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NORMAL CROSSING IMMERSIONS, COBORDISMS AND FLIPS

KARIM ADIPRASITO AND GAKU LIU

ABSTRACT. We study various analogues of theorems from PL topology for cubical complexes. In particular, we characterize when two PL homeomorphic cubulations are equivalent by Pachner moves by showing the question to be equivalent to the existence of cobordisms between generic immersions of hypersurfaces. This solves a question and conjecture of Habegger and Funar.

1. INTRODUCTION

Pachner's influential theorem proves that every PL homeomorphism of triangulated manifolds can be written as a combination of local moves, the so-called Pachner moves (or bistellar moves). Geometrically, they correspond to exchanging, inside the given manifold, one part of the boundary of a simplex by the complementary part. For instance, in a two-dimensional manifold, a triangle can be replaced by three triangles with a common vertex, and two triangles sharing an edge can be replaced by two different triangles sharing the edge connecting the vertices opposite the original edge.

Cubical complexes, popularized for their connection to low-dimensional topology and geometric group theory (see for instance [Gro87]), have analogous moves called *cubical Pachner moves*. However, the situation is not as straightforward as the simplicial case. Indeed, a cubical Pachner move can never change, for instance, the parity of the number of facets. But cubical polytopes, for instance in dimension 3, can have odd numbers of facets, while other cubical polytopes in dimension 3, such as the standard cube, have even number of facets; see [SZ04] for a few particularly notorious constructions.

Hence, there are at least two classes of cubical 2-spheres that can never be connected by cubical Pachner moves. Indeed, Funar [Fun99] provided a conjecture for a complete characterization for the cubical case that revealed an interesting aspect.

Every cubical manifold C (a cubical complex homeomorphic to a manifold) of dimension d has at least an associated immersed normal hypersurface. Consider an edge e of C . An edge e' is *parallel* to e if it is contained in a common square, but does not share a vertex. Take the transitive closure of this relation, and obtain a set of edges E related to e , which we call the *transversal class*.

In essence, this transversal class is the hypersurface, but we can now construct an object that realizes it in a more tangible, geometric way: We can construct a hypersurface crossing the edges of this class, and no others, by considering the derived subdivision $\text{dr } C$ of C , that is, the order complex of the face poset of C . The vertices in this complex correspond

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to faces of C , and the induced subcomplex on those vertices corresponding to the faces containing an element of E is the desired hypersurface, which we call H_E , the **normal hypersurface** corresponding to the transversal class E . We could have defined this for any cubical complex, of course, but for manifolds C , this hypersurface is at the very least an immersed homology manifold of codimension one. We refer to [Fun99] for an in-depth exploration of this construction.

Visually, one may think of the construction like this: draw a $(d - 1)$ -disk orthogonal to e , and continue it through the parallel edges, and continue as we perform the transitive closure iteratively.

Of course, this not the actual hypersurface, but just one connected component: Note that the set of edges $C^{(1)}$ of C is partitioned into transversal classes $E_1, E_2 \dots$.

The collection H_{E_1}, H_{E_2}, \dots forms the **normal hypersurface** of C , and the individual H_{E_i} are its connected components (of course, the immersions may intersect already). Note that the union of all these normal hypersurfaces cover precisely the codimension-one skeleton of the dual decomposition.

Notice further that every self-intersection of that total hypersurface is normal crossing; the intersections are transversal, and combinatorially isomorphic to the intersections of coordinate hyperplanes in euclidean space.

We call two normal crossing hypersurfaces *cobordant* if they are cobordant as normal crossing hypersurfaces: If H, H' are normal crossing hypersurfaces in a manifold M , then they are cobordant if there is a normal crossing hypersurface in $M \times [0, 1]$ that restricts to H and H' in the two boundary components.

Funar conjectured, and we prove:

Theorem 1.1. *For two PL cubulations X_0, X_1 of the same manifold M , the following three conditions are equivalent:*

- (1) *The total normal hypersurfaces Y_0, Y_1 of the cubulations X_0, X_1 of M are cobordant as normal crossing immersions, that is, the natural embedding of Y_i into $M \times \{i\} \subset M \times [0, 1]$ extends to a normal crossing hypersurface Y that has $Y_0 \cup Y_1$ as its boundary.*
- (2) *There is a PL cubulation Y of $M \times [0, 1]$ such that the induced subdivision of $M \times \{0\}$ is X_0 , and the induced subdivision of $M \times \{1\}$ is X_1 .*
- (3) *X_0 and X_1 are related by cubical Pachner moves.*

This resolves a problem of Habegger and Funar [Fun99], see the discussion following his Problem 1. Habegger in fact assumed that only a PL homeomorphism is necessary for two cubulations to be related through cubical Pachner moves; it was Funar that recognized the importance of normal crossing cobordisms for the problem, and in particular also computed them for spheres [FG02]. In fact, Funar proved almost all implications, except one: that (2) implies (3). For surfaces and 1-dimensional manifolds, the problems were solved by Funar [Fun08] and independently sketched by Thurston [Thu93].

2. PRELIMINARIES

2.1. *Basic definitions.* A **polyhedral complex** is a finite collection of polytopes such that the intersection of any two polytopes is a polytope of the collection, and that is closed

under passing to faces of individual elements. The elements of a polyhedral complex are also called **faces**, the maximal elements **facets**. A **simplicial complex** and a **cubical complex** are complexes whose faces are all simplices and cubes, respectively. These are also called **triangulations** and **cubulations**, respectively.

Given a polyhedral complex A and a subcomplex B of A , we let $A \text{ del } B$ be the complex generated by the facets of A that are not in B . Given a complex A and a face τ of A , we let $A - \tau$ be the complex consisting of faces of A that do not contain τ .

Given a polyhedral complex A and a face τ of A , the **star** $\text{St}_\tau A$ of τ in A is the minimal subcomplex containing all faces of A containing τ . A polyhedral complex is a PL ball if it admits a PL homeomorphism to the simplex. A polyhedral manifold is a polyhedral complex homeomorphic to a manifold, and a PL manifold is a polyhedral manifold in which the star of any vertex is a PL ball. In particular, a PL triangulation resp. PL cubulation is a triangulation resp. cubulation in which every star is a PL ball.

The **link** $\text{Lk}_\tau A$ of a face τ in A is, in terms of the poset, the principal filter of τ in A . Geometrically, it can be understood as follows (if we endow the polyhedral complex with the induced length metric):

If τ is a vertex, then its link is the intersection of A with a sufficiently small metric sphere with center τ . Here, a metric sphere is the boundary of a metric ball. Provided this metric sphere is contained entirely in the star of τ , this yields a (spherical) polyhedral complex whose face poset is the filter discussed above.

To obtain the geometric visualization for the link of a general face τ , proceed iteratively, by considering one of its vertices v , and defining $\text{Lk}_\tau A$ to be the link of $\text{Lk}_v \tau$ in $\text{Lk}_v A$. The link of the empty set in a complex is naturally the complex itself.

2.2. *Cubical Pachner moves.* First, it is useful to have different moves available. Let us first recall the definitions of Pachner moves.

If X is a triangulated manifold of dimension d , and S is a simplex of dimension $d + 1$, and D is a d -dimensional cubulated ball that is a subcomplex of ∂S isomorphic to a subcomplex of X , then the process of removing that subcomplex from X and replacing it with $\partial S \text{ del } D$, attached along the common boundary $\partial D = \partial(\partial S \text{ del } D)$, is a (**simplicial**) **Pachner move**.

Definition 2.1 (Cubical Pachner moves). If X is a cubical manifold, that is, a cubulation of a manifold of dimension d , and C is a cube of dimension $d + 1$, and D is a d -dimensional cubulated ball that is a subcomplex of ∂C isomorphic to a subcomplex of X , then the process of removing that subcomplex from X and replacing it with $\partial C \text{ del } D$, attached along the common boundary $\partial D = \partial(\partial C \text{ del } D)$, is a **cubical Pachner move**.

A good way to think about Pachner moves is in terms of cobordisms and shellings. A **relative complex** $P = (A, B)$ is a pair of polyhedral complexes where $B \subseteq A$. The **faces** of P are the faces of A that are not faces of B .

Consider a relative polyhedral d -dimensional complex $P = (A, B)$ and a d -dimensional facet F of P . We say that P **shells** to $P \text{ del } F := ((A \text{ del } F) \cup B, B)$ if $F \cap ((A \text{ del } F) \cup B)$ is shellable of dimension $d - 1$ (or empty) and $P \text{ del } F$ is of the same dimension d as P or has no faces.

Remark 2.2. The restriction that $F \cap ((A \text{ del } F) \cup B)$ is a shellable subcomplex of ∂F of dimension $d - 1$ is not very restrictive in our case, as we consider only simplicial and cubical complexes.

Proposition 2.3. *If S is a simplex of dimension $d + 1$, then any pure¹, d -dimensional subcomplex of ∂S is shellable.*

If C is a cube of dimension $d + 1$, then any $d - 1$ -acyclic, pure, d -dimensional subcomplex D of ∂S is shellable.

Proof. The first part is a simple exercise. The second part, consider a facet F of D . If the antipodal facet of the cube ∂C is in D , remove it first. Then, remove the facets adjacent to F in any order, until finally only F is left. It is easy to see that this is a shelling order. \square

A relative complex (A, B) is **shellable** if iterated shelling steps reduce it to (B, B) . One also says A **shells relative to** B . A complex A is **shellable** if (A, \emptyset) is shellable². With this, we have:

Lemma 2.4. *If a triangulation/cubulation of $(M \times [0, 1], M \times \{0\})$ is shellable then $M \times \{0\}$ and $M \times \{1\}$ are related through Pachner moves.*

Proof. Every shelling step exactly corresponds to a Pachner move: Consider the change in $M \times \{1\}$ in such a step. Observe then that the part of the removed polytope (cube or simplex) in the shelling step that intersects $M \times \{1\}$ is replaced by the remaining facets of the boundary, as desired. \square

Let us note that the converse holds as well, but shall not be needed here: If two cubulations or triangulations are related by (cubical) Pachner moves, then they are related by a shellable cobordism. (That is, there is a triangulation/cubulation of $M \times [0, 1]$ where $M \times \{0\}$ and $M \times \{1\}$ are the given triangulations/cubulation and $(M \times [0, 1], M \times \{0\})$ is shellable.) However, in this, one has to be a bit careful, as a single shelling step, seen as cobordism, is not a triangulation of $M \times [0, 1]$, but often a smaller complex, that is, a polyhedral decomposition of $M \times I_t$, where I_t is a compact interval depending on $t \in M$ that can degenerate to a point. One can extend the notion of cobordisms, or one has to fatten the complex suitably.

Concerning regular subdivisions of convex balls, that is, regions of linearity of a piecewise linear convex function, the following is useful to keep in mind:

Proposition 2.5 ([BM71]). *A regular subdivision of a convex ball D in \mathbb{R}^d is shellable. If, moreover, v is any general position point outside of D , and M is the part of the boundary ∂D of D illuminated by v , that is, those complex generated by those facets that can be connected to v with a line segment that does not intersect D in the interior, then (D, M) is shellable.*

2.3. Cubical stellar and cubical derived subdivisions. We next describe the cubical analogue of stellar subdivisions and derived subdivisions. Recall that a stellar subdivision of a polyhedral complex S picks a face τ of S and considers the complex $S - \tau$. It then glues

¹A polyhedral complex is **pure** if all facets are of the same dimension.

²It is important in this context to distinguish between two polyhedral complexes: \emptyset and $\{\emptyset\}$. The first has no faces, the second has one. The empty face.

$\text{Cone}(\text{St}_S \tau - \tau)$ to $S - \tau$ along $\text{St}_S \tau - \tau$. In geometric situations the cone point is often placed inside τ . Here is the cubical variant.

Definition 2.6 (Cubical stellar subdivision). Let C be a polyhedral complex in \mathbb{R}^d , and let τ be any face of C . Let x_τ denote a point anywhere in the relative interior of τ , and let λ be any number in the interval $(0, 1)$. Define

$$\begin{aligned} \text{c-st}(\tau, C) &:= (C - \tau) \cup \\ &\quad \{\text{conv}(\sigma \cup (\lambda\sigma + (1 - \lambda)x_\tau)) : \sigma \in \text{St}_\tau C - \tau\} \cup \\ &\quad (\lambda\text{St}_\tau C + (1 - \lambda)x_\tau). \end{aligned}$$

The complex $\text{c-st}(\tau, C)$ is the **cubical stellar subdivision**, or **c-stellar subdivision**, of C at τ .

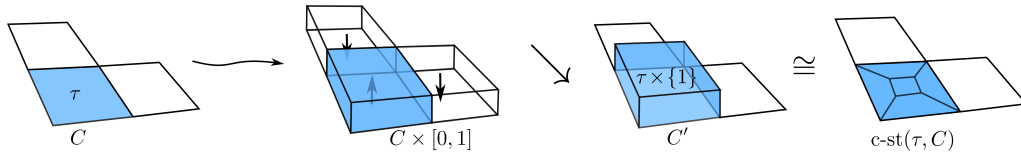


Figure 2.1. A cubical stellar subdivision of a complex C , seen as a subcomplex of the product of C with an interval.

A useful way to think about a cubical stellar subdivision is to think of it as a "truncated" stellar subdivision, that is, the removal of τ from C and coning over the boundary of the hole left over, followed by "cutting off" the cone point. In particular, if the link of τ is regular, then we can indeed achieve this visually by introducing new linear constraints at the apex vertex of the cone.

Finally, we look at cubical derived subdivisions. In the simplicial case, these derived subdivisions arise by performing stellar subdivisions at the faces in order of decreasing dimension. The analogue here is:

Definition 2.7 (Cubical derived subdivision). Let C denote any polyhedral d -complex, with faces ordered by decreasing dimension. A **cubical derived subdivision**, or **c-derived subdivision**, is any subdivision C obtained by first c-stellar subdividing all d -faces of C , then c-stellar subdividing the resulting complex at all the original $(d - 1)$ -faces of C , then all original $(d - 2)$ -faces, and so on up to the faces of dimension 0, that is, the vertices. We denote this by $\text{c-dr}(C)$.

This, of course, is in contrast to the usual (simplicial) derived subdivision, which we denote by $\text{dr}(C)$, and which can be similarly obtained by performing (simplicial, or usual) stellar subdivisions on faces in reverse order of inclusion. It is useful to note that the derived subdivision of a polyhedral complex C is isomorphic to the order complex of the proper part of the face poset of C (where the proper part is the restriction of the poset to all nonempty faces). In particular, every vertex of the derived subdivision $\text{dr}(C)$ corresponds to a nonempty face of C .

(Cubical) derived subdivisions can be iterated, that is, take the (cubical) derived subdivision to the (cubical) derived subdivision of a complex etc. We denote this by an exponent: the i -th (cubical) iterated derived subdivision is denoted by $\text{dr}^i(C)$ resp. $\text{c-dr}^i(C)$.

Again, in the way they were introduced, it is natural to think of stellar subdivisions as cobordisms, obtained by attaching $\text{St}_\tau C \times [0, 1]$ to the complex in question. (Cubical) stellar subdivisions have several nice properties; for example, they preserve shellability.

Lemma 2.8. *Consider a relative complex (X, Y) that is shellable. If (X', Y') is a (cubical) stellar subdivision of this pair, then it is shellable as well.*

Here, the stellar subdivision of a relative complex (X, Y) is defined as the stellar subdivision of both its elements at a face, with the convention that the stellar subdivision of a face not in the complex leaves the complex invariant.

Proof. Restricted to any facet of X , the stellar subdivision is regular, and hence shellability follows from Proposition 2.5. \square

Let us note the following corollary of the main theorem.

Corollary 2.9. *Every cubical stellar subdivision of a PL cubical manifold X can be achieved by cubical Pachner moves.*

Proof. By attaching $\text{St}_F X \times [0, 1]$ to X with the identification $\text{St}_F X \times \{0\} \sim \text{St}_F X$, we obtain a cubical cobordism of X to the cubical stellar subdivision of X at F . This gives the desired result by the main theorem. \square

Let us observe a weaker version, which we can prove immediately without using the main theorem.

Proposition 2.10. *Every cubical stellar subdivision of a PL cubical manifold X can be achieved by cubical Pachner moves provided the link of the subdivided face is shellable.*

Proof. $\text{St}_F X \times [0, 1]$ is shellable relative to $\text{St}_F X \times \{0\}$ by assumption. \square

Note that the converse does not hold: not every cubical Pachner move is obtainable by cubical stellar moves. Specifically, note that cubical stellar moves are topologically “boring” with respect to the associated normal hypersurfaces; they only introduce null-homotopic spheres as hypersurfaces, or remove them. But general cubical Pachner moves can change the topology or intersection patterns of the immersed surfaces. For example, a loop in S^2 with two self-intersections is normal crossing equivalent to the loop with no self intersection, but this cannot be obtained by stellar moves only.

This is contrary to the case of simplicial complexes, where simplicial Pachner moves and stellar subdivisions and inverses are of the same strength, at least for PL manifolds: two PL triangulations of the same manifold are related by Pachner moves if and only if they are related by stellar moves. Beyond PL manifolds, things become more complicated, and stellar subdivisions and inverses can relate triangulations that are unrelated by Pachner moves.

2.4. A non-useful subdivision. There is another kind of subdivision in cubical manifolds. While simple, it is, from the view of cobordisms of normal crossing hypersurfaces, boring. But we record it here for curiosity’s sake. Specifically, consider a cubulation X of a d -manifold M . The normal crossing hypersurface of X divides every d -cube of M into 2^d

smaller d -cubes. It divides every edge into two components, each of which can be canceled out against each other in a cobordism. We obtain:

Proposition 2.11. *The hypersurface of the above subdivision is cobordant to the trivial hypersurface; the empty one.*

2.5. *The cubical cone.* Given a simplicial complex Δ , there is a canonical simplicial complex $\text{Cone}_v \Delta$, the cone over Δ with apex v . It is the unique simplicial complex such that $\text{Lk}_v \text{Cone}_v \Delta = \Delta$ and $\text{St}_v \text{Cone}_v \Delta = \text{Cone}_v \Delta$. It is also called the **cone** over Δ .

It is similarly easy to see that there is a unique cubical complex $\text{cCone}_v \Delta$ with the property $\text{Lk}_v \text{cCone}_v \Delta = \Delta$ and $\text{St}_v \text{cCone}_v \Delta = \text{cCone}_v \Delta$. We call it the **cubical cone**. Note that for a simplex D of dimension d , $\text{Cone}_v \Delta$ is the simplex of dimension $d + 1$ and $\text{cCone}_v D$ is the cube of dimension $d + 1$. We suppress v from the notation whenever it is not necessary; it is used more for the definition than being material.

3. FROM CUBICAL-STELLAR TO STELLAR AND BACK: THE CONFINEMENT MAP COMPARING SIMPLICIAL AND CUBICAL STRUCTURES

Let us compare cubical stellar and stellar subdivision for the briefest moment: Recall, the cubical stellar subdivision of a complex C at a face τ is

$$\begin{aligned} \text{c-st}(\tau, C) &:= (C - \tau) \cup \\ &\quad \{\text{conv}(\sigma \cup (\lambda\sigma + (1 - \lambda)x_\tau)) : \sigma \in \text{St}_\tau C - \tau\} \cup \\ &\quad (\lambda \text{St}_\tau C + (1 - \lambda)x_\tau). \end{aligned}$$

Compared to that, the stellar subdivision is

$$\begin{aligned} \text{st}(\tau, C) &:= (C - \tau) \cup \\ &\quad \{\text{conv}(\sigma \cup x_\tau) : \sigma \in \text{St}_\tau C - \tau\} \end{aligned}$$

In other words, the stellar subdivision is the limit of the cubical stellar subdivision when $\lambda \rightarrow 0$.

In particular, using this contraction we obtain a polyhedral map, called **confinement map**,

$$\varphi_\tau : \text{c-st}(\tau, C) \longrightarrow \text{st}(\tau, C)$$

that is, a map that sends polyhedra to polyhedra and in particular induces a map of posets. Given a polyhedral complex S , let $S^{>0}$ denote the filter generated by all faces of positive dimension. The following is immediately clear:

Lemma 3.1. *The confinement map is injective on $\text{st}(\tau, C)^{>0}$, that is, on the order filter $\text{st}(\tau, C)^{>0}$ and its preimage, the map is one to one as a map of posets.*

Moreover, the preimage of a vertex is a vertex, unless the vertex is x_τ , in which case it is $\text{St}_\tau C$.

In other words, the source of non-injectiveness is the 0-skeleton of $\text{st}(\tau, C)$, that is, the vertices.

One can iterate cubical stellar subdivisions, of course, by applying one after another. And to a cubical stellar subdivision we have an associated simplicial stellar subdivision, and an associated confinement map. Let us analyze how this looks.

Lemma 3.2. *Consider a cubical complex C , and the derived subdivision $C' = \text{dr } C$, as well as the derived cubical subdivision $\Gamma = \text{c-dr } C$ of C . Consider the confinement map*

$$\varphi : \Gamma \longrightarrow C'.$$

Then, for a face τ of C' , which we recall, corresponds to a chain $(t_1 \subsetneq t_2 \subsetneq \cdots \subsetneq t_s)$ of faces in C , we have that

$$\varphi^{-1}(\tau) \cong t_1 \times [0, 1]^{\{t_2\}} \cdots \times [0, 1]^{\{t_{s-1}\}} \times \text{cCone Lk}_{t_s} C.$$

Here, the notation $[0, 1]^{\{t_i\}}$ is used to make the inclusion to faces clear.

Hence, we can also describe the boundary: If $\tau' = (t_2 \subsetneq \cdots \subsetneq t_s)$, then

$$\varphi^{-1}(\tau) \cap \varphi^{-1}(\tau') = (t_1 \subsetneq t_2) \times [0, 1]^{\{t_3\}} \cdots \times [0, 1]^{\{t_{s-1}\}} \times \text{cCone Lk}_{t_s} C$$

and if $\tau' = (t_1 \subsetneq \cdots \subsetneq t_{i-1} \subsetneq t_{i+1} \subsetneq \cdots \subsetneq t_s)$, then

$$\varphi^{-1}(\tau) \cap \varphi^{-1}(\tau') = t_1 \times [0, 1]^{\{t_2\}} \times \cdots \times [0, 1]^{\{t_{i-1}\}} \times [0, 1]^{\{t_{i+1}\}} \times \cdots \times [0, 1]^{\{t_{s-1}\}} \times \text{cCone Lk}_{t_s} C$$

and finally $\tau' = (t_1 \subsetneq \cdots \subsetneq t_{s-1})$, then

$$\varphi^{-1}(\tau) \cap \varphi^{-1}(\tau') = t_1 \times [0, 1]^{\{t_2\}} \cdots \times [0, 1]^{\{t_{s-2}\}} \times (\text{cCone Lk}_{t_s} C \subset \text{cCone Lk}_{t_{s-1}} C)$$

Notice that $\varphi^{-1}(\tau)$ looks *almost* like the star of t_s in C . Both the star and $\varphi^{-1}(\tau)$ are products of $\text{cCone Lk}_{t_s} C$ with a high-dimensional cube. In particular, they are again stars of faces in cubical complexes. We call those complexes **cubical stars**.

Corollary 3.3. *Consider a cubical complex C , and the i -th iterated derived subdivision $C' = \text{dr}^i C$, as well as the derived cubical subdivision $\Gamma = \text{c-dr}^i C$ of C . Consider the confinement map*

$$\varphi : \Gamma \longrightarrow C'.$$

Then, for a face τ of C' , $\varphi^{-1}(\tau)$ is an iterated subdivision of a cubical star. More refinedly, it is of the form

$$\text{c-dr}^j([0, 1]^n \times \text{cCone}_v \text{Lk}_x \text{c-dr}^k C)$$

for suitable j , n , x , and k .

4. COLLAPSES AND MAGNARD-SKOTNICA-TANCER-WHITEHEAD NEIGHBORHOOD THEOREM

Before we start, let us observe a basic proposition, going back to Whitehead [Whi38], but greatly refined by Magnard, Skotnica and Tancer [MST19]. Recall: In a polyhedral complex, an elementary collapse is the removal of a **free** face, that is, a face A contained strictly in only one other face B . Both the free face and the face B are removed together, yielding a smaller cell complex with two faces less. Let us call the face B **cofree**.

A collapse is the combination of elementary collapses. A collapsible (relative) complex (X, Y) is a complex whose faces can all be removed in a sequence of elementary collapses (or in other words, X collapses to Y).

We also introduce the (HRC)-property, following [MST19]: Given a simplicial complex, we say it has the (RC)-property if, after the removal of some facets, it becomes collapsible. The (HRC)-property is then given if every link has the (RC)-property. It is a consequence of the well known fact that shellable contractible complexes are collapsible (see [Don02]) that we have:

Lemma 4.1. *Any shellable complex has the (HRC)-property.*

We recall the theorem of Magnard-Skotnica-Tancer-Whitehead.

Proposition 4.2 (See [MST19]). *Consider a collapsible simplicial (resp. cubical) complex (X, Y) that is pure of dimension d and that has the (HRC)-property.*

Then the second simplicial (resp. cubical) derived subdivision X'' of X shells to the second derived subdivision of Y'' , or in other words, (X'', Y'') is shellable.

Proof. The simplicial case is done in [MST19], and we sketch it here because it makes the transition to the cubical case easier: Define the **neighborhood** $N_A X$ of a subcomplex A in a polyhedral complex X to be the closure of all faces of X intersecting A .

Consider now the vertices of the first derived subdivision; as the original complex admits a collapse, the vertices come in pairs corresponding to those collapsing pairs.

Now, we consider the neighborhoods $(N_{\gamma_i})_{i=1, \dots, m}$ of these pairs $\gamma_i = (A_i, B_i)$ in the second derived subdivision X'' of X . Here, A_i is always the free face, and B_i is the cofree face. Consider in addition the neighborhood N_Y of Y'' .

The basic observation is that the neighborhood N_{γ_i} in X'' can be shelled relative to the complexes consisting of the remaining neighborhoods union N_Y , that is, N_{γ_i} is shellable relative to

$$N_{\gamma_i} \cap \left(N_Y \cup \bigcup_{i' > i} N_{\gamma_{i'}} \right).$$

For this, we remove stars of vertices one by one: First we remove $N_{A_i} = \text{St}_{A_i} X''$ (recall that A_i corresponds to a face in X , and hence to a vertex in X' and X'') and then remove $N_{B_i} = \text{St}_{B_i} X''$. It is easy to see that the (HRC)-property guarantees this can be achieved by shelling steps.

Therefore, the order of collapses gives a shelling of X'' to the neighborhood N_Y .

Finally, extend the partial order by inclusion on Y to a total order δ , $\delta = 1, \dots, k$, and decompose N_Y into the neighborhoods N_δ of the corresponding vertices in Y'' . Since $\text{St}_\delta X$ is of dimension d and shellable relative to $\text{St}_\delta Y$, we have that, N_δ is shellable relative to

$$N_\delta \cap \left(Y'' \cup \bigcup_{\delta' > \delta} N_{\delta'} \right).$$

This gives the shelling in the simplicial case.

For the cubical case, we note that the image under the confinement map has the desired shellability property, as it is simply the second simplicial derived subdivision. It is easy to see that the fibers of the confinement map at faces individually are shellable, as they are cubical derived subdivisions of shellable stars, and derived subdivisions preserve shellability by Lemma 2.8. It follows from the explicit description of fibers of Corollary 3.3 and the intersection of fibers Lemma 3.2 that shellability is preserved when passing from simplicial derived to cubical derived, because the fibers are shellable relative to the intersection with the remaining complex. Hence the second cubical derived subdivision has the desired shelling property as well. \square

We close this section with another simple fact:

Lemma 4.3. *A star $\text{St}_\tau C$ collapses to any of its faces.*

We need an auxiliary lemma.

Lemma 4.4. *Consider a polytope P , and F a face of its boundary. Then P collapses to $\text{St}_F \partial P$.*

Proof. Proposition 2.5 gives this in the context of shellings, but it is an easy and classical fact that a shelling induces a collapse [Don02]. \square

Proof of Lemma 4.3. We prove this statement by induction on the size of $\text{St}_\tau C$. Consider the minimal face T of $\text{St}_\tau C$ containing both τ and the face q we wish to collapse to.

Consider now a maximal face F of $\text{St}_\tau C$, and assume F is not T . Following Lemma 4.4, F collapses to $\text{St}_\tau \partial F$ and hence $\text{St}_\tau C$ collapses down to $(\text{St}_\tau C \setminus F) \cup \text{St}_\tau \partial F$, the complex obtained by removing F and all its faces from $\text{St}_\tau C$, and adding $\text{St}_\tau \partial F$.

Since

$$\text{St}_\tau ((\text{St}_\tau C \setminus F) \cup \text{St}_\tau \partial F) = (\text{St}_\tau C \setminus F) \cup \text{St}_\tau \partial F,$$

we can iterate this process until we collapsed $\text{St}_\tau C$ to T . Now, we use Lemma 4.4 again and collapse the latter to q . \square

We also require an obvious lemma on preservation of collapses:

Lemma 4.5. *If A collapses to a subcomplex B then $A \times [0, 1]$ collapses to $B \times [0, 1]$.*

We note the following consequence of Lemma 3.2:

Lemma 4.6. *Consider a cubical complex C , and the derived subdivision $C' = \text{dr } C$, as well as the derived cubical subdivision $\Gamma = \text{c-dr } C$ of C . Consider the confinement map*

$$\varphi : \Gamma \longrightarrow C'.$$

Then, for a face τ of C' , and a facet τ' of τ , we have that $\varphi^{-1}(\tau')$ collapses to $\varphi^{-1}(\tau') \cap \varphi^{-1}(\tau)$. Moreover, $\varphi^{-1}(\tau)$ collapses to $\varphi^{-1}((\partial\tau) \text{ del } \tau')$.

Proof. The first is a consequence of Lemma 4.3. The second is Lemma 4.5. \square

Finally, let us remind ourselves that similar to the preservations of shellings, we have:

Lemma 4.7. *(Cubical) stellar subdivisions preserve collapsibility.*

For the simplicial case, this is [Wel99]. The cubical case is proven analogously. See also Lemma 2.8.

5. CUBICAL PACHNER THEOREM

We now prove the main theorem. The directions (1) \implies (2) and (3) \implies (1) are shown in [Fun99]; it remains to prove (2) \implies (3). Let M be a manifold with two PL cubulations X_0 and X_1 , and let Y be a PL cubulation of $M \times [0, 1]$ such that $Y \cap (M \times \{0\}) \cong X_0$ and $Y \cap (M \times \{1\}) \cong X_1$.

We have the following lemma, following from the main result of [AB17].

Lemma 5.1. *Let Y be a PL cubical manifold with boundary. After some iterated cubical stellar subdivisions, the links of proper faces have the (HRC)-property. Moreover, the links also have the (HRC)-property for the relative complex $(Y, \partial Y)$. In fact, we can perform such subdivisions until all links are shellable.*

Proof. This is really a simplicial lemma, as links of faces in cubical complexes are simplicial complexes, and cubical stellar subdivisions on the cubical complex result in simplicial stellar subdivisions in the links. Now, instead of proving the (HRC)-property, it suffices to prove that the links of proper faces become shellable after some iterated cubical stellar subdivisions by Lemma 4.1.

We begin by considering the maximal face τ whose link is *not* shellable. We then perform stellar subdivisions at those faces containing it, starting with maximal faces and then going down in the poset. If σ is a face introduced in this procedure (meaning, a face of the subdivision that is not a face in the original complex; recall that a (cubical) stellar subdivision at a face F only affects faces containing F), then the link is shellable by induction (by Lemma 2.8), so it remains to see what happens in the link of τ . In the link of τ , the induced subdivision is a derived subdivision.

Hence, it remains to argue that PL balls and spheres become shellable after sufficiently many derived subdivisions. That is [AB17, Theorem B]. \square

Next, we observe that the cobordism may be chosen so that $M \times [0, 1]$ collapses to $M \times \{0\}$.

Lemma 5.2. *After some iterated cubical stellar subdivision, Y collapses to $Y \cap (M \times \{0\})$.*

Proof. Following [AB17, Theorem C], there exists a simplicial stellar subdivision Y' of Y so that Y' collapses to $Y' \cap (M \times \{0\})$. Indeed, there exists an iterated derived subdivision with that property.

Now, to every ordinary stellar subdivision, we have an associated cubical stellar subdivision.

Recall that this iterated *cubical* stellar subdivision Y'' of the faces of $M \times [0, 1]$ admits a polyhedral map to Y' , that is, a map that sends polyhedra to polyhedra: This is the confinement map of Section 3. We claim passing to the preimage preserves collapsibility:

Now, we use Lemma 4.6 and Lemma 4.7: We have that $\varphi^{-1}(A)$ collapses to $\varphi^{-1}(A \cup B)$ and that $\varphi^{-1}(B)$ collapses to $\varphi^{-1}((\partial B) \text{ del } A)$. Hence, $\varphi^{-1}(\{A, B\})$ collapses to $\varphi^{-1}((\partial B) \text{ del } A)$. Hence, every collapse of Y' induces analogous collapses in Y'' . In particular, Y'' collapses to $Y'' \cap (M \times \{0\})$. \square

Proof of the implication (2) \implies (3) in Theorem 1.1. We may assume by the proof of Lemma 5.1 and Proposition 2.10 that the links of faces in X_0 and X_1 are shellable: Consider a maximal face τ whose link is *not* shellable. Perform stellar subdivisions at faces strictly containing τ until the link is shellable; the faces created in this subdivision clearly have shellable links, as in the proof of Lemma 5.1. Moreover, by Proposition 2.10, these moves are realized by Pachner moves.

Now, we may perform cubical stellar subdivisions on Y until the resulting complex Y'' collapses to the induced subdivision $X_0'' = Y'' \cap (M \times \{0\})$ of X_0 . We may further assume that the links of Y'' are shellable by Lemma 5.1.

Hence, using Proposition 4.2, after sufficiently many cubical derived subdivisions, we have a cubulation Y''' of $M \times [0, 1]$ which shells to $Y''' \cap (M \times \{0\}) = X_0'''$. Hence, $X_1''' = Y''' \cap (M \times \{1\})$ and $X_0''' = Y''' \cap (M \times \{0\})$ are related by cubical Pachner moves. But X_1''' and X_1 are related by cubical Pachner moves by Proposition 2.10, and so are X_0''' and X_0 . Hence Pachner moves transform X_1 into X_1''' , then X_1''' into X_0''' and finally X_0''' into X_0 . \square

REFERENCES

- [AB17] Karim A. Adiprasito and Bruno Benedetti, *Subdivisions, Shellability, and collapsibility of products*, *Combinatorica* **37** (2017), no. 1, 1–30 (English).
- [BM71] Heinz Bruggesser and Peter Mani, *Shellable decompositions of cells and spheres*, *Math. Scand.* **29** (1971), 197–205 (1972).
- [Don02] Xun Dong, *Alexander duality for projections of polytopes*, *Topology* **41** (2002), no. 6, 1109–1121 (English).
- [Fun99] Louis Funar, *Cubulations, immersions, mappability and a problem of Habegger.*, *Ann. Sci. Éc. Norm. Supér. (4)* **32** (1999), no. 5, 681–700.
- [Fun08] ———, *Surface cubications mod flips.*, *Manuscr. Math.* **125** (2008), no. 3, 285–307.
- [FG02] Louis Funar and Rosa Gini, *The graded cobordism group of codimension-one immersions.*, *Geom. Funct. Anal.* **12** (2002), no. 6, 1235–1264.
- [Gro87] Mikhail Gromov, *Hyperbolic groups*, *Essays in group theory*, *Math. Sci. Res. Inst. Publ.*, vol. 8, Springer, New York, 1987, pp. 75–263.
- [MST19] Thomas Magnard, Michael Skotnica, and Martin Tancer, *Shellings and sheddings induced by collapses*, preprint, available at arXiv:1909.13850.
- [SZ04] Alexander Schwartz and Günter M. Ziegler, *Construction techniques for cubical complexes, odd cubical 4-polytopes, and prescribed dual manifolds.*, *Exp. Math.* **13** (2004), no. 4, 385–413.
- [Thu93] William Thurston, *Hexahedral decomposition of polyhedra*, posting on sci.math, ics.uci.edu/~epstein/gina/Thurston-hexahedra.html.
- [Wel99] Volkmar Welker, *Constructions preserving evasiveness and collapsibility*, *Discrete Math.* **207** (1999), 243–255.
- [Whi38] John Henry Constantine Whitehead, *Simplicial spaces, nuclei and m-groups*, *Proc. London Math. Soc.* **S2-45** (1938), 243.

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