mathcal{A}_n \overset{\circ}{=} \mathcal{B}$ hence by [21, Theorem 4.22] $h(f) = \lim_{i \rightarrow \infty} h(f, \mathcal{P}_i)$.

Lemma 7. *Fix $f \in C_\lambda$ and choose an arbitrary interval $J = [a, b] \subset I$. Then*

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } \lambda(f^{-1}(J) \triangle g^{-1}(J)) < \varepsilon \text{ for all } g \in B(f, \delta).$$

Proof. Fix $\varepsilon > 0$ and let $\delta < \frac{\varepsilon}{4}$. If we put $H_\delta = f^{-1}([a + \delta, b - \delta])$, since $f \in C_\lambda$,

$$(1) \quad \lambda(f^{-1}(J)) = \lambda(J) > \lambda(H_\delta) > \lambda(J) - \frac{\varepsilon}{2}.$$

For each $g \in B(f, \delta)$ we can write analogously

$$(2) \quad \lambda(g^{-1}(J)) = \lambda(J) > \lambda(H_\delta) > \lambda(J) - \frac{\varepsilon}{2}.$$

Clearly $H_\delta \subset (f^{-1}(J) \cap g^{-1}(J))$, so (1) and (2) imply

$$\lambda(f^{-1}(J) \triangle g^{-1}(J)) < \varepsilon. \quad \square$$

Lemma 8. For $\ell \in \mathbb{N}$ let positive numbers a_1, \dots, a_ℓ satisfy $\sum_{i=1}^\ell a_i = \eta < 1$. Then

$$-\sum_{j=1}^\ell a_j \log a_j \leq \eta \log \ell - \eta \log \eta.$$

Proof. Let $b_i := a_i/\eta$, then $\sum_{i=1}^\ell b_i = 1$ and thus $-\sum_{i=1}^\ell b_i \log b_i \leq \log \ell$ ([21, Cor 4.2.1]) which is equivalent to the announced inequality. \square

Proposition 9. Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a partition of I consisting of intervals and $n \in \mathbb{N}$. The map

$$f \longrightarrow h_n(f, \mathcal{P}), \quad f \in C_\lambda$$

is continuous.

Proof. Fix $\varepsilon > 0$ and $f \in C_\lambda$. Put $\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{P}) = \{A_1, \dots, A_m\}$, where $\lambda(A_j) > 0$ for each $j \in \{1, \dots, m\}$ and $\sum_{j=1}^m \lambda(A_j) = 1$. It means that there are exactly $m \leq k^n$ nonempty intersections

$$A_j = P_{i_0(j)} \cap f^{-1}(P_{i_1(j)}) \cap \dots \cap f^{-(n-1)}(P_{i_{n-1}(j)}), \quad j \in \{1, \dots, m\},$$

and all other $(k^n - m)$ intersections are empty (mod 0). For g close to f we consider $\bigvee_{i=0}^{n-1} g^{-i}(\mathcal{P})$, and label its positive measure elements by $B_{j,g}$ ($1 \leq j \leq \ell_g \leq k^m$). We choose sufficiently small $\delta > 0$ so that $\ell_g \geq m$ and applying Lemma 7 to the maps $f^0, f^1, \dots, f^{n-1} \in C_\lambda$ we obtain that each set

$$B_{j,g} = P_{i_0(j)} \cap g^{-1}(P_{i_1(j)}) \cap \dots \cap g^{-(n-1)}(P_{i_{n-1}(j)}), \quad j \in \{1, \dots, m\},$$

corresponds to the A_j sufficiently well so that,

$$-\sum_{j=1}^m \lambda(A_j) \log \lambda(A_j) + \sum_{j=1}^m \lambda(B_{j,g}) \log \lambda(B_{j,g}) < \frac{n\varepsilon}{2},$$

and $\sum_{j=m+1}^{\ell_g} \lambda(B_{j,g}) = 1 - \sum_{j=1}^m \lambda(B_{j,g}) = \eta$ for which

$$\eta \log(k^n - m) - \eta \log \eta < \frac{n\varepsilon}{2}$$

for each $g \in B(f, \delta)$. Since $\ell_g \leq k^n$, from Lemma 8 we obtain

$$(3) \quad 0 < -\sum_{j=m+1}^{\ell_g} \lambda(B_{j,g}) \log \lambda(B_{j,g}) \leq \eta \log(k^n - m) - \eta \log \eta < \frac{n\varepsilon}{2}.$$

From (3) we obtain

$$|h_n(f, \mathcal{P}) - h_n(g, \mathcal{P})| = \left| \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} f^{-i}(\mathcal{P})\right) - \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} g^{-i}(\mathcal{P})\right) \right| < \varepsilon. \quad \square$$

Proof of Theorem 1. Fix $\beta > 0$ and let

$$G_\beta := \bigcap_{k \geq 1} \bigcup_{i \geq k} \bigcup_{n \geq k} \{f \in C_\lambda : h_n(f, \mathcal{P}_i) < \beta\}.$$

We claim that

- (i) G_β a dense G_δ set and
- (ii) G_β coincides with the set of maps with entropy less than β .

Once the claim is established the theorem follows since $G := \bigcap_{n \geq 1} G_{1/n}$ is a dense G_δ set and it coincides with the set of zero entropy maps.

We turn to the proof of the claim. Let us first prove (ii). If $h(f) < \beta$ then $h_n(f, \mathcal{P}_i) < \beta$ for each i for all sufficiently large n , so $f \in G_\beta$. Conversely to see that the entropy of the maps in G_β is at most β we fix $f \in G$. By the definition of G_β , for each $f \in G_\beta$ there are sequences $i_k \geq k$ and $n_k \geq k$ such that $h_{n_k}(f, \mathcal{P}_{i_k}) < \beta$ for each $k \geq 1$. Since $h_n(f, \mathcal{P})$ is decreasing in n we conclude $h(f, \mathcal{P}_{i_k}) < \beta$.

Since \mathcal{P}_{i+1} refines \mathcal{P}_i we have $h(f, \mathcal{P}_i)$ is monotonic in i and we conclude $h(f, \mathcal{P}_i) < \beta$ for all i . Thus $h(f) = \lim_i h(f, \mathcal{P}_i) < \beta$ which finishes the proof of (ii).

To prove (i) we remark that the set $\{f \in C_\lambda : h_n(f, \mathcal{P}_i) < \beta\}$ is open by Proposition 9, thus G_β is a G_δ set.

On the other hand the density of G_β follows from [3] Proposition 24. This finishes the proof of the Theorem. \square

3. CONSTRUCTIONS OF ZERO ENTROPY MAPS

Example 10. Suppose $f \in C_\lambda$ be such that $\lambda(\text{Per}(f)) = 1$. If we use the ergodic decomposition theorem [21, p. 153] then

$$(4) \quad h(f) = \int_{E(I, f)} h_\mu(f) d\nu(\mu)$$

where $E(I, f)$ is the set of all ergodic measures invariant for f and ν is a Borel measure on $E(I, f)$ satisfying

$$\lambda = \int_{E(I, f)} \mu d\nu(\mu).$$

Since $\lambda(\text{Per}(f)) = 1$ necessarily ν a.e. measure ergodic μ is a CO-measure (i.e., a measure on a periodic orbit). In [4, Theorem 2] we constructed a dense set of leo maps for which the periodic points have full measure, and thus they all have zero entropy. This yields an alternative proof of the density of G from the proof of Theorem 1.

Example 11. Parry sketched a construction of maps $f \in C_\lambda$ which are piecewise monotone with 2-full branches which are invertible λ -a.e. [17]; a more detailed version of this construction (with k full branches) was given by Friedland and Weiss [13].

4. PROOFS OF THEOREM 2 AND THEOREM 6

Let $C(I)$ denote the set of continuous maps $f : I \rightarrow I$. A function $f \in C(I)$ is said to be *non-decreasing at a point* $x \in I$ if there exists a $\delta > 0$ such that $f(t) \leq f(x)$ when $t \in I \cap (x - \delta, x)$ and $f(t) \geq f(x)$ when $t \in I \cap (x, x + \delta)$. The function f is called *nonincreasing at* x if $-f$ is nondecreasing at x , and f is called *monotone at* x if it is either nondecreasing or nonincreasing at x . If $f \in C(I)$ and $\gamma \in \mathbb{R}$ then we define the map $f_{-\gamma}$ by:

$$f_{-\gamma}(x) := f(x) - \gamma x \text{ for } x \in I.$$

A map f is *monotonic type at* x if there exists a real number γ such that the function $f_{-\gamma}$ is monotone at x . If f is not of monotonic type at any point of I , it will be said to be of *nonmonotonic type*. It is easy to see that f is of nonmonotonic type if and only if for each x , $\overline{D}f(x) = +\infty$ and $\underline{D}f(x) = -\infty$, where $\overline{D}f$, $\underline{D}f$ are the upper and lower derivatives.

The proof of Theorem 2 requires two auxiliary lemmas. The first lemma is a minor modification of Lemma 11 from [5].

Lemma 12. *Let f be a piecewise affine continuous interval with nonzero slopes and such that its derivative does not exist at a finite set E . Then $f \in C_\lambda$ if and only if*

$$(5) \quad \forall y \in [0, 1] \setminus F(E): \sum_{x \in F^{-1}(y)} \frac{1}{|F'(x)|} = 1.$$

A *critical value* of a map f is a local maximum or a local minimum. A *determining value* of a piecewise affine map f is either its critical value or value at a point in which the derivative of f does not exist.

Lemma 13. *The set of piecewise affine maps with rational determining values is dense in C_λ .*

Proof. Consider a dense collection of piecewise affine maps $\{f_m\} \subset C_\lambda$ (Figure 1 left), each f_m has finitely many determining points. Fix m and choose $n = n(m)$ so large that each of the intervals $[k/n, (k+1)/n]$ contains at most one determining value. For each irrational determining value c_i of f_m we consider the map $g_{n,i} \circ f_m$, where $g_{n,i}$ is the map of Figure 1 right. The resulting map has the determining value c_i replaced by k/n and $(k+1)/n$. We progressively remove all irrational determining values of f_m to obtain a map

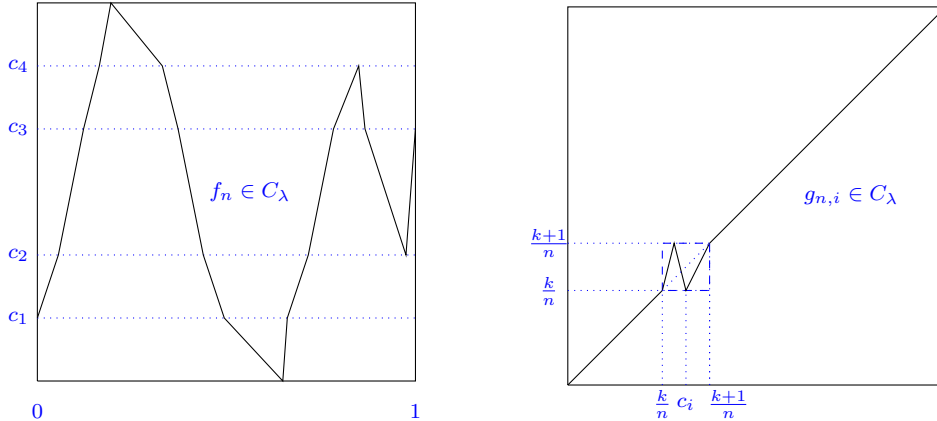


FIGURE 1. Removing irrational critical values.

h_m with only rational determining values. The set $\{h_m\}$ is dense in C_λ since $\rho(f_m, h_m) \rightarrow 0$. \square

The proof of Theorem 2 follows the strategy of the proof of [9, Theorem 2.2]. We use the terminology of the proof from [9]. In particular, for any $n \in \mathbb{N}$, the set A_n denotes the set of functions $f \in C_\lambda$ for which there exist $\gamma \in [-n, n]$ and $x \in [0, 1]$ such that $f_{-\gamma}(t) \leq f_{-\gamma}(x)$ when $t \in [0, 1] \cap (x - 1/n, x)$ and $f_{-\gamma}(t) \geq f_{-\gamma}(x)$ when $t \in [0, 1] \cap (x, x + 1/n)$ (here we changed the notation of [9] replacing their λ by γ to avoid a collision of terminologies).

Proof of Theorem 2. Except for one step, the strategy of the proof is the same as in [9]; this difference is due to the fact of the preserving of Lebesgue measure. The step which differs is showing that each A_n is nowhere dense, this step is the most technical step in the proof in [9]. To show this we proceed as illustrated in Figure 2. We know that piecewise affine maps are dense in C_λ [3]. So from now $n \in \mathbb{N}$ defining A_n is fixed.

I. A choice of f . Using Lemma 13, given an open set $U \subset C_\lambda$ and $\varepsilon > 0$ we can choose a piecewise affine map $f \in U$ whose determining values are rational such that $B(f, \varepsilon) \subset U$. Let q be the greatest common denominator of the determining values. Let $1/\tau$ be a positive integer multiple of q such that $\tau < \varepsilon/3$. For each integer $1 \leq i \leq 1/\tau$ consider the interval $J_i = [(i-1)\tau, i\tau]$. These intervals satisfy

- (i) $\lambda(J_i) = \tau$ for all i ,
- (ii) $J_i^\circ \cap J_j^\circ = \emptyset$ for all $i \neq j$,
- (iii) each J_i has no determining value in its interior,
- (iv) for each i , all points inside a J_i have the same number $n(i)$ of preimages,
- (v) $f^{-1}(J_i^\circ)$ has $n(i)$ connected components $[u_i^k, v_i^k]$, $1 \leq k \leq n(i)$,

- (vi) $f([u_i^k, v_i^k]) = J_i$ for each k ,
- (vii) every window perturbations of f on $[u_i^k, v_i^k]$ remains in $B(f, \tau) \subset B(f, \varepsilon/3)$; furthermore this containment holds if we make a finite number of such window perturbations (see section 2.1 of [7] for the definition of window perturbations).

II. A perturbation \tilde{f} of f . We choose $\delta > 0$ so small that

$$(6) \quad \delta < \min_{i,k} \left(\frac{\tau}{10n}, (v_i^k - u_i^k)/2 \right).$$

If x a common point of two adjacent $[u_i^k, v_i^k]$ intervals then we call the intervals $[x - \delta, x]$ and $[x, x + \delta]$ δ -marked intervals. Each δ -marked interval is contained in a single interval $[u_i^k, v_i^k]$ and its image is contained in a single J_i and δ -marked intervals have pairwise disjoint interiors. We consider a (not necessarily regular) piecewise affine window perturbation $\tilde{f} \in B(f, \tau)$ of f on all $[u_i^k, v_i^k]$'s such that (see Figure 2, left)

- (viii) if $K \subset [u_i^k, v_i^k]$ and $\lambda(K) = \delta$ then K contains either at least three critical points of \tilde{f} or two critical points of \tilde{f} and one of the endpoints u_i^k, v_i^k ,
- (ix) if $[u_i^1, v_i^1]$ is the leftmost interval and f is increasing on $[u_i^1, v_i^1]$ then $\tilde{f}(u_i^1) = f(v_i^1)$, $\tilde{f}(v_i^1) = f(v_i^1)$, the map \tilde{f} has two laps on the δ -marked interval $[v_i^1 - \delta, v_i^1]$,
- (x) if $[u_i^{n(i)}, v_i^{n(i)}]$ is the rightmost interval and f is increasing on $[u_i^{n(i)}, v_i^{n(i)}]$ then $\tilde{f}(u_i^{n(i)}) = f(u_i^{n(i)})$, $\tilde{f}(v_i^{n(i)}) = f(u_i^{n(i)})$, the map \tilde{f} has two laps on the δ -marked interval $[u_i^{n(i)}, u_i^{n(i)} + \delta]$,
- (xi) otherwise $\tilde{f}(u_i^k) = f(u_i^k)$ and $\tilde{f}(v_i^k) = f(v_i^k)$ for each i, k , the map \tilde{f} has two laps on the δ -marked intervals $[u_i^k, u_i^k + \delta]$ and $[v_i^k - \delta, v_i^k]$.

By construction, the map $\tilde{f} \in B(f, \tau) \subset B(f, \varepsilon/3)$ fulfills (5), so $\tilde{f} \in C_\lambda$.

III. Constructing F from \tilde{f} . Finally we construct the map F , adapting the map \tilde{f} , such that (see Figure 2, right)

- (xii) the graphs of \tilde{f} and F coincide outside of the δ -marked intervals,
- (xiii) if $J = [d, e]$ is a δ -marked interval, then $\tilde{f}(d) = \tilde{f}(e)$; the graph of $F|_J$ is a copy of the graph $\tilde{f}|_J$ reflected in the line $y = f(d)$, i.e., $F(x) = 2\tilde{f}(d) - \tilde{f}(x)$; equivalently if $J = [d, e]$ is the union of the two consecutive δ -marked intervals, then $\tilde{f}(d) = \tilde{f}(e)$; the graph of $F|_J$ is a copy of the graph $\tilde{f}|_J$ reversed, i.e., $F(x) = \tilde{f}(d + e - x)$.

By construction, the map F fulfills (5), so $F \in C_\lambda$. By (vii), $F \in B(\tilde{f}, 2\tau) \subset B(f, \varepsilon) \subset U$. Let us show that $F \notin A_n$. Notice that by (6) and (viii) the map

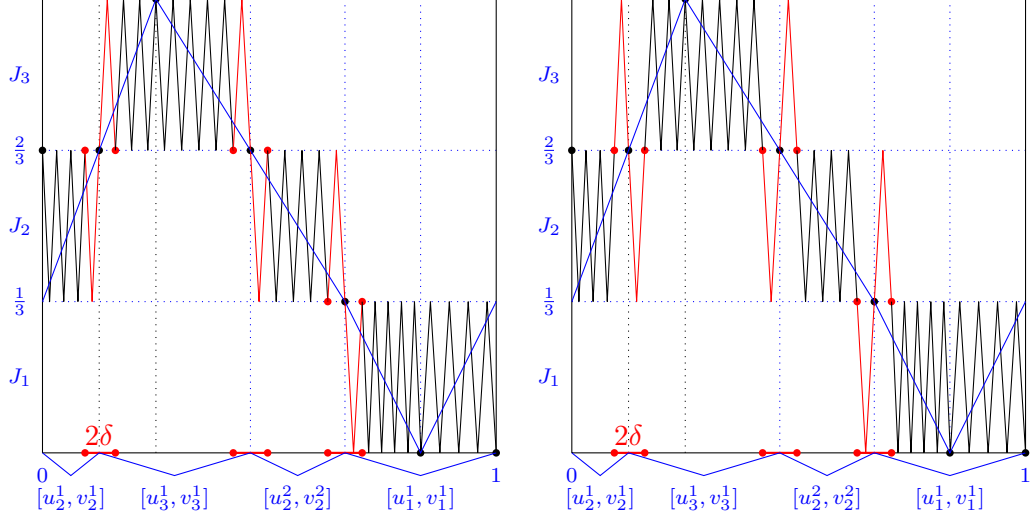


FIGURE 2. Left: the map \tilde{f} in step II. Right: the map F in step III.

F has absolute value of the slope greater than $10n$ on each affine part of its graph.

If $x \in [0, 1]$ and F is decreasing on some one-sided neighborhood V of x then $F_{-\gamma}$, $\gamma \in [-n, n]$, has its derivative less than $-9n$ on the same neighborhood V of x , so either there is $t \in (x, x + \frac{1}{n})$ with $F_{-\gamma}(x) > F_{-\gamma}(t)$ or $t \in (x - \frac{1}{n}, x)$ for which $F_{-\gamma}(t) > F_{-\gamma}(x)$. This is the case when x is an endpoint or a critical point of F or such that F is decreasing on some two-sided neighborhood of x .

It remains to consider the case when $x \in (0, 1)$ and F is increasing on the two-sided neighborhood of x . Let $x \in [u_i^k, v_i^k]$. We distinguish several possibilities.

For convenience denote the endpoints of the interval J_i by $a_i = (i-1)\tau$ and $b_i = i\tau$.

The first case is when $F(x) \in J_i$ and $F(x) \leq (a_i + b_i)/2$. In this case by (viii) there has to be a point $y \in [u_i^k, x)$ for which $x - y < 3\delta$ and $F(y) = b_i$. Then since

$$3\delta < \frac{3\tau}{10n} < \frac{3}{10n} < \frac{1}{n} \quad \text{and} \quad \frac{F(x) - F(y)}{x - y} < -\frac{\tau}{6\delta} < -\frac{5n}{3},$$

we obtain

$$\frac{F_{-\gamma}(x) - F_{-\gamma}(y)}{x - y} < -\frac{2n}{3} \quad \text{for } y \in (x - \frac{1}{n}, x).$$

The second case is $F(x) \in J_i$ and $F(x) > (a_i + b_i)/2$. Again by (viii) there has to be a point $y \in (x, v_i^k)$ for which $y - x < 3\delta$ and $F(y) = a_i$. Analogously as

before

$$\frac{F(x) - F(y)}{x - y} < -\frac{5n}{3},$$

hence

$$\frac{F_{-\gamma}(x) - F_{-\gamma}(y)}{x - y} < -\frac{2n}{3} \text{ for } y \in (x, x + \frac{1}{n}).$$

The last case is when $F(x) \in J_\ell$ with $\ell \neq i$. Then

- if $\ell = i + 1$, one can proceed analogously as above taking y the closest critical point to the right of x where the local minimum is attained,
- if $\ell = i - 1$, one can proceed analogously as above taking y the closest critical point to the left of x where the local maximum is attained.

This finishes the proof. □

Let $f \in C(I)$. For every $c \in \mathbb{R}$ the set $\{x : f(x) = c\}$ is called a *level set* of f . The level sets of a function $f \in C(I)$ are said to be *normal* if there is a countable set E_f in $(\min f, \max f)$ such that the level set $\{x : f(x) = c\}$ of f is

- (i) a non-empty perfect set when $c \notin E_f \cup \{\min f, \max f\}$,
- (ii) a single point when $c = \min f$ or $c = \max f$,
- (iii) the union of a nonempty perfect set P with an isolated point $x \notin P$ when $c \in E_f$.

By replacing Theorem 2.2. from [9] with Theorem 2 we can prove the following theorem using exactly the same arguments as in Theorem 3.3 in [9]. Note that if $f \in C_\lambda$ then $\min f = 0$ and $\max f = 1$.

Proposition 14. *The level sets of a generic map in C_λ are normal.*

Proof of Theorem 6. Theorem 6 follows immediately from Theorem 1, Corollary 3 and Proposition 14. □

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