

# Dynamic online matching with budget refills

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## Abstract

Inspired by sequential budgeted allocation problems, we study the online matching problem with budget refills. In this context, we consider an online bipartite graph  $G = (U, V, E)$ , where the nodes in  $V$  are discovered sequentially and nodes in  $U$  are known beforehand. Each  $u \in U$  is endowed with a budget  $b_{u,t} \in \mathbb{N}$  that dynamically evolves over time. Unlike the canonical setting, in many applications, the budget can be refilled from time to time, which leads to a much richer dynamic that we consider here. Intuitively, adding extra budgets in  $U$  seems to ease the matching task, and our results support this intuition. In fact, for the stochastic framework considered where we studied the matching size built by Greedy algorithm on an Erdős-Rényi random graph, we showed that the matching size generated by Greedy converges with high probability to a solution of an explicit system of ODE. Moreover, under specific conditions, the competitive ratio (performance measure of the algorithm) can even tend to 1. For the adversarial part, where the graph considered is deterministic and the algorithm used is Balance, the  $b$ -matching bound holds when the refills are scarce. However, when refills are regular, our results suggest a potential improvement in algorithm performance. In both cases, Balance algorithm manages to reach the performance of the upper bound on the adversarial graphs considered.

**Keywords:** Matching in bipartite graphs, random graphs, online matching in bipartite graphs, online  $b$ -matching.

## 1. Introduction

Finding matchings in bipartite graphs is a fundamental problem that lies at the intersection of graph theory (Godsil, 1981; Zdeborová and Mézard, 2006), network theory, and combinatorial optimization (Lovász and Plummer, 2009; Schrijver, 2003), with far-reaching implications in a wide range of practical applications, typically in operation research under the name of the "assignment problem" (see also Grove et al. (1995)). Specifically, a Bipartite graph  $G = (U, V, E)$  has two distinct sets of nodes  $U, V$  and a set of edges  $E \subseteq U \times V$ . Such graphs serve as representations for systems where entities from one set are connected to entities in the other. These connections can symbolize relationships, dependencies, or allocations, making bipartite graphs a powerful tool for modeling real-world scenarios. Finding matchings within these graphs involves determining optimal pairings between nodes from the two sets while respecting certain constraints.

Recent practical applications in Internet advertising have sparked a growing interest in the online version of this problem (see Mehta (2013)). In this context, the graph is incrementally unveiled:

nodes in  $U$  are known in advance, while nodes in  $V$  are observed sequentially, along with their associated edges. Upon revealing a node  $t \in V$ , an online algorithm must decide whether to match it with a node  $u \in U$  such that  $(u, t) \in E$  and  $u \in U$  hasn't been matched yet (standard online matching). This decision is irreversible, which leads to a resulting matching that is sub-optimal, and the overall objective is to quantify this sub-optimality. Beyond the foundational Online matching problem, numerous generalizations have been introduced; one of them is called the  $b$ -matching problem (Kalyanasundaram and Pruhs, 1993; Khuller et al., 1994). In this extension, an additional layer of complexity is introduced by assigning budgets to nodes in the set  $U$ . This imposes a new matching constraint: a node  $t \in V$  can only be paired with a node  $u \in U$  if  $u$  has some positive budget, as discussed in Kalyanasundaram and Pruhs (2000); Albers and Schubert (2021, 2022). Several online algorithms have been developed to tackle both the foundational online matching problem and its extensions. For instance, the Greedy algorithm seamlessly matches an incoming vertex in  $V$  with any available neighbor in  $U$ . Another noteworthy algorithm, `BALANCE`, strategically selects a vertex in  $V$  and pairs it with the neighbor in  $U$  that has the highest available current budget. The performance of an online algorithm relies on its competitive ratio — a metric quantifying the ratio between the size of the created matching and the maximum achievable matching in hindsight (see Mehta (2013); Feldman et al. (2009)).

Driven by the evolving dynamics of online advertising, where  $U$  represents the pool of campaigns or ads available to advertisers and nodes in  $V$  denote units of advertising slots arriving sequentially, each with varying eligibility for a subset of campaigns based on their distinct features (such as geographic localization, browsing history, and other relevant information), the advertiser's primary goal is to maximize the number of ads displayed. In practical scenarios, campaigns or ads are not showcased just once but come with a predetermined budget of impressions (for instance, a specific ad may be displayed only 10,000 times each day). This budget is subject to evolution over time, with the possibility of allocating additional resources to certain campaigns once in a while.

**The online matching with budget refill setting.** The formal model we consider is the following: as in classical online matching, a bipartite graph  $G = (U, V, E)$  is sequentially revealed one node in  $V$  after the other. The two sets of nodes are defined by  $U = [n] := \{1, \dots, n\}$  and  $V = [T] := \{1, \dots, T\}$  for  $n, T \in \mathbb{N}^*$  and the set of edges is denoted by  $E \subseteq U \times V$ . The additional complexity over the  $b$ -matching setting is that the available budget of node  $u \in U$  at time  $t \in V$ , denoted  $b_{u,t-1} \in \mathbb{N}$  is not just depleted over time (when node  $u$  is matched), but also sometimes refilled by the *refill dynamic* denoted  $\eta_{u,t} \in \mathbb{N}$ . More precisely, we focus on simple refill dynamics with a constant rate of refill on average over time, which is already a non-trivial budget dynamic improvement. In a nutshell, the space of online matching problems with refills is defined as

$$\mathcal{G}_T = \{(U, V, E, (\eta_{u,t})_{u \in U, t \in V}), |U| = n \leq T, |V| = T, E \subseteq U \times V \text{ and } (\eta_{u,t})_{u \in U, t \in V} \in \mathbb{N}^{n \times T}\}.$$

The challenge of online matching with budget refills is studied through two lenses:

- i) The first one employs an adversarial framework, where the performance of the deterministic algorithm, `BALANCE`, introduced in Kalyanasundaram and Pruhs (2000), is studied.
- ii) The second one adopts a stochastic framework, where the graph is a random variable defined by an Erdős–Rényi random graph model, and the performance of the Greedy algorithm is studied.

The rest of the paper is organized as follows: in section 2, we provide a formal description of the problem studied within the adversarial framework with the main results for this part. section 3 presents the setting for the stochastic case with the main results.

## 2. The adversarial framework

The initial strand of research focused on online matching (without refills) within the adversarial framework, where the algorithm is evaluated on the worst possible instance and vertex arrival order. Notably, the Greedy algorithm, that matches incoming vertices with any available neighbors, demonstrates a competitive ratio of  $1/2$  in the worst-case scenario. However, its performance improves to reach  $1 - 1/e$  competitive ratio when incoming vertices arrive in a random order, as highlighted in [Goel and Mehta \(2008\)](#). Another significant contribution is the Ranking algorithm introduced in [Karp et al. \(1990\)](#), showcasing worst-case optimality by consistently achieving at least  $1 - 1/e$  on any instance ([Karp et al., 1990](#); [Devanur et al., 2013](#); [Birbaum and Mathieu, 2008](#)). Moreover, it exhibits superior performance in scenarios featuring random vertex arrivals ([Mahdian and Yan, 2011](#)).

Beyond traditional online matching, the  $b$ -matching problem assigns fixed budgets  $b \in \mathbb{N}^*$  to nodes in  $U$ , as pioneered in [Kalyanasundaram and Pruhs \(2000\)](#). In this context, [Kalyanasundaram and Pruhs \(2000\)](#) introduced the deterministic Balance algorithm, matching a new vertex in  $V$  with a neighbor in  $U$  that has the highest remaining budget, they proved that Balance achieves an optimal competitive ratio of  $1 - (1/(1 + 1/b))^b$ , tending towards  $1 - 1/e$  as  $b$  grows. Furthermore, [Albers and Schubert \(2021\)](#) explored a broader setting where nodes within  $U$  possess varying budgets  $b_u$ . Through primal-dual methods, they showed that the Balance algorithm achieves a competitive ratio of  $(1/(1 + 1/b_{\min}))^{b_{\min}}$ , where  $b_{\min} = \min_{u \in U} b_u$ .

### 2.1. Model

Studying the online matching problem in an adversarial setting with budget refills requires some restrictions on how powerful the adversary is. Indeed, if the sequence of refills  $(\eta_{u,t})_{u \in U, t \in V}$  was to be chosen in an adversarial fashion, then the result would simply be to set it to 0, reducing the problem to the classical  $b$ -matching problem. To avoid such a reduction, two restrictions are considered on the adversary:

1. The sequence of refills  $(\eta_{u,t})_{u \in U, t \in V}$  is a parameter of the problem, set and known in advance to a refill of one unit every  $m$  time steps.
2. Every node  $t \in V$  has at least one neighbor in  $U$ .

Formally, the subset of graphs from which the oblivious adversary can choose is the following,

$$\mathcal{G}_{T,m} = \{ (U, V, E, (\eta_t)_{t \in V}) \in \mathcal{G}_T : \forall t \in V, \eta_t = [t \bmod m = 0] \text{ and } \exists u \in U, (u, t) \in E \} .$$

The choice of a refill of one unit every  $m$  time steps comes from the motivating application of advertising, where advertisers usually renew their budget monthly or quarterly. Considering a constant value for refills gives a clear and simple setting to disentangle the asymptotic effect of the refills versus the initialization of budgets.

The evolution of budget for  $u \in U$  depends on whether the edge  $(u, t) \in E$  is added to the online matching (this is represented through a binary variable  $x_{u,t} \in \{0, 1\}$ ) and whether  $t$  is a multiple of  $m$  or not; formally, it obeys the following dynamics:

$$b_{u,t} = b_{u,t-1} - x_{u,t} + 1_{t \bmod m=0} \quad \text{and} \quad b_{u,0} = b_0 \text{ for some } b_0 \geq 1.$$

As a consequence, the online *matching* on  $G \in \mathcal{G}_{T,m}$  generated by an algorithm ALG is the subset of edges that can be represented by a binary matrix  $\mathbf{x} \in \{0, 1\}^{n \times T}$  that must satisfy the following constraints:

1.  $\forall (u, t) \in U \times V, (u, t) \notin E \Rightarrow x_{u,t} = 0$  (only edges in  $E$  can be matched).
2.  $\forall t \in V, \sum_{u \in U} x_{u,t} \leq 1$  ( $V$ -nodes can only be matched once).
3.  $\forall (u, t) \in U \times V, b_{u,t-1} < 1 \Rightarrow x_{u,t} = 0$  ( $U$ -nodes need positive budget to be matched).

In online bipartite matching problems, the performance of an algorithm ALG is evaluated by its competitive ratio, which is the ratio between the size of the matching ALG has created and the largest possible matching in hindsight, also referred to as OPT with matrix  $\mathbf{x}^*$ . The rationale is that the optimal matching of some deterministic graph  $G$  can be arbitrarily small. Hence, the constructed matching size alone does not provide any good insight on the “quality” of an algorithm in the adversarial case. Formally, in the adversarial framework, the objective of the algorithm is to obtain the highest worst-case competitive ratio  $\text{CR}^{\text{adv}}(\text{ALG}, \mathcal{G}_{T,m})$ , defined as follows:

$$\text{CR}^{\text{adv}}(\text{ALG}, \mathcal{G}_{T,m}) = \min_{G \in \mathcal{G}_{T,m}} \frac{\text{ALG}(G)}{\text{OPT}(G)}$$

where  $\text{ALG}(G) = \sum_{u \in U} \sum_{t=1}^T x_{u,t}$ , and  $\text{OPT}(G) = \sum_{u \in U} \sum_{t=1}^T x_{u,t}^*$ , are the sizes of the matching generated by ALG and OPT respectively.

As previously highlighted, our analysis focuses on evaluating the Balance algorithm within the mentioned model. We aim to dissect the impact of the initial budget  $b_0$  and the refill process by parameterizing our results with  $T$ , which is both the finite horizon and the size of  $V$ . This choice slightly limits the adversary’s power, as it cannot impact the length of the horizon  $T$  by simply providing no edge for an arbitrary number of time steps. Before delving into the details of our main results within the adversarial setting, here is a summary of our contributions:

- For scenarios with relatively few refills, when  $m$  scales at (approximately) the order of at least  $\sqrt{T}$ , we prove that the impact of refills on the competitive ratio of Balance is negligible. In essence, the competitive ratio remains identical to that of the  $b_0$ -matching problem, i.e. the same problem without refills. Stated otherwise, the dominating effect is the initialization of the budgets. Interestingly, the refill frequency  $1/m$  does not appear in the competitive ratio.
- Conversely, in scenarios with a higher frequency of refills (with  $m$  small compared to  $\sqrt{T}$ ), the narrative is completely different. In such cases, the initial budgets exhibit no discernible influence on the (asymptotic) competitive ratio. Indeed, we establish an upper bound for the competitive ratio of Balance within a specific graph, simultaneously demonstrating that Balance represents the optimal algorithm. Notably, the derived upper bound stands at  $1 - \frac{1-\alpha}{e^{1-\alpha}}$ , wherein  $\alpha \simeq 0.603$ .

## 2.2. Regime $m = \omega(\sqrt{T})$

Intuitively, in the regime  $m = \omega(\sqrt{T})$ , the effect of the performance within only **one** period of length  $m$  can dominate the CR (this is obvious for  $m = T$ , but it happens even for  $m = \omega(\sqrt{T})$ ).

**Theorem 1** *Assuming the initial budgets are  $b_{1,0} = b_{2,0} = \dots = b_{n,0} = b_0 \geq 1$ . If  $m = \omega(\sqrt{T})$  and  $b_0(b_0 + 1)^{b_0} \leq m$ , then,*

$$\sup_{\text{ALG:deterministic}} \text{CR}^{\text{adv}}(\text{ALG}, \mathcal{G}_{T,m}) \leq 1 - \frac{1}{\left(1 + \frac{1}{b_0}\right)^{b_0}} + o_T(1) \quad (1)$$

*The bound is reached for the graph defined in the proof.*

**Sketch of proof.** The complete proof is provided in section A.1. It relies on using  $d = \lfloor \frac{m}{|V_K|} \rfloor$  duplicates of the graph  $G_K = (U_K, V_K, E_K)$  presented in Kalyanasundaram and Pruhs (2000), where the size depends on  $b_0$ . More precisely, it uses  $d$  copies of  $G_K$  at the beginning of the process and during the remaining time  $T - m$  only one node  $\tilde{u}$  from  $U$  is connected with all the remaining edges in  $V$  (see fig. 1 for illustration). Then, the number of edges matched by ALG and OPT during these  $T - m$  last steps is the same, denoted  $\gamma_T$  which is at most  $\lfloor \frac{T}{m} \rfloor$  as it relies on the refills of  $\tilde{u}$  only (see section A.1 for more details). Thus,

$$\text{CR}^{\text{adv}}(\text{ALG}, \mathcal{G}_{T,m}) \leq \frac{d\text{ALG}(G_K) + \gamma_T}{d\text{OPT}(G_K) + \gamma_T}$$

Since  $\gamma_T = o(\sqrt{T})$ , we can conclude that,

$$\text{CR}^{\text{adv}}(\text{ALG}, \mathcal{G}_{T,m}) \leq 1 - \frac{1}{\left(1 + \frac{1}{b_0}\right)^{b_0}} + o_T(1)$$

□

## 2.3. Regime $m = o(\sqrt{T})$

In the regime  $m = o(\sqrt{T})$ , where the refills dominate the initialization, the upper bound on the CR is not as strong. Unlike the previous scenario where it was bounded by  $1 - \frac{1}{e} \approx 0.63$ , it will be bounded only by 0.73. The following theorems establish this upper bound for Balance, and then that no algorithm can achieve significantly better performance.

**Theorem 2** *Assuming the initial budgets are  $b_{1,0} = b_{2,0} = \dots = b_{n,0} = b_0 \geq 1$ . For  $m = o(\sqrt{T})$  and  $mb_0 = o(T)$ , then,*

$$\text{CR}^{\text{adv}}(\text{Balance}, \mathcal{G}_{T,m}) \leq 1 - \underbrace{\frac{(1 - \alpha)}{e^{(1-\alpha)}}}_{\simeq 0.73325\dots} + o_{m,T}(1) \quad (2)$$

where  $\alpha$  is defined by  $\frac{1}{2} = \int_0^\alpha \frac{xe^x}{1-x} dx$ . The bound is reached for the graph defined in the proof.

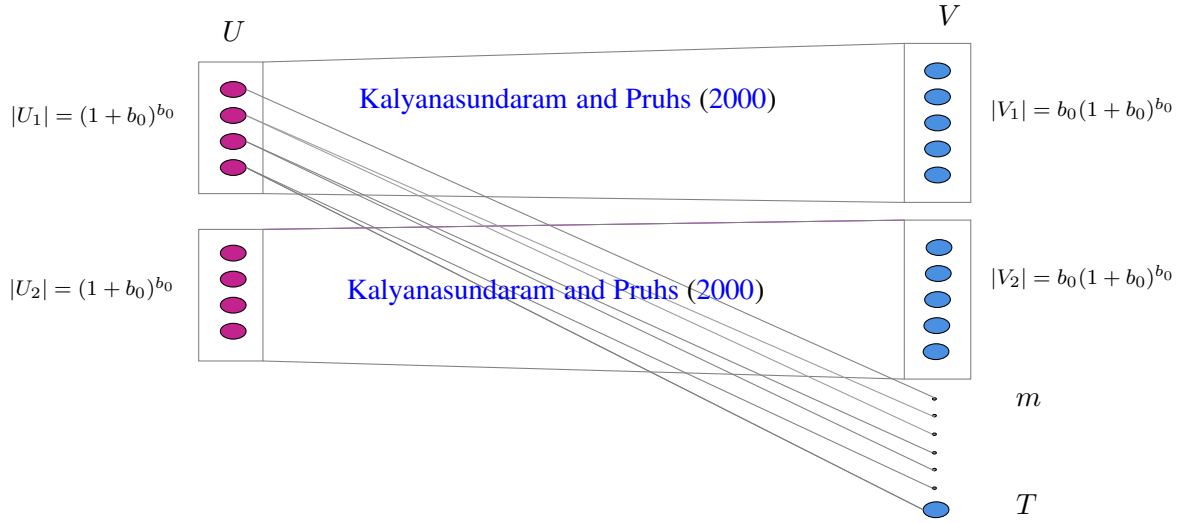


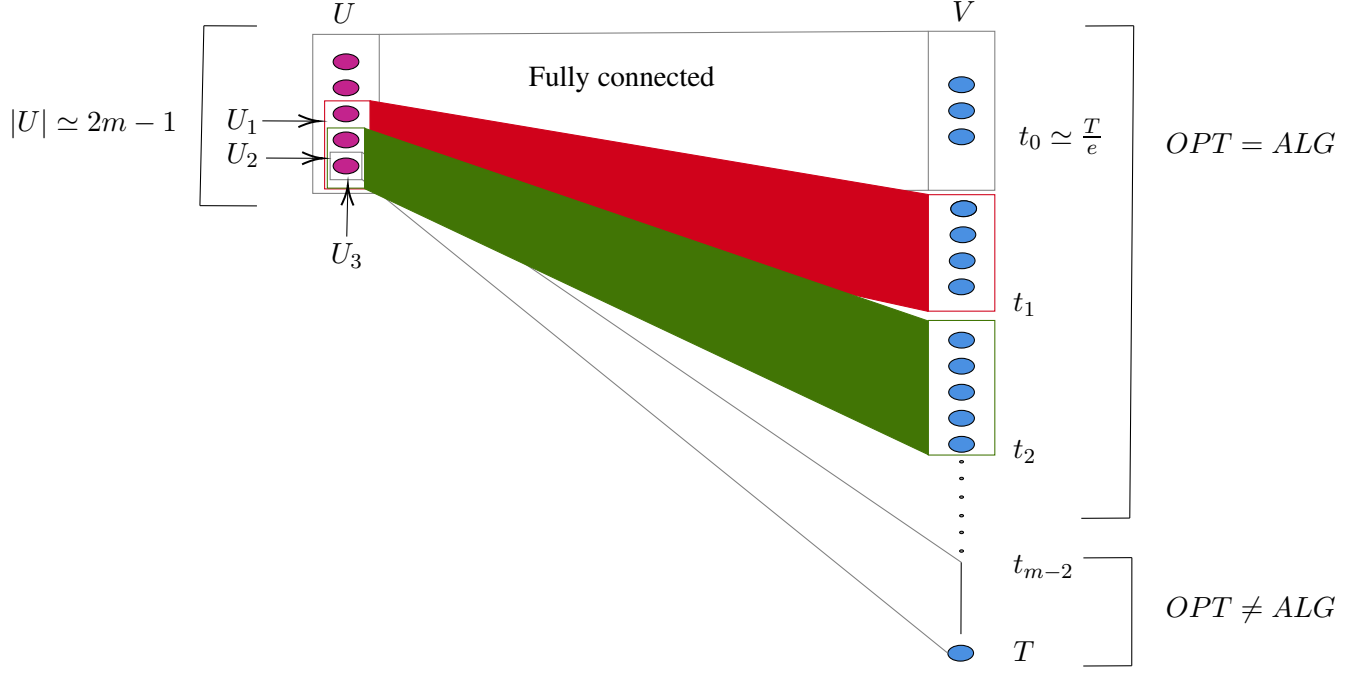
Figure 1: The graph used for the proof of theorem 1

**Sketch of proof.** The full proof is provided in section A.2. It relies on building an adversarial graph  $G^{\text{th.2}} = (U, V, E)$  with the following structure (see fig. 2 for illustration): Initially, for a period of size  $t_0 \simeq \frac{T}{e}$ , the size of  $U$  exceeds  $m$  ( $|U| \simeq 2m - 1$ ), allowing the algorithm to accumulate a significant amount of budget. During this period, ALG and OPT consistently match nodes and accumulate an equal amount of budget on  $U$ , but it is not distributed in the same way. At time  $t_0$ , the adversary removes all but  $m - 1$  nodes from  $U$  (starting with those with the highest budgets). Specifically, When the adversary eliminates nodes, it has no impact on OPT because OPT has perfect hindsight knowledge of the eliminated nodes. Therefore, it can allocate the budget exclusively to nodes that remain available, ensuring that no budget is lost on eliminated nodes at the time of their removal. However, ALG remains unaware of which nodes will be eliminated. Consequently, at the time of elimination, the nodes still have some budget. Subsequently, the remaining nodes are removed one by one, at a rhythm that depends on  $m$ , carefully designed for OPT to widen the gap as much as possible with ALG.

**ALG vs OPT over time:** As previously mentioned, up to time  $t_{m-2}$ , both ALG and OPT have the same performance. It is only between  $t_{m-2}$  and  $T$  that distinctions arise. Hence, the crucial step lies in determining the remaining budget of ALG at time  $t_{m-2}$  denoted  $P_{t_{m-2}}$ . To accomplish this, it's necessary to compute the values of  $t_i$  and then analyze how the remaining budget of ALG evolves over time.

**Intuition behind the choice of " $t_i$ s":** Since the main difference between ALG and OPT lies in the fact that OPT knows the eliminated nodes beforehand, one important quantity to track is the  $t_i$  which represents the time taken to deplete the budget of node  $u_i$  by consistently avoiding matches with  $u_i$  before  $t_{i-1}$  and then matching it at every time step between  $t_{i-1}$  and  $t_i$  (a strategy employed by OPT).  $t_i$  is determined by the following recurrence relationship:

$$t_{i+1} \approx b_0 - 1 + t_i + \frac{b_0 + t_i}{m - 1}$$


 Figure 2: The graph  $G^{\text{th.2}}$  used for the proof of theorem 2

by solving it we get,

$$t_i \approx \left(1 + \frac{1}{m-1}\right)^i (t_0 + mb_0 - m + 1) - mb_0 + m - 1$$

**Intuition about the value of remaining budget:** The remaining budget at time  $t_i$  follows the following recurrence,

$$P_{t_i} \approx \left( \underbrace{P_{t_{i-1}}}_{\text{the remaining budget at time } t_{i-1}} + \underbrace{\frac{(n-i)(t_i - t_{i-1})}{m}}_{\text{the refills received between time } t_i \text{ and } t_{i-1}} - \underbrace{\frac{(t_i - t_{i-1})}{n-i}}_{\text{number of nodes matched}} \right) \frac{n-i-1}{n-i}$$

The expression  $\frac{n-i-1}{n-i}$  represents the ratio of the number of nodes at time  $t_i$  to the number of nodes at time  $t_{i-1}$ .

Therefore, the crux of the proof lies in examining the dynamics and rate of evolution of  $P_{t_i}$  and handling the technicalities related to the approximations of the floor and ceil functions involved in the construction of the different quantities of the problem.  $\square$

### 2.3.1. NO ALGORITHM CAN BEAT Balance

**Theorem 3** Assuming the initial budgets are  $b_{1,0} = b_{2,0} = \dots = b_{n,0} = b_0 \geq 1$ . For  $m = o(\sqrt{T})$ ,

$$\sup_{\text{ALG}} \mathbb{E} \left[ \text{CR}^{\text{adv}}(\text{ALG}, \mathcal{G}_{T,m}) \right] \leq \text{CR}^{\text{adv}}(\text{Balance}, G^{\text{th.2}}) + o_T(1) \quad (3)$$

where the expectation is taken over the randomness from ALG.

The proof is provided in section A.3.

**Sketch of proof.** The intuition is that keeping budgets equalized between the currently available  $U$ -nodes is the best choice an algorithm can make in the adversarial graph used for the proof of theorem 2. This is because the adversary removes  $U$ -nodes one after the other, beginning with those with the highest budget and never providing again a  $U$ -node already removed.  $\square$

Analyzing the online matching problem with budget refills in the adversarial setting provides valuable worst-case guarantees for the competitive ratio of the selected algorithm. However, in practical applications, the environment is rarely completely adversarial and a worst-case guarantee is often far from actual performance. Especially, the graphs built by the adversary exhibit no stationary regime when  $T$  grows, while practical applications often enter such regime after some time. At the cost of additional assumptions, the stochastic framework offers a way to study such regimes more precisely. While the adversarial model ensures robustness, the stochastic approach aligns better with real-world dynamics, where uncertainties and randomness are often inherent.

### 3. The Stochastic framework

Another line of research in online matching has centered on the stochastic version of the problem. For instance, the known i.i.d. model assumes the existence of a probability distribution governing the types of vertices drawn independently and identically at each iteration. With the knowledge of this distribution, algorithms with much better competitive ratios than Ranking were designed (Manshadi et al., 2012; Jaillet and Lu, 2014; Brubach et al., 2016; Huang et al., 2022), the best one to date achieving a competitive ratio of (approximately) 0.711. However, this i.i.d. model, while versatile, remains somewhat contrived. Algorithms for this scenario are optimized for the well-known i.i.d. model and often struggle to accommodate additional knowledge about the graph. Furthermore, since the guarantee is provided for the worst potential input distribution, it might not consistently reflect the average performance of these algorithms. As highlighted in Borodin et al. (2020), in numerous average-case and practical input scenarios, straightforward greedy strategies either outperform or perform equally to state-of-the-art algorithms specifically designed for the known i.i.d. setting. This has led to a call for the formulation of novel stochastic input models that more accurately mirror practical inputs in certain application domains, such as online advertising.

Consequently, another stream of the literature considers standard online algorithms applied to specific classes of random graphs, representing scenarios where certain properties of the underlying graph are known. One seminal example is online matching in Erdős–Rényi graphs introduced in Mastin and Jaillet (2013), where the assumption is that each possible edge exists in  $U \times V$  with a fixed probability, independent of other edges. The most interesting and challenging setting corresponds to the so-called sparse regime where each vertex of  $U$  has an expected degree independent of the size  $n$  of  $V$ , which amounts to taking a probability of connection equal to  $c/n$ . Remarkably, even analyzing the most straightforward Greedy algorithm in these models poses significant challenges and provides valuable insights see (Mastin and Jaillet, 2013; Borodin et al., 2018; Dyer et al., 1993). On another hand, a more general random graph model, known as the configuration model was analyzed in Noiry et al. (2021); Aamand et al. (2022), which specifies a distribution for the degrees of the vertices.



### 3.1. Model

The online matching problem with refills of the budgets in the stochastic setting is studied in the following framework:

1. The random graph is a standard Erdős–Rényi model  $G(n, T, p)$ , i.e., a bipartite graph with  $n$  vertices on one side,  $T$  on the other side and each potential edge  $(u, t) \in U \times V$  occurs independently with probability  $p$ .
2. The regime considered is the sparse one, in the sense that  $p = \frac{a}{n}$  with  $a > 0$ . The motivation comes from online advertising where there are many users compared to the number of ad campaigns, but only a few are eligible.
3. The sequence of refills  $(\eta_{u,t})_{u \in U, t \in V}$  is a realization of independent Bernoulli random variables of parameter  $\beta/n$ , for some  $\beta > 0$ .

We shall denote by  $\mathcal{D}$  the distribution of such random instance (graph and refills).

As emphasized previously, each node in  $U$  is associated with a budget  $b_{u,t} \in \mathbb{N}$ . We add the additional assumption that the maximum budget per node is capped at some  $K \in \mathbb{N}^*$  so that the budget dynamics are now expressed as follows,

$$b_{u,t} = \min(K, b_{u,t-1} - x_{u,t} + \eta_t) \quad \text{with } b_{u,0} = b_0 \geq 1.$$

As before, an online *matching* on  $G$  generated by an algorithm ALG is a subset of edges, represented by a binary matrix  $\mathbf{x} \in \{0, 1\}^{n \times T}$ , satisfying the following constraints:

1.  $\forall (u, t) \in U \times V, (u, t) \notin E \Rightarrow x_{u,t} = 0$  (only edges in  $E$  can be matched).
2.  $\forall t \in V, \sum_{u \in U} x_{u,t} \leq 1$  (no  $V$ -node can be matched twice).
3.  $\forall (u, t) \in U \times V, b_{u,t-1} < 1 \Rightarrow x_{u,t} = 0$  ( $U$ -nodes need some positive budget to be matched).

The reasons behind capping the maximal budget to  $K$  are three-fold. First, in many applications in mind, the budget is capped (either by one, which corresponds to an idle/active state, or by a large number as in the online advertisement motivating example). Second, with the algorithm and the random graph considered, the budget will follow a negatively biased (and non-homogeneous) random walk, so that the maximal budget is sub-linear with arbitrarily high probability (hence this restriction is actually without loss of generality in the random model considered). Third, this capping induces a finite number of quantities to track through time (namely the current proportion of vertices with this or that budget), which greatly simplifies the analysis.

The performance of an algorithm in the stochastic setting can be either measured by the size of the expected matching size it creates or by the ratio between expected matching sizes of ALG and OPT. Formally, the different quantities we shall consider are

$$\text{CR}^{\text{sto}}(\text{ALG}, \mathcal{D}) = \frac{\mathbb{E}_{G \sim \mathcal{D}}[\text{ALG}(G)]}{\mathbb{E}_{G \sim \mathcal{D}}[\text{OPT}(G)]} \quad \text{or matching size} = \mathbb{E}_{G \sim \mathcal{D}}[\text{ALG}(G)]$$

where  $\text{ALG}(G) = \sum_{u \in U} \sum_{t=1}^T x_{u,t}$ , and  $\text{OPT}(G) = \sum_{u \in U} \sum_{t=1}^T x_{u,t}^*$ , are the sizes of the matching generated by ALG and OPT respectively. The dependency on  $T$  is implicitly in the definition of  $G$ , however, we shall explicitly indicate the dependency by considering  $\text{ALG}(G, T)$  and  $\text{OPT}(G, T)$ .

To provide an overview of our contributions within the stochastic setting, we first present a summary of our main results:

- Firstly, we describe the asymptotic performance of Greedy on the Erdős-Rényi model for any value of budget capping (in particular for arbitrarily large ones). We prove this by first introducing a system of ODEs, and second proving that the discrete and random matching process is, with very high probability, close to the continuous and deterministic solution of this system.
- Secondly, we study the stability of the stationary solution of the ODE system. In the specific case of a maximal budget of one (motivated by vertices that are idle after being matched until they become active again after some random time), we also get more specific and technical convergence results.
- Lastly, in terms of competitive ratio, we establish that it converges to 1 as  $K$ ,  $T$  and  $n$  approach infinity.

### 3.2. Main results

Our first main theorem, stated below, identifies the asymptotic size of the matching generated by Greedy on the bipartite Erdős-Rényi model with budget refills. The result shows that with high probability, the size of the matching generated by Greedy is close to the solution of a system of ordinary differential equations.

**Theorem 4** *With probability  $1 - \mathcal{O}(n^{1/4} \exp(-a^3 n^{1/4}))$ , the matching size created by Greedy denoted by  $\text{Greedy}(G, T)$  satisfies,*

$$\text{Greedy}(G, T) = nh(T/n) + \mathcal{O}(n^{3/4})$$

and,

$$\frac{\mathbb{E}[\text{Greedy}(G, T)]}{n} \xrightarrow{n \rightarrow +\infty} h(T/n)$$

where  $h(\tau)$  is solution of the following equation,

$$\dot{h}(\tau) = 1 - e^{-a(1-z_0(\tau))}, \quad \frac{1}{n} \leq \tau \leq \frac{T}{n}$$

and  $z_0(\tau)$  satisfies the following system,

$$\begin{cases} \dot{z}_0(\tau) = -z_0(\tau)\beta + \frac{z_1(\tau)}{1-z_0(\tau)}(1 - e^{-a+az_0(\tau)}) & \text{for } k = 0 \\ \dot{z}_k(\tau) = (z_{k-1}(\tau) - z_k(\tau))\beta + (z_{k+1}(\tau) - z_k(\tau))\frac{1-e^{-a+az_0(\tau)}}{1-z_0(\tau)} & \text{for } 1 \leq k \leq K-1 \\ \dot{z}_k(\tau) = \beta z_{k-1}(\tau) - z_k(\tau)\frac{1-e^{-a(1-z_0(\tau))}}{1-z_0(\tau)} & \text{for } k = K \\ \sum_{k=0}^K z_k(\tau) = 1 \end{cases} \quad (4)$$

**Sketch of proof.** For  $0 \leq k \leq K$ ,  $t \in [T]$ , let  $U_k(t) = \{u \in U : b_{u,t} = k\}$  be the set of nodes with budget equals to  $k$  at time  $t$  and  $Y_k(t) = |U_k(t)|$  the total number of nodes with budget equals  $k$  in  $U$ . The expectation of the one-step change of the variable  $\text{Greedy}(G, t)$  can be expressed as,

$$\mathbb{E}[\text{Greedy}(G, t+1) - \text{Greedy}(G, t) | \text{Greedy}(G, t)] = 1 - \left(1 - \frac{a}{n}\right)^{\sum_{k \geq 1} Y_k(t)} = 1 - \left(1 - \frac{a}{n}\right)^{n - Y_0(t)}$$

As the evolution of  $\text{Greedy}(G, t)$  depends on  $Y_0$ , an analysis of the process  $\mathbf{Y}(t) = (Y_k(t))_{k \geq 0}$  is necessary. The dynamic of this process is described by the following system:

$$\begin{cases} \mathbb{E}[\Delta_0(t)|\mathbf{Y}(t)] = -Y_0(t) [\delta(1 - p\Sigma(t))] + Y_1(t)(1 - \delta)p\Sigma(t) \\ \mathbb{E}[\Delta_1(t)|\mathbf{Y}(t)] = -Y_1(t) [\delta(1 - p\Sigma(t)) + (1 - \delta)p\Sigma(t)] + Y_0(t)\delta + Y_2(t)(1 - \delta)p\Sigma(t) \\ \mathbb{E}[\Delta_k(t)|\mathbf{Y}(t)] = \delta(1 - p\Sigma(t)) [Y_{k-1}(t) - Y_k(t)] + [Y_{k+1}(t) - Y_k(t)] (1 - \delta)p\Sigma(t) \quad \forall k > 1 \end{cases}$$

where  $\forall k \geq 0$ ,  $\Delta_k(t) = Y_k(t+1) - Y_k(t)$ , and  $\Sigma(t) = \frac{1}{p(n-Y_0(t))} (1 - (1-p)^{(n-Y_0(t))})$ .

After establishing the evolution of these processes, the main idea behind the proof of theorem 4 (postponed to section B.1) is to show that  $\text{Greedy}(G, T)$  is closely related to the solution of some ODE (this is sometimes called "the differential equation method" (Wormald, 1999; Warnke, 2019; Enriquez et al., 2019) or "stochastic approximations" (Robbins and Monro, 1951)).  $\square$

Upon establishing that, with high probability  $\text{Greedy}(G, T)$  is close to  $nh(T/n)$ , a function depending on  $z_0(T/n)$ , the solution of eq. (4), the objective is to solve this system to get an exact approximation of the matching size created by Greedy on the Erdős–Rényi model. However, obtaining a closed-form solution of the system of differential equations eq. (4) is quite challenging. To address this complexity, one approach is to explore the system's stationary solution and examine its stability; This means determining whether the solution to eq. (4) converges to this stationary state, and then showing that  $\text{Greedy}(G, T)$  converges towards a function depending on the stationary solution of eq. (4).

More precisely, corollary 5 shows that for  $K \geq 1$ ,  $\text{Greedy}(G, T)$  converges with high probability to  $nh^*$  a function of  $z_0^*$ , the stationary solution of eq. (4) and when  $n \rightarrow \infty$ ,  $\frac{\mathbb{E}(\text{Greedy}(G, T))}{n}$  converges to  $h^*(\psi)$ . Furthermore, corollary 6 demonstrates at a specified rate the convergence of  $\text{Greedy}(G, T)$  to  $nh^*(T/n)$  with high probability and also, for  $n \rightarrow \infty$ ,  $\frac{\mathbb{E}(\text{Greedy}(G, T))}{n}$  converges to  $h^*(T/n)$ . The distinction between these results lies in the type of convergence of  $z_0(t)$  to  $z_0^*$ : In corollary 5,  $z_0(t)$  asymptotically converges to  $z_0^*$ , whereas in corollary 6, the convergence is exponential.

**Corollary 5** For  $K \geq 1$ , with probability at least  $1 - 2e^{-a^2 n^{\frac{3}{8}}/8T}$ ,

$$|\text{Greedy}(G, T) - nh^*(T/n)| \leq o(T)$$

and,

$$\frac{\mathbb{E}[\text{Greedy}(G, T)]}{n} \xrightarrow{n \rightarrow +\infty} h^*(T/n)$$

with  $h^*(x) = \int_{1/n}^x (1 - e^{-a(1-z_0^*)}) d\tau = (x - \frac{1}{n}) (1 - e^{-a(1-z_0^*)})$ , and  $z_0^*$  is the unique solution of  $\sum_{k=0}^K z_0^* \left(\frac{\beta}{g(z_0^*)}\right)^k = 1$  with  $g(z_0^*) = \frac{1 - e^{-a(1-z_0^*)}}{1 - z_0^*}$ .

Section B.2 contains the detailed proof.

**Sketch of proof.** The first step is to compute  $\bar{S}_{z_0^*}$ , the unique stationary solution of eq. (4). Then, to demonstrate that  $\bar{S}_{z_0^*}$  is an asymptotically stable stationary solution, we rely on matrix perturbation theory for the proof of stability. Once stability is established, we further prove that the matching size converges to a function that depends on  $\bar{S}_{z_0^*}$ .  $\square$

**Corollary 6** For  $K = 1$ , with probability at least  $1 - 2e^{-a^2 n^{\frac{3}{2}}/8T}$ ,

$$\left| \mathbb{E}[\text{Greedy}(G, T)] - T(1 - e^{-a(1-z_0^*)}) \right| \leq c \frac{T}{(\log(T))^{3/4}} = o(T)$$

where  $z_0^* = \frac{1}{\beta} - \frac{1}{a} W\left(\frac{a}{\beta} e^{-a(1-\frac{1}{\beta})}\right)$ , with  $W(\cdot)$  the Lambert function, and  $c$  is some universal constant.

The proof is postponed to section B.3.

**Sketch of proof.** For  $K = 1$ , eq. (4) is reduced to a system of two equations. Firstly, we compute  $S_{z_0^*}^1$ , the stationary solution of the reduced system. Then, we prove that  $S_{z_0^*}^1$  is an exponentially stable stationary solution using the perturbation method. Once the exponential stability is established, we further get that the matching size converges to a function depending only on  $S_{z_0^*}^1$ .  $\square$

Our final main result of this section is given below. Firstly, we establish a lower bound on  $\text{CR}^{\text{sto}}$ , which depends on  $(z_0^*, \dots, z_K^*)$  the stationary solution of eq. (4). This lower bound is derived through an exact calculation of the matching size achieved by the Greedy algorithm and an upper bound on the matching size generated by OPT. Subsequently, we demonstrate that the competitive ratio converges to 1 as  $T$ ,  $K$  and  $n$  grows significantly.

**Proposition 7** For  $T, K, n, b_0, \beta \in \mathbb{N}^*$ ,

$$\text{CR}^{\text{sto}}(\text{Greedy}, \mathcal{D}) \geq \frac{Tg(z_0^*)(1 - z_0^*) + nb_0 - n \left( \frac{\beta}{g(z_0^*) - \beta} - \frac{(K+1)\beta^{K+1}}{g(z_0^*)^{K+1} - \beta^{K+1}} \right)}{nb_0 + \beta T} + \mathcal{O}_{K,\beta}(T^{-1/4}) \quad (5)$$

where  $\sum_{k=0}^K z_0^* \left( \frac{\beta}{g(z_0^*)} \right)^k = 1$  with  $g(z_0^*) = \frac{1 - e^{-a(1-z_0^*)}}{1 - z_0^*}$  as defined in corollary 5.

The proof is presented in section B.4

**Sketch of proof.** Initially, we express  $\text{Greedy}(G, T)$  as a function of  $T, z_0(t), \beta, a$ . Then, we use an upper bound on  $\text{OPT}(G, T)$ , which is not very tight as it only takes into account the initial budget and the refills. Subsequently, we approximate  $\text{Greedy}(G, T)$  by a function that depends on the stationary solution  $\bar{S}_{z_0^*}$ .

It is noteworthy that in this context, the matching size  $\text{Greedy}(G, T)$  aligns with that of theorem 4 through the integration of the system eq. (4), up to negligible terms.  $\square$

From proposition 7, the next result shows that when  $K, T, n$  goes to infinity, the competitive ratio approaches 1.

**Theorem 8** For any  $\alpha, \beta > 0$ , the competitive ratio tends to 1, as  $T, K, n$  approach infinity, as

$$\lim_{K, n \rightarrow +\infty} \lim_{T \rightarrow +\infty} \text{CR}^{\text{sto}}(\text{Greedy}, \mathcal{D}) = 1$$

The proof is in section B.5

**Sketch of proof.** The proof relies on calculating  $z_0^*$  as  $K$  approaches infinity. Subsequently, as  $T$  approaches infinity, the limit of  $\text{CR}^{\text{sto}}(\text{Greedy}, \mathcal{D})$  is shown to be  $g(z_0^*)(1 - z_0^*)/\beta$ . Finally, as  $K$  tends towards infinity, the limit converges to 1, with the assurance that this convergence happens with high probability as  $n$  tends to infinity.  $\square$

## 4. Conclusions

This study delves into the online matching problem on a bipartite graph  $G = (U, V, E)$  with budget refills, an intriguing and practical extension of the standard online matching paradigm that involves endowing nodes in  $U$  with budget allocations. Two distinct frameworks, namely the adversarial and the stochastic, are studied, with the primary disparity residing in the types of graphs and the nature of the budget dynamics considered. In the stochastic framework, our exploration of the asymptotic performance of Greedy in random graphs aligns with the initial intuition that periodic budget refills can streamline the online algorithm’s task, occasionally resulting in a competitive ratio CR reaching 1.

In the adversarial case with minimal refills, our analysis demonstrates the negligible impact of refills on the competitive ratio of Balance, aligning its CR with that observed in the  $b$ -matching scenario. However, in cases involving numerous refills, a larger upper bound emerges, suggesting the possibility for algorithms to attain improved performance.

The dynamic nature of the budget constraint poses challenges in applying the primal-dual method to establish a lower bound. A crude lower bound of  $1 - \frac{1}{e}$  is derived from a modified version of Ranking, wherein each node  $u \in U$  is duplicated upon refill, altering the problem to a context where

Ranking secures a minimum performance of  $1 - \frac{1}{e}$ . Nevertheless, this approach leaves a gap compared to the upper-bound of  $1 - \frac{1-\alpha}{e^{1-\alpha}}$  in the case  $m = o(\sqrt{T})$ . While our intuition, based on extensive simulations and optimization, suggests that the upper bound might be precise, this area remains a challenging open question.

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## Appendix A. Adversarial Case

### A.1. Proof of theorem 1

**Theorem 1** *Assuming the initial budgets are  $b_{1,0} = b_{2,0} = \dots = b_{n,0} = b_0 \geq 1$ . If  $m = \omega(\sqrt{T})$  and  $b_0(b_0 + 1)^{b_0} \leq m$ , then,*

$$\sup_{\text{ALG:deterministic}} \text{CR}^{\text{adv}}(\text{ALG}, \mathcal{G}_{T,m}) \leq 1 - \frac{1}{\left(1 + \frac{1}{b_0}\right)^{b_0}} + o_T(1) \quad (1)$$

*The bound is reached for the graph defined in the proof.*

#### Proof.

Let  $b_0, m, T \in \mathbb{N}^*$  such that  $m \geq kb_0$  where  $k \triangleq (1 + b_0)^{b_0}$  and  $m \leq T$ . The bipartite graph of size  $(k, kb_0)$  used in (Kalyanasundaram and Pruhs, 2000, Sec. 2, Thm. 5) is denoted  $(U_0, V_0, E_0)$ . To put the emphasis on which set of nodes the edges are defined on,  $E_0$  will actually be denoted  $E_0(U_0, V_0)$  as the structure of edges will be used on different subsets of nodes of the final graph.

The graph  $G = (U, [T], E)$  with  $U = \{u_1, \dots, u_n\}$  of size  $n \in \mathbb{N}^*$  is built as follows. Intuitively, the first period of length  $m$  is implementing copies of  $E_0$  on disjoint nodes, then one remaining node in  $U$  is the only neighbor of all following time steps. Denoting  $j = \lfloor \frac{m}{kb_0} \rfloor$ ,

$$E = \left( \bigcup_{i=1}^j E_0(U_i, V_i) \right) \cup (\{\tilde{u}\} \times \llbracket jkb_0 + 1, T \rrbracket) \quad (6)$$

where  $U_i = \{u_l : l \in \llbracket (i-1)k + 1, ik \rrbracket\}$ ,  $V_i = \llbracket (i-1)mb_0 + 1, ikb_0 \rrbracket$  and  $\tilde{u}$  is chosen to be a node of  $U_1$  which has been depleted of its initial budget during  $V_1$  (there is at least one).

For each  $i \in [j]$ , on each subset  $V_i$  of time steps, as per Kalyanasundaram and Pruhs (2000, Proof of Thm. 5), ALG matches at most  $b_0(b_0 + 1)^{b_0} - b_0^{b_0+1}$  edges, while OPT manages to match  $b_0(b_0 + 1)^{b_0}$  edges. After time  $jkb_0$ , both ALG and OPT match the same number of edges  $\gamma_T$  which is at most the sum of refills obtained by  $\tilde{u}$  - i.e.  $\lfloor \frac{T}{m} \rfloor$  - (its initial budget is used during period  $V_1$ ). In the end,

$$\text{CR}^{\text{adv}}(\text{ALG}, \mathcal{G}_{T,m}) \leq \frac{j(kb_0 - b_0^{b_0+1}) + \gamma_T}{jkb_0 + \gamma_T} \quad (7)$$

$$\leq \frac{j(kb_0 - b_0^{b_0+1})}{jkb_0} + o_T(1) \quad \text{as } \gamma_T = o(\sqrt{T}) \text{ and } jkb_0 = \omega(\sqrt{T}) \quad (8)$$

$$= 1 - \frac{1}{\left(1 + \frac{1}{b_0}\right)^{b_0}} + o_T(1) \quad \text{def. of } k = (1 + b_0)^{b_0} \quad (9)$$

Similarly, it is straightforward to show that Balance achieves the lower bound of the  $b$ -matching problem on each of the duplicates of  $E_0(U_i, V_i)$ , as these sub-graphs are disjoint. ■



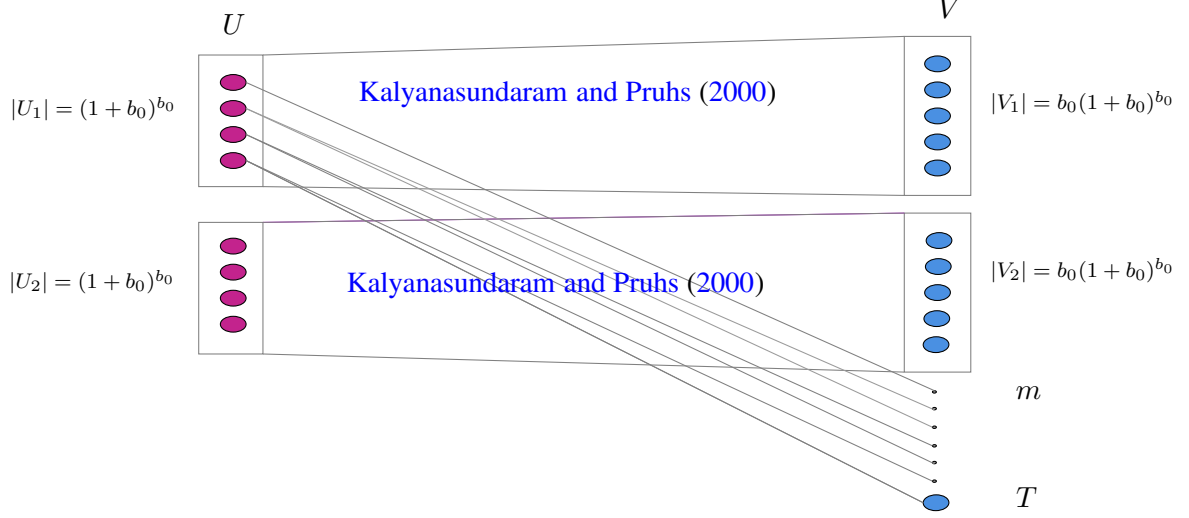


Figure 3: The graph used for the proof of theorem 1

## A.2. Proof of theorem 2

**Theorem 2** Assuming the initial budgets are  $b_{1,0} = b_{2,0} = \dots = b_{n,0} = b_0 \geq 1$ . For  $m = o(\sqrt{T})$  and  $mb_0 = o(T)$ , then,

$$\text{CR}^{\text{adv}}(\text{Balance}, \mathcal{G}_{T,m}) \leq 1 - \underbrace{\frac{(1-\alpha)}{e^{(1-\alpha)}}}_{\simeq 0.73325\dots} + o_{m,T}(1) \quad (2)$$

where  $\alpha$  is defined by  $\frac{1}{2} = \int_0^\alpha \frac{xe^x}{1-x} dx$ . The bound is reached for the graph defined in the proof.

We provide a slightly more detailed result here.

**Theorem 9** Assuming the initial budgets are  $b_{1,0} = b_{2,0} = \dots = b_{n,0} = b_0 \geq 1$ . For  $m = o(\sqrt{T})$  and  $mb_0 = o(T)$ , then,

$$\begin{aligned} \text{CR}^{\text{adv}}(\text{Balance}, \mathcal{G}_{T,m}) \leq & 1 - \frac{mb_0 + t_0}{e(mb_0 + 2t_0)} - \frac{1}{e} \int_0^\alpha \frac{x^2 e^x}{1-x} dx \\ & + \frac{mb_0}{t_0} \left( 1 - \frac{1}{e} + \frac{1}{e} \int_0^\alpha \frac{x(\alpha-x)e^x}{1-x} dx \right) + o_{m,T}(1) \end{aligned} \quad (10)$$

where  $\alpha$  is defined as follows  $\int_0^\alpha \frac{xe^x}{1-x} dx = 1 - \frac{t_0}{mb_0 + 2t_0}$ . The upper bound is reached for the graph defined in the proof.

The proof is organized as follows:

1. Definition of the adversarial graph.
2. Decomposition of  $\text{Balance}(G^{\text{th},2})$ .
3. Several lemmas to treat each term of the decomposition.

**Definition of the adversarial graph for Balance.** For  $b_0, m, T \in \mathbb{N}^*$  such that  $m \leq T$ , the number of  $U$ -nodes is set to  $n = m - 1 + \max\left(\left\lceil \frac{t_0}{b_0 + \lfloor \frac{t_0}{m} \rfloor} \right\rceil, \left\lceil \frac{m \lfloor \frac{t_0}{m} \rfloor}{b_0 + \lfloor \frac{t_0}{m} \rfloor - 1} \right\rceil\right)$  (Note that when  $b_0 \ll \frac{t_0}{m}$ , then  $n \simeq 2m - 1$ ). The graph  $G^{\text{th.2}} = (U, V, E)$  is defined as follows,

$$\begin{cases} U = [n] \\ V = [T] \\ E = (U \times [t_0]) \cup (U_1 \times \llbracket t_0 + 1, t_1 \rrbracket) \cup \dots \cup (U_{m-1} \times \llbracket t_{m-2} + 1, t_{m-1} \rrbracket) \cup (U_{m-1} \times \llbracket t_{m-1} + 1, T \rrbracket) \end{cases}$$

where,

- $U_1$  is the subset of the  $m - 1$  node with the lowest budget at time  $t_0$ , i.e.

$$U_1 \stackrel{\text{unif}}{\sim} \{A \subseteq U : |A| = m - 1, \forall u \in A, u' \in U \setminus A, b_{u,t_0} \leq b_{u',t_0}\}.$$

- for any  $i > 1$ ,  $U_i$  is built by removing the node with the lowest budget at time  $t_{i-1}$  from  $U_{i-1}$  – i.e.  $U_i = U_{i-1} \setminus \{u_i\}$  where

$$u_i \stackrel{\text{unif}}{\sim} \arg \min_{u \in U_{i-1}} b_{u,t_{i-1}}.$$

- for any  $i \geq 1$ ,  $t_i = \inf\{t > t_{i-1} : b_0 + \lfloor \frac{t}{m} \rfloor = (t - t_{i-1})\}$ . *Intuition:  $t_i$  is the time it takes to take the budget of  $u_i$  to 0 by never matching  $u_i$  before  $t_{i-1}$  and matching it at every time step between  $t_{i-1}$  and  $t_i$  (which is what OPT does).*
- $t_0$  is chosen such that  $T - t_{m-1} = o(T)$  (it is possible as proven in lemma 10)

### Proof.

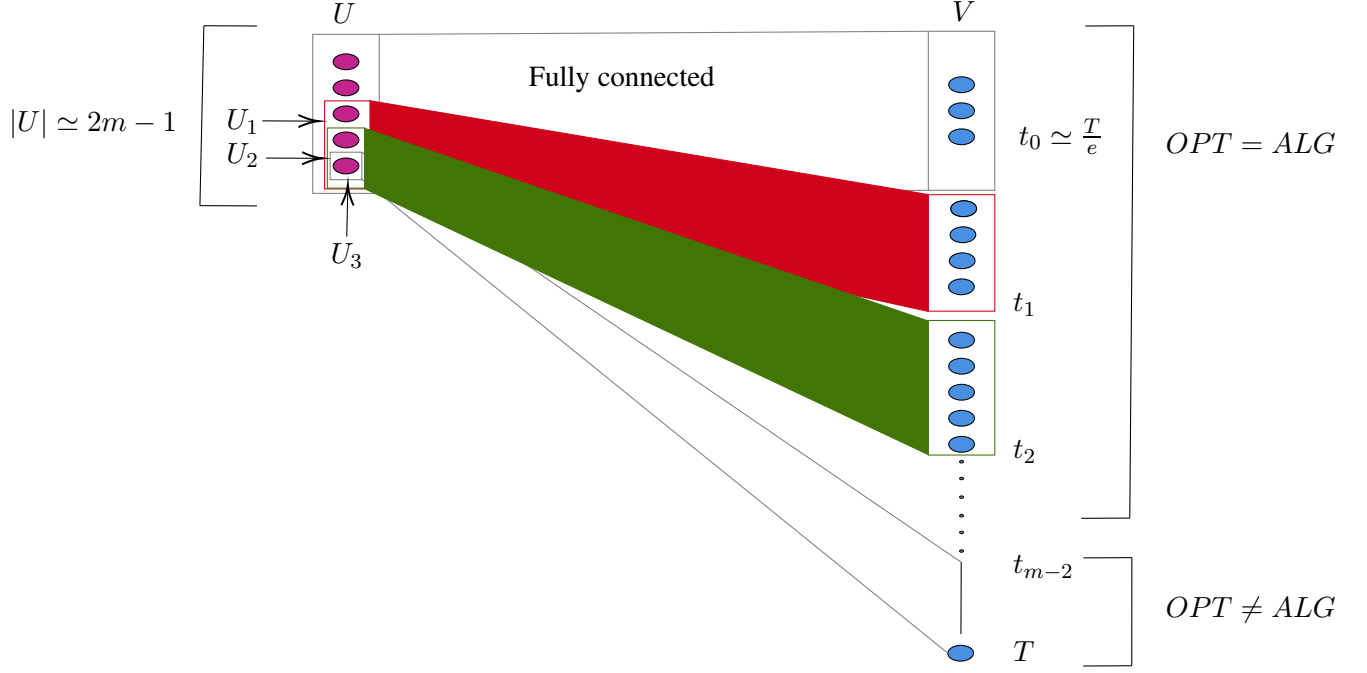
The objective is to compute the performance of Balance and OPT on the graph  $G^{\text{Th. 2}}$  defined above to obtain a bound on the CR.

**Performance of OPT.** Before time  $t_0$ , OPT can use nodes from  $U \setminus U_1$  to match at each time step:  $|U \setminus U_1| = \max\left(\left\lceil \frac{t_0}{b_0 + \lfloor \frac{t_0}{m} \rfloor} \right\rceil, \left\lceil \frac{m \lfloor \frac{t_0}{m} \rfloor}{b_0 + \lfloor \frac{t_0}{m} \rfloor - 1} \right\rceil\right)$ , which ensures that the total budget of nodes in  $U \setminus U_1$  over the period  $[t_0]$  is at least  $t_0$  (accounting for the last refill that cannot necessarily be fully used). As a remark, if  $b_0 = 1$ , this simplifies to  $|U \setminus U_1| = m$ : with a refill every  $m$  timesteps,  $m$  nodes suffice to match at every time step. Thus, at time  $t_0$ , OPT never matched any node from  $U_1$ . Then, by induction and definition of  $t_i$ ,  $\text{OPT}(G^{\text{th.2}}) = t_{m-1}$ .

### Performance of ALG.

$$\text{Balance}(G^{\text{th.2}}) = \underbrace{\sum_{t=1}^{t_0} \sum_{u \in U} x_{u,t}}_{\triangleq A_0} + \sum_{i=1}^{m-1} \underbrace{\sum_{t=t_{i-1}+1}^{t_i} \sum_{u \in U_i} x_{u,t}}_{\triangleq A_i} \quad (11)$$

$$= A_0 + \sum_{i=1}^{m-1} A_i \quad (12)$$


 Figure 4: The graph  $G^{\text{th.2}}$  used for the proof of theorem 2

$$= A_0 + \sum_{i=1}^{m-1} B_{t_{i-1}}^{(i)} - B_{t_i}^{(i)} + (m-i) \left( \left\lfloor \frac{t_i}{m} \right\rfloor - \left\lfloor \frac{t_{i-1}}{m} \right\rfloor \right) \quad (\text{by induction}) \quad (13)$$

$$\text{where } B_t^{(i)} = \sum_{u \in U_i} b_{u,t},$$

$$= A_0 + \sum_{i=1}^{m-1} B_{t_{i-1}}^{(i)} - B_{t_i}^{(i)} + \sum_{i=1}^{m-1} (m-i) \left\lfloor \frac{t_i}{m} \right\rfloor - \sum_{i=0}^{m-2} (m-i-1) \left\lfloor \frac{t_i}{m} \right\rfloor \quad (14)$$

$$= A_0 + \left\lfloor \frac{t_{m-1}}{m} \right\rfloor - (m-1) \left\lfloor \frac{t_0}{m} \right\rfloor + \sum_{i=1}^{m-1} B_{t_{i-1}}^{(i)} - B_{t_i}^{(i)} + \sum_{i=1}^{m-2} \left\lfloor \frac{t_i}{m} \right\rfloor \quad (15)$$

$$= A_0 - (m-1) \left\lfloor \frac{t_0}{m} \right\rfloor + \sum_{i=1}^{m-1} B_{t_{i-1}}^{(i)} - B_{t_i}^{(i)} + \sum_{i=1}^{m-1} \left\lfloor \frac{t_i}{m} \right\rfloor \quad (16)$$

$$= A_0 - (m-1) \left\lfloor \frac{t_0}{m} \right\rfloor + \sum_{i=1}^{m-1} B_{t_{i-1}}^{(i)} - B_{t_{i-1}}^{(i-1)} + B_{t_{i-1}}^{(i-1)} - B_{t_i}^{(i)} + \sum_{i=1}^{m-1} \left\lfloor \frac{t_i}{m} \right\rfloor \quad (17)$$

$$= A_0 - (m-1) \left\lfloor \frac{t_0}{m} \right\rfloor + B_{t_0}^{(0)} - B_{t_{m-1}}^{(m-1)} + \sum_{i=1}^{m-1} B_{t_{i-1}}^{(i)} - B_{t_{i-1}}^{(i-1)} + \sum_{i=1}^{m-1} \left\lfloor \frac{t_i}{m} \right\rfloor \quad (18)$$

$$= A_0 - (m-1) \left\lfloor \frac{t_0}{m} \right\rfloor + B_{t_0}^{(0)} - B_{t_{m-1}}^{(m-1)} + \sum_{i=0}^{m-2} B_{t_i}^{(i+1)} - B_{t_i}^{(i)} + \sum_{i=1}^{m-1} \left\lfloor \frac{t_i}{m} \right\rfloor \quad (19)$$

$$= A_0 - (m-1) \left\lfloor \frac{t_0}{m} \right\rfloor + B_{t_0}^{(1)} - B_{t_{m-1}}^{(m-1)} + \sum_{i=1}^{m-2} B_{t_i}^{(i+1)} - B_{t_i}^{(i)} + \sum_{i=1}^{m-1} \left\lfloor \frac{t_i}{m} \right\rfloor \quad (20)$$

$$= \underbrace{A_0 - (m-1) \left\lfloor \frac{t_0}{m} \right\rfloor}_{\triangleq Q_1} + \underbrace{B_{t_0}^{(1)}}_{\triangleq Q_2} - \underbrace{B_{t_{m-1}}^{(m-1)}}_{\triangleq Q_3} - \underbrace{\sum_{i=1}^{m-2} \left\lfloor \frac{B_{t_i}^{(i)}}{m-i} \right\rfloor}_{\triangleq Q_4} + \underbrace{\sum_{i=1}^{m-1} \left\lfloor \frac{t_i}{m} \right\rfloor}_{\triangleq Q_5} \quad (21)$$

where the last equality comes from  $B_{t_i}^{(i+1)} = B_{t_i}^{(i)} - \left\lfloor \frac{B_{t_i}^{(i)}}{m-i} \right\rfloor$  which in turn comes from the definition of  $U_{i+1}$  (the adversary removes the node with most budget) combined with lemma 11 (Balance equalizes budget among available nodes).

The following lemma proves that  $T = t_{m-1} + o(T)$ ,

**Lemma 10** For  $t_0 \leq \frac{T}{e}$ ,  $T = t_{m-1} + o(T)$ .

**Proof.**

According to lemma 16,

$$\tilde{t}_{m-1} = \left(1 + \frac{1}{m-1}\right)^{m-1} (t_0 + mb_0 - m + 1) - mb_0 + m - 1 \quad (22)$$

Putting everything together gives,

$$T - t_{m-1} = T - t_{m-1} - \tilde{t}_{m-1} + \tilde{t}_{m-1} \quad (23)$$

$$\leq T + \left(1 + \frac{1}{m-1}\right)^{m-1} (t_0 + mb_0 - m + 1) - mb_0 + m - 1 \quad (24)$$

$$\leq T + e(t_0 + mb_0) - mb_0 \quad (25)$$

choosing  $t_0$  such that  $t_0 = \lfloor T/e \rfloor$  along with the fact that  $m = o(\sqrt{T})$ , implies that  $T - t_{m-1} = o(T)$ .  $\blacksquare$

**Computation of the CR.**

$$\text{CR}^{\text{adv}}(\text{Balance}, \mathcal{G}_{T,m}) \quad (26)$$

$$= \frac{Q_1 + Q_2 - Q_3 - Q_4 + Q_5}{t_{m-1} + o(T)} \quad (27)$$

$$= \frac{\mathcal{O}\left(\frac{t_0}{m}\right) + Q_2 - Q_3 - Q_4 + Q_5}{t_{m-1} + o(T)} \quad \text{as } A_0 = t_0 \quad (28)$$

$$= \frac{\mathcal{O}\left(\frac{t_0}{m}\right) + (m-1) \left(b_0 + \left\lfloor \frac{t_0}{m} \right\rfloor - \left\lfloor \frac{t_0}{n} \right\rfloor\right) + \mathcal{O}(m) - Q_3 - Q_4 + Q_5}{t_{m-1} + o(T)} \quad \text{lemma 13} \quad (29)$$

$$= \frac{\mathcal{O}\left(\frac{t_0}{m}\right) + (m-1)\left(b_0 + \left\lfloor \frac{t_0}{m} \right\rfloor - \left\lfloor \frac{t_0}{n} \right\rfloor\right) + \mathcal{O}(m) - Q_4 + Q_5}{t_{m-1} + o(T)} \quad (30)$$

$$= \frac{1}{t_{m-1} + o(T)} \left( \mathcal{O}\left(\frac{t_0}{m}\right) + (m-1)\left(b_0 + \left\lfloor \frac{t_0}{m} \right\rfloor - \left\lfloor \frac{t_0}{n} \right\rfloor\right) - \lfloor \alpha^* m \rfloor \left\lfloor \frac{B_{t_1}^{(1)}}{m} \right\rfloor \right. \\ \left. + \bar{t}_0 \int_{\frac{1}{m}}^{\alpha^*} g_m(x) dx + \frac{g_m(\alpha^*) - g_m(1/m)}{m} + \mathcal{O}(m^2) + Q_5 \right) \quad \text{lemma 23} \quad (31)$$

$$= \frac{1}{t_{m-1} + o(T)} \left( \mathcal{O}\left(\frac{t_0}{m}\right) + (m-1)\left(b_0 + \left\lfloor \frac{t_0}{m} \right\rfloor - \left\lfloor \frac{t_0}{n} \right\rfloor\right) - \lfloor \alpha^* m \rfloor \left\lfloor \frac{B_{t_1}^{(1)}}{m} \right\rfloor \right. \\ \left. + \bar{t}_0 \int_{\frac{1}{m}}^{\alpha^*} g_m(x) dx + \frac{g_m(\alpha^*) - g_m(1/m)}{m} + \mathcal{O}(m^2) \right. \\ \left. + \left( \left(1 + \frac{1}{m-1}\right)^{m-1} - 1 + \frac{1}{m} \right) t_0 + \mathfrak{B}(m, b_0) \right) \quad \text{lemma 19} \quad (32)$$

where  $\alpha^* \in (1/m, 1)$  the solution of

$$\frac{\bar{t}_0}{m} \int_{\frac{1}{m}}^{\alpha^*} \frac{z}{1-z} e^z dz - m\alpha^* - Y_1 = 0, \quad (33)$$

and

$$g_m(x) = \frac{x(\alpha^* - x)}{1-x} \left(1 + \frac{1}{m-1}\right)^{mx}$$

thus  $\mathfrak{B}(m, b_0) \leq (e-2)mb_0 + b_0$  ■

#### A.2.1. PROOF OF LEMMA 11

The following lemma states that  $U$ -nodes that were available exactly at the same time steps in the past should have the same budget within one unit when the algorithm is `Balance`.

**Lemma 11** *Let  $W \subseteq U$  such that  $\forall s \leq t \in V, (\exists u \in W, (u, s) \in E) \Rightarrow (\forall u \in W, (u, s) \in E)$ . For the algorithm `Balance`,*

$$\exists \beta_t \in \mathbb{N}, \forall u \in W, \exists z_{u,t} \in \{0, 1\}, \text{ s.t. } b_{u,t} = \beta_t + z_{u,t} \quad \text{and} \quad \sum_{u' \in W} z_{u',t} < |W| \quad (34)$$

**Proof.**

We first focus on the first part of the result. Let  $t \in \mathbb{N}^*$  and  $W \subseteq U$  such that  $\forall s \leq t \in V, (\exists u \in W, (u, s) \in E) \Rightarrow (\forall u \in W, (u, s) \in E)$ . We need to prove that using the `Balance` algorithm implies that the budgets of the nodes at time  $t$  differ only by one. We will prove it by recursion using the following hypothesis,

$$K(i) : \exists \beta_i \in \mathbb{N}, \forall u \in W, b_{u,i} = \beta_i + z_{u,i} \quad \text{with} \quad z_{u,i} \in \{0, 1\} \quad \text{and} \quad \sum_{u' \in W} z_{u',i} < |W|$$

By assumption,  $\forall u \in W, b_{u,0} = b_0$ , which means that  $K(0)$  holds.

At time  $i$ , `Balance` chooses  $u_i \in \arg \max_{u \in U: (u,i) \in E} b_{u,i}$ . If  $u_i \notin W$ , the result is direct from  $K(i-1)$  with  $\beta_i = \beta_{i-1} + [i \bmod m = 0]$ . Otherwise, there are two cases when  $u_i \in W$ .

**Case**  $\forall u \in W, b_{u,i-1} = \beta_{i-1}$ . Then, by choosing  $\beta_i = \beta_{i-1} - 1 + [i \bmod m = 0]$ , we have  $b_{u,i} = \beta_i$  and for any  $u' \in W \setminus \{u_i\}, b_{u',i} = b_{u',i-1} + [i \bmod m = 0]$ .

**Case**  $\exists u, u' \in W, b_{u,i-1} \neq b_{u',i-1}$ . Then, by choosing  $\beta_i = \beta_{i-1} + [i \bmod m = 0]$ , we have  $b_{u,i} = \beta_i$  and  $\forall u \in W \setminus \{u_i\}, b_{u,i} = b_{u,i-1} + [i \bmod m = 0]$ .

In both cases,  $K(i)$  holds.

■

### A.2.2. PROOF OF LEMMA 13

During the phase  $i$ , between  $t_{i-1}$  and  $t_i$ , the graph is fully-connected to  $U_i$ . Thus,  $\sum_{u \in U_i} b_{u,t}$  follows the following dynamic  $Z_t$ .

Given  $k, m, t, j \in \mathbb{N}^*$ , the dynamic of interest is

$$Z_t = Z_{t-1} - 1_{Z_{t-1} \geq 1} + k1_{t \bmod m = j}. \quad (35)$$

where  $k = |U_i|$ , and  $j$  accounts for the fact that a phase begins at a time  $t_{i-1}$  that is not necessarily a multiple of  $m$ .

**Lemma 12** For  $k, m, t, Z_0, j \in \mathbb{N}^*$ ,

$$Z_t = \begin{cases} (Z_0 - t)_+ + k1_{t=j} & \text{if } t \leq j \\ g(Z_j, k, t - j, m) & \text{if } j < t \leq j + t^* \\ f(k, m, t - j, \tilde{t}) & \text{if } j + \tilde{t} \leq t \end{cases} \quad (36)$$

$$t^* = \begin{cases} Z_j + k \left\lceil \frac{Z_j + 1 - m}{m - k} \right\rceil & \text{if } m > k \\ Z_j & \text{if } m \leq k \text{ and } Z_j < m \\ +\infty & \text{otherwise} \end{cases} \quad \text{and } \tilde{t} = m \left\lceil \frac{t^*}{m} \right\rceil \quad (37)$$

and  $f(k, m, t, \tilde{t}) = \left( 1_{k < m} (k - (t \bmod m))_+ + 1_{k \geq m} \left( k \left( 1 + \left\lfloor \frac{t - \tilde{t}}{m} \right\rfloor \right) - (t - \tilde{t}) \right) \right), g(Z_j, k, t, m) = (Z_j + k \lfloor \frac{t}{m} \rfloor - t)$

**Proof.**

**First, the case when  $j = 0$ .**

For  $k, m, t, Z_0 \in \mathbb{N}^*$ ,  $t^*$  is defined to be the first time at which  $Z_t$  reaches 0.

$$t^* = \min_{t \in \mathbb{N}^*} t \quad \text{s.t. } Z_t = 0 \quad (38)$$

which value is given by lemma 14.

**For any**  $t \leq t^*, 1_{[Z_{t-1} > 0]} = 1$ , thus, by recursion,

$$Z_t = Z_0 - t + k \sum_{t'=1}^t 1_{[t' \bmod m = 0]} \quad (39)$$

$$= Z_0 - t + k \left\lfloor \frac{t}{m} \right\rfloor. \quad (40)$$

For any  $t^* \leq t < \tilde{t}$ ,  $t \bmod m \neq 0$  and thus  $Z_t = 0$  (by recursion starting at  $Z_{t^*} = 0$ ).

For any  $t > \tilde{t}$ , the analysis is split between the case  $k \geq m$  and  $k < m$ . In both cases  $Z_{\tilde{t}} = k$  and we denote  $t = \tilde{t} + \Delta t$ .

First, if  $k \geq m$ , similarly as before, we get  $Z_{\tilde{t}+\Delta t} = (Z_{\tilde{t}} - \Delta t + k \lfloor \frac{\Delta t}{m} \rfloor)$ : for  $k \geq m$ , it is always true that  $Z_{\tilde{t}+\Delta t-1} > 0$  which gives the result by recursion.

Second, if  $k < m$ , the result is proved by recursion.

Recursion hypothesis –  $H(t) = Z_{\tilde{t}+\Delta t} = (k - (t \bmod m))_+$  and by definition of  $\tilde{t}$ ,  $H(\tilde{t})$  holds as  $Z_{\tilde{t}} = k$ .

$$Z_{t+1} = Z_t - 1_{Z_t > 0} + k 1_{t+1 \bmod m = 0} \quad (41)$$

$$= (k - (\tilde{t} + \Delta t) \bmod m)_+ - 1_{(k - (\tilde{t} + \Delta t) \bmod m)_+ > 0} + k 1_{(\tilde{t} + \Delta t + 1) \bmod m = 0} \quad (42)$$

$$= (k - \Delta t \bmod m)_+ - 1_{(k - \Delta t \bmod m)_+ > 0} + k 1_{(\Delta t + 1) \bmod m = 0} \quad (43)$$

- If  $(\Delta t + 1) \bmod m = 0$ , we necessarily have  $Z_{\tilde{t}+\Delta t} = 0$  (as  $k \leq m - 1$ ). Thus  $Z_{\tilde{t}+\Delta t+1} = k = (k - (\tilde{t} + \Delta t + 1) \bmod m)_+$ .
- If  $(\Delta t + 1) \bmod m \neq 0$  and  $Z_{\tilde{t}+\Delta t} > 0$ , we have  $Z_{\tilde{t}+\Delta t+1} = Z_{\tilde{t}+\Delta t} - 1$  which gives the result,
- If  $(\Delta t + 1) \bmod m \neq 0$  and  $Z_{\tilde{t}+\Delta t} = 0$ , we have  $Z_{\tilde{t}+\Delta t+1} = Z_{\tilde{t}+\Delta t}$  which gives the result.

Second, the general case when  $0 \leq j < m$ .

Let  $\tilde{Z}_t = Z_{t+j}$ ,  $t_j^* = \min_{t \in \mathbb{N}^*} t$  s.t.  $\tilde{Z}_t = 0$  and  $\tilde{t}_j = m \lfloor \frac{t_j^*}{m} \rfloor$ . Using the result proved above for  $j = 0$  gives for any  $t > j$

$$\tilde{Z}_{t-j} = g(\tilde{Z}_0, k, t - j, m) 1_{t-j \leq t_j^*} + 1_{t-j \geq \tilde{t}_j} f(k, m, t - j, \tilde{t}) \quad (44)$$

$$\Leftrightarrow Z_t = g(Z_j, k, t - j, m) 1_{t-j \leq t_j^*} + 1_{t-j \geq \tilde{t}_j} f(k, m, t - j, \tilde{t}) \quad (45)$$

and for any  $t \leq j$ ,

$$Z_t = (Z_0 - t)_+ + k 1_{t=j} \quad (46)$$

■

**Lemma 13**  $B_{t_0}^{(1)} = (m - 1) (b_0 + \lfloor \frac{t_0}{m} \rfloor - \lfloor \frac{t_0}{n} \rfloor) + (m - 1 - (t_0 \bmod n))_+$

**Proof.**

By application of lemma 12,  $B_{t_0}^{(0)} = n b_0 + n \lfloor \frac{t_0}{m} \rfloor - t_0$ . By application of lemma 11,

$$B_{t_0}^{(1)} = (m - 1) \left\lfloor \frac{n b_0 + n \lfloor \frac{t_0}{m} \rfloor - t_0}{n} \right\rfloor + (m - 1 - (t_0 \bmod n))_+ \quad (47)$$

$$= (m - 1) \left( b_0 + \left\lfloor \frac{t_0}{m} \right\rfloor - \left\lfloor \frac{t_0}{n} \right\rfloor \right) + (m - 1 - (t_0 \bmod n))_+ \quad (48)$$

■

## A.2.3. PROOF OF LEMMA 19

This section is organized as follows:

1. A characterization of  $t_i$  by a recursive equation.
2. The introduction of  $\tilde{t}_i$  (to approximate  $t_i$ ).
3. The quantification of the approximation error between  $t_i$  and  $\tilde{t}_i$ .
4. A closed-form computation of  $\sum_{i=1}^{m-1} \tilde{t}_i$ .
5. The final result.

**A characterization of  $t_i$ .** The following result allows to characterize the sequence of  $t_i$  by a recursive equation.

**Lemma 14** For  $a, b \geq 0, m, c \geq 2$ , if  $t^* = \inf\{t \in \mathbb{N}^* : b + c \lfloor \frac{t}{m} \rfloor = (t - a)\}$  then,

$$t^* = \begin{cases} a + b + c \left\lceil \frac{a+b+1-m}{m-c} \right\rceil & \text{if } m > c \\ a + b & \text{if } m \leq c \text{ and } a + b < m \\ +\infty & \text{otherwise} \end{cases} \quad (49)$$

**Proof.**

First,

$$t^* = \min_{t \in \mathbb{N}^*} t \quad \text{s.t.} \quad b + c \left\lfloor \frac{t}{m} \right\rfloor - (t - a) = 0 \quad (50)$$

$$= \min_{t \in \mathbb{N}^*} t \quad \text{s.t.} \quad b + a + (c - m) \left\lfloor \frac{t}{m} \right\rfloor - \left\{ \frac{t}{m} \right\} = 0 \quad (\{\cdot\} \text{ denotes the fractional part}) \quad (51)$$

$$= \min_{\substack{k \in \mathbb{N} \\ j \in [0..m-1]}} km + j \quad \text{s.t.} \quad (a + b) - j + (c - m)k = 0 \text{ and } km + j > 0 \quad (52)$$

$$= \min_{k \in \mathbb{N}} a + b + ck \quad \text{s.t.} \quad (a + b) + 1 - m \leq (m - c)k \leq a + b \text{ and } a + b + k > 0 \quad (53)$$

$$(54)$$

Going from eq. (51) to eq. (52) is done by using the Euclidean division of  $t$  by  $m$  as  $t = km + j$ . As eq. (53) is linear in  $k$  with positive coefficients, it is minimized at the lowest feasible value of  $k$  which is

$$k^* = \begin{cases} \left\lceil \frac{a+b+1-m}{m-c} \right\rceil & \text{if } m > c \\ 0 & \text{if } m \leq c \text{ and } a + b < m \\ +\infty & \text{otherwise} \end{cases} \quad (55)$$

The result follows by using the fact that  $t^* = k^*m + j^*$  where  $j^* = a + b + (1 - m)k^*$ . ■

**Corollary 15**  $\forall i \in \mathbb{N}, t_{i+1} = b_0 - 1 + t_i + \left\lceil \frac{b_0 + t_i}{m-1} \right\rceil$ .

**Proof.**

Direct application of lemma 14 from the definition of  $t_i$ . ■



**Introduction of  $\tilde{t}_i$  to approximate  $t_i$ .** To obtain a sequence  $\tilde{t}_i$  close to  $t_i$  with a closed form, the intuition is to "remove" the fractional part and solve the arithmetic-geometric equation.

$$\forall i \in \mathbb{N}^*, \tilde{t}_{i+1} = b_0 - 1 + \tilde{t}_i + \frac{b_0 + \tilde{t}_i}{m-1}, \quad (56)$$

$$\tilde{t}_0 = t_0 \quad (57)$$

The intuitive justification is that we are in the regime  $m = o(\sqrt{T})$ , thus the error introduced by ignoring a term of order  $\left\{ \frac{b_0 + t_i}{m-1} \right\}$  is small (especially if  $t_1 = \Theta(T)$ ).

Now, given that  $\tilde{t}_i$  follows an arithmetic-geometric equation, it admits a closed-form expression:

**Lemma 16** For any  $i \in \mathbb{N}$ ,

$$\tilde{t}_i = \left( 1 + \frac{1}{m-1} \right)^i (t_0 + mb_0 - m + 1) - mb_0 + m - 1 \quad (58)$$

**Proof.**

Let  $i$  be in  $\mathbb{N}$ .

$$\tilde{t}_{i+1} = b_0 - 1 + \tilde{t}_i + \frac{b_0 + \tilde{t}_i}{m-1} \quad (59)$$

$$\Leftrightarrow \tilde{t}_{i+1} = \frac{m}{m-1} \tilde{t}_i + \frac{m}{m-1} b_0 - 1 \quad (60)$$

$$\Leftrightarrow \tilde{t}_{i+1} + mb_0 - m + 1 = \frac{m}{m-1} \tilde{t}_i + \frac{m}{m-1} b_0 - 1 + mb_0 - m + 1 \quad (61)$$

$$\Leftrightarrow \tilde{t}_{i+1} + mb_0 - m + 1 = \frac{m}{m-1} (\tilde{t}_i + b_0 + (m-1)b_0 - m + 1) \quad (62)$$

$$\Leftrightarrow \tilde{t}_{i+1} + mb_0 - m + 1 = \frac{m}{m-1} (\tilde{t}_i + mb_0 - m + 1) \quad (63)$$

$$\Leftrightarrow \tilde{t}_{i+1} = -mb_0 + m - 1 + \left( \frac{m}{m-1} \right)^i (\tilde{t}_0 + mb_0 - m + 1) \quad (64)$$

■

**Quantification of the approximation error.**

**Lemma 17**  $\forall i \in \mathbb{N}^*, t_i - \tilde{t}_i < (m-1) \left( \left( 1 + \frac{1}{m-1} \right)^i - 1 \right)$ .

**Proof.**

$$t_i - \tilde{t}_i = t_{i-1} - \tilde{t}_{i-1} + \left\lceil \frac{t_{i-1} + b_0}{m-1} \right\rceil - \frac{\tilde{t}_{i-1} + b_0}{m-1} \quad \text{eq. (56) and corollary 15} \quad (65)$$

$$= t_{i-1} - \tilde{t}_{i-1} + \frac{t_{i-1} - \tilde{t}_{i-1}}{m-1} + \left\lceil \frac{t_{i-1} + b_0}{m-1} \right\rceil - \frac{t_{i-1} + b_0}{m-1} \quad (66)$$

$$= (t_{i-1} - \tilde{t}_{i-1}) \left( 1 + \frac{1}{m-1} \right) + \left\lceil \frac{t_{i-1} + b_0}{m-1} \right\rceil - \frac{t_{i-1} + b_0}{m-1} \quad (67)$$

Thus, by induction, using that  $t_0 - \tilde{t}_0 = 0$  (by definition).

$$t_i - \tilde{t}_i < (t_{i-1} - \tilde{t}_{i-1}) \left(1 + \frac{1}{m-1}\right) + 1 \quad (68)$$

$$< \left(1 + \frac{1}{m-1}\right)^i (m-1) + 1 - m \quad (69)$$

■

**Closed-form computation of  $\sum_{i=1}^{m-1} \tilde{t}_i$ .**

**Lemma 18**  $\sum_{i=0}^{m-1} \tilde{t}_i = \left( \left(1 + \frac{1}{m-1}\right)^{m-1} - 1 + \frac{1}{m} \right) mt_0 + \mathfrak{A}(m, b_0)$  where

$$\mathfrak{A}(m, b_0) = m(mb_0 - m + 1) \left( \left(1 + \frac{1}{m-1}\right)^{m-1} - 2 + \frac{1}{m} \right). \quad (70)$$

**Proof.**

$$\sum_{i=0}^{m-1} \tilde{t}_i = -m(mb_0 - m + 1) + (t_0 + mb_0 - m + 1) \sum_{i=0}^{m-1} \left(1 + \frac{1}{m-1}\right)^i \quad (71)$$

(72)

$$= -m(mb_0 - m + 1) - m + (t_0 + mb_0 - m + 1) \frac{\left(1 + \frac{1}{m-1}\right)^m - 1}{1 + \frac{1}{m-1} - 1} \quad (73)$$

$$= -m(mb_0 - m + 1) - m + (t_0 + mb_0 - m + 1)(m-1) \left( \left(1 + \frac{1}{m-1}\right)^m - 1 \right) \quad (74)$$

$$\begin{aligned} &= \left( \left(1 + \frac{1}{m-1}\right)^{m-1} - 1 + \frac{1}{m} \right) mt_0 \\ &\quad + \underbrace{m(mb_0 - m + 1) \left( \left(1 + \frac{1}{m-1}\right)^{m-1} - 2 + \frac{1}{m} \right)}_{\triangleq \mathfrak{A}(m, b_0)} \end{aligned} \quad (75)$$

■

**Putting everything together.**

**Lemma 19**  $\sum_{i=1}^{m-1} \lfloor \frac{t_i}{m} \rfloor = \left( \left(1 + \frac{1}{m-1}\right)^{m-1} - 1 + \frac{1}{m} \right) t_0 + \mathfrak{B}(m, b_0)$  where  $\mathfrak{B}(m, b_0) < (e - 2)mb_0 + b_0$

**Proof.**

$$\sum_{i=1}^{m-1} \left\lfloor \frac{t_i}{m} \right\rfloor = \sum_{i=1}^{m-1} \frac{\tilde{t}_i}{m} + \underbrace{\sum_{i=1}^{m-1} \frac{t_i - \tilde{t}_i}{m} - \sum_{i=1}^{m-1} \left\{ \frac{\tilde{t}_i}{m} \right\}}_{\triangleq \mathfrak{E}(m)} \quad (76)$$

where

$$\mathfrak{E}(m) \leq \sum_{i=1}^{m-1} \frac{t_i - \tilde{t}_i}{m} \quad (77)$$

$$< \frac{m-1}{m} \sum_{i=1}^{m-1} \left( \left( 1 + \frac{1}{m-1} \right)^i - 1 \right) \quad (78)$$

$$< \frac{m-1}{m} \left( \frac{\left( 1 + \frac{1}{m-1} \right)^m - 1}{1 + \frac{1}{m-1} - 1} - m \right) \quad (79)$$

$$< \frac{(m-1)}{m} \left( (m-1) \left( \left( 1 + \frac{1}{m-1} \right)^m - 1 \right) - m \right) \quad (80)$$

$$< \frac{(m-1)}{m} \left( m \left( 1 + \frac{1}{m-1} \right)^{m-1} + 1 - 2m \right) \quad (81)$$

$$< (m-1) \left( \left( 1 + \frac{1}{m-1} \right)^{m-1} - 2 \right) + \frac{m-1}{m} \quad (82)$$

Using lemma 18 gives

$$\sum_{i=1}^{m-1} \left\lfloor \frac{t_i}{m} \right\rfloor = \left( \left( 1 + \frac{1}{m-1} \right)^{m-1} - 1 + \frac{1}{m} \right) t_0 + \underbrace{\mathfrak{E}(m) + \mathfrak{A}(m, b_0)}_{\triangleq \mathfrak{B}(m, b_0)} \quad (83)$$

where

$$\mathfrak{B}(m, b_0) < \left( \left( 1 + \frac{1}{m-1} \right)^{m-1} - 2 + \frac{1}{m} \right) m b_0 \quad (84)$$

■

#### A.2.4. PROOF OF LEMMA 23

The objective of this subsection is to compute  $\sum_{i=1}^{m-1} \left\lfloor \frac{B_{t_i}^{(i)}}{m-i} \right\rfloor$ . This section is organised as follows:

1. the introduction of  $Y_i$  to approximate  $\left\lfloor \frac{B_{t_i}^{(i)}}{m-i} \right\rfloor$ ,
2. the bounding of  $Y_i$ ,
3. the quantification of the approximation error between  $Y_i$  and  $\left\lfloor \frac{B_{t_i}^{(i)}}{m-i} \right\rfloor$ ,
4. the final result.

**Introduction of  $Y_i$ .** For any  $i \geq 1$ ,  $Y_i$  is defined by the following recursion:

$$Y_1 = \left\lceil \frac{B_{t_1}^{(1)}}{m-1} \right\rceil \quad (85)$$

$$Y_{i+1} = Y_i - (\tilde{t}_{i+1} - \tilde{t}_i) \left( \frac{1}{m-i-1} - \frac{1}{m} \right) + 1 \quad (86)$$

where  $\tilde{t}_i$  is the approximate time dynamic defined in eq. (56).

**The bounding of  $Y_i$**  .

**Lemma 20** For  $1 \leq i < m-1$ ,

$$Y_1 + i - 1 - \frac{\bar{t}_0}{m} g((i+1)/m) \leq Y_i \leq Y_1 + i - 1 - \frac{\bar{t}_0}{m} g(i/m) \quad (87)$$

where  $g(z) = \int_{\frac{1}{m}}^z \frac{x}{1-x} \exp(x) dx$  and  $\bar{t}_0 = t_0 - mb_0 - m + 1$ .

**Proof.**

By definition of  $Y_i$ , it holds

$$Y_i = Y_1 + i - 1 - \sum_{k=2}^i (\tilde{t}_k - \tilde{t}_{k-1}) \left( \frac{1}{m-k} - \frac{1}{m} \right) \quad (88)$$

$$= Y_1 + i - 1 - \sum_{k=2}^i (\tilde{t}_k - \tilde{t}_{k-1}) \frac{k}{m(m-k)} \quad (89)$$

$$= Y_1 + i - 1 - \frac{\bar{t}_0}{m} \sum_{k=2}^i \left( 1 + \frac{1}{m-1} \right)^k \left( 1 - \frac{m-1}{m} \right) \frac{k}{(m-k)} \quad \text{by eq. (58)} \quad (90)$$

$$= Y_1 + i - 1 - \frac{\bar{t}_0}{m^2} \sum_{k=2}^i \frac{k}{m-k} \left( 1 + \frac{1}{m-1} \right)^k \quad (91)$$

where  $\bar{t}_0 = t_0 - mb_0 - m + 1$ . Moreover, since  $(1 + \frac{1}{m-1}) \geq \exp(\frac{1}{m})$ , this gives

$$\exp\left(\frac{k}{m}\right) \leq \left( 1 + \frac{1}{m-1} \right)^k \leq \exp\left(\frac{k}{m-1}\right) \leq \exp\left(\frac{k}{m}\right) \left( 1 + \frac{2}{m} \right),$$

Since the function  $x \mapsto \frac{x}{1-x} \exp(x)$  is increasing on  $\mathbb{R}_+$ , we get that (for  $i < m-1$ )

$$\underbrace{\int_{\frac{1}{m}}^{\frac{i}{m}} \frac{x}{1-x} \exp(x) dx}_{\triangleq g(i/m)} \leq \underbrace{\frac{1}{m} \sum_{k=2}^i \frac{k}{m-k} \exp\left(\frac{k}{m}\right)}_A \leq \int_{\frac{2}{m}}^{\frac{i+1}{m}} \frac{x}{1-x} \exp(x) dx,$$

Or equivalently, for  $i < m-1$ ,

$$Y_1 + i - 1 - \frac{\bar{t}_0}{m} g((i+1)/m) \leq Y_i \leq Y_1 + i - 1 - \frac{\bar{t}_0}{m} g(i/m) \quad (92)$$

■

**Quantification of approximation error.** The sequence  $Y_i$  only approximates well  $\left\lceil \frac{B_{t_i}^{(i)}}{m-i} \right\rceil$  as long as it stays positive.

**Lemma 21**  $\left\lceil \frac{B_{t_i}^{(i)}}{m-i} \right\rceil - Y_i \leq \mathcal{O}(i)$

**Proof.**

By application of lemma 12 on  $B_t^{(i)}$

$$B_{t_i}^{(i)} = B_{t_{i-1}}^{(i)} - \left\lceil \frac{B_{t_{i-1}}^{(i)}}{m-i+1} \right\rceil + (m-i) \left( 1 + \left\lfloor \frac{t_i}{m} \right\rfloor - \left\lfloor \frac{t_{i-1}}{m} \right\rfloor \right) - (t_i - t_{i-1}) \quad (93)$$

Thus, using the definition of  $Y_i$  in eq. (85),

$$\left\lceil \frac{B_{t_i}^{(i)}}{m-i} \right\rceil - Y_i = \frac{B_{t_i}^{(i)}}{m-i} - Y_i + \mathcal{O}(1) \quad (94)$$

$$\begin{aligned} &= \frac{1}{m-i} \left( B_{t_{i-1}}^{(i)} - \left\lceil \frac{B_{t_{i-1}}^{(i)}}{m-i+1} \right\rceil \right) + \left( 1 + \left\lfloor \frac{t_i}{m} \right\rfloor - \left\lfloor \frac{t_{i-1}}{m} \right\rfloor \right) - \frac{t_i - t_{i-1}}{m-i} - Y_i \\ &+ \mathcal{O}(1) \end{aligned} \quad (95)$$

$$= (t_i - t_{i-1}) \left( \frac{1}{m} - \frac{1}{m-i} \right) - \left( Y_{i-1} - (\tilde{t}_i - \tilde{t}_{i-1}) \left( \frac{1}{m} - \frac{1}{m-i} \right) + 1 \right)$$

$$+ \frac{B_{t_{i-1}}^{(i)}}{m-i+1} + 1 + \mathcal{O}(1) \quad (96)$$

$$= \left\lceil \frac{B_{t_{i-1}}^{(i)}}{m-i+1} \right\rceil - Y_{i-1} + (t_i - t_{i-1} - \tilde{t}_i + \tilde{t}_{i-1}) \left( \frac{1}{m} - \frac{1}{m-i} \right) + \mathcal{O}(1) \quad (97)$$

By induction

$$\left\lceil \frac{B_{t_i}^{(i)}}{m-i} \right\rceil - Y_i = \sum_{j=2}^i (t_j - t_{j-1} - \tilde{t}_j + \tilde{t}_{j-1}) \left( \frac{1}{m} - \frac{1}{m-j} \right) + \mathcal{O}(i) \quad (98)$$

$$= \sum_{j=2}^i \frac{(t_j - t_{j-1}) - (\tilde{t}_j - \tilde{t}_{j+1})}{m} - \sum_{j=2}^i \frac{t_j - t_{j-1} - \tilde{t}_j + \tilde{t}_{j-1}}{m-j} + \mathcal{O}(i) \quad (99)$$

$$= \frac{(t_i - t_1) - (\tilde{t}_i - \tilde{t}_1)}{m} - \sum_{j=2}^i \frac{\left( (t_{j-1} - \tilde{t}_{j-1}) \left( 1 + \frac{1}{m-1} \right) + 1 \right) + \tilde{t}_{j-1} - t_{j-1}}{m-j}$$

$$+ \mathcal{O}(i) \quad \text{by eq. (68)} \quad (100)$$

$$= \frac{t_i - t_1 - \tilde{t}_i + \tilde{t}_1}{m} - \sum_{j=2}^i \frac{t_{j-1} - \tilde{t}_{j-1} + 1}{(m-j)(m-1)} + \mathcal{O}(i) \quad (101)$$

$$\leq \mathcal{O}(i) \quad \text{by lemma 17} \quad (102)$$

■

**Lemma 22**  $i^* = \lfloor \alpha^* m \rfloor$

**Proof.**

The objective is to find  $i^*$  such that

$$Y_1 + i^* - 1 - \frac{\bar{t}_0}{m} g((i^* + 1)/m) < 0 \leq Y_1 + i^* - 1 - \frac{\bar{t}_0}{m} g(i^*/m) \quad (103)$$

Let define  $\alpha^* \in (1/m, 1)$  the solution of

$$\frac{\bar{t}_0}{m} \int_{\frac{1}{m}}^{\alpha} \frac{z}{1-z} e^z dz - m\alpha^* - Y_1 = 0, \quad (104)$$

then  $i^* = \lfloor \alpha^* m \rfloor$  satisfies eq. (103). ■

**Lemma 23**

$$\sum_{i=1}^{m-1} \left\lfloor \frac{B_i^{(i)}}{m-i} \right\rfloor = \lfloor \alpha^* m \rfloor \left\lfloor \frac{B_{t_1}^{(1)}}{m} \right\rfloor - \bar{t}_0 \int_{\frac{1}{m}}^{\alpha^*} g(x) dx + \frac{g(\alpha^*) - g(1/m)}{m} + \mathcal{O}(m^2) \quad (105)$$

where  $\alpha^* \in (1/m, 1)$  the solution of

$$\frac{\bar{t}_0}{m} \int_{\frac{1}{m}}^{\alpha} \frac{z}{1-z} e^z dz - m\alpha^* - Y_1 = 0, \quad (106)$$

and

$$g(x) = \frac{x(\alpha^* - x)}{1-x} \left( 1 + \frac{1}{m-1} \right)^{mx}$$

**Proof.**

Let define  $i^* = \lfloor \alpha^* m \rfloor$ . Then,

$$\sum_{i=1}^{m-1} \left\lfloor \frac{B_i^{(i)}}{m-i} \right\rfloor = \sum_{i=1}^{i^*} Y_i + \sum_{i=1}^{i^*} \left\lfloor \frac{B_i^{(i)}}{m-i} \right\rfloor - Y_i + \sum_{i=i^*+1}^{m-1} \left\lfloor \frac{B_i^{(i)}}{m-i} \right\rfloor \quad (107)$$

$$= \sum_{i=1}^{i^*} Y_i + \sum_{i=1}^{i^*} \left( \frac{t_i - t_1 - \tilde{t}_i + \tilde{t}_1}{m} - \sum_{j=2}^i \frac{t_{j-1} - \tilde{t}_{j-1}}{(m-j)(m-1)} + \mathcal{O}(i) \right) \\ + (m-1)(m-1-i^*) - \sum_{i=i^*+1}^{m-1} (t_i \bmod m) \quad (108)$$

$$= \sum_{i=1}^{i^*} Y_i + i^* \frac{\tilde{t}_1 - t_1}{m} + \sum_{i=1}^{i^*} \frac{t_i - \tilde{t}_i}{m} - \sum_{i=1}^{i^*} \sum_{j=2}^i \frac{t_{j-1} - \tilde{t}_{j-1}}{(m-j)(m-1)} + \mathcal{O}((i^*)^2)$$

$$+ (m-1)(m-1-i^*) - \sum_{i=i^*+1}^{m-1} (t_i \bmod m) \quad (109)$$

$$= \sum_{i=1}^{i^*} Y_i + i^* \frac{\tilde{t}_1 - t_1}{m} + \sum_{i=1}^{i^*} \frac{t_i - \tilde{t}_i}{m} - \sum_{i=1}^{i^*} \sum_{j=2}^i \frac{1}{(m-j)} \left( \left(1 + \frac{1}{m-1}\right)^{j-1} - 1 \right) \quad (110)$$

$$+ (m-1)(m-1-i^*) + \mathcal{O}(m^2) \quad \text{using } i^* = \lfloor \alpha^* m \rfloor \text{ and eq. (69)}$$

$$= \sum_{i=1}^{i^*} Y_i + i^* \frac{\tilde{t}_1 - t_1}{m} + \sum_{i=1}^{i^*} \frac{t_i - \tilde{t}_i}{m} - \frac{1}{(m-1)} \sum_{i=1}^{i^*} \sum_{j=2}^i \left( \left(1 + \frac{1}{m-1}\right)^{j-1} - 1 \right) \quad (111)$$

$$+ (m-1)(m-1-i^*) + \mathcal{O}(m^2)$$

$$= \sum_{i=1}^{i^*} Y_i + i^* \frac{\tilde{t}_1 - t_1}{m} + \frac{i^*(i^*+1)}{2(m-1)} - \frac{1}{(m-1)} \sum_{i=1}^{i^*} \sum_{j=2}^i \left( \left(1 + \frac{1}{m-1}\right)^{j-1} \right) \quad (112)$$

$$- \frac{i^*}{m-1} + (m-1)(m-1-i^*) + \sum_{i=1}^{i^*} \frac{t_i - \tilde{t}_i}{m} + \mathcal{O}(m^2)$$

$$= \sum_{i=1}^{i^*} Y_i + i^* \frac{\tilde{t}_1 - t_1}{m} + \sum_{i=1}^{i^*} \frac{t_i - \tilde{t}_i}{m} + \frac{i^*(i^*+1)}{2(m-i^*)} + (m-1)(m-1-i^*) + \mathcal{O}(m^2) \quad (113)$$

$$= \sum_{i=1}^{i^*} Y_i + \mathcal{O}(m^2) \quad \text{by lemma 17 and } i^* = \lfloor \alpha^* m \rfloor \quad (114)$$

$$= \sum_{i=1}^{i^*} \left( Y_1 + i - 1 - \frac{\bar{t}_0}{m^2} \sum_{k=2}^i \frac{k}{m-k} \left(1 + \frac{1}{m-1}\right)^k \right) + \mathcal{O}(m^2) \quad \text{by eq. (91)} \quad (115)$$

$$= i^* Y_1 + \frac{i^*(i^*+1)}{2} - i^* - \frac{\bar{t}_0}{m^2} \sum_{i=1}^{i^*} \sum_{k=2}^i \frac{k}{m-k} \left(1 + \frac{1}{m-1}\right)^k + \mathcal{O}(m^2) \quad (116)$$

$$= i^* Y_1 - \frac{\bar{t}_0}{m^2} \sum_{i=1}^{i^*} \sum_{k=2}^i \frac{k}{m-k} \left(1 + \frac{1}{m-1}\right)^k + \mathcal{O}(m^2) \quad \text{by } i^* = \lfloor \alpha^* m \rfloor \quad (117)$$

$$= i^* Y_1 - \frac{\bar{t}_0}{m^2} \sum_{k=2}^{i^*} \frac{(i^* - k)k}{m-k} \left(1 + \frac{1}{m-1}\right)^k + \mathcal{O}(m^2) \quad (118)$$

$$= i^* Y_1 - \frac{\bar{t}_0}{m} \sum_{k=2}^{i^*} \frac{\left(\frac{i^* - k}{m} - \frac{k}{m}\right) \frac{k}{m}}{1 - \frac{k}{m}} \left( \left(1 + \frac{1}{m-1}\right)^m \right)^{\frac{k}{m}} + \mathcal{O}(m^2) \quad (119)$$

$$= i^* Y_1 - \frac{\bar{t}_0}{m} \sum_{k=2}^{i^*} g\left(\frac{k}{m}\right) + \mathcal{O}(m^2) \quad \text{with } g(x) = \frac{x(\alpha^* - x)}{1-x} \left(1 + \frac{1}{m-1}\right)^{mx} \quad (120)$$

$$= i^* Y_1 - \frac{\bar{t}_0}{m} \sum_{k=2}^{i^*} g\left(\frac{k}{m}\right) + \mathcal{O}(m^2) \quad (121)$$

$$= i^* \left[ \frac{B_{t_1}^{(1)}}{m-1} \right] - \bar{t}_0 \int_{\frac{1}{m}}^{\alpha^*} g(x) dx + \frac{g(\alpha^*) - g(1/m)}{m} + \mathcal{O}(m^2) \quad (122)$$

Moving from eq. (121) to eq. (122) arises from approximating a Riemann sum by an integral. ■

### A.3. Proof of theorem 3

**Theorem 3** *Assuming the initial budgets are  $b_{1,0} = b_{2,0} = \dots = b_{n,0} = b_0 \geq 1$ . For  $m = o(\sqrt{T})$ ,*

$$\sup_{\text{ALG}} \mathbb{E} \left[ \text{CR}^{\text{adv}}(\text{ALG}, \mathcal{G}_{T,m}) \right] \leq \text{CR}^{\text{adv}}(\text{Balance}, G^{\text{th.2}}) + o_T(1) \quad (3)$$

where the expectation is taken over the randomness from ALG.

#### Proof.

The proof is based on the adversarial graph design defined in section A.2 and organized in two steps:

1. Showing that the sequence of total budget decreases at least as fast as the one of Balance.
2. Using this in eq. (13) to show that  $\text{ALG}(G^{\text{th.2}}) \leq \text{Balance}(G^{\text{th.2}}) + o(T)$ .

ALG is assumed to be any matching algorithm, potentially randomized. The matching built by ALG is denoted  $\mathbf{x}$ , the graph  $G^{\text{th.2}}$  is adversarially defined based on  $\mathbf{x}$  as in section A.2 and we use the rest of the notation defined there. Note that only the choice of nodes in  $U_1, \dots, U_{m-1}$  differs from the graph adversarial to ALG, not the other quantities such as  $t_i$ .

For any  $i \leq m-1$  and  $t \in [T]$ , the total budget of ALG is denoted  $B_t^{(i)} = \mathbb{E} [\sum_{u \in U_i} x_{u,t}]$  where the expectation is taken over the randomness of ALG. We denote  $B_t^{(i),\text{bal}}$  for the sequence generated by Balance in section A.2 for comparison.

**Dynamic of  $B_{t_i}^{(i)}$ .** First, at time  $t_0$ , by application of lemma 12,  $B_{t_0}^{(0)} = B_{t_0}^{(0),\text{bal}}$ . The key element of the proof is to use that

$$\forall y \in \mathbb{R}_+^n, \forall k \leq n, \sum_{j=1}^k y_{(n-j)} \leq \frac{k}{n} \|y\|_1 \quad \text{where } y_{(j)} \geq 0 \text{ is the } i^{\text{th}} \text{ largest coordinate of } y. \quad (123)$$

Thus, by applying eq. (123) on  $b_{\cdot, t_0}$ , after the adversary removes  $n - m + 1$  nodes to build  $U_1$ , we have  $B_{t_0}^{(1)} \leq B_{t_0}^{(1),\text{bal}} + m - 1$ . The term  $r_0 = m - 1$  comes from the fact Balance does not exactly equalize the budgets (see lemma 11) and a randomized balance algorithm could do it more accurately in expectation. By induction, using at each step lemma 12 and eq. (123), we obtain that  $\forall i \leq m-1, B_{t_i}^{(i)} \leq B_{t_i}^{(i),\text{bal}} + i(m-i)$



**Showing that**  $\text{ALG}(G^{\text{th.2}}) \leq \text{Balance}(G^{\text{th.2}}) + o(T)$ . Denoting  $i_t$  the phase of the graph to which time step  $t$  belongs, it is possible to show that,

$$\text{ALG}(G^{\text{th.2}}) = t^* + \sum_{i=i_{t^*}+1}^{m-1} (m-i) \left( \left\lfloor \frac{t_i}{m} \right\rfloor - \left\lfloor \frac{t_{i-1}}{m} \right\rfloor \right) + \mathcal{O}(m) \text{ where } t^* = \max\{t \in [T] : B_t^{(i_t)} \geq 1\}$$

Noting that,  $B_t^{(i_t), \text{bal}} \leq B_t^{(i_t)} + i_t(m - i_t)$ , leads to  $\text{ALG}(G^{\text{Th.2}}) \leq \text{Balance}(G^{\text{Th.2}}) + m^2$ , which gives,

$$\text{CR}^{\text{adv}}(\text{ALG}, \mathcal{G}_{T,m}) \leq \text{CR}(\text{ALG}, G^{\text{Th.2}}) \leq \text{CR}(\text{Balance}, G^{\text{Th.2}}) + o_T(1)$$

where  $\text{CR}(\text{ALG}, G^{\text{th.2}}) = \frac{\text{ALG}(G^{\text{th.2}})}{\text{OPT}(G^{\text{th.2}})}$ . ■

## Appendix B. Stochastic Case

### B.1. Proof of theorem 4

**Theorem 4** *With probability  $1 - \mathcal{O}(n^{1/4} \exp(-a^3 n^{1/4}))$ , the matching size created by Greedy denoted by  $\text{Greedy}(G, T)$  satisfies,*

$$\text{Greedy}(G, T) = nh(T/n) + \mathcal{O}(n^{3/4})$$

and,

$$\frac{\mathbb{E}[\text{Greedy}(G, T)]}{n} \xrightarrow{n \rightarrow +\infty} h(T/n)$$

where  $h(\tau)$  is solution of the following equation,

$$\dot{h}(\tau) = 1 - e^{-a(1-z_0(\tau))}, \quad \frac{1}{n} \leq \tau \leq \frac{T}{n}$$

and  $z_0(\tau)$  satisfies the following system,

$$\begin{cases} \dot{z}_0(\tau) = -z_0(\tau)\beta + \frac{z_1(\tau)}{1-z_0(\tau)}(1 - e^{-a+az_0(\tau)}) & \text{for } k = 0 \\ \dot{z}_k(\tau) = (z_{k-1}(\tau) - z_k(\tau))\beta + (z_{k+1}(\tau) - z_k(\tau))\frac{1-e^{-a+az_0(\tau)}}{1-z_0(\tau)} & \text{for } 1 \leq k \leq K-1 \\ \dot{z}_k(\tau) = \beta z_{k-1}(\tau) - z_k(\tau)\frac{1-e^{-a(1-z_0(\tau))}}{1-z_0(\tau)} & \text{for } k = K \\ \sum_{k=0}^K z_k(\tau) = 1 \end{cases} \quad (4)$$

The proof of theorem 4 is based on Wormald's theorem introduced in [Wormald \(1995, 1999\)](#) and is organized as follows:

1. Definition of the evolution of  $(Y_k(t))_{k \geq 0}$  and  $\text{Greedy}(G, t)$ .
2. Proving that  $(Y_k(t))_{k \geq 0}$  and  $\text{Greedy}(G, t)$  satisfy the hypotheses of the Wormald theorem.
3. Application of Wormald theorem on  $(Y_k(t))_{k \geq 0}$  and  $\text{Greedy}(G, t)$ .

Recall that for all  $u \in U$ ,  $b_{u,t} \in \mathbb{N}$  is given by,

$$b_{u,t} = \min(K, b_{u,t-1} - x_{u,t} + \eta_t) \quad \text{with } b_{u,0} = b_0 \geq 1 \quad \text{and } K \in \mathbb{N}. \quad (124)$$

Here  $\eta_t$  is a realization of a Bernoulli random variable with parameter  $\frac{\beta}{n}$  denoted  $\mathcal{B}\left(\frac{\beta}{n}\right)$ .

First, let's introduce some notations, for  $k \in \mathbb{N}$  and  $t \in [T]$ ,

- $U_k(t) = \{u \in U : b_{u,t} = k\}$  is the set of nodes with budget  $k$ .
- $Y_k(t) = |U_k(t)|$  is the number of nodes with budget equals to  $k$ .
- $\text{Greedy}(G, T) = \sum_{u \in U} \sum_{t=1}^T x_{u,t}$  is the matching size.
- $C(t) = \sum_{k \geq 1} Y_k(t) = n - Y_0(t)$  is the total number of nodes with budget at least equals to 1.

In order to apply Wormald's theorem, it is necessary to track the evolution of  $\text{Greedy}(G, T)$ . To achieve this, we must precisely quantify the one-step change in  $\text{Greedy}(G, t)$  for all  $t \in [T]$ . This crucial step is addressed in the forthcoming lemma,

**Lemma 24** For  $t \in [T]$ , the expectation of the one-step change of  $\text{Greedy}(G, t)$  is given by,

$$\mathbb{E} [\text{Greedy}(G, t+1) - \text{Greedy}(G, t) | \text{Greedy}(G, t)] = 1 - \left(1 - \frac{a}{n}\right)^{\sum_{k \geq 1} Y_k(t)} = 1 - \left(1 - \frac{a}{n}\right)^{n - Y_0(t)}$$

**Proof.**

For  $t \in [T]$ , the matching size at time  $t+1$  is defined as follows,

$$\text{Greedy}(G, t+1) = \text{Greedy}(G, t) + 1_{\{x_{u,t+1}=1, u \in U_k(t+1)\}}$$

Moving to conditional expectation gives,

$$\begin{aligned} \mathbb{E} [\text{Greedy}(G, t+1) - \text{Greedy}(G, t) | \text{Greedy}(G, t)] &= \mathbb{P}(x_{u,t+1} = 1, u \in U_k | \text{Greedy}(G, t)) \\ &= 1 - \left(1 - \frac{a}{n}\right)^{C(t)} \\ &= 1 - \left(1 - \frac{a}{n}\right)^{n - Y_0(t)} \end{aligned}$$

■

Since the evolution of  $\text{Greedy}(G, t)$  depends on  $Y_0$ , we need to quantify the evolution of  $Y_k(t)$ ,  $\forall k \in \mathbb{N}, t \in [T]$ . This is done in the subsequent lemma,

**Lemma 25** For  $t \in [T]$ , denoting  $\Sigma(t) = \frac{1}{pC(t)}(1 - (1-p)^{C(t)})$ , the expectation of the one-step change of  $Y_k$ , when matching is built using Greedy algorithm is given by,

$$\begin{cases} \mathbb{E} [\Delta_0(t) | \mathbf{Y}(t)] = -Y_0(t) [\delta(1-p\Sigma(t))] + Y_1(t)(1-\delta)p\Sigma(t) \\ \mathbb{E} [\Delta_1(t) | \mathbf{Y}(t)] = -Y_1(t) [\delta(1-p\Sigma(t)) + (1-\delta)p\Sigma(t)] + Y_0(t)\delta + Y_2(t)(1-\delta)p\Sigma(t) \\ \mathbb{E} [\Delta_k(t) | \mathbf{Y}(t)] = \delta(1-p\Sigma(t)) [Y_{k-1}(t) - Y_k(t)] + [Y_{k+1}(t) - Y_k(t)] (1-\delta)p\Sigma(t) \quad \forall k > 1 \end{cases} \quad (125)$$

where  $\forall k \geq 0$ ,  $\Delta_k(t) = Y_k(t+1) - Y_k(t)$ .

**Proof.**

For  $t \in [T]$ , the evolution of the number of nodes with budget  $k \in \mathbb{N}$  can be formulated as,

$$\begin{aligned}
 Y_k(t+1) &= Y_k(t) - \sum_{u \in U_k(t)} 1_{\{\{\eta_t=1\} \cap \{x_{u,t}=0\}\} \cup \{\{x_{u,t}=1\} \cap \{\eta_t=0\}\}} + \sum_{u \in U_{k-1}(t)} 1_{\{\{\eta_t=1\} \cap \{x_{u,t}=0\}\}} \\
 &\quad + \sum_{u \in U_{k+1}(t)} 1_{\{\{x_{u,t}=1\} \cap \{\eta_t=0\}\}}
 \end{aligned} \tag{126}$$

We are interested in the conditional expectation of eq. (126) denoted by  $E(t) = \mathbb{E}[Y_k(t+1) - Y_k(t) | \mathbf{Y}(t)]$  where  $\mathbf{Y}(t) = (Y_k(t))_{k \geq 0}$ ,

$$\begin{aligned}
 E(t) &= -\mathbb{E} \left[ \sum_{u \in U_k(t)} 1_{\{\{\eta_t=1\} \cap \{x_{u,t}=0\}\} \cup \{\{x_{u,t}=1\} \cap \{\eta_t=0\}\}} \middle| \mathbf{Y}(t) \right] + \mathbb{E} \left[ \sum_{u \in U_{k-1}(t)} 1_{\{\{\eta_t=1\} \cap \{x_{u,t}=0\}\}} \middle| \mathbf{Y}(t) \right] \\
 &\quad + \mathbb{E} \left[ \sum_{u \in U_{k+1}(t)} 1_{\{\{x_{u,t}=1\} \cap \{\eta_t=0\}\}} \middle| \mathbf{Y}(t) \right]
 \end{aligned} \tag{127}$$

$$\begin{aligned}
 &= - \sum_{u \in U_k(t)} \mathbb{P}(\{\{\eta_t=1\} \cap \{x_{u,t}=0\}\} \cup \{\{x_{u,t}=1\} \cap \{\eta_t=0\}\} | \mathbf{Y}(t)) \\
 &\quad + \sum_{u \in U_{k-1}(t)} \mathbb{P}(\{\{\eta_t=1\} \cap \{x_{u,t}=0\}\} | \mathbf{Y}(t)) + \sum_{u \in U_{k+1}(t)} \mathbb{P}(\{\{x_{u,t}=1\} \cap \{\eta_t=0\}\} | \mathbf{Y}(t))
 \end{aligned} \tag{128}$$

$$\begin{aligned}
 &= - \sum_{u \in U_k(t)} \mathbb{P}(\{\{\eta_t=1\} \cap \{x_{u,t}=0\}\} | \mathbf{Y}(t)) - \sum_{u \in U_k(t)} \mathbb{P}(\{\{\eta_t=0\} \cap \{x_{u,t}=1\}\} | \mathbf{Y}(t)) \\
 &\quad + \sum_{u \in U_{k-1}(t)} \mathbb{P}(\{\{\eta_t=1\} \cap \{x_{u,t}=0\}\} | \mathbf{Y}(t)) + \sum_{u \in U_{k+1}(t)} \mathbb{P}(\{\{x_{u,t}=1\} \cap \{\eta_t=0\}\} | \mathbf{Y}(t))
 \end{aligned} \tag{129}$$

$$\begin{aligned}
 &= - \sum_{u \in U_k(t)} \delta \mathbb{P}(\{x_{u,t}=0\} | \mathbf{Y}(t)) - \sum_{u \in U_k(t)} (1-\delta) \mathbb{P}(\{x_{u,t}=1\} | \mathbf{Y}(t)) \\
 &\quad + \sum_{u \in U_{k-1}(t)} \delta \mathbb{P}(\{x_{u,t}=0\} | \mathbf{Y}(t)) + \sum_{u \in U_{k+1}(t)} (1-\delta) \mathbb{P}(\{x_{u,t}=1\} | \mathbf{Y}(t))
 \end{aligned} \tag{130}$$

Moving from eq. (128) to eq. (129) and then from eq. (129) to eq. (130) is done using independence.

To get the final expression of  $E(t)$ , we need to compute  $\mathbb{P}[\{x_{u,t}=1\} | \mathbf{Y}(t)]$ . By using Bayes formula we can see that,

$$\begin{aligned}
 \mathbb{P}[\{x_{u,t}=1\} | \mathbf{Y}(t)] &= \mathbb{P}[\{(u,t) \in G\} | \mathbf{Y}(t)] \mathbb{P}[\{x_{u,t}=1\} | \mathbf{Y}(t), (u,t) \in G] \\
 &= p \mathbb{P}[\{x_{u,t}=1\} | \mathbf{Y}(t), (u,t) \in G]
 \end{aligned}$$

Now, let's compute  $\mathbb{P} \left[ \{x_{u,t} = 1\} \mid \mathbf{Y}(t), (u, t) \in G \right]$ ,

$$\begin{aligned}
 \mathbb{P} \left[ \{x_{u,t} = 1\} \mid \mathbf{Y}(t), (u, t) \in G \right] &= \sum_{i=1}^{C(t)} \mathbb{P} \left[ \{x_{c,t} = 1, c \in [i], B_c(t) \geq 1\} \mid \mathbf{Y}(t), (u, t) \in G \right] \\
 &= \sum_{i=1}^{C(t)} \mathbb{P} \left[ \{x_{c,t} = 1\} \mid c \in [i], B_c(t) \geq 1, \mathbf{Y}(t), (u, t) \in G \right] \\
 &\mathbb{P} \left[ c \in [i], B_c(t) \geq 1 \mid \mathbf{Y}(t), (u, t) \in G \right] \\
 &= \sum_{i=1}^{C(t)} \frac{1}{i} \mathbb{P} \left[ c \in [i], B_c(t) \geq 1 \mid \mathbf{Y}(t), (u, t) \in G \right] \\
 &= \sum_{i=1}^{C(t)} \frac{1}{i} \binom{C(t)-1}{i-1} p^{i-1} (1-p)^{C(t)-i} \\
 &= \underbrace{\frac{1}{pC(t)} (1 - (1-p)^{C(t)})}_{\Sigma(t)}
 \end{aligned}$$

Thus, we get

$$\mathbb{P} \left[ \{x_{u,t} = 1\} \mid \mathbf{Y}(t) \right] = \frac{1}{C(t)} (1 - (1-p)^{C(t)})$$

Due to Greedy algorithm, here the choice of  $u$  inside the probabilities doesn't depend on  $U_k$ , so putting everything together in  $E(t)$ , and distinguishing cases where  $k = 0$ ,  $k = 1$  and  $k \geq 2$ , we get,

$$\begin{cases}
 \mathbb{E} [\Delta_0(t) \mid \mathbf{Y}(t)] = -Y_0(t) [\delta(1 - p\Sigma(t))] + Y_1(t)(1 - \delta)p\Sigma(t) \\
 \mathbb{E} [\Delta_1(t) \mid \mathbf{Y}(t)] = -Y_1(t) [\delta(1 - p\Sigma(t)) + (1 - \delta)p\Sigma(t)] + Y_0(t)\delta + Y_2(t)(1 - \delta)p\Sigma(t) \\
 \mathbb{E} [\Delta_k(t) \mid \mathbf{Y}(t)] = \delta(1 - p\Sigma(t)) [Y_{k-1}(t) - Y_k(t)] + [Y_{k+1}(t) - Y_k(t)] (1 - \delta)p\Sigma(t) \quad \forall k > 1
 \end{cases}$$

where  $\forall k \geq 0$ ,  $\Delta_k(t) = Y_k(t+1) - Y_k(t)$ . ■

Before establishing the hypotheses of Wormald's theorem, we introduce a technical lemma,

**Lemma 26** For  $n > 0$ ,  $a \leq n/2$  and  $0 \leq w \leq 1$ ,

$$0 \leq e^{-aw} - \left(1 - \frac{a}{n}\right)^{nw} \leq \frac{a}{ne}$$

**Proof.**

Using the following inequalities:  $1 - x \geq e^{-x-x^2}$  for  $x \leq \frac{1}{2}$  and  $1 - x \leq e^{-x}$  for  $x \geq 0$ , we obtain  $e^{-aw} \left(1 - \frac{a^2w}{n}\right) \leq \left(1 - \frac{a}{n}\right)^{nw} \leq e^{-aw}$ . The result follows by rearranging terms and using that  $awe^{-aw} \leq 1/e$ . ■

To apply Wormald's theorem [Wormald \(1999\)](#) in our model, three key hypotheses need to be met: the boundedness hypothesis, the Lipschitz hypothesis, and the trend hypothesis. These hypotheses will be established in the following lemmas for both  $\text{Greedy}(G, t)$  and  $Y_k(t)$ .

**Lemma 27**  $\forall k \geq 0$ , let  $-\epsilon < \tau < \frac{T}{n} + \epsilon$  and  $-\epsilon < z_k < 1 + \epsilon$  where  $\epsilon > 0$ . The functions  $f_k(\tau)$  and  $j_0(\tau)$  defined as follows,

$$f_k(\tau) = \begin{cases} -z_0(\tau)\beta + \frac{z_1(\tau)}{1-z_0(\tau)}(1 - e^{-a+az_0(\tau)}) & \text{for } k = 0 \\ (z_{k-1}(\tau) - z_k(\tau))\beta + (z_{k+1}(\tau) - z_k(\tau))\frac{1-e^{-a+az_0(\tau)}}{1-z_0(\tau)} & \text{for } k \geq 1 \end{cases}$$

$$j_0(\tau) = 1 - e^{-a(1-z_0(\tau))}$$

are Lipschitz with a constant  $L = (\beta + a)(1 + \epsilon)$  and  $L' = ae^{a\epsilon}$  respectively.

**Proof.**

The proof is done for  $k = 0$  and remains the same for  $k \geq 1$ . Let  $-\epsilon < z_0 < 1 + \epsilon$ ,  $-\epsilon < z_1 < 1 + \epsilon$ ,  $-\epsilon < \tau < \frac{T}{n} + \epsilon$  and  $-\epsilon < \tau' < \frac{T}{n} + \epsilon$ .

$$\begin{aligned} |f_0(\tau) - f_0(\tau')| &= \left| -z_0(\tau)\beta + \frac{z_1(\tau)}{1-z_0(\tau)}(1 - e^{-a+az_0(\tau)}) + z_0(\tau')\beta - \frac{z_1(\tau')}{1-z_0(\tau')}(1 - e^{-a+az_0(\tau')}) \right| \\ &\leq \beta |z_0(\tau') - z_0(\tau)| + \left| \frac{z_1(\tau)}{1-z_0(\tau)}(1 - e^{-a+az_0(\tau)}) + \frac{z_1(\tau')}{1-z_0(\tau')}(1 - e^{-a+az_0(\tau')}) \right| \\ &\leq \beta |z_0(\tau') - z_0(\tau)| + a|z_1(\tau) + z_1(\tau')| \quad \text{using } (1 - e^{-ax}) \leq ax \end{aligned}$$

Thus we get,

$$|f_0(\tau) - f_0(\tau')| \leq (\beta + a + 2)(1 + \epsilon)|\tau - \tau'|$$

Therefore, we proved that  $f_0$  is  $L$ -Lipschitz with  $L = (\beta + a + 2)(1 + \epsilon)$ . Now, let's proceed to prove that  $j_0$  is Lipschitz,

$$\begin{aligned} |j_0(\tau) - j_0(\tau')| &= |e^{-a(1-z_0(\tau))} - e^{-a(1-z_0(\tau'))}| \\ &= e^{-a(1-z_0(\tau))} |1 - e^{a(z_0(\tau')-z_0(\tau))}| \\ &\leq e^{-a(1-z_0(\tau))} a |z_0(\tau') - z_0(\tau)| \quad \text{using } 1 - e^{ax} \leq -ax \\ &\leq e^{a\epsilon} a |\tau - \tau'| \end{aligned}$$

Hence  $j_0$  is  $L'$ -Lipschitz with  $L' = e^{a\epsilon} a$ . ■

The next lemma proves the trend hypothesis,

**Lemma 28** For  $t \in [T]$  the functions  $f_k\left(\frac{t}{n}, \frac{Y_0(t)}{n}, \dots, \frac{Y_K(t)}{n}\right)$  and  $j\left(\frac{t}{n}, \frac{Y_0(t)}{n}\right)$  are given by,

$$f_k = \begin{cases} -\frac{Y_0(t)\beta}{n} \left(1 - \frac{1}{n-Y_0(t)}(1 - e^{-a(1-\frac{Y_0(t)}{n})})\right) + \frac{Y_1(t)(n-a)}{n} \frac{1}{n-Y_0(t)} (1 - e^{-a(1-\frac{Y_0(t)}{n})}) & \text{for } k = 0 \\ \frac{-Y_1(t)}{n} \left[ a \left(1 - \frac{1}{n-Y_0(t)}(1 - e^{-a(1-\frac{Y_0(t)}{n})})\right) + \frac{(n-a)}{n-Y_0(t)} (1 - e^{-a(1-\frac{Y_0(t)}{n})}) \right] \\ + \frac{Y_2(t)(n-a)}{n(n-Y_0(t))} (1 - e^{-a(1-\frac{Y_0(t)}{n})}) + \frac{Y_0(t)a}{n} & \text{for } k = 1 \\ \frac{-Y_k(t)}{n} \left[ a \left(1 - \frac{1}{n-Y_0(t)}(1 - e^{-a(1-\frac{Y_0(t)}{n})})\right) + \frac{(n-a)}{n-Y_0(t)} (1 - e^{-a(1-\frac{Y_0(t)}{n})}) \right] \\ + \frac{Y_{k+1}(t)(n-a)}{n(n-Y_0(t))} (1 - e^{-a(1-\frac{Y_0(t)}{n})}) + \frac{Y_{k-1}(t)a}{n} (1 - \frac{1}{n-Y_0(t)}(1 - e^{-a(1-\frac{Y_0(t)}{n})})) & \text{for } k \leq K-1 \\ \frac{Y_{k-1}(t)\beta}{n} \left(1 - \frac{1}{n-Y_0(t)}(1 - e^{-a(1-\frac{Y_0(t)}{n})})\right) - \frac{Y_k(t)(n-a)}{n} \frac{1}{n-Y_0(t)} (1 - e^{-a(1-\frac{Y_0(t)}{n})}) & \text{for } k = K \end{cases}$$

$$j = 1 - e^{-a(1 - \frac{Y_0(t)}{n})}$$

and we have for all  $k \geq 1$ ,

$$\left| \mathbb{E}(\text{Greedy}(G, t+1) - \text{Greedy}(G, t) | \text{Greedy}(G, t)) - j \left( \frac{t}{n}, \frac{Y_0(t)}{n} \right) \right| \leq \frac{a}{en} \quad (131)$$

$$\left| \mathbb{E}(Y_k(t+1) - Y_k(t) | \mathbf{Y}(t)) - f_k \left( \frac{t}{n}, \frac{Y_0(t)}{n}, \dots, \frac{Y_k(t)}{n}, \dots \right) \right| \leq \frac{a}{en} \quad (132)$$

**Proof.**

Let's prove eq. (132) for  $k = 0$  ( the proof is the same for  $k \geq 1$ ),

$$\begin{aligned} M_0 &= \left| \mathbb{E}(Y_0(t+1) - Y_0(t) | \mathbf{Y}(t)) - f_0 \left( \frac{t}{n}, \frac{Y_0(t)}{n}, \frac{Y_1(t)}{n} \right) \right| \\ M_0 &\leq \left| \frac{-Y_0(t)\beta}{n(n - Y_0(t))} (1 - (1 - \frac{a}{n})^{n - Y_0(t)} - 1 + e^{-a(1 - \frac{Y_0(t)}{n})}) \right| \\ &\quad + \left| \frac{Y_1(t)(n - \beta)}{n(n - Y_0(t))} (1 - (1 - \frac{a}{n})^{n - Y_0(t)} - 1 + e^{-a(1 - \frac{Y_0(t)}{n})}) \right| \\ &\leq \frac{a}{ne} \quad (\text{using lemma 26}) \end{aligned}$$

Let's now prove eq. (131),

$$\begin{aligned} P &= \left| \mathbb{E}(\text{Greedy}(G, t+1) - \text{Greedy}(G, t) | \text{Greedy}(G, t)) - j \left( \frac{t}{n}, \frac{Y_0(t)}{n} \right) \right| \\ P &= \left| e^{-a(1 - \frac{Y_0(t)}{n})} - \left( (1 - \frac{a}{n})^{n - Y_0(t)} \right) \right| \\ &\leq \frac{a}{ne} \quad (\text{using lemma 26}) \end{aligned}$$

■

The following lemma shows the Boundness hypothesis,

**Lemma 29** For  $t \in [T]$ ,  $k \geq 0$ ,

$$\begin{aligned} |\text{Greedy}(G, t+1) - \text{Greedy}(G, t)| &\leq \beta' \\ |Y_k(t+1) - Y_k(t)| &\leq \beta \end{aligned}$$

with  $\beta, \beta' > 0$ .

**Proof.**

For  $t \in [T]$  and  $k \geq 0$ ,

$$|\text{Greedy}(G, t+1) - \text{Greedy}(G, t)| = 1_{\{x_{u,t+1}=1, u \in U_k(t+1)\}} \leq 1$$

Hence we have  $\beta' = 1$ .

As seen previously,  $Y_k(t)$  is the number of nodes with budget equals to  $k$  at time  $t$ . So, by the nature of the matching process,

$$|Y_k(t+1) - Y_k(t)| \leq 1$$

Hence  $\beta = 1$ . ■

In the following lemma we approximate with high probability  $Y_k(t), \forall k \geq 0, t \in [T]$  by the solution of a system of differential equations,

**Lemma 30** *With probability  $1 - \mathcal{O}(n^{1/4} \exp(-a^3 n^{1/4}))$ ,*

$$Y_k(T) = n z_k(T/n) + \mathcal{O}(n^{3/4}) \quad \text{for } k \geq 0$$

$\forall \tau \in [\frac{1}{n}, \frac{T}{n}]$ ,  $(z_0, \dots, z_K)$  is the solution of the following system,

$$\begin{cases} \dot{z}_0(\tau) = -z_0(\tau)\beta + \frac{z_1(\tau)}{1-z_0(\tau)}(1 - e^{-a+az_0(\tau)}) & \text{for } k = 0 \\ \dot{z}_k(\tau) = (z_{k-1}(\tau) - z_k(\tau))\beta + (z_{k+1}(\tau) - z_k(\tau))\frac{1-e^{-a+az_0(\tau)}}{1-z_0(\tau)} & \text{for } 1 \leq k \leq K-1 \\ \dot{z}_k(\tau) = \beta z_{k-1}(\tau) - z_k(\tau)\frac{1-e^{-a(1-z_0(\tau))}}{1-z_0(\tau)} & \text{for } k = K \\ \sum_{k=0}^K z_k(\tau) = 1 \end{cases} \quad (133)$$

**Proof.**

For  $\frac{1}{n} \leq \tau \leq \frac{T}{n}$ , let's consider the normalized random variable  $Z_k(\tau) = \frac{Y_k(\tau n)}{n}$  and  $\mathbf{Z}(\tau) = (Z_k(\tau))_{k \geq 0}$ . The conditional expectation of the one-step change of  $Z_k(\tau)$  for different values of  $k$  is given by,

- For  $k = 0$ ,

$$\begin{aligned} \frac{\mathbb{E} \left[ Z_0(\tau + \frac{1}{n}) - Z_0(\tau) \mid \mathbf{Z}(\tau) \right]}{1/n} &= \frac{\mathbb{E} [Y_0(\tau n + 1)/n - Y_0(\tau n)/n \mid \mathbf{Y}(\tau n)/n]}{1/n} \\ &= \frac{\mathbb{E} \left[ -Z_0(\tau) \left[ \delta \left( 1 - \frac{1}{n-nZ_0(\tau)} (1 - (1-p)^{n-nZ_0(\tau)}) \right) \right] \right]}{1/n} \\ &\quad + \frac{\mathbb{E} \left[ Z_1(\tau) (1 - \delta) \frac{1}{n-nZ_0(\tau)} (1 - (1-p)^{n-nZ_0(\tau)}) \right]}{1/n} \end{aligned}$$

when  $n \rightarrow +\infty$ , we get,

$$\dot{z}_0(\tau) = -z_0(\tau)\beta + \frac{z_1(\tau)}{1-z_0(\tau)}(1 - e^{-a+az_0(\tau)})$$

- For  $k = 1$ ,

$$\begin{aligned} \frac{\mathbb{E} \left[ Z_1(\tau + \frac{1}{n}) - Z_1(\tau) \mid \mathbf{Z}(\tau) \right]}{1/n} &= \frac{\mathbb{E} [Y_1(\tau n + 1)/n - Y_1(\tau n)/n \mid \mathbf{Y}(\tau n)/n \forall k \geq 1]}{1/n} \\ &= \frac{\mathbb{E} \left[ -Z_1(\tau) \left[ \delta \left( 1 - \frac{1}{n-nZ_0(\tau)} (1 - (1-p)^{n-nZ_0(\tau)}) \right) \right] \right]}{1/n} \\ &\quad + \frac{\mathbb{E} \left[ -Z_1(\tau) (1 - \delta) \frac{1}{n-nZ_0(\tau)} (1 - (1-p)^{n-nZ_0(\tau)}) \right]}{1/n} \end{aligned}$$

$$+ \frac{\mathbb{E} \left[ Z_0(\tau)\delta + Z_2(\tau)(1 - \delta)\frac{1}{n-nZ_0(\tau)}(1 - (1 - p)^{n-nZ_0(\tau)}) \right]}{1/n}$$

when  $n \rightarrow +\infty$  we get,

$$\dot{z}_1(\tau) = \beta(z_0(\tau) - z_1(\tau)) + (z_2(\tau) - z_1(\tau))\frac{1 - e^{-a+az_0(\tau)}}{1 - z_0(\tau)}$$

•  $k \geq 2$ ,

$$\begin{aligned} \frac{\mathbb{E} \left[ Z_k(\tau + \frac{1}{n}) - Z_k(\tau) | \mathbf{Z}(t) \right]}{1/n} &= \frac{\mathbb{E} \left[ Y_k(\tau n + 1)/n - Y_k(\tau n)/n | \mathbf{Y}(\tau n)/n \right]}{1/n} \\ &= \frac{\mathbb{E} \left[ -Z_k(\tau) \left[ \delta \left( 1 - \frac{1}{n-nZ_0(\tau)} (1 - (1-p)^{n-nZ_0(\tau)}) \right) \right] \right]}{1/n} \\ &+ \frac{\mathbb{E} \left[ -Z_k(\tau) (1 - \delta) \frac{1}{n-nZ_0(\tau)} (1 - (1-p)^{n-nZ_0(\tau)}) \right]}{1/n} \\ &+ \frac{\mathbb{E} \left[ Z_{k-1}(\tau) \delta \left( 1 - \frac{1}{n-nZ_0(\tau)} (1 - (1-p)^{n-nZ_0(\tau)}) \right) \right]}{1/n} \\ &+ \frac{\mathbb{E} \left[ Z_{k+1}(\tau) (1 - \delta) \frac{1}{n-nZ_0(\tau)} (1 - (1-p)^{n-nZ_0(\tau)}) \right]}{1/n} \end{aligned}$$

when  $n \rightarrow +\infty$  we get,

$$\dot{z}_k(\tau) = (z_{k-1}(\tau) - z_k(\tau))\beta + (z_{k+1}(\tau) - z_k(\tau))\frac{1 - e^{-a+az_0(\tau)}}{1 - z_0(\tau)}$$

Applying the Wormald theorem (Wormald, 1995, 1999), with the domain  $D$  defined by  $-\epsilon < \tau < \frac{T}{n} + \epsilon$ ,  $-\epsilon < z_k < 1 + \epsilon$ , for  $\epsilon > 0$ . And taking  $\beta = 1$  for the boundeness hypothesis (see lemma 29),  $\Lambda_1 = a/(en)$  for the trend hypothesis (see lemma 28). The Lipschitz hypothesis is satisfied with Lipschitz constant  $L = (\beta + a)(1 + \epsilon)$  (see lemma 27). Setting  $\lambda = a n^{-1/4}$ , the Wormald theorem gives with probability  $1 - \mathcal{O}(n^{1/4} \exp(-a^3 n^{1/4}))$ ,

$$Y_k(T) = n z_k(T/n) + \mathcal{O}(n^{3/4}) \quad \text{for } k \geq 0$$

with  $(z_0, \dots, z_K)$  the solution of the following system,

$$\begin{cases} \dot{z}_0(\tau) = -z_0(\tau)\beta + \frac{z_1(\tau)}{1-z_0(\tau)}(1 - e^{-a+az_0(\tau)}) & \text{for } k = 0 \\ \dot{z}_k(\tau) = (z_{k-1}(\tau) - z_k(\tau))\beta + (z_{k+1}(\tau) - z_k(\tau))\frac{1-e^{-a+az_0(\tau)}}{1-z_0(\tau)} & \text{for } 1 \leq k \leq K-1 \\ \dot{z}_k(\tau) = \beta z_{k-1}(\tau) - z_k(\tau)\frac{1-e^{-a(1-z_0(\tau))}}{1-z_0(\tau)} & \text{for } k = K \\ \sum_{k=0}^K z_k(\tau) = 1 \end{cases}$$

■



Now we have all the tools to prove theorem 4.

**Proof.**

For  $\frac{1}{n} \leq \tau \leq \frac{T}{n}$ , let's consider the normalized random variable  $H(\tau) = \frac{\text{Greedy}(G, \tau n)}{n}$ , the conditional expectation of the one-step change of  $H(\tau)$  is given by,

$$\begin{aligned} \frac{\mathbb{E} \left[ H \left( \tau + \frac{1}{n} \right) - H(\tau) \mid H(\tau) \right]}{1/n} &= \frac{\mathbb{E} [\text{Greedy}(G, \tau n + 1)/n - \text{Greedy}(G, \tau n)/n \mid \text{Greedy}(G, \tau n)/n]}{1/n} \\ &= 1 - \left( 1 - \frac{a}{n} \right)^{n-n Z_0(\tau)} \end{aligned}$$

when  $n \rightarrow +\infty$  we get,

$$\dot{h}(\tau) = 1 - e^{-a(1-z_0(\tau))}$$

Applying Wormald theorem (Wormald, 1995, 1999), we choose the domain  $D$  defined by  $-\epsilon < \tau < \frac{T}{n} + \epsilon$ ,  $-\epsilon < z_0 < 1 + \epsilon$  for  $\epsilon > 0$ . We have  $\beta' = 1$  for the boundedness hypothesis (see lemma 29),  $\delta = a/(en)$  for the trend hypothesis (lemma 28). The Lipschitz hypothesis is satisfied with Lipschitz constant  $L' = ae^{a\epsilon}$ . Setting  $\lambda = an^{-1/4}$ , the Wormald theorem gives with probability  $1 - \mathcal{O}(n^{1/4} \exp(-a^3 n^{1/4}))$ ,

$$\text{Greedy}(G, T) = nh(T/n) + \mathcal{O}(n^{3/4})$$

where  $h(\tau)$  is solution of the following equation,

$$\dot{h}(\tau) = 1 - e^{-a(1-z_0(\tau))}, \quad 1/n \leq \tau \leq T/n$$

and  $z_0(\tau)$  as defined in the following system,

$$\begin{cases} \dot{z}_0(\tau) = -z_0(\tau)\beta + \frac{z_1(\tau)}{1-z_0(\tau)}(1 - e^{-a+az_0(\tau)}) & \text{for } k = 0 \\ \dot{z}_k(\tau) = (z_{k-1}(\tau) - z_k(\tau))\beta + (z_{k+1}(\tau) - z_k(\tau))\frac{1-e^{-a+az_0(\tau)}}{1-z_0(\tau)} & \text{for } 1 \leq k \leq K-1 \\ \dot{z}_k(\tau) = \beta z_{k-1}(\tau) - z_k(\tau)\frac{1-e^{-a(1-z_0(\tau))}}{1-z_0(\tau)} & \text{for } k = K \\ \sum_{k=0}^K z_k(\tau) = 1 \end{cases}$$

Since  $\text{Greedy}(G, T)$  is bounded and thus uniformly integrable, so convergence in probability implies convergence in mean:

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E}[\text{Greedy}(G, T)]}{n} = h(T/n)$$

■

The next theorem applies an improved version of the Wormald theorem (see Warnke (2019), Enriquez et al. (2019)) on  $\text{Greedy}(G, T)$ ,

**Theorem 31** *With probability at least  $1 - 2e^{-a^2 n^{\frac{3}{2}}/8T}$  we have,*

$$\max_{1 \leq t \leq T} |\text{Greedy}(G, t) - nh(t/n)| \leq 3e^{L'T/n} an^{3/4}$$

with  $L' = ae^{a\epsilon}$  and  $\epsilon > 0$ , here  $h(\tau)$  is solution of the following equation,

$$\dot{h}(\tau) = 1 - e^{-a(1-z_0(\tau))} \quad 1/n \leq \tau \leq T/n$$

and  $z_0(\tau)$  is defined by the following system,

$$\begin{cases} \dot{z}_0(\tau) = -z_0(\tau)\beta + \frac{z_1(\tau)}{1-z_0(\tau)}(1 - e^{-a+az_0(\tau)}) & \text{for } k = 0 \\ \dot{z}_k(\tau) = (z_{k-1}(\tau) - z_k(\tau))\beta + (z_{k+1}(\tau) - z_k(\tau))\frac{1-e^{-a+az_0(\tau)}}{1-z_0(\tau)} & \text{for } 1 \leq k \leq K-1 \\ \dot{z}_K(\tau) = \beta z_{K-1}(\tau) - z_K(\tau)\frac{1-e^{-a(1-z_0(\tau))}}{1-z_0(\tau)} & \text{for } k = K \\ \sum_{k=0}^K z_k(\tau) = 1 \end{cases} \quad (134)$$

**Proof.**

Using the normalized random variable  $H(\tau) = \frac{\text{Greedy}(G, \tau n)}{n}$  with  $\frac{1}{n} \leq \tau \leq \frac{T}{n}$ , let's compute the conditional expectation of the one-step change of  $H(\tau)$ ,

$$\begin{aligned} \frac{\mathbb{E} \left[ H \left( \tau + \frac{1}{n} \right) - H(\tau) \middle| H(\tau) \right]}{1/n} &= \frac{\mathbb{E} [\text{Greedy}(G, \tau n + 1)/n - \text{Greedy}(G, \tau n)/n | \text{Greedy}(G, \tau n)/n]}{1/n} \\ &= 1 - \left( 1 - \frac{a}{n} \right)^{n-n Z_0(\tau)} \end{aligned}$$

when  $n \rightarrow +\infty$  we get,

$$\dot{h}(\tau) = 1 - e^{-a(1-z_0(\tau))}$$

Applying the non-asymptotic version of the Wormald theorem (Warnke, 2019), we choose the domain  $D$  defined by  $-\epsilon < \tau < \frac{T}{n} + \epsilon$ ,  $-\epsilon < z_0 < 1 + \epsilon$  for  $\epsilon > 0$ . We have  $\beta' = 1$  (lemma 29),  $\delta = a/(en)$  for the trend hypothesis (lemma 28). The Lipschitz hypothesis is satisfied with Lipschitz constant  $L' = ae^{a\epsilon}$  (lemma 27). Setting  $\lambda = an^{-1/4}$  we have with probability at least  $1 - 2e^{-a^2 n^{3/2}/8T}$ ,

$$\max_{1 \leq t \leq T} |\text{Greedy}(G, t) - nh(t/n)| \leq 3e^{L'T/n} an^{3/4}$$

■

## B.2. Proof of corollary 5

**Corollary 5** For  $K \geq 1$ , with probability at least  $1 - 2e^{-a^2 n^{3/2}/8T}$ ,

$$|\text{Greedy}(G, T) - nh^*(T/n)| \leq o(T)$$

and,

$$\frac{\mathbb{E}[\text{Greedy}(G, T)]}{n} \xrightarrow{n \rightarrow +\infty} h^*(T/n)$$

with  $h^*(x) = \int_{1/n}^x (1 - e^{-a(1-z_0^*)}) d\tau = \left(x - \frac{1}{n}\right) (1 - e^{-a(1-z_0^*)})$ , and  $z_0^*$  is the unique solution of  $\sum_{k=0}^K z_0^* \left(\frac{\beta}{g(z_0^*)}\right)^k = 1$  with  $g(z_0^*) = \frac{1 - e^{-a(1-z_0^*)}}{1 - z_0^*}$ .

The proof is organized as follows:

1. Finding the stationary solution of eq. (133).
2. Proving that the stationary solution is asymptotically stable.
3. Proving that Greedy( $G, T$ ) converges to a function depending on the stationary solution.

The next result gives a general form for the stationary solution of eq. (133),

**Lemma 32** For  $\frac{1}{n} \leq \tau \leq \frac{T}{n}$ , let  $\bar{S}_{z_0^*} = (z_0^*, \dots, z_k^*, \dots, z_K^*)$  be the stationary solution of the system,

$$\begin{cases} \dot{z}_0(\tau) = -z_0(\tau)\beta + \frac{z_1(\tau)}{1-z_0(\tau)}(1 - e^{-a+az_0(\tau)}) & \text{for } k = 0 \\ \dot{z}_k(\tau) = (z_{k-1}(\tau) - z_k(\tau))\beta + (z_{k+1}(\tau) - z_k(\tau))\frac{1-e^{-a+az_0(\tau)}}{1-z_0(\tau)} & \text{for } 1 \leq k \leq K-1 \\ \dot{z}_k(\tau) = \beta z_{k-1}(\tau) - z_k(\tau)\frac{1-e^{-a(1-z_0(\tau))}}{1-z_0(\tau)} & \text{for } k = K \\ \sum_{k=0}^K z_k(\tau) = 1 \end{cases} \quad (135)$$

$\bar{S}_{z_0^*}$  is unique and satisfies,

$$z_k^* = z_0^* \left( \frac{\beta}{g(z_0^*)} \right)^k \quad \text{for } 0 \leq k \leq K \quad (136)$$

where  $g(z_0^*) = \frac{1-e^{-a(1-z_0^*)}}{1-z_0^*}$ .

**Proof.**

eq. (136) is proved by recurrence. For the uniqueness, according to eq. (135),  $\bar{S}_{z_0^*}$  satisfies  $\sum_{k=1}^K z_k^* = 1$ , using eq. (136) we get that  $P(z_0^*) = \sum_{k=1}^K z_0^* \left( \frac{\beta}{g(z_0^*)} \right)^k - 1$ .  $P(0) = -1$  and  $\lim_{z_0 \rightarrow 1} P(z_0) > 0$ . Moreover  $P$  is continuous and monotonic, this implies that  $P(z_0) = 0$  has a unique solution. Thus,  $\bar{S}_{z_0^*}$  is unique. ■

**Remark 33** Given that  $z_k^*$  follows a geometric progression, for convergence, it's essential that  $\left| \frac{\beta}{g(z_0^*)} \right| \leq 1$ . Therefore, we will proceed with the remaining proofs under this assumption.

The following lemma shows that  $\bar{S}_{z_0^*}$  is an asymptotically stable stationary solution of eq. (135).

**Theorem 34**  $\bar{S}_{z_0^*}$  is an asymptotically stable stationary solution of eq. (135).

**Proof.**

Let  $Z = \begin{pmatrix} z_0(t) \\ \vdots \\ z_K(t) \end{pmatrix}$ , eq. (133) can be seen as  $\dot{Z} = F(Z)$ , where,

$$F(z_0(t), \dots, z_K(t)) = \left( -\beta z_0 + z_1 g(z_0), \dots, \beta z_{k-1}(t) - z_k(t) \frac{1 - e^{-a(1-z_0(t))}}{1 - z_0(t)} \right)$$

The Jacobian of  $F$  at  $\bar{S}_{z_0^*}$  is then given by,

$$DF(\bar{S}_{z_0^*}) = \begin{pmatrix} -\beta + z_1^* g'(z_0^*) & g(z_0^*) & 0 & \dots & \dots & 0 \\ \beta + (z_2^* - z_1^*) g'(z_0^*) & -\beta - g(z_0^*) & g(z_0^*) & 0 & \dots & 0 \\ (z_3^* - z_2^*) g'(z_0^*) & \beta & -\beta - g(z_0^*) & g(z_0^*) & 0.. & 0 \\ \vdots & 0 & \beta & -\beta - g(z_0^*) & g(z_0^*) & ..0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -z_K^* g'(z_0^*) & \dots & \dots & 0 & \beta & -g(z_0^*) \end{pmatrix}$$

Since proving that  $\bar{S}_{z_0^*}$  is asymptotically stable is equivalent to proving that the eigenvalues of  $DF(\bar{S}_{z_0^*})$  are non-positives (Viterbo, 2011). We achieve this using the perturbation method. To do so, we shall write,

$$DF(\bar{S}_{z_0^*}) = M + uv^\top$$

where  $v^\top = (1, 0, \dots, 0)$  and,

$$\begin{aligned} u^\top &= g'(z_0^*)(z_1^*, z_2^* - z_1^*, \dots, -z_K^*) \\ &= z_1^* g'(z_0^*) \left( 1, \frac{\beta}{g(z_0^*)} - 1, \left( \frac{\beta}{g(z_0^*)} - 1 \right) \frac{\beta}{g(z_0^*)}, \left( \frac{\beta}{g(z_0^*)} - 1 \right) \left( \frac{\beta}{g(z_0^*)} \right)^2, \dots \right) \end{aligned}$$

and  $M$  is the matrix with  $g(z_0^*)$  above the diagonal,  $\beta$  below it and its diagonal is  $(-\beta, -\beta - g(z_0^*), \dots, -\beta - g(z_0^*), -g(z_0^*))$ .

$$M = \begin{pmatrix} -\beta & g(z_0^*) & 0 & \dots & \dots & 0 \\ \beta & -\beta - g(z_0^*) & g(z_0^*) & 0 & \dots & 0 \\ 0 & \beta & -\beta - g(z_0^*) & g(z_0^*) & 0.. & 0 \\ \vdots & 0 & \beta & -\beta - g(z_0^*) & g(z_0^*) & ..0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & \beta & -g(z_0^*) \end{pmatrix}$$

Let us denote by  $\Pi_M(\lambda)$  the characteristic polynomial of  $M$ , as a function of  $\lambda$ , so that

$$\Pi_{M+uv^\top}(\lambda) = \Pi_M(\lambda) \left( 1 + v^\top (M - \lambda I)^{-1} u \right)$$

which implies that eigenvalues of  $M + uv^\top$  are either eigenvalues of  $M$  or solutions of

$$1 + v^\top (M - \lambda I)^{-1} u = 0.$$

Since 0 is an eigenvalue of  $M + uv^\top$ , we aim at proving that  $1 + v^\top (M - \lambda I)^{-1} u = 0$  has  $K$  non-positives solutions.

We now claim that the eigenvalues of  $M$  are  $\mu_j := -\beta - g(z_0^*) + 2\sqrt{\beta g(z_0^*)} \cos(\frac{j\pi}{k+1})$  for  $j \in [k]$  and 0. This is a consequence of standard computations along with Theorem 2.2 of Kulkarni et al. (1999). We also denote by  $\omega_j$  the eigenvectors of  $M$  associated to  $\mu_j$  (and  $\omega_0$  to 0) and by  $P$  the matrix whose columns are  $\omega_0, \omega_1, \dots$ . As a consequence,

$$q(\lambda) = 1 + v^\top (M - \lambda I)^{-1} u = 1 + v^\top P \text{diag} \left( \frac{1}{\mu_j - \lambda} \right) P^{-1} u$$

$$= 1 + \left( \frac{\omega_{0,1}}{-\lambda}, \dots, \frac{\omega_{j,1}}{\mu_j - \lambda} \right) P^{-1} u$$

Since we can take any eigenvectors in  $P$ , we can assume that  $\omega_{j,1} \geq 0$  hence it remains to prove that  $P^{-1}u$  is a vector with non-negative coordinates. Notice that this vector is the vector of  $u$  in the basis formed by the eigenvectors of  $M$ .

The computations of  $\omega_j$  are quite standard, and they yield, denoting  $\theta = \sqrt{\frac{\beta}{g(z_0^*)}}$ , for  $1 \leq j \leq K$

$$\begin{aligned} \omega_j = & \left( \theta \sin\left(\frac{j\pi}{K+1}\right), \dots, (-\theta)^{t+1} \sin\left(\frac{(t+1)j\pi}{K+1}\right) + (-\theta)^t \sin\left(\frac{tj\pi}{K+1}\right), \dots, (-\theta)^{K+1} \sin\left(\frac{(K+1)j\pi}{K+1}\right) \right. \\ & \left. + (-\theta)^K \sin\left(\frac{tK\pi}{K+1}\right) \right) \end{aligned}$$

As a consequence,  $u$  and all the eigenvectors  $\omega_j$  are orthogonal to the vectors of ones, which indicates that  $u = \sum_j \alpha_j \omega_j$  for some scalar  $\alpha_j$ . The objective is to prove that they are all positive

The exact forms of  $u$  and  $\omega_j$  give, after a few lines of algebra,

$$\sum_{j=1}^K \alpha_j \sin\left(m \frac{j\pi}{K+1}\right) = (-\theta)^m, \quad \forall m \in [K].$$

This system can be rewritten using the Chebyshev polynomials of second kind (denoted by  $U_n$ ) as,

$$\sum_{j=1}^K \alpha_j U_{m-1}\left(\cos\left(\frac{j\pi}{K+1}\right)\right) \sin\left(\frac{j\pi}{K+1}\right) = (-\theta)^m, \quad \forall m \in [K].$$

Hence in a more compact matrix way it can be seen as,

$$W \begin{pmatrix} \alpha_1 \sin\left(\frac{\pi}{K+1}\right) \\ \vdots \\ \alpha_K \sin\left(\frac{K\pi}{K+1}\right) \end{pmatrix} = -\boldsymbol{\theta},$$

where  $-\boldsymbol{\theta} = \left( (-\theta)^j \right)_{j \in [K]}$  and  $W$  is the matrix whose  $j$ -th column is

$$\left( U_0\left(\cos\left(\frac{j\pi}{K+1}\right)\right), \dots, U_{K-1}\left(\frac{j\pi}{K+1}\right) \right)^\top$$

We introduce now the following polynomial,

$$P_m(X) = \gamma_m \frac{U_m(X)}{X - \cos\left(\frac{m\pi}{K+1}\right)} = \sum_{j=1}^K \beta_{j,m} U_{j-1}(X)$$

where

$$\gamma_m = \frac{1}{\prod_{j \neq m} \left( \cos\left(\frac{m\pi}{K+1}\right) - \cos\left(\frac{j\pi}{K+1}\right) \right) 2^K} = \frac{1}{2^K} \frac{1}{\prod_{j \neq m} -2 \sin\left(\frac{m+j}{2} \frac{\pi}{K+1}\right) \sin\left(\frac{m-j}{2} \frac{\pi}{K+1}\right)}$$

so that the sign of  $\gamma_m$  is  $(-1)^{m-1}$ ,  $P_m(\cos(\frac{j\pi}{K+1})) = 0$  for all  $j \neq m$ , and  $P_m(\cos(\frac{m\pi}{K+1})) = 1$ . We get that

$$\alpha_m = \frac{1}{\sin(\frac{m\pi}{K+1})} \sum_{j=1}^K \beta_{j,m} (-\theta)^j.$$

Using the fact that, by definition of  $P_m$ ,

$$U_K(X) = \sum_{j=1}^K \frac{1}{\gamma_m} \beta_{j,m} U_{j=1}(X) (X - \cos(\frac{m\pi}{K+1}))$$

and the property of Chebishev polynomial,

$$U_k(X) = 2XU_{k-2}(X) - U_{k-3},$$

we can identify the coefficients  $\beta_j$  that satisfy a linear recurrence of order 2 and are defined by

$$\beta_j = 2\gamma_m \frac{\sin(\frac{(K-j+1)m\pi}{K+1})}{\sin(\frac{m\pi}{K+1})}.$$

It remains to compute  $\alpha_m = \sum \beta_j (-\theta)^j$ , and standard computations yield that

$$\alpha_m = -2\gamma_m (-1)^m \frac{(1 - \theta^2)}{1 + 2 \cos(\frac{m\pi}{K+1})\theta + \theta^2} \geq 0$$

Thus, we have proved that  $P^{-1}u$  is a vector with non-negative coordinates. Consequently,  $q(\lambda)$  is an increasing function of  $\lambda$ . As a result,  $M + uv^\top$  has  $K$  eigenvalues of negative real part and one eigenvalue equals to zero. From this, we can conclude that  $\bar{S}_{z_0^*}$  is an asymptotically stationary solution of eq. (135). ■

Given the previous results, we can prove corollary 5,

**Proof.**

Let  $h^*(T/n) = \int_{1/n}^{T/n} (1 - e^{-a(1-z_0^*)}) d\tau = \frac{(T-1)(1-e^{-a(1-z_0^*)})}{n}$ , we have,

$$\begin{aligned} |\text{Greedy}(G, T) - nh^*(T/n)| &= |\text{Greedy}(G, T) - nh(T/n) + nh(T/n) - nh^*(T/n)| \\ &\leq \underbrace{\max_{1 \leq t \leq T} (|\text{Greedy}(G, t) - nh(t/n)| + |nh(T/n) - nh^*(T/n)|)}_{\leq 3e^{L'T/n} an^{3/4} \text{ by theorem 31}} \end{aligned}$$

Let's focus on  $D = |nh(T/n) - nh^*(T/n)|$ , for  $1 \leq T' < T$  and  $\delta > 0$ ,

$$\begin{aligned} D &= |nh(T/n) - nh(T'/n + \delta) + nh(T'/n + \delta) - nh^*(T/n)| \\ &= |nh(T/n) - nh(T'/n + \delta) - nh^*(T/n) + nh^*(T'/n + \delta) + nh(T'/n + \delta) - nh^*(T'/n + \delta)| \\ &= \left| n \int_{T'/n+\delta}^{T/n} (e^{-a(1-z_0^*)} - e^{-a(1-z_0(t))}) dt + n \int_{1/n}^{T'/n+\delta} (e^{-a(1-z_0^*)} - e^{-a(1-z_0(t))}) dt \right| \end{aligned}$$

$$\begin{aligned}
 &= \left| ne^{-a(1-z_0^*)} \int_{T'/n+\delta}^{T/n} (1 - e^{a(z_0(t)-z_0^*)})dt + ne^{-a(1-z_0^*)} \int_{1/n}^{T'/n+\delta} (1 - e^{a(z_0(t)-z_0^*)})dt \right| \\
 &\leq ne^{-a(1-z_0^*)} \left| \int_{T'/n+\delta}^{T/n} (-a(z_0(t) - z_0^*))dt + \int_{1/n}^{T'/n+\delta} (-a(z_0(t) - z_0^*))dt \right| \quad \text{using } (1 - e^{ax} \leq -ax) \\
 &\leq ne^{-a(1-z_0^*)} a \int_{T'/n+\delta}^{T/n} |z_0(t) - z_0^*|dt + ne^{-a(1-z_0^*)} a \int_{1/n}^{T'/n+\delta} |z_0(t) - z_0^*|dt \\
 &\leq ne^{-a(1-z_0^*)} a \int_{T'/n+\delta}^{T/n} \epsilon dt + ne^{-a(1-z_0^*)} a \int_{1/n}^{T'/n+\delta} |z_0(t) - z_0^*|dt \quad \text{using (theorem 34)} \\
 &\leq ne^{-a(1-z_0^*)} a \epsilon \left( \frac{T - T'}{n} - \delta \right) + ne^{-a(1-z_0^*)} a \int_{1/n}^{T'/n+\delta} 1 dt \quad \text{using } 0 \leq z_0 \leq 1 \text{ and } 0 \leq z_0^* \leq 1 \\
 &\leq ne^{-a(1-z_0^*)} a \epsilon \left( \frac{T}{n} \right) + ne^{-a(1-z_0^*)} a \left( \frac{T' - 1}{n} + \delta \right) (1 - \epsilon) \\
 &\leq ne^{-a(1-z_0^*)} a \epsilon \left( \frac{T}{n} \right) + ne^{-a(1-z_0^*)} a \underbrace{\left( \frac{T' - 1}{n} + \delta \right)}_{\leq \frac{T}{n^2}} \\
 &\leq 2e^{-a(1-z_0^*)} a \left( \frac{T}{n} \right) \quad \text{choosing } (\epsilon = \frac{1}{n})
 \end{aligned}$$

Thus,

$$|\text{Greedy}(G, T) - nh^*(T/n)| \leq 3e^{L'T/n} an^{3/4} + 2e^{-a(1-z_0^*)} a \frac{T}{n}$$

with  $L' = ae^{a\gamma}$  where  $\gamma > 0$ . Taking  $n = cT$  with  $c < 1$ , we can see that  $|\text{Greedy}(G, T) - nh^*(T/n)| \leq o(T)$ , ■

### B.3. Proof of corollary 6

**Corollary 6** For  $K = 1$ , with probability at least  $1 - 2e^{-a^2 n^{\frac{3}{8}T}}$ ,

$$\left| \mathbb{E}[\text{Greedy}(G, T)] - T(1 - e^{-a(1-z_0^*)}) \right| \leq c \frac{T}{(\log(T))^{3/4}} = o(T)$$

where  $z_0^* = \frac{1}{\beta} - \frac{1}{a} W\left(\frac{a}{\beta} e^{-a(1-\frac{1}{\beta})}\right)$ , with  $W(\cdot)$  the Lambert function, and  $c$  is some universal constant.

The proof is organized as follows:

1. Finding the stationary solution of eq. (133) for  $K = 1$ .
2. Proving that the stationary solution is exponentially stable.
3. Applying an improved version of the Wormald theorem on  $\text{Greedy}(G, T)$ .

4. Proving that Greedy( $G, T$ ) converges to a function depending on the stationary solution.

Intuitively  $K = 1$  means that the maximum budget reached by each node in  $U$  is equal to 1. From a technical aspect, supposing  $K = 1$  reduces eq. (135) to a system of 2 equations as follows, for  $t \in [\frac{1}{n}, \frac{T}{n}]$

$$\begin{cases} \dot{z}_0(t) &= -\beta z_0(t) + \frac{z_1(t)}{1-z_0(t)}(1 - e^{-a(1-z_0(t))}) \\ \dot{z}_1(t) &= \beta z_0(t) - \frac{z_1(t)}{1-z_0(t)}(1 - e^{-a(1-z_0(t))}) \\ z_0(t) + z_1(t) &= 1 \end{cases} \quad (137)$$

By simplifying eq. (137), we reduce the system to the following equation,

$$\dot{z}_0(t) = -\beta z_0(t) + 1 - e^{-a(1-z_0(t))} \quad (138)$$

The following lemma computes the unique stationary solution of eq. (138),

**Lemma 35** *The stationary solution of eq. (138) is unique and is given for  $\beta, a > 0$  by,*

$$z_0^* = \frac{1}{\beta} - \frac{1}{a} W \left( \frac{a}{\beta} e^{-a(1-\frac{1}{\beta})} \right)$$

where  $W$  is the Lambert function.

**Proof.**

Let's define  $G(z_0) = -\beta z_0 + 1 - e^{-a(1-z_0)}$ , the stationary solution of eq. (138) is the solution of  $G(z_0^*) = 0$  (the homogeneous equation) with  $a > 0$  and  $\beta > 0$ ,

$$\begin{aligned} G(z_0^*) = 0 &\iff 1 - e^{-a(1-z_0^*)} = \beta z_0^* \iff e^{a(\frac{1}{\beta}-z_0^*)} a \left( \frac{1}{\beta} - z_0^* \right) = \frac{a}{\beta} e^{-a(1-\frac{1}{\beta})} \\ &\iff a \left( \frac{1}{\beta} - z_0^* \right) = W \left( \frac{a}{\beta} e^{-a(1-\frac{1}{\beta})} \right) \end{aligned}$$

Where  $W$  is the Lambert function. So, by rearranging the terms in the last equation, the solution of  $G(z_0^*) = 0$  is given by,

$$z_0^* = \frac{1}{\beta} - \frac{1}{a} W \left( \frac{a}{\beta} e^{-a(1-\frac{1}{\beta})} \right)$$

Let's prove the uniqueness of the stationary solution,  $G(0) = 1 - e^{-a} > 0$  and  $G(1) = -\beta < 0$  and we have  $\forall 0 \leq z_0 \leq 1$ ,  $\frac{dG(z_0)}{dz_0} = -(\beta + ae^{-a}e^{-a(1-z_0)}) < 0$ . Thus  $G(z_0) = 0$  has a unique solution. ■

The following theorem proves that  $z_0^*$  is exponentially stable, meaning that  $\forall t \in [\frac{1}{n}, \frac{T}{n}]$ ,  $z_0(t)$  converges to  $z_0^*$  with an exponential rate.

**Theorem 36** *For any  $f_0 \geq 0$  and  $t \in [\frac{1}{n}, \frac{T}{n}]$ , consider the ordinary differential equation (ODE),*

$$\begin{cases} \dot{z}_0(t) &= \beta z_0(t) + 1 - e^{-a(1-z_0(t))} \\ z_0(1/n) &= f_0 \end{cases}$$

*Thus, it implies that  $z_0(t)$  converges to  $z_0^*$  exponentially.*



**Proof.**

The idea here is to prove that for any perturbation that we add to  $z_0^*$ , this perturbation tends to 0 when  $t$  tends to  $+\infty$  exponentially.

Let's consider  $\epsilon(t) : \mathbb{R} \rightarrow \mathbb{R}$ , a perturbation of the stationary solution  $z_0^*$ ,

$$\begin{aligned} \dot{\epsilon}(t) &= -\beta z_0^* - \beta \epsilon(t) + 1 - e^{-a(1-z_0^*(t)-\epsilon(t))} \\ \dot{\epsilon}(t) &= -1 + e^{-a(1-z_0^*)} - \beta \epsilon(t) + 1 - e^{-a(1-z_0^*-\epsilon(t))} \\ \dot{\epsilon}(t) &= e^{-a(1-z_0^*)} \left(1 - e^{a\epsilon(t)}\right) - \beta \epsilon(t) \\ \dot{\epsilon}(t) &\leq -a\epsilon(t) e^{-a(1-z_0^*)} - \beta \epsilon(t) \quad \text{using that } (1 - e^{a\epsilon(t)} \leq -a\epsilon(t)) \\ \dot{\epsilon}(t) &\leq \underbrace{\epsilon(t) \left(-a e^{-a(1-z_0^*)} - \beta\right)}_{\leq 0} \end{aligned}$$

Integrating the last equation , we get,

$$\ln(|\epsilon(t)|) - \ln(|\epsilon(0)|) \leq \left(-a e^{-a(1-z_0^*)} - \beta\right) t \quad (139)$$

$$|\epsilon(t)| \leq |\epsilon(0)| \exp\left(-t \left(a e^{-a(1-z_0^*)} + \beta\right)\right) \quad (140)$$

$$|\epsilon(t)| \leq |f_0 - z_0^*| \exp\left(-t \left(a e^{-a\left(1-\frac{1}{\beta} + \frac{1}{a} W\left(\frac{a}{\beta} e^{-a\left(1-\frac{1}{\beta}\right)}\right)\right)} + \beta\right)\right) \quad (141)$$

$$|\epsilon(t)| \leq |f_0 - z_0^*| \exp\left(-t\beta \left(1 + W\left(e^{-a\left(1-\frac{1}{\beta}\right)}\right)\right)\right) \quad (142)$$

Moving from eq. (141) to eq. (142) is done using  $\exp(-W(x)) = W(x)/x$ .

Thus  $\lim_{t \rightarrow +\infty} \epsilon(t) = 0$  exponentially with the following rate  $\omega = \beta \left(1 + W\left(e^{-a\left(1-\frac{1}{\beta}\right)}\right)\right)$ . ■

**Lemma 37**  $S_{z_0^*}^1 = (z_0^*, z_1^*) = (z_0^*, z_0^* \frac{\beta}{g(z_0^*)})$  is an exponentially stable stationary solution of eq. (137).

**Proof.**

According to lemma 35,  $z_0^*$  is a stationary solution of eq. (138), this implies that  $S_{z_0^*}^1 = (z_0^*, z_1^*) = (z_0^*, z_0^* \frac{\beta}{g(z_0^*)})$  is a stationary solution of eq. (137). As previously demonstrated,  $\forall t \in [\frac{1}{n}, \frac{T}{n}]$ ,  $z_0(t)$  converges to  $z_0^*$  exponentially. This implies that eq. (138) possesses an exponentially stable stationary solution. Given that eq. (138) is a reduced version of eq. (137), we can conclude that  $S_{z_0^*}^1$  is an exponentially stable stationary solution for eq. (137). ■

With all the essential elements assembled, we are now ready to establish the proof for corollary 6

**Proof.**

Let  $h^*(T/n) = \int_{1/n}^{T/n} (1 - e^{-a(1-z_0^*)}) d\tau = \frac{(T-1)(1-e^{-a(1-z_0^*)})}{n}$ , we have,

$$|\text{Greedy}(G, T) - nh^*(T/n)| = |\text{Greedy}(G, T) - nh(T/n) + nh(T/n) - nh^*(T/n)|$$

$$\leq \underbrace{\max_{1 \leq t \leq T} (|\text{Greedy}(G, t) - nh(t/n)|)}_{\leq 3e^{L'T/n} an^{3/4} \text{ by theorem 31}} + |nh(T/n) - nh^*(T/n)|$$

Let's focus on  $|nh(T/n) - nh^*(T/n)|$ ,

$$|nh(T/n) - nh^*(T/n)| = \left| n \int_{1/n}^{T/n} (1 - e^{-a(1-z_0(\tau))}) d\tau - n \int_{1/n}^{T/n} (1 - e^{-a(1-z_0^*)}) d\tau \right| \quad (143)$$

$$= \left| n \int_{1/n}^{T/n} (e^{-a(1-z_0^*)} - e^{-a(1-z_0(\tau))}) d\tau \right| \quad (144)$$

$$= \left| ne^{-a(1-z_0^*)} \int_{1/n}^{T/n} (1 - e^{a(z_0(\tau) - z_0^*)}) d\tau \right| \quad (145)$$

$$\leq \left| ne^{-a(1-z_0^*)} \int_{1/n}^{T/n} -a(z_0(\tau) - z_0^*) d\tau \right| \quad (\text{using } 1 - e^{ax} \leq -ax) \quad (146)$$

$$\leq nae^{-a(1-z_0^*)} \int_{1/n}^{T/n} |z_0(\tau) - z_0^*| d\tau \quad (147)$$

$$\leq nae^{-a(1-z_0^*)} |f_0 - z_0^*| \int_{1/n}^{T/n} \exp\left(-\tau\beta \left(1 + W(e^{-a(1-\frac{1}{\beta}})\right)\right) d\tau \quad (148)$$

$$\leq \frac{nae^{-a(1-z_0^*)} |f_0 - z_0^*|}{\beta \left(1 + W(e^{-a(1-\frac{1}{\beta}})\right)} \left( e^{-\frac{\beta}{n} \left(1 + W(e^{-a(1-\frac{1}{\beta}})\right)} \right) - e^{-\frac{T}{n}\beta \left(1 + W(e^{-a(1-\frac{1}{\beta}})\right)} \right) \quad (149)$$

Moving from eq. (147) to eq. (148) is done using theorem 36. Thus we have,

$$|\text{Greedy}(G, T) - nh^*(T/n)| \leq 3e^{L'T/n} an^{3/4} + \frac{nae^{-a(1-z_0^*)} |f_0 - z_0^*|}{P} \left( e^{-\frac{1}{n}P} - e^{-\frac{T}{n}P} \right)$$

with  $L' = ae^{a\epsilon}$  and  $P = \beta \left(1 + W(e^{-a(1-\frac{1}{\beta}})\right)$ ,  $\epsilon > 0$ .

Now let's focus on  $A = 3e^{L'T/n} an^{3/4} + \frac{nae^{-a(1-z_0^*)} |f_0 - z_0^*|}{P} \left( e^{-\frac{1}{n}P} - e^{-\frac{T}{n}P} \right)$  and considering that  $n = \frac{T}{\alpha \log(T)}$  with  $\alpha > 0$ , we get,

$$\begin{aligned} A &\leq 3e^{L'T/n} an^{3/4} + \frac{nae^{-a(1-z_0^*)} |f_0 - z_0^*|}{P} \left( 1 - e^{-\frac{T}{n}P} \right) \\ &= \frac{ae^{-a(1-z_0^*)} |f_0 - z_0^*|}{P} \left( \frac{T}{\alpha \log(T)} - \frac{T^{1-\alpha P}}{\alpha \log(T)} \right) \\ &\quad + 3a \frac{T^{\frac{3}{4} + \alpha L'}}{(\alpha \log(T))^{\frac{3}{4}}} \\ &\leq \frac{ae^{-a(1-z_0^*)} |f_0 - z_0^*|}{P} \frac{T}{\alpha \log(T)} + 3a \frac{T^{\frac{3}{4} + \alpha L'}}{(\alpha \log(T))^{\frac{3}{4}}} \end{aligned}$$

Taking  $\alpha = \frac{1}{4(P+L')}$ , with  $z_0^* = \frac{1}{\beta} - \frac{1}{a}W\left(\frac{a}{\beta}e^{-a(1-\frac{1}{\beta})}\right)$  and using the fact that  $e^{-W(x)} = \frac{W(x)}{x}$ , we get,

$$\begin{aligned} A &\leq \frac{\beta W\left(e^{-a(1-\frac{1}{\beta})}\right)}{\alpha P} |f_0 - z_0^*| \frac{T}{\log(T)} + 3a(4(L' + P))^{\frac{3}{4}} \frac{T^{\frac{3L'+4P}{4(P+L')}}}{(\log(T))^{\frac{3}{4}}} \\ &= c_1 \frac{T}{\log(T)} + c_2 \frac{T^{\omega'}}{(\log(T))^{\frac{3}{4}}} \\ &\leq c \frac{T}{(\log(T))^{3/4}} \end{aligned}$$

where  $c_1 = \frac{4\beta(P+L')W\left(e^{-a(1-\frac{1}{\beta})}\right)}{P}$ ,  $c_2 = 3a(4(L' + P))^{\frac{3}{4}}$ ,  $c = c_1 + c_2$ ,  $\omega = \frac{3P+4L'}{4(L'+P)}$  and  $\omega' = \frac{3L'+4P}{4(P+L')}$ . ■

#### B.4. Proof of proposition 7

**Proposition 7** For  $T, K, n, b_0, \beta \in \mathbb{N}^*$ ,

$$\text{CR}^{\text{sto}}(\text{Greedy}, \mathcal{D}) \geq \frac{Tg(z_0^*)(1 - z_0^*) + nb_0 - n\left(\frac{\beta}{g(z_0^*)-\beta} - \frac{(K+1)\beta^{K+1}}{g(z_0^*)^{K+1}-\beta^{K+1}}\right)}{nb_0 + \beta T} + \mathcal{O}_{K,\beta}(T^{-1/4}) \quad (5)$$

where  $\sum_{k=0}^K z_0^* \left(\frac{\beta}{g(z_0^*)}\right)^k = 1$  with  $g(z_0^*) = \frac{1-e^{-a(1-z_0^*)}}{1-z_0^*}$  as defined in corollary 5.

**Proof.**

According to eq. (124), we have that for all  $u \in U$ ,

$$b_{u,t} = \min(K, b_{u,t-1} - x_{u,t} + \eta_t) \quad \text{with } b_{u,0} = b_0 \geq 1$$

Which gives,

$$\text{Greedy}(G, T) = \sum_{u \in U} \sum_{t=1}^T x_{u,t} = nb_0 + \underbrace{\sum_{u \in U} \sum_{t=1}^T \mathbb{E}[\eta_t]}_{A_1} - \underbrace{\sum_{u \in U} b_{u,T}}_{A_2} - \underbrace{\sum_{u \in U} \sum_{t=1}^T \mathbb{E}[1_{b_{u,t}=K} 1_{\eta_t=1}]}_{A_3}$$

According to lemma 30, we have w.h.p  $\forall k \geq 0, t \in [T], Y_k(t) = nz_k(t/n) + \mathcal{O}(n^{3/4})$ , let's then compute  $A_1, A_2$  and  $A_3$ ,

$$\begin{aligned} A_1 &= \sum_{u \in U} \sum_{t=1}^T \mathbb{E}[\eta_t] = \beta T \\ A_2 &= \sum_{u \in U} b_{u,T} = n \sum_{k=1}^K kz_k(T/n) + \mathcal{O}\left(\frac{K(K+1)}{2}n^{3/4}\right) \end{aligned}$$

$$A_3 = \beta \sum_{t=1}^T z_K(t/n) + \mathcal{O}(\beta T n^{-1/4})$$

Using the following upper bound on  $\text{OPT}(G, T) \leq nb_0 + \beta T$  and  $n = \mathcal{O}(T)$ , we get that,

$$\text{CR}^{\text{sto}}(\text{Greedy}, \mathcal{D}) \geq \frac{nb_0 + \beta T - n \sum_{k=1}^K k z_k(T/n) - \beta \sum_{t=1}^T z_K(t/n)}{nb_0 + \beta T} + \mathcal{O}(T^{-1/4}) \quad (150)$$

According theorem 34,  $\forall \tau \in [\frac{1}{n}, \frac{T}{n}]$ ,  $(z_0(\tau), \dots, z_K(\tau))$  converges to  $\bar{S}_{z_0^*}$  asymptotically, this implies that,

$$\text{CR}^{\text{sto}}(\text{Greedy}, \mathcal{D}) \geq \frac{nb_0 + \beta T - \beta T z_0^* \left(\frac{\beta}{g(z_0^*)}\right)^K - n z_0^* \sum_{k=1}^K k \left(\frac{\beta}{g(z_0^*)}\right)^k}{nb_0 + \beta T} + \mathcal{O}(T^{-1/4}) \quad (151)$$

From  $\sum_{k=0}^K z_0^* \left(\frac{\beta}{g(z_0^*)}\right)^k = 1$ , we have that  $\left(\frac{\beta}{g(z_0^*)}\right)^K = \frac{g(z_0^*)}{\beta} - \frac{1}{z_0^*} \left(\frac{g(z_0^*)}{\beta} - 1\right)$ , this gives,

$$\begin{aligned} \text{CR}^{\text{sto}}(\text{Greedy}, \mathcal{D}) &\geq \frac{nb_0 + \beta T - \beta T z_0^* \left(\frac{\beta}{g(z_0^*)}\right)^K - n z_0^* \sum_{k=1}^K k \left(\frac{\beta}{g(z_0^*)}\right)^k}{nb_0 + \beta T} + \mathcal{O}(T^{-1/4}) \\ &\geq \frac{nb_0 + \beta T - \beta T z_0^* \left(\frac{g(z_0^*)}{\beta} - \frac{1}{z_0^*} \left(\frac{g(z_0^*)}{\beta} - 1\right)\right) - n z_0^* \sum_{k=1}^K k \left(\frac{\beta}{g(z_0^*)}\right)^k}{nb_0 + \beta T} \\ &\quad + \mathcal{O}(T^{-1/4}) \\ &\geq \frac{nb_0 + g(z_0^*)T(1 - z_0^*) - n z_0^* \sum_{k=1}^K k \left(\frac{\beta}{g(z_0^*)}\right)^k}{nb_0 + \beta T} + \mathcal{O}(T^{-1/4}) \\ &\geq \frac{nb_0 + g(z_0^*)T(1 - z_0^*) - n z_0^* \sum_{k=1}^K k \left(\frac{\beta}{g(z_0^*)}\right)^k}{nb_0 + \beta T} + \mathcal{O}(T^{-1/4}) \end{aligned}$$

Using  $1 - \left(\frac{\beta}{g(z_0^*)}\right)^{K+1} = \frac{1}{z_0^*} \left(1 - \frac{\beta}{g(z_0^*)}\right)$  and  $\sum_{k=1}^K k x^k = x \frac{d}{dx} \left(\frac{1-x^{K+1}}{1-x}\right)$  with  $x = \frac{\beta}{g(z_0^*)}$ ,

$$\begin{aligned} \text{CR}^{\text{sto}}(\text{Greedy}, \mathcal{D}) &\geq \frac{nb_0 + g(z_0^*)T(1 - z_0^*) - n z_0^* \sum_{k=1}^K k \left(\frac{\beta}{g(z_0^*)}\right)^k}{nb_0 + \beta T} + \mathcal{O}(T^{-1/4}) \\ &\geq \frac{nb_0 + g(z_0^*)T(1 - z_0^*) - n \left(\frac{\beta}{g(z_0^*) - \beta} - \frac{(K+1)\beta^{K+1}}{g(z_0^*)^{K+1} - \beta^{K+1}}\right)}{nb_0 + \beta T} + \mathcal{O}(T^{-1/4}) \end{aligned}$$

■

**B.5. Proof of theorem 8**

**Theorem 8** For any  $\alpha, \beta > 0$ , the competitive ratio tends to 1, as  $T, K, n$  approach infinity, as

$$\lim_{K, n \rightarrow +\infty} \lim_{T \rightarrow +\infty} \text{CR}^{\text{sto}}(\text{Greedy}, \mathcal{D}) = 1$$

**Proof.**

$$\lim_{T \rightarrow +\infty} \text{CR}^{\text{sto}}(\text{Greedy}, \mathcal{D}) = \frac{g(z_0^*)(1 - z_0^*)}{\beta}$$

When  $K \rightarrow \infty$ ,  $z_0^*$  satisfies  $\sum_{k=0}^{+\infty} z_0^* \left(\frac{\beta}{g(z_0^*)}\right)^k = 1$ , this gives,

$$\sum_{k=0}^{+\infty} z_0^* \left(\frac{\beta}{g(z_0^*)}\right)^k = z_0^* \frac{1}{1 - \frac{\beta}{g(z_0^*)}} = 1 \implies 1 - e^{-a(1-z_0^*)} = \beta$$

which leads to  $z_0^* = 1 + \frac{\ln(1-\beta)}{a}$ .

Thus,

$$\lim_{K, n \rightarrow +\infty} \lim_{T \rightarrow +\infty} \text{CR}^{\text{sto}}(\text{Greedy}, \mathcal{D}) = \lim_{K, n \rightarrow +\infty} \frac{g(z_0^*)(1 - z_0^*)}{\beta} = 1$$

■