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A Working Note on Generating Uniform Random Rational Numbers

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Abstract

For generating random sequences of rational numbers uniformly distributed between zero and one, an intuitive method consists, for some number m , of randomly choosing numbers among the first m nonnegative integers $\{0, 1, \dots, m - 1\}$ and to divide them by m . In this note we argue that m must be chosen to possess a very high abundancy index $\sigma(m)/m$.

1 Introduction

To apprehend ‘the art of generating random numbers’, we should first turn to Knuth [9, especially §3.1 and §3.5]. Nonetheless, to take a step forward, we adopt and comment on a definition due to L’Ecuyer [11] for formally presenting how to naturally convert the usual uniform random number generators into uniform random (basic) rational number ones (Section 2). Then we will investigate how far the number theoretic properties of their inherited modulus-like parameter m renders their inherited randomness ‘worse’ (Section 3): the refinement of a naturally emerging criterion will lead us to strongly suggest parameters m with a very high abundancy index [10]. Our discussion is fundamentally based on a probability distribution theorem whose the proof is postponed (Section 6), the proof itself being an immediate reminiscence of the proof of the Gauss theorem [6, Art. 39]. Along with this, in order to help in practical choices of the parameter m , we also perform some qualitative and quantitative analyses (Section 4). We ultimately illustrate our results (Section 5).

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2 Random number generators

As an *entrée*, we formally introduce our model by adopting and briefly commenting on a framework due to L'Ecuyer [11].

2.1 Formal random number generators

Definition 1. A [pseudo-]random number generator (RNG) is a structure $\mathcal{G} = (\mathcal{S}, \mu, t, \mathcal{O}, r)$, where \mathcal{S} is a finite set of states (the *state space*), μ is a probability distribution on \mathcal{S} (the *initial distribution*), $t: \mathcal{S} \rightarrow \mathcal{S}$ is the *transition function*, \mathcal{O} is the *observation space* (or the *output space*), and $r: \mathcal{S} \rightarrow \mathcal{O}$ is the *realization function* (or the *output function*).

Remark 2. A generator $\mathcal{G} = (\mathcal{S}, \mu, t, \mathcal{O}, r)$ operates as follows: initialize the zeroth state $s_0 \in \mathcal{S}$ with respect to μ ; then, let the i -th state $s_i \in \mathcal{S}$ evolve according to the recurrence relation $s_{i+1} = t(s_i) \in \mathcal{S}$; at the same time, realize the i -th observation $o_i = r(s_i) \in \mathcal{O}$. The initial state s_0 is called the *seed*. Since \mathcal{S} is finite, the set of observations $r(\mathcal{S}) \subseteq \mathcal{O}$ is finite and the sequences of observations $(o_i)_{i \geq 0}$ are periodic. The deterministic and periodic sequences of observations $(o_i)_{i \geq 0}$ produced by the generator \mathcal{G} are supposed to mimic typical realizations of independent and identically distributed random variables. Of course, to some extent, the imitation cannot be valid. In practice, nevertheless, generators with sufficiently huge periods and sufficiently deluding mimics can be both designed rigorously on a theoretical basis and assessed empirically through batteries of highly torturous statistical tests [9, 11, 12].

2.2 The usual random number generators

Remark 3. Usually the observation space \mathcal{O} is the set of the first m nonnegative integers $\{0, 1, \dots, m-1\}$, for some number m , or the unit interval $[0, 1)$. In the former case, the generator is meant to mimic the discrete uniform distribution $U_{\mathbb{N}}\{m\}$, and, in the latter case, the continuous standard uniform distribution $U_{\mathbb{R}}(0, 1)$.

Remark 4. Actually the generators for $U_{\mathbb{R}}(0, 1)$ -imitation are commonly generators for $U_{\mathbb{N}}\{m\}$ -imitation whose realization function $r: \mathcal{S} \rightarrow \{0, 1, \dots, m-1\}$ is (formally) followed by the real division-by- m function, $\mathbb{R} \ni x \mapsto x/m$. In such a case, the set of observations $r(\mathcal{S}) \subseteq [0, 1)$ takes the form $\{\rangle \frac{0}{m} \langle, \rangle \frac{1}{m} \langle, \dots, \rangle \frac{m-1}{m} \langle\}$, where the real function $\rangle \cdot \langle$ is some (finite) digital representation. Concretely, the radix (conservatively) assumed at the extreme right may be then regarded (liberally) as having been displaced to the extreme left [9, 12], that is to say, $\{0, 1, \dots, m-1\} \ni (d_e d_{e-1} \dots d_1)_b \mapsto (.d_e d_{e-1} \dots d_1)_b \in [0, 1)$ on an e -digit b -base machine when $m = b^e$, the d 's being digits in the b -base digital representation. So we may think of the set of observations $r(\mathcal{S}) \subseteq \mathcal{O}$ as generally taking the form $\{0, 1, \dots, m-1\}$.

Remark 5. The cardinality m of the set of observations $r(\mathcal{S})$ is typically a multiple of the

computer word size¹ WORDSIZE, or a congruence modulus which might be chosen conveniently with respect to WORDSIZE, inherited from the transition function t [9, 11]. To adjust m to a specific positive integer value, the realization function r may be composed with general techniques, such as rejection or scaling methods [9].

2.3 Natural random rational number generators

In this note, we focus on the generators for the rational standard uniform distribution $U_{\mathbb{Q}}(0, 1)$ -imitation that are built as generators for the discrete uniform distribution $U_{\mathbb{N}}\{m\}$ -imitation whose realization function $r: \mathcal{S} \rightarrow \{0, 1, \dots, m-1\}$ is (literally) followed by the division-by- m function $\cdot/m: \mathbb{Z} \rightarrow \mathbb{Q}$ such that

$$k \mapsto \frac{k}{m} = \frac{p}{q} = \left\lfloor \frac{p}{q} \right\rfloor + \frac{p \bmod q}{q} \quad \text{with } (p, q) = \left(\frac{k}{d}, \frac{m}{d} \right) \text{ where } d = \gcd(k, m).$$

Hence the set of observations $r(\mathcal{S}) \subsetneq [0, 1) \cap \mathbb{Q}$ is the set of all basic fractions with denominator m , which becomes, after reduction, the set of *reduced basic fractions*,

$$\mathcal{F}_m = \left\{ \frac{p}{q} \mid (p, q) = \left(\frac{k}{d}, \frac{m}{d} \right) \wedge d = \gcd(k, m), \forall k \in \{0, 1, \dots, m-1\} \right\},$$

a fraction p/q being called *basic* whenever $0 \leq p < q$, and *reduced* whenever $p \perp q$ (p relatively prime to q , i.e., $\gcd(p, q) = 1$) [7]. Obviously, by contrast to its real counterpart in Remark 4, the division function \cdot/m brings here its own part to the final mimicry character of the thus built generators besides the one inherited from their underlying $U_{\mathbb{N}}\{m\}$ -imitation, except trivially when m is a prime number: an extra layer of ‘randomness’ provided by the probability distributions of the thus generated numerators and denominators, which become trivial (the former becomes uniform and the latter degenerate) when m is prime (Example 9). Following Remark 2, the set of observations $r(\mathcal{S}) = \mathcal{F}_m$ is now also meant to be to some extent representative of the set of basic rational numbers $[0, 1) \cap \mathbb{Q}$. Attempts to build ‘good’ representations of $[0, 1) \cap \mathbb{Q}$ naturally lead to Diophantine approximation, a research field full of still open questions [2] which is beyond the scope of this note. On the other hand, ‘bad’ representations are trivially obtained for \mathcal{F}_m whenever m is prime. We consider sets of observations \mathcal{F}_m that are ‘suitable’ (or ‘reasonable’) representations of $[0, 1) \cap \mathbb{Q}$, that is, whose parameter number m favors a rich zoology in general and small denominators in particular.

¹In computing, a *word* is the native unit of data used by a computer. Following Knuth [9], the *word size* (resp., *word length*) is defined as the number of different values (resp., significant digits) that one word can store (resp., align); that is, 2^b (resp., b) on a b -bit binary computer, 10^d (resp., d) on a d -digit decimal machine, or $n!$ (resp., n) for a n -numeral factorial number system [9, Algo. 3.3.2P]. (**CAUTION:** In the literature, *word size* is sometimes used as a synonymous for *word length*!)

3 Number theoretic analysis

The moral lesson of Knuth [9, §3.1], that “random numbers should not be generated with a method chosen at random,” fully applies in this note, particularly in this section. “Some [number] theory should be used.”

3.1 The denominator probability distribution

The object of this note is to show that the set of observations \mathcal{F}_m gets more ‘suitable’ when m gains a higher *abundancy index* $\sigma(m)/m$, where the *divisor function* $\sigma(m)$ means the sum of the divisors of m [13, Seq. [A000203](#)]. Our demonstration essentially hinges on the probability distribution of the denominators of the reduced basic fractions of the sets \mathcal{F}_m , which can be expressed in terms of the *Euler totient function* $\varphi(n)$, defined as the number of numbers not exceeding n and relatively prime to n [13, Seq. [A000010](#)], and the appropriate *Iverson brackets* $[\mathcal{P}]$, which is 1, respectively, 0, if the proposition \mathcal{P} is true, respectively, false [7].

Theorem 6. *For any unreduced basic fraction with denominator m , the probability of its denominator’s reducing to the value q is*

$$[q \setminus m] \frac{\varphi(q)}{m}.$$

Our **proof** is basically an immediate consequence of an intermediate step of the classical proof for the Gauss theorem (Theorem 7) as presented by Apostol [3, Thm. 2.2].

Theorem 7 (Gauss theorem [6, Art. 39]). *For any positive integer n ,*

$$\sum_{d \setminus n} \varphi(d) = n.$$

Remark 8. By virtue of the Gauss theorem (Theorem 7), the probability distribution stated in Theorem 6 is indeed normalized:

$$\sum_{1 \leq q \leq m} [q \setminus m] \frac{\varphi(q)}{m} = \frac{1}{m} \cdot \sum_{q \setminus m} \varphi(q) = \frac{1}{m} \cdot m = 1.$$

Example 9. Whenever m is a prime number p , none of the unreduced basic fractions are reducible except the null fraction, whose denominator always reduces to 1. Thence the probability for such a denominator to reduce to the value q is readily seen to be

$$[q = 1] \frac{1}{p} + [q = p] \frac{p-1}{p}.$$

This is effectively equivalent to the statement claimed by Theorem 6 since $\varphi(1) = 1$ and $\varphi(p) = p - 1$ whenever p is prime. That is to say, the probability distribution is a Bernoulli

distribution: the trivial denominator 1 (the denominator of the null fraction) occurs with probability $1/p$; the nontrivial denominator p (the denominator of the non-null fractions) with probability $(p-1)/p$. Notice that the occurrence of the trivial denominator 1 becomes arbitrarily improbable as p becomes arbitrarily large, so that the probability distribution becomes the degenerate distribution as p tends to infinity. Furthermore, in practice, we might also regard this probability distribution as the degenerate distribution by silently not reducing the denominator of the null fraction. With this in mind, and some abuse of language, let us claim that the probability distribution of the reduced denominators is ‘mostly’ the degenerate distribution. Hence the corresponding set of observations \mathcal{F}_p is trivial in the sense that the probability distribution of the generated denominators is ‘mostly’ the (trivial) degenerate distribution whereas the probability distribution of the generated numerators is the (trivial) uniform distribution.

Example 10. Let us assume m is the size of binary words of length b .¹ Since half of the unreduced basic fractions are irreducible (the ones with an odd numerator), the most probable reduced denominator is *de facto* $m = 2^b$ itself, with probability $\frac{1}{2}$. The reduced denominators are otherwise powers of 2, whose exponent e ranges from 0 to b ; according to Theorem 6, the probability for their binary logarithm to reduce to the value e is

$$[0 \leq e \leq b] \frac{\varphi(2^e)}{2^b} = [e = 0] \frac{1}{2^b} + [1 \leq e \leq b] \frac{1}{2^{b+1-e}}$$

since $\varphi(1) = 1$ and $\varphi(p^n) = p^{n-1}(p-1)$ whenever p is prime and n a positive integer. Thus the probability increases, to reach its maximum value $\frac{1}{2}$ at the event b , as expected: the larger the reduced denominator is, more probable it gets, to the extent that half of the generated denominators occur with the largest possible value. In short, this generator does not favor small denominators: on the contrary. On the other hand, by comparison with the previous Example 9, its zoology is richer since it does not exhibit only one unique denominator, but several denominators: exactly $b+1$, even though all of them are powers of 2 (i.e., not factorizable by any odd prime).

3.2 The totient abundancy index criterion

In practice, the reciprocal of the probability of the greatest reduced denominator of each set \mathcal{F}_m appears naturally as a pre-eminent ‘suitability’ index in the sense that it clearly favors small[er] denominators; especially as it might also be regarded (liberally) as the reciprocal of the upper tail of the concomitant cumulative distribution function. Given that the greatest reduced denominator of each set \mathcal{F}_m is the parameter m itself, since $m-1$ is relatively prime to m , our pre-eminent ‘suitability’ index takes the form $m/\varphi(m)$: the greater $m/\varphi(m)$ becomes, the more probable the reduced denominators are to be smaller than m . Note that the reduced denominator probability readily decomposes so as to exhibit this index:

$$[q \setminus m] \frac{\varphi(q)}{m} = [q \setminus m] \frac{\varphi(q)}{q} \cdot \frac{q}{m} = [q \setminus m] \left(\frac{q}{\varphi(q)} \right)^{-1} \cdot \frac{1}{m/q}$$

where m/q is a positive integer. Thus the index's taking the form $q/\varphi(q)$ keeps regulating the reduced denominator probability, particularly when m/q remains small, viz., for large reduced denominators ($1 \ll q \leq m$). Let us step forward.

Definition 11. We define the *totient abundancy index* of a positive integer n as being the rational number

$$\frac{n}{\varphi(n)}.$$

Since $\varphi(p) = p - 1$ whenever p is prime, we readily have the following statement.

Property 12. For any prime number p ,

$$\frac{p}{\varphi(p)} = \frac{p}{p-1}.$$

From this formula we effortlessly deduce the next useful property through basic analysis.

Property 13. The totient abundancy indices of the prime numbers form a strictly decreasing sequence which is strictly minorized by 1 and majorized by 2, the majorant being attained by 2:

$$1 < \frac{p_b}{\varphi(p_b)} < \frac{p_a}{\varphi(p_a)} \leq 2 \quad \forall p_a, p_b \in \mathbb{P} \mid p_a < p_b.$$

Let us now recall the expression of the Euler totient function $\varphi(n)$ for a general positive integer n as a product over the distinct prime divisors of n (the empty product being defined as 1).

Theorem 14 (Euler product formula [3, Thm. 2.4]). For any positive integer n ,

$$\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right).$$

As an immediate consequence, the totient abundancy index of a positive integer can be written as the product of the totient abundancy index of its distinct prime factors.

Corollary 15. For any positive integer n ,

$$\frac{n}{\varphi(n)} = \prod_{p \mid n} \frac{p}{\varphi(p)}.$$

Proof. Inserting the Euler product formula (Theorem 14) into Definition 11 gives

$$\frac{n}{\varphi(n)} = \prod_{p \mid n} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{p \mid n} \frac{p}{p-1},$$

where Property 12 applies to each factor. □

Example 16. The word size¹ totient abundancy index for some practical machines:

(i) e -digit p -primary machines—binary computers and p -odd-primary toy models:

$$\frac{p^e}{\varphi(p^e)} = \frac{p}{p-1} = 1 + \frac{1}{p-1} = [p=2] 2 + [p \text{ odd prime}] 1 \frac{1}{p-1};$$

(ii) d -digit decimal Chinese circular abaci and (iii) n -numeral vigesimal Mayan abacus bracelets:

$$\frac{10^d}{\varphi(10^d)} = \frac{20^n}{\varphi(20^n)} = \frac{2}{1} \cdot \frac{5}{2^2} = 2 \cdot 1 \frac{1}{4} = 2 \frac{1}{2};$$

(iv) n -numeral sexagesimal Babylonian clay tablets with stylus:

$$\frac{60^n}{\varphi(60^n)} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{2^2} = 2 \cdot 1 \frac{1}{2} \cdot 1 \frac{1}{4} = 3 \frac{3}{4}.$$

3.3 The primorial totient abundancy index criterion

Property 13 and Corollary 15 suggests that we regard the totient abundancy index of a positive integer as also a product of elements of a strictly decreasing sequence strictly minorized by 1 that are functions of its distinct prime factors.

Following this vein, the product of the first n prime numbers or of the prime numbers not exceeding n , for some positive integer n , might come in handy. The latter is better known as the n -primorial number and is denoted by $n\#$ [13, Seq. [A034386](#)]; the former is better defined as the least number with n distinct prime factors. Nevertheless, it is commonly defined as the (n -th prime)-primorial number and denoted as such, viz., $p_n\#$ or $(n)\#$ [13, Seq. [A002110](#)].

Example 17. The word size¹ totient abundancy index for some exotic machines:

(i) d -digit 7-primorial-base and (ii) e -digit 7-factorial-base machines:

$$\frac{7\#^d}{\varphi(7\#^d)} = \frac{7!^e}{\varphi(7!^e)} = 2 \cdot 1 \frac{1}{2} \cdot 1 \frac{1}{4} \cdot 1 \frac{1}{6} = 4 \frac{3}{8} \quad (7\# = (4)\# = 2 \cdot 3 \cdot 5 \cdot 7);$$

(iii) 5-numeral primorial and (iv) 11-numeral factorial number systems [9, Algo. 3.3.2P]:

$$\frac{(5)\#}{\varphi((5)\#)} = \frac{11!}{\varphi(11!)} = 2 \cdot 1 \frac{1}{2} \cdot 1 \frac{1}{4} \cdot 1 \frac{1}{6} \cdot 1 \frac{1}{10} = 4 \frac{13}{16} \quad ((5)\# = 11\# = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11);$$

(v) a machinery with a word size of value $720720 = (6)\# \cdot (2)\# \cdot (1)\#^2$:

$$\frac{720720}{\varphi(720720)} = \frac{(6)\#}{\varphi((6)\#)} = 2 \cdot 1 \frac{1}{2} \cdot 1 \frac{1}{4} \cdot 1 \frac{1}{6} \cdot 1 \frac{1}{10} \cdot 1 \frac{1}{12} = 5 \frac{41}{192}.$$

Whence, as a consequence of Corollary 15 and Property 13, the totient abundancy indices are bounded above by the totient abundancy indices of the primorial numbers as follows, in terms of the number theoretic function $\omega(n)$ (the number of distinct prime factors of n [13, Seq. A001221]).

Corollary 18. *For any positive integer n ,*

$$\frac{n}{\varphi(n)} \leq \frac{(\omega(n))\#}{\varphi((\omega(n))\#)}.$$

Proof. For any positive integer i , let p_i denote the i -th prime number. Suppose n is an arbitrary positive integer with w distinct prime factors $(p_{\nu_i})_{1 \leq i \leq w}$, where $(\nu_i)_{1 \leq i \leq w}$ is the ascending sorted sequence of their natural order, i.e., $p_{\nu_1} < p_{\nu_2} < \dots < p_{\nu_w}$. By Corollary 15,

$$\frac{n}{\varphi(n)} = \prod_{i=1}^{i=w} \frac{p_{\nu_i}}{\varphi(p_{\nu_i})}.$$

Besides, for $1 \leq i \leq w$, $p_i \leq p_{\nu_i}$ since $i \leq \nu_i$ by construction; thus by Property 13,

$$\frac{p_{\nu_i}}{\varphi(p_{\nu_i})} \leq \frac{p_i}{\varphi(p_i)}$$

for $1 \leq i \leq w$. Applying the latter inequalities to the former equality gives

$$\frac{n}{\varphi(n)} \leq \prod_{i=1}^{i=w} \frac{p_i}{\varphi(p_i)}.$$

By Corollary 15, the upper bound product can be expressed in the form

$$\prod_{i=1}^{i=w} \frac{p_i}{\varphi(p_i)} = \frac{p_w\#}{\varphi(p_w\#)} = \frac{(\omega(n))\#}{\varphi((\omega(n))\#)},$$

given that w yields by assumption $\omega(n) = w$. □

At this point, we can legitimately wonder which numbers saturate their totient abundancy index.

Property 19. *A positive integer n with $\omega(n)$ distinct prime factors saturates its totient abundancy index if and only if it is divisible by the least number with $\omega(n)$ distinct prime factors.*

Proof. All along the proof of Corollary 18, the equality holds if and only if the ascending map ν is the identity map ($\nu_i = i$), that is, if and only if the distinct prime factors of n coincide with the first $\omega(n)$ prime numbers, if and only if n is divisible by the least number with $\omega(n)$ distinct prime factors: the $\omega(n)$ -th primorial number $(\omega(n))\#$. □

Therefore the totient abundancy index criterion exhibits a preference for parameters m such that the product of their distinct prime factors is a primorial number. For such parameters m , the totient abundancy index reaches a maximum, which equals the totient abundancy index of the underlying primorial number (Corollary 18 and Property 19).

Definition 20. We say that a totient abundancy index is *primorial* (or *maximal*) whenever it is the totient abundancy index of a primorial number.

In the subsequent Section 4, we will refine the following rough behavioral description of the sequence of the primorial totient abundancy indices.

Property 21. *The primorial totient abundancy indices form a strictly increasing sequence minorized by 2, the minorant being attained by the first primorial, 2:*

$$2 \leq \frac{p_a\#}{\varphi(p_a\#)} < \frac{p_b\#}{\varphi(p_b\#)} \quad \forall p_a, p_b \in \mathbb{P} \mid p_a < p_b.$$

Proof. Given some positive integer i , let p_i be the i -th prime number and p_{i+1} the next one. It is implicitly understood that $(p_i\#)_{i \geq 1}$ forms the ordered sequence of primorial numbers. For the first prime number $p_1 = 2$, $2\#$ being equal to 2, we have

$$\frac{2\#}{\varphi(2\#)} = 2.$$

By Corollary 15, the sequence can be folded into a recurrence relation of the form

$$\frac{p_{i+1}\#}{\varphi(p_{i+1}\#)} = \frac{p_{i+1}}{\varphi(p_{i+1})} \cdot \frac{p_i\#}{\varphi(p_i\#)},$$

and hence the sequence is strictly increasing since, by Property 13,

$$\frac{p_{i+1}}{\varphi(p_{i+1})} > 1. \quad \square$$

Meanwhile, Property 21 allows us to show that the totient abundancy index reaches a record at each primorial number. Let us first demonstrate the same statement but for the distinct prime factor counting function.

Lemma 22. *The distinct prime factor counting function $\omega(n)$ increases to a record at each primorial number:*

$$\omega(n) < \omega(p\#) \quad \forall p \in \mathbb{P}, \forall n \in \mathbb{N}^* \mid n < p\#.$$

Proof. Given some positive integer k , let p_k be the k -th prime number; hence $p_k\#$ is the k -th primorial number and/or the primorial number whose the number of distinct prime factors $\omega(p_k\#)$ is k . Consider an arbitrary positive integer n less than but not equal to the k -th primorial number $p_k\#$. Suppose then that the number of distinct prime factors of n , $\omega(n)$,

is greater than or equal to k , the actual number of distinct prime factors of $p_k\#$, $\omega(p_k\#)$. By construction, the k -th primorial number $p_k\#$ is the least number with k distinct prime factors. Therefore n is greater than or equal to $p_k\#$ since more than k distinct prime factors divide n . However n is less than but not equal to $p_k\#$, and so we have a contradiction! Hence our assumption is incorrect, and the number of distinct prime factors of n must be less than but not equal to k . \square

Now we have all the necessary material to prove that the primorial totient abundance indices are the records of the totient abundance index.

Theorem 23. *The totient abundance index $n/\varphi(n)$ increases to a record at each primorial number:*

$$\frac{n}{\varphi(n)} < \frac{p\#}{\varphi(p\#)} \quad \forall p \in \mathbb{P}, \forall n \in \mathbb{N}^* \mid n < p\#.$$

Proof. Corollary 18 and Property 21 readily yield the chain of inequalities

$$\frac{n}{\varphi(n)} \leq \frac{(\omega(n))\#}{\varphi((\omega(n))\#)} < \frac{p\#}{\varphi(p\#)}$$

since, by Lemma 22, we have $\omega(n) < \omega(p\#)$, while $(\omega(p\#))\#$ is $p\#$. \square

Remark 24. The converse of Theorem 23 leads to defining a primorial number as a number for which there occurs a record of the totient abundance index. An unusual way to define primorial numbers, this approach permits introducing new families of numbers, especially when the involved number theoretic function is intricate. For example: a highly composite number [13, Seq. [A002182](#)] is a number for which there occurs a record of the number theoretic function $\tau(n)$ (the number of divisors of n [13, Seq. [A000005](#)]).

Summary 25. The primorial totient abundance index criterion allows us to isolate numbers whose product of their distinct prime factors reduces to a primorial number as possessing maximal totient abundance indices, viz., as parameters m that favor their largest reduced denominator with minimal odds.

3.4 The abundance index criterion

The following theorem justifies further and *a posteriori* the definition introduced early for the totient abundance index of a positive integer n (Definition 11).

Theorem 26. *The average totient abundance index of the reduced denominators of the unreduced basic fractions with denominator m is the abundance index of m .*

Proof. By virtue of Theorem 6 and Definition 11, we immediately have

$$\sum_{1 \leq q \leq m} [q \setminus m] \frac{\varphi(q)}{m} \frac{q}{\varphi(q)} = \frac{1}{m} \sum_{q \setminus m} q = \frac{\sigma(m)}{m}. \quad \square$$

Therefore, in order to favor small denominators on average, m must be chosen to possess a very high abundancy index.

Remark 27. The abundancy index of a positive integer n , i.e., the ratio $\sigma(n)/n$, was introduced for investigations on *abundant* numbers [10]: a number is deficient, perfect, or abundant if its abundancy index is, respectively, less than, equal to, or greater than 2.

Remark 28. While the set of abundancy indices is dense in $[1, \infty)$ [10], the set of abundant numbers itself is subject to a rich zoology [13, Seq. [A005101](#)]: highly abundant numbers ([A002093](#)), super-abundant numbers ([A004394](#)), colossally abundant numbers ([A004490](#)), extremely abundant numbers ([A217867](#)), primitive abundant numbers ([A091191](#)), etc.

Since the divisors of an arbitrary prime power p^a are the powers of p whose exponent ranges from 0 to a , the divisor sum $\sigma(p^a)$ is the partial geometric series $1 + p + p^2 + \cdots + p^a$, that is to say,

$$\sigma(p^a) = \frac{p^{a+1} - 1}{p - 1}.$$

Hence, we can readily claim the following statement.

Property 29. *For any prime number p and any nonnegative integer a ,*

$$\frac{\sigma(p^a)}{p^a} = \frac{p^{a+1} - 1}{(p - 1)p^a}.$$

Remark 30. Under the same hypothesis, according to Corollary 15 and Property 12, the totient abundancy index satisfies the chain of equalities

$$\frac{p^a}{\varphi(p^a)} = \frac{p}{\varphi(p)} = \frac{p}{p - 1}.$$

Thus, unlike its totient counterpart, the abundancy index is not blind to powers and/or does not expurge them.

Let us now recall the expression for the divisor function $\sigma(n)$ for a general positive integer n as a product over the distinct prime powers that exactly divide n (the empty product being taken to be 1)—we say that d *exactly divides* n (written $d \ll n$) whenever $d \mid n$ and $d \perp n/d$ [7].

Theorem 31 (divisor function product formula [3, §2.13]). *For any positive integer n ,*

$$\sigma(n) = \prod_{p^a \ll n} \frac{p^{a+1} - 1}{p - 1}.$$

As an immediate consequence, the abundancy index of a positive integer can be written as the product of the abundancy index of the distinct prime powers that exactly divide it.

Corollary 32. For any positive integer n ,

$$\frac{\sigma(n)}{n} = \prod_{p^a \parallel n} \frac{\sigma(p^a)}{p^a}.$$

Proof. Dividing the divisor function product formula (Theorem 31) by n and substituting into its right-hand side the equality

$$n = \prod_{p^a \parallel n} p^a$$

yields

$$\frac{\sigma(n)}{n} = \prod_{p^a \parallel n} \frac{p^{a+1} - 1}{(p - 1) p^a},$$

where Property 29 applies to each factor. □

Example 33 (cf. Example 16). The word size¹ abundancy index for some practical machines:

(i) e -digit p -primary machines—binary computers and p -odd-primary toy models:

$$\frac{\sigma(p^e)}{p^e} = \frac{p^{e+1} - 1}{(1 - p) p^e} = [p = 2] \left(2 - \frac{1}{2^e} \right) + [p \text{ odd prime}] \left(1 + \frac{p^e - 1}{(p - 1) p^e} \right);$$

(ii) d -digit decimal Chinese circular abaci:

$$\frac{\sigma(10^d)}{10^d} = \frac{2^{d+1} - 1}{2^d} \cdot \frac{5^{d+1} - 1}{2^2 \cdot 5^d} = \frac{(2^{d+1} - 1)(5^{d+1} - 1)}{2^{d+2} \cdot 5^d};$$

(iii) n -numeral vigesimal Mayan abacus bracelets:

$$\frac{\sigma(20^n)}{20^n} = \frac{2^{2n+1} - 1}{2^{2n}} \cdot \frac{5^{n+1} - 1}{2^2 \cdot 5^n} = \frac{(2^{2n+1} - 1)(5^{n+1} - 1)}{2^{2n+2} \cdot 5^n};$$

(iv) n -numeral sexagesimal Babylonian clay tablets with stylus:

$$\frac{\sigma(60^n)}{60^n} = \frac{2^{2n+1} - 1}{2^{2n}} \cdot \frac{3^{n+1} - 1}{2 \cdot 3^n} \cdot \frac{5^{n+1} - 1}{2^2 \cdot 5^n} = \frac{(2^{2n+1} - 1)(3^{n+1} - 1)(5^{n+1} - 1)}{2^{2n+3} \cdot 3^n \cdot 5^n}.$$

Along the lines of the proof of Corollary 32, let us factorize the abundancy index of a positive integer into the product of its totient abundancy index and a product over the distinct prime powers that exactly divide it.

Corollary 34. For any positive integer n ,

$$\frac{\sigma(n)}{n} = \frac{n}{\varphi(n)} \prod_{p^a \parallel n} \frac{p^{a+1} - 1}{p^{a+1}}.$$

Proof. Repeating the proof of Corollary 32 and then maneuvering further gives

$$\frac{\sigma(n)}{n} = \prod_{p^a \parallel n} \frac{p^{a+1} - 1}{(p-1)p^a} = \prod_{p^a \parallel n} \frac{p^{a+1} - 1}{p^{a+1}} \cdot \frac{p}{p-1} = \prod_{p \mid n} \frac{p}{p-1} \cdot \prod_{p^a \parallel n} \frac{p^{a+1} - 1}{p^{a+1}},$$

where the latter first product is a reminiscence of the proof of Corollary 15. \square

Consequently, every nontrivial abundancy index $\sigma(n)/n$ is strictly bounded by its totient counterpart $n/\varphi(n)$.

Property 35. *For any nontrivial positive integer n ,*

$$\frac{\sigma(n)}{n} < \frac{n}{\varphi(n)}.$$

Proof. According to Corollary 34, the claim follows promptly from the inequality

$$\prod_{p^a \parallel n} \frac{p^{a+1} - 1}{p^{a+1}} < 1. \quad \square$$

3.5 The super-abundancy criterion

Clearly Corollary 34 has Corollary 15 as its totient counterpart. For both, the involved number theoretic index function is written as a product: over the distinct prime powers that exactly divide its argument, for the former; over the distinct prime factors of its argument, for the latter. Formally, since the totient abundancy index is blind to powers (Remark 30), the similitude is strong: for the totient counterpart, the product can be taken over the distinct prime powers that exactly divide the argument. On the other hand, because the abundancy index does not expurge powers whereas its totient counterpart does, the very nature of the products are slightly different: for the latter, the factors of the product belong to a sequence with a straightforward ordering (Property 13); for the former, the involved sequence has, in comparison, a highly intricate ordering for the reason that it mixes the apparent randomness of the prime number sequence with arbitrary powers. In short, analytical studies of the abundancy index are slightly more involved.

Continuing along such lines and having still in mind Remark 24 lead us to recall the definition of a super-abundant number.

Definition 36 (super-abundant numbers [13, Seq. A004394]). A number n is *super-abundant* if its abundancy index $\sigma(n)/n$ is larger than that of any smaller number.

For convenience in notation, let us denote the k -th super-abundant number by A_k^s . Hence, the previously adopted way to phrase Definition 36 is merely as follows.

Property 37. *The abundancy index $\sigma(n)/n$ increases to a record at each super-abundant number A_k^s :*

$$\frac{\sigma(n)}{n} < \frac{\sigma(A_k^s)}{A_k^s} \quad \forall k, n \in \mathbb{N}^* \mid n < A_k^s.$$

Proof. By construction. □

Remark 38. Property 37 has Theorem 23 as totient counterpart.

Continuing along this vein leads us now to the observation that, as concerns the preference for parameters m , the abundancy index criterion favors super-abundant numbers, whereas its totient counterpart promotes numbers the product of whose distinct prime factors reduces to a primorial number. By virtue of a factorization theorem due to Alaoglu and Erdős (Theorem 39), the former's favorite numbers appear to belong to the set formed by the latter's.

Theorem 39 (factorization of super-abundant numbers [1, Thm. 1]). *For any super-abundant number A^s , the prime exponent sequence $(a_i)_{1 \leq i \leq \omega}$ does not increase:*

$$A^s = \prod_{i=1}^{i=\omega} p_i^{a_i} \quad \text{with} \quad a_1 \geq a_2 \geq \dots \geq a_\omega \geq 1,$$

where p_i denotes the i -th prime number.

Example 40. For the 46th super-abundant number $A_{46}^s = 367567200$ [13, Seq. [A004394](#)], the prime number decomposition is

$$367567200 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \quad \text{or} \quad A_{46}^s = p_1^5 p_2^3 p_3^2 p_4 p_5 p_6 p_7$$

with ordinal subscript notations.

The complexity of the super-abundant number sequence has been caught further by Alaoglu and Erdős [1], and certainly even farther by other authors in more recent publications that I am not aware of. Whatever. Theorem 39 is sufficient to claim that the super-abundancy criterion supersedes the primorial totient abundancy index criterion in the sense that the super-abundancy criterion favors numbers that are also favored by the primorial totient abundancy index criterion: each super-abundant number being clearly now a number the product of whose distinct prime factors reduces to a primorial number.

The following immediate corollary renders the superseding of the super-abundancy criterion even more obvious, since it states that super-abundant numbers factorize into primorial numbers.

Corollary 41. *Any super-abundant number A^s can be expressed uniquely in the form*

$$A^s = \prod_{i=1}^{i=\omega} p_i \#^{b_i} \quad \text{with} \quad b_i \geq 0,$$

where p_i denotes the i -th prime number.

Proof. Let A^s be an arbitrary super-abundant number. By Theorem 39, the prime exponent sequence $(a_i)_{1 \leq i \leq \omega}$ of A^s decreases, that is, the associated inverse difference sequence $(b_i)_{1 \leq i \leq \omega}$ is nonnegative; using the Iverson convention, we may write

$$b_i = [i = \omega] a_\omega + [1 \leq i < \omega] (a_i - a_{i+1}) \geq 0.$$

Hence we can proceed by induction over the greatest consecutive prime number sequences to factorize them as primorial powers:

$$\begin{aligned} A^s &= \prod_{k=1}^{\omega} p_k^{a_k} = p_\omega \#^{b_\omega} \prod_{k=1}^{\omega-1} p_k^{(a_k - a_\omega)} = p_\omega \#^{b_\omega} p_{\omega-1} \#^{b_{\omega-1}} \prod_{k=1}^{\omega-2} p_k^{(a_k - a_{\omega-1})} = \\ &\dots = \underbrace{\prod_{l=0}^i p_{\omega-l} \#^{b_{\omega-l}} \prod_{k=1}^{\omega-i-1} p_k^{(a_k - a_{\omega-i})}}_{0 \leq i < \omega} = \dots = \prod_{l=0}^{\omega-1} p_{\omega-l} \#^{b_{\omega-l}}. \end{aligned}$$

Replacing l by $\omega - l$ in the rightmost product yields the desired formula. \square

Example 42. For the 31st super-abundant number $A_{31}^s = 720720$ [13, Seq. [A004394](#)], the prime number decomposition can be recomposed into a primorial number decomposition as follows:

$$720720 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = (2^3 \cdot 3) \cdot (13\#) = (2^2) \cdot (3\# \cdot 13\#) = 2\#^2 \cdot 3\# \cdot 13\#;$$

or, using ordinal subscript notations,

$$A_{31}^s = p_1^4 p_2^2 p_3 p_4 p_5 p_6 = (p_1^3 p_2) \cdot (p_6\#) = (p_1^2) \cdot (p_2\# p_6\#) = p_1\#^2 p_2\# p_6\#.$$

Observe then that the superseding is clearly not reciprocal, since any primorial number greater than 2 multiplied by one of its odd prime factors is obviously not a super-abundant number, by Corollary 41, whereas it fulfills the primorial totient abundancy index criterion by construction.

As a matter of fact, the superseding truly expresses itself through the following property: writing $\text{gpf}(n)$ for the greatest prime factor of n ,

Property 43. *For any super-abundant number A^s , the totient abundancy index $A^s/\varphi(A^s)$ is primorial; we have*

$$\frac{A^s}{\varphi(A^s)} = \frac{\text{gpf}(A^s)\#}{\varphi(\text{gpf}(A^s)\#)}.$$

Proof. Applying Corollary 15 and then Corollary 41, we readily get

$$\frac{A^s}{\varphi(A^s)} = \prod_{p \mid A^s} \frac{p}{\varphi(p)} = \prod_{p \mid \text{gpf}(A^s)\#} \frac{p}{\varphi(p)}. \quad \square$$

In this note, we do not further refine the criterion for choosing a ‘suitable’ parameter m . Having said that, let us nonetheless add as an extra, and a reminder, that the number-of-divisors function $\tau(n)$, encountered in Remark 24, is blind to prime numbers but not to powers.

Theorem 44 (number-of-divisors product formula [3, §2.13]). *For any positive integer n ,*

$$\tau(n) = \prod_{p^a \parallel n} (a + 1).$$

As such, the number of divisors $\tau(n)$ contrasts with the totient abundancy index $n/\varphi(n)$ which is blind to powers but not to prime numbers (Remark 30). Henceforth, among the ‘suitable’ super-abundancy parameters m sharing the same greatest primorial factor $\text{gpf}(m)\#$ (namely the same totient abundancy index $m/\varphi(m)$), the number of divisors $\tau(m)$ may constitute an extra criterion with a view to favoring or not a large number of reduced denominators. (Incidentally, this gives us some insight into what can be defined as ‘suitable’ in future research.)

Summary 45. The super-abundancy index criterion allows us to distinguish the so-called super-abundant numbers as the parameters m that favor a large average totient abundancy index; therefore, as such, as parameters m that promote small reduced denominators on average. Second, the super-abundancy index criterion supersedes the primorial totient abundancy index criterion.

4 Numerical analysis

In what follows, we establish relationships useful for selecting ‘suitable’ parameters m whose totient abundancy index is primorial.

4.1 Asymptotic logarithmic growth

To begin with, let us establish lower and upper bounds for the n -th prime as obtained by Apostol [3, Thm. 4.7].

Property 46 (prime bounds [3, Thm. 4.7, slightly amended]). *For any positive integer n , the n -th prime number p_n satisfies the inequalities*

$$\frac{1}{6} n \ln n < p_n < 12 n \left(\ln n + [n < 2109] \ln \frac{12}{e} \right).$$

Proof. For the complete proof with the Iversonian factor $[n < 2109]$ frozen to 1, we refer to Apostol [3, §4.5]. The Iversonian factor itself, which allows the constant shift $\ln 12/e$ to drop out for sufficiently large n , is inserted as follows. Notice that the logarithmic argument of the

constant shift arises in the derivation as being 6 times the global maximum $2/e$ of the ratio $\ln x/\sqrt{x}$ on the interval $[1, \infty)$, which is attained at e^2 . On its (right) asymptotic branch side, i.e., on the interval $[e^2, \infty)$, the ratio is invertible and its inverse can be expressed in the form $\exp(-2W_{-1}(-\frac{1}{2}x))$, W_{-1} being the secondary branch of the Lambert W function [5]. Hence, beyond the point $\exp(-2W_{-1}(-\frac{1}{12})) = 2108.995656\dots$, the ratio is smaller than $\frac{1}{6}$, that is, the constant shift can be discarded since it is negative. \square

Remark 47. The 2108th and 2109th prime numbers are 18401 and 18413, respectively.

While the techniques developed for proving the prime number theorem [3, §4.1] give sharper bounds [3, Thm. 4.5], “the [previous Property 46] is of interest because of the elementary nature of its proof”—Apostol [3, Intro. §4.5]. Furthermore, along with previously provided material, Property 46 is sufficient to catch (further) the qualitative behavior of the primorial totient abundancy indices.

Property 48 (improved version of Property 21). *The primorial totient abundancy indices form a strictly increasing sequence which is minorized by 2 and divergent, the minorant being attained by the first primorial, 2:*

$$2 \leq \frac{p_a\#}{\varphi(p_a\#)} < \frac{p_b\#}{\varphi(p_b\#)} \quad \forall p_a, p_b \in \mathbb{P} \mid p_a < p_b \quad \text{and} \quad \lim_{\substack{p \rightarrow \infty \\ p \in \mathbb{P}}} \frac{p\#}{\varphi(p\#)} = \infty.$$

Proof. Compared with Property 21, only the divergence property is new: let us repeat the proof of Property 21 accordingly. By Corollary 15, for any integer i greater than 2108, the i -th element of the sequence can be expressed in the form

$$\frac{p_i\#}{\varphi(p_i\#)} = \frac{p_{2108}\#}{\varphi(p_{2108}\#)} \cdot \prod_{k=2109}^{k=i} \frac{p_k}{\varphi(p_k)},$$

where the logarithm of the rightmost multiplicand product is minorized by a divergent term as follows. By expanding the logarithm, then by applying Property 12 to each logarithmic argument, and finally by invoking the inequality $1/x < \ln(x/(x-1))$, we obtain

$$\ln \prod_{k=2109}^{k=i} \frac{p_k}{\varphi(p_k)} = \sum_{k=2109}^{k=i} \ln \frac{p_k}{\varphi(p_k)} = \sum_{k=2109}^{k=i} \ln \frac{p_k}{p_k - 1} > \sum_{k=2109}^{k=i} \frac{1}{p_k}.$$

Next, Corollary 46 permits continuing the chain of equalities/inequalities:

$$12 \sum_{k=2109}^{k=i} \frac{1}{p_k} > \sum_{k=2109}^{k=i} \frac{1}{k \ln k} > \sum_{k=2109}^{k=i} \int_k^{k+1} \frac{dt}{t \ln t} = \int_{2109}^{i+1} \frac{dt}{t \ln t} > \int_{2109}^i \frac{dt}{t \ln t},$$

where integration of the rightmost minorant yields

$$\int_{2109}^i \frac{dt}{t \ln t} = \ln \ln t \Big|_{2109}^i = \ln \log_{2109} i.$$

Returning to where we left off and rearranging things gives

$$\frac{p_i\#}{\varphi(p_i\#)} > \frac{p_{2108}\#}{\varphi(p_{2108}\#)} \sqrt[12]{\log_{2109} i} \quad (i \geq 2109),$$

hence the sequence is indeed divergent. \square

As a matter of fact, the asymptotic behavior of the primorial totient abundancy indices at infinity can be obtained by invoking Mertens's theorem [8, Thm. 429]; let us consider instead an improved version due to Vinogradov [14] that exhibits a sharper bound for the error term, the O notation being employed to express negligible quantities [7].

Theorem 49 (Mertens–Vinogradov estimate [14]). *For any nontrivial positive integer n ,*

$$\prod_{p \leq n} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\ln n} \left(1 + O\left(\frac{1}{\exp\left(C(\ln n)^{3/5}\right)}\right)\right),$$

with γ the Euler constant and C a positive constant.

An immediate equivalent formulation of the Mertens–Vinogradov estimate (Theorem 49), obtained by taking the reciprocal on each side, gives us an estimate for the primorial totient abundancy indices.

Corollary 50. *For any nontrivial positive integer n ,*

$$\frac{n\#}{\varphi(n\#)} = e^\gamma \ln n \left(1 + O\left(\frac{1}{\exp\left(C(\ln n)^{3/5}\right)}\right)\right),$$

with γ the Euler constant and C a positive constant.

Proof. Along with some Iversonian manipulation and Property 12, the reciprocal of the left-hand side of the Theorem 49 estimate is

$$\prod_{p \leq n} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{p \in \mathbb{P}} [p \leq n] \frac{p}{p-1} = \prod_{p \in \mathbb{P}} [p \setminus n\#] \frac{p}{\varphi(p)} = \prod_{p \setminus n\#} \frac{p}{\varphi(p)},$$

that is, by Corollary 15, the expected left-hand side. Second, along with some O manipulation and rearrangements, the reciprocal of the right-hand side of the Theorem 49 estimate readily leads to the desired right-hand side. \square

Definition 51 (cf. Corollary 50). Let the real function

$$\begin{aligned} \tilde{\mathbb{I}}_{\varphi\#}: [2, \infty) &\longrightarrow [e^\gamma \ln 2, \infty) \\ x &\longmapsto e^\gamma \ln x \end{aligned}$$

denote the asymptotic estimate of the primorial totient abundancy index.

Remark 52. The primorial totient abundancy index estimate $\tilde{I}_{\varphi\#}$ is literally a logarithmic growth, whose inverse, exponential growth, has a growth constant equal to $e^{-\gamma}$; as such, for any nontrivial positive integer n and an arbitrary positive integer ν , $\tilde{I}_{\varphi\#}$ yields

$$\tilde{I}_{\varphi\#}(\nu n) - \tilde{I}_{\varphi\#}(n) = e^\gamma \ln 2 \log_2 \nu \quad (e^\gamma \ln 2 = 1.2345\dots).$$

This also implies that, while they might grow rapidly for small numbers, the primorial totient abundancy indices (liberally) reach an evasive asymptotic logarithmic-flat plateau for sufficiently large numbers, that is, they grow more and more slowly: by Corollary 50, for any nontrivial positive integer n and arbitrary positive integer ν , O maneuvers give

$$\frac{n^\nu \#}{\varphi(n^\nu \#)} \Big/ \frac{n \#}{\varphi(n \#)} = \nu + O\left(\frac{1}{\exp\left(C(\ln n)^{3/5}\right)}\right) \quad \text{and} \quad \frac{n \#}{\varphi(n \#)} \Big/ n = O\left(\frac{\ln n}{n}\right),$$

with C a positive constant.

In practice, to consider parameters m significantly large but not insignificantly huge, we need to grasp the quantitative behavior of the evasive asymptotic logarithmic-flat plateau.

4.2 Napierian folding analysis

Notice that by judiciously applying Taylor's theorem, we can easily show that at any given point belonging to its domain, any sufficiently smooth growing/decaying function is tangent to an exponential growth whose algebraic growth constant is its logarithmic derivative at the tangent point.

Proposition 53 (tangent exponential growth). *Given an origin $a \in \mathbb{R}^+$, let $g: [a, \infty) \rightarrow \mathbb{R}^+$ be of class \mathcal{C}^2 , i.e., twice differentiable on $[a, \infty)$. Then, at any point $x \in [a, \infty)$, g is tangent to an exponential growth whose amplitude and algebraic growth constant are $g(x)$ and $g'(x)/g(x)$, respectively:*

$$g(x + \varepsilon) = g(x) \exp\left(\frac{g'(x)}{g(x)} \varepsilon\right) + O(\varepsilon^2) \quad \forall x \in [a, \infty), \forall \varepsilon \in \mathbb{R}.$$

Proof. By applying Taylor's theorem to $\ln \circ g$ at any point $x \in [a, \infty)$, we obtain

$$\ln(g(x + \varepsilon)) = \ln(g(x)) + \frac{g'(x)}{g(x)} \varepsilon + O(\varepsilon^2),$$

where the infinitesimal ε is arbitrarily small. Taking the exponential on each side yields

$$g(x + \varepsilon) = g(x) \exp\left(\frac{g'(x)}{g(x)} \varepsilon\right) (1 + O(\varepsilon^2)),$$

after some O algebra. Some further O manipulation readily gives the expected formula. \square

Next, let us ingenuously observe that any exponential growth algebraically folds by the factor e each time its argument increases by an amount equal to the magnitude of the reciprocal of the growth constant. The number of e -folds appears then convenient to describe the growing, while the algebraic e -folding constant, the reciprocal of the algebraic growth constant, is pertinent to characterizing the growth. Most importantly, their generalization to any sufficiently smooth growing/decaying function is trivial.

Definition 54. Given an origin $a \in \mathbb{R}^+$, suppose $g: [a, \infty) \rightarrow \mathbb{R}^+$ is of class \mathcal{C}^1 , i.e., once differentiable on $[a, \infty)$. At any point $x \in [a, \infty)$, we define the following:

(i) The *number of e -folds* $\mathcal{N}_e[g](x)$ for g by

$$\mathcal{N}_e[g](x) = \int_a^x du \frac{g'(u)}{g(u)} = \ln \frac{g(x)}{g(a)};$$

(ii) The *e -folding length* $\mathcal{F}_e[g](x)$ of g by

$$\mathcal{F}_e[g](x) = \frac{g(x)}{g'(x)}.$$

By construction, we have

Property 55. Given an origin $a \in \mathbb{R}^+$, let $g: [a, \infty) \rightarrow \mathbb{R}^+$ be of class \mathcal{C}^1 , i.e., once differentiable on $[a, \infty)$.

$$(i) \quad \mathcal{N}_e[(x \mapsto x^\nu) \circ g] = \nu \mathcal{N}_e[g] \quad \text{and} \quad \nu \mathcal{F}_e[(x \mapsto x^\nu) \circ g] = \mathcal{F}_e[g] \quad \forall \nu \in \mathbb{R}^*.$$

$$(ii) \quad \mathcal{F}_e[g](x) \cdot [\mathcal{N}_e[g](x)]' = 1 \quad \forall x \in [a, \infty).$$

Let us now investigate the e -folding of the estimate $\tilde{\mathbb{I}}_{\varphi\#}$. The e -fold functions associated to the asymptotic growth $\tilde{\mathbb{I}}_{\varphi\#}$ can be computed effortlessly as follows.

Property 56. For the primorial totient abundancy index estimate $\tilde{\mathbb{I}}_{\varphi\#}$, the e -fold functions satisfy the following at any point $x \in [2, \infty)$.

$$(i) \quad \mathcal{N}_e[\tilde{\mathbb{I}}_{\varphi\#}](x) = \ln \log_2 x; \quad (ii) \quad \mathcal{F}_e[\tilde{\mathbb{I}}_{\varphi\#}](x) = x \ln x.$$

Hence, the primorial totient abundancy indices asymptotically exhibit a strongly non-linear e -folding, that is, a strongly non-exponential growth, and more precisely, a double-logarithmic e -folding. To seize where the asymptotic e -folds occur and/or where the asymptotic e -folding lengths reach some thresholds, let us derive the inverses of the e -fold functions involved.

Property 57. For the primorial totient abundancy index estimate $\tilde{\mathbb{I}}_{\varphi\#}$, and with W_0 being the principal branch of the Lambert W function [5] as usual, the inverses of the e -fold functions satisfy

$$(i) \left(\mathcal{N}_e[\tilde{\mathbb{I}}_{\varphi\#}] \right)^{-1}(x) = 2^{\exp(x)} \quad (0 \leq x); \quad (ii) \left(\mathcal{F}_e[\tilde{\mathbb{I}}_{\varphi\#}] \right)^{-1}(x) = e^{W_0(x)} \quad (2 \ln 2 \leq x).$$

Remark 58. The complete e -fold occurrence points $\tilde{x}_k = 2^{\exp(k)}$, k running over \mathbb{N} , form here a sequence which is the exponentiation of the geometric progression $k \mapsto \ln 2 e^k$, itself the exponentiation of the arithmetic progression $k \mapsto k + \ln \ln 2$. On the other hand, the complete e -fold occurrence points for an exponential growth with κ as algebraic growth constant are the arithmetic progression $k \mapsto \kappa k$. Hence the growth function $\tilde{\mathbb{I}}_{\varphi\#}$ appears of order 3 in an *ad hoc* hierarchy based on iterations starting from the exponential growth. The recurrence relation inherited from the descending progressions are readily obtained as

$$\tilde{x}_{k+1} = (\tilde{x}_k)^e \quad (\tilde{x}_0 = 2, k \in \mathbb{N}).$$

Numerical approximations of the first terms give a crude idea of how rapidly the (hyper-3) progression $(\tilde{x}_k)_{k \in \mathbb{N}}$ increases:

k	\tilde{x}_k	$\tilde{x}_{k+1}/\tilde{x}_k$
0	2	$3.29 \cdot 10^{+0}$
1	6.58	$2.55 \cdot 10^{+1}$
2	167.62	$6.64 \cdot 10^{+3}$
3	1 112 625.71	$2.45 \cdot 10^{+10}$

So the e -folds complete (hyper-)extremely slowly from the very second e -fold.

Now that we have analyzed the e -folding of the estimate, $\tilde{\mathbb{I}}_{\varphi\#}$, let us consider the e -folding of the primorial totient abundancy index $n\#/\varphi(n\#)$ itself. The previous continuous analysis gives way to a discrete analysis in the sequel.

Naturally, *a priori*, the definition of the number of e -folds \mathcal{N}_e has rather taken so far the form given in Definition 54 to legitimate *a posteriori* our preliminary analytical approach based on Proposition 53; thus, having still in mind Property 48, we may readily write

$$\mathcal{N}_e \left[k \mapsto \frac{k\#}{\varphi(k\#)} \right] (n) = \ln \left(\frac{n\#}{\varphi(n\#)} \bigg/ \frac{2\#}{\varphi(2\#)} \right) = \ln \frac{n\#}{2\varphi(n\#)}.$$

Obviously, for the primorial totient abundancy indices $n\#/\varphi(n\#)$, the completion of the e -folds only occurs discretely:

Definition 59. A prime number p is an *e -folding prime number* if the floor of the number of e -folds associated to the (primorial) totient abundancy index of its primorial number $p\#$, to wit, $\lfloor \ln(p\#/2\varphi(p\#)) \rfloor$, is smaller than for any greater prime number.

Property 60. For the primorial totient abundancy index $n\#/\varphi(n\#)$, between each e -folding prime number and its prime postcedent there occurs an e -fold completion.

Proof. By construction. □

Armed with those discrete tools, we can return to our goal of investigating the primorial totient abundancy index sequence. The discrete counterpart of the little table in Remark 58 becomes

k	x_k	x_{k+1}/x_k
0	2	$6.50 \cdot 10^{+0}$
1	13	$3.02 \cdot 10^{+2}$
2	3 929	$1.59 \cdot 10^{+6}$
3	6 241 048 943	$(6.78 \cdot 10^{+17})$

where the nontrivial e -folding prime numbers $(x_k)_{k=1,2,3}$ have been computed with the help of an adapted version of the golden section search algorithm and an efficient implementation of the sieve of Eratosthenes,² the ratio x_4/x_3 (in parentheses) being an estimate based on x_3 and the recurrence relation sketched in Remark 58. The progression $(x_k)_{k \in \mathbb{N}}$ appears to be more rapid, and the second e -fold to complete more slowly. Within the current context, the natural e -folding gap e appears thus definitely enormous, not to say irrelevant. As an informal guideline, let us proceed by dichotomy to figure out a more pertinent e -folding gap, taking for granted the rephrasing of Definition 59 and its concomitant Property 60.

Table 1 gives us the first quarter e -folding prime numbers, which have been computed as before, along with extra data: the quarter e -folding gap $\frac{1}{4}e$ appears more appropriate since the increase of the involved *whatever* e -folding prime numbers is rather smooth than step-like. At first glance, the primorial totient abundancy index $n\#/\varphi(n\#)$ begins to experience the evasive asymptotic logarithmic-flat plateau around the very first e -fold completion: compared with the second one, the third column seems to stagnate from somewhere between the third and the fifth quarter e -folds. Within this neighborhood, the fourth and fifth columns show that the estimate $\tilde{I}_{\varphi\#}$ is yet very approximative, not to say invalid. Nevertheless, the estimate $\tilde{I}_{\varphi\#}$ has the great advantage that it can furnish a quantitative tool: its associated e -folding length $\mathcal{F}_e[\tilde{I}_{\varphi\#}]$. The sixth column shows that the e -folding length $\mathcal{F}_e[\tilde{I}_{\varphi\#}]$ may gain two orders of magnitude from the beginning of the third quarter e -fold to the ending of the fifth one. One way to confirm or not this fact is to bring the e -folding length notion to the primorial totient abundancy index $n\#/\varphi(n\#)$, which can be done by applying Definition 54 to a sufficiently smooth interpolation function built upon its values.

Table 2 shows us the first twenty primorial totient abundancy indices, which appears to wrap over the first eleven eighth e -folds, along with *ad hoc* but illustrative e -folding length interpolations, among other data. Note that, in the current context, the involved e -folding length is as dimensionless as any positive integer. A closer look reveals that the interpolated e -folding length $\widehat{\mathcal{F}}_e$ becomes greater than 50 around the first e -fold completion and greater than 100 around the ninth eighth e -fold completion, while it remains below 25 before the seventh eighth e -fold completion. In other words, the primorial totient abundancy index $n\#/\varphi(n\#)$ tackles the evasive asymptotic logarithmic-flat plateau within the $\frac{1}{4}e$ wide fold-band that spreads from the seventh to the ninth eighth e -fold completions: beyond the

² `primesieve` (version 5.5.0) [open-source software library] available at <http://primesieve.org/>.

Table 1: The first quarter e -folding prime numbers

\mathcal{Q}_e	p	$p\#/\varphi(p\#)$	$\tilde{\mathbb{I}}_{\varphi\#}(p)^a$	$\delta\mathbb{I}_{\varphi\#}(p)^b$	$\mathcal{F}_e[\tilde{\mathbb{I}}_{\varphi\#}](p)$
0	2	2	<u>1.2345453</u>	$3.827 \cdot 10^{-1}$	$1.386 \cdot 10^{+0}$
$0\frac{1}{4}$					
$0\frac{1}{2}$	3	3	<u>1.9567080</u>	$3.478 \cdot 10^{-1}$	$3.296 \cdot 10^{+0}$
$0\frac{3}{4}$	5	$3\frac{3}{4}$	<u>2.8665255</u>	$2.356 \cdot 10^{-1}$	$8.047 \cdot 10^{+0}$
1	13	5.2135417	<u>4.5683606</u>	$1.238 \cdot 10^{-1}$	$3.334 \cdot 10^{+1}$
$1\frac{1}{4}$	41	6.8920994	<u>6.6141408</u>	$4.033 \cdot 10^{-2}$	$1.523 \cdot 10^{+2}$
$1\frac{1}{2}$	137	8.9148540	<u>8.7628423</u>	$1.705 \cdot 10^{-2}$	$6.740 \cdot 10^{+2}$
$1\frac{3}{4}$	613	11.5028469	<u>11.4315728</u>	$6.196 \cdot 10^{-3}$	$3.934 \cdot 10^{+3}$
2	3 929	14.7762868	<u>14.7404051</u>	$2.428 \cdot 10^{-3}$	$3.252 \cdot 10^{+4}$
$2\frac{1}{4}$	42 139	18.9750705	<u>18.9661574</u>	$4.697 \cdot 10^{-4}$	$4.487 \cdot 10^{+5}$
$2\frac{1}{2}$	872 161	24.3649701	<u>24.3628075</u>	$8.876 \cdot 10^{-5}$	$1.193 \cdot 10^{+7}$
$2\frac{3}{4}$	42 509 527	31.2852634	<u>31.2849623</u>	$9.626 \cdot 10^{-6}$	$7.467 \cdot 10^{+8}$
3	6 241 048 943	40.1710738	<u>40.1710449</u>	$7.213 \cdot 10^{-7}$	$1.408 \cdot 10^{+11}$
$3\frac{1}{4}$	3 778 928 404 243	51.5806798	<u>51.5806794</u>	$9.234 \cdot 10^{-9}$	$1.094 \cdot 10^{+14}$

The leftmost column contains the rational indices of the first quarter e -folds, the zeroth e -fold acting as reference, while the next column displays their corresponding quarter e -folding prime number; the remaining columns exhibit derived data as specified in the topmost row. Each thin horizontal separator records the completion of an e -fold; alternatively, each invisible or thin horizontal separator marks the completion of a quarter e -fold (the first quarter e -fold being empty). The leftmost vertical line separates entries from outputs; the second one isolates effective data from estimated material.

^aunderlined digits correspond to faulty digits.

^bestimate relative error: $\delta\mathbb{I}_{\varphi\#}(p) = [p\#/\varphi(p\#) - \tilde{\mathbb{I}}_{\varphi\#}(p)] / (p\#/\varphi(p\#))$.

Table 2: The first primorial totient abundance indices

n	p_n	$p_n\#/\varphi(p_n\#)$	\mathcal{N}_e	$\widehat{\mathcal{F}}_e^a$	\mathcal{Q}_e	
1	2	2	2.0000000	0.000	—	0
2	3	3	3.0000000	0.405	$7.247 \cdot 10^{+0}$	$0\frac{1}{2}$
3	5	$3\frac{3}{4}$	3.7500000	0.629	$1.317 \cdot 10^{+1}$	$0\frac{3}{4}$
4	7	$4\frac{3}{8}$	4.3750000	0.783	$1.994 \cdot 10^{+1}$	$0\frac{7}{8}$
5	11	$4\frac{13}{16}$	4.8125000	0.878	$3.551 \cdot 10^{+1}$	1
6	13	$5\frac{41}{192}$	5.2135417	0.958	$4.411 \cdot 10^{+1}$	
7	17	$5\frac{1657}{3072}$	5.5393880	1.019	$6.256 \cdot 10^{+1}$	
8	19	$5\frac{46843}{55296}$	5.8471318	1.073	$7.226 \cdot 10^{+1}$	$1\frac{1}{8}$
9	23	$6\frac{12487}{110592}$	6.1129105	1.117	$9.227 \cdot 10^{+1}$	
10	29	$6\frac{146525}{442368}$	6.3312288	1.152	$1.228 \cdot 10^{+2}$	
11	31	$6\frac{7196483}{13271040}$	6.5422697	1.185	$1.330 \cdot 10^{+2}$	$1\frac{1}{4}$
12	37	$6\frac{345896111}{477757440}$	6.7239994	1.213	$1.627 \cdot 10^{+2}$	
13	41	$6\frac{17048285191}{19110297600}$	6.8920994	1.237	$1.821 \cdot 10^{+2}$	
14	43	$7\frac{45105549613}{802632499200}$	7.0561970	1.261	$1.916 \cdot 10^{+2}$	
15	47	$7\frac{336451666357}{1605264998400}$	7.2095926	1.282	$2.107 \cdot 10^{+2}$	
16	53	$7\frac{2236061023517}{6421059993600}$	7.3482386	1.301	$2.402 \cdot 10^{+2}$	
17	59	$7\frac{6099138632507}{12842119987200}$	7.4749324	1.318	$2.726 \cdot 10^{+2}$	$1\frac{3}{8}$
18	61	$7\frac{461942296493327}{770527199232000}$	7.5995146	1.335	$2.847 \cdot 10^{+2}$	
19	67	$7\frac{36343824259676909}{50854795149312000}$	7.7146587	1.350	$3.268 \cdot 10^{+2}$	
20	71	$7\frac{2936395088482244539}{3559835660451840000}$	7.8248682	1.364	$3.624 \cdot 10^{+2}$	

The leftmost column presents the ordinal indices; the second column presents the first prime numbers; the third (resp., fourth) column presents the first primorial totient abundance indices in exact mixed-number (resp., rounded decimal) notation; the fifth column presents the number of e -folds; the sixth column presents the e -folding length approximations achieved by applying Definition 54 to a sufficiently smooth interpolation function;^a the last column presents the rational indices of the involved eighth e -folds, the zeroth e -fold acting as reference. Each thin (either dashed or plain) horizontal separator marks the completion of an eighth e -fold, the first, second, third, and fifth eighth e -folds being empty; notably, each thin plain horizontal separator records the completion of an e -fold. The leftmost vertical line separates entries from outputs; the second and third vertical lines bracket quantitative data derived from the Napierian folding analysis; the third vertical line also sets aside qualitative material deduced from the same analysis and reminiscent of Table 1.

^a e -folding length interpolation obtained by applying Definition 54 to an interpolation function taking the form $g(x) = \ln(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)$ whose the least-squares fit treatment was reduced to the quartic polynomial $\exp(g(x))$ least-squares fit [4, §6.6.2] over the sequence $(\exp(p_n\#/\varphi(p_n\#)))_{2 \leq n \leq 20}$; calculation yields $(a_i)_{0 \leq i \leq 4} = (1.5791074, 1.0572385, -1.3717027 \cdot 10^{-2}, 1.8966678 \cdot 10^{-4}, -1.0062851 \cdot 10^{-6})$.

ninth eighth e -fold, the involved primorial factor $n\#$ is rapidly getting insignificantly huge; before the seventh eighth e -fold, it is remaining ridiculously small. Incidentally, this last explanatory paragraph tells us that the eighth e -folding gap $\frac{1}{8}e$ is the appropriate e -folding gap we were looking for.

Summary 61. Significantly large but not insignificantly huge primorial factors arise from the Napierian folding analysis of the primorial totient abundancy index sequence. They map into the $\frac{1}{4}e$ wide band-fold that wrap from the seventh to the ninth eighth e -fold completions and where the growth is neither sharp nor steady; they are

$$11\#, 13\#, 17\#, 19\#, 23\#$$

in primorial notation, the first insignificantly huge primorial factor being $29\#$.

5 Illustration

So far our most ‘suitable’ parameters m are super-abundant numbers A^s whose the (primorial) totient abundancy indices $A^s/\varphi(A^s)$ map into the $\frac{1}{4}e$ wide band-fold centered around the first e -fold completion. However, even if the set of the involved greatest prime/primorial factors is rather small (Summary 61), we may still choose from among numerous candidates: 50, exactly. *Ergo*, at this stage, our investigation has only provided a rough guidance: extra notes that go further might be needed to filter more finely.

Meanwhile, let us consider a subset of the super-abundant numbers, subject to a more stringent definition, the colossally abundant numbers, in order to illustrate our present results.

Definition 62 (colossally abundant numbers [13, Seq. [A004490](#)]). A number n is *colossally abundant* if there exists a strictly positive real exponent ε such that the ratio $\sigma(n)/n^{1+\varepsilon}$ is larger than for any number.

Let us next adapt the convenient notation adopted for super-abundant numbers, viz., let us denote the k -th colossally number by A_k^c . Henceforth, the alternative way discussed in Remark 24 to phrase Definition 62 follows.

Property 63. For each colossally abundant number A_k^c , there exists a strictly positive real exponent ε_k for which the ratio $\sigma(n)/n^{1+\varepsilon_k}$ reaches a global maximum overall number at A_k^c :

$$\frac{\sigma(n)}{n^{1+\varepsilon_k}} \leq \frac{\sigma(A_k^c)}{A_k^{c1+\varepsilon_k}} \quad (\varepsilon_k > 0) \quad \forall k, n \in \mathbb{N}^*.$$

Proof. By construction. □

We promptly have the following corollary.

Corollary 64. Any colossally abundant number is a super-abundant number.

Proof. Let A^c be an arbitrary colossally abundant number and ε its associated exponent. Then, by Property 63, any number n smaller than A^c satisfies the chain of inequalities

$$\frac{\sigma(n)}{n} \leq \left(\frac{n}{A^c}\right)^\varepsilon \frac{\sigma(A^c)}{A^c} < \frac{\sigma(A^c)}{A^c} \quad (n < A^c). \quad \square$$

Now let us focus our attention on the illustration part itself. Table 3 exhibits the first 25 colossally abundant numbers, whose the primorial totient abundancy indices appear to spread a little bit beyond the first five quarter e -folds, along with analytical, qualitative, practical, and quantitative data. Scanning the eighth column, note that the first 14 and 23 colossally abundant numbers can be stored in a `long int` and a `long long int`, respectively—either `unsigned` or `signed`. C99 specifications are assumed. (Amazingly enough, a closer look reveals that each colossally abundant number whose primorial totient abundancy index occurs before the fifth quarter e -fold completion can be held in a `long long int`—either `unsigned` or `signed`. Regarding the second and/or the rightmost columns, still with Summary 61 in mind, readily leads us to preselect the subsequence $(A_k^c)_{9 \leq k \leq 17}$, the first insignificantly huge candidate being A_{18}^c .

Hint 65. In order to filter further, for instance, we can consider the respective number of divisors $\tau(A_k^c)$, presented in the sixth column. At first glance, a magnitude pattern happens around the first e -fold completion: within the eighth $\frac{1}{8}e$ wide band-fold, the number of divisors is of order 100; within the next one, it is of order 1000; and it gains again a factor 10 in the next one. Accordingly, we may want to keep only the ninth $\frac{1}{8}e$ wide band-fold, viz., to reduce the preselected subsequence to $(A_k^c)_{14 \leq k \leq 17}$. And so forth.

Instead of intending to select the ultimate candidate by reducing further the significant set through a cascade of increasingly sophisticated criteria and/or filters as suggested in Hint 65, we might want to apply a rough termination rule, given that the number of candidates is rather small. In accordance with Summary 61, we might then consider picking up the first candidate that completes the ninth eighth e -fold by invoking ...

Rule-of-Thumb 66 (*‘Select the first insignificantly huge candidate’*). *Among a set of numbers whose product of their distinct prime factors is a primorial number, select the first element whose primorial totient abundancy index occurs beyond the ninth eighth e -fold.*

... as exemplified in Example 68.

Remark 67. Rule-of-Thumb 66 might be considered along any cascade of criteria and/or filters whenever the number of remaining significant candidates is sufficiently small.

Let us now complete the illustration part with practical examples.

Example 68 (Rule-of-Thumb). Applying Rule-of-Thumb 66 to the set of colossally abundant numbers, in practice that means to Table 3, leads to selecting as parameter m the 18th colossally abundant number:

$$A_{18}^c = 9316358251200.$$

Table 3: The abundance indices of the first colossally abundant numbers

n	A_n^c	$\sigma(A_n^c)/A_n^c$	$\delta I_\sigma(A_n^c)^a$	$\tau(A_n^c)$	A_n^c	$\lceil \lg(A_n^c) \rceil$	Q_e
1	2#	$1\frac{1}{2}$	$2.500 \cdot 10^{-1}$	2	2	2	0
2	3#	2	$3.333 \cdot 10^{-1}$	4	6	3	$0\frac{1}{2}$
3	3# · 2#	$2\frac{1}{3}$	$2.222 \cdot 10^{-1}$	6	12	4	
4	5# · 2#	$2\frac{4}{5}$	$2.533 \cdot 10^{-1}$	12	60	6	
5	5# · 2# ²	3	$2.000 \cdot 10^{-1}$	16	120	7	$0\frac{3}{4}$
6	5# · 3# · 2#	$3\frac{1}{4}$	$1.333 \cdot 10^{-1}$	24	360	9	
7	7# · 3# · 2#	$3\frac{5}{7}$	$1.510 \cdot 10^{-1}$	48	2 520	12	$0\frac{7}{8}$
8	7# · 3# · 2# ²	$3\frac{88}{105}$	$1.227 \cdot 10^{-1}$	60	5 040	13	
9	11# · 3# · 2# ²	$4\frac{72}{385}$	$1.300 \cdot 10^{-1}$	120	55 440	16	
10	13# · 3# · 2# ²	$4\frac{28}{55}$	$1.351 \cdot 10^{-1}$	240	720 720	20	
11	13# · 3# · 2# ³	$4\frac{32}{55}$	$1.212 \cdot 10^{-1}$	288	1 441 440	21	1
12	13# · 3# ² · 2# ²	$4\frac{100}{143}$	$9.864 \cdot 10^{-2}$	384	4 324 320	23	
13	13# · 5# · 3# · 2# ²	$4\frac{612}{715}$	$6.859 \cdot 10^{-2}$	576	21 621 600	25	
14	17# · 5# · 3# · 2# ²	$5\frac{1721}{12155}$	$7.181 \cdot 10^{-2}$	1 152	367 567 200	29	
15	19# · 5# · 3# · 2# ²	$5\frac{19039}{46189}$	$7.438 \cdot 10^{-2}$	2 304	6 983 776 800	33	$1\frac{1}{8}$
16	23# · 5# · 3# · 2# ²	$5\frac{687881}{1062347}$	$7.613 \cdot 10^{-2}$	4 608	160 626 866 400	38	
17	23# · 5# · 3# · 2# ³	$5\frac{735497}{1062347}$	$6.880 \cdot 10^{-2}$	5 376	321 253 732 800	39	
18	29# · 5# · 3# · 2# ³	$5\frac{27376645}{30808063}$	$6.991 \cdot 10^{-2}$	10 752	9 316 358 251 200	44	
19	31# · 5# · 3# · 2# ³	$6\frac{2420742}{30808063}$	$7.088 \cdot 10^{-2}$	21 504	288 807 105 787 200	49	
20	31# · 7# · 3# · 2# ³	$6\frac{2123886}{11350339}$	$5.429 \cdot 10^{-2}$	32 256	2 021 649 740 510 400	51	
21	31# · 7# · 3# ² · 2# ²	$6\frac{246282}{1031849}$	$4.640 \cdot 10^{-2}$	40 320	6 064 949 221 531 200	53	$1\frac{1}{4}$
22	37# · 7# · 3# ² · 2# ²	$6\frac{15549810}{38178413}$	$4.710 \cdot 10^{-2}$	80 640	224 403 121 196 654 400	58	
23	41# · 7# · 3# ² · 2# ²	$6\frac{126023214}{223616419}$	$4.767 \cdot 10^{-2}$	161 280	9 200 527 969 062 830 400	63	
24	43# · 7# · 3# ² · 2# ²	$6\frac{6886719930}{9615506017}$	$4.818 \cdot 10^{-2}$	322 560	395 622 702 669 701 707 200	69	
25	43# · 7# · 3# ² · 2# ³	$6\frac{429057114}{565618001}$	$4.444 \cdot 10^{-2}$	368 640	791 245 405 339 403 414 400	70	$1\frac{3}{8}$

The leftmost column presents the ordinal indices; the second (resp., seventh) column presents the first colossally abundant numbers in primorial decomposition (resp., plain) form (Corollary 41); the third (resp., fourth) column present their abundance index in exact mixed-number (resp., rounded decimal) notation; the fifth column presents their abundance index relative difference against the totient abundance index (Property 35);^a the sixth column presents the number of divisors (Theorem 44); the eighth column presents the number of significant binary digits; the ninth column presents the rational index of their eighth e -fold (Table 2). Between two consecutive thin (either dashed or plain) horizontal separators is data associated to abundance indices sharing the same primorial totient abundance index (or greatest prime/primorial factor) (Property 43); otherwise, the thin plain horizontal separators indicate eighth e -fold completions (Table 2).
^athe abundance index relative difference against the totient abundance index means $\delta I_\sigma(n) = [n/\varphi(n) - \sigma(n)/n] / (n/\varphi(n))$.

Example 69 (Maple). In Maple 2015³ the *command* `Generate` of the *package* `RandomTools` with the *flavor* `template nonnegative` as *random object* is a “random non-negative rational number” generator whose implementation is based on the imitation investigated in this note. It is documented in the help page `Help:RandomTools/flavor/nonnegative`. By default, the parameter m is the 37607912018th prime number 999999999989. Thus, by default, the *flavor* `nonnegative` is trivial in the sense derived in Example 9. However, an arbitrary parameter m can be passed via an *[argument]-equation* with `denominator` as *keyword parameter*. The help page suggests specifying a highly composite number [13, Seq. [A002182](#)] as parameter m although no proper justification is given: the 38th highly composite number 720720 is proposed as example. Even if the sequence of highly composite numbers and the sequence of super-abundant number share their first 19 terms, the latter is not a subsequence of the former, nor vice versa [13, Seq. [A002182](#)]. It is noteworthy that, as with the super-abundant numbers, the highly composite numbers are the products of primorial numbers [13, Seq. [A112779](#)]. As such, they satisfy the primorial totient abundancy index criterion. Second, as revealed in Example 42 and Table 3, 720720 is also the 31st super-abundant number A_{31}^s and the 10th colossally abundant number A_{10}^c , to wit, the example satisfies the super-abundancy index criterion and has been already encountered. Meanwhile, it is instructive to unveil a part of the magic behind the default parameter m . Besides being the greatest prime number smaller than 10^{12} , viz., $999999999989 = \lfloor 10^{12} \rfloor'$ in *ad hoc* notation, the default parameter m is before everything else the modulus of the Linear Congruential Generator (LCG) [9, §3.2.1] that was used in Maple 9.5 or earlier for implementing the “random number generator” *command* `rand` [12, §7-*LCGs* and UG:§2-*ulcg*]; ever since Maple 10 this LCG has been deprecated in favor of the Mersenne Twister algorithm. Updated information can be reached from the help page `Help:rand`. Whatever. It appears then not illegitimate to think that this material is rather empirical and from an old golden age. Without doubt, the default parameter m must be avoided not only because it is trivial but also because it receives its only legitimacy from being a historical parameter of a now deprecated algorithm. Concerning the absence of a real justification for the implicit highly composite number criterion, it can only be highly regrettable. *Au contraire*, the given example 720720 is legitimate in this note; even so, this choice might not be the most judicious one because the corresponding primorial totient abundancy index only occurs into the eighth $\frac{1}{8}e$ wide band-fold—see Hint 65 for more possible reasons. Continuing along this line, let us recommend instead the number obtained by following Rule-of-Thumb 66 as applied to colossally abundant numbers (Example 68), viz., the 18th colossally abundant number A_{18}^c , which appears to be also the 106th highly composite number [13, Seq. [A002182](#)]. A Maple 2015³ sample statement to generate a uniform random rational number would then take the form

`RandomTools[Generate](nonnegative(denominator=9316358251200))`

in an arbitrary sequence of commands.

³Maple (version 2015) [commercial Computer Algebra System] distributed by the Canadian software company Maplesoft, a division of Waterloo Maple Inc.

6 Proof of the main theorem

The proof is an immediate probabilistic reinterpretation of an intermediate step of the classical proof for the Gauss theorem (Theorem 7) as presented by Apostol [3, Thm. 2.2]. We refer to Graham et al. [7, §4.9, pp. 134–135] for a concrete introduction to the Gauss theorem.

Proof of Theorem 6. Let \mathcal{S}_m be the set of the first m nonnegative integers,

$$\mathcal{S}_m := \{0, 1, \dots, m-1\}. \quad (1)$$

We distribute the m integers of \mathcal{S}_m into disjoint subsets as follows. For each divisor d of m , let $\mathcal{A}_m(d)$ denote the subset which gathers those elements of \mathcal{S}_m whose gcd with m is d ; we write

$$\mathcal{A}_m(d) := \{k \mid \gcd(k, m) = d, \forall k \in \mathcal{S}_m\}. \quad (2)$$

By construction, the subsets $(\mathcal{A}_m(d))_{d \mid m}$ form a disjoint collection,

$$\mathcal{A}_m(d) \cap \mathcal{A}_m(d') = \emptyset \quad (d \neq d'), \quad (3a)$$

whose union is \mathcal{S}_m itself,

$$\bigcup_{d \mid m} \mathcal{A}_m(d) = \mathcal{S}_m. \quad (3b)$$

On the other hand, $\gcd(k, m) = d$ if and only if there exist p and q relatively prime such that $k = pd$ and $m = qd$; hence we have

$$\mathcal{A}_m(d) = \left\{ dp \mid p \perp q, 0 \leq p < q \quad \text{with } q = \frac{m}{d} \right\}. \quad (4)$$

The integer $q = m/d$ is called the *conjugate divisor* to d .

Now let $\tilde{\mathcal{A}}_m(q)$ denote the subset which contains the unreduced numerators associated to the unreduced basic fractions with denominator m whose reduced denominator is q ,

$$\tilde{\mathcal{A}}_m(q) := \{k = pd \mid m = qd \wedge p \perp q, \forall k \in \mathcal{S}_m\}. \quad (5)$$

Basic manipulations and eliminations allow a plain comparison with (4):

$$\tilde{\mathcal{A}}_m(q) = \left\{ dp \mid p \perp q, 0 \leq p < q \quad \text{with } d = \frac{m}{q} \right\}. \quad (6)$$

Hence the subset $\tilde{\mathcal{A}}_m(q)$ is nothing but the subset $\mathcal{A}_m(m/q)$:

$$\tilde{\mathcal{A}}_m(q) = \mathcal{A}_m\left(\frac{m}{q}\right). \quad (7)$$

From (3) and by isomorphism, the subsets $(\tilde{\mathcal{A}}_m(q))_{q \setminus m}$ form then a disjoint collection whose union is \mathcal{S}_m ; we have

$$\tilde{\mathcal{A}}_m(q) \cap \tilde{\mathcal{A}}_m(q') = \emptyset \quad (q \neq q') \quad (8a)$$

while

$$\bigcup_{q \setminus m} \tilde{\mathcal{A}}_m(q) = \mathcal{S}_m. \quad (8b)$$

Therefore, if $\Pr(q; m)$ denotes the probability for an unreduced basic fraction with denominator m to have q as its reduced denominator, we have

$$\Pr(q; m) = [q \setminus m] \frac{|\tilde{\mathcal{A}}_m(q)|}{|\mathcal{S}_m|}. \quad (9)$$

But, while the cardinality of \mathcal{S}_m is trivial, the cardinality of $\tilde{\mathcal{A}}_m(q)$ reduces to the number of numbers not exceeding q and relatively prime to q , that is, $\varphi(q)$:

$$|\tilde{\mathcal{A}}_m(q)| = |\{p \perp q, 0 \leq p < q\}| =: \varphi(q). \quad (10)$$

Hence, (9) becomes

$$\Pr(q; m) = [q \setminus m] \frac{\varphi(q)}{m}. \quad (11)$$

This completes the proof. □

We can hardly resist adding what follows.

Proof of the Gauss theorem (Theorem 7). Expanding the cardinality of the disjoint union (8) and then substituting (10) give

$$\sum_{q \setminus m} \varphi(q) = m$$

as desired. □

7 Notes

The idea of writing this note came when, with the aim of performing tests against a rational geometry software library, I was looking for but could not find any literature on the subject, except notably a few succinct but not convincing words in some Maple 2015³ help pages (cf. Example 69).

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(Concerned with sequences [A000005](#), [A000010](#), [A000203](#), [A001221](#), [A002093](#), [A002110](#), [A002182](#), [A004394](#), [A004490](#), [A005101](#), [A034386](#), [A091191](#), [A112779](#), and [A217867](#).)
