

# STUDY OF WEIGHTED ELLIPTIC COMPOSITION OPERATORS ON THE UNIT BALL OF $\mathbb{C}^N$

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ABSTRACT. We study the general properties, point spectrum and spectrum of a weighted composition operator  $W_{m,\varphi}$  with elliptic symbol  $\varphi$  on the unit ball  $\mathbb{B}_N$  of  $\mathbb{C}^N$ , and general weight  $m \in \text{Hol}(\mathbb{B}_N)$ . We give a complete description of the spectra in the vast majority of cases, and we provide inclusions in general.

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## 1. INTRODUCTION

Let  $N \geq 2$ ,  $\mathbb{B}_N = \{z \in \mathbb{C}^N : |z| < 1\}$  be the unit ball of  $\mathbb{C}^N$  for the euclidean norm, and  $\text{Hol}(\mathbb{B}_N)$  be the set of all complex-valued holomorphic functions on  $\mathbb{B}_N$ . Consider  $\varphi : \mathbb{B}_N \rightarrow \mathbb{B}_N$  a holomorphic map, and  $m \in \text{Hol}(\mathbb{B}_N)$ . The *weighted composition operator, with symbol  $\varphi$  and weight  $m$* , denoted by  $W_{m,\varphi}$ , is defined on  $\text{Hol}(\mathbb{B}_N)$  by

$$W_{m,\varphi}(f) = m(f \circ \varphi).$$

It is a linear, multiplicative operator.

Theory of composition operators is a very popular subject. Most of the vast literature on this topic (see for example the monographs from Cowen, MacCluer in [9], Shapiro in

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[20] or more recently, the articles [5, 7, 8]) considers these operators on Banach spaces of analytic functions, such as Hardy, Dirichlet, or Bergman spaces.

However, in recent years, W. Arendt et al. ([2, 3]) studied one-variable weighted composition operators directly on the Fréchet space  $\text{Hol}(\mathbb{D})$ , where  $\mathbb{D}$  denotes the unit disc of  $\mathbb{C}$ . In this paper, we will only consider *elliptic* symbols, that is  $\varphi$  having a fixed point in the ball. Upon conjugating by some automorphism, we can assume that 0 is a fixed point of  $\varphi$ .

The results of [2, 3] for elliptic symbols are summarized in the following theorem. Denote by  $\mathbb{N}_0$  the set of all non-negative integers, and  $\mathbb{N} = \mathbb{N}_0 \setminus \{0\}$ .

**Theorem 1.1.** *Let  $\varphi$  be an elliptic self-map of  $\mathbb{D}$  such that  $\varphi(0) = 0$ , and  $m \in \text{Hol}(\mathbb{D})$ .*

(i) *If  $\varphi \equiv 0$ , then*

$$\sigma_p(W_{m,\varphi}) = \{0, m(0)\} = \sigma(W_{m,\varphi}).$$

(ii) *If  $m(0) = 0$  and  $\varphi \not\equiv 0$ , then*

$$\sigma_p(W_{m,\varphi}) = \emptyset, \quad \sigma(W_{m,\varphi}) = \{0\}.$$

(iii) *If  $m(0) \neq 0$ ,  $\varphi'(0) = 0$  and  $\varphi \not\equiv 0$ , then*

$$\sigma_p(W_{m,\varphi}) = \{m(0)\}, \quad \sigma(W_{m,\varphi}) = \{0, m(0)\}.$$

(iv) *If  $m(0) \neq 0$  and  $0 < |\varphi'(0)| < 1$ , then*

$$\sigma_p(W_{m,\varphi}) = \{m(0)\varphi'(0)^n : n \in \mathbb{N}_0\}, \quad \sigma(W_{m,\varphi}) = \sigma_p(W_{m,\varphi}) \cup \{0\}.$$

(v) *If  $m(0) \neq 0$  and  $\varphi(z) = \beta z$ ,  $|\beta| = 1$ ,  $\beta^p = 1$ , then if we define the map  $m_p$  by  $m_p(z) = m(z)m(\beta z) \cdots m(\beta^{p-1}z)$ ,*

$$\sigma_p(W_{m,\varphi}) \neq \emptyset \iff \sigma_p(W_{m,\varphi}) = \{m(0)\beta^k : k \in \mathbb{N}_0\} \iff m_p \text{ is a nonzero constant.}$$

*Moreover,  $\sigma(W_{m,\varphi}) = \{\lambda \in \mathbb{C} : \lambda^p \in m_p(\mathbb{D})\}$ .*

This article can be considered as a sequel to [2], [3] and [15]. The aim is to construct a similar theorem in the multidimensional case. Note that the behaviour of the point spectrum  $\sigma_p(W_{m,\varphi})$  and spectrum  $\sigma(W_{m,\varphi})$  of composition operators strongly depends on the properties of the maps  $m$  and  $\varphi$ .

In order to study this function, we need a generalization of Denjoy-Wolff's theorem, by B. MacCluer in [14] and Y. Kubota in [12], presented here as a single result. Let  $\varphi^{[n]}$  be the  $n$ -th iteration of  $\varphi$ .

**Theorem 1.2.** *Let  $\varphi$  be a self-map of  $\mathbb{B}_N$ , and assume that  $\varphi$  has a fixed point in  $\mathbb{B}_N$ .*

(a) *If a subsequence of  $(\varphi^{[n]})_{n \geq 0}$  converges to a constant map  $f \equiv z_0 \in \mathbb{B}_N$ , then the whole sequence converges to  $z_0$ . In this case,  $z_0$  is the unique fixed point of  $\varphi$ .*

(b) *Otherwise, upon conjugating, there exists a subsequence of  $(\varphi^{[n]})$  that converges to a function  $h$  of the form*

$$h(z) = (z_1, \dots, z_r, 0, \dots, 0), \quad \text{with } r \in \{1, \dots, N\}.$$

This theorem classifies the elliptic self-maps of  $\mathbb{B}_N$  into two categories.

**Definition 1.3.** Let  $\varphi$  be a self-map of  $\mathbb{B}_N$ . Assume that  $\varphi$  has a fixed point in  $\mathbb{B}_N$ .

- In the case (a) of Theorem 1.2, we say that  $\varphi$  is *elliptic attractive*, and the *Denjoy-Wolff point* of  $\varphi$  is defined as the  $z_0$ .
- In the case (b) of Theorem 1.2, we say that  $\varphi$  is *elliptic non attractive*.

The paper is organized as follows.

First, in Section 2, we consider the general properties of weighted composition operators on the unit ball: continuity (Proposition 2.1), invertibility (Proposition 2.2), and compactness (Proposition 2.3). We also give a formula about the partial derivatives of composition of functions.

Then, in Section 3, we study the spectral properties of  $W_{m,\varphi}$  if  $m(0) = 0$ . In particular, we prove that the point spectrum is empty (Proposition 3.1), and that the spectrum contains only 0 (Proposition 3.3).

Next, in Section 4, we focus on composition operators with non-vanishing weight at 0 and non-bijective symbol. We split the analysis into three cases. First, we consider symbols  $\varphi$  such that the Jacobian matrix at 0,  $\varphi'(0)$ , is invertible. In this case, the point spectrum is completely described (Proposition 4.5), as well as the spectrum if we assume moreover that  $\varphi'(0)$  is diagonal (Proposition 4.10). Then, we focus on Jacobian matrices such that 0 is their only eigenvalue. In such case, the point spectrum is  $\{m(0)\}$  or  $\{0, m(0)\}$  (Proposition 4.11), and the spectrum is  $\{0, m(0)\}$  (Theorem 4.12). Finally, we give general inclusions for the spectra of  $W_{m,\varphi}$  (Propositions 4.14 and 4.15).

In Section 5, we consider non-vanishing weights and bijective periodic symbols. We give a characterization of when the point spectrum is non-empty (Proposition 5.3), and a description of the spectrum of  $W_{m,\varphi}$  (Theorem 5.5).

Finally, in Section 6, we give results about the point spectrum in two cases: when the weight vanishes at some point in  $\mathbb{B}_N$  and the symbol is bijective aperiodic (Lemma 6.1), and when the symbol is elliptic non-attractive (Proposition 6.2).

## 2. GENERAL PROPERTIES

In this section, we go along the same lines as [2]. We first focus on continuity.

**Proposition 2.1** (Continuity). *The operator  $W_{m,\varphi}$  is continuous on  $\mathcal{L}(\text{Hol}(\mathbb{B}_N))$ .*

*Proof.* Let  $k \in \mathbb{N}$ . Denote by  $\|\cdot\|_{\infty,k}$  the semi-norm on  $\text{Hol}(\mathbb{B}_N)$  defined by

$$\|f\|_{\infty,k} = \sup_{z \in K_k} |f(z)|, \quad \text{with} \quad K_k = \left(1 - \frac{1}{k}\right) \overline{\mathbb{B}_N}.$$

Since  $\varphi$  is continuous and  $K_k$  is compact, there exists  $j \in \mathbb{N}$  such that  $\varphi(K_k) \subset K_j$ . Hence, for all  $f \in \text{Hol}(\mathbb{B}_N)$ ,

$$\|W_{m,\varphi}(f)\|_{\infty,k} = \sup_{z \in K_k} |m(z)f(\varphi(z))| \leq \sup_{z \in K_k} |m(z)| \sup_{w \in K_j} |f(w)| \leq \|m\|_{\infty,k} \|f\|_{\infty,j}. \quad \square$$

We now study the invertibility of weighted composition operators.

**Proposition 2.2** (Invertibility). *The operator  $W_{m,\varphi}$  is invertible if and only if  $m$  does not vanish on  $\mathbb{B}_N$  and  $\varphi$  is bijective.*

*Proof.* If  $\varphi$  is bijective and  $m$  does not vanish on  $\mathbb{B}_N$ , as in [2], we consider the map

$$\eta = \frac{1}{m \circ \varphi^{-1}},$$

which satisfies  $W_{\eta, \varphi^{-1}} \circ W_{m, \varphi} = W_{m, \varphi} \circ W_{\eta, \varphi^{-1}} = \text{Id}$ . Hence,  $W_{m, \varphi}$  is invertible.

Conversely, if there exists  $z_0 \in \mathbb{B}_N$  such that  $m(z_0) = 0$ , then for all  $f \in \text{Hol}(\mathbb{B}_N)$ ,

$$(W_{m, \varphi}(f))(z_0) = 0,$$

so  $W_{m, \varphi}$  is not invertible. Assume now that  $W_{m, \varphi}$  is invertible. Then, for all  $z \in \mathbb{B}_N$ , we know that  $m(z) \neq 0$ . Let  $g \in \text{Hol}(\mathbb{B}_N)$ . Then  $h = mg \in \text{Hol}(\mathbb{B}_N)$ , and there exists  $f \in \text{Hol}(\mathbb{B}_N)$  such that

$$m(f \circ \varphi) = h = mg.$$

Dividing this equality by  $m$  (since  $m$  does not vanish on  $\mathbb{B}_N$ ), we obtain  $f \circ \varphi = g$ . Therefore,  $C_\varphi$  is invertible, so  $\varphi$  is bijective (see [15, Proposition 2.4]).  $\square$

Finally, let us concentrate on the compactness of  $W_{m, \varphi}$ . Recall that for  $f \in \text{Hol}(\mathbb{B}_N)$ , the supremum norm of  $f$  is defined by  $\|f\|_\infty = \sup\{f(z) : z \in \mathbb{B}_N\}$ .

**Proposition 2.3** (Compactness).

*The operator  $W_{m, \varphi}$  is compact if and only if  $m = 0$  or  $\|\varphi\|_\infty < 1$ .*

*Proof.* If  $m = 0$ , then  $W_{m, \varphi} = 0$ , so  $W_{m, \varphi}$  is compact.

If  $m \neq 0$  and  $\|\varphi\|_\infty < 1$ , then  $W_{m, \varphi} = M_m C_\varphi$ , where  $M_m : f \mapsto mf$  is the multiplication operator by  $m$ , and  $C_\varphi : f \mapsto f \circ \varphi$ . Since  $\|\varphi\|_\infty < 1$ , using [15], the operator  $C_\varphi$  is compact. Hence, because the set of all compact operators is an ideal of  $\mathcal{L}(\text{Hol}(\mathbb{B}_N))$ , the operator  $W_{m, \varphi}$  is compact.

Conversely, if  $m \neq 0$  and  $\|\varphi\|_\infty = 1$ , then for  $0 < r < 1$ , we are able to choose  $z_0 \in \mathbb{B}_N$  such that  $|\varphi(z_0)| > r$  and  $m(z_0) \neq 0$ . The same proof as in the non-weighted case [15, Proposition 2.8], using Oka-Weil's theorem ([16, 21]) gives the result.  $\square$

Let us finish this section with a wonderful property concerning the partial derivatives of composed maps. We define two orders on  $\mathbb{N}_0^N$ . If  $\vec{i}, \vec{j} \in \mathbb{N}_0^N$ , set  $|\vec{i}| = i_1 + \dots + i_N$  and

$$\vec{i} \prec \vec{j} \iff |\vec{i}| < |\vec{j}| \quad \text{or} \quad \begin{cases} |\vec{i}| = |\vec{j}| \\ \exists k \in \{1, \dots, N\}, i_1 = j_1, \dots, i_{k-1} = j_{k-1}, i_k < j_k. \end{cases}$$

$$\vec{i} \leq \vec{j} \iff \forall k \in \{1, \dots, N\}, i_k \leq j_k.$$

We write  $\vec{i} \preceq \vec{j}$  if  $\vec{i} \prec \vec{j}$  or  $\vec{i} = \vec{j}$ . The order  $\preceq$  is a well-order. Hence, we can define the *predecessor* and the *successor* of a vector  $\vec{j} \in \mathbb{N}_0^N$ . We denote them respectively by  $\vec{j}_-$  and  $\vec{j}_+$ . For  $\vec{j} \in \mathbb{N}_0^N$ ,  $f \in \text{Hol}(\mathbb{B}_N)$  and  $z \in \mathbb{C}^N$ , let us denote

$$z^{\vec{j}} = \prod_{k=1}^N z_k^{j_k}, \quad f^{(\vec{j})} = \frac{\partial^{|\vec{j}|} f}{\partial z^{\vec{j}}}.$$

Moreover, we define  $\text{Hol}_{\vec{j}}(\mathbb{B}_N) = \{f \in \text{Hol}(\mathbb{B}_N) : \forall \vec{i} \preceq \vec{j}, f^{(\vec{i})}(0) = 0\}$ .

The following lemma, proved in [15] using Faà di Bruno's formula ([13]), will be crucial. Since unitary matrices are automorphisms of the ball fixing 0, using Schür

decomposition ([10]), we assume in the following that the Jacobian matrix of  $\varphi$  at 0 is a triangular matrix, with diagonal entries  $\lambda_1, \dots, \lambda_N$ .

**Lemma 2.4.** *Let  $\vec{j} \in \mathbb{N}_0^N$ . If  $f \in \text{Hol}_{\vec{j}}(\mathbb{B}_N)$ , then*

$$(1) \quad (f \circ \varphi)^{(\vec{j})}(0) = \lambda^{\vec{j}} f^{(\vec{j})}(0).$$

### 3. VANISHING WEIGHT AT 0

When the weight  $m$  satisfies  $m(0) = 0$ , the spectra of  $W_{m,\varphi}$  can be completely described, as it is proved in the following two properties.

**Proposition 3.1.** *Let  $\varphi$  be elliptic, such that  $\varphi(0) = 0$ . If  $m(0) = 0$ , then*

$$\sigma_p(W_{m,\varphi}) \subset \{0\}.$$

*Proof.* Let  $\mu \in \mathbb{C}^*$ . If  $f \not\equiv 0$  satisfies  $m(f \circ \varphi) = \mu f$ , then

- First,  $\mu f(0) = [m(f \circ \varphi)](0) = m(0)f(0) = 0$ , so  $f(0) = 0$  since  $\mu \neq 0$ .
- Assume that for all  $\vec{i} \prec \vec{j}$ ,  $f^{(\vec{i})}(0) = 0$ . Then, by Lemma 2.4 and the Leibniz rule,

$$\mu f^{(\vec{j})}(0) = [m(f \circ \varphi)]^{(\vec{j})}(0) = m(0)(f \circ \varphi)^{(\vec{j})}(0) = 0.$$

Therefore,  $f^{(\vec{j})}(0) = 0$ .

Finally,  $f \equiv 0$ , which is impossible. Thus  $\sigma_p(W_{m,\varphi}) \subset \{0\}$ .  $\square$

**Remark 3.2.** For instance, consider  $N = 2$ ,  $\varphi(z_1, z_2) = (0, z_2)$  and  $m(z_1, z_2) = z_1$ . Then, for  $f(z_1, z_2) = z_1$ , we obtain  $m(f \circ \varphi)(z_1, z_2) = z_1 f(\varphi(z_1, z_2)) = z_1 f(0, z_2) = 0$ . Hence,  $0 \in \sigma_p(W_{m,\varphi})$ .

**Proposition 3.3.** *Let  $\varphi$  be elliptic non automorphic, such that  $\varphi(0) = 0$  and  $\varphi \not\equiv 0$ . If  $m(0) = 0$ , then*

$$\sigma(W_{m,\varphi}) = \{0\}.$$

*Proof.* The proof is exactly the same as in [2, Theorem 4.8].  $\square$

**Remark 3.4.** If  $\varphi \equiv 0$  and  $m(0) = 0$ , we easily show that

$$\sigma_p(W_{m,\varphi}) = \sigma(W_{m,\varphi}) = \{0\}.$$

Indeed, in this case,  $W_{m,\varphi}(f) = f(0)m$ . Thus, for all  $g \in \text{Hol}(\mathbb{B}_N)$  and  $\mu \neq 0$ , the map

$$h = -\frac{1}{\mu} \left[ \frac{g(0)}{\mu} m + g \right]$$

satisfies  $h(0)m - \mu h = g$ , so  $\mu \notin \sigma(W_{m,\varphi})$ . In addition, if we denote  $e_1(z) = z_1$ , we have  $e_1(0) = 0$ . Hence, we obtain  $W_{m,\varphi}(e_1) = 0$ , so  $0 \in \sigma_p(W_{m,\varphi})$ .

## 4. ELLIPTIC ATTRACTIVE SYMBOLS

Throughout the rest of the paper we assume that  $m(0) \neq 0$ . When  $\varphi$  is elliptic attractive, by Denjoy-Wolff theorem,  $\varphi$  is not bijective. Moreover, using [15, Lemma 4.1],  $\|\varphi'(0)\| < 1$ . We begin with the two following results, proved in the one-dimensional case in [2]. The proofs are going along the same lines in our context, the notation  $|\cdot|$  describing the euclidean norm on the ball instead of the modulus on the disc.

**Lemma 4.1.** *Let  $\varphi : \mathbb{B}_N \rightarrow \mathbb{B}_N$  be an elliptic attractive map such that  $\varphi(0) = 0$ . For all  $r \in (0, 1)$ , there exists  $\delta = \delta(r) \in (0, 1)$  such that for all  $|z| \leq r$  and  $n \geq 0$ ,*

$$|\varphi^{[n]}(z)| \leq \delta^n |z|.$$

**Proposition 4.2.** *Let  $\varphi$  be an elliptic attractive map such that  $\varphi(0) = 0$ , and  $m \in \text{Hol}(\mathbb{D})$  such that  $m(0) \neq 0$ . The sequence  $(w_n)_{n \geq 1}$  defined by*

$$w_n(z) = \frac{m_n(z)}{m(0)^n}, \quad m_n(z) = \prod_{k=0}^{n-1} m(\varphi^{[k]}(z))$$

*converges uniformly on all compact subsets of  $\mathbb{B}_N$  to a map  $w$ , which is the only one to satisfy*

$$m(w \circ \varphi) = m(0)w, \quad w(0) = 1.$$

The function  $w$  is called *weighted Koenigs' map of  $\varphi$  and  $m$* .

We now consider different situations, depending on the behaviour of  $\varphi'(0)$ .

**4.1. Invertible Jacobian at 0.** Let us recall Koenigs' theorem in several variables ([6, 15, ?, 19]), generalization of the result proved in 1884 in [11]. Denote by  $\lambda = (\lambda_1, \dots, \lambda_N)$  the diagonal of the matrix  $\varphi'(0)$ .

**Definition 4.3.** We say that the eigenvalues are *resonant* if there exist  $j \in \{1, \dots, N\}$  and  $k_1, \dots, k_N \in \mathbb{N}$  such that  $k_1 + \dots + k_N \geq 2$  and

$$\lambda_1^{k_1} \times \dots \times \lambda_N^{k_N} = \lambda_j.$$

**Theorem 4.4.** *Let  $\varphi : \mathbb{B}_N \rightarrow \mathbb{B}_N$  be an elliptic attractive map, such that  $\varphi(0) = 0$  and  $\varphi'(0)$  is invertible and the eigenvalues of  $\varphi'(0)$  are **not** resonant. Then there exists a holomorphic function  $\kappa : \mathbb{B}_N \rightarrow \mathbb{C}^N$  such that  $\kappa'(0) = \text{Id}$  and*

$$(2) \quad \kappa \circ \varphi = \varphi'(0)\kappa.$$

The map  $\kappa$  is called *Koenigs' function of  $\varphi$* .

Changing the order of the eigenvalues of  $\varphi'(0)$ , this theorem allows us to find functions  $\eta_1, \dots, \eta_N \in \text{Hol}(\mathbb{B}_N) \setminus \{0\}$  such that

$$\eta_k \circ \varphi = \lambda_k \eta_k, \quad k = 1, \dots, N.$$

We obtain the following proposition.

**Proposition 4.5.** *Let  $\varphi : \mathbb{B}_N \rightarrow \mathbb{B}_N$  be an elliptic attractive map, such that  $\varphi(0) = 0$  and  $\varphi'(0)$  is invertible and the eigenvalues of  $\varphi'(0)$  are **not** resonant. Let  $m \in \text{Hol}(\mathbb{B}_N)$  such that  $m(0) \neq 0$ . Then,*

$$\sigma_p(W_{m,\varphi}) = \{m(0)\lambda^{\vec{j}} : \vec{j} \in \mathbb{N}_0^N\},$$

where  $\lambda = (\lambda_1, \dots, \lambda_N)$  is  $\varphi'(0)$  eigenvalues' vector.

*Proof.* Let  $\vec{j} \in \mathbb{N}_0^N$ . Consider  $f = w\eta^{\vec{j}}$ , with  $w$  defined in Proposition 4.2. Then

$$\begin{aligned} m(f \circ \varphi) &= m(w \circ \varphi) \left( \prod_{k=1}^N (\eta_k \circ \varphi)^{j_k} \right) \\ &= m(0)w \left( \prod_{k=1}^N (\lambda_k \eta_k)^{j_k} \right) = m(0)\lambda^{\vec{j}}w\eta^{\vec{j}} = m(0)\lambda^{\vec{j}}f. \end{aligned}$$

Hence, for all  $\vec{j} \in \mathbb{N}_0^N$ ,  $m(0)\lambda^{\vec{j}} \in \sigma_p(W_{m,\varphi})$ .

Conversely, if  $\mu \notin \{m(0)\lambda^{\vec{j}} : \vec{j} \in \mathbb{N}_0^N\}$ , and if  $f \in \text{Hol}(\mathbb{B}_N)$  satisfies  $f \not\equiv 0$  and  $m(f \circ \varphi) = \mu f$ , we show that for all  $\vec{j} \in \mathbb{N}_0^N$ ,  $f^{(\vec{j})}(0) = 0$ .

- Note that  $[m(f \circ \varphi)](0) = m(0)f(0) = \mu f(0)$ . Since  $\mu \neq m(0)$ , we get  $f(0) = 0$ .
- Assume that for all  $\vec{i} \prec \vec{j}$ ,  $f^{(\vec{i})}(0) = 0$ . Then, by Lemma 2.4 and the general Leibniz rule ([17]),

$$\begin{aligned} (3) \quad [m(f \circ \varphi)]^{(\vec{j})}(0) &= \sum_{\vec{\alpha} \leq \vec{j}} \binom{\vec{j}}{\vec{\alpha}} m^{(\vec{j}-\vec{\alpha})}(0) (f \circ \varphi)^{(\vec{\alpha})}(0) \\ &= m(0)(f \circ \varphi)^{(\vec{j})}(0) = m(0)\lambda^{\vec{j}}f^{(\vec{j})}(0). \end{aligned}$$

Indeed, if  $\vec{\alpha} \leq \vec{j}$ , then  $\vec{\alpha} \preceq \vec{j}$ . Therefore, since  $f \in \text{Hol}_{\vec{j}_-}(\mathbb{B}_N) \subset \text{Hol}_{\vec{\alpha}_-}(\mathbb{B}_N)$ , by Lemma 2.4,  $(f \circ \varphi)^{(\vec{\alpha})}(0) = 0$ , unless  $\vec{\alpha} = \vec{j}$ . Finally,  $m(0)\lambda^{\vec{j}}f^{(\vec{j})}(0) = \mu f^{(\vec{j})}(0)$ , and since  $\mu \neq m(0)\lambda^{\vec{j}}$ , we obtain  $f^{(\vec{j})}(0) = 0$ .

We deduce that  $f \equiv 0$ , a contradiction. Hence,  $\sigma_p(W_{m,\varphi}) = \{m(0)\lambda^{\vec{j}} : \vec{j} \in \mathbb{N}_0^N\}$ .  $\square$

If  $\varphi'(0)$  is diagonal, the Koenigs' functions associated with  $\varphi$  satisfy

$$\eta_k \circ \varphi = \lambda_k \eta_k, \quad \eta_k(0) = 0, \quad \frac{\partial \eta_k}{\partial z_\ell}(0) = \begin{cases} 1 & \text{if } \ell > k, \\ 0 & \text{if } \ell = k. \end{cases}$$

Let us start by establishing a property on the maps  $\eta_k$ .

**Lemma 4.6.** *If  $\vec{i}, \vec{j} \in \mathbb{N}_0^N$  satisfy  $\vec{i} \preceq \vec{j}$ , then*

$$(\eta^{\vec{j}})^{(\vec{i})}(0) = \begin{cases} 0 & \text{if } \vec{i} \prec \vec{j}, \\ \vec{j}! & \text{if } \vec{i} = \vec{j}. \end{cases}$$

*Proof.* Note that for all  $k \in \{1, \dots, N\}$ , the Maclaurin series of  $\eta_k$  is

$$\eta_k(z) = \sum_{\ell=1}^k \alpha_{k\ell} z_\ell + o(|z|), \quad \text{with } \alpha_{k\ell} \in \mathbb{C}.$$

Hence, by the multinomial Theorem,

$$\begin{aligned} \eta^{\vec{j}}(z) &= \prod_{k=1}^N \left( \sum_{\ell=1}^k \alpha_{k\ell} z_\ell \right)^{j_k} + o(|z|^{|\vec{j}|}) = \prod_{k=1}^N \left( \sum_{|\vec{p}_k|=j_k} \binom{j_k}{\vec{p}_k} (\alpha_k z)^{\vec{p}_k} \right) + o(|z|^{|\vec{j}|}) \\ &= \sum_{|\vec{p}_1|=j_1} \cdots \sum_{|\vec{p}_N|=j_N} \beta_p z^{\vec{r}} + o(|z|^{|\vec{j}|}), \end{aligned}$$

with  $\beta_p \in \mathbb{C}$ ,  $p_{k\ell} = 0$  if  $k < \ell$ , and

$$\vec{r} = (p_{11} + \cdots + p_{N1}, p_{22} + \cdots + p_{N2}, \cdots, p_{NN}).$$

However, for  $\vec{p}_1, \dots, \vec{p}_N$  satisfying these assumptions,  $|\vec{r}| = |\vec{j}|$ . Also, note that  $p_{11} = j_1$ , so  $\vec{r} \succneq \vec{j}$ . This gives the result for  $\vec{v} \prec \vec{j}$ . Moreover, the only way to get  $\vec{r} = \vec{j}$  is by taking  $p_{k\ell} = j_k \mathbb{1}_{\{k=\ell\}}$ . Hence, we conclude for the case  $\vec{v} = \vec{j}$  since  $\beta_p = 1$ , because  $\alpha_{kk} = 1$  for all  $k$ .  $\square$

Hence, we get the following theorem. For  $\vec{j} \in \mathbb{N}^N$ , let us define

$$X_{\vec{j}} = \text{Vect}(w\eta^{\vec{v}} : \vec{v} \preceq \vec{j}), \quad \text{Hol}_{\vec{j}}(\mathbb{B}_N) = \{f \in \text{Hol}(\mathbb{B}_N) : \forall \vec{v} \preceq \vec{j}, f^{(\vec{v})}(0) = 0\}.$$

**Theorem 4.7.** *Let  $\varphi : \mathbb{B}_N \rightarrow \mathbb{B}_N$  be an elliptic attractive map such that  $\varphi(0) = 0$  and  $\varphi'(0)$  is diagonal, invertible. If  $m(0) \neq 0$  then*

- (i) *For all  $\vec{j} \in \mathbb{N}_0^N$ ,  $\text{Hol}(\mathbb{B}_N) = X_{\vec{j}} \oplus \text{Hol}_{\vec{j}}(\mathbb{B}_N)$ .*
- (ii) *For all  $\vec{j} \in \mathbb{N}_0^N$ , if we define the operators  $P_{\vec{j}}$  and  $Q_{\vec{j}}$  by*

$$P_{\vec{0}}(f) = f(0)w, \quad P_{\vec{j}}(f) = \frac{1}{\vec{j}!} [f - Q_{\vec{j}^-}(f)]^{(\vec{j})}(0) \times w\eta^{\vec{j}}, \quad Q_{\vec{j}}(f) = \sum_{\vec{v} \preceq \vec{j}} P_{\vec{v}}(f),$$

*then  $Q_{\vec{j}}$  is the projection on  $X_{\vec{j}}$  in parallel to  $\text{Hol}_{\vec{j}}(\mathbb{B}_N)$ .*

- (iii) *For all  $\vec{j} \in \mathbb{N}_0^N$ ,  $P_{\vec{j}} \circ W_{m,\varphi} = W_{m,\varphi} \circ P_{\vec{j}} = m(0)\lambda^{\vec{j}}P_{\vec{j}}$ .*

*Proof.* We prove most of the results by induction.

- (i) First, we show that  $X_{\vec{j}} \cap \text{Hol}_{\vec{j}}(\mathbb{B}_N) = \{0\}$ . Indeed, let

$$f = \sum_{\vec{v} \preceq \vec{j}} c_{\vec{v}} w \eta^{\vec{v}} \in \text{Hol}_{\vec{j}}(\mathbb{B}_N).$$

Then,  $c_{\vec{0}} = c_{\vec{0}} w(0) = f(0) = 0$ . Let  $\vec{k} \preceq \vec{j}$  such that for all  $\vec{v} \prec \vec{k}$ ,  $c_{\vec{v}} = 0$ . By the general Leibniz rule,

$$f^{(\vec{k})}(0) = \sum_{\vec{k} \preceq \vec{v} \preceq \vec{j}} c_{\vec{v}} (w\eta^{\vec{v}})^{(\vec{k})}(0) = \sum_{\vec{k} \preceq \vec{v} \preceq \vec{j}} \sum_{\vec{\alpha} \leq \vec{k}} c_{\vec{v}} \binom{\vec{k}}{\vec{\alpha}} w^{(\vec{k}-\vec{\alpha})}(0) (\eta^{\vec{v}})^{(\vec{\alpha})}(0) = c_{\vec{v}} w(0) \vec{k}!.$$

Since  $f \in \text{Hol}_{\vec{j}}(\mathbb{B}_N)$ , we obtain  $c_{\vec{k}} = 0$ . Hence,  $f \equiv 0$ .

Now, let us show that  $X_{\vec{j}} + \text{Hol}_{\vec{j}}(\mathbb{B}_N) = \text{Hol}(\mathbb{B}_N)$ .

- The inclusion  $\subset$  is trivial.

- Note that  $\text{Im}(Q_{\vec{j}}) \subset X_{\vec{j}}$ . Let  $f \in \text{Hol}(\mathbb{B}_N)$ . Then,

$$[f - Q_{\vec{0}}(f)](0) = [f - P_{\vec{0}}(f)](0) = f(0) - f(0)w(0) = 0,$$

because  $w(0) = 1$ . Hence,  $f - Q_{\vec{0}}(f) \in \text{Hol}_{\vec{0}}(\mathbb{B}_N)$ , so  $f \in W_{\vec{0}} + \text{Hol}_{\vec{0}}(\mathbb{B}_N)$ . Assume that  $f - Q_{\vec{j}_-}(f) \in \text{Hol}_{\vec{j}_-}(\mathbb{B}_N)$ . Then,

$$Q_{\vec{j}}(f) = Q_{\vec{j}_-}(f) + P_{\vec{j}}(f) = Q_{\vec{j}_-}(f) + \frac{1}{\vec{j}!} [f - Q_{\vec{j}_-}(f)]^{(\vec{j})}(0) \times w\eta^{\vec{j}}.$$

Let  $\vec{i} \preceq \vec{j}$ . By definition of  $Q_{\vec{j}}$ ,

$$[f - Q_{\vec{j}}(f)]^{(\vec{i})}(0) = [f - Q_{\vec{j}_-}(f)]^{(\vec{i})}(0) - \frac{1}{\vec{j}!} [f - Q_{\vec{j}_-}(f)]^{(\vec{j})}(0) \times (w\eta^{\vec{j}})^{(\vec{i})}(0).$$

If  $\vec{i} \prec \vec{j}$ , using the properties of  $\eta$  and the induction hypothesis, we get  $[f - Q_{\vec{j}_-}(f)]^{(\vec{i})}(0) = (w\eta^{\vec{j}})^{(\vec{i})}(0) = 0$ , so  $[f - Q_{\vec{j}}(f)]^{(\vec{i})}(0) = 0$ . If  $\vec{i} = \vec{j}$ , then

$$[f - Q_{\vec{j}}(f)]^{(\vec{j})}(0) = [f - Q_{\vec{j}_-}(f)]^{(\vec{j})}(0) - \frac{1}{\vec{j}!} [f - Q_{\vec{j}_-}(f)]^{(\vec{j})}(0) \times w(0)\vec{j}! = 0,$$

using Leibniz rule, and noticing that  $w(0) = 1$ .

Finally,  $f - Q_{\vec{j}}(f) \in \text{Hol}_{\vec{j}}(\mathbb{B}_N)$ , so  $f \in X_{\vec{j}} + \text{Hol}_{\vec{j}}(\mathbb{B}_N)$ .

- (ii) Using (i), if  $f \in \text{Hol}(\mathbb{B}_N)$ , then we can write  $f$  as

$$f = g + h, \quad g \in X_{\vec{j}}, \quad h \in \text{Hol}_{\vec{j}}(\mathbb{B}_N),$$

where  $g$  and  $h$  are uniquely determined. Moreover,  $Q_{\vec{j}}(f) = g$ , so  $Q_{\vec{j}}$  is indeed the projection on  $X_{\vec{j}}$  in parallel to  $\text{Hol}_{\vec{j}}(\mathbb{B}_N)$ .

- (iii) First, we show that for all  $\vec{j} \in \mathbb{N}_0^N$ ,  $W_{m,\varphi} \circ P_{\vec{j}} = m(0)\lambda^{\vec{j}}P_{\vec{j}}$ . Indeed,

$$\begin{aligned} (W_{m,\varphi} \circ P_{\vec{j}})(f) &= \frac{1}{\vec{j}!} [f - Q_{\vec{j}_-}(f)]^{(\vec{j})}(0) \times m(w \circ \varphi)(\eta^{\vec{j}} \circ \varphi) \\ &= \frac{1}{\vec{j}!} [f - Q_{\vec{j}_-}(f)]^{(\vec{j})}(0) \times (m(0)w)(\lambda^{\vec{j}}\eta^{\vec{j}}) = m(0)\lambda^{\vec{j}}P_{\vec{j}}(f). \end{aligned}$$

Then, let us prove that for all  $\vec{j} \in \mathbb{N}_0^N$ ,  $Q_{\vec{j}} \circ W_{m,\varphi} = W_{m,\varphi} \circ Q_{\vec{j}}$ . To do so, it suffices to show that  $X_{\vec{j}}$  and  $\text{Hol}_{\vec{j}}(\mathbb{B}_N)$  are invariant by  $W_{m,\varphi}$ .

- Let  $\vec{i} \preceq \vec{j}$ . Since  $m(w\eta^{\vec{i}} \circ \varphi) = m(0)\lambda^{\vec{i}}w\eta^{\vec{i}}$ , we obtain  $m(w\eta^{\vec{i}} \circ \varphi) \in X_{\vec{j}}$ , and  $W_{m,\varphi}(X_{\vec{j}}) \subset X_{\vec{j}}$ .
- Let  $f \in \text{Hol}_{\vec{j}}(\mathbb{B}_N)$ . Then,  $m(f \circ \varphi)(0) = m(0)f(0) = 0$ . In addition, for all  $0 \prec \vec{k} \preceq \vec{j}$ ,  $f \in \text{Hol}_{\vec{k}_-}(\mathbb{B}_N)$ . Hence,

$$[m(f \circ \varphi)]^{(\vec{k})}(0) = m(0)\lambda^{\vec{k}}f^{(\vec{k})}(0) = 0.$$

Finally,  $W_{m,\varphi}(f) \in \text{Hol}_{\vec{j}}(\mathbb{B}_N)$ , so  $W_{m,\varphi}(\text{Hol}_{\vec{j}}(\mathbb{B}_N)) \subset \text{Hol}_{\vec{j}}(\mathbb{B}_N)$ .

We deduce that for all  $f \in \text{Hol}(\mathbb{B}_N)$ , there exist  $g, h \in \text{Hol}_{\vec{j}}(\mathbb{B}_N)$  such that

$$m(f \circ \varphi) = m(Q_{\vec{j}}(f) \circ \varphi) + m(g \circ \varphi) = Q_{\vec{j}}(m(f \circ \varphi)) + h.$$

However, all maps are written in a unique way in  $X_{\vec{j}} \oplus \text{Hol}_{\vec{j}}(\mathbb{B}_N)$ , so

$$(Q_{\vec{j}} \circ W_{m,\varphi})(f) = Q_{\vec{j}}(m(f \circ \varphi)) = m(Q_{\vec{j}}(f) \circ \varphi) = (W_{m,\varphi} \circ Q_{\vec{j}})(f).$$

Finally, we show that  $P_{\vec{j}} \circ W_{m,\varphi} = W_{m,\varphi} \circ P_{\vec{j}}$ . For  $\vec{j} = \vec{0}$ , it is trivial since  $P_{\vec{0}} = Q_{\vec{0}}$ . Otherwise, for  $\vec{j} \in \mathbb{N}_0^N$ , we write

$$P_{\vec{j}} \circ W_{m,\varphi} = (Q_{\vec{j}} - Q_{\vec{j}_-}) \circ W_{m,\varphi} = W_{m,\varphi} \circ (Q_{\vec{j}} - Q_{\vec{j}_-}) = W_{m,\varphi} \circ P_{\vec{j}}. \quad \square$$

We now consider two auxiliary lemmas.

**Lemma 4.8.** *Let  $\varphi$  be an elliptic attractive self-map of  $\mathbb{B}_N$ , such that  $\varphi(0) = 0$ . Let  $g \in \text{Hol}(\mathbb{B}_N)$ , and  $\lambda \neq 0$ . If there exist  $\varepsilon \in (0, 1)$  and  $f \in \text{Hol}(B(0, \varepsilon))$  such that for all  $|z| < \varepsilon$ ,*

$$\lambda f(z) - m(z)f(\varphi(z)) = g(z),$$

*then there exists a function  $\tilde{f} \in \text{Hol}(\mathbb{B}_N)$  such that  $\tilde{f}|_{B(0, \varepsilon)} = f$ , and for all  $z \in \mathbb{B}_N$ ,*

$$\lambda \tilde{f}(z) - m(z)\tilde{f}(\varphi(z)) = g(z).$$

*Proof.* See [15, Lemma 4.7] or [4, Lemma 4.2], for instance. The only thing left is adding  $m$  everywhere.  $\square$

**Lemma 4.9.** *For all  $\lambda_0 > 0$ , there exist  $0 < \varepsilon < 1$  and  $p \in \mathbb{N}_0$  such that for all  $|\lambda| > \lambda_0$  and  $g \in \text{Hol}_p(\mathbb{B}_N)$ ,*

$$\sum_{n \geq 0} \frac{m_n(g \circ \varphi^{[n]})}{\lambda^n}$$

*converges uniformly on  $\overline{B(0, \varepsilon)}$ .*

*Proof.* Let  $\nu > 1$  and  $\|\varphi'(0)\| < \zeta < 1$ . There exists  $p \in \mathbb{N}_0$  such that  $\nu |m(0)| \zeta^{p+1} < \lambda_0$ . Moreover, we know that there exists  $\varepsilon \in (0, 1)$  such that for all  $|z| \leq \varepsilon$ ,

$$|\varphi(z)| \leq \zeta |z| < \varepsilon \quad \text{and} \quad |m(z)| \leq \nu |m(0)|.$$

Using Schwarz's lemma, for all  $n \in \mathbb{N}_0$ ,  $|\varphi^{[n]}(z)| \leq \zeta^n |z| < \varepsilon$ , and since  $g \in \text{Hol}_p(\mathbb{B}_N)$ , there exists  $C > 0$  such that for  $|z| \leq \varepsilon$ ,  $|g(z)| \leq C |z|^{p+1}$ . Finally, for  $|z| \leq \varepsilon$ ,

$$\left| \frac{m_n(z)g(\varphi^{[n]}(z))}{\lambda^n} \right| \leq \frac{C\nu^n |m(0)|^n}{|\lambda|^n} |\varphi^{[n]}(z)|^{p+1} \leq \left( \frac{\nu |m(0)| \zeta^{p+1}}{\lambda_0} \right)^n C \varepsilon^{p+1}.$$

Because  $\nu |m(0)| \zeta^{p+1} < \lambda_0$ , the series converges normally on  $\overline{B(0, \varepsilon)}$ .  $\square$

We finally reach the following result.

**Proposition 4.10.** *Let  $\varphi : \mathbb{B}_N \rightarrow \mathbb{B}_N$  be an elliptic attractive map, such that  $\varphi(0) = 0$  and  $\varphi'(0)$  is diagonal, invertible and the eigenvalues  $\lambda_1, \dots, \lambda_N$  of  $\varphi'(0)$  are **not** resonant. Assume that  $m \in \text{Hol}(\mathbb{D})$  satisfies  $m(0) \neq 0$ . Then*

$$\sigma(W_{m,\varphi}) = \sigma_p(W_{m,\varphi}) \cup \{0\}.$$

*Proof.* The inclusion  $\supset$  is trivial, since  $\varphi \notin \text{Aut}(\mathbb{B}_N)$ . Let us focus on the other one.

Let  $\mu \notin \sigma_p(W_{m,\varphi}) \cup \{0\}$ . By Lemma 4.9, there exist  $0 < \varepsilon < 1$  and  $p \in \mathbb{N}_0$  such that for all  $g \in \text{Hol}_p(\mathbb{B}_N)$ ,

$$\sum_{n \geq 0} \frac{m_n(g \circ \varphi^{[n]})}{\lambda^n}$$

converges uniformly on  $\overline{B(0, \varepsilon)}$ .

In addition, in a same way as in Theorem 4.7,  $\text{Hol}(\mathbb{B}_N) = X_p \oplus \text{Hol}_p(\mathbb{B}_N)$ , with  $X_p = \text{Vect}(w\eta^{\vec{i}} : |\vec{i}| \leq p)$  and  $\text{Hol}_p(\mathbb{B}_N) = \{f \in \text{Hol}(\mathbb{B}_N) : \forall |\vec{i}| \leq p, f^{(\vec{i})}(0) = 0\}$ . The two subspaces are invariant by  $W_{m, \varphi}$ . Hence,

$$W_{m, \varphi} = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix},$$

with  $S \in \mathcal{L}(X_p)$  and  $T \in \mathcal{L}(\text{Hol}_p(\mathbb{B}_N))$ .

- If we consider the basis  $(w\eta^{\vec{j}} : |\vec{j}| \leq p)$  of  $X_p$  and if we denote  $\lambda = (\lambda_1, \dots, \lambda_N)$ ,

$$S = \text{diag}(m(0)\lambda^{\vec{j}} : |\vec{j}| \leq p).$$

Since  $\mu \notin \{m(0)\lambda^{\vec{j}} : |\vec{j}| \leq p\}$ ,  $S - \mu\text{Id}$  is invertible.

- For all  $g \in \text{Hol}_p(\mathbb{B}_N)$ , set

$$h = -\frac{1}{\mu} \sum_{n \geq 0} \frac{m_n(g \circ \varphi^{[n]})}{\lambda^n} \in \text{Hol}(B(0, \varepsilon)).$$

Then, on  $B(0, \varepsilon)$ ,

$$m(h \circ \varphi) - \mu h = \sum_{n \geq 0} \frac{m_n(g \circ \varphi^{[n]})}{\lambda^n} - \sum_{n \geq 0} \frac{m_{n+1}(g \circ \varphi^{[n+1]})}{\lambda^{n+1}} = m_0 g = g.$$

Hence, by Lemma 4.8, there exists  $\tilde{h} \in \text{Hol}(\mathbb{B}_N)$  such that

$$m(\tilde{h} \circ \varphi) - \mu \tilde{h} = g.$$

Moreover,  $\tilde{h} \in \text{Hol}_p(\mathbb{B}_N)$ . Indeed, if we could write  $\tilde{h} = s + t$ , with  $0 \neq s \in W_p$  and  $t \in \text{Hol}_p(\mathbb{B}_N)$ , we would obtain

$$m(\tilde{h} \circ \varphi) - \mu \tilde{h} = (W_{m, \varphi}(s) - \mu s) + (W_{m, \varphi}(t) - \mu t) = g,$$

with  $(W_{m, \varphi}(s) - \mu s) \neq 0$ , since  $s \neq 0$ . This is impossible because  $g \in \text{Hol}_p(\mathbb{B}_N)$ . Finally,  $T - \mu\text{Id}$  is bijective, and invertible.

We conclude that  $W_{m, \varphi} - \mu\text{Id}$  is invertible.  $\square$

**4.2. Non-invertible Jacobian at 0, with 0 as unique eigenvalue.** In this case, we can also describe the spectra of  $W_{m, \varphi}$ .

**Proposition 4.11.** *Let  $\varphi : \mathbb{B}_N \rightarrow \mathbb{B}_N$  be an elliptic attractive map such that  $\varphi(0) = 0$ ,  $\varphi \neq 0$ ,  $\varphi'(0)$  is not invertible, and 0 is the only eigenvalue of  $\varphi'(0)$ . Let  $m \in \text{Hol}(\mathbb{B}_N)$  such that  $m(0) \neq 0$ . Then,*

$$\{m(0)\} \subset \sigma_p(W_{m, \varphi}) \subset \{0, m(0)\}.$$

*Proof.* Let  $\mu \notin \{m(0), 0\}$ . If  $f \neq 0$  satisfies  $m(f \circ \varphi) = \mu f$ , then

- First,  $\mu f(0) = [m(f \circ \varphi)](0) = m(0)f(0)$ , so  $(m(0) - \mu)f(0) = 0$ . Since  $\mu \neq m(0)$ , we obtain  $f(0) = 0$ .

- Assume that for all  $\vec{i} \prec \vec{j}$ ,  $f^{(\vec{i})}(0) = 0$ . Then,

$$\mu f^{(\vec{j})}(0) = [m(f \circ \varphi)]^{(\vec{j})}(0) = m(0)(f \circ \varphi)^{(\vec{j})}(0) = 0.$$

Hence,  $f^{(\vec{j})}(0) = 0$ , since  $\mu \neq 0$ .

Finally,  $f \equiv 0$ , which is impossible. Hence,  $\sigma(W_{m,\varphi}) \subset \{m(0), 0\}$ .

Conversely, the map  $w$  defined in Proposition 4.2 is an eigenvector of  $W_{m,\varphi}$  for the eigenvalue  $m(0)$ . Moreover, we give an example where  $0 \in \sigma_p(W_{m,\varphi})$ : assume that  $\varphi_k = 0$  for some  $k \in \{1, \dots, N\}$ . Then, for  $f(z) = z_k$ ,  $f \circ \varphi = 0$ , so  $W_{m,\varphi}(f) = 0$ .  $\square$

**Theorem 4.12.** *Let  $\varphi : \mathbb{B}_N \rightarrow \mathbb{B}_N$  be an elliptic attractive map such that  $\varphi(0) = 0$  and  $\varphi'(0)$  is not invertible. If  $m(0) \neq 0$ , and 0 is the only eigenvalue of  $\varphi'(0)$ , then*

$$\sigma(W_{m,\varphi}) = \{0, m(0)\}.$$

*Proof.* Note that the diagonal of  $\varphi'(0)$  has only zeroes, so the matrix  $\varphi'(0)$  is nilpotent. Hence, there exists  $n_0 \in \mathbb{N}_0$  such that

$$(\varphi^{[n_0]})'(0) = (\varphi'(0))^{n_0} = 0.$$

Let  $\mu \in \mathbb{C} \setminus \{0, m(0)\}$ . We consider two types of functions.

- Let  $c \in \mathbb{C}^*$ . If  $w$  is the weighted Koenigs' map of  $\varphi$  and  $m$ , then

$$W_{m,\varphi}(w) - \mu w = (W_{m,\varphi}(w) - m(0)w) + (m(0)w - \mu w) = m(0)w - \mu w.$$

Thus, for all  $c \in \mathbb{C}^*$ , if we set  $f_c = cw/(m(0) - \mu)$ , we obtain  $W_{m,\varphi}(f_c) - \mu f_c = cw$ .

- Let  $g \in \text{Hol}(\mathbb{B}_N)$  such that  $g(0) = 0$ . We know that there exists  $0 < \varepsilon < 1$  a constant  $d > 1$  such that for all  $|z| \leq \varepsilon$ ,

$$|g(z)| \leq d|z|, \quad |\varphi(z)| \leq d|z|, \quad |\varphi^{[n_0]}(z)| \leq d|z|^2.$$

Using Schwarz's lemma, for all  $k \in \mathbb{N}_0$ ,  $k \geq 3$  and  $|z| \leq \varepsilon$ ,

$$|\varphi^{[kn_0]}(z)| \leq d^k |z|^{2^k} \leq d^k |z|^{2k+2}.$$

For all  $n \in \mathbb{N}_0$ , if  $n \geq 3n_0$ , we write  $n = kn_0 + p$ , with  $k \geq 3$  and  $0 \leq p < n_0$ . Hence,  $k = (n - p)/n_0 \geq n/n_0 - 1$ . If we set  $\alpha = \sup(|m(z)| : |z| \leq 1/2)$ , we get

$$\begin{aligned} \left| \frac{m_n(z)g(\varphi^{[n]}(z))}{\mu^n} \right| &\leq \frac{d\alpha^n |\varphi^{[n]}(z)|}{|\mu|^n} \leq \frac{d^{p+1}\alpha^n |\varphi^{[kn_0]}(z)|}{|\mu|^n} \\ &\leq \frac{d^{k+p+1}\alpha^n |z|^{2k+2}}{|\mu|^n} \leq d \frac{d^n \alpha^n |z|^{2n/n_0}}{|\mu|^n} = d \left( \frac{d\alpha |z|^{2/n_0}}{|\mu|} \right)^n. \end{aligned}$$

We only have to choose  $\tilde{\varepsilon} \in (0, \varepsilon)$ , so that  $d\alpha\tilde{\varepsilon}^{2/n_0} < |\mu|$ , to have

$$h = -\frac{1}{\mu} \sum_{n \geq 0} \frac{m_n(z)g(\varphi^{[n]}(z))}{\mu^n} \in \text{Hol}(B(0, \tilde{\varepsilon})).$$

Moreover, the same calculations as in Proposition 4.10 gives

$$m(h \circ \varphi) - \mu h = g \text{ on } B(0, \tilde{\varepsilon}).$$

Using Lemma 4.8, there exists  $\tilde{h} \in \text{Hol}(\mathbb{B}_N)$  such that

$$m(\tilde{h} \circ \varphi) - \mu\tilde{h} = g.$$

If  $\eta \in \text{Hol}(\mathbb{B}_N)$ , then  $g = \eta - \eta(0)w$  satisfies  $g(0) = 0$ , and

$$m[(f_{\eta(0)} + \tilde{h}) \circ \varphi] - \mu(f_{\eta(0)} + \tilde{h}) = \eta(0)w + g = \eta.$$

Finally,  $W_{m,\varphi} - \mu\text{Id}$  is invertible, and  $\mu \notin \sigma(W_{m,\varphi})$ , so  $\sigma(W_{m,\varphi}) \subset \{0, m(0)\}$ . To finish the proof, we use Proposition 4.11, and the fact that  $\varphi$  is not bijective.  $\square$

**Remark 4.13.** If  $\varphi \equiv 0$  and  $m(0) \neq 0$ , we show that

$$\sigma_p(W_{m,\varphi}) = \sigma(W_{m,\varphi}) = \{0, m(0)\}.$$

Indeed,  $W_{m,\varphi}(f) = f(0)m$ . Hence, for all  $g \in \text{Hol}(\mathbb{B}_N)$  and  $\mu \notin \{0, m(0)\}$ , the map

$$h = \frac{1}{\mu} \left[ \frac{g(0)}{m(0) - \mu} m - g \right]$$

satisfies  $h(0)m - \mu h = g$ , so  $\mu \notin \sigma(W_{m,\varphi})$ . Moreover,

- Denote  $e_1(z) = z_1$ . Then  $e_1(0) = 0$ , so  $W_{m,\varphi}(e_1) = 0$ , and  $0 \in \sigma_p(W_{m,\varphi})$ .
- Since  $W_{m,\varphi}(m) - m(0)m = m(0)m - m(0)m = 0$ , we get  $m(0) \in \sigma_p(W_{m,\varphi})$ .

**4.3. General results.** In general, we have the following results.

**Proposition 4.14.** *Let  $\varphi : \mathbb{B}_N \rightarrow \mathbb{B}_N$  be an elliptic attractive map such that  $\varphi(0) = 0$  and  $\varphi \not\equiv 0$ . Let  $\lambda_1, \dots, \lambda_p$  be the non-zero eigenvalues of  $\varphi'(0)$ . Let  $m \in \text{Hol}(\mathbb{B}_N)$  such that  $m(0) \neq 0$ . Then,*

$$\{m(0)\} \subset \sigma_p(W_{m,\varphi}) \subset \left\{ m(0) \prod_{k=1}^p \lambda_k^{j_k} : j_1, \dots, j_p \in \mathbb{N}_0 \right\} \cup \{0, m(0)\}.$$

*Proof.* We know that the map  $w$  defined in Proposition 4.2 satisfies  $W_{m,\varphi}(w) = m(0)w$ , so  $m(0) \in \sigma_p(W_{m,\varphi})$ . Moreover, in a same way as in Proposition 4.11, 0 can be an eigenvalue of  $W_{m,\varphi}$ .

Let  $\mu \notin \{0\} \cup \{m(0)\lambda^{\vec{j}} : \vec{j} \in \mathbb{N}_0^p\}$ . If  $\mu \in \sigma_p(W_{m,\varphi})$ , then there exists  $f \in \text{Hol}(\mathbb{B}_N) \setminus \{0\}$  such that  $m(f \circ \varphi) = \mu f$ . Once again by induction, using Lemma 2.4 and the general Leibniz rule, we prove that  $f^{(\vec{j})}(0) = 0$  for all  $\vec{j} \in \mathbb{N}_0^p$ . Hence,  $f \equiv 0$ , a contradiction. Thus,  $\mu \notin \sigma_p(W_{m,\varphi})$ .  $\square$

**Proposition 4.15.** *Let  $\varphi : \mathbb{B}_N \rightarrow \mathbb{B}_N$  be an elliptic attractive map such that  $\varphi(0) = 0$  and  $\varphi \not\equiv 0$ . Let  $m \in \text{Hol}(\mathbb{B}_N)$  such that  $m(0) \neq 0$ . Then,*

$$\{0, m(0)\} \subset \sigma(W_{m,\varphi}) \subset \overline{D(0, |m(0)| \|\varphi'(0)\|)} \cup \{m(0)\}.$$

*Proof.* Since  $\varphi \notin \text{Aut}(\mathbb{B}_N)$ ,  $W_{m,\varphi}$  is not invertible. Hence,  $0 \in \sigma(W_{m,\varphi})$ .

In addition,  $m(0) \in \sigma_p(W_{m,\varphi})$  by Proposition 4.14, so  $m(0) \in \sigma(W_{m,\varphi})$ .

Let  $\mu > |m(0)| \|\varphi'(0)\|$ ,  $\mu \neq m(0)$ . Then, similarly to Lemma 4.9, there exists  $\varepsilon > 0$  such that for all  $h \in \text{Hol}_0(\mathbb{B}_N)$ , the series  $\sum (h \circ \varphi^{[n]}) / \mu^n$  converges uniformly on  $\overline{B(0, \varepsilon)}$ .

For  $g \in \text{Hol}(\mathbb{B}_N)$ , let us define the functions  $h$  and  $f$  by

$$h(z) = g(z) - g(0)w(z), \quad f(z) = \frac{g(0)w(z)}{m(0) - \mu} - \frac{1}{\mu} \sum_{n \geq 0} \frac{m_n(z)(h \circ \varphi^{[n]})(z)}{\mu^n}.$$

Thus,  $f \in \text{Hol}(B(0, \varepsilon))$ . The same calculations as in Theorem 4.12 gives

$$m(z)(f \circ \varphi)(z) - \mu f(z) = g(0)w(z) + h(z) = g(z), \quad |z| < \varepsilon.$$

Hence,  $W_{m,\varphi} - \mu \text{Id}$  is bijective, so  $\mu \notin \sigma(W_{m,\varphi})$ .  $\square$

**Example 4.16.** Consider  $N = 2$ ,  $\varphi(z) = (z_1/3, z_1/3)$  et  $m(z) = 2 + z_1$ .

Point spectrum : Take  $w$  the map defined in Proposition 4.2, and  $f_k(z) = z_1^k$ . Then,

$$m(w \circ \varphi) = m(0)w = 2w, \quad m((wf_k) \circ \varphi) = m(w \circ \varphi)(f_k \circ \varphi) = 2w \frac{f_k}{3^k} = \frac{2}{3^k}(wf_k).$$

Hence, for all  $k \in \mathbb{N}_0$ ,  $2/3^k \in \sigma_p(W_{m,\varphi})$ . Moreover, if  $g(z) = z_1 - z_2$ , then  $g \circ \varphi = 0$ , so  $W_{m,\varphi}(g) = 0$ . Thus,  $0 \in \sigma_p(W_{m,\varphi})$ . Finally, using Proposition 4.14,

$$\sigma_p(W_{m,\varphi}) = \{2/3^k : k \in \mathbb{N}_0\} \cup \{0\}.$$

Spectrum : Let  $\mu \notin \sigma_p(W_{m,\varphi})$ , and  $g \in \text{Hol}(\mathbb{B}_2)$ . We search for  $f \in \text{Hol}(\mathbb{B}_2)$  such that  $m(f \circ \varphi) - \mu f = g$ . Write  $f = \sum a_{ij} z_1^i z_2^j$  and  $g = \sum b_{ij} z_1^i z_2^j$ . Then,

$$m(f \circ \varphi) = \sum_{i,j \geq 0} \frac{a_{ij}}{3^{i+j}} z_1^{i+j} (2 + z_1) = \sum_{i,j \geq 0} (\mu a_{ij} + b_{ij}) z_1^i z_2^j = \mu f + g.$$

Note that there is no  $z_2$  in  $m(f \circ \varphi)$ . Hence, if  $j \geq 1$ , since  $\mu \neq 0$ ,

$$\mu a_{ij} + b_{ij} = 0 \iff a_{ij} = -\mu^{-1} b_{ij}.$$

It remains to consider the case  $j = 0$ . Remark that

$$\begin{aligned} \sum_{i \geq 0} \frac{a_{i0}}{3^i} z_1^i (2 + z_1) &= \sum_{i \geq 0} (\mu a_{i0} + b_{i0}) z_1^i \\ \iff 2a_{00} + \sum_{i \geq 1} \left( \frac{2a_{i0}}{3^i} + \frac{a_{i-1,0}}{3^{i-1}} \right) z_1^i &= \mu a_{00} + b_{00} + \sum_{i \geq 1} (\mu a_{i0} + b_{i0}) z_1^i. \end{aligned}$$

Thus, since  $\mu \neq 2$ ,  $a_{00} = b_{00}/(2 - \mu)$ , and for all  $i \geq 1$ , since  $\mu \neq 2/3^i$ ,

$$a_{i0} = \frac{b_{i0} - 3^{1-i} a_{i-1,0}}{2/3^i - \mu}.$$

Denoting  $C = \min(|2/3^i - \mu| : i \geq 0) > 0$ , since  $3^{i-1} \leq 1$  for all  $i \geq 1$ , we obtain

$$|a_{i0}| \leq \frac{|b_{i0}|}{C} + \frac{|a_{i-1,0}|}{C} \leq \frac{|b_{i0}|}{C} + \frac{|b_{i-1,0}|}{C^2} + \frac{|a_{i-2,0}|}{C^2} \leq \dots \leq \frac{1}{C} \sum_{k=0}^i \frac{|b_{i-k,0}|}{C^k}.$$

However, the series  $\sum \frac{1}{C^k} z_1^k$  has radius of convergence  $C$ , and the series  $\sum |b_{k0}| z_1^k$  has radius of convergence 1, so the map  $f$  is defined on the ball  $B(0, C)$  by product series. Lemma 4.8 gives a function  $\tilde{f} \in \text{Hol}(\mathbb{B}_2)$  such that  $m(\tilde{f} \circ \varphi) - \mu \tilde{f} = g$ , so  $W_{m,\varphi} - \mu \text{Id}$  is invertible. We conclude that  $\sigma(W_{m,\varphi}) = \sigma_p(W_{m,\varphi})$ .

## 5. BIJECTIVE PERIODIC SYMBOLS

In this section, we consider  $\varphi$  a bijective elliptic map with fixed point at 0, that is to say,  $\varphi$  is a unitary matrix. Moreover, conjugating with an automorphism, it follows that

$$\varphi = D, \quad D = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_N}).$$

Inspired by [3], we will focus on *periodic automorphisms*.

**Definition 5.1.** A unitary diagonal matrix  $D = \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_N})$  is *periodic* if for all  $k \in \{1, \dots, N\}$ ,  $\theta_k \in 2\pi\mathbb{Q}$ .

If  $\varphi$  is a periodic automorphism, then there exists  $p \in \mathbb{N}$  such that

$$D^p = \text{Id}, \quad \text{i.e.} \quad W_{m,\varphi}^p(f) = m_p f, \quad \text{where} \quad m_p = \prod_{k=0}^{p-1} (m \circ \varphi^{[k]}).$$

Note that if  $g \in \text{Hol}(\mathbb{B}_N)$  and  $M_g(f) = gf$ , then  $\sigma(M_g) = g(\mathbb{B}_N)$ .

We start by the case where  $m$  vanishes on  $\mathbb{B}_N$ .

**Lemma 5.2.** *If there exists  $z_0 \in \mathbb{B}_N$  such that  $m(z_0) = 0$ , then  $\sigma_p(W_{m,\varphi}) = \emptyset$ .*

*Proof.* First, note that  $m$  and  $\varphi$  are non-constant, so  $0 \notin \sigma_p(W_{m,\varphi})$ .

Let  $\lambda \in \mathbb{C}^*$ . If there exists  $f \in \text{Hol}(\mathbb{B}_N) \setminus \{0\}$  such that  $W_{m,\varphi}(f) = \lambda f$ , then

$$W_{m,\varphi}^p(f) = m_p f = \lambda^p f.$$

Since  $f$  is not identically zero, there exists an open subset  $\Omega$  of  $\mathbb{B}_N$  such that  $f \neq 0$  on  $\Omega$ . Thus,  $m_p = \lambda^p$  on  $\Omega$ , so on  $\mathbb{B}_N$  by uniqueness theorem. However,  $m_p(z_0) = 0$ , so  $\lambda = 0$ , which is impossible. Finally,  $\sigma_p(W_{m,\varphi}) = \emptyset$ .  $\square$

We now characterize when the point spectrum of  $W_{m,\varphi}$  is non-empty.

**Proposition 5.3.** *Let  $\varphi$  be a periodic automorphism of  $\mathbb{B}_N$ , and  $m \in \text{Hol}(\mathbb{B}_N)$ .*

*Denote by  $e = (e^{i\theta_1}, \dots, e^{i\theta_N})$  the eigenvalues of  $\varphi$ , and  $p$  the smallest positive integer such that for all  $k \in \{1, \dots, N\}$ ,  $e^{ip\theta_k} = 1$ . The following assertions are equivalent.*

- (i)  $\sigma_p(W_{m,\varphi}) \neq \emptyset$ .
- (ii)  $m_p$  is a constant map.
- (iii)  $\sigma_p(W_{m,\varphi}) = \{m(0)e^{\vec{j}} : \vec{j} \in \mathbb{N}_0^N\}$ .

*Proof.* (iii  $\implies$  i) is trivial.

(i  $\implies$  ii) : If  $\sigma_p(W_{m,\varphi}) \neq \emptyset$ , then  $m$  does not vanish on  $\mathbb{B}_N$ , by Lemma 5.2. Moreover, by functional calculus,  $\sigma_p(W_{m,\varphi}^p) = (\sigma_p(W_{m,\varphi}))^p := \{\mu^p : \mu \in \sigma_p(W_{m,\varphi})\} \neq \emptyset$ . Hence, there exist  $f \in \text{Hol}(\mathbb{B}_N) \setminus \{0\}$  and  $\lambda \in \mathbb{C}$  such that  $m_p f = \lambda f$ . Since  $f \neq 0$ , the map  $m_p = \lambda$  is constant, using uniqueness theorem. Finally,  $\lambda \neq 0$  because  $m(0) \neq 0$ .

(ii  $\implies$  iii) : If  $m_p$  is constant, then  $m_p \equiv m_p(0) = m(0)^p$ . In addition,

$$W_{m,\varphi}^p(f) = m_p f = m(0)^p f.$$

Thus,  $W_{m,\varphi}^p = m(0)^p \text{Id}$ , so  $\sigma_p(W_{m,\varphi}^p) = \{m(0)^p\}$ . If  $\sigma_p(W_{m,\varphi}) = \emptyset$ , then

$$\sigma_p(W_{m,\varphi}^p) = (\sigma_p(W_{m,\varphi}))^p = \emptyset,$$

which is not the case here. Therefore,  $\sigma_p(W_{m,\varphi}) \neq \emptyset$ .

We show that  $\sigma_p(W_{m,\varphi}) = e^{\vec{j}} \sigma_p(W_{m,\varphi})$ , for all  $\vec{j} \in \mathbb{N}_0^N$ .

- If  $\mu \in \sigma_p(W_{m,\varphi})$ , then there exists  $f \in \text{Hol}(\mathbb{B}_N) \setminus \{0\}$  such that  $m(f \circ \varphi) = \mu f$ . Hence, setting  $f_{\vec{j}}(z) = z^{\vec{j}} f(z)$ , we obtain, for  $z \in \mathbb{B}_N$ ,

$$[m(f_{\vec{j}} \circ \varphi)](z) = m(z) \varphi^{\vec{j}}(z) f(\varphi(z)) = e^{\vec{j}} z^{\vec{j}} \mu f(z) = \mu e^{\vec{j}} f_{\vec{j}}(z).$$

Finally,  $e^{\vec{j}} \mu \in \sigma_p(W_{m,\varphi})$ , so  $e^{\vec{j}} \sigma_p(W_{m,\varphi}) \subset \sigma_p(W_{m,\varphi})$ .

- Conversely, let  $\vec{j} = (j_1, \dots, j_N) \in \mathbb{N}_0^N$ . We set, for  $k \in \{1, \dots, N\}$ ,  $n_k = \lfloor j_k/p \rfloor$ , and  $\vec{\alpha} = (n_1 p - j_1, \dots, n_N p - j_N) \in \mathbb{N}_0^N$ . Then

$$\vec{j} + \vec{\alpha} = (n_1 p, \dots, n_N p),$$

so  $e^{\vec{j} + \vec{\alpha}} = 1$ . Finally, if  $\mu \in \sigma_p(W_{m,\varphi})$ , then  $e^{\vec{\alpha}} \mu \in \sigma_p(W_{m,\varphi})$ , so

$$\mu = e^{\vec{j}} (e^{\vec{\alpha}} \mu) \in e^{\vec{j}} \sigma_p(W_{m,\varphi}).$$

To finish, if  $\mu \in \sigma_p(W_{m,\varphi})$ , we know that  $\mu^p = m(0)^p$ . Thus,  $m(0) = \mu \exp(2i\pi k/p)$ , with  $k \in \{0, \dots, p-1\}$ . Using [15], there exists  $\vec{j} \in \mathbb{N}_0^N$  such that  $e^{\vec{j}} = \exp(2i\pi k/p)$ . We deduce that

$$m(0) = \mu \exp(2i\pi k/p) = \mu e^{k\vec{j}} \in e^{k\vec{j}} \sigma_p(W_{m,\varphi}) = \sigma_p(W_{m,\varphi}).$$

Finally,  $\{m(0)e^{\vec{j}} : \vec{j} \in \mathbb{N}_0^N\} \subset \sigma_p(W_{m,\varphi})$ , and if  $\mu \in \sigma_p(W_{m,\varphi})$ , then there exists  $\ell \in \{1, \dots, p\}$  such that  $\mu = m(0) \exp(2i\pi \ell/p) \in \{m(0)e^{\vec{j}} : \vec{j} \in \mathbb{N}_0^N\}$ . We can conclude that

$$\sigma_p(W_{m,\varphi}) = \{m(0)e^{\vec{j}} : \vec{j} \in \mathbb{N}_0^N\}. \quad \square$$

To obtain the spectrum, we start by a useful lemma.

**Lemma 5.4.** *Let  $\varphi$  be a periodic automorphism of  $\mathbb{B}_N$ , and  $m \in \text{Hol}(\mathbb{B}_N)$ . Then,*

$$\forall \vec{k} \in \mathbb{N}_0^N, \quad m(0)e^{\vec{k}} \in \sigma(W_{m,\varphi}).$$

*Proof.* Consider  $\vec{j} \neq \vec{0}$  the smallest vector (for the order  $\preceq$ ) of  $\mathbb{N}_0^N$  such that  $e^{\vec{j}} = 1$ . We will show that for all  $\vec{i} \prec \vec{j}$ ,

$$z^{\vec{i}} \notin (m(0)e^{\vec{i}} - W_{m,\varphi})\text{Hol}(\mathbb{B}_N).$$

$\vec{i} = \vec{0}$  : Note that for all  $f \in \text{Hol}(\mathbb{B}_N)$ ,

$$[m(0)f - W_{m,\varphi}(f)](0) = m(0)f(0) - m(0)f(\varphi(0)) = 0.$$

Hence,  $1 \notin (m(0) - W_{m,\varphi})\text{Hol}(\mathbb{B}_N)$ , so  $m(0) \in \sigma(W_{m,\varphi})$ .

Now, assume that there exists  $\vec{i} \prec \vec{j}$  such that  $\vec{i} \neq \vec{0}$  and

$$m(0)e^{\vec{i}} f - m(f \circ \varphi) = z^{\vec{i}},$$

for a certain  $f \in \text{Hol}(\mathbb{B}_N)$ ,  $f \neq 0$ .

- Note that  $m(0)e^{\vec{i}}f(0) - m(0)f(\varphi(0)) = [z^{\vec{i}}](0) = 0$ . Moreover,  $m(0) \neq 0$ ,  $e^{\vec{i}} - 1 \neq 0$ . Hence,  $f(0) = 0$ .
- Assume that for all  $\vec{\alpha} \prec \vec{\beta} \prec \vec{i}$ ,  $f^{(\vec{\alpha})}(0) = 0$ . Using the general Leibniz rule,

$$\begin{aligned} & m(0)e^{\vec{i}}f^{(\vec{\beta})}(0) - [m(f \circ \varphi)]^{(\vec{\beta})}(0) \\ &= m(0)e^{\vec{i}}f^{(\vec{\beta})}(0) - \sum_{\vec{\alpha} \prec \vec{\beta}} \binom{\vec{\beta}}{\vec{\alpha}} m^{(\vec{\beta}-\vec{\alpha})}(0)e^{\vec{\alpha}}f^{(\vec{\alpha})}(0) - m(0)e^{\vec{\beta}}f^{(\vec{\beta})}(0) \\ &= m(0)(e^{\vec{i}} - e^{\vec{\beta}})f^{(\vec{\beta})}(0) = [z^{\vec{i}}]^{(\vec{\beta})}(0) = 0. \end{aligned}$$

Since  $e^{\vec{i}} - e^{\vec{\beta}} \neq 0$  (because  $\vec{j}$  is minimal), and  $m(0) \neq 0$ , we get  $f^{(\vec{\beta})}(0) = 0$ .

- In a same way, for  $\vec{\beta} = \vec{i}$ , we obtain

$$m(0)e^{\vec{i}}f^{(\vec{i})}(0) - [m(f \circ \varphi)]^{(\vec{i})}(0) = m(0)(e^{\vec{i}} - e^{\vec{i}})f^{(\vec{i})}(0) = 0 = [z^{\vec{i}}]^{(\vec{i})}(0) = \vec{i}!$$

Therefore, we get a contradiction. Finally,  $z^{\vec{i}} \notin (m(0)e^{\vec{i}} - W_{m,\varphi})\text{Hol}(\mathbb{B}_N)$ , so we have proved that  $m(0)e^{\vec{i}} \in \sigma(W_{m,\varphi})$ . All that remains for us is to see that

$$\{m(0)e^{\vec{i}} : \mathbb{N}_0^N\} = \{m(0)e^{\vec{i}} : \vec{i} \prec \vec{j}\},$$

because  $e^{\vec{j}} = 1$ . Thus,  $\{m(0)e^{\vec{i}} : \mathbb{N}_0^N\} \subset \sigma(W_{m,\varphi})$ .  $\square$

Hence, we prove the following proposition.

**Theorem 5.5.** *Let  $\varphi$  be a periodic automorphism of  $\mathbb{B}_N$ , and  $m \in \text{Hol}(\mathbb{B}_N)$ . If  $p$  is the smallest positive integer such that  $\varphi^p = \text{Id}$ , then*

$$\sigma(W_{m,\varphi}) = \{\lambda \in \mathbb{C} : \lambda^p \in m_p(\mathbb{B}_N)\}.$$

*Proof.* First, we show that  $\vec{j} \in \mathbb{N}_0^N$ ,  $e^{\vec{j}}\sigma(W_{m,\varphi}) \subset \sigma(W_{m,\varphi})$ . To do this, by multiplicativity, we only have to prove it for  $\vec{j} = (0, \dots, 0, 1, 0, \dots, 0)$ , the 1 being in position  $k$ . Note that if  $k \in \{1, \dots, N\}$ ,

$$\Gamma_k : \begin{cases} \text{Hol}(\mathbb{B}_N) & \rightarrow z_k \text{Hol}(\mathbb{B}_N) \\ f & \mapsto z_k f, \end{cases}$$

then  $\Gamma_k$  is bijective. For all  $f \in \text{Hol}(\mathbb{B}_N)$  and  $z \in \mathbb{B}_N$ , if we set  $T = (W_{m,\varphi})|_{z_k \text{Hol}(\mathbb{B}_N)}$ ,

$$(\Gamma_k^{-1} \circ T \circ \Gamma_k)(f)(z) = \frac{m(z)\varphi_k(z)(f \circ \varphi)(z)}{z_k} = m(z)e^{i\theta_k}(f \circ \varphi)(z) = e^{i\theta_k}W_{m,\varphi}(f)(z).$$

Hence,  $\sigma(T) = e^{i\theta_k}\sigma(W_{m,\varphi})$ .

If  $W_{m,\varphi} - \lambda \text{Id}$  is bijective, then for all  $g \in z_k \text{Hol}(\mathbb{B}_N)$ , there exists  $f \in \text{Hol}(\mathbb{B}_N)$  such that

$$m(f \circ \varphi) - \lambda f = g.$$

We prove that  $f \in z_k \text{Hol}(\mathbb{B}_N)$ .

- First,  $m(0)f(\varphi(0)) - \lambda f(0) = (m(0) - \lambda)f(0) = g(0) = 0$ . Since  $\lambda \neq m(0)$  using Lemma 5.4, we get  $f(0) = 0$ .

- Let  $\vec{j} \in \mathbb{N}_0^N$  such that  $j_k = 0$ . Assume that for all  $\vec{i} \prec \vec{j}$  satisfying  $i_k = 0$ ,  $f^{(\vec{i})}(0) = 0$ . Then, by the general Leibniz rule,

$$[m(f \circ \varphi) - \lambda f]^{(\vec{j})}(0) = \sum_{\vec{i} \prec \vec{j}} \binom{\vec{j}}{\vec{i}} m^{(\vec{j}-\vec{i})}(0) e^{\vec{i}} f^{(\vec{i})}(0) + m(0) e^{\vec{j}} f^{(\vec{j})}(0) - \lambda f^{(\vec{j})}(0).$$

But if  $\vec{i} \prec \vec{j}$ , we get  $i_k < j_k = 0$ , so  $i_k = 0$ , and  $f^{(\vec{i})}(0) = 0$ . Thus,

$$[m(f \circ \varphi) - \lambda f]^{(\vec{j})}(0) = (m(0) e^{\vec{j}} - \lambda) f^{(\vec{j})}(0) = g^{(\vec{j})}(0) = 0.$$

Finally, by Lemma 5.4,  $\lambda \neq m(0) e^{\vec{j}}$ , so  $f^{(\vec{j})}(0) = 0$ .

We conclude that  $f \in z_k \text{Hol}(\mathbb{B}_N)$ , so  $T - \lambda \text{Id}$  is bijective. Therefore,

$$\lambda \notin \sigma(W_{m,\varphi}) \implies \lambda \notin \sigma(T) = e^{i\theta_k} \sigma(W_{m,\varphi}).$$

We deduce that  $e^{i\theta_k} \sigma(W_{m,\varphi}) \subset \sigma(W_{m,\varphi})$ . Using multiplicativity, for all  $\vec{j} \in \mathbb{N}_0^N$ ,

$$e^{\vec{j}} \sigma(W_{m,\varphi}) \subset \sigma(W_{m,\varphi}).$$

We can now show the main assertion of the proposition.

If  $\lambda \in \sigma(W_{m,\varphi})$ , then  $\lambda^p \in \sigma(W_{m,\varphi})^p = \sigma(W_{m,\varphi}^p) = \sigma(M_{m_p}) = m_p(\mathbb{B}_N)$ .

Conversely, if  $\lambda^p \in m_p(\mathbb{B}_N)$ , we get  $\lambda^p \in \sigma(W_{m,\varphi})^p$ , so there exists  $\mu \in \sigma(W_{m,\varphi})$  such that  $\lambda^p = \mu^p$ , i.e.  $\lambda = \mu \exp(2ik\pi/p)$ , with  $k \in \{0, \dots, p-1\}$ . Moreover, it was shown in [15] that there exists  $\vec{j} \in \mathbb{N}_0^N$  such that  $\exp(2ik\pi/p) = e^{\vec{j}}$ . Finally,

$$\lambda = \mu \exp\left(\frac{2ik\pi}{p}\right) = \mu e^{\vec{j}} \in e^{\vec{j}} \sigma(W_{m,\varphi}) \subset \sigma(W_{m,\varphi}). \quad \square$$

## 6. OTHER RESULTS

In this last section, we concatenate all the other results about the spectra of  $W_{m,\varphi}$  which do not fit in the cases studies above.

**6.1. Symbol bijective, non-periodic.** If  $\varphi$  is a *non-periodic* holomorphic self-map of  $\mathbb{B}_N$ , then for all  $z \in \mathbb{B}_N$ , at least one coordinate of  $(D^n z)$  does not converge when  $n$  goes to infinity. We obtain the generalization of [3, Proposition 3.6] in the following result.

**Lemma 6.1.** *If there exists  $z_0 \in \mathbb{B}_N$  such that  $m(z_0) = 0$ , then  $\sigma_p(W_{m,\varphi}) = \emptyset$ .*

*Proof.* First, note that  $m$  and  $\varphi$  are non-constant, so if  $W_{m,\varphi}(f) = m(f \circ \varphi) = 0$ , then  $f \equiv 0$ . Hence,  $0 \notin \sigma_p(W_{m,\varphi})$ .

Let  $\lambda \in \mathbb{C}^*$  and  $f \in \text{Hol}(\mathbb{B}_N)$  such that  $m(f \circ \varphi) = \lambda f$ . Then,

$$f(z_0) = m(z_0) f(Dz_0) = 0,$$

so  $m(D^{-1}z_0) f(z_0) = \lambda f(D^{-1}z_0) = 0$ . Iterating this equation, for all  $n \in \mathbb{N}_0$ ,

$$f(D^{-n}z_0) = 0.$$

But  $\{D^{-n}z_0 : n \in \mathbb{N}_0\}$  has an accumulation point. Thus,  $f \equiv 0$ , so  $\lambda \notin \sigma_p(W_{m,\varphi})$ .  $\square$

**6.2. Symbol elliptic non-attractive.** If  $\varphi$  is an elliptic non-attractive self-map of  $\mathbb{B}_N$ , then the iterates of  $\varphi$  do not converge to a point, but to a map  $h : \mathbb{B}_N \rightarrow \mathbb{B}_N$ . In this case, we have the following result.

**Proposition 6.2.** *Let  $\varphi$  be an elliptic non attractive self-map of  $\mathbb{B}_N$ . Assume that  $\varphi(0) = 0$ ,  $\varphi \notin \text{Aut}(\mathbb{B}_N)$  and  $m(0) \neq 0$ . Then,  $\sigma_p(W_{m,\varphi}) \subset D(0, |m(0)|)$ .*

*Proof.* Using the proof of Proposition 4.2, the sequence  $(w_n)$  defined by

$$w_n(z) = \frac{1}{m(0)^n} \prod_{k=0}^{n-1} m(\varphi^{[k]}(z))$$

converges uniformly on all compact subsets of  $\mathbb{B}_N$  to a map  $w$ .

Hence, if there exist  $\lambda \in \mathbb{C}^*$  and  $f \in \text{Hol}(\mathbb{B}_N) \setminus \{0\}$  such that  $m(f \circ \varphi) = \lambda f$ , after  $n$  iterations,

$$w_n(f \circ \varphi^{[n]}) = \left( \frac{\lambda}{m(0)} \right)^n f.$$

Letting  $n \rightarrow \infty$ , there is a subsequence of the left side of this identity that goes to  $w(f \circ h)$ . If  $|\lambda| > |m(0)|$ , then the right side goes to  $\infty$  at least at one point of  $\mathbb{B}_N$ , a contradiction.  $\square$

As in [15], we now consider a particular case: assume that  $\varphi$  has « separable » variables, that is  $\varphi$  can be written as

$$\varphi(z) = (\varphi_1(z_1), \dots, \varphi_N(z_N)),$$

with  $\varphi_k \in \text{Hol}(\mathbb{D})$  elliptic such that  $\varphi(0) = 0$ .

Note that if  $\varphi$  is elliptic non attractive and not bijective, then by Theorem 1.2 and one-variable Denjoy-Wolff's theory, we will assume that some components will be rotations, and the other ones must be elliptic non invertible maps (in one variable) fixing 0. We obtain the following theorem.

**Theorem 6.3.** *Let  $\varphi$  be an elliptic non attractive self-map of  $\mathbb{B}_N$ , such that  $\varphi$  is not invertible,  $\varphi(0) = 0$ , and*

$$\varphi(z_1, \dots, z_N) = (\varphi_1(z_1), \dots, \varphi_p(z_p), \beta_{p+1}z_{p+1}, \dots, \beta_N z_N),$$

with  $\varphi_1, \dots, \varphi_p$  non bijective,  $|\varphi'_1(0)|, \dots, |\varphi'_p(0)| < 1$ , and  $|\beta_{p+1}| = \dots = |\beta_N| = 1$ . Then,

$$\begin{aligned} & \{0\} \cup \left\{ m(0) \prod_{k=1}^p \varphi'_k(0)^{n_k} \times \prod_{k=p+1}^N \beta_k^{n_k} : n_1, \dots, n_N \in \mathbb{N}_0 \right\} \\ & \subset \sigma(W_{m,\varphi}) \subset \left( \bigcup_{n_1, \dots, n_p \geq 0} \left[ \prod_{k=1}^p \varphi'_k(0)^{n_k} \right] m(0)\mathbb{T} \right) \cup \{0\}. \end{aligned}$$

Note that if there exists  $j \in \{1, \dots, n\}$  such that  $\varphi'_j(0) = 0$ , then all the terms

$$m(0) \prod_{k=1}^p \varphi'_k(0)^{n_k} \times \prod_{k=p+1}^N \beta_k^{n_k}$$

with  $n_k \neq 0$  vanish. Then, we will assume that  $0 < |\varphi'_1(0)|, \dots, |\varphi'_p(0)| < 1$ .

The proof of this theorem has already been done without weight in [15]. In dimension 2, considering the map  $\varphi : z \mapsto (\varphi_1(z_1), \beta_2 z_2)$ , the author used the following decomposition of  $\text{Hol}(\mathbb{B}_2)$ , valid for all  $\ell \in \mathbb{N}$ :

$$\text{Hol}(\mathbb{B}_2) = W_\ell \oplus z_1^\ell \text{Hol}(\mathbb{B}_2), \quad W_\ell = \left\{ \sum_{q=0}^{\ell-1} \kappa_1^q(z_1) f_q(z_2) : f_q \in \text{Hol}(\mathbb{D}) \right\}$$

We will consider an other decomposition of  $\text{Hol}(\mathbb{B}_2)$ , using the Koenigs' function  $\kappa_1$  and the weighted Koenigs' function  $w$  of  $\varphi_1$ . The proof of the following property goes along the same lines as in [15], we write it here for sake of completeness.

**Proposition 6.4.** *For all  $\ell \in \mathbb{N}$ , consider*

$$\tilde{W}_\ell = \left\{ w(z_1) \sum_{q=0}^{\ell-1} \kappa_1^q(z_1) f_q(z_2) : f_q \in \text{Hol}(\mathbb{D}) \right\}, \quad X_\ell = z_1^\ell \text{Hol}(\mathbb{B}_2).$$

Then

- (i)  $\text{Hol}(\mathbb{B}_2) = W_\ell \oplus X_\ell$ ,
- (ii)  $C_\varphi(W_\ell) \subset W_\ell$  and  $C_\varphi(X_\ell) \subset X_\ell$ .

*Proof.* We prove (i) by induction, in a same way as in [15].

$m = \underline{1}$ : We have to show that  $\text{Hol}(\mathbb{B}_2) = W_1 \oplus X_1$ .

Let  $f \in \text{Hol}(\mathbb{B}_2)$ . Using the Maclaurin coefficients of  $f$ , we may write

$$f(z) = \sum_{j \geq 0} \sum_{k \geq 0} a_{jk} z_1^j z_2^k = \underbrace{\sum_{k \geq 0} a_{0k} z_2^k}_{=f_0(z_2)} + z_1 \underbrace{\sum_{j \geq 1} \sum_{k \geq 0} a_{jk} z_1^{j-1} z_2^k}_{=F(z)}.$$

Since  $w(0) = 1$ , we may write  $w = 1 + \tilde{w}$ , with  $\tilde{w}(z_1) \in z_1 \text{Hol}(\mathbb{D})$ . Hence,

$$f(z) = \underbrace{w(z_1) f_0(z_2)}_{\in W_1} - \underbrace{\tilde{w}(z_1) f_0(z_2)}_{\in X_1} + F(z).$$

Finally, we get  $\text{Hol}(\mathbb{B}_2) = W_1 + X_1$ .

Now, assume that  $f \in W_1 \cap X_1$ . We can write

$$f(z) = w(z_1) f_0(z_2) = z_1 F(z),$$

with  $f_0 \in \text{Hol}(\mathbb{D})$  and  $F \in \text{Hol}(\mathbb{B}_2)$ . Once again using  $\tilde{w}$ , we get

$$f_0(z_2) = z_1 F(z) - \tilde{w}(z_1) f_0(z_2).$$

Since  $\tilde{w}(0) = 0$ , considering  $z = (0, z_2)$ , we obtain  $f_1(z_2) = 0$  for all  $z_2 \in \mathbb{D}$ . Thus,  $f \equiv 0$ , so  $\text{Hol}(\mathbb{B}_2) = W_1 \oplus X_1$ .

$\ell \rightarrow \ell + 1$ : Assume that  $\text{Hol}(\mathbb{B}_2) = W_\ell \oplus X_\ell$ .

Let  $f \in \text{Hol}(\mathbb{B}_2)$ . By induction hypothesis,

$$f(z) = w(z_1) \sum_{q=0}^{\ell-1} \kappa_1^q(z_1) f_q(z_2) + z_1^\ell \tilde{f}(z),$$

with  $f_0, \dots, f_{\ell-1} \in \text{Hol}(\mathbb{D})$  and  $\tilde{f} \in \text{Hol}(\mathbb{B}_2)$ . Using the same calculations as for  $\ell = 1$ , we may find  $f_\ell \in \text{Hol}(\mathbb{D})$  and  $g \in \text{Hol}(\mathbb{B}_2)$  such that

$$z_1^\ell \tilde{f}(z) = z_1^\ell f_\ell(z_2) + z_1^{\ell+1} g(z).$$

Moreover, since  $\kappa_1(0) = 0$ ,  $\kappa_1'(0) = 1$  and  $w(0) = 1$ , there exists  $h \in \text{Hol}(\mathbb{D})$  such that

$$w(z_1) \kappa_1^\ell(z_1) = z_1^\ell + z_1^{\ell+1} h(z_1).$$

Therefore,

$$z_1^\ell \tilde{f}(z) = w(z_1) \kappa_1^\ell(z_1) f_\ell(z_2) + z_1^{\ell+1} (g(z) - h(z_1) f_\ell(z_2)).$$

Finally, we have

$$f(z) = \underbrace{w(z_1) \sum_{q=0}^{\ell-1} \kappa_1^q(z_1) f_q(z_2)}_{\in W_{\ell+1}} + \underbrace{w(z_1) \kappa_1^\ell(z_1) f_\ell(z_2) + z_1^{\ell+1} (g(z) - h(z_1) f_\ell(z_2))}_{\in X_{\ell+1}},$$

so  $\text{Hol}(\mathbb{B}_2) = W_{\ell+1} + X_{\ell+1}$ .

Now, assume that  $f \in W_{\ell+1} \cap X_{\ell+1}$ . Then,

$$f(z) = w(z_1) \sum_{q=0}^{\ell} \kappa_1^q(z_1) f_q(z_2) = z_1^{\ell+1} \tilde{f}(z),$$

with  $f_0, \dots, f_\ell \in \text{Hol}(\mathbb{D})$  and  $\tilde{f} \in \text{Hol}(\mathbb{B}_2)$ . Once again writing  $w \kappa_1^\ell = z_1^\ell + z_1^{\ell+1} h$ , we obtain

$$\begin{aligned} w(z_1) \sum_{q=0}^{\ell-1} \kappa_1^q(z_1) f_q(z_2) &= z_1^{\ell+1} \tilde{f}(z) - w(z_1) \kappa_1^\ell(z_1) f_\ell(z_2) \\ &= z_1^\ell (z_1 \tilde{f}(z) - f_\ell(z_2) - z_1 h(z_1) f_\ell(z_2)) \in W_\ell \cap X_\ell = \{0\}. \end{aligned}$$

Thus,  $f(z) = w(z_1) \kappa_1^\ell(z_1) f_\ell(z_2) = z_1^{\ell+1} \tilde{f}(z)$ , so

$$z_1^\ell f_\ell(z_2) = z_1^{\ell+1} (\tilde{f}(z) - h(z_1) f_\ell(z_2)).$$

Denote by  $(c_{jk})$  the Maclaurin coefficients of the map  $z \mapsto z_1^\ell f_\ell(z_2)$ , and  $(d_{jk})$  those of the map  $z \mapsto z_1^{\ell+1} (\tilde{f}(z) - h(z_1) f_\ell(z_2))$ . Then,

$$j \neq \ell \implies c_{jk} = 0 \quad \text{and} \quad j = \ell \implies d_{jk} = 0.$$

Finally, since  $c_{jk} = d_{jk}$  for all  $j, k \in \mathbb{N}_0$ , we obtain

$$z_1^\ell f_\ell(z_2) = z_1^{\ell+1} (\tilde{f}(z) - h(z_1) f_\ell(z_2)) = 0.$$

Hence,  $f_\ell \equiv 0$ , so  $f \equiv 0$ . We proved that  $\text{Hol}(\mathbb{B}_2) = W_\ell \oplus X_\ell$ .

To finish, we prove (ii). If  $f \in W_\ell$ , then

$$f = w(z_1) \sum_{q=0}^{\ell-1} \kappa_1^q(z_1) f_q(z_2),$$

with  $f_0, \dots, f_{\ell-1} \in \text{Hol}(\mathbb{D})$ . Thus,

$$m(f \circ \varphi)(z) = m(0)w(z_1) \sum_{q=0}^{\ell-1} \varphi_1'(0)^q \kappa_1^q(z_1) f_q(\beta_2 z_2) = w(z_1) \sum_{q=0}^{\ell-1} \kappa_1^q(z_1) \tilde{f}_q(z_2),$$

where  $\tilde{f}_q(z_2) = m(0)\varphi_1'(0)^\ell f_q(\beta_2 z_2)$ . Thus,  $f \circ \varphi \in W_\ell$ .

If  $f \in X_\ell$ , then  $f(z) = z_1^\ell g(z)$ , with  $g \in \text{Hol}(\mathbb{B}_2)$ . Hence,

$$m(f \circ \varphi)(z) = m(z)\varphi_1(z_1)^\ell g(\varphi_1(z_1), \beta_2 z_2).$$

However, since  $\varphi_1(0) = 0$ , we may write  $\varphi_1(z_1) = z_1\psi(z_1)$ , with  $\psi \in \text{Hol}(\mathbb{D})$ . Therefore,

$$(f \circ \varphi)(z) = z_1^\ell \psi(z_1) m(z) g(\varphi_1(z_1), \beta_2 z_2) \in X_\ell. \quad \square$$

In order to prove the main theorem of this section, we use the two decompositions obtained above.

*Proof of Theorem 6.3.* Let  $g \in \text{Hol}(\mathbb{B}_N)$ . For all  $\ell \in \mathbb{N}$ , using the fact that  $\text{Hol}(\mathbb{B}_2) = \tilde{W}_\ell \oplus X_\ell$ , we may write

$$g(z) = w(z_1) \sum_{q=0}^{\ell-1} \kappa_1^q(z_1) g_q(\tilde{z}) + z_1^\ell G_1(z),$$

with  $g_0, \dots, g_{\ell-1} \in \text{Hol}(\mathbb{B}_{N-1})$  and  $G_1 \in \text{Hol}(\mathbb{B}_N)$ . Now, since  $\text{Hol}(\mathbb{B}_2) = W_\ell \oplus X_\ell$ , going from one coordinate to another, for all  $\ell_1, \dots, \ell_p \in \mathbb{N}$ , we may write  $g$  as

$$g(z) = \sum_{q_1=0}^{\ell_1-1} \cdots \sum_{q_p=0}^{\ell_p-1} w(z_1) \left( \prod_{k=1}^p \kappa_k^{q_k}(z_k) \right) g_{q_1, \dots, q_p}(z_{p+1}, \dots, z_N) + \sum_{k=1}^p z_k^{\ell_k} G_k(z),$$

with  $g_{q_1, \dots, q_p} \in \text{Hol}(\mathbb{B}_{N-p})$  and  $G_k \in \text{Hol}(\mathbb{B}_N)$ . Denote  $\tilde{G}_k = z_k^{\ell_k} G_k$ .

Let  $\lambda \in \mathbb{C}^*$  such that

$$|\lambda| \notin \left\{ m(0) \prod_{k=1}^p |\varphi_k'(0)|^{n_k} : n_1, \dots, n_p \in \mathbb{N}_0 \right\}.$$

- For all  $k \in \{1, \dots, p\}$  and for  $\ell_k$  sufficiently large, the map

$$\tilde{F}_k = \sum_{n \geq 0} \frac{m_n(\tilde{G}_k \circ \varphi^{[n]})}{\lambda^n}$$

is a uniformly convergent series, and is holomorphic on a small ball  $B(0, \varepsilon)$  (cf. Lemma 4.9). Moreover, it satisfies  $m(\tilde{F}_k \circ \varphi) - \lambda \tilde{F}_k = \tilde{G}_k$ . Lemma 4.8 gives  $F_k \in \text{Hol}(\mathbb{B}_N)$  such that  $m(F_k \circ \varphi) - \lambda F_k = \tilde{G}_k$ .

- Denote by  $\phi$  the map defined as  $\phi(z_{p+1}, \dots, z_N) = (\beta_{p+1}z_{p+1}, \dots, \beta_N z_N)$ . Let  $0 \leq q_1 \leq \ell_1 - 1, \dots, 0 \leq q_p \leq \ell_p - 1$ . Since  $|\lambda| \neq m(0) \prod_{k=1}^p |\varphi'_k(0)|^{q_k}$  and  $\phi$  is bijective, there exists  $f_{q_1, \dots, q_p} \in \text{Hol}(\mathbb{D})$  such that

$$m(0) \prod_{k=1}^p \varphi'_k(0)^{q_k} (f_{q_1, \dots, q_p} \circ \phi) - \lambda f_{q_1, \dots, q_p} = g_{q_1, \dots, q_p}.$$

To summarise, the map  $f$  defined as

$$f(z) = \sum_{q_1=0}^{\ell_1-1} \cdots \sum_{q_p=0}^{\ell_p-1} w(z_1) \left( \prod_{k=1}^p \kappa_k^{q_k}(z_k) \right) f_{q_1, \dots, q_p}(z_{p+1}, \dots, z_N) + \sum_{k=1}^p z_k^{\ell_k} F_k(z)$$

satisfies  $m(f \circ \varphi) - \lambda f = g$ , and is holomorphic on  $\mathbb{B}_N$ . Hence,  $W_{m, \varphi} - \lambda \text{Id}$  is bijective.

Conversely, let  $n_1 \in \mathbb{N}$ , and  $f_1, \dots, f_N : \mathbb{D} \rightarrow \mathbb{C}$  holomorphic maps such that

- $m(f_1 \circ \varphi_1) = m(0) \varphi'_1(0)^{n_1} f_1$ ,
- $f_k \circ \varphi_k = \varphi'_k(0) f_k \quad (2 \leq k \leq p)$ ,
- $f_k \circ \varphi_k = \beta_k f_k \quad (p+1 \leq k \leq N)$ .

Such maps exist using the results in one variable ([4, 2, 3]), since  $\varphi$  has separable variables. For  $n_2, \dots, n_N \in \mathbb{N}_0$ , consider  $f : \mathbb{B}_N \rightarrow \mathbb{C}$  defined by

$$f(z) = f_1(z_1) \times \prod_{k=2}^N f_k(z_k)^{n_k}.$$

Then,

$$\begin{aligned} [m(f \circ \varphi)](z) &= m(z) (f_1 \circ \varphi_1)(z) \times \prod_{k=2}^p f_k(\varphi_k(z_k))^{n_k} \times \prod_{k=p+1}^N f_k(\beta_k z_k)^{n_k} \\ &= m(0) \varphi'_1(0)^{n_1} f_1(z) \times \prod_{k=2}^p \varphi'_k(0)^{n_k} f_k(z_k)^{n_k} \times \prod_{k=p+1}^N \beta_k^{n_k} f_k(z_k)^{n_k} \\ &= \left( m(0) \prod_{k=1}^p \varphi'_k(0)^{n_k} \times \prod_{k=p+1}^N \beta_k^{n_k} \right) f(z). \end{aligned}$$

Therefore,  $m(0) \prod_{k=1}^p \varphi'_k(0)^{n_k} \times \prod_{k=p+1}^N \beta_k^{n_k} \in \sigma_p(C_\varphi) \subset \sigma(C_\varphi)$ .  $\square$

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