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# BLOW-UP ANALYSIS FOR THE SOLUTIONS OF AN EQUATION WITH BOUNDARY CONDITION.

SAMY SKANDER BAHOURA

ABSTRACT. Under some conditions we give a blow-up analysis for solutions of an equation with Dirichlet boundary condition.

## 1. INTRODUCTION AND MAIN RESULT

We set  $\Delta = -\nabla^i \nabla_i$  the Laplace-Beltrami operator on a connected compact Riemannian manifold with boundary  $(M, g)$  of dimension  $n \geq 3$  with metric  $g$ .

We consider the following equation:

$$(P) \begin{cases} \Delta u + hu = Vu^{N-1}, & u > 0 & \text{in } M, \\ u = 0 & & \text{on } \partial M. \end{cases}$$

Here, we assume the solutions in the sense of distributions and here also in  $C^{2,\alpha}(\bar{M})$ ,  $\alpha > 0$  and the operator  $\Delta + h$  is coercive in  $H_0^1(M)$  with  $h$  a smooth function and  $0 < a \leq V \leq b < +\infty$ ,  $V$  is a smooth function and  $\|V\|_{C^\alpha} \leq A$  and  $N = \frac{2n}{n-2}$  the critical Sobolev exponent.

In [1-28] we have various estimates and inequalities of type  $\sup + \inf$  and  $\sup \times \inf$ . Here we look to blow-up analysis for regular solutions of the previous equation.

Here we assume:

$$\sup_M u \cdot \int_M u^{N-1} dV_g \leq C \quad (*)$$

We have:

$$\|u\|_N \leq \|u\|_\infty^{\frac{1}{N}} \|u\|_{N-1}^{1-\frac{1}{N}} \leq |M|^{\frac{1}{N}} \|u\|_\infty$$

Thus, the condition (\*) is equivalent to the condition:

$$\|u\|_\infty^{\frac{1}{N}} \|u\|_{N-1}^{1-\frac{1}{N}} \leq C.$$

This condition is an  $L^\infty - L^{N-1}$ -condition which is intermediate between a  $\sup \times \inf$  condition and  $L^N$  condition, see [1] and [7, 8].

Note that in [7, 8] the condition (\*) is satisfied. Thus, this condition is a necessary condition. In [7, 8] it is an exemple where this condition (\*) holds.

Our main results are:

We have the following blow-up analysis on all  $\bar{M}$ :

**Theorem 1.1.** *Assume (\*) and  $\sup_M u_i \rightarrow +\infty$ , then there is a subsequence denoted  $(u_j)$  of (P) and there are  $k$  points inside  $M$ ,  $x_1, \dots, x_k \in M$ , and  $m$  points on  $\partial M$ ,  $y_1, \dots, y_m \in \partial M$ , which are concentration points of  $u_j$ , and  $u_j \rightarrow 0$  in  $C_{loc}^2(\bar{M} - \{x_1, \dots, x_k, y_1, \dots, y_m\})$ .*

To prove this theorem, we use condition (\*) for which there is a concentration of measures inside  $M$  and on  $\partial M$ . As mentionned before, the condition (\*) holds in [7, 8] and thus it is a necessary condition. There is an exemple of blow-up phenomenon which imply the condition (\*). Thus, we must suppose this condition (\*) to know if we have blow-up phenomenon for solutions of the problem (P).

And we have the following compactness result on all  $\bar{M}$ :

**Theorem 1.2.** We assume (\*). Then there is  $\epsilon_n > 0$  such that if  $b \cdot C < \epsilon_n$ , then there is  $c > 0$  such that:

$$\sup_M u_i \leq c, \forall i \in \mathbb{N}.$$

## 2. PROOFS OF THE THEOREMS

Proofs of Theorems 1.1:

We use two measures, on the locally compact space  $M$ ,  $\mu_1$ , and on the boundary  $\partial M$ ,  $\mu_2$ , we use De Giorgi-Nash-Moser iterate scheme, see [1] and the elliptic estimates see [19]. we assume:

$$\sup_M u_i \cdot \int_M u_i^{N-1} dV_g \leq C \quad (*)$$

which imply:

$$\int_M V_i u_i^N dV_g \leq bC$$

Which it is sufficient for the blow-up analysis inside  $M$ .

Step 1:

We look to the concentration of measure inside  $M$ :

$$\int_M (V_i u_i^N \cdot \varphi) dV_g \rightarrow \mu_1(\varphi) \geq 0, \forall \varphi \in C_c^0(M), \varphi \geq 0.$$

Here we use the De Giorgi-Nash-Moser iterate scheme, see [1] and the paper [7,8] :

also, we have with the fact that  $\sup_M u_i \rightarrow +\infty \Rightarrow \int_M u_i^{N-1} dV_g \rightarrow 0$  and up to a subsequence  $u_i \rightarrow 0$  almost everywhere.

a)

$$x_0 \in M, \mu_1(\{x_0\}) < \epsilon_n \leq \frac{\omega_n}{2^n} \Rightarrow \exists \delta > 0, \int_{B(x_0, \delta)} V_i u_i^N dV_g < \epsilon_n \leq \frac{\omega_n}{2^n},$$

$$x_0 \in M, \int_{B(x_0, \delta)} V_i u_i^N dV_g < \epsilon_n \leq \frac{\omega_n}{2^n} \Rightarrow \|u_i\|_{L^\infty(B(x_0, \delta/2))} \leq C'$$

b) there is a finite number of points  $x_j \in M$  such that:

$$\mu_1(\{x_j\}) \geq \epsilon_n, \int_{B(x_j, \delta)} V_i u_i^N dV_g \geq \epsilon_n,$$

For  $\delta \rightarrow 0$  small enough, we have:

$$x_j \in M, \exists (x_j^i)_i, x_j^i \rightarrow x_j, u_i(x_j^i) \rightarrow +\infty.$$

and because  $u_i \rightarrow 0$  almost everywhere on  $M$ , we have on each compact set  $K \subset M - \{x_j, j = 1, \dots, k\}$ :

$$\sup_K u_i \rightarrow 0.$$

Also we have the convergence to a measure: sum of Dirac measures.

Step 2:

We assume the duality theorem for  $v_i = \sup_M u_i \cdot u_i$ , which assert that:

$$\|\nabla v_i\|_{L^q(M)} \leq C_q < +\infty, \forall q, 1 \leq q < \frac{n}{n-1},$$

because  $u_i \rightarrow 0$  on each compact set of  $M$  and the Holder inequality applied for,  $r, \frac{n}{n-1} > r > q \geq 1$ , we obtain near  $\partial M$  (we prove it in charts of points of the boundary):

$$\|\nabla v_i\|_r = o(1), \forall r, 1 \leq r < \frac{n}{n-1}.$$

For the duality theorem:

In fact, we solve  $\Delta z + hz = \operatorname{div} f$ ,  $f \in L^{q'}$  with Dirichlet boundary condition ( $L = \Delta + h$  is coercive, we can solve in Hilbert space), we have:  $z \in L^\infty$  with  $\|z\|_{L^\infty} \leq c_q \|f\|_{q'}$ , see Gilbarg-Trudinger, [19]. We write:

(we smooth the solutions  $f \rightarrow f_j \in C_c^\infty(M)$ ,  $z \rightarrow z_j \in C^\infty$ ,  $f_j \rightarrow_{L^{q'}} f$  and thus  $z_j$  is a Cauchy sequence in  $L^\infty$ ):

$$\int_M f \cdot \nabla v_i = \int_M \nabla z \cdot \nabla v_i + hz v_i = \int_M z v_i u_i^{N-2},$$

thus,

$$\left| \int_M f \cdot \nabla v_i \right| \leq C \|z\|_{L^\infty} \leq C \cdot c_q \|f\|_{q'},$$

Thus,

$$\|\nabla v_i\|_q \leq c'_q < +\infty.$$

(The functions  $z \in L^2(M)$  uniformly, because we multiply the equation of  $z$  by  $z$  and use Cauchy-Schwarz inequality and the Coercivity of the operator).

Also, we use another type of duality theorem:

We solve (in Hilbert space):

$$\Delta \tilde{z} + h\tilde{z} = g \in L^{q'/2}$$

We can take  $g \in L^{q'} \subset L^{q'/2}$ , because  $q' > n > 2$  to take the norm  $L^2$ .

we have (see Gilbarg-Trudinger, [19]):  $\tilde{z} \in L^\infty$  with  $\|\tilde{z}\|_{L^\infty} \leq \tilde{c}_q \|g\|_{q'/2}$ . We write:

We smooth the solutions:

$$\int_M g \cdot v_i = \int_M \nabla \tilde{z} \cdot \nabla v_i + h\tilde{z} v_i = \int_M \tilde{z} v_i u_i^{N-2},$$

thus,

$$\left| \int_M g \cdot v_i \right| \leq C \|\tilde{z}\|_{L^\infty} \leq C \cdot \tilde{c}_q \|g\|_{q'/2},$$

Thus,

$$\|v_i\|_s \leq \tilde{c}'_s < +\infty, 1 \leq s < \frac{n}{n-2}.$$

And,

$$\|v_i\|_q \leq \tilde{c}'_q < +\infty, 1 \leq q < \frac{n}{n-1}.$$

Also, we can use locally, the Poincaré inequality to have a uniform bound of  $\int_M v_i^q$ ,  $1 \leq q < \frac{n}{n-1}$ . (Another fact that this quantity is uniformly bounded).

Let's consider  $\eta$  a cutoff function (in an open set of a chart), then, we apply the th. of duality to  $\nabla(v_i \eta)$  in  $M$ , but with  $\eta$ , we are in the open set of the chart:

$$\Delta(v_i \eta) + h(v_i \eta) = V_i v_i u_i^{N-2} \eta + \nabla v_i \cdot \nabla \eta + v_i \Delta \eta,$$

$$\|\nabla(v_i \eta)\|_q \leq c'_q + C \int_M v_i,$$

but in the chart, we have a Lipschitz domain, we can use the Poincaré inequality:

$$\|v_i \eta\|_q \leq c''_q \|\nabla(v_i \eta)\|_q \leq \tilde{c}_q + C \int_M v_i,$$

Thus, in charts with cutoff functions and partition of unity we obtain:

$$\|v_i\|_q \leq \bar{c}_q + C \int_M v_i, \forall q, 1 \leq q < \frac{n}{n-1}.$$

By contradiction, if  $\int_M v_i \rightarrow +\infty$ , then we consider  $w_i = \frac{v_i}{\int_M v_i}$  is bounded in  $\dot{H}_q^1(M)$  and by the Sobolev embedding, thus converge to a constant  $w \equiv k$ , in  $\dot{H}_r^1(M)$ ,  $1 \leq r < q$ , but

$tr(w_i) = 0 \rightarrow tr(w) = k$ , this is in contradiction with  $\int_M v_i \leq C \|v_i\|_r = o(1) \int_M v_i, 1 = o(1)$ . Thus,  $\int_M v_i \leq C < +\infty, \forall i$ . Thus:

$$\|v_i\|_q \leq \bar{c}_q, \forall q, 1 \leq q < \frac{n}{n-1}.$$

Also, near the boundary, we have (we prove it around points of the boundary in charts):

$$\|v_i\|_r = o(1), \forall r, 1 \leq r < \frac{n}{n-1}.$$

Another fact about the boundedness of  $\int_M v_i$ :

Let  $\eta$  solution of:

$$\Delta \eta + h \eta = 1, \text{ in } M, \text{ and } h = 0 \text{ on } \partial M.$$

Multiplying the equation of  $v_i$  by  $h$  and integrate by parts, we obtain:

$$\left| \int_M h v_i \eta \right| \leq C,$$

Multiplying the equation of  $h$  by  $v_i$  and integrate by parts, we obtain:

$$0 < \int_M v_i \leq C.$$

Step 3:

We consider a concentration of the measure for:

$$\int_M V_i v_i u_i^{N-2} \varphi dx \rightarrow \mu_3(\varphi) \geq 0, \varphi \geq 0, \varphi \in C_c^0(M).$$

We have, if we consider:  $\Delta v_i + (h - V_i u_i^{N-2}) v_i = 0$  and  $v_i \in L^q, q > 1$  uniformly, and around  $x_0$  with  $\mu_1(x_0) < \epsilon_n \Rightarrow u_i \in L^\infty(B(x_0, r)), r > 0$  uniformly. thus we apply the Harnack inequality to  $v_i$ , we have  $v_i \in L^\infty(B(x_0, r')), r > r' > 0$  uniformly. Thus,  $\mu_3(x_0) < \epsilon_n$ .

Thus,

$$\mu_1(x_0) < \epsilon_n \Leftrightarrow \mu_3(x_0) < \epsilon_n.$$

and,

$$\mu_1(x_j) \geq \epsilon_n > 0 \Leftrightarrow \mu_3(x_j) \geq \epsilon_n > 0.$$

Thus,

$$\text{supp} \mu_1 = \text{supp} \mu_3 = \{x_1, \dots, x_k\}.$$

Thus,  $v_i \rightarrow v$  in  $W_0^{1,q}(M)$  and locally uniformly on  $M$  with  $v$  solution in the sense of distributions of:

$$\Delta v + h v = \sum_{j=1}^k \mu_{3,j} \delta_{x_j}, \mu_{3,j} = \mu_3(x_j) \geq \epsilon_n > 0.$$

But, the operator  $\Delta + h$  is coercive and thus, we have the existence of the Green function with Dirichlet boundary condition,  $G$  of this operator. Comparing  $v$  and  $G_0(\cdot) = \sum_{j=1}^k \mu_{3,j} G(x_j, \cdot)$ , we have:  $v = G_0$  almost everywhere.

Thus,

$$v_i \rightarrow G_0, \text{ in } W_0^{1,q}(M) \cap C^2(M - \{x_1, \dots, x_k\}), \text{ with } G_0 \in C^2(\bar{M} - \{x_1, \dots, x_k\}).$$

Thus, as in dimension 2, ( $G_0$  is smooth up to the boundary), we have:

$$\|\nabla(v_i - G_0)\|_q = o(1), \|v_i - G_0\|_q = o(1), 1 \leq q < n/(n-1).$$

Thus, we have the same situation as in dimension 2, except that we have singularities inside  $M$ .

Also, we have near the boundary (because for the tests functions near the boundary we have estimates of type,  $\eta_{\epsilon'}$  with  $\|\nabla\eta_{\epsilon'}\|_{\infty} \leq C/(\epsilon')^2$ ):

$$|\Omega_{\epsilon'}| \leq (\epsilon')^n \Rightarrow \|\nabla v_i\|_q + \|v_i\|_q = O((\epsilon')^n), \quad i \geq i_0(\epsilon'), \quad n \geq 3.$$

Step 4:

Now, we look to the case of the boundary:

By the maximum principle and the fact that  $\Delta + h$  is coercive and the Dirichlet boundary condition, we have:  $\partial_{\nu} u_i \geq 0$ .

By the Stokes formula and the duality theorem, see the papers of dimension 2, we have concentration phenomenon on the boundary  $\partial M$  of  $v_i = \sup_M u_i \cdot u_i$ :

$$\int_{\partial M} \partial_{\nu} (\sup_M u_i \cdot u_i) d\sigma \leq C'',$$

$$\int_{\partial M} \partial_{\nu} (\sup_M u_i \cdot u_i) \cdot \varphi d\sigma \rightarrow \mu_2(\varphi) \geq 0, \quad \forall \varphi \in C^0(\partial M), \varphi \geq 0.$$

We use the equation with  $v_i$  and we consider a test function  $\eta_{\epsilon}$  by a chart near  $y_0 \in \partial M$ , and a Dirichlet problem:

$$\Delta \tilde{\eta}_{\epsilon} + h \tilde{\eta}_{\epsilon} = 0 \text{ in } M, \quad \tilde{\eta}_{\epsilon} = \eta_{\epsilon} \text{ on } \partial M.$$

For this as in dimension 2, we solve:

$$\Delta \bar{\eta}_{\epsilon} + h \bar{\eta}_{\epsilon} = \Delta \eta_{\epsilon} + h \eta_{\epsilon} \text{ in } M, \quad \bar{\eta}_{\epsilon} = 0 \text{ on } \partial M,$$

and we take  $\tilde{\eta}_{\epsilon} = -\bar{\eta}_{\epsilon} + \eta_{\epsilon}$ .

$$\Delta(v_i \tilde{\eta}_{\epsilon}) = V_i v_i u_i^{N-2} \tilde{\eta}_{\epsilon} + \nabla v_i \cdot \nabla \tilde{\eta}_{\epsilon},$$

and we integrate.

$$\int_M \Delta(v_i \tilde{\eta}_{\epsilon}) = \int_{\partial M} \partial_{\nu}(v_i \tilde{\eta}_{\epsilon}) = \int_{\partial M} \partial_{\nu}(v_i) \tilde{\eta}_{\epsilon} \rightarrow \mu_2(\tilde{\eta}_{\epsilon}) \leq \mu_2(J_{\epsilon}) < \epsilon_n,$$

and near the boundary, we have:

$$\|\nabla v_i\|_r = o(1), \quad \|v_i\|_r = o(1), \quad \forall r, \quad 1 \leq r < \frac{n}{n-1}.$$

Thus,

$$\sup_M u_i \int_{B(y_0, \delta)} V_i u_i^{N-1} dV_g \leq \int_M V_i v_i u_i^{N-2} \tilde{\eta}_{\epsilon} < \epsilon_n,$$

In fact we have by the duality theorem in a small neighborhood of  $y_0 \in \partial M$  (and the holder inequality) and the convergence inside  $M$ :

$$y_0 \in \partial M, \mu_2(\{y_0\}) < \epsilon_n \Rightarrow \sup_M u_i \int_{B(y_0, \delta)} V_i u_i^{N-1} dV_g < \epsilon_n \Rightarrow \int_{B(y_0, \delta)} V_i u_i^N dV_g < \epsilon_n$$

First we consider the equation satisfied by  $v_i$  to use the measure  $\mu_2$ , after we consider the equation satisfied by  $u_i$ .

and by the De Giorgi-Nash-Moser iterate scheme applied to  $u_i$  with respect to initial equation, we have:

$$y_0 \in \partial M, \int_{B(y_0, \delta)} V_i u_i^N dV_g < \epsilon_n \leq \frac{\omega_n}{2^n} \Rightarrow \|u_i\|_{L^{\infty}(B(y_0, \delta/2))} \leq C'$$

and,

There is a finite number of points  $y_j \in \partial M$ , such that:

$$\mu_2(\{y_j\}) \geq \epsilon_n > 0.$$

$$\mu_2(\{y_j\}) \geq \epsilon_n > 0 \Rightarrow \forall \delta > 0, \sup_M u_i \cdot \int_{B(y_j, \delta/2)} V_i u_i^{N-1} dV_g = \int_{B(y_j, \delta)} V_i v_i u_i^{N-2} \geq \frac{\epsilon_n}{2},$$

For  $\delta \rightarrow 0$  small enough and because  $\|v_i\|_1 = o(1)$  near the boundary, we have:

$$y_j \in \partial M, \exists (y_j^i)_i, y_j^i \rightarrow y_j, u_i(y_j^i) \rightarrow +\infty.$$

and, for all compact  $\tilde{K} \subset \bar{M} - \{x_j, j = 1, \dots, k, y_j, j = 1, \dots, m\}$

$$\sup_{\tilde{K}} u_i \rightarrow 0.$$

Also, we have the convergence to a measure on the boundary: sum of Dirac measures on the boundary.

By the Harnack inequality on the boundary see, Gilbarg-Trudinger [19], we have  $v_i \rightarrow G_0$  in  $C^2(\bar{M} - \{x_1, \dots, x_k, y_1, \dots, y_m\})$ , up to the boundary.

We can see that locally, we have the following asymptotic estimate outside the blow-up points:

$$u_j(x) \approx \frac{1}{\sup_M u_j}.$$

And near the boundary and outside the blow-up points:

$$u_j(x) \approx \frac{o(1)}{\sup_M u_j}.$$

And,

$$\sup_M u_j \cdot u_j(\cdot) \rightarrow G_0 = \sum_{j=1}^k \mu_{3,j} G(x_j, \cdot)$$

in  $C^2(\bar{M} - \{x_1, \dots, x_k, y_1, \dots, y_m\})$  with  $\mu_{3,j} = \mu_3(x_j) \geq \epsilon_n > 0$ .

Proofs of Theorems 1.2:

In the proof of the theorem 1.1. we have determined  $\epsilon_n > 0$  the mass's reference. By contradiction, if  $\sup_M u_i \rightarrow +\infty$ , there are concentration of measures on  $M$  and on  $\partial M$  with the total mass  $b \cdot C \geq \epsilon_n > 0$ , which is not possible. Thus, the compactness result.

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DEPARTMENT OF MATHEMATICS, PIERRE AND MARIE CURIE UNIVERSITY, 75005, PARIS, FRANCE.  
*Email address:* samybahoura@gmail.com