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Dark and superdark theorems with applications to helical beams (beams with a topological charge) which are not vortex beams

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ABSTRACT

Relying on the time averaged Poynting vector expressed in terms of beam shape coefficients, dark and superdark theorems are discussed generalizing a previously published dark theorem. The superdark theorem allows one to characterize helical beams (that is to say beams possessing a topological charge) in terms of beam shape coefficients, and to demonstrate that beams with a topological charge equal to ± 2 are not necessarily vortex beams, although it is erroneously traditional to state that helical beams are all vortex beams.

1. Introduction

Electromagnetic arbitrary shaped beams may be encoded by beam shape coefficients (BSCs) usually denoted $g_{n, TM}^m$ and $g_{n, TE}^m$, in which n ranges from 1 to infinity, m from $(-n)$ to $(+n)$, while TM stands for ‘‘Transverse Magnetic’’ and TE for ‘‘Transverse Electric’’. These BSCs are for instance used in light scattering theories, such as generalized Lorenz–Mie theories (GLMTs) for homogeneous spheres [1], for multilayered spheres [2,3], for assemblies of spheres and aggregates [4,5], for spheres with an eccentrically located inclusion [6–9], or in Extended Boundary Condition Method (EBCM) when used for arbitrary shaped beams [10–14], see [15] for a textbook and [16–18] for recent reviews.

One quantity of interest in such a framework is the time-averaged Poynting vector \mathbf{S} (or S_i). The longitudinal z -component of this vector has been evaluated in terms of BSCs [19] in order to study the properties of longitudinal axisymmetric beams, defined as beams for which the longitudinal component S_z does not depend on the azimuthal angle φ , at least in a well chosen coordinate system. More recently, we have been interested with helical beams, a terminology here used to denote beams exhibiting a non zero topological charge. Some of these beams (but not all of them, as we shall discuss below) are vortex beams, that is to say beams in which the energy flowing along a certain axis is equal to zero (many examples will be provided in the bulk of the paper). These regions of zero intensity are due to phase singularities produced by a characteristic phase variation of $e^{il\varphi}$ about the beam axis.

This has been a motivation to extend [19] in order to evaluate all components of \mathbf{S} [20] and to introduce a darkness theorem allowing one to state on the vortex or non-vortex nature of helical beams [21], and to introduce new families of symmetries associated with Poynting vector properties [22].

The darkness theorem mentioned above is actually a longitudinal z -darkness theorem, i.e. it concerns conditions in which the z -component of the Poynting vector is zero, which provides a sufficient condition to test whether helical beams are or are not vortex beams (i.e. if the z -component of the Poynting vector is not zero on the symmetry axis of the helical beam, then this helical beam is not a vortex beam).

The present paper generalizes the darkness theorem to x - and y -darkness theorems and then introduces a superdarkness theorem providing conditions to state whether a helical beam is or is not a vortex beam. The paper is then organized as follows. Section 2 recalls a background required for the sequel. Section 3 introduces prolegomena which will afterward facilitate later discussions, and exhaustively considers the case of high-order Bessel beams. Section 4 will discuss a thorough list of examples which, using prolegomena, can be expounded using fairly short arguments. Section 5 is a conclusion which will introduce a possibly open problem.

2. Background material

The laser beam under study is assumed to have a time dependence of the form $\exp(i\omega t)$ which is omitted below as is the traditional practice in the case of monochromatic beams, and is supposed to propagate along the z -axis. We then consider two coordinate systems, one Cartesian coordinate system (x, y, z) and its associated spherical coordinate system (r, θ, φ) . The electric and magnetic fields in the GLMT framework are expressed in terms of BSCs, e.g. Eqs. (3.39)–(3.50) in [15], from which we can deduce the components of the time averaged Poynting

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vector according to:

$$\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \frac{1}{2} \text{Re} \begin{pmatrix} E_y H_z^* - E_z H_y^* \\ E_z H_x^* - E_x H_z^* \\ E_x H_y^* - E_y H_x^* \end{pmatrix} \quad (1)$$

in which the star denotes a complex conjugation. The results are given in Eqs. (23)–(37) of [20]. In the present paper, we only need the expressions for what will be called the on-axis location, i.e. when the support of the Poynting vector is located on the propagation z -axis, i.e. corresponding to $\theta = 0$ (not to be confused with the on-axis formulation or configuration to be introduced later). The results are provided by Eqs. (41)–(53) of [20]. Because they provide the basic background expressions to deal with the darkness and superdarkness theorems, they are repeated below, assuming for normalization that the electric and magnetic force strengths E_0 and H_0 satisfy $E_0 H_0^*/2 = 1$. We then have:

$$S_x(\theta = 0) = \frac{k}{r} \text{Re} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} c_n^{pw} c_p^{pw*} i(\psi_n'' + \psi_n) \Omega_p (\mu_{np} \psi_p' - \nu_{np} \psi_p) \quad (2)$$

in which:

$$\mu_{np} = g_{n, TM}^0 (g_{p, TE}^{1*} - g_{p, TE}^{-1*}) - g_{n, TE}^0 (g_{p, TM}^{1*} - g_{p, TM}^{-1*}) \quad (3)$$

$$\nu_{np} = g_{n, TM}^0 (g_{p, TM}^{1*} + g_{p, TM}^{-1*}) + g_{n, TE}^0 (g_{p, TE}^{1*} + g_{p, TE}^{-1*}) \quad (4)$$

$$S_y(\theta = 0) = \frac{k}{r} \text{Re} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} c_n^{pw} c_p^{pw*} (\psi_n'' + \psi_n) \Omega_p (\eta_{np} \psi_p' + \lambda_{np} \psi_p) \quad (5)$$

in which:

$$\eta_{np} = g_{n, TM}^0 (g_{p, TE}^{1*} + g_{p, TE}^{-1*}) - g_{n, TE}^0 (g_{p, TM}^{1*} + g_{p, TM}^{-1*}) \quad (6)$$

$$\lambda_{np} = g_{n, TM}^0 (g_{p, TM}^{-1*} - g_{p, TM}^{1*}) + g_{n, TE}^0 (g_{p, TE}^{-1*} - g_{p, TE}^{1*}) \quad (7)$$

and:

$$S_z(\theta = 0) = \frac{-2}{r^2} \text{Re} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} i c_n^{pw} c_p^{pw*} \Omega_n \Omega_p \quad (8)$$

$$\begin{aligned} & [(g_{n, TM}^1 \psi_n' + g_{n, TE}^1 \psi_n)(g_{p, TE}^1 \psi_p' - g_{p, TM}^1 \psi_p) \\ & - (g_{n, TM}^{-1} \psi_n' - g_{n, TE}^{-1} \psi_n)(g_{p, TM}^{-1} \psi_p + g_{p, TE}^{-1} \psi_p')] \end{aligned}$$

in which:

$$\Omega_n = \frac{-n(n+1)}{2} \quad (9)$$

and c_n^{pw} are plane wave coefficients introduced in [23], reading as:

$$c_n^{pw} = \frac{1}{k} (-i)^{n+1} \frac{2n+1}{n(n+1)} \quad (10)$$

while ψ_n , with the argument kr (k being the wave-number) being omitted, denotes Riccati–Bessel functions, and a prime (double-prime) denotes the derivation (double derivation) of the function with respect to the argument.

In the on-axis location, it is then very important to remark from the above equations that the Poynting vector components only depend on the BSCs with $m = 0, \pm 1$. Furthermore, retrospectively, it has been found fairly difficult to proceed from Eq. (23) of [20] to Eq. (2) above. An efficient demonstration is then provided in the Appendix (the component S_y being processed similarly).

3. Darkness and superdarkness theorems

We begin with a recall of the darkness theorem, now called the z -darkness theorem which is not new, but refined, before dealing with the new material concerning the x -, y -darkness theorems and the superdarkness theorem.

3.1. z -darkness (longitudinal darkness)

A beam is called z -dark on an axis if and only if the z -component of the Poynting vector is zero on this axis. We then rely on the on-axis location of Eq. (8) which may be rewritten as:

$$\begin{aligned} S_z(\theta = 0) = & \frac{-2}{r^2} \text{Re} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} i c_n^{pw} c_p^{pw*} \Omega_n \Omega_p \\ & [(g_{n, TM}^1 g_{p, TE}^{1*} - g_{n, TM}^{-1} g_{p, TE}^{-1*}) \psi_n' \psi_p' \\ & - (g_{n, TM}^1 g_{p, TM}^{1*} + g_{n, TM}^{-1} g_{p, TM}^{-1*}) \psi_n' \psi_p \\ & + (g_{n, TE}^1 g_{p, TE}^{1*} + g_{n, TE}^{-1} g_{p, TE}^{-1*}) \psi_n \psi_p' \\ & + (g_{n, TE}^{-1} g_{p, TM}^{-1*} - g_{n, TE}^1 g_{p, TM}^{1*}) \psi_n \psi_p] \end{aligned} \quad (11)$$

A sufficient condition for the beam to be z -dark is then given by the nullity of 4 BSC-expressions which, actually, reduces to 3 expressions reading as:

$$g_{n, TM}^1 g_{p, TE}^{1*} - g_{n, TM}^{-1} g_{p, TE}^{-1*} = 0 \quad (12)$$

$$g_{n, TM}^1 g_{p, TM}^{1*} + g_{n, TM}^{-1} g_{p, TM}^{-1*} = 0 \quad (13)$$

$$g_{n, TE}^1 g_{p, TE}^{1*} + g_{n, TE}^{-1} g_{p, TE}^{-1*} = 0 \quad (14)$$

In particular, for $n = p$, we obtain:

$$g_{n, TM}^1 g_{n, TE}^{1*} - g_{n, TM}^{-1} g_{n, TE}^{-1*} = 0 \quad (15)$$

$$|g_{n, TM}^1|^2 + |g_{n, TM}^{-1}|^2 = 0 \quad (16)$$

$$|g_{n, TE}^1|^2 + |g_{n, TE}^{-1}|^2 = 0 \quad (17)$$

Let us consider Eq. (17) whose l.h.s is a summation of two positive terms which is zero only if and only if each of these terms is zero, i.e. if and only if $g_{n, TE}^{\pm 1}$ is zero. Similarly, Eq. (16) is satisfied if and only if $g_{n, TM}^{\pm 1}$ is zero. Eq. (15) is then automatically satisfied as well.

These results provide an immediate obvious sufficient condition for a beam being a z -dark beam, namely ‘‘A beam is a z -dark beam if, for some axis, we have $g_{n, TE}^{\pm 1} = g_{n, TM}^{\pm 1} = 0$ ’’. That the condition is sufficient but not necessary may be shown by considering a special case, namely when all the BSCs are zero excepted for $g_{1, TM}^1 = g_{1, TE}^1$, so that $g_{1, TM}^1 g_{1, TE}^{1*}$ is a real number, leading to $S_z(\theta = 0) = 0$.

3.2. x - and y -darknesses (transverse darkness)

A beam is called x -dark (y -dark) on an axis if and only if the x -component (the y -component) of the Poynting vector is zero on this axis. From Eqs. (2)–(4), a sufficient condition for a beam to be x -dark is then (for some axis):

$$g_{n, TM}^0 (g_{p, TE}^{1*} - g_{p, TE}^{-1*}) - g_{n, TE}^0 (g_{p, TM}^{1*} - g_{p, TM}^{-1*}) = 0 \quad (18)$$

$$g_{n, TM}^0 (g_{p, TM}^{1*} + g_{p, TM}^{-1*}) + g_{n, TE}^0 (g_{p, TE}^{1*} + g_{p, TE}^{-1*}) = 0 \quad (19)$$

That the condition is sufficient but not necessary may be shown by considering the special case when all BSCs are zero excepted for $g_{1, TM}^0 = g_{1, TE}^0$, so that $g_{1, TM}^0 g_{1, TE}^{0*}$ is a real number, implying that $S_z(\theta = 0) = 0$. In practice, we may deal with two sufficient conditions as follows: (i) a beam is x -dark if $g_{n, TM}^0 = g_{n, TE}^0 = 0$ and (ii) a beam is x -dark if $g_{n, TM}^{\pm 1} = g_{n, TE}^{\pm 1} = 0$.

Similarly, the necessary condition for a beam to be y -dark is given by Eqs. (6)–(7) according to:

$$g_{n, TM}^0 (g_{p, TE}^{1*} + g_{p, TE}^{-1*}) - g_{n, TE}^0 (g_{p, TM}^{1*} + g_{p, TM}^{-1*}) = 0 \quad (20)$$

$$g_{n, TM}^0 (g_{p, TM}^{-1*} - g_{p, TM}^{1*}) + g_{n, TE}^0 (g_{p, TE}^{-1*} - g_{p, TE}^{1*}) = 0 \quad (21)$$

leading to the same two sufficient conditions for a beam to be y -dark, namely (i) a beam is y -dark if $g_{n, TM}^0 = g_{n, TE}^0 = 0$ and (ii) a beam is y -dark if $g_{n, TM}^{\pm 1} = g_{n, TE}^{\pm 1} = 0$, after having shown, similarly as above that the conditions are sufficient but not necessary.

Therefore, more generally, (i) a beam is transversely dark if $g_{n, TM}^0 = g_{n, TE}^0 = 0$ and (ii) a beam is transversely dark if $g_{n, TM}^{\pm 1} = g_{n, TE}^{\pm 1} = 0$.

3.3. Superdarkness

A beam is said to be superdark (on an axis) if and only if it is simultaneously x -, y - and z -dark (on this axis). The interest of this definition, in the context of the present work, is that a vortex beam is superdark on its axis. In other words, a beam which is not superdark on any axis is not a vortex beam.

4. Prolegomena and Bessel beams

This section introduces prolegomena which will facilitate several discussions in the sequel and discusses the case of Bessel beams to illustrate the use of the darkness and superdarkness theorems.

4.1. Prolegomena

We recall that the radial electric component E_r and the radial magnetic component H_r of the electric and magnetic fields, respectively, may be expressed as, see Eqs. (3.39), (3.42), (3.45) and (3.48) of [15]:

$$E_r = kE_0 \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} c_n^{pw} g_{n, TM}^m [\psi_n''(kr) + \psi_n(kr)] P_n^{|m|}(\cos \theta) e^{im\varphi} \quad (22)$$

$$H_r = kH_0 \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} c_n^{pw} g_{n, TE}^m [\psi_n''(kr) + \psi_n(kr)] P_n^{|m|}(\cos \theta) e^{im\varphi} \quad (23)$$

while, conversely, the BSCs may be expressed in terms of the radial components of the fields according to Eqs. (3.13) and (3.20) of [15]:

$$g_{n, TM}^m = \frac{1}{E_0 c_n^{pw}} \frac{2n+1}{4\pi n(n+1)} \frac{(n-|m|)!}{(n+|m|)!} \frac{r}{j_n(kr)} \int_0^\pi \int_0^{2\pi} E_r P_n^{|m|}(\cos \theta) e^{-im\varphi} \sin \theta d\theta d\varphi \quad (24)$$

$$g_{n, TE}^m = \frac{1}{H_0 c_n^{pw}} \frac{2n+1}{4\pi n(n+1)} \frac{(n-|m|)!}{(n+|m|)!} \frac{r}{j_n(kr)} \int_0^\pi \int_0^{2\pi} H_r P_n^{|m|}(\cos \theta) e^{-im\varphi} \sin \theta d\theta d\varphi \quad (25)$$

in which j_n 's denote spherical Bessel functions of the first kind also denoted as $\Psi_n^{(1)}$ when using Meixner and Schäfke notation, and P_n^m 's denote associated Legendre functions using Hobson's notation, see [24]. Also, note that, when the beam description exactly satisfies Maxwell's equations, the integrals in Eqs. (24) and (25) generate a term equal to $j_n(kr)/r$ so that the BSCs are coordinate-independent as they should.

Now, let us assume the case when the electric field radial component is written as:

$$E_r = A_{l+1}^E(r, \theta) e^{i(l+1)\varphi} + A_{l-1}^E(r, \theta) e^{i(l-1)\varphi} \quad (26)$$

We insert Eq. (26) into Eq. (24), and use the orthogonality relation:

$$\int_0^{2\pi} e^{i(m-m')\varphi} d\varphi = 2\pi \delta_{mm'} \quad (27)$$

to obtain:

$$g_{n, TM}^m = \frac{1}{E_0 c_n^{pw}} \frac{2n+1}{2n(n+1)} \frac{(n-|m|)!}{(n+|m|)!} \frac{r}{j_n(kr)} \times [\delta_{l+1-m} \int_0^\pi A_{l+1}^E(r, \theta) P_n^{|m|}(\cos \theta) \sin \theta d\theta$$

$$+ \delta_{l-1-m} \int_0^\pi A_{l-1}^E(r, \theta) P_n^{|m|}(\cos \theta) \sin \theta d\theta]$$

which implies that all TM -BSCs are zero excepted $g_{n, TM}^{l\pm 1}$ according to:

$$g_{n, TM}^{l+1} = \frac{1}{E_0 c_n^{pw}} \frac{2n+1}{2n(n+1)} \frac{(n-|l+1|)!}{(n+|l+1|)!} \frac{r}{j_n(kr)} \times \int_0^\pi A_{l+1}^E(r, \theta) P_n^{|l+1|}(\cos \theta) \sin \theta d\theta \quad (29)$$

$$g_{n, TM}^{l-1} = \frac{1}{E_0 c_n^{pw}} \frac{2n+1}{2n(n+1)} \frac{(n-|l-1|)!}{(n+|l-1|)!} \frac{r}{j_n(kr)} \times \int_0^\pi A_{l-1}^E(r, \theta) P_n^{|l-1|}(\cos \theta) \sin \theta d\theta \quad (30)$$

Conversely, let us assume that all BSCs are zero but $g_{n, TM}^{l\pm 1}$ according to:

$$g_{n, TM}^m = \delta_{l+1-m} G_{n, TM}^{l+1} + \delta_{l-1-m} G_{n, TM}^{l-1} \quad (31)$$

Inserting Eq. (31) into Eq. (22), we readily recover Eq. (26) in which:

$$A_{l+1}^E = kE_0 \sum_{n=1}^{\infty} c_n^{pw} G_{n, TM}^{l+1} (\psi_n'' + \psi_n) P_n^{|l+1|} \quad (32)$$

$$A_{l-1}^E = kE_0 \sum_{n=1}^{\infty} c_n^{pw} G_{n, TM}^{l-1} (\psi_n'' + \psi_n) P_n^{|l-1|} \quad (33)$$

Similarly, let us assume that the magnetic field reads as:

$$H_r = A_{l+1}^H(r, \theta) e^{i(l+1)\varphi} + A_{l-1}^H(r, \theta) e^{i(l-1)\varphi} \quad (34)$$

Then, as above for the electric field, we establish that all TE -BSCs are zero but $g_{n, TE}^{l\pm 1}$. Reciprocally, if we assume that all TE -BSCs are zero but $g_{n, TE}^{l\pm 1}$, then we recover Eq. (34).

It is very important to note that these prolegomena are valid only for beams which exactly satisfy Maxwell's equations. For beams which do not satisfy Maxwell's equations, the process of computing BSCs leads to a remodeling of the beam. Then, if all non-zero remodeled BSCs are for $g_{n, TM}^{l\pm 1}$ and $g_{n, TE}^{l\pm 1}$, Eqs. (26) and (34) hold for the remodeled beams, not necessarily (although possibly) for the original beams. Therefore, in the case of beams which do not satisfy Maxwell's equations, we shall not necessarily refer to these prolegomena.

4.2. (1,0)-polarized circularly symmetric Bessel beams

After their introduction by Durning [25] and Durning et al. [26], due in particular for their properties of being self-healing and non-diffracting, there has been a huge development in the study and applications of Bessel beams, e.g. [27–33], [34–40], [41–51] and references therein. More related papers are quoted and discussed in review papers, e.g. [52] (with 370 references), [17] (with 481 references) and [53] (planned to have more than 600 references). These review papers may serve for all examples discussed in the present paper and will not be mentioned again any more. Another under-mentioned property of Bessel beams is that the speed of propagation of such beams may be made much smaller than the speed of light [54,55].

In this subsection, we consider x -polarized circularly symmetric Bessel beams. We shall begin with what is called an off-axis configuration. We still consider a beam propagating in the z -direction but described in a Cartesian coordinate system which is off-axis of the beam axis in which the description of the beam is the simplest (here, in practice, the beam axis of the helical beam). We shall afterward simplify the expressions to the case of an on-axis configuration in which the off-axis parameters are simplified to zero. This should not be confused with an on-axis location which means that we take $\theta = 0$.

In an off-axis configuration, the electric field components are given by Eqs. (7)–(9) of [56], and read as:

$$E_x^{(1,0)} = E_0 g(\alpha_0) e^{-ik(z-z_0) \cos \alpha_0} \{(-j)^l (1 + \cos \alpha_0)$$

$$\begin{aligned} & \times \sum_{p=-\infty}^{+\infty} J_p(k_\rho \rho_0) J_{p+l}(k_\rho \rho) e^{i(l+p)\varphi} e^{-ip\varphi_0} \\ & - \frac{1 - \cos \alpha_0}{2} [(-i)^{l+2} \sum_{p=-\infty}^{+\infty} J_p(k_\rho \rho_0) J_{p+l+2}(k_\rho \rho) e^{i(l+2+p)\varphi} e^{-ip\varphi_0} \\ & + (-i)^{l-2} \sum_{p=-\infty}^{+\infty} J_p(k_\rho \rho_0) J_{p+l-2}(k_\rho \rho) e^{i(l-2+p)\varphi} e^{-ip\varphi_0}] \end{aligned} \quad (35)$$

$$E_y^{(1,0)} = E_0 g(\alpha_0) e^{-ik(z-z_0) \cos \alpha_0} \frac{\cos \alpha_0 - 1}{2i} \quad (36)$$

$$\begin{aligned} & \times [(-i)^{l+2} \sum_{p=-\infty}^{+\infty} J_p(k_\rho \rho_0) J_{p+l+2}(k_\rho \rho) e^{i(l+2+p)\varphi} e^{-ip\varphi_0} \\ & - (-i)^{l-2} \sum_{p=-\infty}^{+\infty} J_p(k_\rho \rho_0) J_{p+l-2}(k_\rho \rho) e^{i(l-2+p)\varphi} e^{-ip\varphi_0}] \end{aligned}$$

$$E_z^{(1,0)} = -E_0 g(\alpha_0) e^{-ik(z-z_0) \cos \alpha_0} \sin \alpha_0 \quad (37)$$

$$\begin{aligned} & \times [(-i)^{l+1} \sum_{p=-\infty}^{+\infty} J_p(k_\rho \rho_0) J_{p+l+1}(k_\rho \rho) e^{i(l+1+p)\varphi} e^{-ip\varphi_0} \\ & + (-i)^{l-1} \sum_{p=-\infty}^{+\infty} J_p(k_\rho \rho_0) J_{p+l-1}(k_\rho \rho) e^{i(l-1+p)\varphi} e^{-ip\varphi_0}] \end{aligned}$$

in which J_p 's denotes p -order Bessel function of the first kind, α_0 denotes the axicon angle of the beam (also called half-cone angle), off-axis configuration is defined by ρ_0 and φ_0 , ρ denotes the transverse coordinate with respect to the z -direction of propagation and k_ρ is the transverse wave-number. Also, $g(\alpha_0)$ allows one to specify different kinds of Bessel beams already discussed in the literature [54,56–58]. When $g(\alpha_0) = (1 + \cos \alpha_0)/4$, we obtain a Davis circularly symmetric beam [54,59] while, when $g(\alpha_0) = 1/2$, we obtain Bessel beams discussed in [28,60,61].

We now consider an on-axis configuration in which $\rho_0 = 0$ and φ_0 is undefined (it is then convenient to take it equal to 0) and, in this configuration, we consider a location on the axis for which we also have $\rho = 0$. In other words, we consider an on-axis location in an on-axis configuration. Recalling that $J_p(0) = \delta_{p0}$, we then obtain the expressions of the electric field components for an on-axis configuration of the beam, on the axis of the beam where $\theta = 0$, reading as:

$$E_x^{(1,0)} = E_0 g(\alpha_0) e^{-ik(z-z_0) \cos \alpha_0} \{(-i)^l (1 + \cos \alpha_0) J_l(0) e^{il\varphi} - \frac{1 - \cos \alpha_0}{2} [(-i)^{l+2} J_{l+2}(0) e^{i(l+2)\varphi} + (-i)^{l-2} J_{l-2}(0) e^{i(l-2)\varphi}]\} \quad (38)$$

$$E_y^{(1,0)} = E_0 g(\alpha_0) e^{-ik(z-z_0) \cos \alpha_0} \frac{\cos \alpha_0 - 1}{2i} \times [(-i)^{l+2} J_{l+2}(0) e^{i(l+2)\varphi} - (-i)^{l-2} J_{l-2}(0) e^{i(l-2)\varphi}] \quad (39)$$

$$E_z^{(1,0)} = -E_0 g(\alpha_0) e^{-ik(z-z_0) \cos \alpha_0} \sin \alpha_0 \times [(-i)^{l+1} J_{l+1}(0) e^{i(l+1)\varphi} + (-i)^{l-1} J_{l-1}(0) e^{i(l-1)\varphi}] \quad (40)$$

TM -BSCs are deduced from the radial component E_r as shown in Eq. (24). However, for a location on the axis of the helical beam, i.e. $\theta = 0$, we have $E_r = E_z$ given by Eq. (40) which has exactly the structure of Eq. (26), although with $\theta = 0$. Therefore, according to prolegomena, the only non-zero TM -BSCs are for $m = l \pm 1$. The general expression for the TM -BSCs generated by the off-axis fields of Eqs. (35)–(37) is given in Eq. (15) of [56], rewritten in Eqs. (15) and (17) of [62]. They have been obtained by using the quadrature method (the original method introduced to evaluate BSCs, e.g. [15]) which, in the present case, may be evaluated analytically. When specified to the case of an on-axis configurations, these BSCs become, see Eqs. (46)–(47) of [22]:

$$g_{n, TM}^{l+1} = -g(\alpha_0) \exp(ik_z z_0) A_n^{l+1} \quad (41)$$

$$g_{n, TM}^{l-1} = -g(\alpha_0) \exp(ik_z z_0) B_n^{l-1} \quad (42)$$

in which:

$$A_n^{l+1} = (-1)^{(l+1-|l+1|)/2} \frac{(n-l-1)!}{(n+|l+1|)!} [\tau_n^{l+1}(\cos \alpha_0) + (l+1)\pi_n^{l+1}(\cos \alpha_0)] \quad (43)$$

$$B_n^{l-1} = (-1)^{(l-1-|l-1|)/2} \frac{(n-l+1)!}{(n+|l-1|)!} [\tau_n^{l-1}(\cos \alpha_0) - (l-1)\pi_n^{l-1}(\cos \alpha_0)] \quad (44)$$

in which expressions for the generalized Legendre functions are given in Eqs. (61)–(62) of Appendix.

The expressions for the magnetic field may be deduced from Eqs. (35) to (37) by using the Maxwell-Ampère equation. These expressions are not provided in [56], but the TE -BSCs are given in Eq. (17) of [56] for an off-axis configuration, rewritten in Eqs. (16) and (18) of [62]. When specified to the case of an on-axis configuration, these BSCs become, see Eqs. (48)–(49) of [22]:

$$g_{n, TE}^{l+1} = ig(\alpha_0) \exp(ik_z z_0) A_n^{l+1} \quad (45)$$

$$g_{n, TE}^{l-1} = -ig(\alpha_0) \exp(ik_z z_0) B_n^{l-1} \quad (46)$$

According to the prolegomena, Eqs. (41)–(42) and (45)–(46) ensure that on the axis of the beam in the on-axis configuration, the magnetic field would satisfy Eq. (34).

We are now providing a discussion that we shall call **the darkness discussion** to be repeatedly used in the sequel. The assumption for this discussion is that we have BSCs which are all equal to 0 but for $g_{n, TM}^{l\pm 1}$ and $g_{n, TE}^{l\pm 1}$. Then we may consider different cases.

(i) Let $l = 0$. The non-zero BSCs are then $g_{n, TM}^{\pm 1}$ and $g_{n, TE}^{\pm 1}$. Since $g_{n, TM}^0$ and $g_{n, TE}^0$ are zero, the beam is x -dark and y -dark according to a sufficient condition for x - and y -darknesses. However, since $g_{n, TM}^{\pm 1}$ and $g_{n, TE}^{\pm 1}$ are not zero, the beam is possibly not z -dark. Therefore it is possibly not superdark and possibly not a vortex beam. This was expected since such beams have no topological charge, i.e. they are actually not helical beams.

(ii) Let $l = +1$. The non-zero BSCs are then $g_{n, TM}^2$, $g_{n, TM}^0$, $g_{n, TE}^2$ and $g_{n, TE}^0$. Since $g_{n, TM}^{\pm 1}$ and $g_{n, TE}^{\pm 1}$ are zero, the beam is x -, y -, and z -dark according to sufficient conditions. It is therefore superdark and we are dealing with a helical beam which is a vortex beam. Similarly, beams with $l = -1$ are vortex beams.

(iii) Let $l = +2$. The non-zero BSCs are then $g_{n, TM}^3$, $g_{n, TM}^1$, $g_{n, TE}^3$ and $g_{n, TE}^1$. Since $g_{n, TM}^0$ and $g_{n, TE}^0$ are zero, the beam is x -dark and y -dark according to sufficient conditions. But, since $g_{n, TM}^1$ and $g_{n, TE}^1$ are not zero, the beam is possibly not z -dark (recall Eq. (11) and Eqs. (12)–(14)). Therefore the beam is possibly not superdark and therefore it is possibly not a vortex beam. Similarly, beams with $l = -2$ are not necessarily vortex beams. In this case, an explicit check is required. Let us consider in particular, the case discussed above of (1, 0) x -polarized circularly symmetric beams, for $l = 2$. From Eqs. (42) and (46), we have $g_{1, TM}^1 \neq 0$, $g_{1, TE}^1 \neq 0$, with $g_{1, TM}^1 g_{1, TE}^{1*}$ being imaginary, leading to $S_z(\theta = 0) \neq 0$, meaning that we are indeed not dealing with a vortex beam.

(iv) Let $|l| > 2$. We readily check that such beams are vortex beams.

In summary, helical beams with BSCs equal to zero but for $g_{n, TM}^{l\pm 1}$ and $g_{n, TE}^{l\pm 1}$ are not necessarily vortex beams for $l = \pm 2$. In such cases, direct checks are required.

4.3. Bessel beams. Other polarizations

Bessel beams with other polarizations have been discussed in [57, 58]. BSCs for an off-axis configuration, once again obtained using the quadrature technique with analytical evaluations, are summarized in Eqs. (3)–(4) of [63] for an off-axis configuration. For an on-axis configuration, they simplify to:

$$g_{n, TM}^m = ig(\alpha_0) (-1)^{(m-|m|)/2} \frac{(n-m)!}{(n+|m|)!} \exp(ik_z z_0) \{ [p_x \tau_n^m(\cos \alpha_0) + p_y m \pi_n^m(\cos \alpha_0)] \} \quad (47)$$

$$\begin{aligned}
& \times [i^{l-m+1} e^{i(l-m+1)\phi_0} J_{l-m+1}(0) + i^{l-m-1} e^{i(l-m-1)\phi_0} J_{l-m-1}(0)] \\
& + [ip_y \tau_n^m(\cos \alpha_0) - p_x m \pi_n^m(\cos \alpha_0)] \\
& \times [-i \cdot i^{l-m+1} e^{i(l-m+1)\phi_0} J_{l-m+1}(0) + i \cdot i^{l-m-1} e^{i(l-m-1)\phi_0} J_{l-m-1}(0)] \} \\
g_{n,TE}^m &= g(\alpha_0) (-1)^{(m-|m|)/2} \frac{(n-m)!}{(n+|m|)!} \exp(ik_z z_0) \{ \\
& [ip_x m \pi_n^m(\cos \alpha_0) + p_y \tau_n^m(\cos \alpha_0)] \\
& \times [i^{l-m+1} e^{i(l-m+1)\phi_0} J_{l-m+1}(0) + i^{l-m-1} e^{i(l-m-1)\phi_0} J_{l-m-1}(0)] \\
& + [-p_x \tau_n^m(\cos \alpha_0) + ip_y m \pi_n^m(\cos \alpha_0)] \\
& \times [-i \cdot i^{l-m+1} e^{i(l-m+1)\phi_0} J_{l-m+1}(0) + i \cdot i^{l-m-1} e^{i(l-m-1)\phi_0} J_{l-m-1}(0)] \}
\end{aligned} \quad (48)$$

in which p_x and p_y define the polarization of the beam according to $(p_x, p_y) = (1, 0)$ for x -polarization, $(0, 1)$ for y -polarization, $(1, i)$ for left circular polarization, $(1, -i)$ for right circular polarization, $(\cos \beta, \sin \beta)$ for radial polarization, and $(-\sin \beta, \cos \beta)$ for azimuthal polarization in which β is an azimuthal angle, see as well [28,64]. Because $J_k(0) = \delta_{k0}$, Eqs. (47)–(48) show that the only non-zero BSCs are those for which $m = l \pm 1$, according to:

$$\begin{aligned}
g_{n,TM}^{l+1} &= ig(\alpha_0) (-1)^{(l+1-|l+1|)/2} \frac{(n-l-1)!}{(n+|l+1|)!} \exp(ik_z z_0) \\
& \times (ip_x + p_y) [\tau_n^{l+1}(\cos \alpha_0) + (l+1) \pi_n^{l+1}(\cos \alpha_0)]
\end{aligned} \quad (49)$$

$$\begin{aligned}
g_{n,TM}^{l-1} &= ig(\alpha_0) (-1)^{(l-1-|l-1|)/2} \frac{(n-l+1)!}{(n+|l-1|)!} \exp(ik_z z_0) \\
& \times (ip_x - p_y) [\tau_n^{l-1}(\cos \alpha_0) - (l-1) \pi_n^{l-1}(\cos \alpha_0)]
\end{aligned} \quad (50)$$

$$\begin{aligned}
g_{n,TE}^{l+1} &= g(\alpha_0) (-1)^{(l+1-|l+1|)/2} \frac{(n-l-1)!}{(n+|l+1|)!} \exp(ik_z z_0) \\
& \times (ip_x + p_y) [\tau_n^{l+1}(\cos \alpha_0) + (l+1) \pi_n^{l+1}(\cos \alpha_0)]
\end{aligned} \quad (51)$$

$$\begin{aligned}
g_{n,TE}^{l-1} &= -g(\alpha_0) (-1)^{(l-1-|l-1|)/2} \frac{(n-l+1)!}{(n+|l-1|)!} \exp(ik_z z_0) \\
& \times (ip_x - p_y) [\tau_n^{l-1}(\cos \alpha_0) - (l-1) \pi_n^{l-1}(\cos \alpha_0)]
\end{aligned} \quad (52)$$

The darkness discussion then implies that all these helical beams (i.e. with $l \neq 0$) are vortex beams excepted possibly when $l = \pm 2$. A simple direct check, similarly as for the previous case, may be made by considering only $g_{1,TM}^1 \neq 0$ and $g_{1,TE}^1 \neq 0$, for $l = 2$, and $g_{1,TM}^{-1} \neq 0$ and $g_{1,TE}^{-1} \neq 0$, for $l = -2$, leading to $S_z(\theta = 0) \neq 0$, meaning again that we are indeed not dealing with a vortex beam. Simple direct checks may be made similarly for the other examples of the next section, although we shall be more concise to deal with them.

5. Other examples

5.1. Frozen waves

We begin with frozen waves which are made of a discrete or continuous superposition of Bessel beams, with the same topological charge but with different wave-numbers, e.g. [65–73] for what is likely to be a fairly exhaustive list of references on the issue.

Let us consider BSCs given by Eqs. (6)–(9) in [69]. In the case of an on-axis configuration, using the fact that $J_k(0) = \delta_{k0}$ as in the previous example of Bessel beams, it is found that the only non-zero BSCs are for $m = l \pm 1$ (with the notations of the present paper). Therefore, from the darkness discussion, all these helical beams are vortex beams, excepted possibly for $l = \pm 2$. Similarly as previously, the fact that these beams for $l = \pm 2$ are not vortex beams can be confirmed by simple direct checks.

Furthermore, from the prolegomena, we expect that the fields should have the structure of Eqs. (26) and (34). This is in conflict with the structure of Eqs. (2)–(4) for the electric field in [69]. This mismatch is however due to a typo, namely the r.h.s.'s of these equations should be multiplied by $\exp(i\nu\varphi)$, in the notation of [69], so that E_z of Eq. (4) of [69], i.e. E_r for $\theta = 0$, would actually possess the structure of

Eq. (26) as required. The magnetic field (details omitted in [69]) would be treated similarly.

We may similarly consider [74] in which BSC of frozen waves are evaluated by using the Integral Localized Approximation (ILA), e.g. [75,76] for a review. BSCs are given in Eqs. (10)–(13) for an off-axis configuration, so that, once again, due to $J_k(0) = \delta_{k0}$, the BSCs in the on-axis configuration are those for which $m = l \pm 1$, in agreement with the structure of Eqs. (6) and (9) for the electric and magnetic fields respectively. Therefore, here again, all these helical beams are vortex beams, but possibly for $l = \pm 2$. Again, the fact that these beams for $l = \pm 2$ are not vortex beams can be confirmed by simple direct checks.

5.2. Lommel beams

Lommel beams are built from a combination of Bessel beams of various topological charges, e.g. [77–79]. In [78], Eq. (1), they are expressed using basic electric modes in cylindrical coordinates (r, φ, z) reading as:

$$E_v = e^{ik_z z} \sum_{p=0}^{\infty} (-1)^p c^{2p} \exp[i(l+2p)\varphi] J_{v+2p}(k_t r) \quad (53)$$

in which c is a parameter, $k_z = k \cos \alpha_0$ and $k_t = k \sin \alpha_0$ are the longitudinal and transverse wave-numbers, E_v represents the x - or y -component of the electric field which is obtained from a scalar wave equation solution, implying that the obtained beam does not satisfy Maxwell's equations (called a non-Maxwellian beam). BSCs are however obtained by using the ILA. This process, in the present case, generates a remodeling of the beam in which the original non-Maxwellian beam is converted to a beam which exactly satisfies Maxwell's equations (called a Maxwellian beam). These BSCs for x - and y -polarizations are provided in Eqs. (7) and (9) of [78] for an off-axis configuration. They have been worked out for $c = 0$ in the case of an on-axis configuration in [22]. It is then found that the non-zero BSCs $g_{n,TM}^m$ and $g_{n,TE}^m$ are for $m = l \pm 1$. The darkness discussion then implies that we are facing remodeled helical beams which are vortex beams excepted possibly for $l = \pm 2$. As before, the fact that these beams for $l = \pm 2$ are not vortex beams can be confirmed by simple direct checks.

5.3. Bessel-Gauss beams

Bessel beams possess the unpleasant feature of being unphysical because they propagate an infinite amount of energy. Therefore physical beams have been obtained from Bessel beams by using some kind of apodization. The most famous example is the one of Bessel–Gauss beams in which the apodization is carried out by using a Gaussian form, e.g. [70,79–91].

The present subsection will rely on the results of Valdivia et al. [92] which evaluate BSCs using three different methods (quadratures, finite series, and localized approximations). Because the original Bessel–Gauss beams do not exactly satisfy Maxwell equations, each of these methods provide a remodeling of the beam in which the non-Maxwellian beam is converted to a Maxwellian beam. We shall focus on the localized approximation and on the finite series methods.

Using a localized approximation more specifically ILA already used above for Bessel and Lommel beams, it is found that all the BSCs in an on-axis configuration are zero excepted for $m = l \pm 1$, see [92] and Eqs. (112)–(113) in Section 5.1 of [22]. Similarly, in the case of a remodeling using finite series, all the BSCs in an on-axis configuration are zero excepted for $m = l \pm 1$, see Eqs. (19)–(20) and (30)–(31) in [92], and (84)–(87) in Section 4.4 of [22]. Therefore, according to the darkness discussion, all the corresponding helical beams are vortex beams but possibly for $l = \pm 2$. Once again, the fact that these beams for $l = \pm 2$ are not vortex beams can be confirmed by simple direct checks. In particular, such checks may be very expedient if we simply use Eq. (88) of [22] because it implies directly that $g_{n,TM}^{l\pm 1} g_{n,TE}^{l\pm 1*}$ is purely imaginary, implying that $S_z(\theta = 0) \neq 0$.

Bessel–Gauss beams BSCs are discussed as well in [93] using an angular spectrum representation, where relevant equations confirming our previous discussions are Eqs. (20), (32)–(33), (39) and (47) to be specified for an on-axis configuration.

5.4. Laguerre-Gauss beams

Another example concerns Laguerre–Gauss beams, either freely propagating for which we have a paraxial non-Maxwellian description or focused by a lens for which we have a Maxwellian description. For examples of paper devoted to Laguerre–Gauss beam descriptions and applications, the reader may consult (to name a few) [94–98], [99–108].

Concerning Laguerre–Gauss beams freely propagating, BSCs have been evaluated by using three different remodeling methods, namely quadratures, localized approximation, and finite series [109,110] with a focus on the use of finite series in [111,112]. The description of such beams follows Siegman's books [113,114]. Once again, it is found that the BSCs of the remodeled beam in an on-axis configuration, using a localized approximation, are non-zero only for $m = l \pm 1$, see Eqs. (35)–(37) in [110]. Similarly, see as well Eqs. (10)–(11), (20)–(21) in [109], (68)–(70), (101)–(106) in [111], and (3.2)–(3.3) in [112]. In [109,110], TE -coefficients have not been worked out. But direct checks may be carried out in an expedient way using Eqs. (104)–(16) in [111] and (3.2)–(3.3) in [112].

Concerning Laguerre-beams focused by a lens, we possess a Maxwellian description, see [115–117]. The use of localized approximations is discussed in [118]. Eqs. (36)–(37) then show that the radial component E_r of the electric field satisfies Eq. (26). The non-zero BSCs are then found to be for $m = -l + 1$ and $m = -l - 1$, see Eqs. (47)–(48) in [118], although TE -coefficients have not been worked out. These m -values are not the ones we expected, i.e. the ones mentioned in the darkness discussion, but they as well imply that we are facing helical beams which possibly are not vortex beams for $l = +2$ and -2 , where we indeed have $m = -1, -3$ and $m = 3, 1$ respectively. A subtlety when dealing with these papers for Laguerre–Gauss beams focused by a lens is that they relied on a time-dependent term of the form $\exp(-i\omega t)$ called the N-convention, while the GLMT is traditionally used with a time-dependent term of the form $\exp(i\omega t)$ called the P-convention. The relation of BSCs between N- and P-conventions reads as, e.g. Eq. (8) in [119], and [120,121]:

$$g_{n,X}^m(P) = (-1)^{n+1} g_{n,X}^{-m}(N)^*, \quad X = TM, TE \quad (54)$$

The use of finite series implies as well that Laguerre–Gauss beams focused by a lens are possibly not vortex beams for $l = \pm 2$, see Eqs. (17), (21), (32) and (36) in [119] and (101)–(106) in [120]. TE -coefficients have not been worked out in [119] but a direct check can easily be carried out, in an expedient way, by using Eqs. (104)–(106) in [120].

5.5. Miscellaneous

In [122], various kinds of non-diffracting beam (higher-order Bessel beams, Mathieu beams, parabolic beams) are examined and expressed in terms of circularly symmetric Bessel beams, see Eqs. (11) and (12)a, leading again to BSCs which are non-zero for $m = l \pm 1$. In [123], BSCs of Hermite–Gaussian beams are expressed in terms of Laguerre–Gaussian beam BSCs. While the original BSCs may only assume non-zero values for $m = l \pm 1$, it is found that Hermite–Gaussian modes may as well assume non-zero values for $m = -(l \pm 1)$. Such a situation has already been observed above in the case of Laguerre–Gauss beams focused by a lens studied by using a localized approximation. It corresponds to BSCs which are not zero for $m = \pm 1$ when $l = \pm 2$, i.e. for $l = +2$ and -2 , we indeed have $m = -1, -3$ and $m = 3, 1$ respectively.

6. Conclusion

Helical beams (beams exhibiting a topological charge) became very common since a significant amount of time and several papers have been published using them, either for describing them or for using them for various applications. Most usually, such beams are said to be vortex beams. It however happens that helical beams are not necessarily vortex beams, i.e. they do not necessarily have a zero intensity (singularity) on their axis. To deal with this issue, a previously published darkness theorem (now renamed z-darkness theorem) is generalized to x -, y -darkness theorems and to a superdarkness theorem. They provide sufficient conditions to state that beams are dark or super-dark.

They then serve to identify helical beams which, by using properties of the Poynting vector expressed in terms of BSCs which have been introduced in a GLMT-framework to encode the beam description, could possibly be non vortex beams. Many examples are examined and it is shown, in these examples, that helical beams having a topological charge $l = \pm 2$ are not vortex beams. Therefore, we may state that helical beams are not necessarily vortex beams although this fact is largely ignored in the literature.

In the best of our knowledge, we would not dare to generalize this result and state that the conclusion, based on special cases, holds in any case. This word of caution is due to the fact that, since decades, we may observe in the literature new proposals for new kinds of beams and, in particular, it is difficult to predict which kinds of beams will be newly invented. There is therefore on this issue an open question for future work. In particular, an open question is that whether we could face non vortex helical beams for topological charges other than $l = \pm 2$.

CRedit authorship contribution statement

G erard Gouesbet: Conceptualization, Methodology, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix

Eq. (23) of [20] reads as:

$$S_x = \text{Re} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} \sum_{p=1}^{\infty} \sum_{q=-p}^{+p} c_n^{pw} c_p^{pw*} e^{i(m-q)\varphi} \left[\frac{k \sin \varphi}{r} (\psi_n'' + \psi_n) A_{nmpq} + \frac{ik \cos \theta \cos \varphi}{r} (\psi_n'' + \psi_n) B_{nmpq} + \frac{i \sin \theta \cos \varphi}{r^2} C_{nmpq} \right] \quad (55)$$

in which:

$$A_{nmpq} = [\alpha_{nmpq} \psi_p' \tau_p^{|q|} + q \beta_{nmpq} \psi_p \pi_p^{|q|}] P_n^{|m|} \quad (56)$$

$$B_{nmpq} = [-q\alpha_{nmpq}\psi'_p\pi_p^{|q|} - \beta_{nmpq}\psi_p\tau_p^{|q|}]P_n^m \quad (57)$$

$$C_{nmpq} = \alpha_{nmpq}(m\psi'_n\psi'_p\pi_n^{|m|}\tau_p^{|q|} + q\psi_n\psi_p\tau_n^{|m|}\pi_p^{|q|}) + \beta_{nmpq}(mq\psi'_n\psi'_p\pi_n^{|m|}\pi_p^{|q|} - \psi_n\psi'_p\tau_n^{|m|}\tau_p^{|q|}) \quad (58)$$

in which:

$$\alpha_{nmpq} = g_{p, TM}^{q*} g_{n, TE}^m - g_{n, TM}^m g_{p, TE}^{q*} \quad (59)$$

$$\beta_{nmpq} = g_{n, TM}^m g_{p, TM}^{q*} + g_{p, TE}^{q*} g_{n, TE}^m \quad (60)$$

and $\pi_n^m = \pi_n^m(\cos \theta)$, $\tau_n^m = \tau_n^m(\cos \theta)$ are generalized Legendre functions reading as:

$$\pi_n^m(\cos \theta) = \frac{P_n^m(\cos \theta)}{\sin \theta} \quad (61)$$

$$\tau_n^m(\cos \theta) = \frac{dP_n^m(\cos \theta)}{d\theta} \quad (62)$$

in which P_n^m 's are associated Legendre functions which, using Hobson's convention, read as:

$$P_n^m(\cos \theta) = (-1)^m (\sin \theta)^m \frac{d^m P_n(\cos \theta)}{(d \cos \theta)^m} \quad (63)$$

in which P_n 's are usual Legendre polynomials.

For $\theta = 0$, we use:

$$k\pi_n^k(\theta = 0) = \tau_n^k(\theta = 0) = 0, \quad k \neq 1 \quad (64)$$

$$\pi_n^1(\theta = 0) = \tau_n^1(\theta = 0) = \Omega_n = -n(n+1)/2 \quad (65)$$

$$P_n^{|m|}(\theta = 0) = P_n(\theta = 0)\delta_{|m|0} = \delta_{|m|0} \quad (66)$$

so that the summation over q in Eq. (55) reduces to $q = \pm 1$, and we obtain:

$$S_x(\theta = 0) = \frac{k}{r} \text{Re} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} c_n^{pw} c_p^{pw*} (\psi_n'' + \psi_n) \times [e^{-i\varphi}(\sin \varphi A_{n0p1} + i \cos \varphi B_{n0p1}) + e^{i\varphi}(\sin \varphi A_{n0p-1} + i \cos \varphi B_{n0p-1})] \quad (67)$$

in which:

$$A_{n0p1} = -B_{n0p1} = (\alpha_{n0p1}\psi'_p + \beta_{n0p1}\psi_p)\Omega_p \quad (68)$$

$$A_{n0p-1} = B_{n0p-1} = (\alpha_{n0p-1}\psi'_p - \beta_{n0p-1}\psi_p)\Omega_p \quad (69)$$

This leads to:

$$S_x(\theta = 0) = \frac{k}{r} \text{Re} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} c_n^{pw} c_p^{pw*} (\psi_n'' + \psi_n) \times [A_{n0p1} e^{-i\varphi}(\sin \varphi - i \cos \varphi) + A_{n0p-1} e^{i\varphi}(\sin \varphi + i \cos \varphi)] \quad (70)$$

Eq. (70) is afterward simplified by noting that the φ -terms reduce to $(-i)$ and $(+i)$ respectively, and rearranged by implementing the expressions for A_{n0p1} and A_{n0p-1} , and also the expressions for α_{n0p1} , α_{n0p-1} , β_{n0p1} and β_{n0p-1} readily leading to Eq. (2). The derivation of Eq. (5) from Eq. (24) of [20] is treated similarly.

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