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POSITIVE FORMULA FOR THE PRODUCT OF CONJUGACY CLASSES ON THE UNITARY GROUP

QUENTIN FRANÇOIS AND PIERRE TARRAGO

ABSTRACT. The convolution product of two generic conjugacy classes of the unitary group U_n is described by a probability distribution on the space of central measures which admits a density. Relating the convolution to the quantum Littlewood-Richardson coefficients and using recent results describing those coefficients, we give a manifestly positive formula for this density. In the same vein as the hive model of Knutson and Tao, this formula is given in terms of a subtraction-free sum of volumes of explicit polytopes. As a consequence, this expression also provides a positive formula for the volume of moduli spaces of SU_n -valued flat connections on the three-holed two dimensional sphere, which was first given by Witten in terms of an infinite sum of characters.

1. INTRODUCTION

Irreducible characters and conjugacy classes are two dual facets of the space of central measures on a compact group. By the underlying group multiplication, the space of central measure is given a convolution product which allows to expand the product of two characters along the basis of irreducible characters. The same holds in the case of conjugacy classes, for which the expansion is described by a measure on the space of conjugacy classes. It is then a classical fact that this expansion involves in both cases non-negative quantities, for which it is in general hard to give manifestly positive formulas (see [24] and [11] for the two corresponding open problems in the case of the symmetric group S_n).

In the framework of compact Lie groups, the problem of finding positive combinatorial formulas for the expansion of characters has been solved in the case of the unitary group U_n by Littlewood and Richardson [20] and in full generality by Lusztig [21], Kashiwara [13] and Littelmann [19]. The case of conjugacy classes is still open in full generality, and the goal of the present paper is to address this problem in the case of U_n . Conjugacy classes of U_n are indexed by the symmetrized torus $\mathcal{H} = [0, 1]^n/S_n$, and the decomposition of the product of two conjugacy classes α and β is defined as a probability distribution $\mathbb{P}[\cdot|\alpha, \beta]$ on \mathcal{H} . As for any compact Lie group, $\mathbb{P}[\cdot|\alpha, \beta]$ can be described in a weak sense as a complex weighted sum of characters, each of which being seen as a function on \mathcal{H} . In the case of a simple connected Lie group and when the conjugacy classes α and β have maximal dimension, in which case α and β are called regular, $\mathbb{P}[\cdot|\alpha, \beta]$ has a density $d\mathbb{P}[\cdot|\alpha, \beta]$ with respect to the Lebesgue measure on \mathcal{H} . The present paper gives then an explicit positive formula for this density as a subtraction-free sum of volume of some explicit polytopes.

The convolution structure of conjugacy classes is also intimately related to the moduli spaces of G -valued flat connections on punctured Riemannian surfaces. This has been first observed by Witten [31], who used this relation to express the volume of such moduli spaces as an infinite sum on characters on the corresponding Lie group G . This formula has then been later expressed [23] as an alternating sum of certain volumes which now have an explicit description (see also [28] for an alternative method using residues). An asymptotic formula for this volume in the large n limit has also been given by Levy [18] as a special case of the more evolved study of the Yang-Mills measure. The formula obtained in this paper for the convolution of conjugacy classes directly translates into a manifestly positive expression of the volume of the moduli space of flat connections on the sphere for $G = SU_n$ in terms of volumes of explicit polytopes.

Conjugacy classes on U_n are naturally related through the exponential map to co-adjoint orbits on the underlying Lie algebra \mathfrak{u}_n of Hermitian matrices, which are indexed by the quotient space \mathbb{R}^n/S_n . In the same way as the multiplicative structure yields a convolution product on conjugacy classes of the Lie group U_n , the additive structure yields a convolution product on co-adjoint orbits of \mathfrak{u}_n . For co-adjoint orbits of maximal dimension, a positive formula for the density of the convolution product has already been obtained following the work of Knutson and Tao, see [16, 7]. In particular, using those formulas with the results of [23] already allowed to simplify the original expression of Witten for

the volume of SU_n -valued flat connections on the sphere from an infinite sum of complex coefficients to a finite alternate sum of explicit quantities.

The convolution structure on co-adjoint orbits can be seen as a semi-classical limit of the ring of characters on the Lie group [14]. Thanks to this relation, the formula for the positive density in the \mathfrak{u}_n case is up to an explicit factor a limit of Littlewood-Richardson coefficients with growing partitions. Such a coefficient had been expressed by Berenstein and Zelevinsky [3] as the number of integers points in a convex polytope whose boundaries depend on the involved partitions. A reformulation of this expression by Knutson and Tao turned this convex polytope into a simple convex body, called discrete hive. Using the discrete hive model, the semiclassical limit yields then a positive formula for the density of convolutions of regular co-adjoint orbits, which is then given by the volume of certain polytopes called continuous hives. Besides their apparent beauty, having positive formulas for such convolution products offered new tools to tackle difficult probabilistic problems concerning invariant measures on the Lie algebras. For example, the hive model for convolution of coadjoint orbits of U_n lies at the heart of the recent results of Narayanan and Sheffield [25] on large deviations for the spectrum of sums of conjugation invariant Hermitian random matrices.

When replacing co-adjoint orbits on \mathfrak{u}_n by conjugacy classes of U_n , the convolution product has to be seen instead as a semi-classical limit of the fusion ring of the Lie algebra with growing level [31] (see also [8] for a more probabilistic approach). For each given level, the fusion ring is a specialization of the ring of characters at roots of unity and the structure coefficients of the multiplication of characters in this quotient ring are non-negative integers called fusion coefficients. There exist no general effective combinatorial expression for those coefficients, but in the case of U_n this fusion ring is isomorphic to the quantum cohomology ring of Grassmannian on \mathbb{P}^1 (see [32] for a geometric explanation of this fact). Through this isomorphism, characters of the fusion ring are Schubert classes of the quantum cohomology.

Using a reinterpretation of the quantum cohomology of Grassmannian in terms of complex bundles on \mathbb{P}^1 [27], Buch, Kresch and Tamvakis [6] related the coefficients appearing in the multiplication of Schubert cycles in the quantum cohomology ring of Grassmannian, called quantum Littlewood-Richardson coefficients, to the structure coefficients of the cohomology ring of the two-steps flag variety. Using this relation and proving a conjectural formula of Knutson on the two-steps flag variety, those authors and Purbhoo [5] gave a positive formula for the quantum Littlewood-Richardson coefficients in terms of certain puzzles (see also [4] for an equivariant version). Such puzzles are generalization of puzzles that already appeared in the work of Knutson, Tao and Woodward [17] in a reformulation of the hive model.

In the present paper, we mimick the program achieved in the co-adjoint case to give a positive formula for the density of the convolution of two regular conjugacy classes on U_n in terms of volumes of explicit polytopes. One of the main difficulties in the present framework is the absence of convex formulations for the quantum Littlewood-Richardson coefficients. Based on the puzzle expression of [5], we give an expression of these coefficients as the counting of integers points in a finite union of convex bodies indexed by a smaller tiling model. The expression we found for the boundaries of those convex bodies are not fully explicit. Fortunately, they are explicit enough to operate an asymptotic integer counting which simplifies the boundary, yielding then a positive formula for the convolution of two regular conjugacy classes as the volume of a union of convex bodies with explicit boundaries, see Theorem 2.4 and the notations before.

Before this manuscript, several results have been achieved in the description on the convolution of conjugacy classes of U_n . One of the most important result, independently obtained by Belkale [2] and Agnihotri and Woodward [1] is the description of the support of the convolution of two non necessarily regular conjugacy classes (see also [29]). This support happens to be a convex set with boundary described by certain quantum Littlewood-Richardson coefficient, in the same vein as the solution to the Horn problem given by Klyachko [15] and Knutson and Tao [16]. The regularity of the convolution of two conjugacy classes has also been investigated, with for example results of [33] showing that the convolution of two conjugacy classes has an L^2 density when the image yields an open set in \mathcal{H} .

Finally, let us mention that the exponential maps yields a much deeper relation between the convolution of conjugacy classes and the one of co-adjoint orbits. This has been formalized [9, 30] through the introduction of the wrapping map Φ from measures on \mathbb{R}^n/S_n to measures on \mathcal{H} . This maps happens to intertwine convolution on both spaces; the wrapping map does not preserve positiveness in general,

which prevents from deducing positive formula for the convolution of conjugacy classes from the one for co-adjoint orbits (see also [23] for a concrete deduction of the formula for the conjugacy classes as an alternating sum). However, this map preserves positiveness for measures of \mathbb{R}^n/S_n having support on \mathcal{H} . Up to a scaling, this allows to recover the positive formula for the convolution of co-adjoint orbits as a special case of the formula for conjugacy classes, exactly as Littlewood-Richardson coefficients are particular cases of their quantum version. At the level of volume of moduli spaces of flat connections, this phenomenon had been already noticed by Jeffrey from purely symplectic arguments [12].

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2. NOTATIONS AND STATEMENT OF THE RESULT

Fix $n \geq 3$ throughout this manuscript (the case $n = 2$ can be handled by direct computation) and denote by U_n the unitary group of size n . Then, recall that the set of conjugacy classes of U_n is homeomorphic to the quotient space $\mathcal{H} = [0, 1]^n/S_n$, where S_n acts on $[0, 1]^n$ by permutation of the coordinates. For $\theta = (\theta_1 \geq \theta_2 \geq \dots \geq \theta_n) \in \mathcal{H}$, denotes by $\mathcal{O}(\theta)$ the corresponding conjugacy class defined by

$$\mathcal{O}(\theta) := \left\{ U e^{2i\pi\theta} U^*, U \in U_n \right\}, \text{ where } e^{2i\pi\theta} = \begin{pmatrix} e^{2i\pi\theta_1} & 0 & \dots & & \\ 0 & e^{2i\pi\theta_2} & & & \\ \vdots & & \ddots & & \\ & & & \ddots & \\ & & & & e^{2i\pi\theta_n} \end{pmatrix}.$$

If the product of two conjugacy classes can be directly defined in the finite group setting, some extra care are needed in the continuous framework and several equivalent definitions are possible depending on the point of view (symplectic, measure theoretic or probabilistic). Let us adopt the probabilistic point of view and say that the product structure on conjugacy classes translates into a convolution product $*$: $\mathcal{M}_1(\mathcal{H}) \times \mathcal{M}_1(\mathcal{H}) \rightarrow \mathcal{M}_1(\mathcal{H})$ on the space of probability distributions on \mathcal{H} such that for $\theta, \theta' \in \mathcal{H}$, $\delta_\theta * \delta_{\theta'}$ is the random variable $p(U_\theta U_{\theta'})$, where U_θ (resp. $U_{\theta'}$) is sampled uniformly on $\mathcal{O}(\theta)$ (resp. $\mathcal{O}(\theta')$) and $p : U_n \rightarrow \mathcal{H}$ maps an element of U_n to the renormalized arguments of its spectrum in \mathcal{H} .

Let us denote by $\mathcal{H}_{reg} = \{\theta \in \mathcal{H}, \theta_1 > \theta_2 > \dots > \theta_n\}$ the set of regular conjugacy classes of U_n , namely the ones of maximal dimension in U_n . For $\alpha, \beta \in \mathcal{H}_{reg}$, $\delta_\alpha * \delta_\beta$ admits a density $d\mathbb{P}[\cdot | \alpha, \beta]$ with respect to the Lebesgue measure on $\{\gamma \in \mathcal{H}, \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i - \sum_{i=1}^n \gamma_i \in \mathbb{N}\}$ (see Section 3 for a concrete proof of this classical result).

The toric hive cones \mathcal{C}_g . The main result of the present manuscript is a positive formula for $d\mathbb{P}[\cdot | \alpha, \beta]$ in terms of the volume of polytopes similar to the hive model of Knutson and Tao [16]. Namely, for $\alpha, \beta, \gamma \in \mathcal{H}$ such that $\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i = \sum_{i=1}^n \gamma_i + d$ for some $d \in \mathbb{N}$, introduce the toric hive

$$R_{d,n} = \{(v_1, v_2) \in \llbracket 0, n \rrbracket^2, d \leq v_1 + v_2 \leq n + d\},$$

which can be described as a discrete hexagon through the map $(v_1, v_2) \mapsto v_1 + v_2 e^{i\pi/3}$, see Figure 1 for a particular case and its hexagon representation.

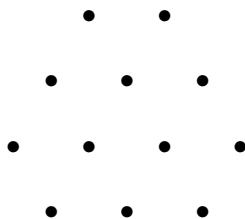


FIGURE 1. The set $R_{1,3}$ represented through the map $(v_1, v_2) \mapsto v_1 + v_2 e^{i\pi/3}$.

Boundary of the toric hive. For any set S and any function $f : R_{d,n} \rightarrow S$, we denote by f^A (resp f^B , f^C) the vector $(f(n + d - i \vee n, i))_{0 \leq i \leq n}$ (resp. $(f((d - i) \vee 0, n + d - i \wedge n))_{0 \leq i \leq n}$, resp. $(f(n - i, i + d - n \vee 0))_{0 \leq i \leq n}$). The vectors f^A , f^B and f^C correspond respectively to the east, north-west and south-east boundaries of $R_{d,n}$ through the hexagonale representation, see Figure 2.

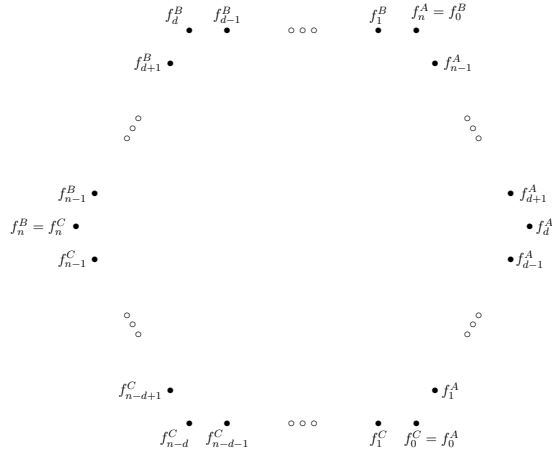


FIGURE 2. The set boundary vectors f^A , f^B and f^C .

Toric rhombus concavity. Let us call a lozenge of $R_{d,n}$ any sequence $(v^1, v^2, v^3, v^4) \in R_{d,n}^4$ corresponding to one of the three configurations of Figure 3.

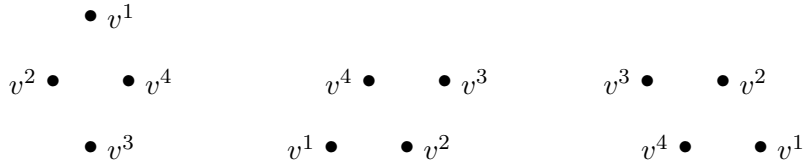


FIGURE 3. The three possible lozenges (v^1, v^2, v^3, v^4) (beware of the position of the vertices which can not be permuted).

Definition 2.1 (Regular labeling). *A labeling $g : R_{d,n} \rightarrow \mathbb{Z}_3$ is called regular whenever*

- $g_i^A = i[3]$, $g_i^B = n + i[3]$ and $g_i^C = i[3]$,
- on any lozenge $\ell = (v^1, v^2, v^3, v^4)$,

$$(g(v^2) = g(v^4)) \Rightarrow \{g(v^1), g(v^3)\} = \{g(v^2) + 1, g(v^2) + 2\}.$$

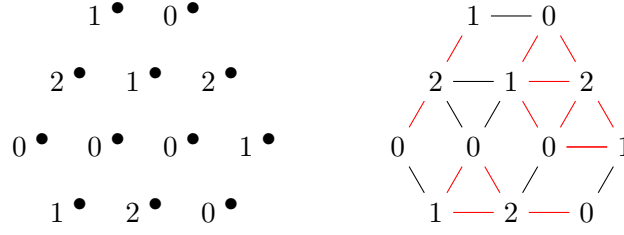
A lozenge (v^1, v^2, v^3, v^4) on which g takes the value $(a, a + 1, a + 2, a + 1)$ for some $a \in \{0, 1, 2\}$ is called rigid.

The support of a regular labeling $g : R_{d,n} \rightarrow \mathbb{Z}_3$ is the subset $\text{Supp}(g) \subset R_{d,n}$ of vertices of $R_{d,n}$ which are not a vertex v_4 of a rigid lozenge (v^1, v^2, v^3, v^4) .

Remark that by the boundary condition of a regular labeling, any vertex v_4 of a rigid lozenge of g can not be on the boundary of $R_{d,n}$, so that the latter is always contained in $\text{Supp}(g)$.

Although given in a compact form, there may be better ways of considering a regular labeling for growing n . By seeing $R_{d,n}$ through its embedding in \mathbb{C} , a regular embedding is equivalent to a tiling of $R_{d,n}$ with either black or red triangle of size 1 or lozenge of size 1 with alternating colors on its boundaries, such that the six boundary edges of $R_{d,n}$ are alternatively colored red and black, starting with the color red on the south edge $\{(v_1, v_2) \in R_{d,n}, v_2 = 0\}$. The bijection from the former representation to the latter is given by assigning the red (resp. black) color to any edge of the form $(v, v + e^{2i\pi\ell})$, $0 \leq \ell \leq 2$ along which the labels of g increase by 1 (resp. decrease by 1), see Figure 4 for an example with $n = 4, d = 1$ and Proposition 7.9 for a proof of this fact.

Definition 2.2 (Toric hive cone). *A function $f : R_{d,n}$ is called rhombus concave with respect to a regular labeling $g : R_{d,n} \rightarrow \mathbb{Z}_3$ when $f(v_2) + f(v_4) \geq f(v_1) + f(v_3)$ on any lozenge $\ell = (v^1, v^2, v^3, v^4)$, with equality if ℓ is rigid with respect to g .*


 FIGURE 4. A regular labeling on $R_{d,n}$ and its colored representation

For any regular labeling g , the toric hive cone \mathcal{C}_g with respect to g is the cone

$$\mathcal{C}_g = \{f|_{\text{Supp}(g)}, f : R_{d,n} \rightarrow \mathbb{R} \text{ toric rhombus concave with respect to } g\}.$$

As we will see later, for any regular labeling g , $\text{Supp}(g)$ has cardinal $(n-1)(n-2)/2 + 3n$ and so is the dimension of \mathcal{C}_g . As such, we recover the usual dimension of the classical hive cone from [16]. The latter is then a particular case of toric hive cone for $d = 0$. An example of a toric hive in the case $n = 3, d = 1$ is given in Figure 5.

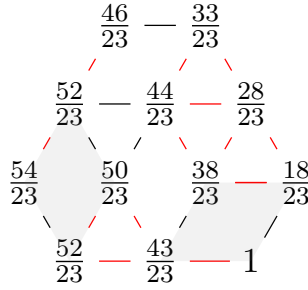


FIGURE 5. A toric rhombus concave function for $n = 3, d = 1$: the regular labeling is depicted through colored edges and the shaded lozenge are the rigid ones yielding the equality cases in the toric rhombus concavity.

Definition 2.3 (Polytope $P_{\alpha,\beta,\gamma}^g$). Let $n \geq 3$ and let $\alpha, \beta, \gamma \in \mathcal{H}_{reg}$ be such that $\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i = \sum_{i=1}^n \gamma_i + d$ with $d \in \mathbb{N}$. Let g be a regular labeling. Then, $P_{\alpha,\beta,\gamma}^g$ is the polytope of $\mathbb{R}^{\text{Supp}(g) \setminus \partial R_{d,n}}$ consisting of functions in \mathcal{C}_g such that

$$f^A = \left((d-i)^+ + \sum_{s=1}^i \alpha_s \right)_{0 \leq i \leq n}, \quad f^B = \left(\sum_{s=1}^n \alpha_s + \sum_{s=1}^i \beta_s \right)_{0 \leq i \leq n}, \quad f^C = \left(d + \sum_{s=1}^i \gamma_s \right)_{0 \leq i \leq n}.$$

An example of an element of $P_{\alpha,\beta,\gamma}^g$ for $n = 3$ and $d = 1$ is depicted in Figure 5, for $\alpha = \left(\frac{18}{23} \geq \frac{10}{23} \geq \frac{5}{23}\right)$, $\beta = \left(\frac{13}{23} \geq \frac{6}{23} \geq \frac{2}{23}\right)$ and $\gamma = \left(\frac{20}{23} \geq \frac{9}{23} \geq \frac{2}{23}\right)$.

Statement of the results: Our main result gives then a formula for the density of the convolution of regular conjugacy classes in terms of volume of polytopes coming from \mathcal{C}_g for regular labeling g .

Theorem 2.4. Let $n \geq 3$ and let $\alpha, \beta, \gamma \in \mathcal{H}_{reg}$ be such that $\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i = \sum_{i=1}^n \gamma_i + d$ with $d \in \mathbb{N}$. Then,

$$(2.1) \quad d\mathbb{P}[\gamma|\alpha, \beta] = \frac{(2\pi)^{(n-1)(n-2)/2} \prod_{k=1}^{n-1} k! \Delta(e^{2i\pi\gamma})}{n! \Delta(e^{2i\pi\alpha}) \Delta(e^{2i\pi\beta})} \sum_{g: R_{d,n} \rightarrow \mathbb{Z}_3 \text{ regular}} \text{Vol}_g(P_{\alpha,\beta,\gamma}^g),$$

where $\Delta(e^{2i\pi\theta}) = 2^{n(n-1)/2} \prod_{i < j} \sin(\pi(\theta_i - \theta_j))$ for $\theta \in \mathcal{H}$, Vol_g denotes the volume with respect to the Lebesgue measure on $\mathbb{R}^{\text{Supp}(g) \setminus \partial R_{d,n}}$.

Note that the case $n = 2$ admits explicit formulas which do not need such a machinery. Some polytopes $P_{\alpha,\beta,\gamma}^g$ may be empty and thus may not contribute to $d\mathbb{P}[\gamma|\alpha, \beta]$. Numerical experiments for $n = 3$ suggest that there are $\alpha, \beta \in \mathcal{H}_{reg}$ for which all regular labeling g yields a non-empty polytope $P_{\alpha,\beta,\gamma}^g$.

for some $\gamma \in \mathcal{H}_{reg}$. However, for a fixed triple $(\alpha, \beta, \gamma) \in \mathcal{H}_{reg}$ there seems to be generically only a strict subset of $\{P_{\alpha, \beta, \gamma}^g\}_{g \text{ regular}}$ which are not empty and contribute. Finally, remark that we only considered the case of regular conjugacy classes to ensure the existence of a density for the convolution product. Such a hypothesis is regularly assumed (see for example [31, 23]). However, we expect similar results to hold in cases where less than $n/2$ coordinates of \mathcal{H} are equal, in which case the convolution product is still expected to have a density.

As a direct application of this theorem, we deduce a positive formula for the volume of flat connections on the punctured two-dimensional sphere with prescribed holonomies around the holes. Denote by Σ_0^3 the sphere with three removed marked points a, b, c . We denote by $\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)$ the moduli space of flat SU_n -valued connections on Σ_0^3 for which the holonomies around a, b, c respectively belongs to $\mathcal{O}_\alpha, \mathcal{O}_\beta$ and \mathcal{O}_γ . In the specific case of the punctured sphere, this moduli space can be alternatively described as

$$\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma) = \{(U_1, U_2, U_3) \in \mathcal{O}_\alpha \times \mathcal{O}_\beta \times \mathcal{O}_\gamma, U_1 U_2 U_3 = Id_{SU_n}\} / SU_n,$$

where SU_n acts diagonally by conjugation, see [1], see also [23, Section 3] for an introduction to SU_n -valued flat connections and a proof of the latter equality. As a corollary of Theorem 2.4, we thus get an expression of the volume of $\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)$ as a sum of volumes of explicit polytopes.

Corollary 2.5. *Let $n \geq 3$ and consider the usual volume on SU_n . For $\alpha, \beta, \gamma \in \mathcal{H}_{reg}$ such that $|\alpha|_1, |\beta|_1, |\gamma|_1 \in \mathbb{N}$, then $Vol[(\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma))] \neq 0$ only if $\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i + \sum_{i=1}^n \gamma_i = n + d$ for some $d \in \mathbb{N}$, in which case*

$$Vol[(\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma))] = \frac{2^{(n+1)[2]} (2\pi)^{(n-1)(n-2)}}{n! \Delta(e^{2i\pi\gamma}) \Delta(e^{2i\pi\alpha}) \Delta(e^{2i\pi\beta})} \sum_{g: R_{d,n} \rightarrow \mathbb{Z}_3 \text{ regular}} Vol_g(P_{\alpha, \beta, \tilde{\gamma}}^g),$$

where $\tilde{\gamma} = (1 - \gamma_n, \dots, 1 - \gamma_1)$ and the polytope $P_{\alpha, \beta, \tilde{\gamma}}^g$ are defined in Theorem 2.4.

Note that the choice of normalization for the volume of SU_n slightly differs from the one used in [31] for numerical applications. A consequence of this corollary is that the volume is a piecewise polynomial in α, β, γ , up to the normalization factor coming from the volume of the conjugacy classes. Such a phenomenon is a reflect of the underlying symplectic structure which had already been observed in [23]. A same phenomenon occurs in the co-adjoint case, see [7, 10]. Remark that the results of Corollary 2.5 only address the genus zero case when the Lie group is SU_n . It would be very interesting to have similar expressions in higher genus cases or for different compact Lie groups. An equally interesting problem would be to get such explicit expressions already at the level of the fusion coefficients.

Sketch of the proof of Theorem 2.4 and Corollary 2.5. Let us give the structure of the paper while sketching the proof of our main statement. The first step of the prood is the semi-classical approximation of the density of the convolution product by a limit of quantum Littlewood-Richardson coefficients. Such approximation scheme is done in Section 3. All the work of the remaining part of the manuscript consists in turning known expressions for the quantum Littlewood-Richardson coefficients into integers points counting in convex bodies, for which the convergence towards volumes of polytope is straightforward, see [16]. Section 4 introduces the puzzle expression for those coefficients obtained in [5] and gives a first simplification of the puzzle formulation by only keeping the position of certain pieces of the puzzles. It is then deduced in Section 5 an expression of the coefficients as the counting of integers points in a family of convex polytopes indexed by certain two-colored tilings which are reminiscent of Figure 4. Up to this point, those polytopes are degenerated and non-rational polytopes in a higher dimension space, which prevents any proper asymptotic counting in the semi-classical limit. By a combinatorial work on the underlying two-colored tiling, we give in Section 6 a parametrization of the integer points of those polytopes in terms of integer points of genuine convex bodies. Remark that the results of Section 5 hold more generally for any coefficient of the two-step flag variety, a fact which is not true anymore from Section 6. The asymptotic counting of integers points in convex bodies is then much more tractable, and the conclusion of the proof of Theorem 2.4 and Corollary 2.5 is done in Section 7.

3. DENSITY FORMULA VIA THE QUANTUM COHOMOLOGY OF THE GRASSMANNIAN

Let $n \geq 1$ and consider $\alpha, \beta \in \mathcal{H}^2$: $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ where

$$1 \geq \alpha_1 > \alpha_2 > \dots > \alpha_n \geq 0 \text{ and } 1 \geq \beta_1 > \beta_2 > \dots > \beta_n \geq 0.$$

Suppose furthermore that

$$(3.1) \quad \sum_{i=1}^n \alpha_i = k \text{ and } \sum_{i=1}^n \beta_i = k'$$

for some $k, k' \in \mathbb{Z}$. Let $A = Ue^{2i\pi\alpha}U^*$, $B = Ve^{2i\pi\beta}V^*$, where U, V are independent Haar distributed matrices on $U(n)$ so that A and B are respectively uniformly distributed on the conjugacy classes $\mathcal{O}(\alpha)$ and $\mathcal{O}(\beta)$. The goal of this section is to prove Theorem 3.8 stated below which gives a density formula (3.28) for the probability $d\mathbb{P}[\gamma|\alpha, \beta]$ that $AB \in \mathcal{O}(\gamma)$ for $\gamma \in \mathcal{H}$ in terms of structure constants defined in Section 3.2. Such a semi-classical convergence had been already suggested and proven several times in different forms (see [31] for a similar approach with fusion coefficient and [8] for a convergence in distribution). In Section 3.1, we recall in Proposition 3.1 a classical expression of the density in terms of characters of irreducible representations of $SU(n)$. In Section 3.2, we link the density of Proposition 3.1 to the structure constants and derive Theorem 3.8.

3.1. A first density formula. This part aims at recalling a proof of the formula (3.3) which gives the value of $d\mathbb{P}[\gamma|\alpha, \beta]$ as an infinite sum of characters. A similar treatment of the convolution of orbit measures in the general context of Lie algebras can be found in [8, Sec. 7].

Let us denote by dg the *normalized* Haar measure on $U(n)$ and for $\theta \in \mathcal{H}$, φ_θ the map

$$\begin{aligned} \varphi_\theta : U(n) &\rightarrow \mathcal{O}(\theta) \subset SU(n) \\ U &\mapsto Ue^{2i\pi\theta}U^*. \end{aligned}$$

Let us write

$$(3.2) \quad m_\theta = \varphi_\theta \# dg$$

the push-forward of dg by φ_θ . The measure m_θ is a measure on $\mathcal{O}(\theta)$ called the orbital measure. For any function $f : \mathcal{O}(\theta) \rightarrow \mathbb{R}$,

$$\int_{\mathcal{O}(\theta)} f dm_\theta = \int_{U(n)} f(ge^{2i\pi\theta}g^{-1})dg.$$

Recall that the irreducible representations of the compact group $SU(n)$ are parameterized by $\lambda \in \mathbb{Z}_{\geq 0}^n$ and we denote by $(\rho_\lambda, V_\lambda)$ the corresponding representation where $\rho_\lambda : SU(n) \rightarrow V_\lambda$ and $\chi_\lambda : \text{End}_{V_\lambda} \rightarrow \mathbb{C}$, $x \mapsto \text{Tr}[x]$ is the associated character.

Proposition 3.1 (Induced density of eigenvalues). *Let $(\alpha, \beta) \in \mathcal{H}^2$ and let $A, B \in \mathcal{O}(\alpha) \times \mathcal{O}(\beta)$ be two independent random variables sampled from m_α and m_β respectively. Let $C = AB \in \mathcal{O}(\gamma)$ for some random $\gamma \in [0, 1]^n/S_n$. The density of $\gamma = \gamma_1 > \dots > \gamma_n \geq 0$ is given by the absolute convergent series*

$$(3.3) \quad d\mathbb{P}[\gamma|\alpha, \beta] = \frac{|\Delta(e^{2i\pi\gamma})|^2}{(2\pi)^{n-1}n!} \sum_{\lambda \in \mathbb{Z}_{\geq 0}^n} \frac{1}{\dim V_\lambda} \chi_\lambda(e^{2i\pi\alpha}) \chi_\lambda(e^{2i\pi\beta}) \chi_\lambda(e^{-2i\pi\gamma}).$$

Another expression of the density (3.3) is given in (3.26).

Definition 3.2 (Fourier Transform on $SU(n)$). *Let m be a measure on $SU(n)$. The Fourier transform \widehat{m} of m is defined as*

$$(3.4) \quad \widehat{m} : \mathbb{Z}_{\geq 0}^n \rightarrow \text{End}_{V_\lambda}$$

$$(3.5) \quad \lambda \mapsto \int_{SU(n)} \rho_\lambda(g) dm(g).$$

From this one can deduce the Fourier transform of m_θ .

Lemma 3.3 (Fourier Transform of m_θ). *One has, for $\lambda \in \mathbb{Z}_{\geq 0}^n$,*

$$(3.6) \quad \widehat{m}_\theta(\lambda) = \frac{\chi_\lambda(e^{2i\pi\theta})}{\dim V_\lambda} \text{id}_{V_\lambda},$$

where id_{V_λ} is the identity element of V_λ .

Proof. For any $\lambda \in \mathbb{Z}_{\geq 0}^n$, the function $g \mapsto \rho_\lambda(g)$ is invariant by conjugation, so that

$$(3.7) \quad \widehat{m}_\theta(\lambda) = \int_{U(n)} \rho_\lambda(g e^{2i\pi\theta} g^{-1}) dg$$

satisfies $\widehat{m}_\theta(\lambda) = c \cdot \text{id}_{V_\lambda}$. One computes the value of c by taking the trace which gives

$$(3.8) \quad c = \frac{\chi_\lambda(e^{2i\pi\theta})}{\dim V_\lambda}.$$

□

Definition 3.4 (Convolution of measures). *Let m, m' be two measures on $SU(n)$. Let $m \boxtimes m'$ be the product measure on $SU(n) \times SU(n)$. Define $\text{mult} : SU(n) \times SU(n) \rightarrow SU(n)$ to be the multiplication on $SU(n)$: $\text{mult}(g_1, g_2) = g_1 g_2$. The convolution of m and m' denote by $m * m'$ is defined as*

$$(3.9) \quad m * m' = \text{mult}\#(m \boxtimes m').$$

which means that for any function f on $SU(n)$,

$$\int_{SU(n)} f(g) d(m * m')(g) = \int_{SU(n)} \int_{SU(n)} f(g_1 g_2) dm(g_1) dm(g_2).$$

For $(\alpha, \beta) \in \mathcal{H}^2$, we write $m_{\alpha, \beta} := m_\alpha * m_\beta$ the convolution of m_α and m_β .

By Definition 3.4, the measure $m_{\alpha, \beta}$ is the law on $SU(n)$ of $C = A \cdot B$ where A and B are sampled from measures μ_α and μ_β on $\mathcal{O}(\alpha)$ and $\mathcal{O}(\beta)$ respectively. Recall that for two measures m, m' on $SU(n)$, then,

$$(3.10) \quad \widehat{m * m'}(\lambda) = \widehat{m}(\lambda) \widehat{m'}(\lambda).$$

In particular

$$(3.11) \quad \widehat{m}_{\alpha, \beta}(\lambda) = \widehat{m}_\alpha(\lambda) \widehat{m}_\beta(\lambda).$$

Recall that we are interested in the measure $\mu_{\alpha, \beta}$. By (3.11), one knows how to compute its Fourier transform. Let us define the inverse Fourier transform.

Definition 3.5 (Inverse Fourier Transform). *Let $f : \mathbb{Z}_{\geq 0}^n \rightarrow \text{End}_{V_\lambda}$ be a function such that*

$$(3.12) \quad \|f\|^2 = \sum_{\lambda \in \mathbb{Z}_{\geq 0}^n} \dim V_\lambda \cdot \text{Tr}[f(\lambda) f^*(\lambda)] < \infty.$$

The inverse Fourier transform of f is

$$(3.13) \quad f^\vee : SU(n) \rightarrow \mathbb{C}$$

$$(3.14) \quad g \mapsto \sum_{\lambda \in \mathbb{Z}_{\geq 0}^n} \dim V_\lambda \cdot \text{Tr}[\rho_\lambda(g^{-1}) f(\lambda)].$$

In order to apply inverse Fourier transform to $\widehat{m}_{\alpha, \beta}$, one needs to check condition (3.12). This is the purpose of the next lemma.

Lemma 3.6 (Product Fourier transform is L^2). *For $(\alpha, \beta) \in \mathcal{H}^2$,*

$$(3.15) \quad \sum_{\lambda \in \mathbb{Z}_{\geq 0}^n} \dim(V_\lambda) \text{Tr}[\widehat{m}_{\alpha, \beta}(\lambda) \widehat{m}_{\alpha, \beta}(\lambda)^*] < \infty.$$

Proof. Using (3.11) together with (3.6), one has

$$(3.16) \quad \sum_{\lambda \in \mathbb{Z}_{\geq 0}^n} \dim(V_\lambda) \text{Tr}[\widehat{m}_{\alpha, \beta}(\lambda) \widehat{m}_{\alpha, \beta}(\lambda)^*] = \sum_{\lambda \in \mathbb{Z}_{\geq 0}^n} \frac{1}{\dim(V_\lambda)^2} |\chi_\lambda(e^{2i\pi\alpha})|^2 |\chi_\lambda(e^{2i\pi\beta})|^2.$$

Using Weyl's character formula [7, eq. (21)],

$$(3.17) \quad \chi_\lambda(e^{i\theta}) = \frac{\det[e^{i\theta_r \lambda'_s}]_{1 \leq r, s \leq n}}{\Delta(e^{i\theta})}$$

where $\lambda' = \lambda + \rho$ with $\rho = (n-1, \dots, 0)$ and where $\Delta(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$ is the Vandermonde determinant. Recall that by assumption, $\alpha_i \neq \alpha_j$ for $i \neq j$ and the same holds for β , so that the expressions $\Delta(e^{2i\pi\alpha})$ and $\Delta(e^{2i\pi\beta})$ are well-defined. The previous sum becomes

$$(3.18) \quad \frac{1}{|\Delta(e^{2i\pi\alpha})\Delta(e^{2i\pi\beta})|^2} \sum_{\lambda_1 \geq \dots \geq \lambda_{n-1} \geq \lambda_n = 0} \frac{|\det[e^{2i\pi\alpha_r \lambda'_s}]|^2 |\det[e^{2i\pi\beta_r \lambda'_s}]|^2}{\dim(V_\lambda)^2}$$

$$(3.19) \quad \leq \frac{n^{2n}}{|\Delta(e^{2i\pi\alpha})\Delta(e^{2i\pi\beta})|^2} \sum_{\lambda_1 \geq \dots \geq \lambda_{n-1} \geq \lambda_n = 0} \frac{1}{\dim(V_\lambda)^2}$$

where we used Hadamard's inequality for the upper bound on determinants. From the identity

$$(3.20) \quad \dim(V_\lambda) = \frac{\Delta(\lambda')}{sf(n-1)},$$

which can be found in [7, eq. (19)] and where $sf(n) = \prod_{1 \leq j \leq n} j!$, it suffices to show that

$$(3.21) \quad V_n = \sum_{\lambda_1 > \dots > \lambda_n > 0} \frac{1}{\Delta(\lambda)^2}$$

converges. Let us write

$$\begin{aligned} \sum_{\lambda_1 > \dots > \lambda_n > 0} \frac{1}{\Delta(\lambda)^2} &= \sum_{\lambda_1 > \dots > \lambda_n > 0} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^{-2} \\ &= \sum_{\lambda_2 > \dots > \lambda_n > 0} \left[\sum_{\lambda_1 > \lambda_2} \prod_{2 \leq j \leq n} (\lambda_1 - \lambda_j)^{-2} \right] \prod_{2 \leq i < j \leq n} (\lambda_i - \lambda_j)^{-2} \\ &\leq \sum_{\lambda_2 > \dots > \lambda_n > 0} \left[\sum_{\lambda_1 > \lambda_2} (\lambda_1 - \lambda_2)^{-2(n-1)} \right] \prod_{2 \leq i < j \leq n} (\lambda_i - \lambda_j)^{-2} \end{aligned}$$

the innermost sum is bounded by $\sum_{k \geq 1} k^{-2(n-1)} \leq \sum_{k \geq 1} k^{-2} = c$ for $n \geq 2$. Thus,

$$V_n \leq cV_{n-1}$$

so that $V_n \leq c^{n-1}$ which proves the convergence. \square

Lemma 3.6 shows that the Fourier transform of $\mu_\alpha * \mu_\beta$ is in L^2 , so that one can take its inverse Fourier Transform. This leads to the following result.

Lemma 3.7 (Inverse Fourier of Convolution). *Let $(\alpha, \beta) \in \mathcal{H}^2$ and $g \in SU(n)$. Then,*

$$(3.22) \quad (\widehat{m}_{\alpha, \beta})^\vee(g) = \sum_{\lambda \in \mathbb{Z}_{\geq 0}^n} \frac{1}{\dim V_\lambda} \chi_\lambda(e^{2i\pi\alpha}) \chi_\lambda(e^{2i\pi\beta}) \chi_\lambda(g^{-1})$$

Proof. Using (3.14) together with (3.11) yields

$$(3.23) \quad (\widehat{m}_{\alpha, \beta})^\vee(g) = \sum_{\lambda \in \mathbb{Z}_{\geq 0}^n} \dim V_\lambda \cdot Tr[\rho_\lambda(g^{-1}) \widehat{m}_\alpha(\lambda) \widehat{m}_\beta(\lambda)]$$

$$(3.24) \quad = \sum_{\lambda \in \mathbb{Z}_{\geq 0}^n} \dim V_\lambda \cdot Tr \left[\frac{\chi_\lambda(e^{2i\pi\alpha}) \chi_\lambda(e^{2i\pi\beta})}{\dim V_\lambda^2} \rho_\lambda(g^{-1}) \right]$$

$$(3.25) \quad = \sum_{\lambda \in \mathbb{Z}_{\geq 0}^n} \frac{1}{\dim V_\lambda} \chi_\lambda(e^{2i\pi\alpha}) \chi_\lambda(e^{2i\pi\beta}) \chi_\lambda(g^{-1})$$

\square

Proof of Proposition 3.1. The induced density on γ is given by the density (3.22) multiplied by the Jacobian of diagonalization map $g \mapsto Ve^{2i\pi\gamma}V^*$ with $\gamma = (\gamma_1, \dots, \gamma_n)$ such that $\sum \gamma_i \in \mathbb{Z}$. Since this Jacobian is $\frac{|\Delta(e^{2i\pi\gamma})|^2}{(2\pi)^{n-1}n!}$, see [7, eq. (17)], we obtain the desired expression. \square

Writing the density (3.3) using (3.17) and the fact that $\dim V_\lambda = \frac{\Delta(\lambda')}{sf(n-1)}$ yields

$$(3.26) \quad \begin{aligned} d\mathbb{P}[\gamma|\alpha, \beta] &= \frac{\Delta(e^{2i\pi\gamma})sf(n-1)}{(2\pi)^{n-1}n!\Delta(e^{2i\pi\alpha})\Delta(e^{2i\pi\beta})} \sum_{\lambda \in \mathbb{Z}_{\geq 0}^n} \frac{1}{\Delta(\lambda')} \det[e^{2i\pi\alpha_r\lambda'_s}] \det[e^{2i\pi\beta_r\lambda'_s}] \det[e^{-2i\pi\gamma_r\lambda'_s}] \\ &= \frac{sf(n-1)(2\pi)^{(n-1)(n-2)/2}\Delta(e^{2i\pi\gamma})}{\Delta(e^{2i\pi\alpha})\Delta(e^{2i\pi\beta})n!} J[\gamma|\alpha, \beta], \end{aligned}$$

where

$$(3.27) \quad J[\gamma|\alpha, \beta] = \frac{1}{(2\pi)^{n(n-1)/2}} \sum_{\lambda \in \mathbb{Z}_{\geq 0}^n} \frac{1}{\Delta(\lambda')} \det[e^{2i\pi\alpha_r\lambda'_s}] \det[e^{2i\pi\beta_r\lambda'_s}] \det[e^{-2i\pi\gamma_r\lambda'_s}].$$

is called the volume function for the unitary Horn problem.

3.2. Link with quantum cohomology of the Grassmannian. The goal of this section is to link the density (3.26) with structure constants of the quantum cohomology ring of Grassmannians $QH^\bullet(\mathbb{G}r)$. These constants are the Gromov-Witten invariants, which are related to characters via [26, Cor. 6.2]. We refer the reader to [22] and [27] for an introduction to the subject.

For $N \geq n$, the ring $QH^\bullet(\mathbb{G}r(n, N))$ has an additive basis $(g^d \otimes \sigma_\lambda, d \geq 0, \lambda \in \mathbb{Z}_{N-n}^n, (N-n \geq \lambda_1 \geq \dots \geq \lambda_n \geq 0))$. We will denote $c_{\lambda, \mu}^{\nu, d}$ the structure constants of this ring so that

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu, d \geq 0} c_{\lambda, \mu}^{\nu, d} g^d \otimes \sigma_{\nu^\vee}.$$

where the sum is over partitions $(\nu, d) \in \mathbb{Z}_{N-n}^n$ such that $|\lambda| + |\mu| + |\nu| = n(N-n) + Nd$. The structure coefficients $c_{\lambda, \mu}^{\nu, d}$ are the degree d Gromov-Witten invariants associated to the Schubert cycles $\sigma_\lambda, \sigma_\mu, \sigma_{\nu^\vee}$, see [26, Cor. 6.2]. The main result of this section is Theorem 3.8 below.

Theorem 3.8 (Density as limit of quantum coefficients). *Let $(\alpha, \beta, \gamma) \in \mathcal{H}^3$. For each $N \geq 1$, let $(\lambda_N, \mu_N, \nu_N)$ be three partitions in \mathbb{Z}_{N-n}^n such that $|\lambda_N| + |\mu_N| + |\nu_N| = n(N-n) + dN$ for some $d \in \mathbb{Z}_{\geq 0}$ and such that $\frac{1}{N}\lambda_N = \alpha + o(1)$, $\frac{1}{N}\mu_N = \beta + o(1)$ and $\frac{1}{N}\nu_N = \gamma + o(1)$ as $N \rightarrow +\infty$. Then,*

$$(3.28) \quad \lim_{N \rightarrow \infty} N^{-(n-1)(n-2)/2} c_{\lambda_N, \mu_N}^{\nu_N, d} = J[\gamma|\alpha, \beta] = \frac{\Delta(e^{2i\pi\alpha})\Delta(e^{2i\pi\beta})n!}{sf(n-1)(2\pi)^{(n-1)(n-2)/2}\Delta(e^{2i\pi\gamma})} d\mathbb{P}[\gamma|\alpha, \beta].$$

The rest of this section is devoted to the proof of Theorem 3.8. In subsection 3.2.1 we prove a determinantal formula for coefficients $c_{\lambda, \mu}^{\nu, d}$ along with some results on the quantities involved in the expression. In subsection 3.2.2 we prove Theorem 3.8 using Lemmas 3.16 and 3.15.

3.2.1. Determinantal expression for $c_{\lambda, \mu}^{\nu, d}$. For $1 \leq n \leq N$, set

$$I_{n, N} = \left\{ (I_1, \dots, I_n) \in \left(\mathbb{Z} + \left(\frac{1}{2} \right)^{(n-1)[2]} \right)^n : -\frac{n-1}{2} \leq I_n < \dots < I_1 \leq N - \frac{n+1}{2} \right\}.$$

Lemma 3.9 (Determinantal expression for $c_{\lambda, \mu}^{\nu, d}$). *Let λ, μ, ν such that $|\lambda| + |\mu| + |\nu| = n(N-n) + Nd$. Then,*

$$(3.29) \quad c_{\lambda, \mu}^{\nu, d} = \frac{1}{N^n} \sum_{I \in I_{n, N}} \frac{\det[e^{\frac{2i\pi I_r(\lambda_s + (s-1))}{N}}] \det[e^{\frac{2i\pi I_r(\mu_s + (s-1))}{N}}] \det[e^{-\frac{2i\pi I_r((\nu^\vee)_s + (s-1))}{N}}]}{\Delta(\xi^I)}$$

Proof. Let $\xi = \exp(2i\pi/N)$ and for $I \in I_{n, N}$, set $\xi^I = (\xi^{I_1}, \dots, \xi^{I_n})$. Let $S_\lambda(x_1, \dots, x_n) = \frac{\det[x_r^{\lambda_s + (s-1)}, 1 \leq r, s \leq n]}{\Delta(x)}$ be the Schur function corresponding to the partition λ . Using [26, Corollary 6.2]:

$$(3.30) \quad c_{\lambda, \mu}^{\nu, d} = \frac{1}{N^n} \sum_{I \in I_{n, N}} S_{\lambda}(\xi^I) S_{\mu}(\xi^I) S_{\nu}(\xi^I) \frac{|\Delta(\xi^I)|^2}{S_{(N-n)}(\xi^I)}.$$

Moreover, by [26, eq. (4.3)], one has

$$\frac{S_{\nu}(\xi^I)}{S_{(N-n)}(\xi^I)} = \overline{S_{\nu^{\vee}}(\xi^I)}$$

so that

$$(3.31) \quad \begin{aligned} c_{\lambda, \mu}^{\nu, d} &= \frac{1}{N^n} \sum_{I \in I_{n, m}} S_{\lambda}(\xi^I) S_{\mu}(\xi^I) S_{\nu^{\vee}}(\overline{\xi^I}) |\Delta(\xi^I)|^2 \\ &= \frac{1}{N^n} \sum_{I \in I_{n, m}} \frac{\det[e^{\frac{2i\pi I_r(\lambda_s + (s-1))}{N}}] \det[e^{\frac{2i\pi I_r(\mu_s + (s-1))}{N}}] \det[e^{-\frac{2i\pi I_r((\nu^{\vee})_s + (s-1))}{N}}]}{\Delta(\xi^I)} \end{aligned}$$

□

We are interested in the asymptotic behaviour of the previous expression as $N \rightarrow \infty$. Define

$$(3.32) \quad F(I, \lambda, \mu, \nu, N) = \frac{\det[e^{\frac{2i\pi I_r(\lambda_s + (s-1))}{N}}] \det[e^{\frac{2i\pi I_r(\mu_s + (s-1))}{N}}] \det[e^{-\frac{2i\pi I_r((\nu^{\vee})_s + (s-1))}{N}}]}{\Delta(\xi^I)}.$$

Lemma 3.10 (Translation invariance). *Let $I \in (\frac{1}{2}\mathbb{Z})^n$ and $a \in \frac{1}{2}\mathbb{Z}$. We still assume that $|\lambda| + |\mu| + |\nu| = N(N-n) + Nd$ for some $d \in \mathbb{Z}_{\geq 0}$. Then,*

$$(3.33) \quad F(I + a, \lambda, \mu, \nu, N) = F(I, \lambda, \mu, \nu, N).$$

Proof. Since

$$\det \left[\exp \left(\frac{2i\pi(I_r + a)(\lambda_s + s - 1)}{N} \right) \right] = \det \left[\exp \left(\frac{2i\pi I_r(\lambda_s + s - 1)}{N} \right) \right] \exp \left(a \frac{2i\pi(|\lambda| + \sum_{l=0}^{n-1} l)}{N} \right),$$

the numerator of $F(I + a, \lambda, \mu, \nu, N)$ is the one of $F(I, \lambda, \mu, \nu, N)$ times the factor

$$\exp \left(a \frac{2i\pi}{N} (|\lambda| + |\mu| - |\nu^{\vee}| + n(n-1)/2) \right) = \exp \left(a \frac{i\pi n(n-1)}{N} \right)$$

since $|\lambda| + |\mu| - |\nu^{\vee}| = 0 [N]$. The Vandermonde in the denominator of $F(I + a, \lambda, \mu, \nu, N)$ is

$$\begin{aligned} \Delta(\xi^{I+a}) &= \prod_{1 \leq r < s \leq n} \left(\exp \left(\frac{2i\pi(I_r + a)}{N} \right) - \exp \left(\frac{2i\pi(I_s + a)}{N} \right) \right) \\ &= \prod_{1 \leq r < s \leq n} \left(\exp \left(\frac{2i\pi I_r}{N} \right) - \exp \left(\frac{2i\pi I_s}{N} \right) \right) \exp \left(a \frac{i\pi n(n-1)}{N} \right) \\ &= \Delta(\xi^I) \exp \left(a \frac{i\pi n(n-1)}{N} \right) \end{aligned}$$

so that the quotient cancels the common additional factor appearing in the numerator and denominator of $F(I + a, \lambda, \mu, \nu, N)$. □

From Lemma 3.10, we can shift I by $\frac{n-1}{2}$. In the following, we will assume that $0 \leq I_n < \dots < I_1 \leq N-1$ and that the I 's are in \mathbb{Z} . Denote by $J_{n, N}$ the set

$$J_{n, N} = \{I \in \{0, \dots, N-1\}^n, i_1 > i_2 > \dots > i_n\}.$$

Definition 3.11 (Action Φ_N and orbits). *The translation action Φ_N of \mathbb{Z} on $J_{n, N}$ is given by*

$$\begin{aligned} \Phi_N : \mathbb{Z} \times J_{n, N} &\rightarrow J_{n, N} \\ (l, I = (i_1 > \dots > i_n)) &\mapsto I + (l, \dots, l) [N] \end{aligned}$$

where the tuple $I + (l, \dots, l)[N]$ consist of elements $i_1 + l[N], \dots, i_n + l[N]$ sorted in the decreasing order.

Lemma 3.10 shows that F is invariant under the action of Φ_N .

In order to give some properties of orbits of Φ_m , let us recall the lexicographic order on $\mathbb{Z}_{\geq 0}^n$. For $I, J \in \mathbb{Z}_{\geq 0}^n$, let $r^* = \inf_{1 \leq r \leq n} I_r \neq J_r$, with the convention that $r^* = 0$ if $I = J$. We say that $I > J$ if $I_{r^*} > J_{r^*}$ and $I < J$ if $I_{r^*} < J_{r^*}$. This defines a total order on $\mathbb{Z}_{\geq 0}^n$.

Let $Orbits(N)$ denote the orbits of Φ_N . For $\Omega_N \in Orbits(N)$ an orbit of Φ_N , denote $\min(\Omega_N)$ its minimal element with respect to the lexicographic order. Then, necessarily, $(\min(\Omega_N))_n = 0$, otherwise $\Phi(-1, \min(\Omega_N))$ would be an element of Ω_N strictly inferior to $\min(\Omega_N)$. For an ordered n -tuple I of $\{0, \dots, N-1\}^n$, let $\Omega(I, N)$ denote its orbit under the action of Φ_N .

Lemma 3.12 (Orbit structure for large N). *Let $I = (I_1 > \dots > I_{n-1} > I_n = 0)$. Then, for N large enough, the orbit of I under the action of Φ_N has cardinal N and I is its minimal element:*

$$(3.34) \quad \exists M = M(I), \forall N \geq M : I = \min(\Omega(I, N)) \text{ and } |\Omega(I, N)| = N.$$

Proof. Let $G_{I,N}$ denote the stabilizer of I under Φ_N . Then, $N\mathbb{Z} \subset G_{I,N}$ so that $G_{I,N} = p_N\mathbb{Z}$ for some $p_N \geq 1$ such that $p_N|N$. Denote $d(N) = N - I_1 \geq 1$. Then, for $p_N\mathbb{Z}$ to be the stabilizer of I , one must have $d(N) \leq p_N$ (recall that $I_n = 0$). However, as I_1 is fixed, for $N > 2I_1$, $\frac{N}{2} < d(N) \leq p_N$ which implies that $p_N = N$ since $p_N|N$ which gives an orbit of cardinal N .

Let us show that I is minimal in its orbit $\Omega(I, N)$ when N is large enough. Take N such that $I_1 \leq \frac{N}{2}$. The only points J in the orbit of I such that $J_n = 0$ different from I are

$$\{\Phi_N(-I_{n-1}, I), \Phi_N(-I_{n-2}, I), \dots, \Phi_N(-I_1, I)\}$$

These tuples are all strictly greater than I since $N - (I_k - I_{k+1}) - I_1 \geq N - 2I_1 > 0$. Since the minimal element of an orbit must have $I_n = 0$, the only possibility is I . \square

Lemma 3.13 (Orbit decomposition). *Let n, N be fixed. Then,*

$$(3.35) \quad \sum_{I \in I_{n,N}} F(I, \lambda, \mu, \nu, N) = \sum_{I: I_n=0} 1_{I=\min(\Omega(I,N))} |\Omega(I, N)| F(I, \lambda, \mu, \nu, N).$$

Proof of Lemma 3.13. By the translation invariance of Lemma 3.10,

$$\sum_{I \in I_{n,N}} F(I, \lambda, \mu, \nu, N) = \sum_{0 \leq I_n < \dots < I_1 \leq N-1} F(I, \lambda, \mu, \nu, N)$$

We decompose the elements $0 \leq I_n < \dots < I_1 \leq N-1$ along orbits of the action defined in Section 3.2.1.

$$\begin{aligned} \sum_{0 \leq I_n < \dots < I_1 \leq N-1} F(I, \lambda, \mu, \nu, N) &= \sum_{\Omega \in Orbits(N)} \sum_{I \in \Omega} F(I, \lambda, \mu, \nu, N) \\ &= \sum_{\Omega \in Orbits(N)} |\Omega| F(\min(\Omega), \lambda, \mu, \nu, N) \\ &= \sum_{I_1 > \dots > I_n=0} 1_{I=\min(\Omega(I,N))} |\Omega(I, m)| F(I, \lambda, \mu, \nu, N) \end{aligned}$$

\square

We will need the following result which asserts that I_1/N cannot be arbitrary close to one.

Lemma 3.14 (Uniform spacing of I_1). *Let n be fixed. Then,*

$$\forall N \geq n, \forall \Omega \in Orbits(N) : \frac{(\min(\Omega))_1}{N} \leq 1 - \frac{1}{n}.$$

Proof of Lemma 3.14. Let $N \geq n$ and consider $\Omega \in Orbits(N)$. Denote $I = \min(\Omega)$ its minimal element. Assume for the sake of contradiction that $I_1 > N - \frac{N}{n}$. Divide the circle in n pieces that we denote P_1, \dots, P_n of length $\frac{N}{n}$ starting from $I_n = 0$, that is, viewing the circle as $[0, 1]$,

$$P_j = \left[j - 1 \frac{N}{n}, j \frac{N}{n} \right], 0 \leq j \leq n - 1.$$

Since $I_1 > N - \frac{N}{n}$ and $I_n = 0$, I_1 and I_n both belong to the last piece P_n . There are $n - 2$ remaining elements $I_2 > \dots > I_{n-1}$ to be placed inside the $n - 1$ unused segments P_1, \dots, P_{n-1} and P_n . Thus, there exists $1 \leq j \leq n - 1$ such that $I \cap P_j = \emptyset$. Take the maximal such j and consider $r = \max\{l \in [1, n] : I_l \geq j \frac{N}{n}\}$ the first element after P_j starting from 0 counterclockwise on the circle. We claim that

$$J = \Phi_N(-I_r, I) < I$$

Indeed, since P_j is empty, $J_1 \leq N - \frac{N}{n} < I_1$. This contradicts the fact that I is minimal in the orbit Ω_N .

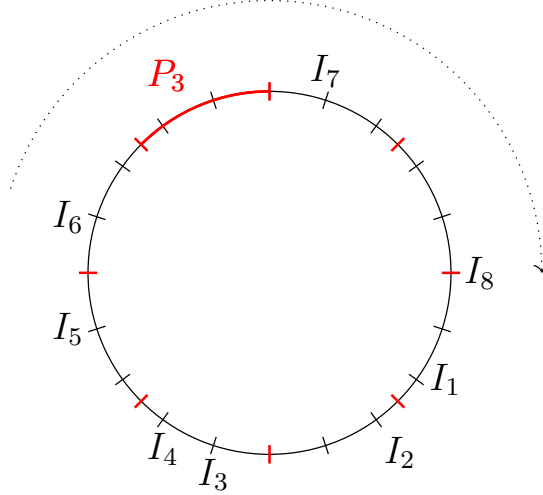


FIGURE 6. Illustration of the argument for $N = 20$ and $n = 8$. Red ticks are the $j \frac{N}{n}$ for $0 \leq j \leq n - 1$ delimiting the P_j 's. Here $j = 3$ is the maximal index for which P_j is empty, see the red arc for P_3 and $r = 6$ with $I_r = 9$. The rotation $\Phi_N(-I_r, I) = \Phi_N(-9, I) = J$ is represented by the dotted arrow. J has $J_1 = I_7 - 9[20] = 15$ which is strictly inferior to $I_1 = 18$ leading to a contradiction as I should be minimal in its orbit.

□

3.2.2. Convergence of scaled coefficients.

Lemma 3.15 (Control of $F(I, \lambda, \mu, \nu, N)$). *Let n, N be fixed with $n \geq 3$. Then,*

$$(3.36) \quad \frac{N^{-n+1}}{N^{(n-1)(n-2)/2}} |F(I, \lambda, \mu, \nu, N)| \leq C_I.$$

for some C_I such that $\sum_{I: I_n=0} C_I < \infty$.

Proof of Lemma 3.15. One has

$$|F(I, \lambda, \mu, \nu, N)| = \left| \frac{\det[e^{\frac{2i\pi I_r \lambda'_s}{N}}] \det[e^{\frac{2i\pi I_r \mu'_s}{N}}] \det[e^{-\frac{2i\pi I_r (\nu^\vee)'_s}{N}}]}{\Delta(\xi^I)} \right| \leq \frac{(n!)^{3n}}{|\Delta(\xi^I)|}.$$

On the other hand,

$$\begin{aligned} \frac{1}{|\Delta(\xi^I)|} &= \prod_{1 \leq r < s \leq n} \left| \exp\left(\frac{2i\pi I_r}{N}\right) - \exp\left(\frac{2i\pi I_s}{N}\right) \right|^{-1} \\ &= \prod_{1 \leq r < s \leq n} \left| 2 \sin\left(\frac{\pi(I_r - I_s)}{N}\right) \right|^{-1}. \end{aligned}$$

Recall that on $[0, \pi - c]$, one has $\sin(x) \geq \frac{\sin(\pi - c)}{\pi - c} x$. Using Lemma 3.14 for I minimal in its orbit,

$$\forall 1 \leq r < s \leq n : \pi \frac{I_r - I_s}{N} \leq \pi \frac{I_1}{N} \leq \pi \left(1 - \frac{1}{n}\right),$$

so that

$$\sin\left(\frac{\pi(I_r - I_s)}{N}\right) \geq c_n \frac{I_r - I_s}{N}$$

with $c_n = \frac{\sin\pi(1-1/n)}{(1-1/n)}$. Thus,

$$\frac{1}{|\Delta(\xi^I)|} \leq \left(\frac{N}{c_n}\right)^{n(n-1)/2} \prod_{1 \leq r < s \leq n} \frac{1}{I_r - I_s}.$$

It remains to prove that $\sum_{I: I_n=0} \frac{1}{\Delta(I)} < \infty$. We will proceed by induction on n . For $n = 3$, the sum is

$$\sum_{I_1 > I_2 > I_3=0} \frac{1}{(I_1 - I_2)I_1 I_2} = \sum_{I_2 \geq 1} \frac{1}{I_2} \sum_{I_1 \geq I_2+1} \frac{1}{(I_1 - I_2)I_1}.$$

Moreover, for $I_2 \geq 1$,

$$\sum_{I_1 \geq I_2+1} \frac{1}{(I_1 - I_2)I_1} \leq \frac{1}{I_2 + 1} + \int_{I_2+1}^{\infty} \frac{1}{t(t - I_2)} dt = \frac{1}{I_2 + 1} + \frac{\ln(I_2 + 1)}{I_2}$$

which proves the convergence for $n = 3$ since $\sum_{I_2 \geq 1} \frac{1}{I_2} \left(\frac{1}{I_2+1} + \frac{\ln(I_2+1)}{I_2} \right) = C < \infty$. For $n \geq 4$,

$$\sum_{I_1 > \dots > I_{n-1} > I_n=0} \prod_{1 \leq r < s \leq n} (I_r - I_s)^{-1} = \sum_{I_2 > \dots > I_{n-1} > I_n=0} \prod_{2 \leq r < s \leq n} (I_r - I_s)^{-1} \sum_{I_1 > I_2} \prod_{2 \leq s \leq n} (I_1 - I_s)^{-1}$$

and, since

$$\sum_{I_1 > I_2} \prod_{2 \leq s \leq n} (I_1 - I_s)^{-1} \leq \sum_{I_1 > I_2} (I_1 - I_2)^{-(n-1)} \leq c_3 = \frac{\pi^2}{6},$$

we have

$$\sum_{I_1 > \dots > I_{n-1} > I_n=0} \prod_{1 \leq r < s \leq n} (I_r - I_s)^{-1} \leq c_3 \sum_{I_2 > \dots > I_{n-1} > I_n=0} \prod_{2 \leq r < s \leq n} (I_r - I_s)^{-1} \leq c_3^{n-3} C < \infty.$$

Therefore,

$$\frac{N^{-n+1}}{N^{(n-1)(n-2)/2}} |F(I, \lambda, \mu, \nu, N)| \leq \frac{(n!)^{3n}}{|\Delta(\xi^I)|} = C_I.$$

with $\sum_{I: I_n=0} C_I < \infty$ as wanted. \square

Lemma 3.16 (Pointwise convergence). *Let $(\alpha, \beta, \gamma) \in \mathcal{H}^3$. For $N \geq 1$, let $(\lambda_N, \mu_N, \nu_N)$ be three partition in \mathbb{Z}_{N-n}^n such that $|\lambda_N| + |\mu_N| + |\nu_N| = n(N - n) + dN$ for some $d \in \mathbb{Z}_{\geq 0}$ and such that $\frac{1}{N}\lambda_N = \alpha + o(1)$, $\frac{1}{N}\mu_N = \beta + o(1)$ and $\frac{1}{N}\nu_N = \gamma + o(1)$ as $N \rightarrow +\infty$. Let $I = I_1 > \dots > I_{n-1} > I_n = 0$ be fixed. Then,*

(3.37)

$$\lim_{N \rightarrow \infty} \frac{N^{-n} \delta_{I=\min(\Omega(I, N))} |\Omega(I, N)|}{N^{(n-1)(n-2)/2}} F(I, \lambda_N, \mu_N, \nu_N, N) = \lim_{N \rightarrow \infty} \frac{N^{-n+1}}{N^{(n-1)(n-2)/2}} F(I, \lambda_N, \mu_N, \nu_N, N)$$

$$(3.38) \quad = (2\pi)^{-n(n-1)/2} \frac{1}{\Delta(I)} \det[e^{2i\pi\alpha_r I_s}] \det[e^{2i\pi\beta_r I_s}] \det[e^{-2i\pi\gamma_r I_s}].$$

Proof of Lemma 3.16. The first equality is derived from Lemma 3.12 which implies that for any I and N large enough

$$\delta_{I=\min(\Omega(I, N))} |\Omega(I, N)| = N.$$

For a fixed $n \geq 3$, by continuity,

$$\begin{aligned}
 \lim_{N \rightarrow \infty} \det \left[\exp \left(\frac{2i\pi I_r(\lambda_{N,s} + s - 1)}{N} \right) \right] &= \det[\exp(2i\pi I_r \alpha_s)] \\
 \lim_{N \rightarrow \infty} \det \left[\exp \left(\frac{2i\pi I_r(\mu_{N,s} + s - 1)}{N} \right) \right] &= \det[\exp(2i\pi I_r \beta_s)] \\
 \lim_{N \rightarrow \infty} \det \left[\exp \left(-\frac{2i\pi I_r(\nu_{N,s} + s - 1)}{N} \right) \right] &= \det[\exp(-2i\pi I_r \gamma_s)] \\
 \lim_{N \rightarrow \infty} \frac{N^{-n+1}}{N^{(n-1)(n-2)/2} \Delta(\xi^I)} &= \left(\frac{1}{2\pi} \right)^{n(n-1)/2} \frac{1}{\Delta(I)}
 \end{aligned}$$

where for the last convergences, we used that $\sin\left(\frac{\pi(I_r - I_s)}{N}\right) \sim \frac{\pi(I_r - I_s)}{N}$ for a fixed subset I . The four convergences above imply the result. \square

Proof of Theorem 3.8. From (3.31), together with Lemma 3.13, one has

$$c_{\lambda_N, \mu_N}^{\nu_N, d} = N^{-n} \sum_{I: I_n=0} \delta_{I=\min(\Omega(I, N))} |\Omega(I, N)| F(I, \lambda_N, \mu_N, \nu_N, N).$$

By Lemma 3.16 and Lemma 3.15 using the dominated convergence theorem we have that

$$\lim_{N \rightarrow \infty} N^{-(n-1)(n-2)/2} c_{\lambda_N, \mu_N}^{\nu_N, d} = \sum_{I: I_n=0} \frac{(2\pi)^{-n(n-1)/2}}{\Delta(I)} \det[e^{2i\pi\alpha_r I_s}] \det[e^{2i\pi\beta_r I_s}] \det[e^{-2i\pi\gamma_r I_s}] = J[\gamma|\alpha, \beta].$$

where $J[\gamma|\alpha, \beta]$ was defined in (3.27) such that

$$d\mathbb{P}[\gamma|\alpha, \beta] = \frac{sf(n-1)(2\pi)^{(n-1)(n-2)/2} \Delta(e^{2i\pi\gamma})}{\Delta(e^{2i\pi\alpha}) \Delta(e^{2i\pi\beta}) n!} J[\gamma|\alpha, \beta].$$

\square

4. PUZZLES OF THE QUANTUM COHOMOLOGY OF GRASSMANNIANS AND THEIR SKELETON

The main goal of this section is a rewriting of the puzzle formula of [5] for the expression of quantum LR-coefficient in terms of a more compact form approaching the hive model yielding the classical LR-coefficients, see [16].

4.1. Puzzles and the quantum-LR coefficients. We will mainly work on puzzles describing the two-step flag cohomology from [5], in the special case where they describe the quantum Littlewood-Richardson coefficients previously introduced in Section 3.

Definition 4.1. *The triangular grid of size N , denoted by T_N , is the planar graph whose vertices are the set $V_N = \{r + se^{i\pi/3}, r, s \in \mathbb{N}, r + s \leq N\}$ and edges are the set $E_N = \{(x, y), x, y \in T_N, |y - x| = 1\}$.*

The faces of T_N are triangles which are called direct (resp. reversed) if the corresponding vertices $(x^1, x^2, x^3) \in V_n^3$ can be labelled in such a way that $x^2 - x^1 = 1$ and $x_3 - x_1 = e^{i\pi/3}$ (resp. $x_3 - x_1 = e^{-i\pi/3}$).

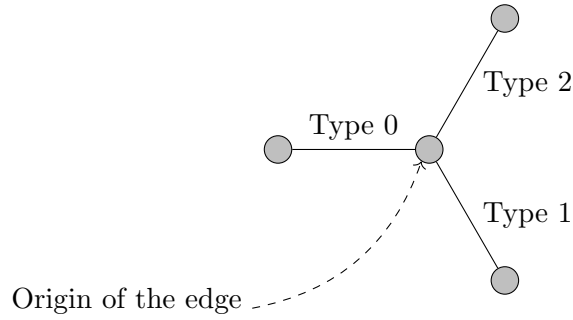
Remark that the set of edges can be partitioned into three subset depending on their orientation. If $x = r + se^{i\pi/3} \in T_N$ we define three coordinates

$$x_0 = N - (r + s), \quad x_1 = r, \quad x_2 = s,$$

and we usually denote an element of T_N by those three coordinates to emphasize the threefold symmetry of the triangle. We say that an edge $e = (x, x + v)$ is of type ℓ , $\ell \in \{0, 1, 2\}$ when $v = e^{i\pi + 2\ell i\pi/3}$.

Definition 4.2. *For $x \in T_n$ and $\ell \in \{0, 1, 2\}$ such that $x + e^{i\pi + 2\ell i\pi/3} \in T_N$, the coordinates of the edge $e = (x, x + e^{i\pi + 2\ell i\pi/3})$ of type ℓ is the triple (e_0, e_1, e_2) given by $e_i = x_i$. We define the height $h(e)$ of an edge of type ℓ by*

$$h(e) = e_\ell.$$

FIGURE 7. Type of an edge in T_N

If $e = (x, y)$ is of type ℓ , we have

$$(4.1) \quad y_\ell = x_\ell + 1, \quad y_{\ell-1} = x_{\ell-1}, \quad y_{\ell+1} = x_{\ell+1} - 1.$$

Remark that the height of an edge does not characterize its position, since for example the translations of an edge of type 1 by $e^{i\pi/3}$ will have the same height.

Let us consider the following set of puzzle pieces, which are considered as the assignment of a label in $\{0, \dots, 7\}$ to edges of T_n around a triangular face. Each piece can be rotated by a multiple of $\frac{\pi}{3}$ but not reflected.

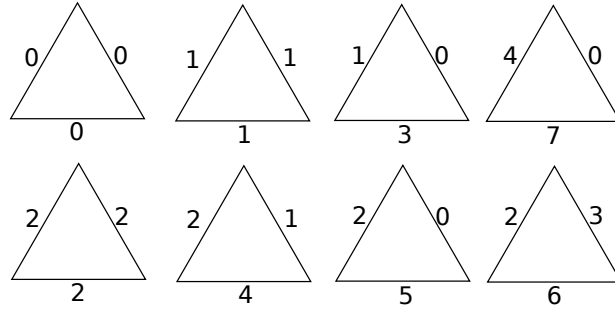


FIGURE 8. Possible pieces of the puzzle

Definition 4.3. A triangular puzzle of size $N \geq 1$ is a map $P : E_N \rightarrow \{0, \dots, 7\}$ such that the value around each triangular face belongs to the set of possible puzzle piece displayed in Figure 8.

The boundary coloring ∂P of a puzzle P is the sequence $(\omega_0, \omega_1, \omega_2)$ such that ω_ℓ is the sequence $(P(e))_{e \in \partial_\ell T_N}$, where $\partial_\ell T_N$ is the sequence of boundary edges of T_N of type ℓ ordered by their height.

For any triple $(\omega_0, \omega_1, \omega_2)$ of words in $\{0, 1, 2\}^N$, we denote by $P(\omega_0, \omega_1, \omega_2)$ the set of puzzles whose boundary coloring is $(\omega_0, \omega_1, \omega_2)$.

Based on a conjecture of Knutson and a previous work of Buch [6] relating the quantum cohomology of Grassmannians to the classical cohomology of the two steps flag manifold, it has been proven in [5] that such puzzles describe the quantum Littlewood-Richardson coefficient in the following way.

Theorem 4.4 ([5]). Let $1 \leq n \leq N$ and $\lambda^0, \lambda^1, \lambda^2$ be partitions of length n with first part smaller than $N - n$ such that $|\lambda^1| + |\lambda^2| = |\lambda^0| + Nd$. Then, $\langle \sigma_{\lambda^1}, \sigma_{\lambda^2}, \sigma_{\lambda^0}^\vee \rangle$ is equal to $\#P(\omega_0, \omega_1, \omega_2)$ where $\omega_\ell, \ell \in \{0, 1, 2\}$ are constructed as follows :

- (1) for $\ell \in \{1, 2\}$, set $\omega_\ell(\lambda_i^\ell + (n - i)) = 0$ for $1 \leq i \leq n$ and $\omega_\ell(i) = 2$ otherwise,
- (2) set $\omega_0(N - 1 - (\lambda_i^0 + (n - i))) = 0$ for $1 \leq i \leq n$ and $\omega_0(i) = 2$ otherwise,
- (3) for $\ell \in \{0, 1, 2\}$, replace the d last occurrences of 0 and the d first occurrences of 2 in ω_ℓ by 1.

The set of pieces can be further simplified in two steps. First, gluing two pieces with edges having label 2 along an edge labeled 4, 5 or 6 yields lozenges with edge labelled 2 and either 0, 1 or 3; then concatenating consecutively such lozenges having same label 0, 1 or 3 and considering also the triangle with all edges labeled 2 yield the pieces of Figure 10, which are called pieces of type II. Let us then call pieces of type I any piece displayed in Figure 9 which consists of the first three triangles of Figure 8 and pieces obtained by concatenating two pieces with label (2, 4, 1) and an arbitrary even number of

pieces with label $(4, 7, 0)$. We will first show that a puzzle P is completely characterize by the position of pieces of type I.

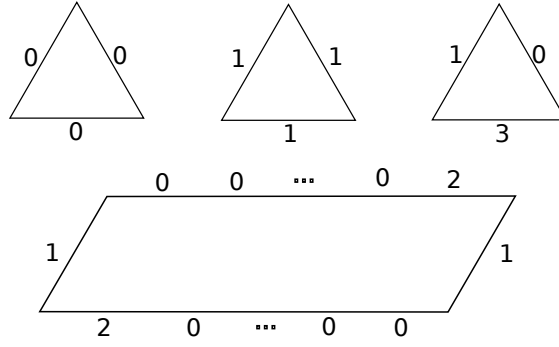


FIGURE 9. Puzzle pieces of type I.

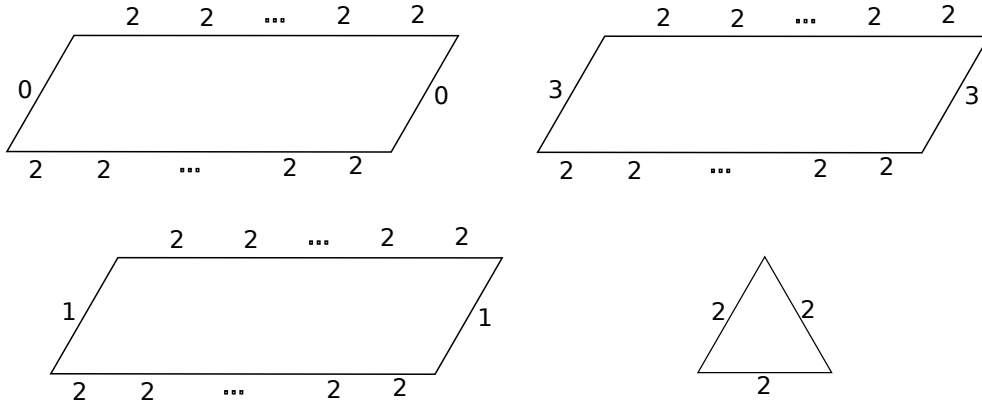


FIGURE 10. Puzzle pieces of type II.

4.2. Edge, vertex and face partitions. Let us first mention a first general result on height of edges on the border of a same triangle.

Lemma 4.5 (Triangle sum). *Suppose that f is a triangle of T_N with edges e^0, e^1, e^2 . Then,*

$$(4.2) \quad \begin{aligned} h(e^0) + h(e^1) + h(e^2) &= N - 1 \text{ if } f \text{ is direct} \\ h(e^0) + h(e^1) + h(e^2) &= N - 2 \text{ if } f \text{ is reversed.} \end{aligned}$$

Moreover, $e_{i-1}^i = e_{i-1}^{i-1}$ (resp. $e_{i-1}^i = e_{i-1}^{i+1}$), for $i \in \{0, 1, 2\}$ if f is direct (resp. reversed).

Proof. A direct triangle has edges $e^0 = (N - (i + j + 1), i + 1, j)$, $e^1 = (N - (i + j + 1), i, j + 1)$ and $e^2 = (N - (i + j), i, j)$ for some $0 \leq i, j \leq N - 1$ with $i + j \leq N - 1$, so that

$$h(e^0) + h(e^1) + h(e^2) = N - 1 + 0 + 0 = N - (i + j + 1) + i + j = N - 1.$$

A reversed triangle has edges $e^0 = (N - (i + j + 1), i, j + 1)$, $e^1 = (N - (i + j), i - 1, j + 1)$ and $e^2 = (N - (i + j), i, j)$ for some $1 \leq i, j \leq N - 1$ with $i + j \leq N - 1$, so that

$$h(e^0) + h(e^1) + h(e^2) = N - 1 + 0 + 0 = N - (i + j + 1) + i - 1 + j = N - 2.$$

It is also clear from the coordinates of the edges that $e_{i-1}^i = e_{i-1}^{i-1}$ (resp. $e_{i-1}^i = e_{i-1}^{i+1}$) for $i \in \{0, 1, 2\}$ if the triangle is direct (resp. reversed). \square

Definition 4.6. *The edge set of a puzzle P is the set \mathcal{E} of edges labeled 0, 1, 3 of either a type I piece or on the boundary.*

The vertex partition of \mathcal{E} is the covering \mathcal{P}_v of \mathcal{E} whose sets of size greater than one consist of all the edges colored $\{0, 1, 3\}$ of a same type I piece and singletons consists of edges of \mathcal{E} on the boundary of T_N not belonging to a type I piece.

The edge partition of \mathcal{E} is the set partitions \mathcal{P}_e of \mathcal{E} whose blocks of size greater than one consist of edges of a common type II piece.

The face partition \mathcal{P}_f is the set partition of V_n whose blocks are the connected components of the subgraph of T_N obtained by only keeping the edges colored 2.

Remark that \mathcal{P}_e can also have singletons. An element $e \in \mathcal{E}$ is a singleton of \mathcal{P}_e if and only if it is a common edge of two type I pieces. However, no element of \mathcal{E} can be a singleton of both \mathcal{P}_v and \mathcal{P}_e , since a border edge colored 0 or 1 not belonging to a type I piece has to belong to a type II piece.

Lemma 4.7. *A block of order 3 in \mathcal{P}_v consists of three edges e^0, e^1, e^2 of type 0, 1, 2 such that*

- *either $e_{i-1}^i = e_{i-1}^{i-1}$, $i \in \{0, 1, 2\}$ and $\sum_{i=0}^2 h(e^i) = N - 1$, or $e_{i-1}^i = e_{i-1}^{i+1}$, $i \in \{0, 1, 2\}$ and $\sum_{i=0}^2 h(e^i) = N - 2$,*
- *$(c(e^0), c(e^1), c(e^2))$ is either $(0, 0, 0)$, $(1, 1, 1)$ or any cyclic permutation of $(0, 1, 3)$.*

A block of order $2(r+1)$, $r \geq 2$ in \mathcal{P}_v consists of $2(r+1)$ edges $\{e^0, f^1, \dots, f^r, e^{0'}, f^{1'}, \dots, f^{r'}\}$ such that

- *$e^0, e^{0'}$ are of type i when $f^1, \dots, f^r, e^{0'}, f^{1'}, \dots, f^{r'}$ are of type $i+1 \pmod 3$ for some $i \in \{0, 1, 2\}$*
- *$h(e^0) = h(e^{0'})$ and $f_i^1 = \dots = f_i^r = f_i^{1'} + 1 = \dots = f_i^{r'} + 1 = h(e^0) + 1$,*
- *$h(f^s) = h(f^{s'}) = h(f^1) + (s-1) = e_{i+1}^{0'}$,*
- *the edges f^0 and $f^{r+1'}$ of type $i+1$ with $f_i^{r+1} = f_i^{r+1'} + 1 = h(e^0) + 1$ and $h(f^0) = e_{i+1}^{0'} - 1$ and $h(f^{r+1'}) = e_{i+1}^0 - 1$ are not in \mathcal{E} .*

Any edge $e \in \mathcal{E}$ belongs to at most two blocks of \mathcal{P}_v .

Proof. In the case of a block of order 3, $\{e^0, e^1, e^2\}$ for a triangle of T_N and the results on the height of the edges is given by Lemma 4.5. The results on the color of the edges is given by the possible coloring of edges of Type I pieces from Figure 9.

In the case of a block B of order $2(r+1)$, the edges correspond to the boundary edges not colored 2 of a puzzle piece of the last shape of Figure 9. We can thus first label cyclically the boundary edges colored 0 and 1 as $\{e^0, f^1, \dots, f^r, e^{0'}, f^{1'}, \dots, f^{r'}\}$ so that e^0 and $e^{0'}$ are colored 1 and of type $i \in \{0, 1, 2\}$ and $f^i, f^{i'}$, $1 \leq i \leq r$ are colored 0 and are of type $i+1$. Then, remark that such a piece is the concatenation of $r+1$ direct triangle T_1, \dots, T_{r+1} and $r+1$ reversed triangle T'_1, \dots, T'_{r+1} such that T_i and T'_i (resp. T'_i and T_{i+1}) share an edge of type $i+1$ (resp. i_1), e^0 (resp. $e^{0'}$ is the edge of type i of T_1 (resp. T'_{r+1}) end f^i (resp. $f^{i'}$) is the edge of type $i+1$ of T'_i (resp. T_{i+1}). The relations giving the height and the labels of the edges are then direct consequences of Lemma 4.5. \square

Definition 4.8. *A pair $B = \{e^1, e^2\} \subset E_N$ of edges is called admissible if e^1 and e^2 have same type $j \in \{0, 1, 2\}$ and same height.*

The strip S_B of an admissible pair $B = \{e^1, e^2\}$ of type j is the set of all edges $e = (x, y) \in E_n$ such that x, y belong to the parallelogram delimited by e^1 and e^2 . Namely, if $e_j^1 = e_j^2$ and $e_{j-1}^2 \leq e_{j-1}^1$, S_B is the set of edges (x, y) such that $e_j^1 \leq x_j, y_j \leq e_j^1 + 1$, $e_{j-1}^2 \leq x_{j-1}, y_{j-1} \leq e_{j-1}^1$.

The boundary $\partial_2 S_B$ of a strip S_B consists of all edges of S_B type $j+1$.

Remark that such a definition is still valid if $e^1 = e^2$, in which case $B = \{e^1\}$ is always admissible and $S_B = \{e^1\}$. In particular, for any $B \in \mathcal{P}_e$ for a puzzle P , S_B consists of all edges appearing on the type II piece bordered by elements of B : B coincide exactly the boundary edges of S_B which are not labeled 2 and $\partial_2 S_B$ consists of the boundary edges of the type II piece which are labeled 2 (that is, all boundary edges except for B).

Moreover, given a pair $\{e^1, e^2\}$ of edges of same type and same height, one can rephrase the condition of belonging to the strip S_B depending on the type of the edge we consider:

- *if e has type j , $e \in S_B$ if and only if $e_j = e_j^1$ and $e_{j-1} \in [e_{r-1}^2, e_{r-1}^1]$,*
- *if e has type $j+1$, $e \in S_B$ if and only if $e_j \in \{e_j^1, e_j^1 + 1\}$ and $e_{j-1} \in [e_{r-1}^2, e_{r-1}^1]$,*
- *if e has type $j-1$, $e \in S_B$ if and only if $e_j = e_j^1 + 1$ and $e_{j-1} \in [e_{r-1}^2, e_{r-1}^1]$.*

Lemma 4.9. *Let P be a puzzle and \mathcal{P}_e the corresponding edge partition. Any pair $\{e^1, e^2\} \in \mathcal{P}_e$ (with possibly $e^1 = e^2$) is admissible and is such that $c(e^1) = c(e^2) \in \{0, 1, 3\}$. For any different blocks $B, B' \in \mathcal{P}_e$, $S_B \cap S_{B'} \subset \partial_2 S_B \cap \partial_2 S_{B'}$.*

Proof. Recall that blocks of \mathcal{P}_e of size greater than one corresponds to edges colored 0, 1 or 3 on the border of a same Type II piece from 10. In particular the blocks of size greater than one are only pairs, and the first part of the lemma is a direct consequence of the possible type II pieces of Figure 10.

For the second part of the lemma, remark first that for any block $\{e^1, e^2\}$ in \mathcal{P}_e , $S_B \cap \mathcal{E} = B$: first, boundary edges of S_B are either colored 0, 1 or 3 and in B or colored 2 and not in \mathcal{E} . Then, interior edges of S_B which are colored 0, 1 or 3 are boundary edges of two pieces with the same labelling of the second row of Figure 8 and thus are not in \mathcal{E} (remark that the second piece of the second row of Figure 8 is used in the last type I piece of Figure 8 but is surrounded by pieces with different boundary labels).

Consider two different block $B, B' \in \mathcal{P}_e$. Let $e \in S_B \cap S_{B'}$. Remark that any edge of S_B or $S_{B'}$ is not on the border of the strip if and only if it is neither in $B \cup B'$ nor colored 2. Hence, if e is colored 2, then $e \in \partial_2 S_B \cap \partial_2 S_{B'}$. Suppose by contradiction that e is not colored 2. Since $B \cap S_{B'} = S_B \cap B' = \emptyset$ and e is not colored 2, e belongs to the interior of both S_B and $S_{B'}$: hence, the two triangular puzzle pieces whose boundary is e belong to S_B and $S_{B'}$, and we deduce that there is an edge f^1 colored 0, 1 or 3 which belong to $S_B \cap S_{B'}$. If $f^1 \notin B \cup B'$, f^1 belong to the interior of S_B and $S_{B'}$, and thus there exists an edge $f^2 \in S_B \cap S_{B'}$ with $f_{i+1}^2 = f_{i+1}^1 + 1$. Let us repeat the process until there is an edge $f^s \in S_B \cap S_{B'}$ which belongs to either B or B' . This means that $B \cap S_{B'}$ or $B' \cap S_B$ is not empty, which contradicts the fact proven previously that $S_B \cap \mathcal{E} = B$ and $S_{B'} \cap \mathcal{E} = B'$. Hence, any edge not colored 2 do not belong to $S_B \cap S_{B'}$. \square

We say that two admissible pairs $B = \{e^1, e^2\}$ and $B' = \{e^3, e^4\}$ of E_N cross when

- either $B \cap S_{B'} \neq \emptyset$ or $B' \cap S_B \neq \emptyset$,
- or $B = \{e^1, e^2\}, B' = \{e^3, e^4\}$ are blocks of size two which are of respective type $j, j + 1$ for some $j \in \{0, 1, 2\}$, and

$$e_j^3 \leq e_j^1 = e_j^2 \leq e_j^4 \quad e_{j+1}^1 \leq e_{j+1}^3 = e_{j+1}^4 < e_{j+1}^2.$$

Lemma 4.10. *For any distinct pairs $B, B' \subset E_N$, the two following properties are equivalent :*

- (1) B and B' do not cross,
- (2) $S_B \cap S_{B'} \subset \partial_2 S_B \cap \partial_2 S_{B'}$.

Proof. Let B, B' be admissible pairs of E_N .

If B is a singleton, then $\partial_2 S_B = \emptyset$ and thus $\partial_2 S_B \cap \partial_2 S_{B'} = \emptyset$. Moreover, since B is a singleton, $B = S_B$. Hence, the non-crossing condition is equivalent to $S_B \cap S_{B'} = \emptyset$, and B and B' do not cross if and only if $S_B \cap S_{B'} \subset \emptyset = \partial_2 S_B \cap \partial_2 S_{B'}$.

If $B = \{e^1, e^2\}$ and $B' = \{e^3, e^4\}$ are pairs of the same type $j \in \{0, 1, 2\}$, the non-crossing condition means that $B \cap S_{B'} = \emptyset$ and $B' \cap S_B = \emptyset$. Suppose that B and B' cross, and without loss of generality, assume that $B \cap S_{B'} \neq \emptyset$. Since $B \cap \partial_2 S_B = \emptyset$, we deduce that $B \cap S_{B'} \not\subset \partial_2 S_B \cap \partial_2 S_{B'}$. Hence, $S_B \cap S_{B'} \not\subset \partial_2 S_B \cap \partial_2 S_{B'}$.

Reciprocally, suppose that $S_B \cap S_{B'} \not\subset \partial_2 S_B \cap \partial_2 S_{B'}$, and assume without loss of generality that $S_B \cap S_{B'} \cap (S_B \setminus \partial_2 S_B) \neq \emptyset$. Since $\partial_2 S_B$ is the set of edges of S_B of type $j + 1$, there exists an edge $e = (x, y)$ of type j or $j - 1$ in $S_B \cap S_{B'}$. If e is of type j , this means that $e_j = e_j^1 = e_j^2$ and $e_{j+1} \in [e_{j+1}^1, e_{j+1}^2]$. Similarly, $e_j = e_j^3 = e_j^4$ and $e_{j+1} \in [e_{j+1}^3, e_{j+1}^4]$. Hence, $[e_{j+1}^1, e_{j+1}^2] \cap [e_{j+1}^3, e_{j+1}^4] \neq \emptyset$, and the extremity of one of these interval is contained in the other. Assume without loss of generality that $e_{j+1}^1 \in [e_{j+1}^3, e_{j+1}^4]$. Then, e_1 is an edge of type j such that $e_j^1 = e_j^3 = e_j^4$ and $e_{j+1}^1 \in [e_{j+1}^3, e_{j+1}^4]$, thus $e^1 \in S_{B'}$ and thus $B \cap S_{B'} \neq \emptyset$. If $e = (x, y)$ is of type $j - 1$, then $x_{j-1} = y_{j-1} - 1$ and $x_j = y_j + 1$, see (4.1). Hence, the conditions $x_j, y_j \in \{e_j^1, e_j^2 + 1\}$ and $x_{j-1}, y_{j-1} \in [e_{j-1}^2, e_{j-1}^1]$ from Definition 4.8 yield that $e' = (x', x)$ with $x'_j = y_j$ and $x'_{j-1} = x_j$ is an edge of type j which belongs to S_B . Similarly, $e' \in S_{B'}$, and the previous reasoning allows to conclude that $B \cap S_{B'} \neq \emptyset$ or $B' \cap S_B \neq \emptyset$.

Suppose finally that $B = \{e^1, e^2\}$ and $B' = \{e^3, e^4\}$ are pairs of respective type j and $j + 1$ for $j \in \{0, 1, 2\}$. Remark first that edges of $\partial_2 S_B$ have type $j + 1$ and edges of $\partial_2 S_{B'}$ have type $j + 2$, so that $\partial_2 S_B \cap \partial_2 S_{B'} = \emptyset$.

Suppose that B and B' cross. First, if $B \cap S_{B'} \neq \emptyset$ or $B' \cap S_B \neq \emptyset$, then $S_B \cap S_{B'} \neq \emptyset$ and thus $S_B \cap S_{B'} \not\subset \partial_2 S_B \cap \partial_2 S_{B'}$. Suppose that $B \cap S_{B'} = B' \cap S_B = \emptyset$, and thus

$$e_j^3 \leq e_j^1 = e_j^2 \leq e_j^4 \quad e_{j+1}^1 \leq e_{j+1}^3 = e_{j+1}^4 < e_{j+1}^2.$$

Consider the edge $e = (x, y)$ of type $j + 1$ with $x_j = e_j^1$ and $x_{j+1} = e_{j+1}^3$. Since e is of type $j + 1$, $y_j = x_j$ and $y_{j+1} = x_{j+1} + 1$. First, since $x_{j+1}, y_{j+1} \in \{e_{j+1}^3, e_{j+1}^3 + 1\}$ and $x_j = y_j = e_j^1 \in [e_j^3, e_j^4]$, $e \in S_{B'}$. Then, remark that

$$x_{j-1} = N - (x_j + x_{j+1}) = N - e_j^1 - e_{j+1}^3, \quad e_{j-1}^1 = N - e_j^1 - e_{j+1}^1, \quad e_{j-1}^2 = N - e_j^2 - e_{j+1}^2.$$

From the equality $e_j^1 = e_j^2$ and $e_{j+1}^1 \leq e_{j+1}^3 < e_{j+1}^2$, we deduce that $x_{j-1} \in]e_{j-1}^2, e_{j-1}^1]$. Since $y_{j-1} = N - y_j - y_{j+1} = N - x_j - x_{j+1} - 1 = x_{j-1} - 1$, $y_{j-1} \in [e_{j-1}^2, e_{j-1}^1]$. The two latter inclusions together with $y_j = x_j = e_j^1$ yield that $e \in S_B$. In particular, $S_B \cap S_{B'} \neq \emptyset$ and thus $S_B \cap S_{B'} \not\subset \partial_2 S_B \cap \partial_2 S_{B'}$.

Suppose that $S_B \cap S_{B'} \neq \emptyset$. If $B \cap S_{B'} \neq \emptyset$ or $B' \cap S_B \neq \emptyset$, then B and B' cross. When $B \cap S_{B'} = B' \cap S_B = \emptyset$, $S_B \cap S_{B'} \neq \emptyset$ if and only if $\partial_2 S_B \cap \partial_2 S_{B'} \neq \emptyset$. One implication is straightforward. For the other implication, remark that there is necessarily an edge $e \in S_B \cap S_{B'}$ which belongs to B or $\partial_2 S_B$, and since $B \cap S_{B'} = \emptyset$, $e \in \partial_2 S_B$. In particular, since B has type j , edges of $\partial_2 S_B$ have type $j + 1$ and thus e has type $j + 1$.

The edge $e = (x, y)$ of type $j + 1$ belongs to $\partial_2 S_B$ if and only

$$\begin{cases} x_j = y_j \in \{e_j^1, e_j^1 + 1\}, \\ x_{j-1} \in [e_{j-1}^2, e_{j-1}^1] \text{ and } y_{j-1} = x_{j-1} - 1 \in [e_{j-1}^2, e_{j-1}^1]. \end{cases}$$

Similarly e of type $j + 1$ to belong to $S_{B'}$ if and only if

$$\begin{cases} x_{j+1} = y_{j+1} - 1 = e_{j+1}^3 = e_{j+1}^4, \\ x_j = y_j \in [e_j^3, e_j^4]. \end{cases}$$

Hence, $e \in \partial_2 S_B \cap S_{B'}$ if and only if

$$\begin{cases} x_j = y_j \in \{e_j^1, e_j^1 + 1\} \cap [e_j^3, e_j^4], x_{j+1} = y_{j+1} - 1 = e_{j+1}^3 \\ N - x_j - e_{j+1}^3 \in]N - e_j^2 - e_{j+1}^2, N - e_j^1 - e_{j+1}^1]. \end{cases}$$

If $x_j = e_j^1$, the latter conditions implies

$$\begin{cases} e_j^3 \leq e_j^1 = e_j^2 \leq e_j^4, \\ e_{j+1}^1 \leq e_{j+1}^3 = e_{j+1}^4 < e_{j+1}^2. \end{cases}$$

If $x_j = e_j^1 + 1$, the latter conditions yield

$$\begin{cases} e_j^3 - 1 \leq e_j^1 = e_j^2 \leq e_j^4 - 1, x_{j+1} = y_{j+1} - 1 = e_{j+1}^3 \\ e_{j+1}^1 - 1 \leq e_{j+1}^3 = e_{j+1}^4 < e_{j+1}^2 - 1. \end{cases}$$

If $e_j^3 - 1 = e_j^2$ and $e_{j+1}^1 - 1 \leq e_{j+1}^3 < e_{j+1}^2 - 1$, then $e_j^3 = e_j^2 + 1$ and

$$e_{j-1}^3 = N - e_j^3 - e_{j+1}^3 \in]N - e_j^2 - e_{j+1}^2, N - e_j^1 - e_{j+1}^1] =]e_{j-1}^2, e_{j-1}^1],$$

so that $e^3 \in S_B$ by the condition following Definition 4.8. Likewise, if $e_{j+1}^3 = e_{j+1}^1 - 1$ and $e_j^1 \in [e_j^3, e_j^4]$, then $e^1 \in S_{B'}$. Hence, the fact that $B \cap S_{B'} = B' \cap S_B = \emptyset$ strengthens the above condition to imply

$$\begin{cases} e_j^3 \leq e_j^1 = e_j^2 \leq e_j^4, \\ e_{j+1}^1 \leq e_{j+1}^3 = e_{j+1}^4 < e_{j+1}^2. \end{cases}$$

□

From the latter lemma, we deduce a description of puzzles in terms of their type I pieces. L

Proposition 4.11. *The map $\Phi : P \mapsto (\mathcal{E}, c)$ is a bijection from the set of puzzles to the set of subsets of E_N with a coloring $c : \mathcal{E} \rightarrow \{0, 1, 3\}$ such that there exist a covering \mathcal{P}_v and a partition \mathcal{P}_e of \mathcal{E} with*

- blocks of \mathcal{P}_v are of size 3 or $2(r + 1)$, $r \geq 1$ and satisfy the properties of Lemma 4.7,
- blocks of \mathcal{P}_e are either singleton or pairs satisfying the properties of Lemma 4.9,
- If e belong to only one block of \mathcal{P}_v , then e is not a singleton of \mathcal{P}_e and if $e \neq e'$ belong to a same block of \mathcal{P}_v , then $\{e, e'\} \notin \mathcal{P}_e$.

Proof. Let us build the candidate reversed bijection, and consider a subset $\mathcal{E} \subset E_N$ with a coloring $c : \mathcal{E} \rightarrow \{0, 1, 3\}$ and a covering \mathcal{P}_v and a partition \mathcal{P}_e satisfying the conditions of Proposition 4.11. For each pair $B = (e, e') \in \mathcal{P}_e$ of type i and color c , color all edges of type i (resp. $i + 1$, resp. $i + 2$) of the strip $S_B \setminus \{e, e'\}$ with the color c (resp. 2 , resp. $c + 5$). Remark that such a coloring is possible, since by properties of Lemma 4.9, any edge belonging to $S_B \cap S_{B'}$ for two strips of respective types i, i' must be included in the boundary $\partial_2 S_B \cap \partial_2 S_{B'}$, which consists then of edges of same type $i + 1 = i' + 1$ colored 2 by the above rule.

Then, consider any block $B \in \mathcal{P}_v$ of order $2(r+1)$ whose edges e, e' colored 1 are of type i . Remark then that by the properties of Lemma 4.7, $B = \{e, e'\}$ is an admissible pair of edges of \mathcal{E} of same color 1. Moreover, all edges but two of the boundary $\partial_2 S_B$ of the strip S_B consists of edges of \mathcal{E} colored 0.

Let i be the type of B , and suppose by contradiction that there is an edge e^0 of type i or $i-1$ inside the strip S_B which is contained in a strip $S_{\tilde{B}}$ for some $\tilde{B} \in \mathcal{P}_e$. Suppose first that $\tilde{B} \neq \{e^0\}$. Then, since e^0 shares a vertex with at least three edges of type $i+1$ colored 0, $S_{\tilde{B}}$ must contain of those three edges, called f^1 ; since \mathcal{P}_e is non-crossing and $f^1 \in \mathcal{E} \cap S_{\tilde{B}}$, $f^1 \in \tilde{B}$ and thus \tilde{B} is of type $i+1$. Since $e^0 \in S_{\tilde{B}}$ and \tilde{B} is of type $i+1$, the other edge f^2 of S_B with same height as the one f^1 must also belong to $S_{\tilde{B}}$, and by the non-crossing condition we have $\tilde{B} = \{f^1, f^2\}$. This contradicts the fact that two elements of the same block of \mathcal{P}_v do not belong to the same block of \mathcal{P}_e .

If e^0 is a singleton of \mathcal{P}_e , then $e \in \mathcal{E}$ and belongs to at least two blocks of \mathcal{P}_v . Hence, there must be another edge e^1 between e^0 and e in the strip S_B which belongs to \mathcal{E} . Iterating the process yields an edge \tilde{e} such that e and \tilde{e} belong to a same block \tilde{B} in \mathcal{P}_v . Then, \tilde{e} cannot be of type $i-1$, for otherwise this block would be a block of order 3, contradicting the fact that the edge f of type $i+1$ with $f_i = e_i, f_{i+1} = e_{i+1} - 1$, which would then belong to this block \tilde{B} , does not belong to \mathcal{E} . Similarly, if \tilde{e} is of type i , then e and \tilde{e} would be boundary edges of type I piece with $2(r'+1)$ edges, with $1 \leq r' < r$. This is impossible since the boundary $\partial_2 S_{\{e, \tilde{e}\}}$ contains at most one edge which is not in \mathcal{E} . Hence, no edge of type i or $i-1$ inside S_B belongs to some strip S_B for $B \in \mathcal{P}_e$ and thus none of those edges has been colored 2 in the previous labelling. Therefore, one can color all type i edge in $S_{\tilde{B}}$ different from e, e' with the label 7 and all type $i-1$ edge in $S_{\tilde{B}}$ with label 6.

Finally, the edge f of type $i+1$ with $f_i = e_i, f_{i+1} = e_{i+1} - 1$ can not be part of a strip $S_{\tilde{B}}$ for some $\tilde{B} = (f^1, f^2)$ of type $i+1$ or $i-1$, for otherwise e would also belong to $S_{\tilde{B}}$ and \mathcal{P}_e would not be non-crossing. Hence, either $f \in \partial_2 S_B$ for some strip S_B and f has been labelled 2 in the first coloring step, or f has not been colored before and thus f can be colored 2.

Finally, color all remaining edge with the label 2. One then check that the labels on the boundary of any triangle of the puzzle satisfy the conditions of Figure 8, so that the labelling of edges of E_n yields a genuine puzzle P . It is then straightforward to check that $P_v = \mathcal{P}_v$ and $P_e = \mathcal{P}_e$. The map Φ is thus surjective.

For the injectivity, remark that the data of \mathcal{P}_v alone gives the list and position of type I pieces of the puzzle, which uniquely characterizes it. \square

4.3. Graph of a puzzle.

Definition 4.12. *The graph of a puzzle P is the graph \mathcal{G}_P whose set of vertices is P_v , set of edges is P_e and set faces is P_f .*

The endpoints of an edge $B_e \in P_e$ are the vertices $B_v, B_{v'} \in P_v$ such that $B_e \cap B_v \neq \emptyset$ and $B_e \cap B_{v'} \neq \emptyset$.

The boundary of a face $B_f \in P_f$ are the edges $B \in P_e$ such that there is $e \in B, v \in B_f$ such that v is an endpoint of e . A face $B_f \in P_f$ is called an outer face (resp. inner face) if there is an element (resp. no element) $v \in B_f$ on the border of T_N .

Remark that elements of P_v and P_e are sets of edges of T_N while elements of P_f are set of vertices of T_N . Moreover, any edge $B \in P_e$ has a type $\ell \in \{0, 1, 2\}$ and a color $c \in \{0, 1, 3\}$, which is the type and the color of the edges of T_N in B .

Let $B_f \in \mathcal{P}_f$ and denote by ∂B_f the set of edges on the boundary of B_f . Then, there is a natural cyclic order on ∂B_f such that $\partial B_f = (B_1 < \dots < B_p)$ where B_i and B_{i+1} share a vertex of \mathcal{P}_v and the edges of ∂B_f are read in the clockwise order around the region B_f .

Lemma 4.13. *Let $B_f \in \mathcal{P}_f$. Then, the sequence of type of edges on the boundary of B_f is a subsequence of $(0, 1, 2, 0, 1, 2)$ (up to cyclic rotation), and two consecutive edges $B < B'$ on the boundary of B_f sharing a vertex $B_v \in \mathcal{P}_v$ are*

- of type $(\ell, \ell+1)$ if B_v is a block of size three and the color of B and B' are either both 0, both 1 or $(3, 0)$, $(0, 1)$ or $(1, 3)$, or a block of size $2r$, $r \geq 2$ and B, B' have respective color $(0, 1)$.
- of type $(\ell, \ell+2)$ if B_v is a block of size $2r$, $r \geq 2$ and B and B' have color $(0, 1)$,
- of type ℓ if B_v is a block of size $2r$, $r \geq 3$ and B and B' have color 0.

Proof. Let (B_1, \dots, B_p) be the previously defined cyclic ordering of the edges around B_f such that B_i, B_{i+1} share a vertex in \mathcal{P}_v . Let $1 \leq i \leq p$ and denote by ℓ the type of B_i . Since B_i, B_{i+1} share the

vertex B_v , there exist $e^i \in B_i$ and $e^{i+1} \in B_{i+1}$ such that $e^i, e^{i+1} \in B_v$. Since e^i, e^{i+1} are not colored 2, the type and colors of e^i (resp. e^{i+1}) are the ones of B_i (resp. B_{i+1}).

If B_v is a block of size 3, then it is a triangle vertex whose boundary colors in the clockwise order are either $(0, 0, 0)$, always $(1, 1, 1)$, or $(1, 3, 0)$ up to a rotation. Since the angle between e^i and e^{i+1} is $-\pi/3$, the type of e_{i+1} is $\ell + 1$, and the colors B_i, B_{i+1} are either $(0, 0)$, $(1, 1)$, $(3, 0)$, $(0, 1)$ or $(1, 3)$.

If B_v is a block of size $2(r + 1)$, $r \geq 1$, then 3 configurations can occur depending on the colors of the consecutive edges :

- if e^i is colored 1 and e^{i+1} is colored 0, then the type of e^{i+1} is $\ell + 1$,
- if e^i is colored 0 and e^{i+1} is colored 1, then the type of e^{i+1} is $\ell - 1$,
- if e^i and e^{i+1} are both colored 0 then the edges are adjacent and have same type ℓ . Remark that in this case, the vertex B_v must have at least 6 edges.

Finally, remark that the angle between two consecutive edges B_i, B_{i+1} is equal to $(1 - r_i/3)\pi$ if the difference of the type from B_i to B_{i+1} is r_i (with $r_i = 3$ if B_i and B_{i+1} have both type ℓ). Since the sum of the angles must be equal to the $(p - 2)\pi$ if B_f is an inner face and smaller otherwise, we must have $\sum_{i=1}^p (1 - r_i/3) \leq p - 2$, so that

$$\sum_{i=1}^p r_i \leq 6.$$

We deduce that the sequence of types of edges of the boundary must be a subsequence of $(0, 1, 2, 0, 1, 2)$, up to cyclic permutation. \square

5. DISCRETE TWO-COLORED DUAL HIVE MODEL

In this section, we associate to each puzzle of size N a two-colored hive in the same spirit as in [16]. Beware that because of the rigid crossing from Figure 9, the discrete hives won't be actual hives as in [16] but rather a dual hive. Let us fix in this section the number n of edges colored 0 or 1 on one edge of the puzzles. This number is the same on each edge of the puzzle and is part of the boundary data of the puzzle. The two colored dual hive associater to a puzzle will then be a decoration of the triangular grid of T_n instead of T_N .

Definition 5.1 (Discrete boundary). *The boundary ∂T_n of the discrete triangular grid T_n is the intersection of T_n with the boundary of the triangle.*

We denote by F_n the set of faces of T_n . Remark that the faces are all triangular, and we decompose F_n as $F_n = F_n^+ \cup F_n^-$ depending on whether the triangles are direct or reversed. For $e \in E_n, f \in T_n$, we write $e \in f$ when e is an edge on the boundary of f . We denote by $E_n^{(\ell)}$ the set of edges of type ℓ and all the notation introduced for T_N are still valid for T_n . Any union of two triangles sharing an edge e is called a lozenge, and e is then called the middle edge of the lozenge.

Remark that the boundary ∂T_n can be decomposed into three subgraphs $\partial T_n^{(i)}$, $0 \leq i \leq 2$, where each subgraph $\partial T_n^{(i)}$ consists of edges of type i .

Definition 5.2 (Discrete two-colored dual hive). *A two-colored discrete dual hive of size n is given by the following combinatorial data on T_n :*

- a color map $C : E_n \rightarrow \{0, 1, 3, m\}$, such that the boundary colors around each triangular face in the clockwise order is either $(0, 0, 0)$, $(1, 1, 1)$, $(1, 0, 3)$ or $(0, 1, m)$ up to a cyclic rotation.
- a label map $L : E_n \rightarrow \mathbb{N}$, with the two following conditions:
 - (1) for all $f \in F_n$ with boundary edges e^0, e^1, e^2 , $L(e^0) + L(e^1) + L(e^2) = N - 1$ except when $f \in F_n^-$ with boundary colors different from $\{0, 1, m\}$, in which case $L(e^0) + L(e^1) + L(e^2) = N - 2$,
 - (2) if e, e' are edges of same type $\ell \in \{0, 1, 2\}$ on the boundary of a same lozenge, then
 - (a) $L(e) = L(e')$ if the middle edge is colored m ,
 - (b) $L(e) \geq L(e')$ if $e'_{\ell+1} = e_{\ell+1} + 1$ and none of the edge different of e' are colored m ,
 - (c) $L(e) > L(e')$ if $e_\ell > e'_\ell$ or if $e'_{\ell+1} = e_{\ell+1} + 1$ and one of the boundary edges different from e' is colored m .

The boundary value $[(c^{(0)}, c^{(1)}, c^{(2)}), (l^{(0)}, l^{(1)}, l^{(2)})]$ of a two-colored discrete dual hive is the restriction of (C, L) to ∂T_n , where $c^{(i)} \in \{b, r, m^-, m^+\}^n$ (resp. $l^{(i)} \in \mathbb{N}^n$) is the restriction of C (resp. L) to $\partial T_n^{(i)}$, for $0 \leq i \leq 2$.

Remark 5.3. *As a corollary of the Condition (2) on the label map, we have $L(e) > L(e')$ for any pair of edges e, e' of same type ℓ such that $e_\ell > e'_\ell$ and $e_{\ell+1} \leq e'_{\ell+1}$. Indeed, it suffices to show this for e, e' such that $e_\ell = e'_\ell + 1$ and $e'_{\ell+1} \in \{e_{\ell+1}, e_{\ell+1} + 1\}$. The case $e'_{\ell+1} = e_{\ell+1}$ is given by Condition (2.c), and we now suppose that $e'_{\ell+1} = e_{\ell+1} + 1$. Let e'' be such that $e''_{\ell+1} = e'_{\ell+1}$ and $e''_\ell = e_\ell + 1$. If the middle edge of the lozenge with boundary e', e'' is not colored m , then by Condition (2.c) we have $L(e'') > L(e')$, and then by (3.a) or (3.b), we get $L(e) \geq L(e'') > L(e')$. If the middle edge of the lozenge with boundary e', e'' is colored m , then $L(e') = L(e'')$. Then, the middle edge colored m of this lozenge is then a boundary edge of the lozenge with boundary e'', e different from e'' and e , so that (3.c) implies that $L(e) > L(e'') = L(e')$.*

For $(c, l) = (c^{(0)}, c^{(1)}, c^{(2)}, l^{(0)}, l^{(1)}, l^{(2)}) \in \{b, r\}^{3n} \times \mathbb{N}^{3n}$, we denote by $H(c, l)$ the set of two-colored discrete dual hives with boundary value (c, l) . For any boundary date (c, l) , denote by $\omega^{c, l}$ the triple of words of length N such that $\omega_\ell^{c, l}(l^{(\ell)}(i)) = c^{(\ell)}(i)$ for $\ell \in \{0, 1, 2\}$ and $1 \leq i \leq n$ and $\omega_\ell^{c, l}(j) = 2$ for other $0 \leq j \leq N - 1$.

Proposition 5.4. *For any valid pair $(c, l) \in \{b, r\}^{3n} \times \mathbb{N}^{3n}$,*

$$|H(c, l)| = |P(\omega^{c, l})|.$$

Remark that Proposition 5.8 actually gives more generally the expression of the structure coefficients of the two-step flag variety in terms of counting of integers points into a convex polytope.

We prove this proposition by exhibiting a bijection $\Phi : P(\omega^{c, l}) \rightarrow H(c, l)$ which is of interest for the sequel. Given a puzzle $P \in P(\omega^{c, l})$, let \mathcal{G}_P be the corresponding graph introduced in Section 4.3.

Let us first transform the \mathcal{G}_P into a new graph $\widehat{\mathcal{G}}_P$ by blowing up each vertex $v \in \mathcal{P}_v$ of size $2(r+1)$ as follows. First, by Lemma 4.7, a vertex $v \in \mathcal{P}_v$ of size $2(r+1)$ has adjacent edges (B^1, \dots, B^{2r+2}) (indexed in the cyclic order around the vertex with first edge colored 1) such that B^1, B^{r+2} have type $\ell \in \{0, 1, 2\}$ and are colored 1 and $B^i, i \notin \{1, r+2\}$ have type $\ell+1$ and are colored 0. Introduce $2r-1$ new edges $\tilde{B}^1, \dots, \tilde{B}^{2r-1}$ of type $\ell-1, \ell, \dots, \ell-1$ and colored $m, 1, \dots, m$ and transform v into $2r$ vertices v^1, \dots, v^{2r} such that the edges adjacent to v^{2j+1} are $(\tilde{B}^{2j}, B^{2r+2-j}, \tilde{B}^{2j+1})$ and edges adjacent to v^{2j+2} are $(\tilde{B}^{2j+1}, \tilde{B}^{2j+2}, B^{j+1})$ (with the convention $\tilde{B}^0 = B^1$ and $\tilde{B}^{2r} = B^{r+2}$. We define the height of \tilde{B}^i as $h(\tilde{B}^{2i}) = h(B^1)$ and $h(\tilde{B}^{2i-1}) = N - 1 - h(B^1) - h(B^{2i})$ for $1 \leq i \leq r-1$.

The resulting graph has then only vertices of degree 3 or singletons (which correspond to edges of T_N on the boundary of the triangle).

Lemma 5.5. *Let B_f be a face in $\widehat{\mathcal{G}}_P$. Then, the the boundary of B_f has*

- 6 edges if no edge of the boundary of B_f is a boundary edge of \mathcal{G}_P ,
- 4 edges if two edges of the boundary of B_f are boundary edges of \mathcal{G}_P of the same type,
- 2 edges if two edges of the boundary of B_f are boundary edges of \mathcal{G}_P of different type.

Proof. By Lemma 4.13 and the blowing-up of vertices of degree larger than 4, two edges B, B' on the boundary of B_f sharing a vertex are of type $(\ell, \ell+1)$, and the edge type of boundary edges of B_f is a subsequence of $(0, 1, 2, 0, 1, 2)$ (up to a cyclic rotation). The only possibility for B_f to have less than 6 edges on the boundary is then having edges which have a singleton as boundary vertex. The edge of \mathcal{E} corresponding to this singleton is necessary a boundary edge of the graph and thus B_f is a connected component of \mathcal{P}_2 touching the boundary of T_N . There are then two possibilities : either B_f contain one of the three extreme vertex of T_N , in which case the boundary edges of B_f have different type and by convexity, B_f has only two edges in $\widehat{\mathcal{G}}_P$, or B_f contains only boundary vertices which are not extreme points of T_N , in which case the boundary edges have same type ℓ and the boundary of B_f consists of four edges of type $(\ell, \ell+1, \ell+2, \ell)$. \square

Construction of a discrete two-colored dual hive from a puzzle. The resulting planar graph $\widehat{\mathcal{G}}_P$ is thus a graph with only trivalent vertices and hexagonal inner faces. From each side of the triangle T_N , there are $n-2$ faces $B \in \mathcal{P}_f$ which have degree 4 (one for each pair of consecutive boundary edge labeled 0 or 1 on a same side of T_N) and from each extreme vertex of T_N there is a face of degree 2.

Let us denote by $\tilde{\mathcal{G}}_P$ the dual graph, namely the graph whose vertices are faces of $\widehat{\mathcal{G}}_P$, faces are vertices of $\widehat{\mathcal{G}}_P$ and such that there is one edge between each neighboring faces of $\tilde{\mathcal{G}}_P$ (which correspond then vertices of $\widehat{\mathcal{G}}_P$).

Lemma 5.6. *There is an isomorphism from $\tilde{\mathcal{G}}_P$ to T_n mapping edges of type ℓ of $\tilde{\mathcal{G}}_P$ to edges of type ℓ of T_n .*

Proof. Since vertices of $\hat{\mathcal{G}}_P$ are trivalent, faces of $\tilde{\mathcal{G}}_P$ are triangular. Similarly, inner faces of $\hat{\mathcal{G}}_P$ have degree 6, and thus inner vertices of $\tilde{\mathcal{G}}_P$ have degree 6. Hence, $\tilde{\mathcal{G}}_P$ is isomorphic to a polygon H of the planar triangular grid.

Since the sequence of degrees of the $3n$ outer faces of $\hat{\mathcal{G}}_P$ is

$$(2, \underbrace{4, \dots, 4}_{n \text{ times}}, 2, \underbrace{4, \dots, 4}_{n \text{ times}}, 2, \underbrace{4, \dots, 4}_{n \text{ times}}),$$

the same holds for the sequence of degrees of outer vertices of $\tilde{\mathcal{G}}_P$. Remark that for each vertex of degree 4 (resp. 2) in $\tilde{\mathcal{G}}_P$, the angle of the boundary at the corresponding vertex in H is π (resp. $5\pi/3$). We deduce that the boundary of the $\tilde{\mathcal{G}}_P$ is isomorphic to the one of T_n , and thus $\tilde{\mathcal{G}}_P$ is isomorphic to T_n . Let us denote by Φ the corresponding bijection between set of edges.

Remark that around every triangle of $\tilde{\mathcal{G}}_P$ the type of the edges is $(\ell, \ell+1, \ell+2)$ (this is true for faces coming from trivalent vertex of \mathcal{G}_P and true by construction for faces coming from the blowing up of higher order vertices of $\hat{\mathcal{G}}_P$). We deduce that all edges with the same type in $\tilde{\mathcal{G}}_P$ are sent through Φ to edges with the same orientation in T_n . Up to composing Φ with an internal rotational symmetry of T_n , we can thus assume that Φ preserves the type of the edges. \square

Each edge B of $\tilde{\mathcal{G}}_P$ has then a color $c(B)$ and a height $h(B)$ coming from the dual edge of $\hat{\mathcal{G}}_P$. Composing with Φ , this yields maps $C : E_n \rightarrow \mathbb{N}$ and $L : E_n \rightarrow \mathbb{N}$ with $C = \Phi \circ c$ and $L = \Phi \circ h$.

Lemma 5.7. *The resulting pair of maps (C, L) is a discrete two-colored dual hive.*

Proof. Lemma 4.7 and the he color rules introduced before in case of blowing-up of even degree vertices yield that edges around each trivalent vertex of $\tilde{\mathcal{G}}_P$ are colored $(0, 0, 0)$, $(1, 1, 1)$, $(1, 0, 3)$ or $(0, 1, m)$ in the clockwise order, which translate into the same color rule around each triangular face of T_n .

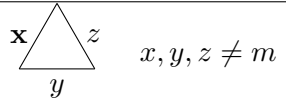
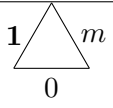
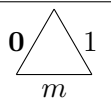
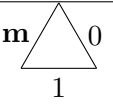
It remains to prove that the map L on E_n satisfies the two conditions of Definition 5.2. The sum condition (1) around a triangle is a consequence of Lemma 4.5 in case no edge is colored m , and the direct deduction of the blowing up of vertices of even degree in case one of the edges is colored m .

The condition (2) is checked case by case. By Lemma 4.7 and the definition of L on edges colored 1 coming from the blowing-up of even vertices, $L(e) = L(e')$ for any opposite edges e, e' of a lozenge with middle edge colored m , yielding the condition (2.a).

Without loss of generality, suppose that $e \in E_n$ has type ℓ and let s be a lozenge such that e is a border edge of type ℓ of s and the opposite edge e' is a translation of e such that $h(e) \geq h(e')$ and $e_{\ell+1} \leq e'_{\ell+1}$. Hence, either $h(e) = h(e')$, $e_{\ell+1} = e'_{\ell+1} - 1$ and the middle edge f of s is of type $k = \ell - 1$ or $h(e') = h(e) - 1$, $e'_{\ell+1} = e_{\ell+1}$ and the middle edge f is of type $k = \ell + 1$. Suppose now that at least one of the edges of the lozenge is colored m and the middle edge f is not dual to an edge of $\hat{\mathcal{G}}_P$ coming from the blowing-up of an even vertex. Hence, f corresponds to a strip S_f of k . Moreover, there exist \tilde{e} and \tilde{e}' of type ℓ with $h(\tilde{e}) = L(e)$ and $h(\tilde{e}') = L(e')$, $\tilde{f}, \tilde{f}' \in S_f$ of type k such that \tilde{e} and \tilde{f} (resp. \tilde{e}' and \tilde{f}') comes from a same vertex v of P_v (either directly or after a blowing-up). Then S_f corresponds to a genuine strip S_f with ending edges \tilde{f}, \tilde{f}' and there exists \tilde{e}, \tilde{e}' of type ℓ with $h(\tilde{e}) = L(e)$, $h(\tilde{e}') = L(e')$ and \tilde{e} (resp. \tilde{e}') belongs to a same vertex as \tilde{f} (resp. \tilde{f}') in P_v . Remark that $\tilde{f}_\ell \geq \tilde{f}'_\ell$ for otherwise there would an edge of type different from f and labeled 0 or 1 in the strip S_f , which is not possible from Figure 10. Then, if \tilde{e} is of type ℓ and \tilde{f} is of type $\ell - 1$ coming from a same triangle of T_n , we have the following relation between \tilde{f}_ℓ and $h(\tilde{e})$ depending on the orientation of the triangle and the colors of the boundary edges (the color and position of \tilde{f} is bold)

From those relation and the fact that $\tilde{f}_\ell \geq \tilde{f}'_\ell$, we deduce that $h(\tilde{e}) \geq h(\tilde{e}')$ in the case $e'_{\ell+1} = e_{\ell+1} + 1$ and that $h(\tilde{e}) > h(\tilde{e}')$ if one of the boundary edge of s different from e is colored m . The case $e_\ell = e'_\ell + 1$ is done similarly, yielding always $h(\tilde{e}) > h(\tilde{e}')$.

Finally, if the middle edge is coming from the blowing up of an even vertex and is not colored m , then this edge is necessarily colored 1, and thus e and e' are colored 0 and the opposite edges are colored 1. The strict inequality is directly deduced from construction of m and Lemma 4.7 giving the height of edges colored 0 in an even vertex. \square

				
Coloring of a direct triangle	$x, y, z \neq m$			
\tilde{f}_ℓ	$h(\tilde{e}) + 1$	$h(\tilde{e}) + 2$	$h(\tilde{e}) + 1$	$h(\tilde{e}) + 1$

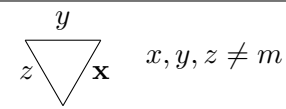
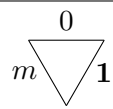
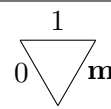
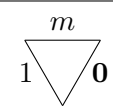
				
Coloring of a reverse triangle	$x, y, z \neq m$			
\tilde{f}_ℓ	$h(\tilde{e}) + 1$	$h(\tilde{e})$	$h(\tilde{e}) + 1$	$h(\tilde{e})$

FIGURE 11. Coordinates of an edge in function of the coloring and height of the next edge in a triangle (\tilde{f} correspond to the bold edge and \tilde{e} corresponds to the horizontal edge).

We deduce that T_n with the labelling (C, L) is a genuine discrete two-colored dual hive, which we denote by $\Phi(P)$.

Proposition 5.8. *The map Φ is bijective.*

Proof. Let us construct the reverse bijection. Let H be a two-colored dual hive. We first define the candidate vertex partition (without the coloring for now) \mathcal{P}_v as follows :

- for each triangle face $t = (t^0, t^1, t^2) \in T_n$, with t^ℓ of type ℓ , with boundary colors in $(0, 0, 0)$, $(1, 1, 1)$ or $(1, 0, 3)$ (up to a cyclic order), we define a block $B_t \subset E_N$ with edges e^0, e^1, e^2 , with e^ℓ of type ℓ , and such that :

$$h(e^\ell) = L(t^\ell), e_{\ell+1}^\ell = L(t^{\ell+1}) + 1.$$

- for each long rhombus with boundary $u = (s^0, t^1, \dots, t^r, s^1, t^{r+1}, \dots, t^{2r})$ with s^i of type ℓ and t^i of type $\ell + 1$ and $t_i^1 = s_i^0 + 1$ such that $C(s^0) = C(s^1) = 1$, $C(t^i) = 0$, $L(t^i) = L(t^{2r+1-i}) = L(t^1) - (i - 1)$ for $1 \leq i \leq 2r$, and which is not included in an other rhombus satisfying such property, we define a block $B_u \subset E_N$ with edges $(e^0, e^1, f^1, \dots, f^{2r})$ with $h(v) = L(v)$ for $v \in B_u$ and

$$e_{\ell+1}^0 = L(t^1) + 2, e_{\ell+1}^1 = L(t^r), f_{\ell+2}^i = f_{\ell+2}^{2r+1-i} - 1 = e_{\ell+2}^0 + i, 1 \leq i \leq r.$$

Remark that for $1 \leq i \leq r$, $f_{\ell+2}^i = f_{\ell+2}^{2r+1-i} - 1 = t(g^i)$, where g^i is the edge of type $\ell + 2$ colored m adjacent to f^i or f^{2r+1-i} .

- for each edge s of type ℓ on the boundary of T_n , we define a singleton in $B_e \subset E_N$ consisting of the unique edge e of type ℓ with $h(e) = L(s)$.

Moreover, by Lemma 5.9 below, edges e, e' coming from different edge $t, t' \in T_n$ by the previous constructions are distinct. Let $\mathcal{E} = \bigcup_{B \in \mathcal{P}_v} B$ We can thus define a coloring c on \mathcal{E} by setting $c(e) = C(t)$ when e is constructed from t above. By construction and the property (1) of Definition 5.2, the covering \mathcal{P}_v satisfies all the properties of Lemma 4.7. With the above constructions and the conditions (1) and (2.a) of a two-color dual hive, we can then check that all the relations from Figure 11 is still satisfied when the bold edge is an element of \mathcal{E} .

Define then a relation \sim on \mathcal{E} by saying that $e \sim e'$ if e, e' are coming from a same edge of T_n through the previous construction, and denote by \mathcal{P}_e the set partition coming from this relation. By the properties of a two-colored dual hive, any long rhombus considered before of border edges of type $\ell, \ell + 1$ has its inner middle edges of type $\ell - 1$ colored m , so that none of the triangles inside this long rhombus yields block of \mathcal{P}_v through the first step. Hence, each edge of T_n yields at most 2 edges of \mathcal{E} . Remark that an edge of T_n is either adjacent to two faces or to one face and the boundary of T_n , so that \mathcal{P}_e consists of pairs or singleton, and in the latter case the singleton belongs to two blocks of \mathcal{P}_v . If $e \sim e'$, by the above construction $c(e) = c(e'), L(e) = L(e')$ and e, e' have a same type, so that \mathcal{P}_e only has admissible pairs (which can be reduced to a singleton). If $e \neq e'$ belong to a same block of \mathcal{P}_v they come from different edges of T_n and thus $\{e, e'\} \notin \mathcal{P}_e$.

In view of applying Proposition 4.11, it suffices to prove that two pairs of \mathcal{P}_e do not cross. Suppose that $B = \{e^1, e^2\}$ and $B' = \{e^3, e^3\}$ are two blocks of \mathcal{P}_e . If they are of same type ℓ , then Lemma 5.9 yields that $B \cap S_{B'} = B' \cap S_B = \emptyset$. If B are of different type ℓ and $\ell + 1$, Lemma 5.10 yields that the

second condition of crossing strips is never satisfied, and the first condition may only be satisfied in the case (4) of Lemma 5.10 where $t'_\ell \geq t_\ell$ and $t'_{\ell+1} \geq t_{\ell+1}$, when $e_{\ell+1}^2 = e_{\ell+1}^3$. But in the latter case, by Definition 4.8, edges of type $\ell + 1$ of the strip S_B have $\ell + 1$ -coordinate strictly smaller than $e_{\ell+1}^2$, so that $B' \cap S_B = \emptyset$. Likewise, edges of the strip $S_{B'}$ of type ℓ have $\ell + 1$ -coordinate strictly larger than $\min(e_{\ell+1}^3, e_{\ell+1}^4)$ so that $B \cap S_{B'} = \emptyset$. Hence, S_B and $S_{B'}$ do not cross.

Pairs of \mathcal{P}_e are admissible and any different pairs $B, B' \in \mathcal{P}_e$ do not cross, thus partition \mathcal{P}_e satisfies the properties of Lemma 4.9. Finally, by Proposition 4.11 applied to $(\mathcal{P}_v, \mathcal{P}_e)$, there exists a unique puzzle P such that the corresponding vertex and edge partitions are respectively \mathcal{P}_v and \mathcal{P}_e . Denote by $\Psi(H)$ this puzzle. It is then straightforward to prove that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are identity maps. \square

Lemma 5.9. *Let $t \neq t' \in T_n$ of same type ℓ in blocks of \mathcal{P}_v , and suppose without loss of generality that $t_{\ell+1} > t'_{\ell+1}$ or $t_{\ell+1} = t'_{\ell+1}$ and $h(t') > h(t)$. Then, if $h(t) < h(t')$ and $t_{\ell+1} > t'_{\ell+1}$, the edges e^1, e^2 (resp. e^3, e^4) of T_N associated to t (resp. t') satisfy*

$$h(e^1) = h(e^2) < h(e^3) = h(e^4),$$

if $h(t) < h(t')$ and $t_{\ell+1} = t'_{\ell+1}$,

$$\min(e_{\ell-1}^1, e_{\ell-1}^2) > \max(e_{\ell-1}^3, e_{\ell-1}^4).$$

and if $h(t) \geq h(t')$ and $t_{\ell+1} > t'_{\ell+1}$,

$$\min(e_{\ell+1}^1, e_{\ell+1}^2) > \max(e_{\ell+1}^3, e_{\ell+1}^4).$$

Proof. Since t and t' are in blocks of \mathcal{P}_v , neither t nor t' are colored m . If $h(t) < h(t')$ and $t_{\ell+1} > t'_{\ell+1}$, $L(t) < L(t')$ by Remark 5.3. Since $h(e^1) = h(e^2) = L(t)$ and $h(e^3) = h(e^4) = L(t')$, this implies

$$h(e^1) = h(e^2) < h(e^3) = h(e^4).$$

If $t_{\ell+1} = t'_{\ell+1}$ and $h(t') > h(t)$, then by Condition (2.c) of Definition 5.2, $L(t') > L(t)$ except if the middle edge of all lozenges between t and t' are colored m . In the latter case, let s (resp. s') be the edge of type $\ell - 1$ such that t, s form a reverse triangle (resp. t', s' form a direct triangle). Since $s_\ell = t_\ell + 1$ and $s'_\ell = t'_\ell$, the inequality $t'_\ell > t_\ell$ implies that $s'_\ell > s_\ell$. Similarly, since $s_{\ell-1} = t_{\ell-1} - 1$ and $s'_{\ell-1} = t'_{\ell-1}$, we have

$$s_{\ell-1} = t_{\ell-1} - 1 = N - t_\ell - t_{\ell+1} - 1 > N - t'_\ell - t'_{\ell+1} - 1 \geq t'_{\ell-1} \geq s'_{\ell-1}.$$

Hence, $L(s') \leq L(s)$. Let us introduce the third edge r (resp. r') of the triangle with edges s, t (resp. s', t'). By Figure 11 and Condition (1) from Definition 5.2, we get that $e_{\ell-1}^3 = N - e_\ell^3 - e_{\ell+1}^3 = N - L(t') - L(r') - 1 = L(s')$ and $e_{\ell-1}^2 = N - e_\ell^2 - e_{\ell+1}^2 = N - L(t) - L(r) = L(s) + 1$ and thus $e_{\ell-1}^2 = L(s) + 1 > L(s) = e_{\ell-1}^3$, so that

$$e_{\ell-1}^1 \geq e_{\ell-1}^2 > e_{\ell-1}^3 \geq e_{\ell-1}^4.$$

If $h(t) \geq h(t')$ and $t_{\ell+1} > t'_{\ell+1}$, then $t'_{\ell+2} > t_{\ell+2}$. Suppose without loss of generality that $e_{\ell+1}^1 \geq e_{\ell+1}^2$ and $e_{\ell+1}^3 \geq e_{\ell+1}^4$. Let us consider the edges s, s' of type $\ell + 1$ such that (t, s, r) and (t', s', r') are respectively direct and reverse triangles of T_n so that the corresponding edge of t and the piece containing the direct triangle is e^2 and the corresponding edge for t' and the reverse triangle is e^3 . Since $h(s) = t_{\ell+1} - 1$ and $s_{\ell+2} = t_{\ell+2} + 1$ and $h(s') = t'_{\ell+1} - 1$ and $s'_{\ell+2} = t'_{\ell+2}$, $h(s) > h(s')$ and $s'_{\ell+2} \geq s_{\ell+2}$, so that $L(s) > L(s')$ by Remark 5.3. Then, since (t, u, s) is a direct triangle, Figure 11, $e_{\ell+1}^2 = L(s) + 1$, except if $c(t) = 1, c(s) = 0$ and $c(u) = m$ where $e_{\ell+1}^2 = L(s) + 2$. Likewise, since (t', s', u') is a reverse triangle, $e_{\ell+1}^3 = L(s') + 1$ except if $c(t) = 1, c(s) = 0, c(u) = m$ or $c(t) = 0, c(s) = m, c(u) = 1$ where $e_{\ell+1}^3 = L(s')$. Hence, in any case,

$$e_{\ell+1}^3 \leq L(s') + 1 < L(s) + 1 \leq e_{\ell+1}^2$$

and

$$e_{\ell+1}^1, e_{\ell+1}^2 > e_{\ell+1}^3, e_{\ell+1}^4.$$

\square

Lemma 5.10. *Let $t, t' \in T_n$ be of respective type $\ell, \ell + 1$ yielding edges in \mathcal{E} , and denote by e^1, e^2 (resp. e^3, e^4) the edges of T_N corresponding to t (resp. t'). Then,*

(1) if $t'_\ell > t_\ell$ and $t'_{\ell+1} < t_{\ell+1}$, then

$$e_\ell^1 = e_\ell^2 < e_\ell^3 \leq e_\ell^4.$$

(2) if $t'_\ell < t_\ell$ and $t'_{\ell+1} \geq t_{\ell+1}$, then

$$e_\ell^3 \leq e_\ell^4 < e_\ell^1 = e_\ell^2.$$

(3) if $t'_\ell \leq t_\ell$ and $t'_{\ell+1} < t_{\ell+1}$,

$$e_{\ell+1}^3 \leq e_{\ell+1}^4 < e_{\ell+1}^1 \leq e_{\ell+1}^2.$$

(4) if $t'_\ell \geq t_\ell$ and $t'_{\ell+1} \geq t_{\ell+1}$,

$$e_{\ell+1}^1 \leq e_{\ell+1}^2 \leq e_{\ell+1}^3 = e_{\ell+1}^4.$$

Proof. The proof of the four assertion are similar.

(1) Suppose that $t'_\ell > t_\ell$ and $t'_{\ell+1} < t_{\ell+1}$. Let (t', s', r') and (t', s'', r'') be the reverse and direct triangle belonging to pieces yielding respectively e_ℓ^3 and e_ℓ^4 , with s', s'' of type ℓ . Since then $e_\ell^4 \geq e_\ell^3$, it suffices to show that $e_\ell^3 > e_\ell^2$. Since $s'_\ell = t'_\ell - 1$ and $s'_{\ell+1} = t'_{\ell+1} + 1$, we have $s'_\ell \geq t_\ell$ and $t_{\ell+1} \geq s'_{\ell+1}$, so that $L(s') \geq L(t)$. Since t' is of type $\ell + 1$ not and colored m and (t', s', r') is a reverse triangle, Figure 11 and Condition (1) from Definition 5.2 yield that either $e_{\ell+2}^3 = L(r') + 1$ and $L(r') + L(s') + L(t') = N - 2$ or $e_{\ell+2}^3 = L(r')$ and $L(r') + L(s') + L(t') = N - 1$. In any case, $e_{\ell+2}^3 = N - 1 - L(t') - L(s')$, so that

$$e_\ell^3 = N - e_{\ell+1}^3 - e_{\ell+2}^3 = N - L(t') - (N - 1 - L(r') - L(s')) = L(s') + 1 > L(t) = e_\ell^2 = e_\ell^1.$$

(2) Suppose that $t'_\ell < t_\ell$ and $t'_{\ell+1} \geq t_{\ell+1}$, and let s' be the edge of type ℓ such that (t', s', r') is a direct triangle. Since $s'_\ell = t'_\ell$ and $s'_{\ell+1} = t'_{\ell+1} + 1$, $s'_\ell < t_\ell$ and $s'_{\ell+1} > t_{\ell+1}$, so that $L(t) > L(s')$ by Remark 5.3. Since (t', s', r') is a direct triangle, Figure 11 and Condition (1) from Definition 5.2 yield by a same reasoning as above that $e_{\ell+2}^4 \geq L(r') + 1 = N - L(s') - L(t')$, so that, using that $e_{\ell+1}^4 = L(t')$,

$$e_\ell^4 = N - e_{\ell+1}^4 - e_{\ell+2}^4 \leq L(s') < L(t) = e_\ell^1 = e_\ell^2.$$

(3) Suppose that $t'_\ell \leq t_\ell$ and $t'_{\ell+1} < t_{\ell+1}$, and let s be the edge of type $\ell + 1$ such that (t, s) is part of a direct triangle. Then, $s_\ell = t_\ell$ and $s_{\ell+1} = t_{\ell+1} - 1$, so that $s_\ell \geq t'_\ell$ and $s_{\ell+1} \geq t'_{\ell+1}$. We deduce that $s_{\ell+2} \leq t'_{\ell+2}$, and thus $L(s) \geq L(t')$. Since $e_{\ell+1}^1 \geq L(s) + 1$ by Figure 11, we thus have

$$e_{\ell+1}^3 = e_{\ell+1}^4 = L(t') \leq L(s) < e_{\ell+1}^1 \leq e_{\ell+1}^2.$$

(4) Suppose that $t'_\ell \geq t_\ell$ and $t'_{\ell+1} \geq t_{\ell+1}$, and let s be the edge of type $\ell + 1$ such that (t, s, r) is a reverse triangle. Then, $s_\ell = t_\ell + 1$ and $s_{\ell+1} = t_{\ell+1} - 1$. Hence, $t'_{\ell+1} > s_{\ell+1}$ and $t'_{\ell+2} = N - t'_{\ell+1} - t'_\ell \leq N - s_{\ell+1} - 1 - s_\ell + 1 \leq s_{\ell+2}$ and the inequality is strict except when $t'_\ell = t_\ell$ and $t'_{\ell+1} = s_{\ell+1}$. Hence, by Remark 5.3 ins the case of strict inequality and Condition (2.b) and (2.c) from Definition 5.2, $L(t') > L(s)$, except when $t'_\ell = t_\ell, t'_{\ell+1} = s_{\ell+1}$ and $C(u) = m$, in which case $L(t') = L(s)$. In the first case, by Figure 11 we have $e_{\ell+1}^2 \leq L(s) + 1 \leq L(t')$. In the second case, since $C(r) = m$ we have $e_{\ell+1}^2 = L(s) \leq L(t')$, so that in any case

$$e_{\ell+1}^1 \leq e_{\ell+1}^2 \leq e_{\ell+1}^3 = e_{\ell+1}^4.$$

□

6. COLOR SWAP

From this section to the end of the paper, we will assume that the color map $c^{(i)}$ one each border of a two-color hive considered is

$$c^{(i)} = (\underbrace{1, \dots, 1}_{d \text{ times}}, \underbrace{0, \dots, 0}_{n-d \text{ times}}, 2, \underbrace{1, \dots, 1}_{d \text{ times}}).$$

Such a two-color hive is called regular.

6.1. Gash propagation.

Definition 6.1 (Gash). *Let H be a dual hive. We define a gash to be a pair of edges $(e_1, e_2) \in E_n^2$ sharing one vertex such that $C(e_1) = 0$, $C(e_2) = 1$ and such that e_1 and e_2 are both of the same type with $h(e_1) > h(e_2)$. We call the type of a gash g the common type of its two edges.*

Note that this definition only depends on the color map C of H . Let g be a gash on $\partial T_n^{(2)}$ for some dual hive H . Then g is necessarily of type 2. East to g , there are only six possible configurations given in Figure 12. In this section, we show that such a gash g can be moved across the color map C using local moves until reaching configuration (v) or (vi) of Figure 12.

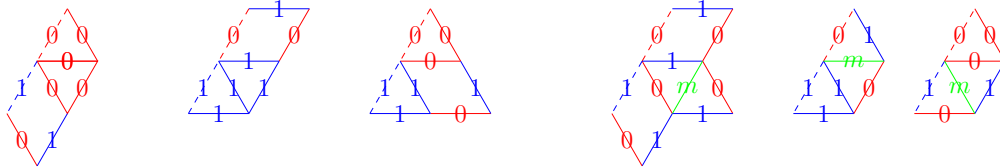


FIGURE 12. The six possible adjacent configurations to a gash of type 1 shown in dashed edges. The same holds for a gash of other types up to rotations.

Definition 6.2 (Gash propagation in (i), (ii) and (iii).). *Let $g = (e_1, e_2)$ be a gash and let c be the configuration of the color map C at g .*

- (1) *If c is either (i) or (ii), the propagation of g through c is the gash $g' = (e'_1, e'_2)$ given by the pair of edges having the same type as (e_1, e_2) .*
- (2) *If c is (iii), the two other pairs of edges of the same type in the triangle (iii) can be taken as the propagation of g .*

6.1.1. *Propagation through configuration (iv).* The configuration (iv) does not give an immediate way to continue the gash propagation. In this section, we give a way to propagate any gash having such configuration adjacent to it.

Definition 6.3 (Openings and arrows). *Consider either two edges (e_1, e_2) of the same color 0 or 1 sharing a vertex and forming an angle of $\frac{2\pi}{3}$. We call this a 0 or 1 opening depending on the color. There are only two possible completions of an opening as shown in Figure 13 for the case of a 0 opening. Note that in completions (0.II) and (1.II), the two new edges (e_3, e_4) both of the same color as (e_1, e_2) form another opening of the same color. From an opening, denote $r \geq 0$ the number of successive completions of type (II) before the first completion of type (I). The resulting configuration is called an arrow of color 0 or 1 depending of any of its $r + 1$ openings and r is the length of this arrow.*

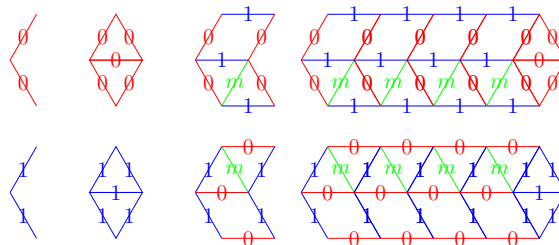


FIGURE 13. First row from left to right : an 0 opening, completion (0.I), completion (0.II) and an 0 arrow of length 4. The second row is the analog for color 1.

Let us call a lozenge rigid if the inner edge is colored m . Remark that configuration (iv) corresponds to two openings neighbors to a same rigid lozenge. Each of the arrows on these openings has some finite length, say $l_1, l_2 \geq 0$. The region delimited by the two arrows is completely determined and consists of $l_1 \times l_2$ rigid lozenges, see Figure 14.

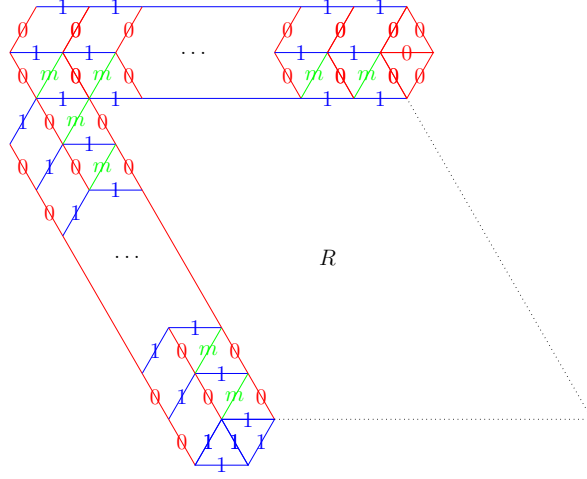


FIGURE 14. The region R in between two arrows is fully determined and consists in rigid lozenges with the same orientation as in the arrows.

Once R is filled with rigid lozenges, there are two new openings, one of each color at the end of the two arrows. We call a (l_1, l_2) region the configuration where two openings are on consecutive sides of a region filled with rigid lozenges such that there are l_1 , respectively l_2 , edges of type 2 colored 0, respectively of type 0 colored 1. Note that the opposite side of configuration (iv) from the gash can be seen as a $(1, 1)$ region. When $(l_1, l_2) = (0, 0)$, we define the region to be the configuration on the right in Figure 15.

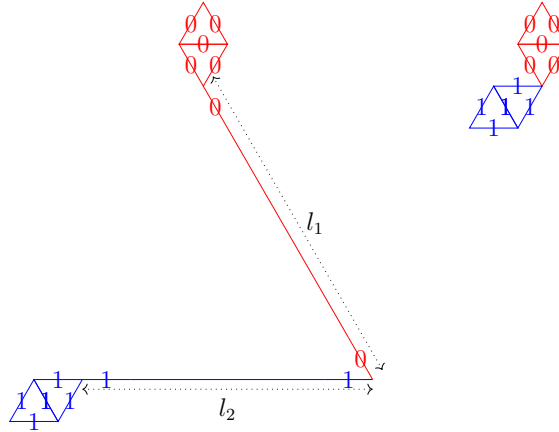


FIGURE 15. A (l_1, l_2) region (left) and a $(0, 0)$ region (right).

Lemma 6.4 (Growing region procedure). *Consider a (l_1, l_2) region for any $l_1, l_2 \geq 1$. Let $l'_1, l'_2 \geq 0$ be the length of the arrows from the openings of color 0 and 1 respectively. Then, there is a unique color map completion which results in a $(l_1 - 1 + l'_2, l_2 - 1 + l'_1)$ region.*

Proof. Let us first assume that $(l_1, l_2, l'_1, l'_2) \neq (1, 1, 0, 0)$. The reasoning is similar to the one before Figure 14. The arrow of color 0 creates an empty region R_0 similar to R of size $(l_1 - 1) \times l'_1$ which can only be filled using rigid lozenges because of its boundary conditions. Likewise, the arrow of color 1 creates an empty region R_1 of size $(l_2 - 1) \times l'_2$ which can also only be filled using rigid lozenges. The regions R_0 and R_1 define a region R_2 which by the same argument can only be filled using rigid lozenges. After filling every region $R_i, 0 \leq i \leq 2$, we obtain a $(l_1 - 1 + l'_2, l_2 - 1 + l'_1)$ region. In the case where $(l_1, l_2, l'_1, l'_2) = (1, 1, 0, 0)$, we obtain a $(0, 0)$ region. \square

Lemma 6.5 (General form of region (iv)). *Let c be a configuration (iv) where we view its east boundary as a $(1, 1)$ region. There exists a finite sequence $\left((l_1^{(1)}, l_2^{(1)}), (l_1^{(2)}, l_2^{(2)}), \dots, (l_1^{(r)}, l_2^{(r)}) \right)$, $r \geq 1$ such that $(l_1^{(i+1)}, l_2^{(i+1)})$ is obtained from $(l_1^{(i)}, l_2^{(i)})$ by the growing procedure of Lemma 6.4 and such that $(l_1^{(r)}, l_2^{(r)}) = (0, 0)$.*

Proof. Lemma 6.4 shows that at each step of the arrow growing procedure, the region obtained is also (l_1, l_2) region. Since the hive is finite, the procedure has to terminate which only happens by a $(0, 0)$ region. \square

Definition 6.6 (Propagation through (iv)). *Let g be a gash on a configuration (iv) as in Figure 12. Let R be the region obtained by the arrow growing procedure of Lemma 6.5. The propagation of g is the gash g' which is constituted by the pair of edges of different colors sharing a vertex both of the same type as g in the last $(0, 0)$ region.*

6.1.2. *From a gash to a rigid lozenge.* Since one know how to propagate a gash, we now define the following procedure.

Proposition 6.7 (Gash to rigid lozenge). *Let H be a hive and assume that there exists a gash g on $\partial_\lambda H$. Apply the following steps:*

- (1) *Start from the south-most gash g on $\partial T_n^{(2)}$.*
- (2) *Propagate g via (i), (ii) or (iv) configurations. If the propagation encounters a configuration (v) or (vi), STOP. If the propagation encounters a configuration t of type (iii), apply (2'):*
 - (2') *Propagate the gash of type 0 from south border of t using (i), (ii) or (iv). If the propagation encounters configuration (v) or (vi), STOP. Otherwise, when arriving at downward (iii) or $\partial T_n^{(0)}$, go to step (3).*
- (3) *Consider the gash g' of type 2 on the east border of t . Continue gash propagation from g' with the additional rule that whenever propagation meets a downward (iii) the next gash in the propagation is the gash of type 1 or 2 on its east side.*

The above algorithm terminates on a configuration that is either (v) or (vi) in H .

For the proof of Proposition 6.7, we will need the following lemma which shows that the propagation cannot end by a (iii) and that it ends with a configuration after the last (iii) met.

Lemma 6.8 (Last (iii) configuration). *Suppose a gash g of type 1 arrives on an upward (iii) configuration t . Consider the resulting gash g' of type 2 on the east border of t . If g' does not encounter any (iii) configuration, then it ends at a (v) or (vi) configuration.*

Before proving Lemma 6.8, we need Lemma 6.9 which shows that any triangular region having two of its sides with edges colored 1 has all its edges colored 1.

Lemma 6.9 (Regular equilateral triangles). *Let $C : E_n \rightarrow \{0, 1, 3, m\}$ be a color map on edges of T_n . Let R be any subset of edges of E_n such that ∂R is an equilateral triangle of size $s \geq 1$. Assume that the south and west boundary edges of R have color $C(e) = 1$. Then, every edge in R has color 1.*

Proof. Let us first show a general fact about a shape described below that we call a cup. For $r \geq 1$, we call a cup of length r the union of r consecutive type 0 edges together with one edge of type 2, respectively of type 1, forming an angle of $\frac{2\pi}{3}$ with the west-most, respectively east-most type 0 edge. See Figure 16 for an example. Suppose that edges of a cup are all colored 1. Let us show by induction on r that the only possible color of edges in the convex hull of the cup is 1.

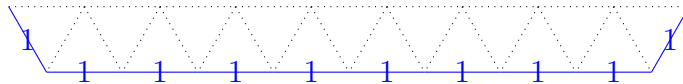


FIGURE 16. A cup of length $r = 8$. Edges in the convex hull of the cup are dotted

Two adjacent edges of different types form a type 1 arrow. If this arrow had completion (1.II) of Figure 13 all the r edges of type 0 would belong in some lozenge with middle edge colored 3 which is incompatible with the 1 color of the type 1 edge at the east of the cup. Thus, the arrow completion is (1.I). Then, the rest of the convex hull is a cup of length $r - 1$ which completes the proof of cup completion.

Considering R , the south-west corner triangular face has all of its edges colored 1 as two of them lie on the boundary of R . North-east to this face, there is a cup of length 1 having edges of type 1. By the previous, its only completion is edges of color 1. At each completion of a cup of size r , the north-east region is a cup of size $r + 1$. Filling cups of sizes $1, 2, \dots, s - 1$ with edges of color 1 fills R proves the claim. \square

By a similar argument, Lemma 6.9 also holds if two boundaries of R are colored 0 by replacing 3 lozenges by rigid lozenges in the proof.

Proof of Lemma 6.8. Assume for the sake of contradiction that g' does not encounter any configuration (v) or (vi). By assumption, g' propagates via configurations (i), (ii) or (iv) which preserve its type until reaching the boundary $\partial T_n^{(1)}$. Recall that $\partial T_n^{(1)}$ and $\partial T_n^{(0)}$ are regular. Therefore, the south east corner of H consists of a d equilateral triangle having all of its edges colored 1 by Lemma 6.9. Let us call R_d this region. Thus there cannot be a (i) configuration on $\partial T_n^{(1)}$ so that the gash's path cannot end by (i). Let $l \geq 0$ be the number of configurations (ii) from the last either (i) or (iv) configuration met. If this was configuration (i), then we would have a 0 colored edge e coming from the last configuration (i) in R_d . Likewise, if it was configuration (iv), consider the south-east most edge e having color 0 in the last arrow of type 1 with non-zero length (circled). Then, e would be contained in R_d . Both cases would imply that R_d has a 0 colored edge and would contradict the regularity assumption on H . See Figures 17 and 18.

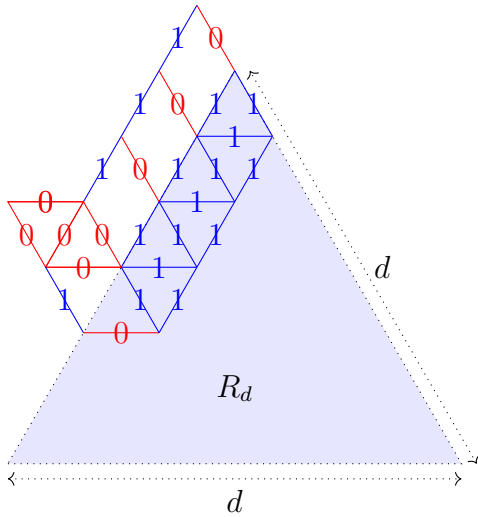


FIGURE 17. The last (i) configuration has a 0 colored edge inside R_d .

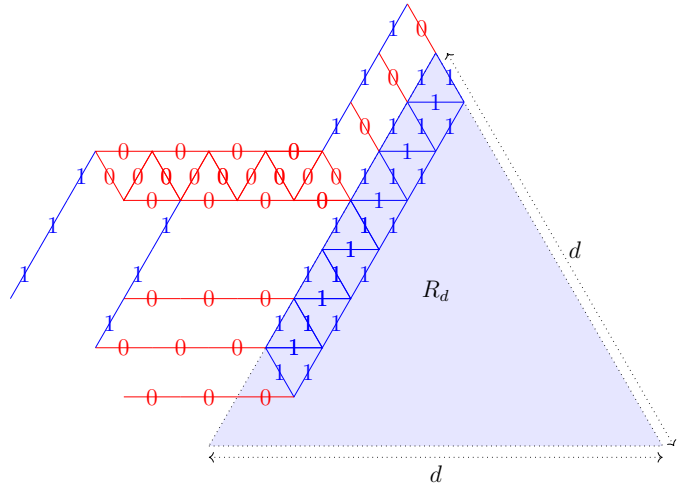


FIGURE 18. The last (iv) configuration has a 0 colored edge inside R_d .

\square

Proof of Proposition 6.7. Assume that no configuration (v) or (vi) is encountered in step (2') when propagating a gash of type 0. Let us show that a configuration (v) or (vi) is encountered in the steps (2) and (3).

Consider a gash g of type 1 in H . The gash propagates a finite number of times via (i), (ii) and (iv) configurations. As they both preserve the type of g , the propagation path must either end with (v) or (vi) or meet an upward (iii) configuration t from its west border. By Lemma 6.8, the gash g' on the east border of t either meets a downward (iii) or a configuration (v) or (vi). From a downward (iii), g' propagates to a gash g'' of type 1 associated to its east side. As there can only be a finite number of propagations $g \rightarrow g' \rightarrow g''$. Consider the last (iii) configuration t met either upward or downward. If t is upward, Lemma 6.8 shows that the path ends in a configuration (v) or (vi). Moreover, t cannot be downward as the next gash in the propagation would have type 1 which cannot reach $\partial T_n^{(1)} \cup \partial T_n^{(0)}$ due to its type preserved by configurations (i), (ii) and (iv). Therefore, the path from t ends in a configuration (v) or (vi). \square

6.2. Hexagon rotation.

Definition 6.10 (ABC hexagons). Let C be a color map and let h be a hexagon, that is, the union of six triangular faces sharing one vertex in T_n . We say that h is an ABC hexagon (for the color map C) if the color map C restricted to h is any of the three configurations in Figure 19 up to rotations which preserves the value of C on the boundary $\partial h \cap E_n$. Note that type B has three possible rotations whereas A and C only have two. Denote E_h the edges of E_n which are in h .

A rotation of an ABC hexagon h is the replacement of the values of C by the ones obtained from a rotation of h that preserve its boundary.

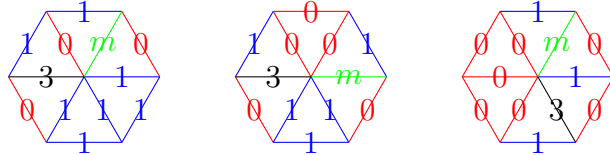


FIGURE 19. The three types of hexagons : A (left), B (center) and C (right).

6.3. Color swap path. In Proposition 6.7, we showed that any gash g on $\partial T_n^{(2)}$ can be propagated to find a configuration (v) or (vi) having a rigid lozenge in it. In this section, we show that via hexagon rotations of Definition 6.10, one can bring this rigid lozenge on $\partial T_n^{(2)}$ at the location of g . We first show that such a lozenge can travel through a configuration (iv) in Lemma 6.11 below.

Lemma 6.11 (Color swap in region (iv)). Let g be a gash having a configuration (iv) adjacent to it. Let g' be the propagation of g given by Definition 6.6. Suppose that g' is adjacent to a configuration c of type (v) or (vi) . Using hexagon rotations, one can place c so that g is adjacent to c .

Proof. We will do the proof in the case of configuration (vi) but the same can be adapted in the case of configuration (v) by changing the types of the hexagons moved in the process. Denote R the region induced by the configuration (iv) . Suppose that R consists only in the region delimited by two non-zero length arrows, one of type 0 and one of type 1 as in Figure 14. Notice that using hexagon rotations (of type A if the arrow is type 1 and type C if arrow is type 0) an arrow can be reversed i.e the resulting configuration is the arrow rotated by an angle π . We start by reversing the top type 0 arrow. Note that this creates an arrow of the same type below it. By induction, reverse all the l_1 arrows in the region. Note that the hexagon whose diagonal is g' is now of type B. Rotate it such that the rigid lozenge l from the configuration (vi) east to g' is vertical. Using type A rotations, one can bring l to the south-west corner of R in a type B hexagon. Doing a B hexagon rotation and then type C ones, one can place l adjacent to g forming a configuration (vi) . The procedure is illustrated in Figures 20 to 25 below where for the sake of clarity we omit the m colors from Figure 21, keeping track only of the rigid lozenge originally in (vi) denoted M .

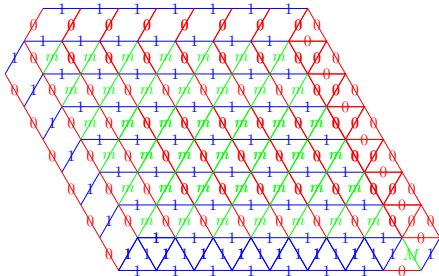


FIGURE 20. Original configuration delimited by two arrows.

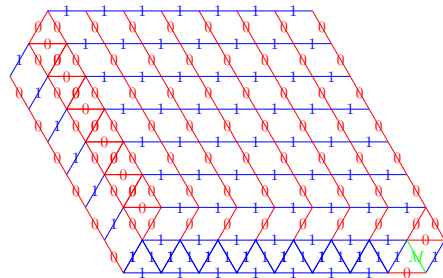


FIGURE 21. Reverting type 0 arrows.

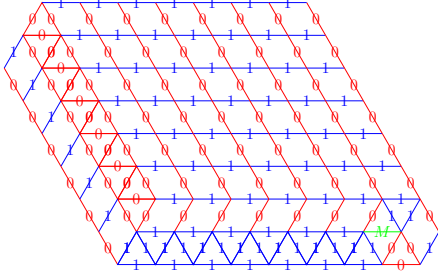


FIGURE 22. Type B hexagon rotation.

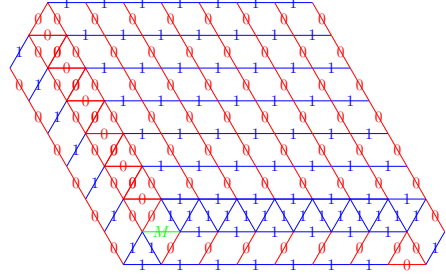


FIGURE 23. Type A hexagon rotations.

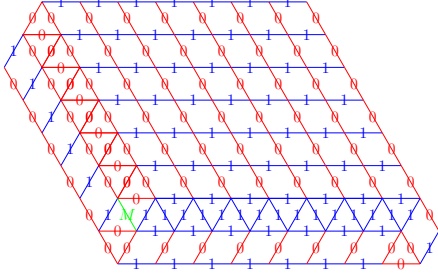


FIGURE 24. Type B hexagon rotation.

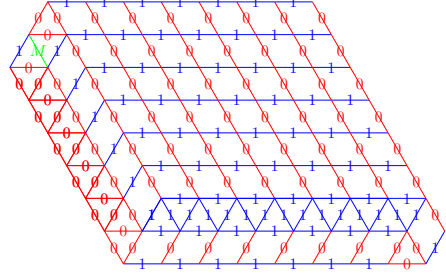


FIGURE 25. Type C hexagon rotations.

In the general case, by Lemma 6.5 the region R consists of an amalgamation of regions (l_1, l_2) similar to the one in Figure 20. Applying the previous procedure to each of those regions, one can place c so that g is adjacent to c . Notice that all the hexagon rotations in this proof keep the boundaries of the region fixed. \square

Proposition 6.12 (Color swap path). *Let g be a gash on $\partial T_n^{(2)}$. Consider the first (v) or (vi) configuration encountered in the algorithm of Proposition 6.7. This configuration has a rigid lozenge c and let us call color swap path the resulting path between g and c . Using hexagonal rotations on this path, one can move c so that g is adjacent to c .*

Proof. If a configuration of type (v) or (vi) is adjacent to the exit gash of a step (i) or (ii), the resulting hexagon is of ABC type and can be rotated so that c is now on the entry gash. Lemma 6.11 shows that the same holds if c is on the exit gash of a region (iv). One can also check that if c is on the bottom side or east side of a configuration (iii), one can place it on the west border via a hexagon rotation. Each of those rotations keep boundaries of the path fixed. Therefore, one can apply hexagon rotations on the path from g to c which result in g and c being adjacent. \square

6.4. Color map reduction. In this section, we only consider regular boundary conditions for T_n that is, the color map $C : E_n \rightarrow \{0, 1, 3, m\}$ is given by $1 \dots 10 \dots 01 \dots 1$ on every boundary of T_n , where there are d ones one each side of the zeros. The goal of this section is to show that any color map having regular boundaries can be mapped via hexagon rotation to the simple color map of Definition 6.13.

Definition 6.13 (Simple color map). *Let $k, d \geq 0$ and $n = k + d$. The color map $C_0 : E_n \rightarrow \{0, 1, 3, m\}$ called the simple color map is defined by*

- (1) C_0 is given by $1 \dots 10 \dots 01 \dots 1$ on every boundary of E_n , where there are d ones around the zeros,
- (2) $C_0(e) = 1$ for every edge $e \in E_n$ in any corner equilateral triangle of side length d in T_n ,
- (3) $C_0(e) = 1$ for every edge $e \in E_n$ in the vertical lozenge of side length d in T_n , two of its sides being the d north-most colored edges of $\partial_\lambda T_n$ and $\partial_\mu T_n$,
- (4) $C_0(e) = 0$ for every edge $e \in E_n$ in the equilateral triangle having one of its side consisting of the type 0 edges $\partial_\nu E_n$,
- (5) $C_0(e) = m$, respectively $C_0(e) = 3$ for every horizontal edge $e \in E_n$ in the $(k - d) \times d$ rhombus in-between the triangle of (4) and the 0 labels of $\partial_\lambda T_n$, respectively $\partial_\mu T_n$.

Here is an example of the simple color map C_0 for $n = 5$ and $d = 2$.

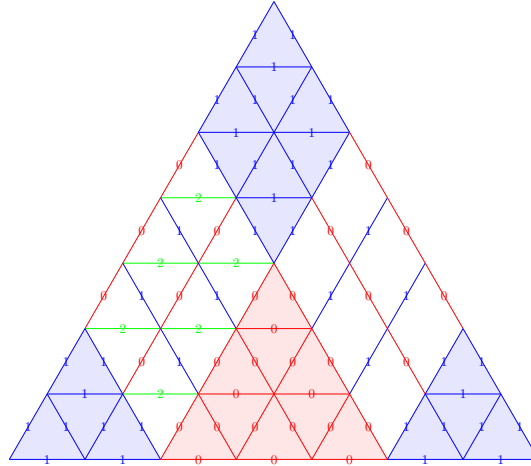


FIGURE 26. The color map C_0 for $n = 5$ and $d = 2$. Uncolored edges have color 3.

Proposition 6.14 (Color map reduction). *Let $C : E_n \rightarrow \{0, 1, 3, m\}$ be a color map with regular boundaries. Using hexagon rotations, one can map C to C_0 the simple color map of Definition 6.13.*

Proof. Let C be a color map on E_n . Since C is regular, the d equilateral triangles on each corner of T_n consists in triangular faces having boundary colors $C(e) = 1$ by Lemma 6.9. Consider the south-west such corner. Its north-most 1 colored edge of type 1 forms with the south-most 0 colored edge on $\partial T_n^{(2)}$ an angle of $\frac{2\pi}{3}$ that we will call a 01 opening. A 01 opening has only three possible completions showed in Figure 27 depending on the color of the horizontal edge in the angle.

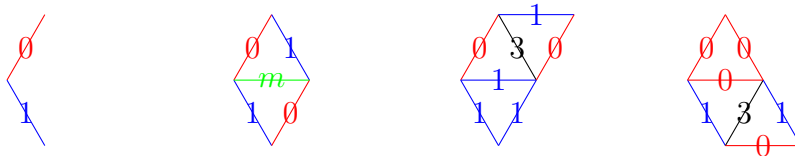


FIGURE 27. A 01 opening (left) and its possible completions

If both the two edges belong to a same rigid lozenge as in the first completion in Figure 27, the south-east edge of type 1 and color 0 of that lozenge forms a 01 opening with the 1 colored edge below it. If all these successive 01 openings are part of rigid lozenge, then there are d such rigid lozenges and the triangular face south to the last one has all of its edges colored 0 as two of its edges are. Then, the second south-most 0 label on $\partial T_n^{(2)}$ is part of a 01 opening. Consider the first 01 opening o not filled with a rigid lozenge with respect to that exploration order. Since o is not part of a rigid lozenge, the region east to o is one of two rightmost possibilities of Figure 27. Both of them give rise to a 01 gash g on their east side. The region west to o consists of vertical rigid lozenges on top of each other.

By Proposition 6.12 there exists a rigid lozenge in the region east to g that can be moved to g via hexagon rotations. Therefore, the 01 opening o can be filled using a rigid lozenge without changing the region west to it. Therefore, C is equivalent up to hexagon rotations to the color map where the $(k - d) \times d$ rhombus east to $\partial T_n^{(2)}$ consists only of rigid lozenges. From this configuration, there is only one possible color map to complete the rest of the hive E_n which is the simple color map C_0 . \square

Since the number of piece of each type is preserved under hexagon rotations, one derive the following enumerations.

Corollary 6.15 (Tiles enumeration). *Let H be a hive with regular boundaries. Let $hc(H)$, respectively, $sc(H)$, be the number of m , respectively 3 colored edges inside H . Then,*

$$(6.1) \quad hc(H) = d(n - d) = sc(H).$$

Moreover, for $i \in \{0, 1\}$, denote $nt^{(i)}(H)$, respectively $st^{(i)}(H)$, the number of north-pointing, respectively south-pointing, triangular pieces of size 1 with color i on each side. Then,

$$(6.2) \quad nt^{(i)}(H) = \frac{n(i)(n(i) + 1)}{2} \quad \text{and} \quad st^{(i)}(H) = \frac{n(i)(n(i) - 1)}{2}.$$

where $n(0) = n - d$ and $n(1) = 2d$ which gives $nt^{(0)}(H) = \frac{(n-d)(n-d+1)}{2}$, $st^{(0)}(H) = \frac{(n-d)(n-d-1)}{2}$, $nt^{(1)}(H) = d(2d + 1)$, $st^{(1)}(H) = d(2d - 1)$.

6.5. Quasi dual hives. The goal of this section is to extend hexagon rotations to hives. As of now, hexagon rotations map one color map to another. To also change label maps, we need to relax the inequality constraints of Definition 5.2 leading to quasi hives of Definition 6.17.

Definition 6.16 (Hexagonal dual hives). *Let $n, d \geq 1$. Denote $E_{n,d} = \{\{u, v\} \in R_{n,d}^2 : d(u, v) = 1\}$ the set of edges of the discrete hexagon $R_{n,d}$. We define $H_{hex}(\lambda, \mu, \nu, N)$ as the set of maps (C, L) , $C : E_{n,d} \rightarrow \{0, 1, 3, m\}$ and $L : E_{n,d} \rightarrow \frac{1}{N}\mathbb{Z}$ satisfying conditions of Definition 5.2 restricted to edges $e \in E_{n,d}$.*

By Lemma 6.9, any dual hive $H = (C, L) \in H(\lambda, \mu, \nu, N)$ with regular boundary conditions has the following property : every equilateral triangle of size d anchored in a corner of T_n has edges with color $C(e) = 1$. Giving values of the label map L restricted to any of these corner is thus equivalent via Proposition 5.8 to giving a classical Knutson-Tao puzzle. Since one of the boundaries of these puzzle is the empty partition by the regularity of boundaries, there is only one such puzzle and thus one such label map. Removing these fixed triangular corners, we have a bijection

$$(6.3) \quad H(\lambda, \mu, \nu, N) \rightarrow H_{hex}(\lambda, \mu, \nu, N)$$

for regular boundaries. The inverse map is given by extending C from the hexagon $R_{n,d}$ to the triangle T_{n+d} by adding color 1 edges in corners and completing the labels L in the unique possible way in corner triangles. In this section, we now view hives in the hexagon $R_{n,d}$ and we will omit the subscript writing $H(\lambda, \mu, \nu, N)$ for notation convenience.

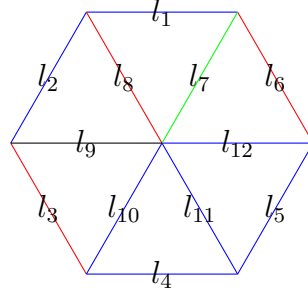
Definition 6.17 (Quasi label map, quasi dual hive). *Let $C : E_{n,d} \rightarrow \{0, 1, 3, m\}$ be a color map and $N \geq 1$. A quasi label map is a map $L : E_{n,d} \rightarrow \frac{1}{N}\mathbb{Z}$ satisfying the equality conditions of Definition 5.2, that is, equality condition on every triangular face and rigid lozenges with respect to the color map C . Denote $\tilde{L}^C(\lambda, \mu, \nu, N)$ the set of such label maps.*

A quasi dual hive is the data of a color map C and a quasi label map L . We denote by $\tilde{H}(\lambda, \mu, \nu, N)$ the set of quasi dual hives with boundary conditions (λ, μ, ν) .

The difference with label maps of dual hives is that one does not impose the inequality constraints of Definition 5.2. Note that a dual two-colored hive is a quasi dual hive that is, $H(\lambda, \mu, \nu, N) \subset \tilde{H}(\lambda, \mu, \nu, N)$.

Lemma 6.18 (Boundary value determine interior). *Let $H = (C, L)$ be a quasi dual hive and h an ABC hexagon for its color map C . Then, the values of L on E_h^o the interior edges of h are uniquely determined by values of L on boundary edges of h and by the position of the rigid lozenge in h . Moreover, the values of L on E_h^o are affine combinations of the values of L on boundary edges of h .*

Proof. Suppose that the values of L on ∂h are given by l_1, \dots, l_6 and values on E_h^o by l_7, \dots, l_{12} . We will do the proof for a type A hexagon and the other types B and C can be treated using similar arguments. Without loss of generality up to some permutation of the indexes suppose that $C(e_7) = m$, see Figure 28.

FIGURE 28. Labels of edges in h a type A hexagon.

By the equality conditions on opposite sides of the rigid lozenge, one has $l_{12} = l_1$ and $l_8 = l_6$. Let us show that the values l_7, l_9, l_{10}, l_{11} are uniquely determined in the region. Writing equality conditions on the four triangular faces gives

$$\begin{aligned} l_9 + l_8 + l_2 &= 1 - \frac{1}{N} \\ l_{10} + l_9 + l_3 &= 1 - \frac{2}{N} \\ l_{11} + l_{10} + l_4 &= 1 - \frac{1}{N} \\ l_{12} + l_{11} + l_5 &= 1 - \frac{2}{N} \\ l_{12} + l_7 + l_6 &= 1 - \frac{1}{N} \end{aligned}$$

which implies

$$(6.4) \quad l_1 + l_3 + l_5 = l_2 + l_4 + l_6 - \frac{2}{N}.$$

Suppose that the former holds. The fourth equations of the system is redundant with the others and (6.4) so that $l_7, l_8, l_9, l_{10}, l_{11}, l_{12}$ is solution to

$$(6.5) \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l_7 \\ l_8 \\ l_9 \\ l_{10} \\ l_{11} \\ l_{12} \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{N} - l_1 - l_6 \\ l_6 \\ 1 - \frac{1}{N} - l_2 - l_6 \\ 1 - \frac{2}{N} - l_3 \\ 1 - \frac{1}{N} - l_4 \\ l_1 \end{pmatrix}.$$

□

We now give a definition of hexagon rotations that incorporates the label map of a quasi hive.

Definition 6.19 (Rotation map). *Let C, C' be two color maps that differ by a hexagon rotation $h \rightarrow h'$ that is, $C'(e) = C(\text{rot}(e))$ where $\text{rot} : E_{n,d} \rightarrow E_{n,d}$ is the permutation of edges induces by the rotation mapping h to h' . Define*

$$(6.6) \quad \text{Rot}[C \rightarrow C'] : \tilde{L}^C(\lambda, \mu, \nu, N) \rightarrow \tilde{L}^{C'}(\lambda, \mu, \nu, N)$$

$$(6.7) \quad L \mapsto \text{Rot}[C \rightarrow C'](L) = L'$$

by setting $L'(e) = L(e)$ for $e \in E_{n,d} \setminus E_h^\circ$. Note that Lemma 6.18 ensures that the values of $L'(e)$ for $e \in E_h^\circ$ are uniquely determined by the new coloring C' .

For notation convenience, we will call a hexagon rotation the application of a map of the form of Definition 6.19 for some ABC hexagon h inside a quasi hive.

Lemma 6.20 (Rotation is affine bijection). *Let C, C' be two color maps that differ by a hexagon rotation. The map $R[C \rightarrow C'] : \tilde{L}^C(\lambda, \mu, \nu, N) \rightarrow \tilde{L}^{C'}(\lambda, \mu, \nu, N)$ is an affine bijection with integer coefficients. Its inverse is $R[C' \rightarrow C]$.*

Proof. For every edge e not interior to h , one has $L'(e) = L(e)$. In particular, $L'(e) = L(e)$ for $e \in \partial h$. Using the same arguments as in the proof of Lemma 6.18, one shows that the values $\{L'(e), e \in h^\circ\}$ are integer combinations of values $\{L(e) = L'(e), e \in \partial h\}$. \square

Figure 29 shows an example of a hexagon rotation and the affine map $L \mapsto L'$.

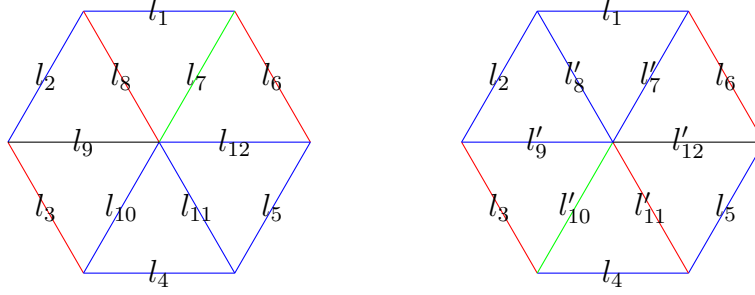


FIGURE 29. Action of a rotation on labels of inner edges.

Using face summation constraints together with equality constraints in the rigid lozenge in h' one has

$$\begin{aligned} l'_9 &= l_4 & l'_7 &= 1 - \frac{1}{N} - l_6 - l'_{12} = \frac{1}{N} + l_3 + l_5 - l_6 \\ l'_{11} &= l_3 & l'_8 &= 1 - \frac{2}{N} - l_1 - l'_7 = \frac{1}{N} + l_3 + l_5 - l_6 \\ l'_{12} &= 1 - \frac{2}{N} - l_3 - l_5 & l'_{10} &= 1 - \frac{1}{N} - l_4 - l'_3. \end{aligned}$$

which is a affine combination of values of L with integer coefficients.

For two color maps C, C' that differ by more than one hexagon rotation, we denote $Rot[C \rightarrow C']$ the composition of maps in 6.19 for each hexagon rotation needed to go from C to C_0 and then from C_0 to C' where existence of such paths was given by Proposition 6.14.

Lemma 6.21 (Simple quasi hive). *Let $H = (C_0, L) \in \tilde{H}^{C_0}(\lambda, \mu, \nu, N)$ be a quasi hive having the simple color map C_0 . Let $I \subset E_{n,d}$ be the set of edges of $E_{n,d}$ that are not in a rigid lozenge of C_0 except the edges of type 2 between a rigid lozenge and a triangular face with colors $(0, 0, 0)$ and edges of type 1 between a rigid lozenge and a triangular face with colors $(1, 1, 1)$, see Figure 30 below for an example. Denote I_0 the edges of I of type 0 where we remove the east-most such edge on each row. There are D such edges so that $I_0 = (e_1, \dots, e_D)$.*

Then, for any $z = (z_1, \dots, z_D) \in (\frac{1}{N}\mathbb{Z})^D$, there exists a unique label map $\Phi^{C_0}(z) \in L^{C_0}(\lambda, \mu, \nu, N)$ such that for all $1 \leq i \leq D$: $\Phi^{C_0}(z)(e_i) = z_i$. Moreover, for all $1 \leq i \leq D$, $\Phi^{C_0}(z)(e_i)$ is given by an affine combination with integer coefficients of $(z_1, \dots, z_D, \lambda, \mu, \nu, \frac{1}{N})$.

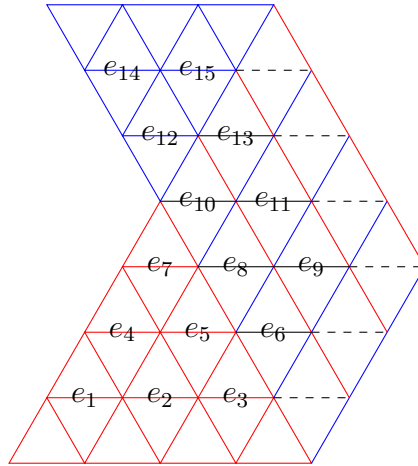


FIGURE 30. The region I for $n = 7$ and $d = 3$. $I_0 = \{e_i, 1 \leq i \leq 15 = D\}$. Dashed edges are edges of type 0 in $I \setminus I_0$.

Proof. Let $L : \partial E_{n,d} \rightarrow \frac{1}{N}\mathbb{Z}$ be a function defined on the boundaries of $E_{n,d}$ satisfying the boundary condition (λ, μ, ν, N) . We will show that L can be extended to a quasi label $E_{n,d}$ in a unique way. For any edge of type 2 part of a rigid lozenge, there exists an edge $e_\partial(e) \in \partial E_{n,d}$ of the same type obtained by translation of e by a multiple of $e^{i\frac{2\pi}{3}}$. Likewise, for any edge of type 1 part of a rigid lozenge, there exists an edge $e_\partial(e) \in \partial E_{n,d}$ of the same type obtained by translation of e by a multiple of $e^{i\frac{\pi}{3}}$. Assign $L(e) = L(e_\partial(e))$ for each such edge e . By the equality condition on opposite edges of rigid lozenges, any quasi label map has the same values on these edges.

It remains to extend L to edges $e \in I$. The values of ∂I are already determined uniquely by the boundary conditions. Set $L(e_i) = z_i$ for $1 \leq i \leq D$. We call a band the following configuration of adjacent faces where the west-most face has both its edges of type 0 and 2 labeled, the east-most face has its type 1 edge labeled and all other faces in between have their type 0 edge labeled.

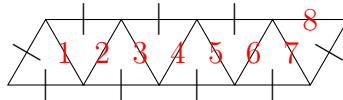


FIGURE 31. A band of size four. Marked edges are the already labeled edge. The edge labels of other edges are determined in the order of the red numbers by face summation constraints.

We claim that there is a unique labeling of the edges in the band such that the face summation constraints hold. The west-most face of the band has two of its three edges labeled so that the third label is determined uniquely. The south pointing triangular face east to it has two out of three edges labeled so that the third one is also fixed. By inductively propagating east, one labels the edges of the band. Note that the label of the last east-most edge of type 0 is determined. This is why we do not require to fix the values of type 0 edges in $I \setminus I_0$.

The previous paragraph shows that one can extend L to all edges in the south most band of I . Notice then that the region above is also a band. By induction, one extends L to the $n-d$ south most bands of I . The remaining region consists of bands turned upside down which can also be uniquely determined by the same reasoning. The resulting map L on $E_{n,d}$ satisfies the face summation constraints so that $L \in \tilde{L}^{C_0}(\lambda, \mu, \nu, N)$. Since the band completions are unique at each step, the labels of any other quasi label map $L' \in \tilde{L}^{C_0}(\lambda, \mu, \nu, N)$ with same values on I_0 would agree with L . \square

Definition 6.22 (Label map associated to edge coordinates). Define $\Phi^{C_0}(z) \in L^{C_0}(\lambda, \mu, \nu)$ to be the unique label map constructed in Lemma 6.21 from specifying edge coordinates $z \in (\frac{1}{N}\mathbb{Z})^D$. Let

$C : E_{n,d} \rightarrow \{0, 1, 3, m\}$ be any color map and let $D = (n-1)(n-2)/2$. Define the following map

$$\begin{aligned} \Phi^C : \left(\frac{1}{N}\mathbb{Z}\right)^D &\rightarrow L^C(\lambda, \mu, \nu, N) \\ z = (z_1, \dots, z_D) &\mapsto \Phi^C(z) = \text{Rot}[C_0 \rightarrow C](\Phi^{C_0}(z)). \end{aligned}$$

Proposition 6.23 (Quasi hive structure). *The map Φ^C of Definition 6.22 is bijective. Moreover, for every $z \in \left(\frac{1}{N}\mathbb{Z}\right)^D$ and edge $e \in E_{n,d}$, $\Phi^C(z)(e)$ is an affine combination of $(z_1, \dots, z_D, \lambda, \mu, \nu, \frac{1}{N})$ with integer coefficients.*

Proof. The map Φ^C is the composition of two bijections : $z \mapsto \Phi^{C_0}(z) \in L^{C_0}(\lambda, \mu, \nu, N)$ and $L \mapsto \text{Rot}[C_0 \rightarrow C](L)$ which are both affine in $(z_1, \dots, z_D, \lambda, \mu, \nu, \frac{1}{N})$ by Lemma 6.20 and Lemma 6.21. Its inverse is given by $(\Phi^C)^{-1}(L) = \text{Rot}[C_0 \rightarrow C](L)|_{I_0}$ where to a label map $L \in L^{C_0}(\lambda, \mu, \nu, N)$, $L|_{I_0} = (L(e_1), \dots, L(e_D))$ are the labels of the edges in I_0 . \square

So far we have defined the maps $\text{Rot}[C \rightarrow C']$ and Φ^C from $\tilde{L}^C(\lambda, \mu, \nu, N) \rightarrow \tilde{L}^{C'}(\lambda, \mu, \nu, N)$ and $\left(\frac{1}{N}\mathbb{Z}\right)^D \rightarrow \tilde{L}^C(\lambda, \mu, \nu, N)$ respectively. We will now extend their definitions to quasi hives.

Definition 6.24 (Extension to dual hives). *Let C, C' be two color maps. We extend the maps $\text{Rot}[C \rightarrow C']$ and Φ^C of Definitions 6.19 and 6.22 to quasi hives by*

$$(6.8) \quad \text{Rot}[C \rightarrow C'] : \tilde{H}^C(\lambda, \mu, \nu, N) \rightarrow \tilde{H}^{C'}(\lambda, \mu, \nu, N)$$

$$(6.9) \quad H = (C, L) \mapsto \text{Rot}[C \rightarrow C'](H) = (C', \text{Rot}[C \rightarrow C'](L)),$$

and

$$(6.10) \quad \Phi^C : \left(\frac{1}{N}\mathbb{Z}\right)^D \rightarrow \tilde{H}^C(\lambda, \mu, \nu, N)$$

$$(6.11) \quad z = (z_1, \dots, z_D) \mapsto (C, \Phi^C(z)).$$

7. CONVERGENCE TO A VOLUME OF HIVES

7.1. Limit dual hives.

Definition 7.1 (Limit dual hive). *For $\alpha, \beta, \gamma \in (\mathbb{R}_{\geq 0})^3$, the limit dual hive $H(\alpha, \beta, \gamma, \infty)$ is the set of pairs (C, L) on the hexagon edges $E_{n,d}$ such that :*

- (1) $C : E_{n,d} \rightarrow \{0, 1, 3, m\}$ is a color map,
- (2) $L : E_{n,d} \rightarrow \mathbb{R}_{\geq 0}$ is the label map satisfying
 - (a) $L(e_1) + L(e_2) + L(e_3) = 1$ for every face F_n ,
 - (b) L satisfies the same inequalities as label maps for dual hives stated in (2) of Definition 5.2 replacing the strict inequalities in (2c) by a large one.
 - (c) The values of L on $\partial E_{n,d}$ are given by (α, β, γ) .

As in the discrete case, if C is a color map, we denote by $H^C(\lambda, \mu, \nu, \infty) \subset H(\lambda, \mu, \nu, \infty)$ the subset of limit dual hives of having color map C . As in the previous section, we denote by $\tilde{H}(\lambda, \mu, \nu, \infty)$ the set of pairs (C, L) as above where we remove the inequality conditions (2b) on L .

Remark 7.2 (Maps Φ^C and Rot on limit dual hives). *Note that Lemmas 6.18, 6.20 and 6.21 hold for limit quasi dual hives $H = (C, L) \in \tilde{H}(\lambda, \mu, \nu, \infty)$ extending $z \in \left(\frac{1}{N}\mathbb{Z}\right)^D$ to $z \in \mathbb{R}^D$. Using the same construction as in Section 6, we define $\text{Rot}[C \rightarrow C'] : \tilde{H}^C(\lambda, \mu, \nu, \infty) \rightarrow \tilde{H}^{C'}(\lambda, \mu, \nu, \infty)$ and $\Phi^C : \mathbb{R}^D \rightarrow \tilde{H}^C(\lambda, \mu, \nu, \infty)$ which are affine bijections with integer coefficients.*

7.2. Convergence to a volume. The goal of this part is to prove Proposition 7.3 which expresses the limit quantum cohomology coefficients as the volume involving limit dual hives. The proof relies on Lemma 7.4 Lemma 7.6 below.

Proposition 7.3 (Convergence to volume of dual hives).

$$(7.1) \quad \lim_{n \rightarrow \infty} N^{-D} c_{\lambda_N, \mu_N}^{\nu_N, d} = \sum_C \text{Vol}(u \in \mathbb{R}^D, \Phi^C[u] \in H^C(\alpha, \beta, \gamma, \infty)).$$

Recall that

$$(7.2) \quad N^{-D} c_{\lambda_N, \mu_N}^{\nu_N, d} = N^{-D} |H(\lambda_N, \mu_N, \nu_N, N)| = N^{-D} |\tilde{H}(\lambda_N, \mu_N, \nu_N, N) \cap Ineq(n)|$$

$$(7.3) \quad = \sum_C \int_{\mathbb{R}^D} \sum_{z=(z_1, \dots, z_d) \in (\frac{1}{N}\mathbb{Z})^D: \Phi^C(z) \in \tilde{H}(\lambda_N, \mu_N, \nu_N, N) \cap Ineq(n)} \mathbf{1}(u)_{z+[-\frac{1}{N}, \frac{1}{N}]^D} du.$$

where $Ineq(n)$ are the inequality constraints given in (2b) and (2c) of Definition 5.2.

Lemma 7.4 (Pointwise convergence). *For any color map C and $N \geq 1$, define*

$$f_N^C : \mathbb{R}^D \rightarrow \mathbb{R} \\ u \mapsto \sum_{z=(z_1, \dots, z_d) \in (\frac{1}{N}\mathbb{Z})^D: \Phi^C(z) \in \tilde{H}(\lambda_N, \mu_N, \nu_N, N) \cap Ineq(n)} \mathbf{1}(u)_{\{z+[-\frac{1}{N}, \frac{1}{N}]^D\}}.$$

Then, for any $u \in \mathbb{R}^D$:

$$(7.4) \quad \lim_{N \rightarrow \infty} f_N^C(u) = \mathbf{1}(u)_{\{\Phi^C[u] \in H^C(\lambda, \mu, \nu, \infty)\}}.$$

Remark 7.5. *Note that a priori, $\Phi^C[u] \in \tilde{H}^C(\lambda, \mu, \nu, \infty)$ is a label map such that (C, L) is a limit hive of Definition 7.1 without inequality constraints (2c) and (2d). Here, the right hand side is more restrictive as it requires that $(C, \Phi^C[u]) \in H^C(\lambda, \mu, \nu, \infty) = \tilde{H}^C(\lambda, \mu, \nu, \infty) \cap Ineq(n)$.*

Proof. Take u such that $\Phi^C[u]$ in the interior of $H^C(\lambda, \mu, \nu, \infty)$. We want to show that $f_N^C(u) = 1$ for $N \geq N_0$ which means that one can find a sequence $(z^{(N)}, N \geq N_0) = ((z_1^{(N)}, \dots, z_D^{(N)}), N \geq N_0)$ such that for each $N : u \in z_N \pm 1/N$. Let us take the label map

$$(7.5) \quad L^{(N)} : e \mapsto \Phi_N^C(\lfloor Nu \rfloor / N)(e) \in \frac{1}{N}\mathbb{Z}.$$

associated to $z^{(N)} = \lfloor Nu \rfloor / N : L^{(N)} = \Phi_N^C(z^{(N)})$ where Φ_N^C is associated to the boundaries $(\lambda_N, \mu_N, \nu_N)$. One has $|z^{(N)} - u| \leq 1/N$ by construction. We need to check that $L^{(N)} \in \tilde{H}(\lambda_N, \mu_N, \nu_N, N) \cap Ineq(n)$.

By definition, Φ_N^C is a quasi label map with boundary conditions $(\lambda_N, \mu_N, \nu_N)$ so that $L^{(N)} \in \tilde{H}(\lambda_N, \mu_N, \nu_N, N)$. Let us check the inequality constraints of $Ineq(n)$. Take any pair of edges (e, e') subject to an inequality. Since $\Phi^C[u] \in H^C(\lambda, \mu, \nu, \infty)$ is in the interior, this equality is sharp for $\Phi^C[u]$ that is

$$(7.6) \quad \exists \epsilon_{e, e'} > 0 : \Phi^C[u](e) \leq \Phi^C[u](e') + \epsilon_{e, e'}.$$

Since $\lim_{N \rightarrow +\infty} L^{(N)}(e) = \Phi^C[u](e)$ for any edge e , there exists $N_0(e, e')$ such that for $N \geq N_0(e, e')$:

$$(7.7) \quad L^{(N)}(e) \leq L^{(N)}(e').$$

Taking $N_0 = \max(N_0(e, e'), (e, e') \in E_{n, d}^2)$, $L^{(N)}$ satisfies the inequality constraints for $N \geq N_0$. Therefore,

$$(7.8) \quad \forall N \geq N_0 : L^{(N)} \in \tilde{H}(\lambda_N, \mu_N, \nu_N, N) \cap Ineq(n).$$

so that $\lim_{N \rightarrow +\infty} f_N^C(u) = 1$ as desired. For $\Phi^C[u] \notin H^C(\lambda, \mu, \nu, \infty)$, at least one of the two conditions (2c), (2d) of Definition 7.1 is violated. It cannot be that conditions (2a), (2b) do not hold by construction of Φ^C . Let (e, e') be a pair of edge such that either (2c) or (2d) is violated : $\Phi^C[u](e) < \Phi^C[u](e')$. Using that $\lim_{N \rightarrow +\infty} L^{(N)}(e) = \Phi^C[u](e)$, one has that for N large enough $L^{(N)}(e) < L^{(N)}(e')$ so that $L^{(N)} \notin \tilde{H}(\lambda_N, \mu_N, \nu_N, N) \cap Ineq(n)$. Therefore, we have that (7.4) holds. \square

Lemma 7.6 (Uniform bound). *Let $f_N^C(u)$ be as in (7.4). Then, there exists a compact $K^C \subset \mathbb{R}^D$ such that for every $N \geq 1$,*

$$(7.9) \quad |f_N^C(u)| \leq \mathbf{1}_{\{u \in K^C\}}.$$

Proof of Lemma 7.6. If $C = C_0$, the values $z \in \mathbb{Z}^D$ such that $\Phi_N^{C_0}(z) \in \tilde{H}^{C_0}(\lambda_N, \mu_N, \nu_N, N) \cap \text{Ineq}(n)$ are in $[0, 1]^D$ since (z_1, \dots, z_D) are the values of $(\Phi_N^{C_0}(z)(e_1), \dots, \Phi_N^{C_0}(z)(e_D))$ for some horizontal edges $(e_1, \dots, e_D) \in E_{n,d}$ which are in $[0, 1]$.

If $C \neq C_0$, then there exists a composition of rotation $\text{Rot}[C \rightarrow C_0]$ such that $\text{Rot}[C \rightarrow C_0](e) \in [0, 1]$ for each horizontal edge e so that $z \in \cup_e (z \mapsto \text{Rot}[C \rightarrow C_0](e)^{-1}[0, 1])$ which is compact.

If $C \neq C_0$ by definition

$$(7.10) \quad \Phi^C(z) = \text{Rot}[C_0 \rightarrow C](\Phi^{C_0}(z))$$

where $\Phi^{C_0}(z)$ is the label map of the quasi hive with simple color map C_0 having horizontal edge labels z . For $z \in \mathbb{Z}^D$ such that $\Phi_N^{C_0}(z) \in \tilde{H}^{C_0}(\lambda_N, \mu_N, \nu_N, N) \cap \text{Ineq}(n)$, we know from $\text{Ineq}(n)$ that values $\{\Phi_N^{C_0}(z)(e), e \in E_{n,d}\}$ are in $[0, 1]^{E_{n,d}}$. Applying the affine hence continuous $\text{Rot}[C \rightarrow C_0]$ we get that $\text{Rot}[C \rightarrow C_0]\Phi^C(z) = \Phi^{C_0}(z) \in \text{Rot}[C \rightarrow C_0]([0, 1]^{E_{n,d}})$ which is compact. In particular, the labels $(z_i = \Phi^{C_0}(z)(e_i), 1 \leq i \leq D)$ of horizontal edges e_i in I are in compact sets hence bounded. \square

Proof of Proposition 7.3. By Lemma 7.4 Lemma 7.6, using dominated convergence in (7.3):

$$(7.11) \quad \lim_{n \rightarrow \infty} N^{-D} c_{\lambda_N, \mu_N}^{\nu_N, d} = \sum_C \text{Vol}(u \in \mathbb{R}^D, \Phi^C[u] \in H^C(\alpha, \beta, \gamma, \infty)).$$

\square

7.3. Volume preserving map. This subsection aims at proving that there is a volume preserving map between dual hives $H(\alpha, \beta, \gamma, \infty)$ and hives $P_{\alpha, \beta, \gamma}^g$ of Definition 2.3.

Definition 7.7 (Dual hive to hive). *Let C be a color map. Define*

$$(7.12) \quad S^C = \{v_4 : l = (v_1, v_2, v_3, v_4) \text{ is a } m \text{ lozenge}\} \subset R_{d,n}$$

and

$$(7.13) \quad \tilde{P}_{\alpha, \beta, \gamma}^C := \{f : R_{d,n} \setminus S^C \rightarrow \mathbb{R}, f|_{\partial R_{d,n}} \text{ given by } \alpha, \beta, \gamma\}.$$

The choice of v_4 and the coloring on the boundary ensures that v_4 is never on the boundary of $R_{d,n}$. We will assign to any $(C, L) \in \tilde{H}^C(\alpha, \beta, \gamma, \infty)$ an element in $\tilde{P}_{\alpha, \beta, \gamma}^C$:

$$(7.14) \quad \Psi^C : \tilde{H}^C(\alpha, \beta, \gamma, \infty) \rightarrow \tilde{P}_{\alpha, \beta, \gamma}^C$$

$$(7.15) \quad (C, L) \mapsto \Psi^C[L]$$

where $\Psi^C[L](v) = 0$ on the south-east corner of $R_{d,n}$ and adding $L(e)$ when crossing edges of type 1, 2 and $1 - L(e)$ for edges of type 0 as in Figure

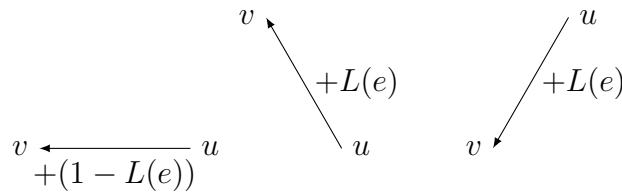


FIGURE 32. Values at vertices when traversing an edge $e = (u, v)$.

That the map Ψ^C is well defined is due to the face summation constraint $L(e_1) + L(e_2) = 1 - L(e_0)$ around every face $f \in F_n$ having edges (e_0, e_1, e_2) of respective types 0, 1, 2 for the label maps L of dual hives in $\tilde{H}^C(\alpha, \beta, \gamma, \infty)$.

Remark 7.8 (Extension of function of $\tilde{P}_{\alpha, \beta, \gamma}^C$). *Let $f \in \tilde{P}_{\alpha, \beta, \gamma}^C$. Then f can be uniquely extended to a map $f : R_{d,n} \rightarrow \mathbb{R}$ by setting $f(v_4) = f(v_3) + f(v_1) - f(v_2)$ for any $v_4 \in S^C$.*

Proposition 7.9 (From color maps to regular labelings). *Let $C : E_{n,d} \rightarrow \{0, 1, 3, m\}$ be a regular color map. Define*

$$g[C] : R_{d,n} \rightarrow \mathbb{Z}_3$$

$$v \mapsto g[C](v)$$

where the value at vertex $v \in R_{d,n}$ is set as follows. If $v = A_0$ the south-east most vertex in $R_{n,d}$, set $g[C](v) = 0$. Orient the edges in $E_{d,n}$ around north-pointing faces clockwise and edges around south-pointing faces counterclockwise. For any oriented edge $e = (u, v)$, set

$$(7.16) \quad g[C](v) = \begin{cases} g[C](u) + 1 & \text{if } C((u, v)) = 1 \\ g[C](u) + 2 & \text{if } C((u, v)) = 0 \\ g[C](u) & \text{if } C((u, v)) \in \{3, m\}. \end{cases}$$

Then, $C \mapsto g[C]$ is a bijection between color maps on $E_{n,d}$ and regular labelings on $R_{n,d}$. For any g regular labeling, its inverse is given by

$$C[g] : E_{d,n} \rightarrow \{0, 1, 3, m\}$$

$$e = (u, v) \mapsto C[g](e)$$

where

$$(7.17) \quad C[g](e) = \begin{cases} 1 & \text{if } g(v) - g(u) = 1 \\ 0 & \text{if } g(v) - g(u) = 2 \\ 3 & \text{if } g(v) = g(u) \text{ and } F(e) \text{ has cyclic difference } (1, 2, 0) \\ m & \text{if } g(v) = g(u) \text{ and } F(e) \text{ has cyclic difference } (2, 1, 0). \end{cases}$$

Proof. Let us show that $g[C]$ is well defined. Since C is a color map, the only colors around a triangular face in $E_{n,d}$ are up to cyclic permutations $(0, 0, 0)$, $(1, 1, 1)$, $(1, 0, 3)$ and $(0, 1, m)$. One checks that summing the clockwise differences of values of g going from a vertex to itself around any such color triple gives a zero contribution in \mathbb{Z}_3 . Therefore, the value of $g[C](v)$ does not depend on the choice of the edge $e = (u, v) \in E_{n,d}$ since any different neighbor choice u' such that $(u', v) \in E_{n,d}$ differ by two triangular faces, see Figure 33 where (X, Y, Z) denote the differences of values of g . We have $g(u) + X = g(u') + Y'[3]$ since $g(u) + X = g(u) - Y - Z = g(w) - Z = g(w) + X' + Y' = g(u') + Y'$.

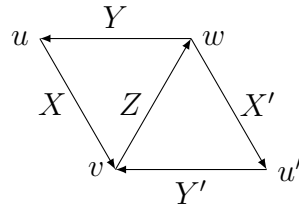


FIGURE 33. The equality $g(u) + X = g(u') + Y'[3]$

That $(g[C]^A, g[C]^B, g[C]^C)$ has the right boundary conditions is due to the fact that C is regular. It remains to check the lozenge condition on $g[C]$ from Definition 2.1. Take any lozenge $l = (v^1, v^2, v^3, v^4)$ and suppose that $g[C](v^2) = g[C](v^4)$. Note that from Figure 3, the edge between v^2 and v^4 is always oriented from v^4 to v^2 . The edge $e = (v^4, v^2)$ has color either 3 or m . Since C is a color map, the two faces adjacent to e have either $(1, 0, 3)$ or $(0, 1, m)$ colors. The face with vertices (v^1, v^2, v^4) , respectively (v^3, v^2, v^4) is always north pointing, respectively south pointing, see Figure 3. If $C(e) = 3$, $g[C](v^1) = g[C](v^2) + 1$ and $g[C](v^3) = g[C](v^2) + 2$ whereas if $C(e) = m$, $g[C](v^1) = g[C](v^2) + 2$ and $g[C](v^3) = g[C](v^2) + 1$. In both cases, $\{g[C](v^1), g[C](v^3)\} = \{g[C](v^2) + 1, g[C](v^2) + 2\}$ and thus $g[C]$ is a regular labeling.

Let us show that $g \mapsto C[g]$ maps a regular labeling g to a color map. Since g is regular, $C[g]$ also is by the same argument as above. Let us show that the only cyclic colors configurations around any triangular face are $(0, 0, 0)$, $(1, 1, 1)$, $(1, 0, 3)$ and $(0, 1, m)$. Take any triangular face and denote X, Y, Z the clockwise differences of values of g . Then, $X + Y + Z = 0[3]$ so that

$(X, Y, Z) \in \{(1, 1, 1), (2, 2, 2), (0, 1, 2), (0, 2, 1)\}$. Note that we exclude $(0, 0, 0)$ by the lozenge condition in Definition 2.1 since a rigid lozenge cannot have three vertices with equal values. These possible height differences give the clockwise colors $\{(1, 1, 1), (0, 0, 0), (3, 1, 0), (m, 0, 1)\}$ respectively. Therefore, $C[g]$ is a color map and by construction $g \mapsto C[g]$ is the inverse of $C \mapsto g[C]$. \square

Lemma 7.10 (Image of limit dual hives are toric concave functions). *For any regular color map C ,*

$$(7.18) \quad \Psi^C(H^C(\alpha, \beta, \gamma, \infty)) = P_{\alpha, \beta, \gamma}^{g[C]}$$

where $g[C]$ is the regular labeling associated to C as in Proposition 7.9 and $P_{\alpha, \beta, \gamma}^{g[C]}$ is the polytope defined in Definition 2.3.

Proof. The image $\Psi^C[(C, L)]$ of any limit hive $(C, L) \in H^C(\alpha, \beta, \gamma, \infty)$ can be extended by Remark 7.8 to a function $f : R_{d,n} \rightarrow \mathbb{R}$ such that by construction $\Psi^C[(C, L)] = f|_{R_{d,n} \setminus S^C} = f|_{\text{Supp}(g[C])}$. Let us check that $f \in P_{\alpha, \beta, \gamma}^{g[C]}$. By definition of the extension, f satisfies the equality constraints over any rigid lozenge in $R_{n,d}$. For any other lozenge $l = (v_1, \dots, v_4)$, the inequality $f(v_2) + f(v_4) \geq f(v_1) + f(v_3)$ is equivalent to the inequality (2c) of Definition 7.1.

Conversely, to any function $f \in P_{\alpha, \beta, \gamma}^{g[C]}$, associate the label map $L[f] : E_{n,d} \rightarrow \mathbb{R}_{\geq 0}$, $e = (u, v) \mapsto L[f](e) = f(v) - f(u)$ if e has type 1 or 2 and $L[f](e) = 1 - (f(v) - f(u))$ if e has type 0. We have that $\Psi^C(C, L) = f$. moreover, the equality and inequality constraints in Definition 7.1 are equivalent to the rhombus concavity of f so that $(C, L) \in H^C(\alpha, \beta, \gamma, \infty)$. \square

With the definitions above, we have an affine map $\Psi^C \circ \Phi^C : \mathbb{R}^D \rightarrow \tilde{P}_{\alpha, \beta, \gamma}^C$. In the rest of this section, we write $\det(\Psi^C \circ \Phi^C)$ for the determinant of the linear part of this application.

Proposition 7.11 (Volume preservation by duality). *Let C be a color map. Then, the map*

$$(7.19) \quad \Psi^C \circ \Phi^C : \mathbb{R}^D \rightarrow \tilde{P}_{\alpha, \beta, \gamma}^C$$

satisfies

$$(7.20) \quad |\det(\Psi^C \circ \Phi^C)| = 1$$

and thus

$$(7.21) \quad \text{Vol}(u \in \mathbb{R}^D, \Phi^C(u) \in H^C(\lambda, \mu, \nu, \infty)) = \text{Vol}(u \in \mathbb{R}^D, \Psi^C \circ \Phi^C[u] \in P_{\alpha, \beta, \gamma}^{g[C]}) = \text{Vol}(P_{\alpha, \beta, \gamma}^{g[C]}).$$

Proof. For $C = C_0$, enumerate e_1, \dots, e_D the horizontal edges in C_0 (except for the ones on the east band).

$$[\Psi^{C_0} \circ \Phi^{C_0}(u)](v) = \sum_{e_i} (1 - u_i)$$

where the sum is over edges e_i of type 0 connecting v to the east boundary of $E_{n,d}$ with inverse

$$[\Psi^{C_0} \circ \Phi^{C_0}(f)]^{-1}(i) = 1 - (f(v) - f(v'))$$

where v, v' the both endpoint of e_i with the correct orientation. Since $\Psi^{C_0} \circ \Phi^{C_0}$ and $[\Psi^{C_0} \circ \Phi^{C_0}(f)]^{-1}$ have integer coefficients, $\det(\Psi^{C_0} \circ \Phi^{C_0}) = 1$.

If C is general, introduce for each hexagon rotation $C \rightarrow C'$ the map $\tilde{R}_{C \rightarrow C'} : \tilde{P}_{\alpha, \beta, \gamma}^C \rightarrow \tilde{P}_{\alpha, \beta, \gamma}^{C'}$ by

- (1) For $f \in \tilde{P}_{\alpha, \beta, \gamma}^C$, extend f uniquely to a function $f : R_{d,n} \rightarrow \mathbb{R}$,
- (2) Let us describe how the hexagon rotation $C \rightarrow C'$ maps f to another function $f' : R_{n,d} \rightarrow \mathbb{R}$. The value of the center vertex c of h is uniquely determined by the position of the rigid lozenge in h and the values of f on ∂h . Indeed, if (v, v', v'') are the three other vertices of the rigid lozenge such that $C((c, v'')) = m$, then $f(c) = f(v) + f(v') - f(v'')$. Note that $(v, v', v'') \in \partial h^3$. For every vertex $u \in R_{n,d}$ other than the center vertex c of h , we set $f'(u) = f(u)$. In the hexagon rotation $C \rightarrow C'$, the position of the rigid lozenge changes

and we set $f'(c) = f(w) + f(w') - f(w'')$ where $(v, v', v'') \in \partial h^3$ are the new vertices of the rigid lozenge on $\partial h' = \partial h$.

(3) The map $\tilde{R}_{C \rightarrow C'}(f)$ is defined as the restriction of f' of the previous step to $R_{n,d} \setminus S^{C'}$.

Note that the map $\tilde{R}_{C \rightarrow C'}$ is an affine bijection with integer coefficients whose inverse is given by $\tilde{R}_{C' \rightarrow C}$. Let us check that the following diagram is commutative

$$(7.22) \quad \begin{array}{ccc} \tilde{P}_{\alpha, \beta, \gamma}^C & \xrightarrow{\tilde{R}_{C \rightarrow C'}} & \tilde{P}_{\alpha, \beta, \gamma}^{C'} \\ \Psi^C \uparrow & & \uparrow \Psi^{C'} \\ \tilde{H}^C(\alpha, \beta, \gamma, \infty) & \xrightarrow{\text{Rot}[C, C']} & \tilde{H}^{C'}(\alpha, \beta, \gamma, \infty) \end{array}$$

Let $H = (C, L) \in \tilde{H}^C(\alpha, \beta, \gamma, \infty)$ having an ABC hexagon h with center vertex c . Denote C' the color map obtained after any rotation $h \mapsto h'$. For every vertex $u \neq c \in R_{n,d}$, one has $\tilde{R}_{C \rightarrow C'}(\Psi^C)(u) = \Psi^{C'}(u)$. Moreover, if $u \neq c$ then one can find a path of edges from the south-east most vertex of $R_{n,d}$ to u without any edge incident to c . Since the labels of these edges are not changed by $\text{Rot}[C \rightarrow C']$, $\Psi^{C'}(L')(u) = \Psi^C(L)(u) = \tilde{R}_{C \rightarrow C'}(\Psi^C(L))(u)$ as desired. It remains to check that the same property holds for $u = c$. Denote (v, v', v'') , respectively (w, w', w'') the vertices on ∂h such that (c, v, v', v'') , respectively (w, w', w'') , are vertices of the rigid lozenge in h , respectively h' and that $C((u, v'')) = C'((u, w'')) = m$. Then,

$$\tilde{R}_{C \rightarrow C'}(\Psi^C(L))(u) = \Psi^C(L)(w) + \Psi^C(L)(w') - \Psi^C(L)(w'')$$

and

$$\Psi^{C'}(\text{Rot}[C, C'](L))(u) = \Psi^{C'}(L')(u)$$

Since the values of Ψ do not depend on the chosen path, let us choose the following four paths. Take any path $p = (e_1, \dots, e_r) \in (E_{n,d})^r$ from the south-east origin to w such that for each $1 \leq i \leq r$, e_i is not an interior edge of h . then,

- (1) To evaluate $\Psi^C(L)(w)$, we choose the path p ,
- (2) To evaluate $\Psi^C(L)(w'')$, we append the edge $(w, w'') \in \partial h$ to p ,
- (3) To evaluate $\Psi^C(L)(w')$, we append edges $(w, w''), (w'', w') \in (\partial h)^2$ to p ,
- (4) To evaluate $\Psi^{C'}(L')(u)$, we append (w, u) to p

which gives

$$\begin{aligned} \Psi^C(L)(w) &= \sum_{1 \leq i \leq r} L(e_i) \\ \Psi^C(L)(w'') &= \sum_{1 \leq i \leq r} L(e_i) + L((w, w'')) \\ \Psi^C(L)(w') &= \sum_{1 \leq i \leq r} L(e_i) + L((w, w'')) + L((w'', w')) \\ \Psi^{C'}(L')(u) &= \sum_{1 \leq i \leq r} L'(e_i) + L'((w, u)). \end{aligned}$$

Since we have chosen edges $e_i \in p$ not interior to h , $L(e_i) = L'(e_i)$ for $1 \leq i \leq r$. The commutativity of the diagram is thus equivalent to

$$L'((w, u)) = L((w, w'')) + L((w'', w')) - L((w, w'')) = L((w'', w'))$$

i.e

$$L'((w, u)) = L((w, w''))$$

Notice that $(w, u), (w'', w')$ are two edges of the same type in the rigid lozenge in C' which implies that $L'((w, u)) = L'((w', w')) = L((w', w''))$ where the last equality is due to the fact that $(w', w'') \in \partial h$ so its label is unchanged by $\text{Rot}[C, C']$. The commutativity of (7.22) is thus showed.

Using (7.22), we have for any sequence of hexagon rotations $C_0 \rightarrow C_1 \rightarrow \dots \rightarrow C$,

$$\prod \tilde{R}_{C_i \rightarrow C_{i+1}} \Psi^{C_0} \circ \Phi^{C_0} = \Psi^C \prod \text{Rot}[C_i \rightarrow C_{i+1}] \Phi^{C_0} = \Psi^C \circ \Phi^C.$$

On the left hand side, every map is affine with integer coefficient and with inverse having integer coefficients, so the same is true on the right hand-side, and thus

$$\det(\Psi^C \circ \Phi^C) = 1.$$

The first equality of (7.21) is due to Lemma 7.10 and the second is a consequence of $\det(\Psi^C \circ \Phi^C) = 1$. \square

7.4. Proof of Theorem 2.4 and Corollary 2.5.

Proof of Theorem 2.4. By Theorem 3.8 we had

$$(7.23) \quad d\mathbb{P}[\gamma|\alpha, \beta] = \frac{\Delta(e^{2i\pi\gamma})}{\Delta(e^{2i\pi\alpha})\Delta(e^{2i\pi\beta})} \lim_{N \rightarrow \infty} N^{-(n-1)(n-2)/2} c_{\lambda_N, \mu_N}^{\nu_N, d}.$$

By Proposition 7.3 and Proposition 7.11,

$$(7.24) \quad d\mathbb{P}[\gamma|\alpha, \beta] = \frac{\Delta(e^{2i\pi\gamma})}{\Delta(e^{2i\pi\alpha})\Delta(e^{2i\pi\beta})} \sum_C \text{Vol}[u \in \mathbb{R}^D, \Phi^C[u] \in H^C(\alpha, \beta, \gamma, \infty)]$$

$$(7.25) \quad = \frac{\Delta(e^{2i\pi\gamma})}{\Delta(e^{2i\pi\alpha})\Delta(e^{2i\pi\beta})} \sum_C \text{Vol}(P_{\alpha, \beta, \gamma}^{g[C]})$$

$$(7.26) \quad = \frac{\Delta(e^{2i\pi\gamma})}{\Delta(e^{2i\pi\alpha})\Delta(e^{2i\pi\beta})} \sum_{g: R_{d,n} \rightarrow \mathbb{Z}_3 \text{ regular}} \text{Vol}_g(P_{\alpha, \beta, \gamma}^g).$$

\square

Proof of Corollary 2.5. From [31, Eq. (4.116)], we have

$$\text{Vol}[\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)] = \frac{\#Z(SU_n)\text{Vol}(SU_n)}{\text{Vol}((\mathbb{R}/2\pi\mathbb{Z})^{n-1})^3} \sum_{\lambda \in \mathbb{Z}_{\geq 0}^n} \frac{1}{\dim V_\lambda} \chi_\lambda(e^{2i\pi\alpha}) \chi_\lambda(e^{2i\pi\beta}) \chi_\lambda(e^{2i\pi\gamma}),$$

where $Z(SU_n)$ is the center of SU_n . From (3.3), we deduce that

$$\text{Vol}[\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)] = \frac{\#Z(SU_n)\text{Vol}(SU_n)(2\pi)^{n-1}n!}{\text{Vol}((\mathbb{R}/2\pi\mathbb{Z})^{n-1})^3 |\Delta(e^{2i\pi\gamma})|^2} d\mathbb{P}[-\gamma|\alpha, \beta].$$

Corollary 2.5 is then deduced from Theorem 2.4 and the fact that $Z(SU_n) = 2^{(n+1)[2]}$, $\text{Vol}(SU_n) = \frac{(2\pi)^{n(n+1)/2-1}}{\prod_{k=1}^n k!}$ and $\text{Vol}((\mathbb{R}/2\pi\mathbb{Z})^{n-1}) = (2\pi)^{n-1}$. \square

REFERENCES

- [1] S Agnihotri and C Woodward. Eigenvalues of products of unitary matrices and quantum schubert calculus. *Mathematical Research Letters*, 5(6):817–836, 1998.
- [2] Prakash Belkale. Quantum generalization of the horn conjecture. *Journal of the American Mathematical Society*, 21(2):365–408, 2008.
- [3] Arkady Berenstein and Andrei Zelevinsky. Tensor product multiplicities, canonical bases and totally positive varieties. *Inventiones Mathematicae*, 143(1):77, 2001.
- [4] Anders Skovsted Buch. Mutations of puzzles and equivariant cohomology of two-step flag varieties. *Annals of mathematics*, pages 173–220, 2015.
- [5] Anders Skovsted Buch, Andrew Kresch, Kevin Purbhoo, and Harry Tamvakis. The puzzle conjecture for the cohomology of two-step flag manifolds. *J. Algebr. Comb.*, 44(4):973–1007, 2016.
- [6] Anders Skovsted Buch, Andrew Kresch, and Harry Tamvakis. Gromov-witten invariants on grassmannians. *Journal of the American Mathematical Society*, 16(4):901–915, 05 2003.
- [7] Robert Coquereaux and Jean-Bernard Zuber. From orbital measures to littlewood-richardson coefficients and hive polytopes. *Annales de l'Institut Henri Poincaré D*, 5(3):339–386, 07 2018.
- [8] Manon Defosseux. Fusion coefficients and random walks in alcoves. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 52(4), 11 2016.
- [9] Anthony Haynes Dooley and Norman John Wildberger. Harmonic analysis and the global exponential map for compact lie groups. *Functional Analysis and Its Applications*, 27(1):21–27, 1993.
- [10] Pavel Etingof and Eric Rains. Mittag-leffler type sums associated with root systems. *arXiv preprint arXiv:1811.05293*, 2018.

- [11] D.M Jackson. Some combinatorial problems associated with products of conjugacy classes of the symmetric group. *Journal of Combinatorial Theory, Series A*, 49(2):363–369, 1988.
- [12] Lisa C Jeffrey. Extended moduli spaces of flat connections on riemann surfaces. *Mathematische Annalen*, 298:667–692, 1994.
- [13] Masaki Kashiwara and Toshiki Nakashima. Crystal graphs for representations of the q-analogue of classical lie algebras. *Journal of Algebra*, 165:295–345, 1994.
- [14] A. Kirillov. *Lectures on the Orbit Method*. American Mathematical Society, 07 2004.
- [15] Alexander A Klyachko. Stable bundles, representation theory and hermitian operators. *Selecta Mathematica*, 4:419–445, 1998.
- [16] Allen Knutson and Terence Tao. The honeycomb model of gl_n tensor products i: Proof of the saturation conjecture. *Journal of the American Mathematical Society*, 12(4):1055–1090, 4 1999.
- [17] Allen Knutson, Terence Tao, and Christopher Woodward. The honeycomb model of $gl_n(\mathbb{C})$ -tensor products ii: Puzzles determine facets of the littlewood-richardson cone. *Journal of the American Mathematical Society*, 17(1):19–48, 2004.
- [18] Thierry Lévy. *Yang-Mills measure on compact surfaces*. American Mathematical Soc., 2003.
- [19] Peter Littelmann. A littlewood-richardson rule for symmetrizable kac-moody algebras. *Inventiones mathematicae*, 116(1):329–346, 1994.
- [20] Dudley Ernest Littlewood and Archibald Read Richardson. Group characters and algebra. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, 233(721-730):99–141, 1934.
- [21] George Lusztig. *Introduction to quantum groups*. Springer Science & Business Media, 2010.
- [22] Dusa McDuff and Dietmar Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *Colloq. Publ., Am. Math. Soc.* Providence, RI: American Mathematical Society (AMS), 2nd ed. edition, 2012.
- [23] Eckhard Meinrenken and Chris Woodward. Moduli spaces of flat connections on 2-manifolds, cobordism, and witten’s volume formulas. In *Advances in geometry*, pages 271–295. Springer, 1999.
- [24] FD Murnaghan. On the direct product of irreducible representations of the symmetric group. *Proceedings of the National Academy of Sciences*, 23(9):488–490, 1937.
- [25] Hariharan Narayanan and Scott Sheffield. Large deviations for random hives and the spectrum of the sum of two random matrices. *The Annals of Probability*, 52(3):1093–1152, 2024.
- [26] Konstanze Rietsch. Quantum cohomology rings of grassmannians and total positivity. *Duke Mathematical Journal*, 110(3), 12 2001.
- [27] Anders Skovsted Buch. Quantum cohomology of grassmannians. *Compositio mathematica*, 137:227–235, 2003.
- [28] Andras Szenes. Iterated residues and multiple bernoulli polynomials. *International Mathematics Research Notices*, 1998(18):937–956, 1998.
- [29] Constantin Teleman and Christopher Woodward. Parabolic bundles, products of conjugacy classes, and gromov-witten invariants. 53(3):713–748, 2003.
- [30] Norman John Wildberger. Hypergroups and harmonic analysis. In *Miniconference on probability and analysis*, volume 29, pages 238–254. Australian National University, Mathematical Sciences Institute, 1992.
- [31] Edward Witten. On quantum gauge theories in two dimensions. *Communications in Mathematical Physics*, 141(1):153 – 209, 1991.
- [32] Edward Witten. The verlinde algebra and the cohomology of the grassmannian. *arXiv preprint hep-th/9312104*, 1993.
- [33] Alex Wright. Sums of adjoint orbits and l2-singular dichotomy for $su(m)$. *Advances in Mathematics*, 227(1):253–266, 2011.

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