Strongly Locally Irregular Graphs and Decompositions

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Abstract

Locally irregular graphs are graphs in which no two adjacent vertices have the same degree, while locally irregular decompositions are edge-partitions of graphs into locally irregular graphs. These notions were introduced as a possible antonym to graph regularity, and appeared naturally along the study of related objects and problems, such as proper labellings, other types of distinguishing labellings, and the 1-2-3 Conjecture.

In the current work, we introduce and study a stronger version of locally irregular graphs and decompositions, which we call strongly locally irregular graphs and decompositions, where the notion of irregularity is tweaked, in that we require adjacent vertices to have degrees differing by at least 2. We prove results of multiple natures, showing both similarities and discrepancies between our modified notions and the original ones. In particular, we prove that graphs admitting strongly locally irregular decompositions are not easy to characterise (unless P=NP), which is not the case for locally irregular ones. Among other results, we also establish bounds on the least number of strongly locally irregular graphs decomposing certain graph classes, and compare them with similar bounds known for locally irregular decompositions. While some of our bounds do not differ much with some for the original context, some others do significantly. We also establish, for our notions, NP-hardness results that go beyond such results known for the original notions.

Keywords: locally irregular graph; locally irregular decomposition; 1-2-3 Conjecture.

1. Introduction

In this work, we introduce and study a stronger version of so-called **locally irregular graphs** and **locally irregular decompositions**. So that we can state and define these modified notions properly, we start by recalling the original ones.

A graph G is said *locally irregular* (l.i. for short) if no two of its adjacent vertices have the same degree; that is, $d(u) \neq d(v)$ for every edge $uv \in E(G)$. A *decomposition* $\mathcal{D}$ of G is a partition $E_1, \ldots, E_k$ of the edge set $E(G)$ of G, where $k \in \mathbb{N}^*$. We say $\mathcal{D}$ is *locally irregular* (l.i. for short) if, for every part E_i of $\mathcal{D}$, the graph $G[E_i]$ is indeed l.i. In other words, if, for every vertex u of G and every $i \in \{1, \ldots, k\}$, we denote by $d_i(u)$ the degree of u in $G[E_i]$, then, for every edge $uv \in E_i$, we require that $d_i(u) \neq d_i(v)$. Note that a locally irregular decomposition into k parts can equivalently be seen as a k-edge-colouring where every colour yields a l.i. graph. Now, in case G does not admit any l.i. decomposition, we say G is an *exception*; otherwise, G is *decomposable*, and we are interested in the smallest $k \geq 1$ such that G has l.i. k-edge-colourings. This k is denoted by $\chi_{\text{l.i.}}(G)$.

These notions were first introduced in [1], in particular to make more formal some phenomena observed for the so-called **1-2-3 Conjecture**. Without entering too much into the details (refer e.g. to [21] for more details), the 1-2-3 Conjecture, posed in [12], asks whether for almost all graphs we can assign labels $1, 2, 3$ to the edges so that no two

adjacent vertices share the same sum of the labels on their incident edges. A connection between this conjecture and l.i. graphs, arises when noticing that the 1-2-3 Conjecture, equivalently, asks whether for almost all graphs we can replace each edge with at most three parallel edges so that a l.i. multigraph results. Also, it is not too complicated to observe that assigning labels 1 and 2 to the edges of a regular graph so that no two adjacent vertices are incident to the same sum of labels, is actually equivalent to designing a l.i. 2-edge-colouring. Such connections motivated the authors of [1] to introduce l.i. decompositions; since then, these objects have been studied as such, for their own sake, in later works (see below).

One first, important question on l.i. decompositions, is the characterisation of exceptions. So that we can state the next result, we first need to introduce the following class $\mathfrak{T}$ of graphs. The smallest graph of $\mathfrak{T}$ is the triangle K_3. Given a graph $G \in \mathfrak{T}$ having a vertex u of degree 2, a larger member of $\mathfrak{T}$ is then obtained from G by attaching at u

- an even-length path, or
- an odd-length path at the other end of which is attached a triangle.

Exceptions, now, comprise exactly three classes of graphs:

Theorem 1.1 (Baudon, Bensmail, Przybyło, Woźniak [1]). *A connected graph is an exception if and only if it is an odd-length path, an odd-length cycle, or a member of $\mathfrak{T}$.*

Now that it should be clear for which graphs G the parameter $\chi_{\text{l.i.}}(G)$ is well defined, a legitimate question to investigate is how large can $\chi_{\text{l.i.}}(G)$ be. Observing that $\chi_{\text{l.i.}}(G) \leq 3$ holds for several common decomposable graphs G (including complete graphs, complete bipartite graphs, regular graphs with large degree, and others), the authors of [1] asked whether 3 is a general upper bound on $\chi_{\text{l.i.}}(G)$ for all decomposable graphs G. This presumption was later supported by the fact that it holds for decomposable trees [4] (in which work it was also proved that deciding whether $\chi_{\text{l.i.}}(G) \leq 2$ holds for a decomposable graph G is NP-complete), graphs with large minimum degree [18], and split graphs [14].

Another legitimate question in the same line is whether expressing bounds on $\chi_{\text{l.i.}}(G)$ through absolute, small constants for any decomposable graph G is indeed the way to favour. A first answer to that question was provided in [7], in which the authors proved that $\chi_{\text{l.i.}}(G) \leq 328$ holds for every decomposable graph G. Later on, this upper bound, through improved arguments, was improved to $\chi_{\text{l.i.}}(G) \leq 220$ in [16].

More recently, a counterexample to the conjecture in [1] was exhibited in [22]. Namely, the authors exhibited a single connected graph B with $\chi_{\text{l.i.}}(B) = 4$. Although B is the only known graph with this property, we arrive at the following, refined conjecture:

Conjecture 1.2 (Sedlar, Škrekovski [22]). *If G is a decomposable graph, then $\chi_{\text{l.i.}}(G) \leq 4$.*

Conjecture 1.2 was further studied in [23], where it was proved to hold for decomposable cacti, in [13], in which it was investigated for various graph classes, in [15], in which it was considered for decomposable subcubic graphs, and in [19], in which it was studied for regular graphs with large degree. This is where investigations on the topic stand to date.

As mentioned earlier, our main intent in this work is to introduce and study a stronger version of l.i. graphs and l.i. decompositions. It turns out that a few such other variants have been introduced recently, such as variants in digraphs [2, 5, 8], and a weaker variant where in the notion of local irregularity considered, every vertex is allowed to have a certain number of neighbours with the same degree [20]. In the notions we consider here, we ask

adjacent vertices to have degrees differing in a neater way. Namely, we say a graph is *strongly locally irregular* (s.l.i. for short) if every two adjacent vertices have their degrees differing by at least 2. Likewise, we say a decomposition is *strongly locally irregular* (s.l.i. for short) if each of its parts yields a s.l.i. graph. We call a graph admitting no s.l.i. decomposition at all a *strong exception*, while we call it *strongly decomposable* otherwise. Last, for every strongly decomposable graph G, we denote through the parameter $\chi_{\text{s.l.i.}}(G)$ the smallest $k \geq 1$ such that G admits s.l.i. decompositions into k parts.

Considering these notions is primarily motivated by the investigations from [3], in which the authors investigate a "doubled" variant of the 1-2-3 Conjecture (in which graphs are labelled so that no two adjacent vertices are incident to sums differing by at most 1); in a way, our new notions are for this variant what l.i. graphs and l.i. decompositions are for the original 1-2-3 Conjecture. We are also interested in figuring out how such a tweak of l.i. graphs and l.i. decompositions affects the general behaviour of these objects.

This work is organised as follows. We start in Section 2 by discussing strong exceptions, showing that they cannot be characterised easily (unless P=NP). Then, in Section 3, we wonder about the largest possible value of $\chi_{\text{s.l.i.}}(G)$ for a graph G, and exhibit graphs G with $\chi_{\text{s.l.i.}}(G)$ being as large as 16. We then focus on common graph classes along Section 4, covering complete bipartite graphs, subcubic graphs, complete graphs, and trees, for each of which we provide more or less tight bounds on the parameter $\chi_{\text{s.l.i.}}$. Next, in Section 5, we prove that determining $\chi_{\text{s.l.i.}}(G)$ for a graph G is NP-complete in general. We end up in Section 6 with some words on possible directions we could consider as further work.

An important point for us throughout this work, is to compare every result we get on s.l.i. decompositions with what is currently known for l.i. decompositions. For this reason, most of the upcoming sections begin with some reminder of the best results we know of on particular aspects of l.i. decompositions. Be aware that a more general comparison between the two will also eventually be given in concluding Section 6.

2. Characterising strong exceptions

A natural, first question to investigate, is what graphs we are dealing with, when studying s.l.i. decompositions. We start by deducing a set of obvious strong exceptions, from the connections between l.i. graphs/decompositions and s.l.i. graphs/decompositions.

Observation 2.1. *Every s.l.i. graph is l.i., and every s.l.i. decomposition is l.i. Consequently, every exception is a strong exception. Furthermore, if G is a strongly decomposable graph, then $\chi_{\text{l.i.}}(G) \leq \chi_{\text{s.l.i.}}(G)$.*

A corollary of Theorem 1.1 and Observation 2.1 is then:

Corollary 2.2. *Every odd-length path or cycle, and member of $\mathfrak{T}$, is a strong exception.*

As will be exposed throughout what follows, the strong exceptions exhibited in Corollary 2.2 are far from covering all strong exceptions. To begin with, a few others can be deduced from e.g. the following observation which will be useful later on.

Observation 2.3. *If G is a connected s.l.i. graph with edges, then $\Delta(G) \geq 3$.*

Proof. Assume this is wrong, *i.e.*, $\Delta(G) \leq 2$, and let uv be any edge of G. Then $d(u), d(v) \in \{1, 2\}$, and $|d(u) - d(v)| \leq 1$. This contradicts that G is s.l.i. Thus, we need $\Delta(G) \geq 3$. □

A direct consequence is:

Observation 2.4. *If G is a graph with $\Delta(G) \leq 2$, then G is a strong exception.*

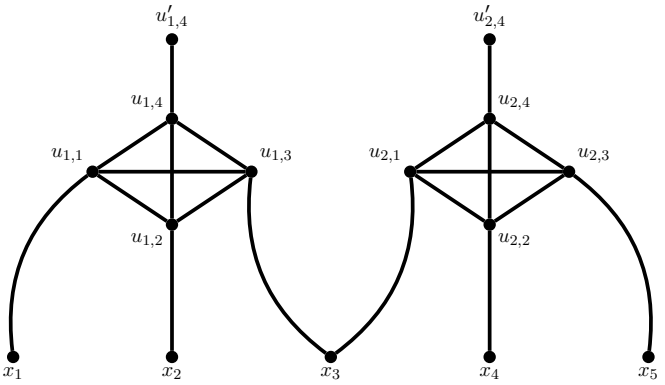


Figure 1: Illustration of (part of) the reduction in the proof of Theorem 2.6, for a formula F containing two clauses $C_1 = (x_1 \vee x_2 \vee x_3)$ and $C_2 = (x_3 \vee x_4 \vee x_5)$.

Thus, in our context, not only paths and cycles of odd length are troublesome, as actually all paths and cycles are strong exceptions. In light of Observation 2.3, it is actually not that of a surprise that graphs with very small vertex degrees cannot be decomposed in a s.l.i. way. Exploiting the fact that adjacent vertices of small degree can prevent a graph from being strongly decomposable, we can actually establish that, contrarily to exceptions (recall Theorem 1.1), strong exceptions can be of arbitrarily large maximum degree.

Theorem 2.5. *There are strong exceptions with arbitrarily large maximum degree.*

Proof. Consider any graph H with maximum degree Δ, and let u be any vertex of H. Let now G be the graph obtained from H by attaching a pendant path of length 2 at u; that is, by adding two edges uv and vw where v and w are two new vertices (of degree 2 and 1, respectively). We claim G is a strong exception (with maximum degree at least Δ). Indeed, assume this is wrong, and let ϕ be a s.l.i. edge-colouring of G. Then, two main situations can occur:

- $\phi(uv) = \phi(vw) = i$; thus, $d_i(w) = 1 = d_i(v) - 1$ (since $d_i(v) = 2$), and ϕ is not s.l.i.
- $\phi(uv) = i \neq j = \phi(vw)$; thus, $d_j(w) = 1 = d_j(v)$, and ϕ is, again, not s.l.i.

Thus, ϕ cannot be s.l.i., which is a contradiction. So G is a strong exception. □

Let us remark that, in the proof of Theorem 2.5, graph H can essentially be anything, meaning the result remains true when restricted to particular graph classes. As a particular case, Theorem 2.5 remains true even for trees, planar graphs, etc.

Through constructions such as the one introduced in the proof of Theorem 2.5, one can actually design quite a few configurations implying a graph must be a strong exception. This leads to the more general question of what strong exceptions look like, and whether there is a nice way to characterise them. Through the next result, we prove this is not the case (unless P=NP), as we show that determining whether a graph is strongly decomposable is an NP-complete problem, which stands as a significant contrast to Theorem 1.1.

Theorem 2.6. *Determining if a given graph is strongly decomposable is NP-complete.*

Proof. Since the problem is clearly in NP, we focus on proving it is NP-hard, which we do by reduction from the Cubic Monotone 1-in-3 SAT problem (which was proved to be NP-hard in [17]). Recall that an instance of Cubic Monotone 1-in-3 SAT consists in a 3CNF

formula F defined over clauses $C_1, \ldots, C_n$ and variables $x_1, \ldots, x_n$, where all variables appear (in positive form) in three distinct clauses each, all clauses contain exactly three distinct (positive) variables each, and the question is whether F can be *1-in-3 satisfied*, that is, whether there is a truth assignment to the variables such that every clause contains exactly one true variable. From an instance F of Cubic Monotone 1-in-3 SAT, we build, in polynomial time, a graph G such that F is 1-in-3 satisfiable if and only if G is strongly decomposable.

The construction of G goes as follows (see Figure 1 for a partial illustration). For every variable x_i of F, we add, to G, a *variable vertex* v_i. Now, for every clause C_j of F, we add, to G, four new vertices $u_{j,1}, u_{j,2}, u_{j,3}, u_{j,4}$ forming a clique, as well as the edge $u_{j,4}u'_{j,4}$, where $u'_{j,4}$ is a new vertex (of degree 1). Last, whenever variable x_i belongs to some clause C_j in F, assuming x_i is the kth variable appearing in F (where $k \in \{1, 2, 3\}$), we add the edge $v_i u_{j,k}$ to G. Clearly, the construction of G is achieved in polynomial time.

Before proving we have the desired equivalence between F and G, we need to observe a few facts first. In what follows, assuming we have a decomposition $\mathcal{D}$ of G, we say a variable vertex v_i is *strong* if all three edges incident to v_i belong to a same part of $\mathcal{D}$, while we say v_i is *weak* if the three edges incident to v_i belong to three distinct parts of $\mathcal{D}$.

Claim 2.7. *In any s.l.i. decomposition of G, every variable vertex is either weak or strong.*

Proof of the claim. Let $\mathcal{D}$ be a s.l.i. decomposition of G, and let v be a variable vertex of G. Assume further that exactly two edges incident to v belong to the ith part of $\mathcal{D}$, including, w.l.o.g., some edge $vu_{j,1}$. Since $u_{j,1}$ has degree 4 in G, note that, for $\mathcal{D}$ to be s.l.i., we must have $d_i(u_{j,1}) = 4$, and thus $u_{j,1}u_{j,2}$, $u_{j,1}u_{j,3}$, and $u_{j,1}u_{j,4}$ must also belong to the ith part of $\mathcal{D}$. Note then that $u_{j,4}u'_{j,4}$ cannot also belong to the ith part, as then we would get either $d_i(u_{j,4}) = 2$ while $d_i(u'_{j,4}) = 1$ or $d_i(u_{j,4}) \in \{3, 4\}$ (while $d_i(u_{j,1}) = 4$). Thus $u_{j,4}u'_{j,4}$ belongs to some i'th part of $\mathcal{D}$ with $i' \neq i$, and, to avoid any conflict between $u_{j,4}$ and $u'_{j,4}$, the i'th part must also contain $u_{j,4}u_{j,2}$ and $u_{j,4}u_{j,3}$. Since $\mathcal{D}$ is s.l.i., note that this prevents $u_{j,2}u_{j,3}$ from belonging to the i'th part (as otherwise we would necessarily have a conflict between $u_{j,2}$ and $u_{j,3}$). For the same reason, $u_{j,2}u_{j,3}$ also cannot belong to the ith part. So $u_{j,2}u_{j,3}$ must belong to some i''th part of $\mathcal{D}$ with $i'' \notin \{i, i'\}$, but, since $u_{j,2}$ and $u_{j,3}$ have degree 4 in G, and each of these two vertices is incident to at most two edges belonging to the i''th part, we deduce we must have a conflict between $u_{j,2}$ and $u_{j,3}$ in the i''th part. This contradicts that $\mathcal{D}$ is s.l.i. Thus, v must be weak or strong. ⋄

Claim 2.8. *Let $C_j = (x_{j_1} \vee x_{j_2} \vee x_{j_3})$ be any clause of F. Then, in any s.l.i. decomposition of G, exactly one of v_{j_1}, v_{j_2}, and v_{j_3} is strong, while the other two are weak.*

Proof of the claim. Let $\mathcal{D}$ be a s.l.i. decomposition of G. By Claim 2.7, recall that each of v_{j_1}, v_{j_2}, and v_{j_3} must be weak or strong. By definition, note that if, say, $v_{j_1}u_{j,1}$ belongs to the ith part of $\mathcal{D}$, then:

- if v_{j_1} is strong, then $d_i(v_{j_1}) = 3$; this implies, since $\mathcal{D}$ is s.l.i., that none of $u_{j,1}u_{j,2}$, $u_{j,1}u_{j,3}$, and $u_{j,1}u_{j,4}$ can belong to the ith part;
- if v_{j_1} is weak, then $d_i(v_{j_1}) = 1$; this implies, since $\mathcal{D}$ is s.l.i., that at least two of $u_{j,1}u_{j,2}$, $u_{j,1}u_{j,3}$, and $u_{j,1}u_{j,4}$ must belong to the ith part.

From this, the claim follows from the following arguments:

- If v_{j_1}, v_{j_2}, and v_{j_3} are strong, then, by the remark above, $\mathcal{D}$, when restricted to $G' = G[\{u_{j,1}, u_{j,2}, u_{j,3}, u_{j,4}, u'_{j,4}\}]$, must be s.l.i. Note that G' is a complete graph of

order 4 with a pendant vertex attached, thus a graph with 7 edges. Since G' is clearly not s.l.i., the smallest s.l.i. graph is the star with three leaves, which has 3 edges, and the smallest two s.l.i. graphs are the stars with three and four leaves, which have 3 and 4 edges, it is not too hard to be convinced that $\mathcal{D}$ should form a decomposition of G' into two stars, one with three leaves and one with four. However, it can be checked that G' admits no such decomposition. Thus, $\mathcal{D}$ cannot be s.l.i.

- If, say, v_{j_1} is weak while v_{j_2}, and v_{j_3} are strong, then, by the remarks above, no edge of $G' = G[\{u_{j,1}, u_{j,2}, u_{j,3}, u_{j,4}, u'_{j,4}\}]$ can belong to the parts of $\mathcal{D}$ containing the edges $v_{j_2}u_{j,2}$ and $v_{j_3}u_{j,3}$, while the part, say it is the ith one, containing $v_{j_1}u_{j,1}$ must contain two or three edges of G' incident to $u_{j,1}$. Note indeed that this would yield $d_i(u_{j,1}) = 3$ or $d_i(u_{j,1}) = 4$, respectively.

  - Since G' has maximum degree 4, in the first case (two edges of G' incident to $u_{j,1}$ in the ith part), note that no other edge of G' can belong to the ith part (as otherwise it would not yield a s.l.i. graph), implying that $\mathcal{D}$ must form a s.l.i. decomposition of G'' being isomorphic to either $G' - \{u_{j,1}u_{j,2}, u_{j,1}u_{j,3}\}$ or $G' - \{u_{j,1}u_{j,2}, u_{j,1}u_{j,4}\}$ (depending on which two edges incident to $u_{j,1}$ belong to the ith part). In the first case, G'' is a triangle with two pendant vertices attached at a single vertex, while, in the second case, G'' is a triangle with two pendant vertices attached at two distinct vertices. In both cases, $\mathcal{D}$ should thus form a s.l.i. decomposition of G'', which is impossible since G'' is not s.l.i., and cannot be decomposed into smaller s.l.i. graphs.
  - In the second case (all three edges of G' incident to $u_{j,1}$ belong to the ith part), note that no more edge of $G'' = G' - \{u_{j,1}u_{j,2}, u_{j,1}u_{j,3}, u_{j,1}u_{j,4}\}$ can belong to the ith part; indeed, since $d_i(u_{j,1}) = 4$, each of $u_{j,2}$, $u_{j,3}$, and $u_{j,4}$ can be incident to at most one more edge of G'' in the ith part, and it can be checked that a conflict arises in case we have any more edge ($u_{j,4}u'_{j,4}$, or any edge joining two vertices in $\{u_{j,2}, u_{j,3}, u_{j,4}\}$). Thus, $\mathcal{D}$ must form a s.l.i. decomposition of G'', which is a triangle with a pendant vertex attached. Again, G'' is not s.l.i., and due to its number of edges (4), it can be checked it admits no s.l.i. decomposition. Thus, again, $\mathcal{D}$ is not s.l.i.

- Assume last v_{j_1}, v_{j_2}, and v_{j_3} are weak, and that, w.l.o.g., $v_{j_1}u_{j,1}, v_{j_2}u_{j,2}, v_{j_3}u_{j,3}$ belong to the ith part, i'th part, and i''th part, respectively, of $\mathcal{D}$.

  We note first that not all three edges of $G' = G[\{u_{j,1}, u_{j,2}, u_{j,3}, u_{j,4}, u'_{j,4}\}]$ incident to $u_{j,1}$ can belong to the ith part. Indeed, to begin with, it cannot be that, say, $i = i'$, as, since we must have $d_{i'}(u_{j,2}) \in \{3, 4\}$ by a remark above, this would imply there is a conflict in the i'th colour. Thus $i \neq i'$, and since we must have $d_{i'}(u_{j,2}) \in \{3, 4\}$, we necessarily have $u_{j,2}u_{j,3}$ and $u_{j,2}u_{j,4}$ in the i'th part. But then, again, to avoid any conflict in the ith and i'th parts, we cannot have $i'' \in \{i, i'\}$, and it is now impossible to have $d_{i''}(u_{j,3}) \in \{3, 4\}$ as required.

  So we can now assume that we have $d_i(u_{j,1}) = d_{i'}(u_{j,2}) = d_{i''}(u_{j,3}) = 3$. Note that if some of i, i', and i'' are equal, for instance $i = i'$, then $u_{j,1}, u_{j,2}$ cannot be adjacent in the ith part since $d_i(u_{j,1}) = d_{i'}(u_{j,2})$. But then, since both $u_{j,1}$ and $u_{j,2}$ have degree exactly 4, this would imply that the edge $u_{j,1}u_{j,2}$ is isolated in some part of $\mathcal{D}$, which is a contradiction to $\mathcal{D}$ being s.l.i. So, all of i, i', and i'' must be distinct. Now, because there are exactly 10 edges in the subgraph G'' induced by $\{u_{j,1}, u_{j,2}, u_{j,3}, u_{j,4}, u'_{j,4}, v_{j_1}, v_{j_2}, v_{j_3}\}$, while $d_i(u_{j,1}) = d_{i'}(u_{j,2}) = d_{i''}(u_{j,3}) = 3$ and

i, i', i'' are pairwise distinct, the restrictions of the ith, i'th, and i''th parts of $\mathcal{D}$ to G'' must each consists of only one connected component, one of which, say that in the ith part, must cover exactly four edges, including $u_{j,4}u'_{j,4}$. But then we deduce that $d_i(u_{j,1}) = 3$ and $d_i(u_{j,4}) = 2$ while $u_{j,1}u_{j,4}$ belongs to the ith part, a contradiction.

Thus, in all cases, we get to a contradiction.

Thus, if $\mathcal{D}$ is s.l.i., then we must have the claimed property. ◇

Back to the proof of Theorem 2.6, we are now ready to prove that we have the desired equivalence between F and G.

- Assume first that we have a s.l.i. decomposition $\mathcal{D}$ of G. By Claim 2.7, each variable vertex v_i is either weak or strong. We consider the truth assignment ϕ to the variables of F where each variable x_i is assigned truth value *true* if v_i is strong by $\mathcal{D}$, and truth value *false* otherwise, if v_i is weak. Now, for every clause $C_j = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$ of F, by Claim 2.8 we have that, due to vertices $u_{j,1}$, $u_{j,2}$, and $u_{j,3}$, exactly one of v_{i_1}, v_{i_2}, and v_{i_3} is strong while the other two are weak. So, by ϕ, clause C_j contains exactly one true variable and two false ones. Thus, ϕ satisfies F in a 1-in-3 way.

- Consider now a truth assignment ϕ to the variables that satisfies F in a 1-in-3 way. We construct a s.l.i. decomposition $\mathcal{D}$ of G in the following way. First, for every variable x_i of F that is set to *true* by ϕ, we add, to $\mathcal{D}$, a part containing the three edges incident to v_i in G. Note that every such part yields a star with three leaves, thus a s.l.i. graph. Now, for every variable x_i of F that is set to *false* by ϕ, we add, to $\mathcal{D}$, three partial parts each containing one of the three edges incident to v_i in G.

  It now remains to decompose the remaining edges of G, and this in such a way that every part of $\mathcal{D}$ that currently contains only one edge (to make sure some v_i will eventually be weak) is extended to a larger part yielding a s.l.i. graph. This is done as follows. Consider each clause $C_j = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$ of F. Since ϕ 1-in-3 satisfies F, we have that, say, x_{i_1} and x_{i_2} are set to *false* while x_{i_3} is set to *true*. In G, recall that, currently, $v_{i_1}u_{j,1}$ is the only edge of its part P_1, and $v_{i_2}u_{j,2}$ is the only edge of its part P_2, while $v_{i_3}u_{j,3}$ belongs to a part already inducing a s.l.i. graph. We then modify $\mathcal{D}$ as follows. First, we add to P_1 the edges $u_{j,1}u_{j,2}$ and $u_{j,1}u_{j,3}$ so that the part now yields a s.l.i. graph (a star with three leaves). Second, we add to P_2 the edges $u_{j,2}u_{j,3}$ and $u_{j,2}u_{j,4}$ so that it also yields a s.l.i. graph. Last, we add to $\mathcal{D}$ the part $\{u_{j,4}u_{j,1}, u_{j,4}u_{j,3}, u_{j,4}u'_{j,4}\}$, that also yields a s.l.i. graph.

  Note that proceeding this way for every clause C_j of F leads to $\mathcal{D}$ being a full decomposition of G. Also, since ϕ 1-in-3 satisfies F, it can be checked that all parts of $\mathcal{D}$ actually yield stars with three leaves. Thus, $\mathcal{D}$ is s.l.i.

We thus have the desired equivalence, and the claim holds. □

The fact that Theorem 2.6 holds is not too surprising, given that the reduced graphs we construct in the proof have very small vertex degrees, while, as going to be exposed more firmly throughout the next sections, vertex with small degrees have a great influence on how s.l.i. decompositions must go. To give further evidence that this is indeed for a fact, in what follows we provide an easy proof that graphs with large enough minimum vertex degrees are always strongly decomposable. We prove this mainly through the following auxiliary result, which is not new (it appears e.g. in [6], and probably in earlier works); for the sake of completeness, we provide a short proof anyway.

Lemma 2.9. *Every graph G has an orientation H with $d^+_H(v) \geq \left\lfloor \frac{d_G(v)}{2} \right\rfloor$ for all $v \in V(G)$.*

Proof. We can assume G is connected. Let G' be the graph obtained from G by adding a vertex r, and the edge vr for all vertices v of G with odd degree. In case all vertices of G were already of even degree, we simply set $G' = G$ instead. Both ways, G' has a Eulerian tour E. We follow E from start to finish, starting from any vertex, and orient edges on the way as they are traversed, to obtain an orientation H' of G'. Note that by the definition of a Eulerian tour, we have $d^+_{H'}(v) = d^-_{H'}(v) = d_{G'}(v)/2$ for all vertices $v \in V(G')$. It is easy to see, now, that if we consider H, the orientation of G being the restriction of H' to the edges of G, then we have $d^+_H(v) \geq \lfloor d_G(v)/2 \rfloor$, as claimed, for all $v \in V(G)$. □

Theorem 2.10. *Every graph G with $\delta(G) \geq 6$ is strongly decomposable.*

Proof. By Lemma 2.9, G has an orientation H such that $d^+_H(v) \geq 3$ for all vertices $v \in V(G)$. We consider the decomposition $\mathcal{D}$ of G obtained by considering every vertex v, and adding to $\mathcal{D}$ a part containing all (at least three) edges incident to v that correspond to arcs outgoing from v in H. As a result, every part of $\mathcal{D}$ yields a star with at least three edges, which implies that $\mathcal{D}$ is a s.l.i. decomposition of G. □

3. Graphs G with large value of $\chi_{\text{s.l.i.}}(G)$

In this section, we wonder about the maximum value that $\chi_{\text{s.l.i.}}(G)$ can reach, for a strongly decomposable graph G. Recall that, when it comes to l.i. decompositions, at the moment the worst connected graph B we know has $\chi_{\text{l.i.}}(B) = 4$. There is, however, only one such known graph B (exhibited in [22], obtained essentially from an edge uv by attaching two triangles at u and two triangles at v), and it is not too hard to see that this graph is a strong exception, essentially because it contains two adjacent vertices of degree 2 (which is an obvious obstruction to being strongly decomposable). Thus, through the relationship in Observation 2.1, the best we can obtain is that there exist graphs G with $\chi_{\text{s.l.i.}}(G) \geq 3$.

It is actually possible to construct strongly decomposable graphs G with $\chi_{\text{s.l.i.}}(G)$ being larger than 3, and, for that, we will exploit the fact that having adjacent vertices of small degree forces s.l.i. decompositions to behave in a certain way. Namely:

Observation 3.1. *Let G be a strongly decomposable graph having a vertex v of degree 3 adjacent to a vertex u of degree 1. Then, in every s.l.i. decomposition of G, all three edges incident to v must belong to the same part.*

Proof. Assume uv belongs to the ith part of some s.l.i. decomposition of G. Then $d_i(u) = 1$, and, so that we do not have a conflict between u and v, we must have $d_i(v) \geq 3$, which requires that all edges incident to v belong to the ith part. □

A notable consequence of Observation 3.1 is that if we have a graph with vertices of degree 1 that is either a strong exception or admits a s.l.i. k-edge-colouring, then we can construct larger graphs with the same property. This is achieved through exploiting the following operation: given a graph G with a vertex u, by *attaching a claw S* at u we mean adding the edges uv, vu', and vu'' to G, where v, u', and u'' are new vertices (thus of degree 3, 1, and 1, respectively). It has indeed the following properties:

Observation 3.2. *Let G be a graph with a vertex u of degree 1, and H be the graph obtained from G by attaching two claws S and S' at u. If G is a strong exception, then H also is. Otherwise, we have $\chi_{\text{s.l.i.}}(H) = \max\{3, \chi_{\text{s.l.i.}}(G)\}$.*

Proof. Consider how a hypothetical s.l.i. decomposition $\mathcal{D}$ of H would behave w.r.t. the claws attached at u. By Observation 3.2, for each of S and S' the three edges must belong to the same part of $\mathcal{D}$. Assume the edges of S belong to the ith part while those of S' belong to the i'th one (where possibly $i = i'$). Since u is adjacent, in the ith part and the i'th part, to a vertex v with $d_i(v) = 3$ and similarly to a vertex v with $d_{i'}(v) = 3$, we deduce, since $d(u) = 3$, that we must have $i \neq i'$ and $d_i(u) = d_{i'}(u) = 1$. Thus a part of $\mathcal{D}$ must contain the three edges of S, another part of $\mathcal{D}$ must contain the three edges of S', and the rest of $\mathcal{D}$ must form a s.l.i. decomposition of the rest of the graph, which is nothing but G. Thus, H is not decomposable in a s.l.i. way if G is not. Likewise, we can extend any s.l.i. k-edge-colouring of G to one of H, unless $k \leq 2$. Put differently, due to the arguments above, we necessarily have $\chi_{\text{s.l.i.}}(H) \geq 3$. The claim thus holds. □

Apart from providing a way to build strong exceptions from existing ones, Observations 3.1 and 3.2 can also serve to build arbitrarily large strongly decomposable graphs G with somewhat large values of $\chi_{\text{s.l.i.}}(G)$. We start off by providing a construction for trees.

Theorem 3.3. *There exist arbitrarily large trees T with $\chi_{\text{s.l.i.}}(T) = 5$.*

Proof. Let us start from the tree T obtained from an edge uv by attaching four claws S_1, S_2, S_3, S_4 at u, and attaching four claws S_5, S_6, S_7, S_8 at v. In any s.l.i. decomposition of T, recall that, by Observation 3.1, it must be that the center of each of $S_1, \ldots, S_8$ must have its three incident edges covered by a same part.

Suppose now that T admits a s.l.i. 4-edge-colouring. Note then that, since $d(u) = 5$, there must be an $i \in \{1, 2, 3, 4\}$ such that $d_i(u) \geq 2$, which requires that some claw attached at u, say S_1 w.l.o.g., has its three edges being assigned colour i. Now, so that we do not have a conflict between u and the center of S_1, we must then have $d_i(u) = 5$, which requires that uv is assigned colour 1. Mimicking these exact same arguments, there must be a $j \in \{1, 2, 3, 4\}$ such that $d_j(v) \geq 2$, and actually we must have $d_j(v) = 5$, which implies $i = j$, and thus that we have a conflict between u and v, a contraction. Thus, $\chi_{\text{s.l.i.}}(T) \geq 5$.

To see, however, that $\chi_{\text{s.l.i.}}(T) \leq 5$ holds, it suffices to consider e.g. the 5-edge-colouring of T where the three edges of S_1 are assigned colour 1, the three edges of S_2 are assigned colour 2, the three edges of S_3 are assigned colour 3, the three edges of S_4 are assigned colour 4, and all other edges are assigned colour 5. Thus, we have $\chi_{\text{s.l.i.}}(T) = 5$.

Last, since T has vertices of degree 1, and attaching claws to a graph results (without creating new cycles) in a graph with vertices of degree 1, by repeatedly attaching, starting from T, pairs of claws to vertices of degree 1, by Observation 3.2 we get arbitrarily large trees T' that satisfy $\chi_{\text{s.l.i.}}(T') = 5$. □

To describe the next construction, we need to tweak the way we attach claws by a bit. Namely, given a graph G with two vertices u and v, by *attaching* a claw at u and v we mean adding the edges uw, vw, and wx to G, where w and x are new vertices (thus of degree 3 and 1, respectively). Exploiting this operation, we can play with adjacent vertices with small degree to prove the following result:

Theorem 3.4. *There exist arbitrarily large graphs G with $\chi_{\text{s.l.i.}}(G) = 16$.*

Proof. We begin with the graph G obtained from five isolated vertices u, v_1, v_2, v_3, and v_4 by attaching four claws S_1, S_2, S_3, S_4 to u and v_1, four claws S_5, S_6, S_7, S_8 to u and v_2, four claws $S_9, S_{10}, S_{11}, S_{12}$ to u and v_3, and four claws $S_{13}, S_{14}, S_{15}, S_{16}$ to u and v_4 (see Figure 2). Then u has degree 16, while all of v_1, v_2, v_3, and v_4 have degree 4.

Consider now any s.l.i. edge-colouring ϕ of G. By Observation 3.1, recall that the center of each of $S_1, \ldots, S_{16}$ must have its three incident edges assigned the same colour

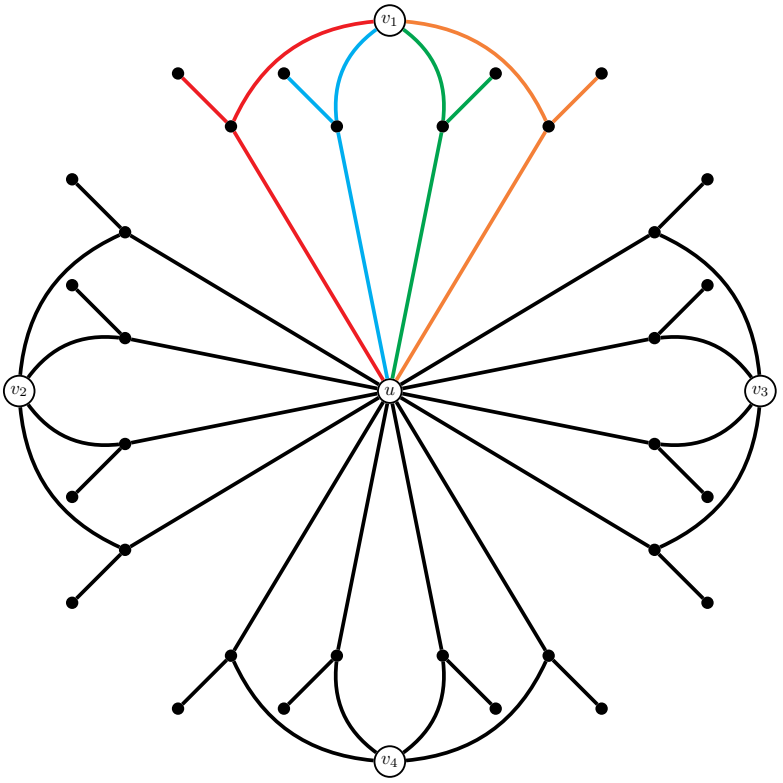


Figure 2: A graph G with $\chi_{\text{s.l.i.}}(G) = 16$, and a partial s.l.i. 16-edge-colouring of G.

by ϕ. This guarantees also that there is no i such that $d_i(v_1) \in \{2, 3, 4\}$, as otherwise there would be a conflict in colour i. Obviously, this also holds for v_2, v_3, and v_4. Thus, all of S_1, S_2, S_3, and S_4 must have their centers incident to four different colours, and similarly for all of S_5, S_6, S_7, and S_8, for all of S_9, S_{10}, S_{11}, and S_{12}, and for all of S_{13}, S_{14}, S_{15}, and S_{16}. This implies that, for any i, we must have $d_i(u) \leq 4$, and actually, so that we do not have a conflict in colour i, we must have $d_i(u) = 1$. From this, we get $\chi_{\text{s.l.i.}}(G) \geq 16$.

To see that equality actually holds, it suffices to consider the 16-edge-colouring ϕ of G where S_i has its three incident edges assigned colour i, for all $i \in \{1, \ldots, 16\}$. Clearly, ϕ is s.l.i. Now, since $\delta(G) = 1$, by repeatedly attaching pairs of claws to vertices of degree 1, by Observation 3.2 we get arbitrarily large graphs with the same property as G. □

4. Common classes of graphs

In this section, we study the parameter $\chi_{\text{s.l.i.}}(G)$ for graphs G in common graph classes. In particular, we focus on complete bipartite graphs, subcubic graphs, complete graphs, and trees. In each case, we wonder about both strong exceptions and the parameter $\chi_{\text{s.l.i.}}$, and compare the results we obtain with what is currently known for l.i. decompositions.

4.1. Complete bipartite graphs

In this section, we determine the value of $\chi_{\text{s.l.i.}}(K_{p,q})$ for every complete bipartite graph $K_{p,q}$. Recall, see [1], that we have $\chi_{\text{l.i.}}(K_{p,q}) = 1$ if $p \neq q$, and $\chi_{\text{l.i.}}(K_{p,q}) = 2$ otherwise, if $p = q$. Although there are a few differences for s.l.i. decompositions, we prove that $\chi_{\text{l.i.}}(K_{p,q})$ and $\chi_{\text{s.l.i.}}(K_{p,q})$ are always quite close, for any $K_{p,q}$.

Theorem 4.1. *Let p, q be two integers with $1 \leq p \leq q$. Then, we have:*

- *if* $q > p + 1$, *then* $\chi_{\text{s.l.i.}}(K_{p,q}) = 1$;
- *if* $q = p + 1$, *then:*
  - *if* $p = 1$, *then* $K_{p,q}$ *is a strong exception;*
  - *otherwise,* $\chi_{\text{s.l.i.}}(K_{p,q}) = 2$;
- *if* $q = p$, *then:*
  - *if* $p \leq 2$, *then* $K_{p,q}$ *is a strong exception;*
  - *if* $p = 3$, *then* $\chi_{\text{s.l.i.}}(K_{p,q}) = 3$;
  - *otherwise,* $\chi_{\text{s.l.i.}}(K_{p,q}) = 2$.

Proof. If $q > p + 1$, then $K_{p,q}$ is s.l.i., and thus $\chi_{\text{s.l.i.}}(K_{p,q}) = 1$.

Now assume $q = p + 1$. Since $K_{p,q}$ is not s.l.i., we have $\chi_{\text{s.l.i.}}(K_{p,q}) \geq 2$. If $p = 1$, then $K_{p,q}$ is a path of length 2, and thus a strong exception by Observation 2.4. Now, if $p \geq 2$, then let v be any vertex in the part of size p of $K_{p,q}$. Note that for E being the set of edges incident to v, the subgraphs of $K_{p,q}$ induced by E and $E(K_{p,q}) \setminus E$ are s.l.i. Thus, $(E, E(K_{p,q}) \setminus E)$ forms a s.l.i. decomposition of $K_{p,q}$ and we have $\chi_{\text{s.l.i.}}(K_{p,q}) = 2$.

Last consider when $p = q$. If $p \leq 2$, then $\Delta(K_{p,q}) \leq 2$, and $K_{p,q}$ is a strong exception by Observation 2.4. Now consider when $p \geq 3$. Again, since $K_{p,q}$ is not s.l.i., we have $\chi_{\text{s.l.i.}}(K_{p,q}) \geq 2$. If $p \geq 4$, then we pick two vertices v and v' in the part of size p of $K_{p,q}$, and set E as the set of edges incident to either v or v'. Note that E and $E(K_{p,q}) \setminus E$ induce s.l.i. graphs in $K_{p,q}$; thus, $\chi_{\text{s.l.i.}}(K_{p,q}) = 2$. Last, if $p = 3$, then $K_{p,q}$ is 3-regular. Recall that, by Observation 2.3, in every s.l.i. decomposition of $K_{p,q}$, each part must contain a vertex of degree 3. Due to the structure of $K_{p,q}$, note that if, in some part, a vertex v has degree 3, then its three neighbours must have degree 1. Thus, any s.l.i. decomposition of $K_{p,q}$ must form a decomposition into claws. So we have $\chi_{\text{s.l.i.}}(K_{p,q}) \geq 3$, and since a decomposition into three claws of $K_{p,q}$ clearly exists, we actually have $\chi_{\text{s.l.i.}}(K_{p,q}) = 3$. □

4.2. Subcubic graphs

We saw earlier through the proof of Theorem 2.6 that for graphs with low maximum degree (4 there), s.l.i. decompositions are decompositions into very limited structures, which makes it easy to understand how s.l.i. decompositions behave in such contexts. Following this idea, in this section we focus on subcubic graphs, *i.e.*, graphs with maximum degree at most 3. For context, let us remind that, regarding l.i. decompositions, the best results we know of are that decomposable subcubic graphs admit l.i. 4-edge-colourings [16], and that some even admit l.i. 3-edge-colourings (see e.g. [15]). At the moment, however, it is still unknown whether subcubic graphs G with $\chi_{\text{l.i.}}(G) = 4$ exist (as the only known connected graph with this property, exhibited in [22], has maximum degree 5).

First, we note that s.l.i. decompositions of subcubic graphs behave in a narrow way:

Observation 4.2. *The only connected subcubic s.l.i. graph is the claw. Therefore, any s.l.i. decomposition of a subcubic graph is a decomposition into claws.*

Proof. This follows mainly from the fact that, in a connected subcubic graph, the vertex degrees lie in $\{1, 2, 3\}$. Thus, for any edge uv of such a graph, so that $d(u)$ and $d(v)$ differ by at least 2, we must have $\{d(u), d(v)\} = \{1, 3\}$. □

We note also that Theorem 2.6 does not hold for subcubic graphs. This might indicate that subcubic graphs that are strong exceptions should be easier to recognise, and, indeed, on the way below we will get to some understanding of their structure. However, although their structure is not hard to comprehend, it is not that natural to describe through easy words. The upcoming results and tools will actually allow for some kind of nice description.

We continue with other obstructions for s.l.i. decompositions in subcubic graphs.

Observation 4.3. *Any strongly decomposable graph cannot contain:*

- *a vertex of degree* 2 *adjacent to a vertex of degree* 1*;*
- *two adjacent vertices of degree* 2*.*

Proof. This follows directly from Observation 4.2. □

From Observation 4.2, we also deduce something reminiscent of Claim 2.7. Reusing our terminology, by a s.l.i. decomposition of a subcubic graph, a vertex of degree 3 is said *strong* if its three incident edges belong to the same part. Otherwise, if its three incident edges belong to three distinct parts, then it is said *weak*. Then:

Observation 4.4. *In any s.l.i. decomposition of a subcubic graph, every vertex of degree* 3 *is either weak or strong. Also:*

- *every vertex of degree* 3 *with a neighbour of degree* 1 *or* 2 *is strong;*
- *for every two adjacent vertices of degree* 3*, one is weak while the other is strong.*

Proof. This is, again, a direct consequence of Observation 4.2. □

From Observation 4.4, we get lots of ways to build arbitrarily large connected subcubic strong exceptions. For instance, a subcubic graph is a strong exception whenever it has a path $v_1 \dots v_k$ where $k \geq 2$ even, all v_i's are vertices of degree 3, and v_1 and v_k are adjacent to vertices of degree 1. Likewise, a subcubic graph is a strong exception whenever it has a cycle $v_1 \dots v_k v_1$ where $k \geq 3$ is odd and all v_i's have degree 3.

From these thoughts, in some sense the main idea here is that any s.l.i. decomposition of a subcubic graph leads to a bipartition $W \cup S$ of the vertices of degree 3, where W contains the weak ones while S contains the strong ones, and this bipartition necessarily fulfils a number of properties. More particularly, things are highly forced in connected subcubic graphs having vertices of degree 1 or 2, while for connected cubic graphs there is a bit more room. We consider these two cases separately through what follows.

Connected non-cubic subcubic graphs

For convenience, below we say a connected graph G is *strictly subcubic* if $\delta(G) \leq 2$ and $\Delta(G) = 3$ (that is, G is of maximum degree exactly 3, but not cubic). As a starting point, we note that deciding whether G is strongly decomposable can be decided easily.

Theorem 4.5. *Determining if a given connected strictly subcubic graph is a strong exception can be done in polynomial time.*

Proof. Let G be a connected strictly subcubic graph. By Observation 4.2, a s.l.i. decomposition of G would essentially be a decomposition into claws, which equivalently can be seen as a bipartition (W, S) of the vertices of degree 3 of G such that W contains the weak ones, while S contains the strong ones. Recall also that this bipartition must comply with the properties described in Observation 4.4.

Let us remind also that because G is strictly subcubic, it contains vertices of degree at most 2. By Observation 4.3, the neighbours of all such vertices are vertices of degree 3. Also, we know they must be strong, by any s.l.i. decomposition of G.

We now build two subsets (with possibly non-empty intersection) W and S of the set of vertices of degree 3 of G, as follows. We start by considering every vertex of degree at most 2 of G, and add its at most two neighbours to S. While doing so and onwards, whenever adding a vertex of degree 3 to S (resp. W), we also add all its neighbours of degree 3 to W (resp. S), if they are not already present in this set. Clearly, this process must end at some point. Now, if $W \cap S = \varnothing$, then (W, S) forms a partition of the set of vertices of degree 3 of G, from which we can deduce a decomposition into claws of G (by just making sure the vertices of S are strong, so that those of W are weak), and thus a s.l.i. decomposition of G. Otherwise, by the previous remarks we can assert that G is a strong exception (since the vertices of degree at most 2, as well as how the vertices of degree 3 are structured, imply we cannot partition the vertices of degree 3 into weak and strong ones). Since this whole process can clearly be achieved in polynomial time, the claim follows. □

If a connected strictly subcubic graph G is strongly decomposable, then, by the arguments in the proof of Theorem 4.5, the bipartition (W, S) of the set of its vertices of degree 3 into weak and strong vertices of degree 3, respectively, is unique. We thus denote these two sets by $W(G)$ and $S(G)$. To determine $\chi_{\text{s.l.i.}}(G)$, the question, now, is about considering every vertex of $S(G)$, assigning the same colour (from a minimum pool) to all the edges incident to it, and so that no vertex of $W(G)$ is incident to two edges assigned the same colour. This can thus be regarded as a pure colouring problem, which we could investigate by building a graph with vertex set the vertices of $S(G)$, in which two vertices are adjacent if they have a common neighbour (of degree 2, or of degree 3 thus belonging to $W(G)$).

These concerns somewhat connect to exact square colourings and injective colourings of graphs (see e.g. [10] for more details). Recall that an *injective colouring* of a graph G is a vertex-colouring where no vertex has two neighbours assigned the same colour, while an *exact square colouring* is a vertex-colouring where every two vertices at distance exactly 2 are assigned distinct colours. The smallest $k \geq 1$ such that G admits injective k-colourings is denoted by $\chi_{\text{i}}(G)$, while the smallest $k \geq 1$ such that G admits exact square k-colourings is denoted by $\chi^{[\#2]}(G)$. As reminded in [10], an injective colouring is always an exact square colouring, and, by arguments above, it is not too complicated to see that, for a connected strictly subcubic graph, an exact square k-colouring ϕ yields a s.l.i. k-edge-colouring (just consider every vertex v of $S(G)$, and assign colour $\phi(v)$ to its three incident edges). Strictly speaking, note that we could actually only care about vertex-colourings where the distance-2 constraint is verified only for vertices of $S(G)$. Still, since it was proved that subcubic graphs G satisfy $\chi_{\text{i}}(G) \leq 7$ (see [11]), we deduce:

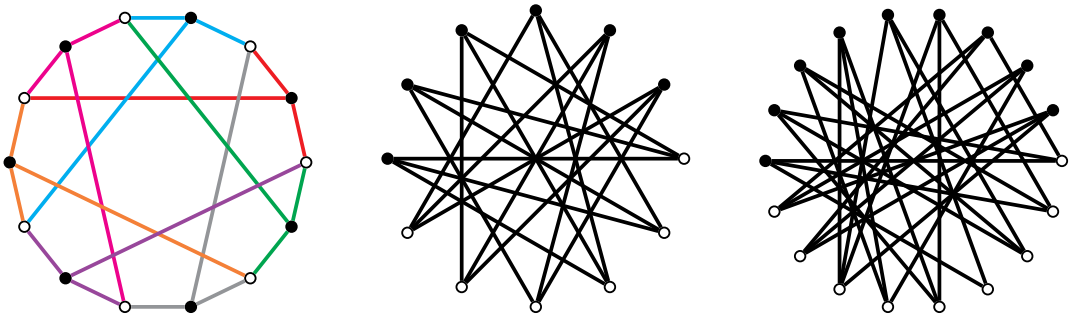
Theorem 4.6. *If G is a connected strongly decomposable strictly subcubic graph, then*

$$\chi_{\text{s.l.i.}}(G) \leq \chi^{[\#2]}(G) \leq \chi_{\text{i}}(G) \leq 7.$$

We now focus on cubic graphs throughout what follows. In particular, on the way, we will get to discussing the tightness of the upper bound established in Theorem 4.6.

Connected cubic graphs

For connected cubic graphs G, things are slightly different in that, since there are no vertices of degree 1 and 2, the bipartition of the vertices of degree 3 of G (thus of actually the whole vertex set) into $W(G)$ and $S(G)$ is not fixed. However, note that, after deciding



(a) Heawood graph (b) G_1 (c) G_2

Figure 3: Three strongly decomposable cubic (bipartite) graphs. Vertices in black and white show the partition classes of the bipartition, and, in (a), edge colours depict a s.l.i. 7-edge-colouring.

that a given vertex of degree 3 of G must be strong (or weak) by some s.l.i. edge-colouring, the whole bipartition into $W(G)$ and $S(G)$ is then forced, since G is connected.

Before getting to that point, let us first observe that the characterisation of connected strongly decomposable cubic graphs is easier to establish. Indeed, due to previous remarks:

Observation 4.7. *A cubic graph is strongly decomposable if and only if it is bipartite.*

Now, previous remarks and Theorem 4.6 here adapt to:

Theorem 4.8. *Let G be a connected strongly decomposable cubic graph, thus bipartite (by Observation 4.7) with bipartition (U, V). Then $\chi_{\text{s.l.i.}}(G)$ is the minimum number of colours needed to colour, in an exact square way in G, either only the vertices of U, or only those of V. Regardless, 7 colours always suffice, by Theorem 4.6. Thus, $\chi_{\text{s.l.i.}}(G) \leq 7$.*

From this, interesting side questions and remarks arise. For instance, one can wonder whether, for a connected strongly decomposable cubic graph G with bipartition (U, V), the choice of which of U and V should play the role of $S(G)$ is important. First off, we observe there are instances where this is not the case, as the two possible ways to choose $S(G)$ lead to the same number of needed colours. This is well illustrated by the Heawood graph (depicted in Figure 3 (a)), which was proved to be the only connected subcubic graph G with $\chi_{\text{i}}(G) = 7$ (see [11]), and has the property that any two vertices in any of the partition classes have a common neighbour. By previous arguments, we then get that for G being the Heawood graph, we have $\chi_{\text{s.l.i.}}(G) = 7$ (but, due to Theorem 4.6, and, as reported in [10], at most 6 colours suffice for all other connected subcubic graphs).

This phenomenon, however, is not systematic, in that there exist connected strongly decomposable cubic graphs G with bipartition (U, V) for which the choice of which part is $S(G)$ is important, in order for the number of colours by a s.l.i. edge-colouring to be minimised. As an example, using computer programs, we were able to observe that, for some pairs $\{\alpha, \beta\}$, there are (many) strongly decomposable cubic bipartite graphs G for which choosing one partition class as $S(G)$ implies we need to use α colours at best, while choosing the other partition class implies we need to use β colours at best. More precisely, there exist such graphs for all pairs $\{3, 4\}$, $\{3, 5\}$, $\{4, 5\}$, and $\{5, 6\}$ as $\{\alpha, \beta\}$. For instance, the graphs G_1 and G_2, depicted in Figures 3 (b) and (c), illustrate this phenomenon, for $\{\alpha, \beta\}$ being $\{3, 5\}$ and $\{5, 6\}$, respectively.

4.3. Complete graphs

In the field we are dealing with, it is always interesting to investigate complete graphs at some point, as they form a peculiar class of regular graphs (which are of interest, since

they are in some sense among the least irregular graphs), with lots of constraints because of their density. As a reminder, it was proved in [1] that we have $\chi_{\text{l.i.}}(K_n) \leq 3$ for any complete graph K_n with $n \geq 4$ (recall K_3 is an exception), and that we actually have equality since decomposing a regular graph into two l.i. graphs is equivalent to labelling its edges with 1 and 2 so that adjacent vertices are incident to distinct sums of labels, while this latter thing cannot be achieved in complete graphs (see e.g. [9]).

To establish our main results in this section, we will somewhat build on the proof that $\chi_{\text{l.i.}}(K_n) \leq 3$ holds for every $n \geq 4$, which involves a nice trick which we will also employ. Thus, let us remind these ideas first. To begin with, it can easily be checked that K_4 admits a l.i. 3-edge-colouring into three paths of length 2. Note that this edge-colouring of K_4 has the property that there is no vertex having all of its incident edges assigned colour 1. This implies that if we add a universal vertex and assign colour 1 to all its incident edges, then we get a l.i. 3-edge-colouring of K_5, this time with the extra property that there is no vertex having all its incident edges assigned colour 2. So if we add a universal vertex and assign colour 2 to all its incident edges, then we get a l.i. 3-edge-colouring of K_6, with the extra property that there is no vertex having all its incident edges assigned colour 1. And so on: by repeatedly adding universal vertices with their incident edges assigned some well-chosen, same colour, we can construct l.i. 3-edge-colourings of any larger complete graph.

So that we can employ such arguments as is for s.l.i. decompositions, we need to investigate a few things. First, we need to come up with a first (ideally small) complete graph admitting s.l.i. decompositions. Or, in some sense, we need to identify complete graphs that are strong exceptions. And then we need to investigate in what contexts we can add a universal vertex to a graph (given with a s.l.i. edge-colouring) with all its incident edges assigned the same colour, so that a s.l.i. edge-colouring results.

We start off by establishing that any complete graph K_n with $n \leq 5$ is a strong exception.

Theorem 4.9. *K_n is a strong exception for every $n \in \{2, 3, 4, 5\}$.*

Proof. For $n \in \{2, 3\}$, this follows from the fact that K_n is an exception (recall Theorem 1.1).

Let us now focus on K_4. Towards a contradiction, assume K_4 admits a s.l.i. decomposition $\mathcal{D}$. Since $\Delta(K_4) = 3$, by Observation 4.2 we have that $\mathcal{D}$ must form a decomposition into claws. Denoting by E the three edges of any part of $\mathcal{D}$, we thus deduce that $\mathcal{D}$ must also yield a s.l.i. decomposition of $K_4 - E$, which is isomorphic to K_3. Since K_3 admits no s.l.i. decomposition, we reach a contradiction.

We last consider K_5. Towards a contradiction, assume there is a s.l.i. decomposition $\mathcal{D}$ of K_5. Set $V(K_5) = \{v_1, \ldots, v_5\}$. We consider two main cases:

- We assume first all edges incident to, say, v_1 belong to some ith part E_i of $\mathcal{D}$. Then $d_i(v_1) = 4$, and, so that $K_5[E_i]$ is s.l.i., we must have $d_i(v_2), d_i(v_3), d_i(v_4) \leq 2$. If $d_i(v_2) = d_i(v_3) = d_i(v_4) = 1$, then note that $\mathcal{D}$ would yield a s.l.i. decomposition of $K_5 - E_i$, which is isomorphic to the strong exception K_4, a contradiction. So E_i must also contain, say, v_2v_3, and so that v_2 and v_3 are not in conflict in the ith part, actually we must have $\{d_i(v_2), d_i(v_3)\} = \{2, 4\}$. But then we must have a conflict in $K_5[E_i]$ between v_1 and one of v_2 and v_3, a contradiction.

- Since, by Observation 2.3, for every part E_i of $\mathcal{D}$ it must be that $K_5[E_i]$ contains a vertex of degree at least 3, but the previous case does not apply, hence the decomposition must be a decomposition into claws. Since K_5 has 10 edges, we reach a contradiction.

Thus, in all cases we deduce that $\mathcal{D}$ cannot be s.l.i., and the claim holds. □

Theorem 4.10. $\chi_{\text{s.l.i.}}(K_6) = 5$.

Proof. We first prove that $\chi_{\text{s.l.i.}}(K_6) \leq 5$. Set $V(K_6) = \{v_0, \ldots, v_5\}$, and consider the $\{0, \ldots, 4\}$-edge-colouring ϕ of K_6 obtained as follows:

- for every $i \in \{0, \ldots, 4\}$, assign colour i to $v_i v_5$;
- for every $i \in \{0, 1, 2\}$, assign colour i to $v_i v_{i+1}$ and $v_i v_{i+2}$;
- assign colour 3 to $v_3 v_4$ and $v_3 v_0$;
- assign colour 4 to $v_4 v_0$ and $v_4 v_1$.

Then ϕ forms a decomposition of K_6 into five claws, thus a s.l.i. decomposition.

We now prove that $\chi_{\text{s.l.i.}}(K_6) \geq 5$. It can be checked that the only connected s.l.i. graphs with order at most 6 are the stars with 3, 4, and 5 edges, and $K_{2,4}$ which has 8 edges. From this, we deduce that, in a s.l.i. decomposition of K_6, no part can yield a non-connected graph. The other way round, every part must yield a connected s.l.i. graph, being one of the four aforementioned small graphs. Note that removing the edges of a $K_{2,4}$ from K_6 leaves an isolated edge, so no part of a s.l.i. decomposition can yield a $K_{2,4}$. Likewise, removing the edges of a star with 5 edges from K_6 results in K_5, which is a strong exception by Theorem 4.9; so, again, no part can yield a star with 5 edges. From these arguments, we deduce that any s.l.i. decomposition of K_6 must be into parts each yielding a star on 3 or 4 edges. Since K_6 has 15 edges, actually any s.l.i. 4-edge-colouring of K_6 must have three colours yielding a star on 4 edges, and one colour yielding a star on 3 edges. Then, since all vertices of K_6, call them now $v_1, \ldots, v_6$ for simplicity, have degree 5, by analysing which vertices are the centers of these stars, we deduce that such a 4-edge-colouring must behave as follows:

- Assume that $v_1v_6, v_1v_5, v_1v_4, v_1v_3$ are assigned colour 1, yielding a star on 4 edges.
- Then, because v_1v_2 is not assigned colour 1, clearly v_2 must be the center of some star (as, besides v_1v_2, there remain four uncoloured edges incident to v_2), and this star must contain v_1v_2 as all other edges incident to v_1 are already coloured.
  - Assume first that $v_2v_1, v_2v_6, v_2v_5, v_2v_4$ are assigned colour 2, yielding a star on 4 edges. Then v_3 is necessarily the center of the last star on 4 edges, since each of v_4, v_5, and v_6, at this point, is incident to only three uncoloured incident edges. So $v_3v_2, v_3v_4, v_3v_5, v_3v_6$ are assigned colour 3, yielding the last star on 4 edges. But then the only three remaining uncoloured edges, v_4v_5, v_5v_6, v_6v_4, yield a triangle, not a star on 3 edges, a contradiction.
  - Assume second that v_2v_1, v_2v_6, v_2v_5 are assigned colour 2, yielding a star on 3 edges. Since only v_3 and v_4 are now incident to four uncoloured edges, we have that v_3 and v_4 must be the centers of the two remaining stars on 4 edges, which is impossible since they are joined by an edge. This is thus another contradiction.

Thus, no s.l.i. 4-edge-colouring of K_6 can exist, and we have $\chi_{\text{s.l.i.}}(K_6) = 5$. □

We can now prove a somewhat tight result for all complete graphs K_n with $n \geq 6$. As mentioned earlier, this is done through adapting, to our context, a colouring tool from [1]:

	After 1st step				After 2nd step				After 3rd step			
v_i	$d_1(v_i)$	$d_2(v_i)$	$d_3(v_i)$	$d_4(v_i)$	$d_1(v_i)$	$d_2(v_i)$	$d_3(v_i)$	$d_4(v_i)$	$d_1(v_i)$	$d_2(v_i)$	$d_3(v_i)$	$d_4(v_i)$
v_1	1	1	1	1	2	2	2	2	4	4	2	6
v_2	1	1	1	1	2	2	2	2	4	4	4	4
v_3	2	1	1	1	4	2	2	1	6	4	4	3
v_4	0	3	1	1	1	4	2	2	3	6	4	4
v_5	0	0	4	1	1	1	5	2	3	3	7	4
v_6	0	0	0	5	1	1	1	6	3	3	3	8
v_7	-	-	-	-	6	1	1	1	8	3	3	3
v_8	-	-	-	-	0	7	1	1	2	9	3	3
v_9	-	-	-	-	0	0	8	1	2	2	10	3
v_{10}	-	-	-	-	1	0	0	8	3	2	2	10
v_{11}	-	-	-	-	-	-	-	-	11	2	2	2
v_{12}	-	-	-	-	-	-	-	-	1	12	2	2
v_{13}	-	-	-	-	-	-	-	-	1	1	12	3
v_{14}	-	-	-	-	-	-	-	-	1	1	1	14
v_{15}	-	-	-	-	-	-	-	-	14	1	1	1
v_{16}	-	-	-	-	-	-	-	-	0	15	1	1
v_{17}	-	-	-	-	-	-	-	-	0	0	15	2
v_{18}	-	-	-	-	-	-	-	-	0	0	0	17

Table 1: Resulting coloured degrees in the proof of Theorem 4.13.

Observation 4.11. *Let G be a graph with order n, and ϕ be a s.l.i. k-edge-colouring of G. If, for some $i \in \{1, \ldots, k\}$, we have $d_i(v) < n - 2$ for all $v \in V(G)$, then, through adding a universal vertex to G with all n incident edges assigned colour i, we get a s.l.i. k-edge-colouring of the resulting graph.*

Theorem 4.12. $\chi_{\text{s.l.i.}}(K_n) \leq 5$ *for all* $n \geq 6$.

Proof. By Theorem 4.10, K_6 admits a s.l.i. 5-edge-colouring ϕ, and, actually, it admits a ϕ such that no vertex v satisfies $d_1(v), d_2(v), d_3(v) \geq |V(K_6)| - 2 = 4$. So, by Observation 4.11, by adding a universal vertex with its incident edges assigned colour 1, we get a s.l.i. 5-edge-colouring of K_7, with the extra property that no vertex v satisfies $d_2(v), d_3(v) \geq |V(K_7)| - 2 = 5$. Then, by adding a universal vertex with its incident edges assigned colour 2, we get a s.l.i. 5-edge-colouring of K_8, with the extra property that no vertex v satisfies $d_3(v) \geq |V(K_8)| - 2 = 6$. Then, by adding a universal vertex with its incident edges assigned colour 3, we get a s.l.i. 5-edge-colouring of K_9, with the extra property that no vertex v satisfies $d_1(v) \geq |V(K_9)| - 2 = 7$. Then, by adding a universal vertex with its incident edges assigned colour 1, we get a s.l.i. 5-edge-colouring of K_{10}, with the extra property that no vertex v satisfies $d_2(v) \geq |V(K_{10})| - 2 = 8$. Then, by adding a universal vertex with its incident edges assigned colour 2, we get a s.l.i. 5-edge-colouring of K_{11}, with the extra property that no vertex v satisfies $d_3(v) \geq |V(K_{11})| - 2 = 9$. Then, by adding a universal vertex with its incident edges assigned colour 3, we get a s.l.i. 5-edge-colouring of K_{12}, with the extra property that no vertex v satisfies $d_1(v) \geq |V(K_{12})| - 2 = 10$. And so on. Since these arguments keep on applying, the claim holds. □

The bound in Theorem 4.12 can be perceived as tight, because of Theorem 4.10. However, one can legitimately wonder whether the bound can be improved for large enough complete graphs. We prove this is the case, for complete graphs on at least 18 vertices.

Theorem 4.13. $\chi_{\text{s.l.i.}}(K_n) \leq 4$ *for all* $n \geq 18$.

Proof. This is proved similarly as Theorem 4.12, by first proving that $\chi_{\text{s.l.i.}}(K_{18}) \leq 4$, and that K_{18} even admits s.l.i. 4-edge-colourings from which we can then repeatedly apply Observation 4.11 for a set of three colours, so that a s.l.i. 4-edge-colouring of any larger complete graph is obtained as a result.

We obtain a s.l.i. 4-edge-colouring of K_{18} in the following, constructive way. The main idea is to start from two isolated vertex v_1 and v_2. During successive steps, we will then add $v_3, v_4, \ldots, v_{18}$ one at a time, where each v_i added this way is adjacent to all of

$v_1, \ldots, v_{i-1}$. All their incident edges added this way will be coloured with colours $1, 2, 3, 4$ so that they induce a s.l.i. 4-edge-colouring. More precisely, this will be done so that, after adding v_{18}, we can add v_1v_2 and assign some colour to it so that, altogether, we get a s.l.i. 4-edge-colouring of K_{18} (see Table 1 for the successive coloured degrees we get below).

We split the process into three main steps, starting from v_1 and v_2 being isolated.

- **First step.**

  First, we add v_3 and assign colour 1 to its incident edges. Second, we add v_4 and assign colour 2 to its incident edges. Third, we add v_5 and assign colour 3 to its incident edges. Fourth, we add v_6 and assign colour 4 to its incident edges.

At this point, note that the main problem we have is that v_1v_3 and v_2v_3 are assigned colour 1, but we have $d_1(v_1) = d_1(v_2) = 1$ and $d_1(v_3) = 2$ (thus a conflict in colour 1). The main goal of the second step is to get rid of this conflict.

- **Second step.**

  First, we add v_7 and assign colour 1 to its incident edges. Second, we add v_8 and assign colour 2 to its incident edges. Third, we add v_9 and assign colour 3 to its incident edges. Fourth, we add v_{10} and assign colour 4 to its five incident edges, except for the edge going to v_3 to which we assign colour 1.

The third step, now, is to make sure $d_3(v_1)$ and $d_3(v_2)$ differ by at least 2, so that, eventually, no conflict arises when adding the edge v_1v_2 and assigning colour 3 to it.

- **Third step.**

  First, we add v_{11} and assign colour 1 to its incident edges. Second, we add v_{12} and assign colour 2 to its incident edges. Third, we add v_{13} and assign colour 3 to its incident edges, except for the edge going to v_1 which we assign colour 4. Fourth, we add v_{14} and assign colour 4 to its incident edges. Last, we add $v_{15}, v_{16}, v_{17}, v_{18}$ in the exact same way. That is, so that v_{17} has all its incident edges assigned colour 3, but the one going to v_1 which is assigned colour 4.

By then adding v_1v_2 and assigning colour 3 to that edge, it can be checked we get a s.l.i. 4-edge-colouring ϕ of K_{18}. Furthermore, by ϕ, no vertex of K_{18} is incident to at least 16 edges assigned colour 1, so, by Observation 4.11, by adding a universal vertex with all its incident edges assigned colour 1, we get a s.l.i. 4-edge-colouring of K_{19}. By that edge-colouring, no vertex of K_{19} is incident to at least 17 edges assigned colour 2, so we can proceed the same way to K_{20} (with colour 2). Then, there is no vertex incident to at least 18 edges assigned colour 3, so we can again extend to K_{21} with colour 3. From here on, we can then repeat this process, adding universal vertices with incident edges assigned colours 1, 2, and 3 alternately, to get a s.l.i. 4-edge-colouring of a bigger complete graph each step. Thus, the claimed result holds. □

4.4. Trees

We now turn our attention to trees. Regarding l.i. decompositions, trees have been investigated mainly in [4]. Namely, the authors proved that every decomposable tree T (*i.e.*, being different from an odd-length path) satisfies $\chi_{\text{l.i.}}(T) \leq 3$, and that there is an easy characterisation of trees T with $\chi_{\text{l.i.}}(T) = 3$. In particular, for trees T with maximum degree at least 5 we have $\chi_{\text{l.i.}}(T) \leq 2$, and, more generally speaking, determining $\chi_{\text{l.i.}}(T)$ for a given tree T can be done in polynomial time.

One issue we encountered when trying to establish bounds on $\chi_{\text{s.l.i.}}(T)$ for strongly decomposable trees T, is that it is not obvious which trees are actually strongly decomposable. As a consequence, removing, e.g. to invoke inductive arguments, even the simplest structure from a strongly decomposable tree might result in a strong exception, and, even when it does, this does not grant lots of information on what the tree actually looks like. For this reason, our main purpose in this section is only to establish some reasonable constant upper bound on $\chi_{\text{s.l.i.}}(T)$ for strongly decomposable trees T. For comparison, remind that we exhibited trees T with $\chi_{\text{s.l.i.}}(T) = 5$ earlier, recall Theorem 3.3.

The main result we prove here, thus reads as follows:

Theorem 4.14. *If T is a strongly decomposable tree, then $\chi_{\text{s.l.i.}}(T) \leq 16$.*

Proof. The proof is by induction on the order of T. Since the claim obviously holds when T has order at most 3, we focus on proving the general case. We consider two main cases.

Assume first that $\Delta(T) \leq 15$. Since T is strongly decomposable, it admits a s.l.i. edge-colouring ϕ. If ϕ is a 16-edge-colouring, then we are done. Otherwise, consider the following arguments. Since T is connected, and ϕ assigns more than one colour, there must be a vertex r having incident edges assigned, say, colour 1 and colour 2. Root T at r, and decompose T into T' and T'', where T' contains the edges of $S(e, r)$ for all edges e incident to r assigned colour 1, where $S(e, r)$ denotes the subtree of T with root r induced by the descendants of r having e on their unique shortest path to r; and $T'' = T - E(T')$. Note that $(E(T'), E(T''))$ forms a partition of $E(T)$, and $V(T') \cap V(T'') = \{r\}$. Also, it should be clear that ϕ, when restricted to any of T' and T'', remains a s.l.i. edge-colouring. Thus, both T' and T'' are strongly decomposable. Now, since T' and T'' are smaller than T, by the induction hypothesis there are s.l.i. 16-edge-colouring ϕ' and ϕ'' of T' and T'', respectively. And, since $\Delta(T) \leq 15$, we have $d(r) \leq 15$, and, thus $d_{T'}(r) + d_{T''}(r) \leq 15$. Thus, it is possible to permute colours (around r) by ϕ' and ϕ'', if needed, so that, when combined and considered in T, they yield a s.l.i. 16-edge-colouring of T. Indeed, it suffices to make sure that the edges incident to r being part of T' are assigned, by ϕ', colours different from those assigned by ϕ'' to the edges incident to r being part of T''.

Assume now that $\Delta(T) \geq 16$, and let r be a vertex with maximum degree Δ (thus at least 16). We denote by $u_1, \ldots, u_\Delta$ the neighbours of r, and by $T_1, \ldots, T_\Delta$ the trees $S(ru_1, r), \ldots, S(ru_\Delta, r)$, respectively. Since T is strongly decomposable, it admits a s.l.i. decomposition $\mathcal{D}$. If $\mathcal{D}$ has at most 16 parts, then we are done. Otherwise, we denote by $\mathcal{D}_1, \ldots, \mathcal{D}_\Delta$ the restrictions of $\mathcal{D}$ to $T_1, \ldots, T_\Delta$, respectively. Note that, for any $i \in \{1, \ldots, \Delta\}$, when considering $\mathcal{D}_i$ in T_i, the degrees in each part for all vertices are preserved, except maybe for r. More precisely, recall that r is incident to exactly one edge in T_i, namely ru_i, so there is exactly one part $V_j \in \mathcal{D}_i$ such that $d_{V_j}(r) = 1$. We set $\varepsilon_i = d_{V_j}(u_i)$.

- If $\varepsilon_i \geq 3$, then note that $\mathcal{D}_i$ actually forms a s.l.i. decomposition of T_i; thus T_i is strongly decomposable, and by the induction hypothesis we can actually assume $\mathcal{D}_i$ has at most 16 parts.

- If $\varepsilon_i \leq 2$, then note that $\mathcal{D}_i$ is not a s.l.i. decomposition of T_i, since, assuming $ru_i \in V_j$, there is a conflict between r and u_i in part V_j. In this case, consider H, the tree obtained from T_i by attaching four leaves at r. Note that H is strongly decomposable: a s.l.i. decomposition is obtained from $\mathcal{D}_i$ by adding all new four edges to V_j. Also, H is smaller than T (since $d_H(r) \leq 5 < 16 \leq d_T(r)$). Thus, by the induction hypothesis, H admits a s.l.i. decomposition into at most 16 parts. Abusing the notation, we consider it back in T_i, and still denote it $\mathcal{D}_i$.

In the second case above, note that we still have not guarantee that $\mathcal{D}_i$ is a s.l.i. decomposition of T_i. In particular, we may still have $\varepsilon_i \leq 2$ for a new $\mathcal{D}_i$. However, even in this case, we have the guarantee now that $\mathcal{D}_i$ has at most 16 parts.

Anyhow, assume now all $\mathcal{D}_i$'s have at most 16 parts. Switching to the colouring point of view, our goal now is to permute colours by $\mathcal{D}_1, \dots, \mathcal{D}_\Delta$ so that a s.l.i. edge-colouring of T results. By *setting* some $\mathcal{D}_i$ *to colour* j (for some $j \in \{1, \dots, 16\}$), we mean permuting colours (by $\mathcal{D}_i$) so that ru_i is assigned colour j. Note that permuting/setting colours mainly affect the coloured degrees of r. In particular, if some $\mathcal{D}_i$ is set to colour j, then we need to guarantee $d_j(r) \notin \{\varepsilon_i - 1, \varepsilon_i, \varepsilon_i + 1\}$. Note that guaranteeing this even when $\varepsilon_i \leq 2$ will also fix the fact that $\mathcal{D}_i$, strictly speaking, is, locally, not a s.l.i. edge-colouring of T_i.

Our goal in what follows, is to set the $\mathcal{D}_i$'s to some colours in a particular fashion that will either result in a s.l.i. edge-colouring of T at some point, or reveal possible, restricted values for all ε_i's. More precisely, in that latter case, we will reveal that:

- for some $i_1 \in \{1, \dots, \Delta\}$, we have $\varepsilon_{i_1} \in \{\Delta - 1, \Delta\}$;
- for some $i_2 \in \{1, \dots, \Delta\} \setminus \{i_1\}$, we have $\varepsilon_{i_2} \in \{\Delta - 2, \Delta - 1, \Delta\}$;
- for some $i_3 \in \{1, \dots, \Delta\} \setminus \{i_1, i_2\}$, we have $\varepsilon_{i_3} \in \{\Delta - 3, \Delta - 2, \Delta - 1\}$;
- for all $x \in \{\lfloor \frac{\Delta-3}{2} \rfloor + 1, \dots, \Delta - 3\}$, there are exactly two indexes $j, j' \in \{1, \dots, \Delta\}$ such that $\varepsilon_{i_j}, \varepsilon_{i_{j'}}$ lie in $\{x - 1, x, x + 1\}$, except when Δ is even and there is only one $j \in \{1, \dots, \Delta\}$ such that ε_{i_j} lies in $\{\lfloor \frac{\Delta-3}{2} \rfloor, \lfloor \frac{\Delta-3}{2} \rfloor + 1, \lfloor \frac{\Delta-3}{2} \rfloor + 2\}$ (the "smallest" set).

W.l.o.g., to make it clearer, we will reveal (unless we get a s.l.i. 16-edge-colouring of T early) that ε_1 lies in $\{\Delta - 1, \Delta\}$, ε_2 lies in $\{\Delta - 2, \Delta - 1, \Delta\}$, ε_3 lies in $\{\Delta - 3, \Delta - 2, \Delta - 1\}$, ε_4 and ε_5 lie in $\{\Delta - 4, \Delta - 3, \Delta - 2\}$, ε_6 and ε_7 lie in $\{\Delta - 5, \Delta - 4, \Delta - 3\}$, and so on, up to ε_Δ (and perhaps $\varepsilon_{\Delta-1}$ depending on the parity of Δ) lies in $\{\lfloor \frac{\Delta-3}{2} \rfloor, \lfloor \frac{\Delta-3}{2} \rfloor + 1, \lfloor \frac{\Delta-3}{2} \rfloor + 2\}$.

A crucial point is that, looking at the values above, all ε_i's are revealed to have value at least $\lfloor \frac{\Delta-3}{2} \rfloor$. In particular, because $\Delta \geq 16$, we have $\lceil \frac{\Delta}{4} \rceil < \lfloor \frac{\Delta-3}{2} \rfloor - 1$. This implies that if at most $\lceil \frac{\Delta}{4} \rceil$ revealed $\mathcal{D}_i$'s are set to the same colour, then we cannot get a conflict in that colour. Also, $\lfloor \frac{\Delta-3}{2} \rfloor \geq 6$, so we will reveal that none of the ε_i's have value 2 or less.

As a starting point, let us set all $\mathcal{D}_i$'s to colour 1. If the resulting edge-colouring of T is not s.l.i., then, since $d_1(r) = \Delta$, it must be that, say, $\varepsilon_1 \in \{\Delta - 1, \Delta\}$. Next set $\mathcal{D}_1$ to colour 2 and all other $\mathcal{D}_i$'s to colour 1. As mentioned above, we cannot get any conflict in colour 2. So if the resulting edge-colouring of T is not s.l.i., then it must be because of colour 1; since $d_1(r) = \Delta - 1$, it must be that, say, $\varepsilon_2 \in \{\Delta - 2, \Delta - 1, \Delta\}$. Now set $\mathcal{D}_1$ and $\mathcal{D}_2$ to colour 2 and all other $\mathcal{D}_i$'s to colour 1. Again, we cannot have a conflict in colour 2, so if the edge-colouring of T is not s.l.i. then, say, $\varepsilon_3 \in \{\Delta - 3, \Delta - 2, \Delta - 1\}$, since $d_1(r) = \Delta - 2$.

At this point, there remain $\Delta - 3$ ε_i's to reveal. As mentioned earlier, we will reveal them in pairs, except perhaps for the very last one. So we will run $\lceil \frac{\Delta-3}{2} \rceil$ steps, where each step will reveal two unrevealed ε_i's (except perhaps for the last step). So, at the beginning of the ith step, for every $i \in \{1, \dots, \lceil \frac{\Delta-3}{2} \rceil\}$, there will remain exactly $\Delta - 3 - 2(i-1)$ unrevealed edges (*i.e.*, such that the associated ε_i's remain to be revealed), and, during the step, all (at most two) revealed ε_i will be revealed to lie in $\{\Delta - 4 - (i-1), \Delta - 4 - (i-1) + 1, \Delta - 4 - (i-1) + 2\}$.

We carry each ith step as follows.

- We first consider the $\Delta - 3 - 2(i-1)$ unrevealed edges, and set the associated $\mathcal{D}_i$'s to colour 1. In colour 1, we also set the largest

$$(\Delta - 4 - (i-1) + 1) - (\Delta - 3 - 2(i-1)) = i - 1$$

revealed $\mathcal{D}_i$'s (that is, $\mathcal{D}_1, \ldots, \mathcal{D}_{i-1}$) to colour 1. Recall that there are $3+2(i-1) > i-1$ such $\mathcal{D}_i$'s, so there are enough of them. As a result, we get $d_1(r) = \Delta - 4 - (i-1) + 1$.

We claim we cannot get a conflict in colour 1 between r and its neighbours in $T_1, \ldots, T_{i-1}$ (for which we know the possible degrees in colour 1, given by $\varepsilon_1, \ldots, \varepsilon_{i-1}$). For $i = 1$, note that we did not have to set $\mathcal{D}_i$'s to colour 1. For $i = 2$, we had to set $\mathcal{D}_1$ to colour 1, but since we have $d_1(r) = \Delta - 4 - 1 + 1 = \Delta - 4$ and $\varepsilon_1 = \{\Delta - 1, \Delta\}$, we cannot have a conflict between r and u_1. For $i = 3$, we had to set $\mathcal{D}_1$ and $\mathcal{D}_2$ to colour 1, but since we have $d_1(r) = \Delta - 4 - 2 + 1 = \Delta - 5$ and $\varepsilon_2 = \{\Delta - 2, \Delta - 1, \Delta\}$, we cannot have a conflict between r and either of u_1 and u_2. Likewise, for $i = 4$, we had to set $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ to colour 1, but since we have $d_1(r) = \Delta - 4 - 3 + 1 = \Delta - 6$ and $\varepsilon_3 = \{\Delta - 3, \Delta - 2, \Delta - 1\}$, we cannot have a conflict between r and either of u_1, u_2, and u_3. Now, for any $i \geq 5$, we had to set all of $\mathcal{D}_1, \ldots, \mathcal{D}_{i-1}$ to colour 1, but since we have $d_1(r) = \Delta - 4 - (i-1) + 1 = \Delta - 2 - i$ and $\varepsilon_{i-1} = \{\Delta - 4 - (\lfloor \frac{i-4}{2} \rfloor - 1), \Delta - 4 - (\lfloor \frac{i-4}{2} \rfloor - 1) + 1, \Delta - 4 - (\lfloor \frac{i-4}{2} \rfloor - 1) + 2\}$, we cannot have $\Delta - 2 - i \geq \Delta - 4 - (\lfloor \frac{i-4}{2} \rfloor - 1) - 1 = \Delta - 4 - \lfloor \frac{i-4}{2} \rfloor$ since $i \geq 5$. So, in colour 1, we cannot have a conflict between r and $u_1, \ldots, u_{i-1}$.

Regarding the other, revealed $\Delta - 3 - 2(i-1) - (i-1)$ edges, we set the corresponding $\mathcal{D}_i$'s to colour 2, 3, 4, and 5, chosen arbitrarily so that each colour is set for at most $\lceil \frac{\Delta}{4} \rceil$ of the $\mathcal{D}_i$'s. As mentioned earlier, this means we cannot get any conflict in any of colour 2, 3, 4, and 5. So, if the resulting edge-colouring is not s.l.i., then it must be in colour 1, between r and some unrevealed u_i, for which we reveal ε_i lies in $\{\Delta - 4 - (i-1), \Delta - 4 - (i-1) + 1, \Delta - 4 - (i-1) + 2\}$.

- Still part of the ith step, we then repeat these arguments, taking into account that there is one more revealed edge. That is, we first consider the $\Delta - 3 - 2(i-1) - 1$ remaining unrevealed edges, and set the associated $\mathcal{D}_i$'s to colour 1. In colour 1, we also set the largest

$$(\Delta - 4 - (i-1) + 1) - (\Delta - 3 - 2(i-1) - 1) = i$$

largest revealed $\mathcal{D}_i$'s (that is, $\mathcal{D}_1, \ldots, \mathcal{D}_i$) to colour 1. This way, once again we obtain $d_1(r) = \Delta - 4 - (i-1) + 1$.

We claim that, again, we cannot have any conflict between r and $u_1, \ldots, u_i$ in colour 1. For $i = 1$, we had to set $\mathcal{D}_1$ to colour 1, but since we have $d_1(r) = \Delta - 4 + 1 = \Delta - 3$ and $\varepsilon_1 = \{\Delta - 1, \Delta\}$, we cannot have a conflict between r and u_1. For $i = 2$, we had to set $\mathcal{D}_1$ and $\mathcal{D}_2$ to colour 1, but since we have $d_1(r) = \Delta - 4 - 1 + 1 = \Delta - 4$ and $\varepsilon_2 = \{\Delta - 2, \Delta - 1, \Delta\}$, we cannot have a conflict between r and u_1 or u_2. For $i = 3$, we had to set $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ to colour 1, but since we have $d_1(r) = \Delta - 4 - 2 + 1 = \Delta - 5$ and $\varepsilon_3 = \{\Delta - 3, \Delta - 2, \Delta - 1\}$, we cannot have a conflict between r and either of u_1, u_2, and u_3. Now, for any $i \geq 4$, we had to set all of $\mathcal{D}_1, \ldots, \mathcal{D}_i$ to colour 1, but since we have $d_1(r) = \Delta - 4 - (i-1) + 1 = \Delta - 2 - i$ and $\varepsilon_i = \{\Delta - 4 - (\lfloor \frac{i-3}{2} \rfloor - 1), \Delta - 4 - (\lfloor \frac{i-3}{2} \rfloor - 1) + 1, \Delta - 4 - (\lfloor \frac{i-3}{2} \rfloor - 1) + 2\}$, we cannot have $\Delta - 2 - i \geq \Delta - 4 - (\lfloor \frac{i-3}{2} \rfloor - 1) - 1 = \Delta - 4 - \lfloor \frac{i-3}{2} \rfloor$ since $i \geq 4$. So, again, in colour 1, we cannot have a conflict between r and $u_1, \ldots, u_i$.

Last, regarding the other, revealed $\Delta - 3 - 2(i-1) - 1 - i$ edges, we set the corresponding $\mathcal{D}_i$'s to colour 2, 3, 4, and 5, chosen arbitrarily so that each colour is set by at most $\lceil \frac{\Delta}{4} \rceil$ of the $\mathcal{D}_i$'s. Same arguments as earlier then apply, to deduce that if the resulting edge-colouring of T is not s.l.i., then we reveal another ε_i lies in $\{\Delta - 4 - (i-1), \Delta - 4 - (i-1) + 1, \Delta - 4 - (i-1) + 2\}$.

Eventually, once all ε_i's are revealed to have value at least $\lfloor \frac{\Delta-3}{2} \rfloor \geq 6$, a s.l.i. 16-edge-colouring of T is then obtained by setting the $\mathcal{D}_i$'s to colour 1, 2, 3, 4, and 5, in any way but so that each colour is set to at most $\lceil \frac{\Delta}{4} \rceil$ $\mathcal{D}_i$'s. □

A significant consequence of the fact that the parameter $\chi_{\text{s.l.i.}}$ is bounded above by an absolute constant, 16, in strongly decomposable trees, is that we can provide a polynomial-time algorithm for determining $\chi_{\text{s.l.i.}}(T)$ for a strongly decomposable tree T, which contrasts with the more general case, see upcoming Theorem 5.1.

Before describing our next ideas, let us introduce a few terminology first, partly borrowed from [4]. In what follows, every tree T is assumed rooted at some root vertex r. In the usual way, this defines a (virtual) root-to-leaves orientation of T where every non-root vertex as a unique *parent* (its unique neighbour being closer to r), every non-leaf or root vertex has *children* (its neighbours being farther from r) and *descendants* (vertices obtained through iterating the child relationship). A tree T with root r fulfilling $d(r) = 1$ is called a *shrub*. Now, given any vertex u of T with $d = d(u) \geq 1$ children $v_1, \ldots, v_d$, we denote by $T(u, v_1), \ldots, T(u, v_d)$ the d shrubs hanging at u, where, for $i \in \{1, \ldots, d\}$, the shrub $T(u, v_i)$ is the subtree of T with root u induced by u, v_i, and all descendants of v_i.

Let us now discuss what a s.l.i. edge-colouring ϕ of a tree T looks like locally. Assume T is rooted at some vertex r, and let u be any non-root vertex of T with $d \geq 1$ children $v_1, \ldots, v_d$. Then note that, for any $i \in \{1, \ldots, d\}$, the restriction of ϕ to $T(u, v_i)$ provides the same degrees in any colour, but maybe for u. Indeed, assuming $\phi(uv_i) = 1$ w.l.o.g., we necessarily have $d_1(u) = 1$ by ϕ in $T(u, v_i)$, while we might have $d_1(u) > 1$ by ϕ in T. This implies ϕ is not necessarily s.l.i. in $T(u, v_i)$; however, in that case, we know for sure this is because of a conflict between u and v_i in colour 1. By these arguments, a s.l.i. edge-colouring of T can thus essentially be seen as a combination of edge-colourings of its subshrubs that are "almost" s.l.i. Formally, for a shrub H with root u having unique child v, an edge-colouring ψ is *loosely s.l.i.* if, assuming $\psi(uv) = i$, the only possible conflict to ψ be s.l.i. is between u and v in colour i. In particular, a s.l.i. edge-colouring of H is loosely s.l.i (but obviously the contrary is not always true). Thus, now, assuming $u_1, \ldots, u_d$ are the $d \geq 1$ children of r in T, the question of whether T admits s.l.i. k-edge-colourings falls down to the question of whether $T(r, u_1), \ldots, T(r, u_d)$ admit loosely s.l.i. k-edge-colourings which, when combined in T, yield a s.l.i. k-edge-colouring (that is, so that r is not involved in any conflict). In turn, this yields to the question of determining, for a shrub H with root u having unique child v, whether H admits loosely s.l.i. k-edge-colourings ψ, and, assuming it does, what are the possible values of $d_1(v)$ when assuming $\psi(uv) = 1$ w.l.o.g. It turns out this can be determined in polynomial time when k is constant.

Lemma 4.15. *Let H be a shrub with order $n \geq 2$ and root u having unique child v, and $k \geq 1$ be a fixed integer. Then, in time $\mathcal{O}(n^k)$, we can determine $L(H, k)$, the list of values α such that $d_1(v) = \alpha$ by some loosely s.l.i. k-edge-colouring ψ of H with $\psi(uv) = 1$.*

Proof. The proof is by induction on n. If $n = 2$, then the unique k-edge-colouring ψ of H to consider is that where $\psi(uv) = 1$, which is loosely s.l.i., yields $d_1(v) = 1$, and thus $L(H, k) = \{1\}$. So, let us now proceed with the more general case.

In general, we thus have a larger shrub with root u having unique child v, having itself $d \geq 1$ children $w_1, \ldots, w_d$. Note that each of $H_1, \ldots, H_d$, where $H_i = H(v, w_i)$ for every $i \in \{1, \ldots, d\}$, is a shrub with root v having unique child w_i, for which, by the induction hypothesis, we can compute, in polynomial time, $L_i = L(H_i, k)$. The main question, now, for every value $\alpha \in \{1, \ldots, d\}$, is whether we can assign colour 1 to uv, and k-edge-colour all H_i's in a loosely s.l.i. way so that a loosely s.l.i. k-edge-colouring ψ of H with $d_1(v) = \alpha$

results. If this can be decided in polynomial time for all α's, then as a result we will get the desired $L(H,k)$ in polynomial time. We claim this can be done solely from the L_i's.

Let $\pi = (a_1, \ldots, a_k)$ be an integer partition of d into k integers (*i.e.*, $d = a_1 + \cdots + a_d$), where $a_1 \geq 1$ (but all other a_i's can be 0). We claim we can determine in polynomial time whether there is a loosely s.l.i. k-edge-colourings ψ of H where $d_i(v) = a_i$ for every $i \in \{1, \ldots, k\}$, and $\psi(uv) = 1$. Since the number of such π's to consider is $\mathcal{O}(n^k)$, this implies $L(H,k)$ can be determined in polynomial time (just repeat the process exhaustively for all admissible π's). So the question now, is whether there are loosely s.l.i. k-edge-colourings $\psi_1, \ldots, \psi_d$ of $H_1, \ldots, H_d$ which, when combined and assigning label 1 to uv, yield the desired ψ. This mainly requires that, for any $i \in \{1, \ldots, d\}$, if $\psi_i(vw_i) = x$, then $d_x(v) = a_x$ by ψ does not lie in $\{d_x(w_i) - 1, d_x(w_i), d_x(w_i) + 1\}$ by ψ_i. And this relies solely on the fact that there is some y in L_i such that $a_x \notin \{y - 1, y, y + 1\}$.

This last question can be modelled and solved in polynomial time through a flow network formalism. Namely, given $\pi = (a_1, \ldots, a_k)$ as above and the L_i's, we consider the flow network N with the following vertices and arcs:

- for every $i \in \{1, \ldots, d\}$, we have a vertex c_i in N (modelling child w_i);
- for every $i \in \{1, \ldots, k\}$, we have a vertex p_i in N (modelling element a_i);
- N has source s and sink t;
- for every $i \in \{1, \ldots, d\}$, (s, c_i) is an arc with capacity 1 of N;
- for every $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, k\}$ with $L_i \cap \{a_j - 1, a_j, a_j + 1\} \neq \varnothing$, (c_i, p_j) is an arc with capacity 1 of N;
- (p_1, t) is an arc of capacity $a_1 - 1$ of N, while, for every $i \in \{2, \ldots, k\}$, (p_i, t) is an arc of capacity a_i.

We claim the desired ψ exists if and only if N has a flow with value d. This is because there is a straight equivalence between flows with value d of N and loosely s.l.i. k-edge-colourings ψ of H such that $d_i(v) = a_i$ for every $i \in \{1, \ldots, k\}$ (and $\psi(uv) = 1$). Indeed, consider f, a flow of N with value d. By the Flow-Integrity Theorem, since all capacities in N are integers, we can assume f is a flow with integers values. Now decompose f into a collection of path flows, and focus on any one of these path flows P. Since all arcs incident to s in N have capacity 1, clearly P has value 1. Assuming now P traverses some arc (c_i, p_j), we regard this as permuting colours by some loosely s.l.i. k-edge-colouring ψ_i of H_i so that vw_i is assigned colour j. Repeating this for all path flows decomposing f, we deduce permutations of colours to the ψ_i's, which, together with assigning colour 1 to uv, yields some k-edge-colouring ψ of H. Now, due to the capacities in N, and the fact that f has value d, we deduce that, by ϕ, we have $d_i(v) = a_i$ for all $i \in \{1, \ldots, k\}$. Also, due to the arcs of the form (c_i, p_j) we have added, if P is a path flow going through some arc (c_i, p_j), then we know there is some ψ_i we can consider for which $d_j(w_i) \notin \{a_j - 1, a_j, a_j + 1\}$, thus so that we have no conflict in colour j (assigned by ψ to vw_i, after permuting colours) between v and w_i by ϕ. By such arguments we get the resulting ϕ is indeed loosely s.l.i., and, *vice versa*, it is not too complicated to reverse those arguments to prove that a loosely s.l.i. k-edge-colouring of H realising π yields a flow with value d of N.

Since the Maximum Flow problem is well known to be solvable in polynomial time (recall e.g. Ford and Fulkerson's Algorithm), we can thus decide in polynomial time whether there is a loosely s.l.i. k-edge-colouring of H realising π. Besides, since the number of such

π's to consider is polynomial (given that k is a constant), altogether we get that $L(H, k)$ can be determined in polynomial time, as claimed. □

Theorem 4.16. *For any tree T and any fixed integer $k \geq 1$, we can determine in time $\mathcal{O}(n^k)$ whether $\chi_{\text{s.l.i.}}(T) \leq k$ holds.*

Proof. If $\Delta(T) \leq 2$, then by Observation 2.4 we deduce T is a strong exception. Otherwise, root T at any vertex r of degree $d \geq 3$. Denote $v_1, \ldots, v_d$ the neighbours (children) of r, and by $T_1, \ldots, T_d$ the shrubs $T(r, v_1), \ldots, T(r, v_d)$ decomposing T. By Lemma 4.15, for every $i \in \{1, \ldots, d\}$, we can compute, in polynomial time, the list L_i of values α such that $d_1(v_i) = \alpha$ by some loosely s.l.i. k-edge-colouring ψ_i of T_i with $\psi_i(rv_i) = 1$. Our goal, now, by earlier arguments, is to determine whether there are loosely s.l.i. k-edge-colourings $\psi_1, \ldots, \psi_d$ of $T_1, \ldots, T_d$, which, when combined and have their colours permuted, yield a k-edge-colouring ϕ of T such that if any edge rv_i is assigned some colour j then r and v_i are not in conflict in colour j. Similarly as in the proof of Lemma 4.15, this can be determined through a flow formalism (the main difference being, in the construction of N, that (p_1, t) is now an arc of capacity a_1, since r has no parent). So, as earlier, the claim follows. □

Now, combining Theorems 4.14 and 4.16, we get:

Corollary 4.17. *For any tree T, we can determine $\chi_{\text{s.l.i.}}(T)$ in polynomial time.*

Proof. By Theorem 4.14, we have $\chi_{\text{s.l.i.}}(T) \leq 16$. So, to determine $\chi_{\text{s.l.i.}}(T)$, we just have to wonder whether $\chi_{\text{s.l.i.}}(T) = k$ holds for all successive values of $k \in \{1, \ldots, 15\}$, which can be done in polynomial time by Theorem 4.16. This process runs in polynomial time. □

Corollary 4.17 thus meets a similar result for l.i. decompositions established in [4]. As a side fact, we deduce also that determining whether a given tree is a strong exception can be done in polynomial time. Let us raise as well that all our arguments in the proofs of Lemma 4.15 and Theorem 4.16 are constructive (in particular the part involving flows), so we also deduce that, for any strongly decomposable tree T, a s.l.i. $\chi_{\text{s.l.i.}}(T)$-edge-colouring can be constructed in polynomial time.

5. On the complexity of determining $\chi_{\text{s.l.i.}}(G)$ for a graph G

In this section, we reuse some forcing mechanisms we exhibited in Subsection 4.2 to prove that determining whether $\chi_{\text{s.l.i.}}(G) \leq 2$ holds for a given graph G is NP-complete. For context, let us remind that the similar result for l.i. decompositions is also known to hold; that is, it was proved in [4] that determining whether $\chi_{\text{l.i.}}(G) \leq 2$ holds for a graph G is NP-complete. A significant difference in our case, however, is that we prove our result to hold for bipartite graphs G, while it is still unknown to date whether the l.i. counterpart of the problem is NP-complete for bipartite graphs. Let us remind too that determining the value of $\chi_{\text{s.l.i.}}(T)$ for a given tree T can be done in polynomial time, recall Corollary 4.17.

Before proceeding with the proof of our main result in this section, we first introduce the main gadgets (partly depicted in Figure 4) we will employ. Let us start from a path $P = u_1u_2\ldots$ of any even length at least 2, and, for every even $i \geq 2$, add the edge u_iv_i where v_i is a new vertex of degree 1. Reusing ideas from Subsection 4.2, in the sequel, for a graph G and a s.l.i. 2-edge-colouring ϕ of G, we say a vertex u is *strong* if all its incident edges are assigned the same colour, and, assuming $d(u) = d$ and uv is an edge with $d(v) = 1$, to make it more precise we call u a *d-forcing vertex* and uv a *d-forcing edge*. Due to e.g. Observation 3.1, note that, in P, every u_i with i even is a 3-forcing vertex,

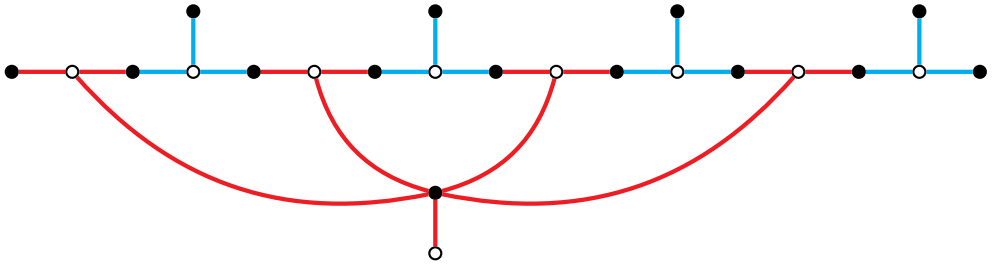


Figure 4: Creating a 5-forcing vertex/edge from 3-forcing vertices/edges. Vertices in black and white show the partition classes of the bipartition, while edges in red and blue depict a s.l.i. 2-edge-colouring.

every u_iv_i with i even is a 3-forcing edge, and if ϕ is a s.l.i. 2-edge-colouring of P, then all u_i's with $i \equiv 0 \bmod 4$ have their three incident edges assigned the same colour, which is different from the colour assigned to the three incident edges of all u_i's with $i \equiv 2 \bmod 4$. We thus call the u_i's and u_iv_i's with $i \equiv 0 \bmod 4$ *even*, while we call those with $i \equiv 2 \bmod 4$ *odd*. To sum up, since these arguments apply whatever the length of P is, we can generate arbitrarily many even 3-forcing vertices/edges and odd 3-forcing vertices/edges. Note that we cannot tell for sure what colours the edges incident to the even 3-forcing vertices and odd 3-forcing vertices necessarily are by any s.l.i. 2-edge-colouring, but we know for sure they are different. We note as well that all these vertices and edges form a bipartite graph, all 3-forcing vertices being part of the same partition class of the bipartition.

From these ideas, we can actually build arbitrarily many even and odd d-forcing vertices/edges for any odd $d \geq 5$. For instance (see Figure 4), to create a new 5-forcing vertex (and edge), pick, in P, four even (or similarly odd) 3-forcing edges $a_1b_1, a_2b_2, a_3b_3, a_4b_4$ (where the a_i's have degree 3 while the b_i's have degree 1), identify all the b_i's into a single vertex b, and add an edge bc where c is a new vertex of degree 1. As a result, $d(b) = 5$, and the graph remains bipartite. If we now consider a s.l.i. 2-edge-colouring ϕ of the resulting graph, then, since $a_1b_1, a_2b_2, a_3b_3, a_4b_4$ are either all even or all odd, they all get assigned the same colour by ϕ, say 1 w.l.o.g., and we have that $d_1(a_1) = d_1(a_2) = d_1(a_3) = d_1(a_4) = 3$ holds by arguments above. Thus, so that we do not get a conflict between b and the a_i's in colour 1, we must have $\phi(bc) = 1$ so that $d_1(b) = 5$. Thus b is a 5-forcing vertex, bc is a 5-forcing edge, and by applying this process with $a_1b_1, a_2b_2, a_3b_3, a_4b_4$ being either all odd or all even, we can, again, generate arbitrarily many such 5-forcing vertices and edges w.r.t. to one colour or another of a s.l.i. 2-edge-colouring. We call even those originating from even 5-forcing edges, and odd the other ones, originating from odd 5-forcing edges.

Through the same ideas, for any successive odd $d \geq 7$, we can generate a d-forcing vertex and a d-forcing edge from $(d-1)$-forcing edges, by simply picking $(d-1)$-forcing edges of the same type (either all even or all odd) $a_1b_1, \ldots, a_{d-1}b_{d-1}$, and, assuming the b_i's have degree 1, identifying the b_i's to a single vertex b, and adding the edge bc where c is a new vertex of degree 1. By the same arguments as earlier, it is easy to check indeed that b is a d-forcing vertex, bc is a d-forcing edge, and, through having the a_ib_i's being all even or all odd, we can again generate arbitrarily many such d-forcing vertices and edges w.r.t. to one colour or another of a s.l.i. 2-edge-colouring. In terms of bipartition, we note that any resulting graph obtained this way remains bipartite at any point, and it can be checked that, for sure, all resulting d-forcing vertices with $d \equiv 1 \bmod 4$ belong to the same partition class, while those with $d \equiv 3 \bmod 4$ belong to the other.

To generate arbitrarily many forcing vertices and edges this way, we thus create, from P, a bipartite graph we call a *thread*, which can thus be arbitrarily long, and that admits s.l.i. 2-edge-colourings such that some of the forcing vertices and edges are w.r.t. one

colour while the others are for the second one (even/odd forcing vertices/edges).

We are now ready to prove our main result.

Theorem 5.1. *Determining if $\chi_{\text{s.l.i.}}(G) \leq 2$ holds for a bipartite graph G is NP-complete.*

Proof. Since the problem is clearly in NP, we focus on proving its NP-hardness. We do it by reduction from the Cubic Monotone 1-in-3 SAT problem again (recall the definition of the problem in the proof of Theorem 2.6). From a 3CNF formula F, we construct, in polynomial time, a bipartite graph G such that F can be satisfied in a 1-in-3 way if and only if G admits s.l.i. 2-edge-colourings.

We start from G being the bipartite graph modelling the structure of F. That is, for every variable x_i of F we have a *variable vertex* v_i in G, for every clause C_j of F we have a *clause vertex* c_j in G, and whenever some variable x_i belongs to some clause C_j of F we have the *formula edge* v_ic_j in G. Next, we add to G some gadget thread P, which we only assume for now is long enough so that we have sufficiently many forcing vertices/edges whenever we need some below (what is important is that the needed length of P is a polynomial function of the size of F, see below). Then:

- For every clause vertex c_j, we pick an even 9-forcing edge uv (where v is the vertex of degree 1) of P, and identify c_j and v (in what follows, we say uv is *attached* at c_j). Likewise, we pick four even 5-forcing edges of P and attach them at c_j.
- For every variable vertex v_i, we first add an edge v_iv_i' where v_i' is a new vertex of degree 1. We then pick nine even 11-forcing edges of P and attach them at v_i, and similarly for nine odd 11-forcing edges of P.

This concludes the construction of G, which is clearly obtained in polynomial time. In particular, so that P can generate the needed forcing edges, note that its length must only be a linear function of n, the number of clauses and variables of F. Also, G is bipartite by earlier arguments, since the variable and clause vertices induce a bipartite graph, and we have attached to clause vertices 9-forcing edges and 5-forcing edges while we have attached to variable vertices 11-forcing edges, and we have $5 \equiv 9 \not\equiv 11 \bmod 4$.

To see now that we have the desired equivalence between F and G, let us discuss how a s.l.i. 2-edge-colouring ϕ of G behaves.

- First, for every clause vertex c_j, due to the even 9-forcing edge and four even 5-forcing edges of P attached at c_j, and since $d(c_j) = 8$, by ϕ there must be some colour $i \in \{1, 2\}$ for which we must have $d_i(c_j) \in \{5, 6, 7, 8\}$. More precisely, assuming all even 5-forcing vertices and all even 9-forcing vertices have degree 5 and 9 in colour 1 by ϕ, w.l.o.g., to avoid any conflict in colour 1 involving c_j it must be that $d_1(c_j) = 7$, which requires that two formula edges incident to c_j are assigned colour 1 while the last one is assigned colour 2.
- Second, for every variable vertex v_i, due to v_iv_i', the nine even 11-forcing edges of P attached at v_i, and the nine odd 11-forcing edges attached at v_i, by ϕ we must have $d_1(v_i), d_2(v_i) \in \{9, 10, 11, 12, 13\}$ (recall $d(v_i) = 22$). However, due to the even and odd 11-forcing vertices adjacent to v_i, so that we do not have any conflict involving v_i in colour 1 or 2, we must have $d_1(v_i), d_2(v_i) \in \{9, 13\}$, which requires v_iv_i' and all three formula edges incident to v_i are assigned the same colour by ϕ. In particular, this implies v_i and v_i' can never cause a conflict.

Through these arguments, it is not too complicated to see that we have the desired equivalence between F and G. Indeed:

- Assume first we have a s.l.i. 2-edge-colouring ϕ of G. By earlier arguments, we know all even 5-forcing vertices of P have their incident edges assigned the same colour, say 1 w.l.o.g., by ϕ. Then every clause vertex is incident to exactly one formula edge assigned label 2, while every variable vertex has all its incident formula edges assigned the same colour. Consider now the truth assignment ψ to the variables of F where, for every variable vertex v_i, we set x_i to *true* if all three edges incident to v_i are assigned colour 2, and to *false* otherwise. Seeing now the fact that any formula edge v_ic_j of G is assigned colour 2 (or 1) models that x_i brings truth value *true* (*false*, resp.) to clause C_j of F by a truth assignment, we deduce that ψ satisfies F in a 1-in-3 way. In particular, the fact that all variable vertices of G have their three incident edges assigned the same colour by ϕ models that, by a truth assignment, any variable brings the same truth value to all clauses containing it.

- Assume second we have a 1-in-3 truth assignment ψ to the variables of F. To get a s.l.i. 2-edge-colouring ϕ of G, we first consider every clause C_j of F, denote x_i its true variable by ψ, and assign colour 2 by ϕ to the variable edge v_ic_j of G; lastly, we assign colour 1 to all uncoloured formula edges. For every variable vertex v_i, assuming its three incident formula edges are assigned colour i by ϕ (this necessarily holds since ψ is a truth assignment), we then assign colour i to v_iv_i'. We last extend ϕ to the edges of the thread P is the obvious way, starting from the forcing edges incident to the clause vertices being all assigned colour 1, and then extending to the rest of the edges. Then, by arguments above, the resulting 2-edge-colouring ϕ of G is s.l.i. In particular, we note that every clause vertex c_j satisfies $d_1(c_j) = 7$ and $d_2(c_j) = 1$, while every variable vertex v_i satisfies $d_1(v_i), d_2(v_i) \in \{9, 13\}$, so we cannot have any conflict between variable vertices and clause vertices.

We thus have the desired equivalence and the result holds. □

6. Conclusion

In this work, we introduced a stronger version of l.i. graphs and l.i. decompositions, namely s.l.i. graphs and s.l.i. decompositions, being motivated mainly by the fact that they stand as a natural strengthening of the original notions, and provide a decompositional point of view over some variant of the 1-2-3 Conjecture.

One of our main intents was to compare our stronger notions with the original ones, and, in particular, to investigate how imposing a slightly stronger distinction condition makes s.l.i. decompositions differ from l.i. ones. A first significant difference we observed, is that, unless P=NP, strong exceptions (w.r.t. s.l.i. decompositions) get way harder to describe than exceptions (w.r.t. l.i. decompositions), compare Theorems 1.1 and 2.6. This makes certain proof schemes harder to apply in our context, such as inductive arguments, as, when removing some structure from a strongly decomposable graph, if the resulting graph is a strong exception, we do not get much from this piece of information, since a strong exception can essentially be anything. Still, we managed to provide bounds on $\chi_{\text{s.l.i.}}(G)$ for certain graphs G. In particular, we provided somewhat tight results for complete bipartite graphs and subcubic graphs (recall Theorem 4.1 and Subsection 4.2), for complete graphs we provided almost tight results (recall Theorems 4.12 and 4.13), while, for trees, we are still a bit far from understanding everything (compare Theorems 3.3 and 4.14). Finally,

although we did not manage to establish a constant upper bound on $\chi_{\text{s.l.i.}}(G)$ for all strongly decomposable graphs G, we managed to observe that, for some graphs G, it is possible for $\chi_{\text{s.l.i.}}(G)$ to be as large as 16 (recall Theorem 3.4).

Following our investigations, we believe lots of directions could be worth studying further. For instance, the following directions seem appealing to us.

- It would be of prime interest to establish (or prove we cannot) a constant upper bound on $\chi_{\text{s.l.i.}}(G)$ for all strongly decomposable graphs G. By Theorem 3.4, we know that such a general bound, if it exists, would have value at least 16. Maybe one first step could be to establish bounds in terms of other graph parameters. As an example, note that, from the proof of Theorem 2.10, we get that every graph G with $\delta(G) \geq 6$ satisfies $\chi_{\text{s.l.i.}}(G) \leq |V(G)|$. It could be interesting to adapt and improve this upper bound in general, for all strongly decomposable graphs.

- Still about Theorem 2.10, we wonder whether it can be improved by a bit. By results we provided, such as Observation 4.7, we know that strong exceptions can have minimum degree 3. Thus, from here, one could ask whether there are strong exceptions with minimum degree 4 or 5.

- Regarding Theorems 4.12 and 4.13, we wonder whether sufficiently large complete graphs K_n satisfy $\chi_{\text{s.l.i.}}(K_n) \leq 3$. Through an approach similar to that we used to prove Theorem 4.13, we were able to prove such a result for sufficiently large complete graphs with an edge missing. But we are still not sure about full complete graphs.

- Regarding out results in Subsection 4.2, it could be interesting next to wonder about subquartic graphs, *i.e.*, graphs with maximum degree at most 4. It can be observed that some of the results and observations we raised adapt and generalise to subquartic graphs. However, a bit more work has to be done to fully understand s.l.i. decompositions in these graphs. In particular, note that our proof of Theorem 2.6 actually implies that strongly decomposable subquartic graphs cannot be described easily (unless P=NP), which might be a troublesome point.

- About cubic graphs, it might be interesting to wonder a bit more about the pairs $\{\alpha, \beta\}$ we mentioned at the very end of the section. In particular, are there strongly decomposable cubic graphs with other values of $\{\alpha, \beta\}$? What can be told about the proportion of cubic graphs realising any given pair $\{\alpha, \beta\}$?

- Regarding trees, there is still some gap between Theorems 3.3 and 4.14. We believe that, through some efforts, it might be possible to push the approach in the proof of Theorem 4.14 further, in order to prove that all strongly decomposable trees admit s.l.i. 5-edge-colourings.

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