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## Dealing with the irregular in inverse and imaging problems

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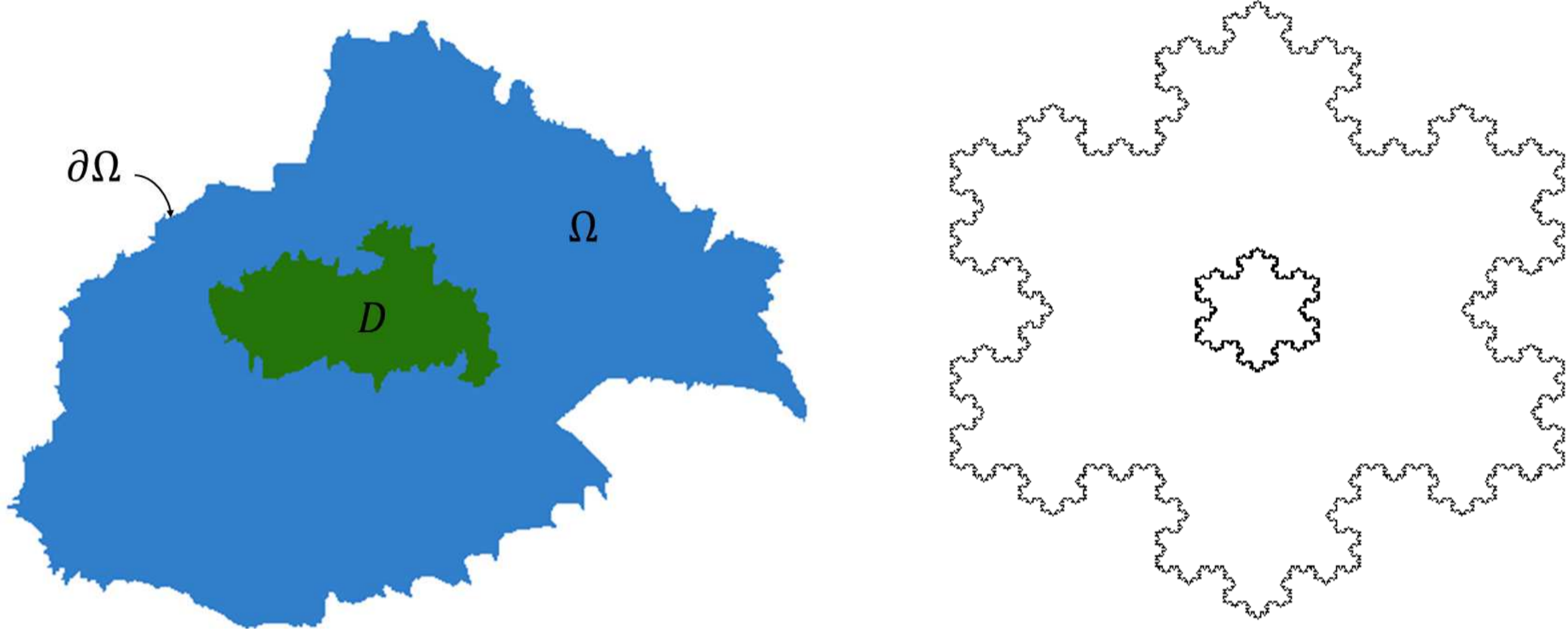
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## 1 Description of the problem

In medical imaging, cancer tumours are known to have irregular boundaries, which are best modelled by **non-Lipschitz shapes such as fractals**. We consider an imaging problem set on a domain  $\Omega$  of  $\mathbb{R}^n$ ,  $n \geq 2$ , inside which is an inclusion  $D$  characterized by its conductivity  $k$ :



$$\begin{cases} \nabla \cdot \left( (1 + (k-1)\mathbb{1}_D) \nabla u \right) = 0 & \text{on } \Omega, \\ \frac{\partial_i u}{\partial \nu} \Big|_{\partial \Omega} = g, \\ \int_{\Omega} u \, dx = 0. \end{cases} \quad (1)$$

Here,

- $\Omega$  is a bounded domain,
- $\partial \Omega$  is connected,
- $D \subset \subset \Omega$  is a domain,
- $k > 0$  and  $k \neq 1$ ,
- $\mathbb{1}_D$  is the indicator function of  $D$ .

We wish to **identify the inclusion  $D$  from boundary measurement on  $\partial \Omega$**  when  $\Omega$  (and  $D$ ) can be an irregular or fractal domain.

• **Domain regularity:**  $\Omega$  is a **(two-sided) extension domain**:

- $\Omega$  is an  $H^1$ -extension domain:  $\exists$  a linear continuous extension operator  $H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$ ,
  - $\Omega^c := \mathbb{R}^n \setminus \bar{\Omega}$  is an  $H^1$ -extension domain,
  - the Hausdorff dimension of  $\partial \Omega$  is (locally) in  $[n-1, n]$ .
- $\partial \Omega$  can have a **changing Hausdorff dimension**.
  - $\Omega$  cannot have a cusp (conditions (i) and (ii)).

## 2 Trace and weak normal derivatives

• **Trace operators** – Dirichlet boundary conditions [2]:

$$\text{Tr}_i u(x) = \lim_{r \rightarrow 0} \frac{1}{\lambda^{(n)}(\Omega \cap B_r(x))} \int_{\Omega \cap B_r(x)} u \, dx, \quad u \in H^1(\Omega), \quad x \in \partial \Omega \text{ q.e.},$$

is the **interior trace operator**, where  $\lambda^{(n)}$  is Lebesgue's measure on  $\mathbb{R}^n$ .

- $\text{Ker}(\text{Tr}_i) = H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$
- $\text{Ker}(\text{Tr}_i)^\perp$  is the space of **1-harmonic functions on  $\Omega$** .
- $\mathcal{B}(\partial \Omega) := \text{Tr}_i(H^1(\Omega))$  is a Hilbert space for the norm:

$$\|f\|_{\mathcal{B}(\partial \Omega)} := \min \{ \|v\|_{H^1(\Omega)} \mid \text{Tr}_i v = f \} = \|\tilde{v}\|_{H^1(\Omega)},$$

where  $\tilde{v}$  is 1-harmonic on  $\Omega$  and  $\text{Tr}_i \tilde{v} = f$ .

Similarly, we define the **exterior trace operator**  $\text{Tr}_e : H^1(\Omega^c) \rightarrow \mathcal{B}(\partial \Omega)$ .

• **Weak normal derivatives** – Neumann boundary conditions [3]: for  $u \in H^1(\Omega)$  with  $\Delta u \in L^2(\Omega)$ , denoting  $\mathcal{B}'(\partial \Omega) := \mathcal{L}(\mathcal{B}(\partial \Omega), \mathbb{R})$ ,

$$\forall v \in H^1(\Omega), \quad \left\langle \frac{\partial_i u}{\partial \nu}, \text{Tr}_i v \right\rangle_{\mathcal{B}'(\partial \Omega), \mathcal{B}(\partial \Omega)} = \int_{\Omega} (\Delta u) v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx,$$

is the **weak interior normal derivative** of  $u$ .

Similarly, we define the **weak exterior normal derivative**  $\frac{\partial_e u}{\partial \nu} \in \mathcal{B}'(\partial \Omega)$  for  $u \in H^1(\Omega^c)$  with  $\Delta u \in L^2(\Omega^c)$ .

## 3 Layer potential operators

$$\text{Transmission problem:} \quad \begin{cases} -\Delta u = 0 & \text{on } \mathbb{R}^n \setminus \partial \Omega, \\ \text{Tr}_i u - \text{Tr}_e u = -f & \text{on } \partial \Omega, \\ \frac{\partial_i u}{\partial \nu} - \frac{\partial_e u}{\partial \nu} = g & \text{on } \partial \Omega. \end{cases} \quad (2)$$

• If  $\Omega$  is **Lipschitz**,  $f, g \in L^2(\partial \Omega, \sigma)$  with  $\sigma$  Lebesgue's measure on  $\partial \Omega$  and  $\int_{\partial \Omega} g \, d\sigma = 0$ , then  $u = \mathcal{S}_{\partial \Omega} g - \mathcal{D}_{\partial \Omega} f$  [4], where:

$$\begin{aligned} \mathcal{S}_{\partial \Omega} g(x) &= \int_{\partial \Omega} G(x-y) g(y) \, \sigma(dy), \quad x \in \mathbb{R}^n, \\ \mathcal{D}_{\partial \Omega} f(x) &= \int_{\partial \Omega} \frac{\partial G}{\partial \nu_y}(x-y) f(y) \, \sigma(dy), \quad x \in \mathbb{R}^n \setminus \partial \Omega, \end{aligned}$$

with  $G$  Green's function for  $-\Delta$  on  $\mathbb{R}^n$ , are the **single** and **double layer potential operators** respectively.

The Neumann-Poincaré operator is the singular integral:

$$\mathcal{K}_{\partial \Omega} f(x) = \frac{1}{\omega_n} \text{p.v.} \int_{\partial \Omega} \frac{\langle y-x, \nu_y \rangle}{|x-y|^n} f(y) \, \sigma(dy),$$

with  $\omega_n$  the area of the unit sphere and  $\nu_y$  the outward normal to  $\partial \Omega$ .

It holds:

$$\mathcal{K}_{\partial \Omega} = \frac{1}{2} (\text{Tr}_i + \text{Tr}_e) \circ \mathcal{D}_{\partial \Omega}. \quad (3)$$

• If  $\Omega$  is an **extension domain**,  $f \in \mathcal{B}(\partial \Omega)$ ,  $g \in \mathcal{B}'(\partial \Omega)$  with  $\langle g, 1 \rangle_{\mathcal{B}', \mathcal{B}} = 0$  (i.e.,  $g \in \mathcal{B}'_0(\partial \Omega)$ ), then  $u = \mathcal{S}_{\partial \Omega} g - \mathcal{D}_{\partial \Omega} f$ , where

$$\begin{aligned} \mathcal{S}_{\partial \Omega} : g \in \mathcal{B}'_0(\partial \Omega) &\mapsto u_{\mathcal{S}} \text{ weak sol. to (2) for } f = 0, \\ \mathcal{D}_{\partial \Omega} : f \in \mathcal{B}(\partial \Omega) &\mapsto u_{\mathcal{D}} \text{ weak sol. to (2) for } g = 0, \end{aligned}$$

are the **generalized layer potential operators**.

The Neumann-Poincaré operator is defined as  $\mathcal{K}_{\partial \Omega} : \mathcal{B}(\partial \Omega) \rightarrow \mathcal{B}(\partial \Omega)$  by (3).

## 4 Imaging results

We generalize imaging results given in [1] in the Lipschitz case.

• **Representation formula:** let  $\Omega$  be an **extension domain** and  $g \in \mathcal{B}'_0(\partial \Omega)$ . Then, the unique weak solution to (1) is:

$$u = H - \mathcal{S}_{\partial D} \varphi \quad \text{with} \quad H = \mathcal{S}_{\partial \Omega} g - \mathcal{D}_{\partial \Omega} f,$$

where  $f = \text{Tr}_i u|_{\partial \Omega} \in \mathcal{B}(\partial \Omega)$  and  $\varphi \in \mathcal{B}'_0(\partial D)$  such that:

$$\left( \frac{k+1}{2k-1} I + \mathcal{K}_{\partial D}^* \right) \varphi = \frac{\partial H}{\partial \nu} \Big|_{\partial D}.$$

• **Monotonous identification:** let  $D_1 \subset D_2 \subset \subset \Omega$  be **extension domains** of  $\mathbb{R}^n$ . If for some  $g \in \mathcal{B}'_0(\partial \Omega) \setminus \{0\}$ , the solutions  $u_1$  and  $u_2$  to problem (1) for  $D = D_1$  and  $D = D_2$  resp. satisfy  $\text{Tr}_i u_1|_{\partial \Omega} = \text{Tr}_i u_2|_{\partial \Omega}$ , then  $D_1 = D_2$ .

• **Identification of disks:** let  $\Omega$  be an **extension domain of  $\mathbb{R}^2$**  and  $D_1, D_2 \subset \subset \Omega$  be **disks**. If for some  $g \in \mathcal{B}'_0(\partial \Omega) \setminus \{0\}$ , the solutions  $u_1$  and  $u_2$  to problem (1) for  $D = D_1$  and  $D = D_2$  resp. satisfy  $\text{Tr}_i u_1|_{\partial \Omega} = \text{Tr}_i u_2|_{\partial \Omega}$ , then  $D_1 = D_2$ .

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