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Quaternionic representation and design of depolarizers

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Abstract

Quaternions have been used to represent polarization states and polarization operators. But so far, only polarizers, dichroic or non-depolarizing devices have been represented in that way. We propose a quaternionic representation of perfect as well as partial depolarizers. It leads us to design actual setups for producing natural (unpolarized) light from an arbitrary partially or completely polarized wave or for reducing the degree of polarization of a wave. We conclude by making the link with Azzam's polarization orthogonalization problem.

Key words: Azzam's orthogonalization issue, depolarizer, minquat, polarization optics, polarizer, quaternion.

1. Introduction

Quaternions have been used to represent polarization states as well as polarization operators (birefringent and dichroic devices, polarizers) [1–3]. Their introduction in polarization optics results from essential properties of polarized light; in particular, a Lorentz quadratic form (Lorentz metric), built on appropriate magnitudes, plays a preponderant role and leads us to consider polarization states as endowed with the structure of Minkowski's space. Polarization operators are then described as Lorentz proper rotations whose group is classically related to quaternions [3–5]. A simpler and more direct introduction, less profound, can be developed from the Stokes vector theory [1, 2], so that the considered quaternionic representation of polarization optics can be seen as an alternative to the Stokes-Mueller formalism. It can also be regarded like an analytic tool complementing the Poincaré sphere.

In the present article we extend the above mentioned formalism to depolarizers. The quaternionic representation of a depolarizer will lead to an actual design of such a device.

2. Quaternionic representation of polarized light

2.1. Stokes vectors and Poincaré sphere

For a short and simple introduction to the quaternionic representation of polarization states, we refer to Stokes vectors and to the Poincaré sphere [6, 7]. A Stokes vector is written $\mathbf{X} = {}^t(X_0, X_1, X_2, X_3)$, where subscript t means “transpose” and where the X_μ 's, called Stokes parameters, are positive quantities, homogeneous to irradiances (up to a multiplicative dimensional factor, equal to the impedance of the propagation medium) or “vibratory intensities” [8]. Stokes parameters generally refer to a plane wave with a given direction of propagation; polarization states may be regarded as complex vectors

in a plane wave-surface. A Stokes vector \mathbf{X} associated with a real (physical) lightwave \mathcal{X} is such that $(X_0)^2 - (X_1)^2 - (X_2)^2 - (X_3)^2 \geq 0$. The lightwave is completely polarized if $(X_0)^2 = (X_1)^2 + (X_2)^2 + (X_3)^2$; it is unpolarized (natural light) if $X_1 = X_2 = X_3 = 0$; and partially polarized if $(X_0)^2 > (X_1)^2 + (X_2)^2 + (X_3)^2$. The degree of polarization of the lightwave is ρ ($0 \leq \rho \leq 1$), such that ($X_0 \neq 0$)

$$\rho = \frac{\sqrt{(X_1)^2 + (X_2)^2 + (X_3)^2}}{X_0}. \quad (1)$$

In the following we write “polarization state” of a lightwave, in the general meaning of partial-polarization state, including completely polarized states, also called pure states, and unpolarized states, as particular cases. Since Stokes parameters measured on a lightwave are proportional to the wave power, by abuse we also call “power” the parameter X_0 , when we refer to a lightwave.

If $\mathbf{X} = {}^t(X_0, X_1, X_2, X_3)$ is the Stokes vector associated with a completely polarized lightwave, then $\mathbf{X}_\perp = {}^t(X_0, -X_1, -X_2, -X_3)$ is associated with an orthogonally (and completely) polarized lightwave.

Stokes vectors can be added [7]: if \mathbf{X} and \mathbf{Y} respectively represent the polarization states of lightwaves \mathcal{X} and \mathcal{Y} , then $\mathbf{X} + \mathbf{Y}$ represents the polarization state of the incoherent superposition of waves \mathcal{X} and \mathcal{Y} , symbolically written $\mathcal{X} + \mathcal{Y}$.

Pure states are represented on the so-called Poincaré sphere [6, 7]. For a pure state, since $(X_0)^2 = (X_1)^2 + (X_2)^2 + (X_3)^2$, the knowledge of X_1 , X_2 and X_3 completely characterizes the polarization state. The sphere is referred to 3 orthogonal axes, each axis corresponding to a Stokes parameter X_j ($j = 1, 2, 3$) (Fig. 1). A pure polarization state is represented by a point P on the sphere, and since the radius of the Poincaré sphere is generally taken equal to 1, that is $X_0 = 1$, Stokes parameters X_1 , X_2 and X_3 are the Cartesian coordinates of P on the sphere.

The intersection of the sphere with the plane X_1 - X_2 is called equator. Points on the equator correspond to rectilinear polarization states. In general, the point $P(1, 0, 0)$ represents the “horizontal” polarization (in the wave surface), and point $P'(-1, 0, 0)$ the “vertical” one. (Lightwave direction of propagation is also assumed to be horizontal; it is orthogonal to both horizontal and vertical polarization directions.) The North pole $(0, 0, 1)$ represents a left-handed circular polarization, and the South pole $(0, 0, -1)$ a right-handed circular polarization. If P represents the polarization of the wave \mathcal{P} , the point P_\perp , opposite to P on the sphere, represents the wave \mathcal{P}_\perp that is orthogonally polarized to \mathcal{P} (with the same power).

The Poincaré sphere is helpful in managing birefringent media, since the effect of a birefringent device on polarization states is a rotation on the sphere. The angle of the

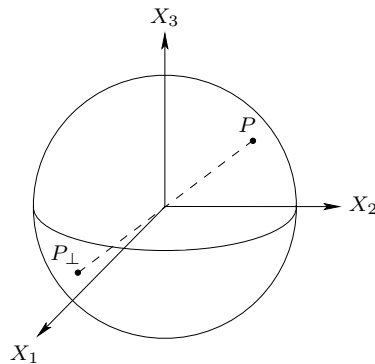


Fig. 1. The Poincaré sphere. Points P and P_\perp , opposite on the sphere, represent two orthogonally polarized lightwaves.

rotation is the birefringence (that is, the phase shift between eigenvibrations) and the axis is $E_{\perp}E$, where E represents the fast eigenvibration, on the sphere, and E_{\perp} the slow eigenvibration of the device ($E_{\perp}E$ constitutes a diameter of the sphere).

2.2. Quaternions

We now consider the 4-dimensional complex vector space \mathbb{C}^4 , whose canonical basis is

$$\mathbf{e}_0 = (1, 0, 0, 0), \quad \mathbf{e}_1 = (0, 1, 0, 0), \quad \mathbf{e}_2 = (0, 0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 0, 1), \quad (2)$$

and we define a quaternionic product by

$$(\mathbf{e}_0)^2 = -(\mathbf{e}_1)^2 = -(\mathbf{e}_2)^2 = -(\mathbf{e}_3)^2 = \mathbf{e}_0, \quad (3)$$

$$\mathbf{e}_0 \mathbf{e}_j = \mathbf{e}_j = \mathbf{e}_j \mathbf{e}_0, \quad j = 1, 2, 3, \quad (4)$$

$$\mathbf{e}_1 \mathbf{e}_2 = \mathbf{e}_3 = -\mathbf{e}_2 \mathbf{e}_1, \quad \text{and circular permutations on } j = 1, 2, 3. \quad (5)$$

The quaternionic product is not commutative; it is distributive with respect to addition. Equipped with the quaternionic product, \mathbb{C}^4 becomes the algebra of complex quaternions, denoted \mathbb{H}_c .

A complex quaternion is written $q = q_0 \mathbf{e}_0 + q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 + q_3 \mathbf{e}_3$, where the q_{μ} 's are complex numbers.

Some noteworthy quaternions

The Hamilton conjugate of $q = q_0 \mathbf{e}_0 + q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 + q_3 \mathbf{e}_3$ is $q^* = q_0 \mathbf{e}_0 - q_1 \mathbf{e}_1 - q_2 \mathbf{e}_2 - q_3 \mathbf{e}_3$, and its complex conjugate is $\bar{q} = \bar{q}_0 \mathbf{e}_0 + \bar{q}_1 \mathbf{e}_1 + \bar{q}_2 \mathbf{e}_2 + \bar{q}_3 \mathbf{e}_3$. Then $q^* = \bar{q}^*$.

The norm of q is $N(q) = q q^* = q^* q = (q_0)^2 + (q_1)^2 + (q_2)^2 + (q_3)^2$. The norm of quaternions is multiplicative: if q and r are quaternions, then $N(qr) = N(q)N(r)$.

A quaternion is unitary (or a unit quaternion) if $N(q) = 1$. A quaternion q is a pure quaternion if $q_0 = 0$. For example the \mathbf{e}_j 's, $j = 1, 2, 3$, are unit pure quaternions. A quaternion $\mathbf{e}_n = n_1 \mathbf{e}_1 + n_2 \mathbf{e}_2 + n_3 \mathbf{e}_3$, with $(n_1)^2 + (n_2)^2 + (n_3)^2 = 1$, is a unit pure quaternion; it is such that $(\mathbf{e}_n)^2 = -\mathbf{e}_0$.

The exponential of the quaternion q is

$$\exp q = \sum_{j=0}^{+\infty} \frac{q^j}{j!}. \quad (6)$$

If $q = \psi \mathbf{e}_n$, where ψ is a complex number and \mathbf{e}_n a unit pure quaternion, we have $(\mathbf{e}_n)^2 = -\mathbf{e}_0$, and we obtain

$$\exp \mathbf{e}_n \psi = \mathbf{e}_0 \sum_{j=0}^{+\infty} \frac{(-1)^j \psi^{2j}}{(2j)!} + \mathbf{e}_n \sum_{j=0}^{+\infty} \frac{(-1)^j \psi^{2j+1}}{(2j+1)!} = \mathbf{e}_0 \cos \psi + \mathbf{e}_n \sin \psi. \quad (7)$$

Matrix representation

Complex quaternions can be represented by 2×2 matrices, with the help of Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (8)$$

The correspondence is

$$\mathbf{e}_0 \mapsto \sigma_0, \quad \mathbf{e}_j \mapsto -i\sigma_j, \quad j = 1, 2, 3, \quad (9)$$

so that

$$q = q_0\mathbf{e}_0 + q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3 \mapsto \sigma = q_0\mathbf{e}_0 - iq_1\sigma_1 - iq_2\sigma_2 - iq_3\sigma_3. \quad (10)$$

We have

$$\sigma = \begin{pmatrix} q_0 - iq_1 & -q_3 - iq_2 \\ q_3 - iq_2 & q_0 + iq_1 \end{pmatrix}. \quad (11)$$

If $\det \sigma$ denotes the determinant of matrix σ , then $N(q) = \det \sigma$.

2.3. Representing polarization states and operators by quaternions

Polarization states

We represent the Stokes vector $\mathbf{X} = {}^t(X_0, X_1, X_2, X_3)$ by the quaternion

$$X = X_0\mathbf{e}_0 + i(X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3). \quad (12)$$

According to Synge, such a quaternion is called a minquat [3,9]. It can also be written

$$X = X_0(1 + i\rho\mathbf{e}_n), \quad (13)$$

where ρ is the degree of polarization of the corresponding polarization state (if $X_0 \neq 0$) and where

$$\mathbf{e}_n = \frac{1}{\rho X_0}(X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3), \quad (14)$$

if $\rho \neq 0$. (If $X_1 = X_2 = X_3 = 0$, then $\rho = 0$ and \mathbf{e}_n is not determined.)

Polarized and unpolarized components

With abuse, we also called ‘‘polarization state’’ a minquat such as $X_0(1 + i\rho\mathbf{e}_n)$, which represents a lightwave polarization-state.

The minquat $X = X_0(1 + i\mathbf{e}_n)$ is a completely polarized state to which $X_\perp = X_0(1 - i\mathbf{e}_n)$ is orthogonally polarized.

The (completely) polarized component of $X = X_0(1 + i\rho\mathbf{e}_n)$ is $X_c = \rho X_0(1 + i\mathbf{e}_n)$, which owns a power ρX_0 ; the unpolarized component of state X is $X_u = (1 - \rho)X_0$. If $X' = X_0(1 - i\rho\mathbf{e}_n)$, then $X'_c = X_{c\perp}$ and $X'_u = X_u$: the polarized components of X and X' are orthogonal to each other, whereas their unpolarized components are equal.

Birefringent and dichroic devices

A birefringent device—generally an elliptical one—has two (unitary) eigenvibrations, represented by two opposite points on the Poincaré sphere. Those two points form a diameter of the sphere, called the birefringent axis. The phase shift between the slow and fast eigenvibrations, say φ , is called the birefringence. On the Poincaré sphere, the effect of a birefringent device is a rotation of angle φ around the birefringent axis (oriented from the slow vibration representative point towards the fast vibration representative one).

In the following, we identify a pure unit quaternion $\mathbf{e}_n = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$ with the vector joining the center O of the sphere to the point of coordinates (n_1, n_2, n_3) . The birefringent axis (as previously defined) is then perfectly determined by a unit pure quaternion, say \mathbf{e}_n , also called by abuse the axis of the birefringent.

Finally, the birefringent device of axis e_n and birefringence φ is represented by the unit quaternion $\exp(e_n\varphi/2)$. For an incident polarization state represented by the minquat X , the outgoing state is represented by X' such that [1–3]

$$X' = \exp\left(e_n\frac{\varphi}{2}\right) X \exp\left(-e_n\frac{\varphi}{2}\right). \quad (15)$$

(Equation (15) is shown to express a rotation operating on minquats.)

A dichroic device whose axis is e_n , dichroism δ and isotropic absorption α ($\alpha > 0$), is represented by $\sqrt{\alpha}\exp(ie_n\delta/2)$ and its effect on X is expressed by

$$X' = \alpha \exp\left(ie_n\frac{\delta}{2}\right) X \exp\left(ie_n\frac{\delta}{2}\right). \quad (16)$$

If $\alpha = 1$, the device is a pure dichroic device. Then $X'_0 > X_0$, which means that an eigenstate is amplified, whereas the orthogonal state is attenuated. Active cavities of some lasers may be regarded as pure dichroic devices. (Choosing the word “absorption” for α does not sound very appropriate!)

Polarizers

We complete the quaternionic representation of polarization optics by describing polarizers. A pure (or ideal) polarizer lets a specific completely polarized wave pass and stops the orthogonally polarized waves. If the passing state corresponds to the minquat $X_0(1 + ie_n)$, the polarizer is represented by $(1 + ie_n)/2$. If the incident state is X , the outgoing state is

$$X' = \frac{1}{4}(1 + ie_n) X (1 + ie_n). \quad (17)$$

We also say that e_n is the axis of the polarizer (on the Poincaré sphere).

For example, if $X = X_0(1 + ie_n)$, we obtain $X' = X$; if $X = X_0(1 - ie_n)$, then $X' = 0$.

If $X = X_0(1 + i\rho e_m)$ and if θ is the angle between e_n and e_m (on the Poincaré sphere), then

$$X' = \frac{X_0}{2}(1 + \rho \cos\theta)(1 + ie_n). \quad (18)$$

The outgoing wave is completely polarized along the passing state of the polarizer (i.e. along the axis e_n on the Poincaré sphere), and its power is $X_0(1 + \rho \cos\theta)/2$, which is a generalized form of Malus law.

3. Representation of depolarizers

3.1. Complete depolarizers

The state outgoing from a polarizer is completely polarized (or it is 0), whatever the incident state maybe. At first, a complete (or perfect) depolarizer device could be thought of as the inverse operator of a pure polarizer: whatever the input state is, the output should be a completely-unpolarized wave (natural light). But a polarizer is not an invertible operator and the previous representation of a polarizer cannot be used to represent a depolarizer (a quaternion of the form $1 + ie_n$ has no inverse in \mathbb{H}_c).

On the other hand, we notice that a dichroic device is a partial polarizer. If we send an unpolarized wave, represented by $X = X_0$, on such a device, the output state corresponds to a partially polarized wave whose polarized component is along the dichroic axis, say e_n (on the Poincaré sphere). The quaternion $\sqrt{\alpha}\exp(ie_n\delta/2)$ is invertible: its inverse is

$(1/\sqrt{\alpha}) \exp(-ie_n \delta/2)$. But the corresponding operator does not conform to the notion of partial depolarizer, because it has not identical effects on two arbitrary states whose degrees of polarization are equal, as a perfect depolarizer should have. Indeed, the inverse operator of a dichroic device is another dichroic device, multiplied by an isotropic factor.

A perfect depolarizer is such that for every incident wave the outgoing wave is unpolarized. For every input minquat X , the output minquat should take the form $X' = X'_0$.

Let us consider the (linear) mapping ψ defined for every quaternion q by

$$\psi(q) = \frac{1}{4}(q - e_1 q e_1 - e_2 q e_2 - e_3 q e_3). \quad (19)$$

We have $\psi(1) = 1$ and $\psi(e_1) = \psi(e_2) = \psi(e_3) = 0$, and, by linearity, for every minquat $X = X_0(1 + i\rho e_n)$, we obtain $\psi(X) = X_0$, and that is exactly the effect of a perfect depolarizer. In conclusion, the mapping, defined on minquats by

$$D : X \mapsto D(X) = \frac{1}{4}(X - e_1 X e_1 - e_2 X e_2 - e_3 X e_3), \quad (20)$$

represents a perfect depolarizer.

More generally, let $[e_u, e_v, e_w]$ be an orthonormal basis of the subspace of pure quaternions. Then the mapping defined for every minquat X by

$$D' : X \mapsto D'(X) = \frac{1}{4}(X - e_u X e_u - e_v X e_v - e_w X e_w), \quad (21)$$

represents a perfect depolarizer.

Remark 1. If $X = X_0(1 + i\rho e_n)$, then $-e_1 X e_1 - e_2 X e_2 - e_3 X e_3 = X_0(3 - i\rho e_n)$, whose polarized component $\rho X_0(1 - i e_n)$ is orthogonal to $\rho X_0(1 + i e_n)$, the polarized component of X . According to Eq. (20), the depolarizing effect is obtained by adding two lightwaves with orthogonal polarized components. (The same conclusion holds with $[e_u, e_v, e_w]$ in place of $[e_1, e_2, e_3]$.)

3.2. Partial depolarizers

A partial depolarizer is a device that reduces the degree of polarization of every incident wave by a given factor. More precisely, let γ be a real number with $0 \leq \gamma \leq 1$. A partial depolarizer depolarizes by a factor $1 - \gamma$, if the output corresponding to the input minquat $X = X_0(1 + i\rho e_n)$, is $X_0(1 + i\gamma\rho e_n)$; the degree of polarization ρ becomes $\gamma\rho$. (A perfect depolarizer depolarizes by a factor 1, that is, $1 - \gamma = 1$, and $\gamma = 0$: for every incident wave the outgoing wave is unpolarized.)

In Eq. (21)—and similarly in Eq. (20)—the polarized component of X is balanced by the polarized component of $-e_u X e_u - e_v X e_v - e_w X e_w$. To obtain a partially polarized lightwave, not necessarily an unpolarized one, that is, a wave whose polarized component is not necessarily zero, we reduce the weight of $-e_u X e_u - e_v X e_v - e_w X e_w$ by introducing a factor β ($0 \leq \beta \leq 1$), and we consider the mapping

$$D_p : X \mapsto D_p(X) = \frac{1}{1 + 3\beta} [X - \beta(e_u X e_u + e_v X e_v + e_w X e_w)]. \quad (22)$$

Then

$$D_p[X_0(1 + i\rho e_n)] = X_0 \left(1 + \frac{1 - \beta}{1 + 3\beta} i\rho e_n \right). \quad (23)$$

The degree of polarization of $D_p[X_0(1 + i\rho e_n)]$ is $\rho(1 - \beta)/(1 + 3\beta)$. Thus, to reduce the degree of polarization from ρ to $\gamma\rho$, we choose β such that

$$\frac{1 - \beta}{1 + 3\beta} = \gamma, \quad (24)$$

and we obtain

$$\beta = \frac{1 - \gamma}{1 + 3\gamma}. \quad (25)$$

We report the value of β in Eq. (22) and eventually write

$$D_p(X) = \frac{1 + 3\gamma}{4}X - \frac{1 - \gamma}{4}(e_u X e_u + e_v X e_v + e_w X e_w), \quad (26)$$

for the general representation of a (generally partial) depolarizer whose depolarizing factor is $1 - \gamma$.

4. Designing depolarizers

4.1. Perfect depolarizers

For the sake of simplicity, we consider $[e_u, e_v, e_w] = [e_1, e_2, e_3]$, and we begin by the design of a perfect depolarizer ($\gamma = 0$). The mapping $X \mapsto X$ is the identity. Since $\exp(e_1\pi/2) = e_1$, according to Eq. (15) the mapping $X \mapsto -e_1 X e_1$ corresponds to the effect of a half-wave plate whose axis is the X_1 axis (on the Poincaré sphere), or e_1 with quaternions. According to Eq. (20), we obtain a perfect depolarizer device by sending the input polarized wave on three half-wave plates (with respective axes along e_1 , e_2 and e_3) and by making the incoherent sum of the three corresponding outgoing waves, plus the incident wave. The process is schematized on Fig. 2.

The half-wave plate whose axis is e_1 (X_1) on the Poincaré sphere has its fast and slow vibrations horizontal and vertical. They are at $\pm 45^\circ$ for the half-wave plate whose axis is along e_2 . The third half-wave plate is optically active: its eigenvibrations are circularly polarized as schematically indicated on Fig. 2 (the rotatory power is half the birefringence, that is, $\pi/2$). Beam splitter BS 0 has a transmission factor $T_0 = 0.75$, whereas its reflection factor is 0.25. For BS 1, we have $T_1 = 0.67$ and $R_1 = 0.33$; for BS 2, we have $T_2 = 0.50$ and $R_2 = 0.50$. Beams 0, 1, 2, 3 thus have identical powers (0.25

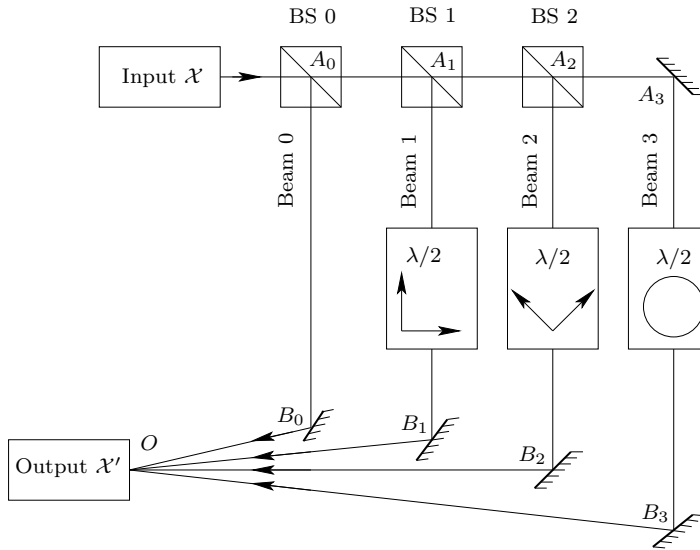


Fig. 2. Setup for a perfect depolarizer. For each half-wave plate the eigenvibrations are indicated.

times the input power). For the wave superposition at O to be incoherent, the optical path differences $[A_3B_3O] - [A_2B_2O]$, $[A_2B_2O] - [A_1B_1O]$ and $[A_1B_1O] - [A_0B_0O]$ must be greater than the coherence length of the light source.

4.2. Partial depolarizers

We assume $[e_u, e_v, e_w] = [e_1, e_2, e_3]$. For the design of a partial depolarizer, we just adapt beam splitter BS 0. Its transmission factor becomes $T_0 = 3(1 - \gamma)/4$ and its reflection factor $R_0 = (1 + 3\gamma)/4$.

Remark 2. A more general partial polarizer would a priori reduce the degree of polarization, but also change the polarization of the polarized component of the incident wave. Such a device actually involves a partial depolarizer in the previous meaning (that is, without changing the polarization of the polarized component) composed with an operator of a distinct kind. For example let us consider a partial depolarizer D_p (in the meaning of the previous section) and a birefringent of axis e_m and birefringence φ . The whole device, denoted D'_p , is such that, for $X = X_0(1 + i\rho e_n)$

$$\begin{aligned} D'_p(X) &= \exp\left(e_m \frac{\varphi}{2}\right) D_p(X) \exp\left(-e_m \frac{\varphi}{2}\right) \\ &= \exp\left(e_m \frac{\varphi}{2}\right) D_p[X_0(1 + i\rho e_n)] \exp\left(-e_m \frac{\varphi}{2}\right) \\ &= \exp\left(e_m \frac{\varphi}{2}\right) [X_0(1 + i\gamma\rho e_n)] \exp\left(-e_m \frac{\varphi}{2}\right) \\ &= X_0(1 + \gamma\rho e_p), \end{aligned} \tag{27}$$

where

$$e_p = \exp\left(e_m \frac{\varphi}{2}\right) e_n \exp\left(-e_m \frac{\varphi}{2}\right), \tag{28}$$

is deduced from e_n in the rotation of angle φ around e_m .

It should be clear that the depolarizer D_p and the birefringent device commute, that is,

$$\exp\left(e_m \frac{\varphi}{2}\right) D_p(X) \exp\left(-e_m \frac{\varphi}{2}\right) = D_p \left[\exp\left(e_m \frac{\varphi}{2}\right) X \exp\left(-e_m \frac{\varphi}{2}\right) \right]. \tag{29}$$

The depolarizing effect is due to D_p only, and changing the direction of polarization (from e_n to e_p) is due to the birefringent device only.

5. Link with Azzam's orthogonalization problem

We recall the Azzam's problem [10]: given an arbitrary completely-polarized wave, find a device that transforms it into an orthogonally-polarized wave. A half-wave plate with appropriate orientation provides the simplest solution [11]. However, we may extend the issue to partially polarized lightwaves and look for a device that transforms the polarized component of a partially polarized state into its orthogonal state (with the same power). The solution is plain, because if a half-wave plate transforms the completely-polarized state $X_0(1 + i e_n)$ into the orthogonal state $X_0(1 - i e_n)$, it also transforms $\rho X_0(1 + i e_n)$ into $\rho X_0(1 - i e_n)$ (and it preserves the unpolarized component $(1 - \rho)X_0$). Every solution of the Azzam's issue also works for a partially polarized wave.

We now point out that a given half-wave plate is a solution to the Azzam's issue only for incident polarization-states whose representative points lie on the great circle whose plane is orthogonal to the plate axis (on the Poincaré sphere). In the following, we

provide a “weak” solution to the extended Azzam’s issue that holds for every incident polarization-state, but without preserving the power of the polarized component.

If $X = X_0(1 + i\rho e_n)$, then $-e_1 X e_1 - e_2 X e_2 - e_3 X e_3 = X_0(3 - i\rho e_n)$, whose polarized component is $\rho X_0(1 - i e_n)$, orthogonal to the polarized component of X (see Remark 1). But the power of $X_0(3 - i\rho e_n)$ is three times the power of X . Unless we use light amplifiers, we have to limit ourselves to passive devices.

A passive device performing the mapping

$$A : X \mapsto A(X) = -\frac{1}{3}(e_1 X e_1 + e_2 X e_2 + e_3 X e_3), \quad (30)$$

thus provides a weak solution to the extended Azzam’s problem, that holds for every incident partially-polarized state $X = X_0(1 + i\rho e_n)$: the polarized component of state $A(X) = X_0(3 - i\rho e_n)/3$ is $\rho X_0(1 - i e_n)/3$ and is orthogonal to $\rho X_0(1 + i e_n)$, the polarized component of X . There is a price to pay for that result: the unpolarized component of $A(X)$ is $[1 - (\rho/3)]X_0$ and is greater than $(1 - \rho)X_0$, the unpolarized component of X (because $\rho > 0$). Part of the incident polarized power has been transformed into unpolarized power.

An actual device can be deduced from the setup of Fig. 2: we only have to suppress BS 0 and beam 0.

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- [11] The Azzam’s issue is almost trivial when solved with the Poincaré sphere, if we look for a birefringent device as a solution. Let P denote the point on the sphere that represents the incident polarization state and let P_\perp represent the orthogonal state. Points P and P_\perp are opposite on the sphere. Every (elliptical) birefringent transforms P into a point P' that is on a circle containing P and whose plane is orthogonal to the birefringent axis. For obtaining P' at P_\perp , the circle is necessarily a great circle and the rotation angle is π . The birefringent is a half-wave plate (an elliptical one in general). Let \mathcal{C} be a great circle passing by P and P_\perp . The end points of the diameter of the sphere, orthogonal to the plane containing \mathcal{C} , provide the eigenstates of the half-wave plate. If P and P_\perp are not the North and South poles of the Poincaré sphere, then only one circle \mathcal{C} (passing by P and P_\perp) is in a plane orthogonal to the equatorial plane, and there is only one solution with a rectilinear half-wave plate (having rectilinear eigenvibrations). If P and P_\perp are the North and South poles, then every rectilinear half-wave plate is a solution. We eventually remark that a half-wave plate is solution of the Azzam’s issue for all polarization states whose representative points are on the great circle whose plane is orthogonal to the plate axis.