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# BLOW-UP ANALYSIS FOR THE SOLUTIONS OF AN EQUATION WITH BOUNDARY CONDITION.

SAMY SKANDER BAHOURA

ABSTRACT. Under some conditions we give a blow-up analysis for solutions of an equation with Dirichlet boundary condition.

## 1. INTRODUCTION AND MAIN RESULT

We set  $\Delta = -\nabla^i \nabla_i$  the Laplace-Beltrami operator on a connected compact Riemannian manifold with boundary  $(M, g)$  of dimension  $n \geq 3$  with metric  $g$ .

We consider the following equation:

$$(P) \begin{cases} \Delta u + hu = Vu^{N-1}, & u > 0 & \text{in } M, \\ u = 0 & & \text{on } \partial M. \end{cases}$$

Here, we assume the solutions in the sense of distributions and here also in  $C^{2,\alpha}(\bar{M})$ ,  $\alpha > 0$  and the operator  $\Delta + h$  is coercive in  $H_0^1(M)$  with  $h$  a smooth function and  $0 < a \leq V \leq b < +\infty$ ,  $V$  is a smooth function and  $\|V\|_{C^\alpha} \leq A$  and  $N = \frac{2n}{n-2}$  the critical Sobolev exponent.

In [1-26] we have various estimates and inequalities of type  $\sup + \inf$  and  $\sup \times \inf$ . Here we look to blow-up analysis for regular solutions of the previous equation.

Here we assume:

$$\sup_M u \cdot \int_M u^{N-1} dV_g \leq C \quad (*)$$

This condition is an  $L^\infty - L^{N-1}$ -condition which is intermediate between a  $\sup \times \inf$  condition and  $L^N$  condition, see [1] and [7].

Note that in [7] the condition (\*) is satisfied. Thus, this condition is a necessary condition. In [7] it is an exemple where this condition (\*) holds.

Our main result is:

**Theorem 1.1.** *Assume (\*) and  $\sup_M u_i \rightarrow +\infty$ , then there is a subsequence denoted  $(u_j)$  of (P) and there are  $k$  points inside  $M$ ,  $x_1, \dots, x_k \in M$ , and  $m$  points on  $\partial M$ ,  $y_1, \dots, y_m \in \partial M$ , which are concentration points of  $u_j$ , and  $u_j \rightarrow 0$  in  $C_{loc}^2(\bar{M} - \{x_1, \dots, x_k, y_1, \dots, y_m\})$ .*

To prove this theorem, we use condition (\*) for which there is a concentration of measures inside  $M$  and on  $\partial M$ . As mentionned before, the condition (\*) hold in [7] and thus it is a necessary condition. There is an exemple of blow-up phenomenon which imply the condition (\*). Thus, we must suppose this condition (\*) to know if we have blow-up phenomenon for solutions of the problem (P).

## 2. PROOFS OF THE THEOREM

Proofs of Theorems 1.1:

We use two measures, on the locally compact space  $M$ ,  $\mu_1$ , and on the boundary  $\partial M$ ,  $\mu_2$ , we use De Giorgi-Nash-Moser iterate scheme, see [1] and the elliptic estimates see [18].

we assume:

$$\sup_M u_i \cdot \int_M u_i^{N-1} dV_g \leq C \quad (*)$$

which imply:

Step 1:

$$\int_M V_i u_i^N dV_g \leq C$$

$$\int_M (V_i u_i^N \cdot \varphi) dV_g \rightarrow \mu_1(\varphi) \geq 0, \forall \varphi \in C_c^0(M), \varphi \geq 0.$$

Which is sufficient for the blow-up analysis inside  $M$ . Here we use the De Giorgi-Nash-Moser iterate scheme, see [1] and the paper [7]:

also, we have with the fact that  $\sup_M u_i \rightarrow +\infty \Rightarrow \int_M u_i^{N-1} dV_g \rightarrow 0$  and up to a subsequence  $u_i \rightarrow 0$  almost everywhere.

a)

$$x_0 \in M, \mu_1(\{x_0\}) < \epsilon_n \leq \frac{\omega_n}{2^n} \Rightarrow \exists \delta > 0, \int_{B(x_0, \delta)} V_i u_i^N dV_g < \epsilon_n \leq \frac{\omega_n}{2^n},$$

$$x_0 \in M, \int_{B(x_0, \delta)} V_i u_i^N dV_g < \epsilon_n \leq \frac{\omega_n}{2^n} \Rightarrow \|u_i\|_{L^\infty(B(x_0, \delta/2))} \leq C'$$

b) there is a finite number of points  $x_j \in M$  such that:

$$\mu_1(\{x_j\}) \geq \epsilon_n, \int_{B(x_j, \delta)} V_i u_i^N dV_g \geq \epsilon_n,$$

For  $\delta \rightarrow 0$  small enough, we have:

$$x_j \in M, \exists (x_j^i)_i, x_j^i \rightarrow x_j, u_i(x_j^i) \rightarrow +\infty.$$

and because  $u_i \rightarrow 0$  almost everywhere on  $M$ , we have on each compact set  $K \subset M - \{x_j, j = 1, \dots, k\}$ :

$$\sup_K u_i \rightarrow 0.$$

Step 2:

We assume the duality theorem for  $v_i = \sup_M u_i \cdot u_i$ , which assert that:

$$\|\nabla v_i\|_{L^q(M)} \leq C_q < +\infty, \forall q, 1 \leq q < \frac{n}{n-1},$$

because  $u_i \rightarrow 0$  on each compact set of  $M$  and the Holder inequality applied for,  $r, \frac{n}{n-1} > r > q \geq 1$ , we obtain near  $\partial M$  (we prove it in charts of points of the boundary):

$$\|\nabla v_i\|_r = o(1), \forall r, 1 \leq r < \frac{n}{n-1}.$$

For the duality theorem:

In fact, we solve  $\Delta z + hz = \text{div} f, f \in L^{q'}$  with Dirichlet boundary condition ( $L = \Delta + h$  is coercive, we can solve in Hilbert space), we have:  $z \in L^\infty$  with  $\|z\|_{L^\infty} \leq c_q \|f\|_{q'}$ . We write:

$$\int_M f \cdot \nabla v_i = \int_M \nabla z \cdot \nabla v_i + hz v_i = \int_M z v_i u_i^{N-2},$$

thus,

$$\left| \int_M f \cdot \nabla v_i \right| \leq C \|z\|_{L^\infty} \leq C \cdot c_q \|f\|_{q'},$$

Thus,

$$\|\nabla v_i\|_q \leq c'_q < +\infty.$$

We use locally, the Poincaré inequality to have a uniform bound of  $\int_M v_i^q, 1 \leq q < \frac{n}{n-1}$ .

Let's consider  $\eta$  a cutoff function (in an open set of a chart), then, we apply the previous theorem of duality to  $v_i \eta$  in  $M$ , but with  $\eta$ , we are in the open set of the chart:

$$\Delta(v_i \eta) + h(v_i \eta) = V_i v_i u_i^{N-2} \eta + \nabla v_i \cdot \nabla \eta,$$

$$\|\nabla(v_i \eta)\|_q \leq c'_q$$

but in the chart, we have a Lipschitz domain, we can use the Poincaré inequality:

$$\|v_i \eta\|_q \leq c_q'' \|\nabla(v_i \eta)\|_q \leq \tilde{c}_q,$$

Thus, in charts with cutoff functions and partition of unity we obtain:

$$\|v_i\|_q \leq \bar{c}_q, \forall q, 1 \leq q < \frac{n}{n-1}.$$

Also, near the boundary, we have (we prove it around points of the boundary in charts):

$$\|v_i\|_r = o(1), \forall r, 1 \leq r < \frac{n}{n-1}.$$

By the Stokes formula and the duality theorem, see the papers of dimension 2, we have concentration phenomenon on the boundary  $\partial M$  of  $v_i = \sup_M u_i \cdot u_i$ :

$$\int_{\partial M} \partial_\nu(\sup_M u_i \cdot u_i) d\sigma \leq C'',$$

$$\int_{\partial M} \partial_\nu(\sup_M u_i \cdot u_i) \cdot \varphi d\sigma \rightarrow \mu_2(\varphi) \geq 0, \forall \varphi \in C^0(\partial M), \varphi \geq 0.$$

We use the equation with  $v_i$  and we consider a test function  $\eta_\epsilon$  by a chart near  $y_0 \in \partial M$ , and a Dirichlet problem:

$$\Delta \tilde{\eta}_\epsilon + h \tilde{\eta}_\epsilon = 0 \text{ in } M, \tilde{\eta}_\epsilon = \eta_\epsilon \text{ on } \partial M.$$

For this as in dimension 2, we solve:

$$\Delta \bar{\eta}_\epsilon + h \bar{\eta}_\epsilon = \Delta \eta_\epsilon + h \eta_\epsilon \text{ in } M, \bar{\eta}_\epsilon = 0 \text{ on } \partial M,$$

and we take  $\tilde{\eta}_\epsilon = -\bar{\eta}_\epsilon + \eta_\epsilon$ .

$$\Delta(v_i \tilde{\eta}_\epsilon) = V_i v_i u_i^{N-2} \tilde{\eta}_\epsilon + \nabla v_i \cdot \nabla \tilde{\eta}_\epsilon,$$

and we integrate.

$$\int_M \Delta(v_i \tilde{\eta}_\epsilon) = \int_{\partial M} \partial_\nu(v_i \tilde{\eta}_\epsilon) = \int_{\partial M} \partial_\nu(v_i) \tilde{\eta}_\epsilon \rightarrow \mu_2(\tilde{\eta}_\epsilon) \leq \mu_2(J_\epsilon) < \epsilon_n,$$

and near the boundary, we have:

$$\|\nabla v_i\|_r = o(1), \|v_i\|_r = o(1), \forall r, 1 \leq r < \frac{n}{n-1}.$$

Thus,

$$\sup_M u_i \int_{B(y_0, \delta)} V_i u_i^{N-1} dV_g \leq \int_M V_i v_i u_i^{N-2} \tilde{\eta}_\epsilon < \epsilon_n,$$

In fact we have by the duality theorem in a small neighborhood of  $y_0 \in \partial M$  (and the holder inequality) and the convergence inside  $M$ :

$$y_0 \in \partial M, \mu_2(\{y_0\}) < \epsilon_n \Rightarrow \sup_M u_i \int_{B(y_0, \delta)} V_i u_i^{N-1} dV_g < \epsilon_n \Rightarrow \int_{B(y_0, \delta)} V_i u_i^N dV_g < \epsilon_n$$

First we consider the equation satisfied by  $v_i$  to use the measure  $\mu_2$ , after we consider the equation satisfied by  $u_i$ .

and by the De Giorgi-Nash-Moser iterate scheme applied to  $u_i$  with respect to initial equation, we have:

$$y_0 \in \partial M, \int_{B(y_0, \delta)} V_i u_i^N dV_g < \epsilon_n \leq \frac{\omega_n}{2^n} \Rightarrow \|u_i\|_{L^\infty(B(y_0, \delta/2))} \leq C'$$

and,

There is a finite number of points  $y_j \in \partial M$ , such that:

$$\mu_2(\{y_j\}) \geq \epsilon_n > 0.$$

$$\mu_2(\{y_j\}) \geq \epsilon_n > 0 \Rightarrow \forall \delta > 0, \sup_M u_i \cdot \int_{B(y_j, \delta/2)} V_i u_i^{N-1} dV_g = \int_{B(y_j, \delta)} V_i v_i u_i^{N-2} \geq \frac{\epsilon_n}{2},$$

For  $\delta \rightarrow 0$  small enough and because  $\|v_i\|_1 = o(1)$  near the boundary, we have:

$$y_j \in \partial M, \exists (y_j^i)_i, y_j^i \rightarrow y_j, u_i(y_j^i) \rightarrow +\infty.$$

and, for all compact  $\tilde{K} \subset \bar{M} - \{x_j, j = 1, \dots, k, y_j, j = 1, \dots, m\}$

$$\sup_{\tilde{K}} u_i \rightarrow 0.$$

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