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Dealing with the irregular in inverse and imaging problems

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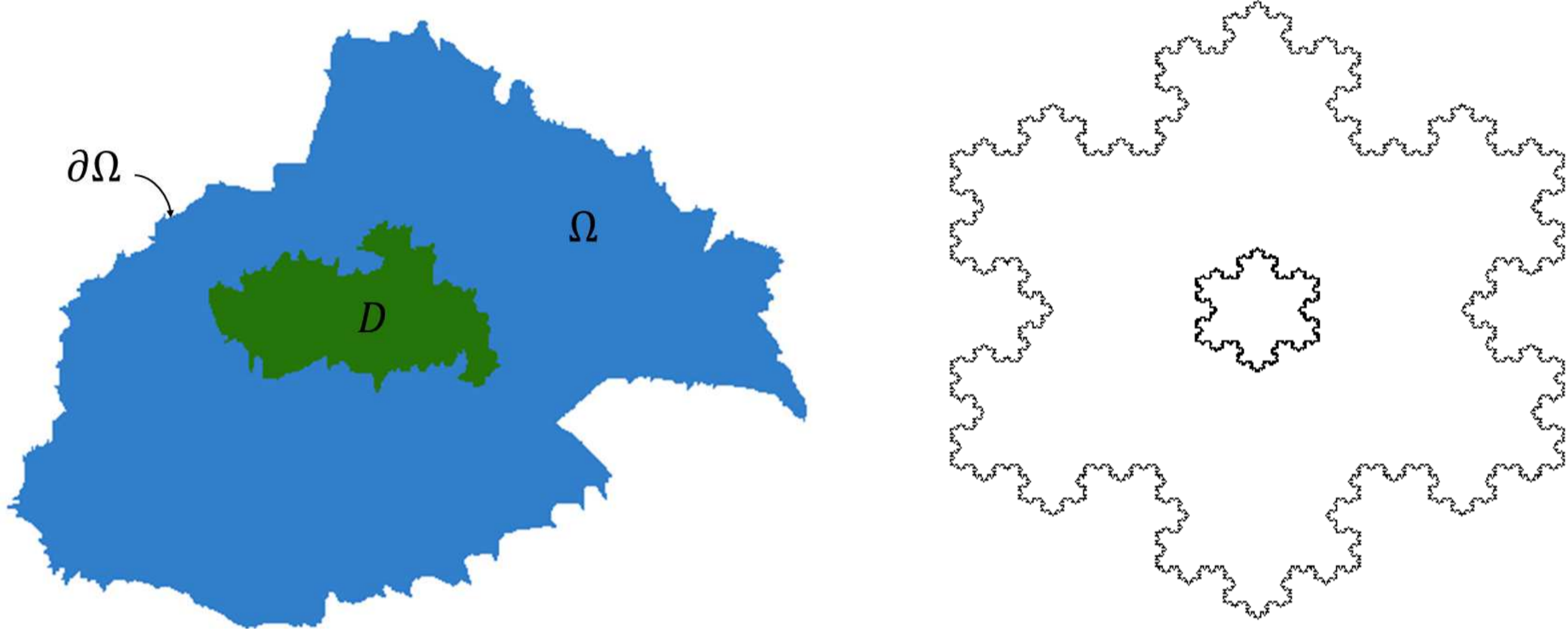
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1 Description of the problem

In medical imaging, cancer tumours are known to have irregular boundaries, which are best modelled by **non-Lipschitz shapes such as fractals**. We consider an imaging problem set on a domain Ω of \mathbb{R}^n , $n \geq 2$, inside which is an inclusion D characterized by its conductivity k :



$$\begin{cases} \nabla \cdot \left((1 + (k-1)\mathbb{1}_D) \nabla u \right) = 0 & \text{on } \Omega, \\ \frac{\partial_i u}{\partial \nu} \Big|_{\partial \Omega} = g, \\ \int_{\Omega} u \, dx = 0. \end{cases} \quad (1)$$

Here,

- Ω is a bounded domain,
- $\partial \Omega$ is connected,
- $D \subset \subset \Omega$ is a domain,
- $k > 0$ and $k \neq 1$,
- $\mathbb{1}_D$ is the indicator function of D .

We wish to **identify the inclusion D from boundary measurement on $\partial \Omega$** when Ω (and D) can be an irregular or fractal domain.

• **Domain regularity:** Ω is a **(two-sided) extension domain**:

- Ω is an H^1 -extension domain: \exists a linear continuous extension operator $H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$,
 - $\Omega^c := \mathbb{R}^n \setminus \bar{\Omega}$ is an H^1 -extension domain,
 - the Hausdorff dimension of $\partial \Omega$ is (locally) in $[n-1, n]$.
- ▶ $\partial \Omega$ can have a **changing Hausdorff dimension**.
▶ Ω cannot have a cusp (conditions (i) and (ii)).

2 Trace and weak normal derivatives

• **Trace operators** – Dirichlet boundary conditions [2]:

$$\text{Tr}_i u(x) = \lim_{r \rightarrow 0} \frac{1}{\lambda^{(n)}(\Omega \cap B_r(x))} \int_{\Omega \cap B_r(x)} u \, dx, \quad u \in H^1(\Omega), \quad x \in \partial \Omega \text{ q.e.},$$

is the **interior trace operator**, where $\lambda^{(n)}$ is Lebesgue's measure on \mathbb{R}^n .

- ▶ $\text{Ker}(\text{Tr}_i) = H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$
- ▶ $\text{Ker}(\text{Tr}_i)^\perp$ is the space of **1-harmonic functions on Ω** .
- ▶ $\mathcal{B}(\partial \Omega) := \text{Tr}_i(H^1(\Omega))$ is a Hilbert space for the norm:

$$\|f\|_{\mathcal{B}(\partial \Omega)} := \min \{ \|v\|_{H^1(\Omega)} \mid \text{Tr}_i v = f \} = \|\tilde{v}\|_{H^1(\Omega)},$$

where \tilde{v} is 1-harmonic on Ω and $\text{Tr}_i \tilde{v} = f$.

Similarly, we define the **exterior trace operator** $\text{Tr}_e : H^1(\Omega^c) \rightarrow \mathcal{B}(\partial \Omega)$.

• **Weak normal derivatives** – Neumann boundary conditions [3]: for $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$, denoting $\mathcal{B}'(\partial \Omega) := \mathcal{L}(\mathcal{B}(\partial \Omega), \mathbb{R})$,

$$\forall v \in H^1(\Omega), \quad \left\langle \frac{\partial_i u}{\partial \nu}, \text{Tr}_i v \right\rangle_{\mathcal{B}'(\partial \Omega), \mathcal{B}(\partial \Omega)} = \int_{\Omega} (\Delta u) v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx,$$

is the **weak interior normal derivative** of u .

Similarly, we define the **weak exterior normal derivative** $\frac{\partial_e u}{\partial \nu} \in \mathcal{B}'(\partial \Omega)$ for $u \in H^1(\Omega^c)$ with $\Delta u \in L^2(\Omega^c)$.

3 Layer potential operators

$$\text{Transmission problem:} \quad \begin{cases} -\Delta u = 0 & \text{on } \mathbb{R}^n \setminus \partial \Omega, \\ \text{Tr}_i u - \text{Tr}_e u = -f & \text{on } \partial \Omega, \\ \frac{\partial_i u}{\partial \nu} - \frac{\partial_e u}{\partial \nu} = g & \text{on } \partial \Omega. \end{cases} \quad (2)$$

• If Ω is **Lipschitz**, $f, g \in L^2(\partial \Omega, \sigma)$ with σ Lebesgue's measure on $\partial \Omega$ and $\int_{\partial \Omega} g \, d\sigma = 0$, then $u = \mathcal{S}_{\partial \Omega} g - \mathcal{D}_{\partial \Omega} f$ [4], where:

$$\begin{aligned} \mathcal{S}_{\partial \Omega} g(x) &= \int_{\partial \Omega} G(x-y) g(y) \, \sigma(dy), \quad x \in \mathbb{R}^n, \\ \mathcal{D}_{\partial \Omega} f(x) &= \int_{\partial \Omega} \frac{\partial G}{\partial \nu_y}(x-y) f(y) \, \sigma(dy), \quad x \in \mathbb{R}^n \setminus \partial \Omega, \end{aligned}$$

with G Green's function for $-\Delta$ on \mathbb{R}^n , are the **single** and **double layer potential operators** respectively.

The Neumann-Poincaré operator is the singular integral:

$$\mathcal{K}_{\partial \Omega} f(x) = \frac{1}{\omega_n} \text{p.v.} \int_{\partial \Omega} \frac{\langle y-x, \nu_y \rangle}{|x-y|^n} f(y) \, \sigma(dy),$$

with ω_n the area of the unit sphere and ν_y the outward normal to $\partial \Omega$.

It holds:

$$\mathcal{K}_{\partial \Omega} = \frac{1}{2} (\text{Tr}_i + \text{Tr}_e) \circ \mathcal{D}_{\partial \Omega}. \quad (3)$$

• If Ω is an **extension domain**, $f \in \mathcal{B}(\partial \Omega)$, $g \in \mathcal{B}'(\partial \Omega)$ with $\langle g, 1 \rangle_{\mathcal{B}', \mathcal{B}} = 0$ (i.e., $g \in \mathcal{B}'_0(\partial \Omega)$), then $u = \mathcal{S}_{\partial \Omega} g - \mathcal{D}_{\partial \Omega} f$, where

$$\begin{aligned} \mathcal{S}_{\partial \Omega} : g \in \mathcal{B}'_0(\partial \Omega) &\mapsto u_{\mathcal{S}} \text{ weak sol. to (2) for } f = 0, \\ \mathcal{D}_{\partial \Omega} : f \in \mathcal{B}(\partial \Omega) &\mapsto u_{\mathcal{D}} \text{ weak sol. to (2) for } g = 0, \end{aligned}$$

are the **generalized layer potential operators**.

The Neumann-Poincaré operator is defined as $\mathcal{K}_{\partial \Omega} : \mathcal{B}(\partial \Omega) \rightarrow \mathcal{B}(\partial \Omega)$ by (3).

4 Imaging results

We generalize imaging results given in [1] in the Lipschitz case.

• **Representation formula:** let Ω be an **extension domain** and $g \in \mathcal{B}'_0(\partial \Omega)$. Then, the unique weak solution to (1) is:

$$u = H - \mathcal{S}_{\partial D} \varphi \quad \text{with} \quad H = \mathcal{S}_{\partial \Omega} g - \mathcal{D}_{\partial \Omega} f,$$

where $f = \text{Tr}_i u|_{\partial \Omega} \in \mathcal{B}(\partial \Omega)$ and $\varphi \in \mathcal{B}'_0(\partial D)$ such that:

$$\left(\frac{k+1}{2k-1} I + \mathcal{K}_{\partial D}^* \right) \varphi = \frac{\partial H}{\partial \nu} \Big|_{\partial D}.$$

• **Monotonous identification:** let $D_1 \subset D_2 \subset \subset \Omega$ be **extension domains** of \mathbb{R}^n . If for some $g \in \mathcal{B}'_0(\partial \Omega) \setminus \{0\}$, the solutions u_1 and u_2 to problem (1) for $D = D_1$ and $D = D_2$ resp. satisfy $\text{Tr}_i u_1|_{\partial \Omega} = \text{Tr}_i u_2|_{\partial \Omega}$, then $D_1 = D_2$.

• **Identification of disks:** let Ω be an **extension domain of \mathbb{R}^2** and $D_1, D_2 \subset \subset \Omega$ be **disks**. If for some $g \in \mathcal{B}'_0(\partial \Omega) \setminus \{0\}$, the solutions u_1 and u_2 to problem (1) for $D = D_1$ and $D = D_2$ resp. satisfy $\text{Tr}_i u_1|_{\partial \Omega} = \text{Tr}_i u_2|_{\partial \Omega}$, then $D_1 = D_2$.

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