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GAUSSIAN MEASURE ON THE DUAL OF $U(N)$, RANDOM PARTITIONS, AND TOPOLOGICAL EXPANSION OF THE PARTITION FUNCTION

THIBAUT LEMOINE AND MYLÈNE MAÏDA

ABSTRACT. We study a Gaussian measure $\mathcal{G}(N, q)$ with parameter $q \in (0, 1)$ on the dual of the unitary group of size N : we prove that a random highest weight under $\mathcal{G}(N, q)$ is the coupling of two independent q -uniform random partitions α, β and a random integer $n \sim \mathcal{G}(1, q)$. We prove deviation inequalities for the q -uniform measure, and use them to show that the coupling of random partitions under $\mathcal{G}(N, q)$ vanishes in the limit $N \rightarrow \infty$. We also prove that the partition function of this measure admits an asymptotic expansion in powers of $1/N$, and that this expansion is topological, in the sense that its coefficients are related to the enumeration of ramified coverings of elliptic curves. It provides a rigorous proof of the gauge/string duality for the Yang-Mills theory on a 2D torus with gauge group $U(N)$, advocated by Gross and Taylor [[GT93b](#), [GT93a](#)].

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1. INTRODUCTION

There is a longstanding study of random discrete models involving algebraic or combinatorial structures with growing size, in particular distributions on Young diagrams. Multiple motivations hide behind these distributions: understanding the eigenvalue distribution of random matrices [BOO00], estimating matrix integrals in quantum field theory [NO06, Oko06], describing the representations of infinite unitary or symmetric groups [BO05, BK08], or even statistical physics [Zyg22].

In this paper, we are mainly interested in measures on the set $\widehat{\mathrm{U}(N)}$ of highest weights of the unitary group, which label the irreducible representations of $\mathrm{U}(N)$ and correspond to nonincreasing N -tuples of integers $\lambda = (\lambda_1, \dots, \lambda_N)$. For any $q \in (0, 1)$, we denote by $\mathcal{G}(N, q)$ the probability measure on $\widehat{\mathrm{U}(N)}$ such that, for any subset Λ of $\widehat{\mathrm{U}(N)}$,

$$\mathbb{P}(\lambda \in \Lambda) \propto \sum_{\mu \in \Lambda} q^{c_2(\mu)},$$

where $c_2(\lambda)$ is the *Casimir number* of the representation λ , given by

$$c_2(\lambda) = \frac{1}{N} \sum_{i=1}^N \lambda_i(\lambda_i + N + 1 - 2i),$$

which is (up to a sign) an eigenvalue of the Laplace operator on $\mathrm{U}(N)$.

Writing $q = e^{-\beta}$ for some inverse temperature $\beta > 0$ makes explicit the fact that $\mathcal{G}(N, q)$ is a Gibbs measure of the form

$$\mathbb{P}(\lambda \in \Lambda) \propto \sum_{\mu \in \Lambda} e^{-\beta H(\mu)},$$

where the Hamiltonian $H : \widehat{\mathrm{U}(N)} \rightarrow \mathbb{R}$ is the function that assigns to any highest weight its Casimir number. As c_2 is a quadratic form in the coordinates of λ , it is even a (discrete) Gaussian measure. Gibbs measures of the same kind also appear in the character expansion of some unitary matrix models [GM05].

The main quantity that we will investigate is the *partition function* of the model, which is given by

$$Z_N(q) = \sum_{\lambda \in \widehat{\mathrm{U}(N)}} q^{c_2(\lambda)}.$$

In the case of $\widehat{U(1)} = \mathbb{Z}$, the partition function is nothing but the Jacobi theta function

$$\theta(q) = \sum_{n \in \mathbb{Z}} q^{n^2},$$

which is analytical on the unit disk $\{q \in \mathbb{C} : |q| < 1\}$. We will see that the measure $\mathcal{G}(1, q)$ will play a particular role in the study of the general measure $\mathcal{G}(N, q)$, corresponding to some coupling of two random partitions.

A natural question, addressed in many earlier works for measures on integer partitions – see [LS77, VK77, Ver96, BOO00, Oko01, Rom15, CLW23] and references therein – is to understand the limiting distribution of the model. We want to treat a similar question, but with highest weights instead of partitions.

1.1. Coupling of random partitions. The highest weights of $U(N)$ can be related to triples (α, β, n) where α, β are integer partitions and n is an integer. We denote by \mathfrak{P} the set of all integer partitions and for any $\alpha = (\alpha_1 \geq \dots \geq \alpha_r > 0) \in \mathfrak{P}$, we denote by $\ell(\alpha) = r$ its number of parts. As pointed out in the literature [Lit44b, Lit44a, Sta84, Koi89], given two partitions α and β and an integer n , for any $N \geq \ell(\alpha) + \ell(\beta)$, one can build a highest weight $\lambda_N(\alpha, \beta, n)$ of $U(N)$ according to the following scheme:

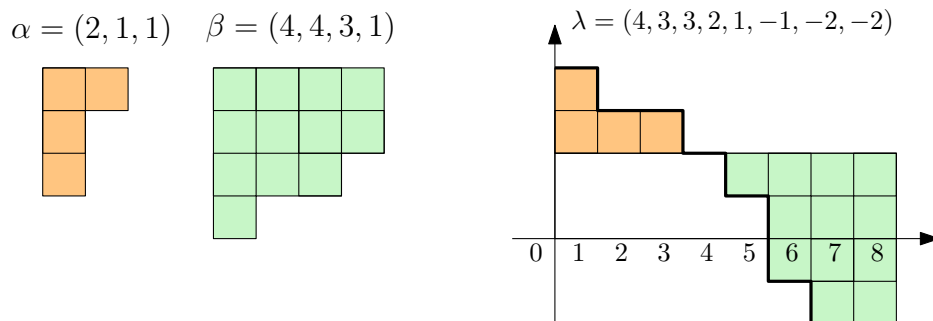


FIGURE 1. Construction of the highest weight $\lambda_N(\alpha, \beta, n)$ for $N = 8$, $\alpha = (2, 1, 1)$, $\beta = (4, 4, 3, 1)$ and $n = 2$.

It turns out that, for any $N \geq 1$, if we add the restriction that the triple (α, β, n) should belong to

$$\Lambda_N = \{(\alpha, \beta, n) \in \mathfrak{P}^2 \times \mathbb{Z} : \ell(\alpha) \leq A_N, \ell(\beta) \leq B_N\},$$

with

$$A_N := \lfloor (N+1)/2 \rfloor - 1 \text{ and } B_N := N - \lfloor (N+1)/2 \rfloor, \quad (1)$$

then, there is a converse construction, namely a bijection $\Phi_N : \widehat{U(N)} \rightarrow \Lambda_N$. More details will be given in Section 3 (see in particular Proposition 3.3). We will express the pushforward $(\Phi_N)_* \mathcal{G}(N, q)$ in terms of $\mathcal{G}(1, q)$ and a q -deformation of the uniform measure on the set of partitions.

A random partition $\alpha \in \mathfrak{Y}$ is q -uniform if

$$\mathbb{P}(\alpha \in \Lambda) \propto \sum_{\mu \in \Lambda} q^{|\mu|}, \quad \forall \Lambda \subset \mathfrak{Y}, \quad (2)$$

and we denote by $\mathcal{U}(q)$ the q -uniform distribution on \mathfrak{Y} . We will show that for any $N \geq 1$ and any $q \in (0, 1)$, there exists a function $F_{N,q} : \Lambda_N \rightarrow \mathbb{R}$ such that, for all bounded measurable $f : \mathfrak{Y}^2 \times \mathbb{Z} \rightarrow \mathbb{R}$,

$$\mathbb{E}[f \circ \Phi_N(\lambda)] = \mathbb{E}[f(\alpha, \beta, n) F_{N,q}(\alpha, \beta, n)],$$

where $\lambda \sim \mathcal{G}(N, q)$, α, β are two independent random partitions with distribution $\mathcal{U}(q)$ and n is a random integer with distribution $\mathcal{G}(1, q)$, independent from α and β . The precise result is stated in Proposition 3.5. A remarkable fact is that the coupling between α, β , and n , quantified through $F_{N,q}$, vanishes asymptotically as $N \rightarrow \infty$:

Theorem 1.1. *For any $q \in (0, 1)$, we have the following convergence in distribution:*

$$(\Phi_N)_* \mathcal{G}(N, q) \xrightarrow[N \rightarrow \infty]{\mathcal{L}} \mathcal{U}(q)^{\otimes 2} \otimes \mathcal{G}(1, q). \quad (3)$$

In other terms, if α, β, n are such that $\lambda_N(\alpha, \beta, n)$ has distribution $\mathcal{G}(N, q)$, then as N tends to infinity they become independent, with $\alpha, \beta \sim \mathcal{U}(q)$ and $n \sim \mathcal{G}(1, q)$.

1.2. Asymptotic expansion of the partition function. Building on the formulation of highest weights as couplings of partitions, on their asymptotic independence and on deviation inequalities under $\mathcal{U}(q)$ that we establish in Section 2.2, we will provide an asymptotic expansion of $Z_N(q)$ in powers of $\frac{1}{N}$ up to any order. By symmetry, coefficients of odd powers exactly vanish, leading to an expansion in powers of $\frac{1}{N^2}$:

Theorem 1.2. *For any $q \in (0, 1)$, the partition function $Z_N(q)$ admits the following expansion for any $p \geq 1$:*

$$Z_N(q) = \sum_{k=0}^p \frac{a_{2k}(q)}{N^{2k}} + O(N^{-2(p+1)}). \quad (4)$$

All coefficients are explicit and will be provided in a more detailed way in Theorem 4.5. We will now make the link with two-dimensional Yang–Mills theory and show that, up to a simple correspondance of parameters, the partition function $Z_N(q)$ is the partition function of the latter on a torus.

1.3. Two-dimensional Yang–Mills theory. The two-dimensional Yang–Mills theory with gauge group $U(N)$ in the large N limit has been popularized by the breakthrough work from 't Hooft [tH74] to describe strong interaction in the large coupling regime. The probabilistic counterpart of this quantum field theory is the Yang–Mills holonomy process. It is a stochastic process (H_ℓ) indexed by a space of loops on a compact connected orientable surface $\Sigma_{g,t}$ and with values in a compact Lie group G . It has been constructed by Sengupta [Sen97] and Lévy [Lév00] based on the fact that its finite-dimensional marginals correspond to the discrete Yang–Mills measure on an area-preserving cellular embedding of a graph in $\Sigma_{g,t}$. A

recent overview of this process and its relationship with the quantum Yang–Mills theory considered by physicists can be found in the lecture notes by Lévy [Lév20]. In the case when $G = U(N)$ and if its Lie algebra is endowed with the inner product $\langle X, Y \rangle = N\text{Tr}(XY^*)$, using the character expansion of the heat kernel on $U(N)$, the partition function of the model can be expressed as

$$Z_N^{\text{YM}}(g, t) = \sum_{\lambda \in \widehat{U(N)}} e^{-\frac{t}{2}c_2(\lambda)} d_\lambda^{2-2g}, \quad (5)$$

where d_λ denotes the dimension of the irreducible representation of $U(N)$ with highest weight λ . In the case $g = 1$, we have

$$Z_N^{\text{YM}}(1, t) = Z_N(e^{-\frac{t}{2}}).$$

Therefore, the asymptotic expansion given by Theorem 1.2 can be immediately translated into an asymptotic expansion for the partition function $Z_N^{\text{YM}}(1, t)$ on a torus.

As a direct corollary of our asymptotic expansion, we recover the convergence, for any value of t of the Yang–Mills partition function on a torus to a finite value, which was proved by the first author through other means. Let us define the Euler function ϕ as follows: for any $q \in \mathbb{C}$ such that $|q| < 1$,

$$\phi(q) = \prod_{m=1}^{\infty} (1 - q^m). \quad (6)$$

Theorem 1.3 ([Lem22]). *For any $t > 0$, setting $q_t = e^{-\frac{t}{2}}$, we have*

$$\lim_{N \rightarrow \infty} Z_N^{\text{YM}}(1, t) = \frac{\theta(q_t)}{\phi(q_t)^2}. \quad (7)$$

In a forthcoming paper, we will also address the asymptotic expansion of the partition function in higher genus, and we expect that the tools developed for the partition function might also work for the general distribution of the Yang–Mills holonomy field, leading to an improvement of some recent results obtained by the first author with Dahlqvist [DL23, DL22].

1.4. On the notion of topological expansion. In theoretical physics, in particular in the so-called *matrix models*, there is a formal counterpart of the matrix integrals called the *formal matrix integrals*, consisting of the sums of power series expansions which are not required to converge [Eyn11]. These power series expansions are often called *topological expansions* when their coefficients are related to topological invariants, or to the enumeration of coverings or topological maps. The idea is that, whether the power series converges or not, computing these coefficients become more tractable because of this topological/geometric interpretation. As discussed by Eynard [Eyn11, Eyn16], there has been for a long time a confusion between asymptotic expansions of matrix integrals and topological expansions of formal matrix integrals, and many results from the theoretical physics literature, for instance the seminal papers [tH74, BIPZ78], are rigorous at the level

of *formal matrix integrals*, which is sufficient from a combinatorial perspective. The remaining question is whether these formal matrix integrals coincide with the convergent matrix integrals; it has been answered in a few special cases, see for instance [BI99, GZ02, GM05, GMS06, GN15]. Concerning the Yang–Mills partition function $Z_N^{\text{YM}}(g, t)$, it is known to admit a formal series expansion by major series of works in theoretical physics [GT93b, GT93a, BT94, Ram96, CMR97]. Our claim is that the asymptotic expansion in Theorem 1.2 matches the formal power series expansions due to Gross and Taylor [GT93b, GT93a]. It is not only a nontrivial result, but also contradicts the expectations from Gross and Taylor, who stated that the topological expansion that they exhibited diverges. The precise statement of this topological expansion is given in Theorem 6.3. In a slightly simplified setting, one can consider the so-called *chiral partition function* obtained by isolation of only one of the coupled partitions (α, β)

$$Z_N^{\text{chir}}(1, t) = \sum_{\substack{\alpha \in \mathfrak{Q} \\ \ell(\alpha) \leq N}} e^{-\frac{t}{2} c_2(\alpha)}.$$

Here, we are able to prove the following.

Theorem 1.4. *The chiral partition function of the Yang–Mills measure on a torus of area $t > 0$ admits the following asymptotic expansion, for any $p \in \mathbb{N}$:*

$$Z_N^{\text{chir}}(1, t) = \sum_{k=0}^p \frac{t^{2k}}{(2k)! N^{2k}} \mathcal{F}_{1, k+1}(e^{-\frac{t}{2}}) + O(N^{-2p-2}), \quad (8)$$

where $\mathcal{F}_{1, k+1}$ denotes the generating function of the number of (generically) ramified coverings of the torus with genus $k + 1$.

If we denote by ρ_t a properly defined uniform measure on the space \mathcal{R} of the ramified coverings of the torus of area t , Equation (8) can be turned into an expression in terms of *random ramified coverings*, as conjectured in [GT93b]:

$$Z_N^{\text{chir}}(1, t) = \int_{\mathcal{R}} e^{-\frac{t}{2} \deg(R)} N^{\chi(R)} \rho_t(dR), \quad (9)$$

where $\deg(R)$ and $\chi(R)$ are respectively the *degree* and *Euler characteristic* of the ramified covering R . The precise statement is detailed in Section 6. Such integrals appear in string theory, where the ramified coverings of the torus are seen as worldsheets, i.e. string embeddings, and the interplay between Yang–Mills theory and string theory is the *gauge/string duality* developed (at least at the level of formal power series) by Gross and Taylor. The only other proofs of such duality, for convergent matrix integrals, seem to have been found in the plane [Lév08, PPSY23].

1.5. Organisation of the paper. In Section 2, we provide a systematic study of the measure $\mathcal{U}(q)$ on \mathfrak{Q} , and we give deviation inequalities for this measure that will be used in the remainder of the paper. In Section 3, we introduce the measure $\mathcal{G}(N, q)$. After a quick description of $\mathcal{G}(1, q)$, we will provide the relationship between $\mathcal{U}(q)$, $\mathcal{G}(1, q)$ and $\mathcal{G}(N, q)$. A key feature is the construction of an explicit bijection between the set of highest weights

and the subset $\Lambda_N \subset \mathfrak{Y}^2 \times \mathbb{Z}$. In Section 4, we will prove here the asymptotic decoupling described in Theorem 1.1 and the asymptotic expansion of Theorem 1.2. In Section 6, we will show that in the case of the torus, the asymptotic expansion provided by Theorem 1.2 is in fact a topological expansion and we will also discuss the chiral case. Preliminary material for topological expansions will be gathered in Section 5, in particular we will define and study the Frobenius measures in relation with the enumeration of ramified coverings.

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2. q -UNIFORM MEASURE ON PARTITIONS

In this section we study in detail the measure $\mathcal{U}(q)$ and prove deviation inequalities needed for the asymptotic expansions.

2.1. Definitions and main properties. Before going into the study of the measure $\mathcal{U}(q)$, we recall some well-known facts about integer partitions. An *integer partition*¹ $\alpha \vdash n$ of size n is a finite family of integers $\alpha = (\alpha_1, \dots, \alpha_{\ell(\alpha)}) \in \mathbb{N}^{\ell(\alpha)}$ with $\alpha_1 \geq \dots \geq \alpha_{\ell(\alpha)} > 0$, $|\alpha| := \alpha_1 + \dots + \alpha_{\ell(\alpha)} = n$. The integers $\alpha_1, \dots, \alpha_{\ell(\alpha)}$ are called the *parts* of the partition and $\ell(\alpha)$ is called its *length*. We denote by \mathfrak{Y}_n the set of partitions of size n , and $\mathfrak{Y} = \bigsqcup_n \mathfrak{Y}_n$ the set of all integer partitions. A partition $\alpha \vdash n$ is usually represented by a *Young diagram*, which is a collection of n boxes in $\ell(\alpha)$ left-justified rows, where the i -th row contains α_i boxes. An example is displayed in Figure 2 below. In this representation, each box is labelled



FIGURE 2. The Young diagram of the partition $(5, 4, 1)$, in the English convention (on the left), and the French convention (on the right).

(i, j) , where $1 \leq i \leq \ell(\alpha)$ (resp. $1 \leq j \leq \alpha_i$) is the number of the row (resp. column) that contains the box. The *content* of the box $\square = (i, j)$ is defined by

$$c(\square) = j - i.$$

¹It is more usual to denote integer partitions by λ , but we rather keep the notations λ, μ for highest weights and use α, β for partitions, in order to avoid confusions, unless specific cases where a highest weight is also a partition.

The *total content* of a partition α is the sum of the contents of each boxes in the corresponding Young diagram:

$$K(\alpha) = \sum_{\square \in \alpha} c(\square). \quad (10)$$

It is well known that integer partitions of size n are in bijection with the irreducible representations of the symmetric group \mathfrak{S}_n (see e.g. [Sag01] for a nice introduction). Consequently, for each partition $\alpha \vdash n$, one can define the character of the irreducible representation of \mathfrak{S}_n as the trace of the corresponding representation, and we shall denote it by χ_α .

Let us now define our family of measures $\mathcal{U}(q)$.

Definition 2.1. Let $q \in (0, 1)$ be a real parameter. A random partition $\alpha \in \mathfrak{Y}$ is distributed according to the q -uniform measure $\mathcal{U}(q)$ if

$$\mathbb{P}(\alpha \in \Lambda) = \phi(q) \sum_{\beta \in \Lambda} q^{|\beta|}, \quad \forall \Lambda \subset \mathfrak{Y}, \quad (11)$$

where ϕ is the Euler function defined in (6).

The fact that $\mathcal{U}(q)$ is a probability measure comes from the classical equality

$$\sum_{\alpha \in \mathfrak{Y}} q^{|\alpha|} = \sum_{n \geq 0} p(n)q^n = \phi(q)^{-1}, \quad \forall q \in \mathbb{C}, \quad |q| < 1, \quad (12)$$

with $p(n)$ the number of partitions of the integer n .

One can also remark that $\mathcal{U}(q)$ is related to the uniform measure on partitions, when we condition the random partition to have a fixed size.

Proposition 2.2. Let $q \in (0, 1)$ be fixed, and $\alpha \in \mathfrak{Y}$ be a random partition with distribution $\mathcal{U}(q)$. The distribution of $|\alpha|$ is given by

$$\mathbb{P}(|\alpha| = n) = \phi(q)p(n)q^n, \quad \forall n \in \mathbb{N}. \quad (13)$$

Moreover, conditionally to $|\alpha| = n$, α is uniform on \mathfrak{Y}_n .

Proof. The expression of $\mathbb{P}(|\alpha| = n)$, for $n \in \mathbb{N}$, follows from the definition of $p(n)$. Then, we have for any $\beta \in \mathfrak{Y}$ and any $n \in \mathbb{N}$,

$$\mathbb{P}(\alpha = \beta \mid |\alpha| = n) = \frac{\mathbb{P}(\{\alpha = \beta\} \cap \{|\alpha| = n\})}{\mathbb{P}(|\alpha| = n)} = \frac{1}{p(n)},$$

which corresponds indeed to the uniform measure on \mathfrak{Y}_n . □

This property of the q -uniform measure described in Proposition 2.2 is reminiscent from the Poissonized Plancherel measure: recall that the distribution of α is the Plancherel measure on \mathfrak{Y}_n when

$$\mathbb{P}(\alpha = \beta) = \frac{\chi_\beta(1)^2}{n!}, \quad \forall \beta \vdash n,$$

and the distribution of α is the Poissonized Plancherel measure on \mathfrak{Y} of parameter t when

$$\mathbb{P}(\alpha = \beta) = e^{-t} t^{|\beta|} \frac{\chi_\beta(1)^2}{|\beta|!}.$$

In this case the distribution of $|\alpha|$ is the Poisson measure of parameter t and conditionally to $|\alpha| = n$, α follows the Plancherel distribution on \mathfrak{Y}_n . Another common point between the Poissonized Plancherel and q -uniform measures is that in both cases, the random point process $\{\alpha_i + 1/2 - i\}_{i \in \mathbb{Z}}$ is determinantal [BOO00, BO00]. Moreover, as observed by Borodin [Bor07], there is a deep relationship between the Poissonized Plancherel measure and $\mathcal{U}(q)$ in the sense that both measures are special limits of a more general family of measures introduced by Nekrasov and Okounkov in the study of Seiberg–Witten theory [NO06]. We will discuss in Section 5 another relationship between both measures, coming from enumeration of surface coverings, and which appears more natural to us.

2.2. Deviation inequalities. We require two estimates related to the generating function of $p(n)$. For any integers n, k we set

$$p_k(n) = \#\{\alpha \vdash n : \ell(\alpha) = k\},$$

as well as

$$p_{\leq k}(n) = p_1(n) + p_2(n) + \cdots + p_k(n).$$

It is clear that, for n fixed, the sequence $(p_{\leq k}(n))_{k \geq 1}$ is increasing, bounded, and is stationary equal to $p(n)$ for $k \geq n$. We start with the following lemma:

Lemma 2.3. *For any $M \geq 1$ and any $q \in \mathbb{C}$ such that $|q| < 1$, we have the following inequality:*

$$\sum_{n \in \mathbb{N}} (p(n) - p_{\leq M}(n)) q^n \leq 5.44 q^{M+1} \sum_{m=0}^{\infty} (1 + e^{\pi \sqrt{\frac{2m}{3}}}) e^{m \log q}. \quad (14)$$

Proof. For any $M \geq n$, $p_{\leq M}(n) = p(n)$, we have

$$\sum_{n \in \mathbb{N}} (p(n) - p_{\leq M}(n)) q^n = \sum_{n \geq M+1} (p(n) - p_{\leq M}(n)) q^n.$$

Then, we know from [Oru16, Corollary 1] that for any $1 \leq k \leq n-1$,

$$p_k(n) \leq \frac{5.44}{n-k} e^{\pi \sqrt{\frac{2(n-k)}{3}}}.$$

It follows that for any $n \geq M+1$,

$$\sum_{k=M+1}^{n-1} p_k(n) \leq \sum_{k=M+1}^{n-1} f(n-k),$$

where $f(x) = \frac{5.44}{x} e^{\pi \sqrt{\frac{2x}{3}}}$. By a change of variable $\ell = n-k$, it reads

$$\sum_{k=M+1}^{n-1} p_k(n) \leq \sum_{\ell=1}^{n-M-1} f(\ell).$$

However, as f is increasing on $[1, \infty)$, we can bound each term from above by $f(n-M-1)$. We finally obtain

$$\sum_{k=M+1}^{n-1} p_k(n) \leq (n-M-1) f(n-M-1).$$

As a conclusion,

$$p(n) - p_{\leq M}(n) = p_n(n) + \sum_{k=M+1}^{n-1} p_k(n) \leq 1 + (n - M - 1)f(n - M - 1).$$

We can put it back in the sum over n , yielding

$$\sum_{n \in \mathbb{N}} (p(n) - p_{\leq M}(n))q^n \leq \sum_{n \geq M+1} (1 + (n - M - 1)f(n - M - 1))q^n.$$

We can perform a change of variable $m = n - M - 1$:

$$\sum_{n \in \mathbb{N}} (p(n) - p_{\leq M}(n))q^n \leq \sum_{m \geq 0} (1 + mf(m))q^{m+M+1} = q^{M+1} \sum_{m \geq 0} (1 + mf(m))q^m.$$

□

Note that the sum in the previous lemma is finite for $|q| < 1$ by the integral test for convergence. In the following lemma, we make clear that it decays exponentially in terms of M .

Lemma 2.4. *For any $q \in \mathbb{C}$ such that $|q| < 1$, there exists a constant C_q such that, for any $M \geq 1$,*

$$\sum_{n=M+1}^{\infty} p(n)q^n \leq C_q \cdot e^{\frac{1}{2} \log q \cdot M}. \quad (15)$$

Proof. We use the standard upper bound

$$p(n) < e^{c\sqrt{n}}, \quad \forall n \geq 1,$$

with $c = \pi\sqrt{2/3}$ (see [Apo76, Theorem 14.5] for instance). It yields

$$\sum_{n=M+1}^{\infty} p(n)q^n \leq \sum_{n=M+1}^{\infty} e^{c\sqrt{n} + n \log q}.$$

By a direct change of index, we have

$$\sum_{n=M+1}^{\infty} p(n)q^n \leq \sum_{n=1}^{\infty} e^{c\sqrt{n+M} + (n+M) \log q}.$$

As $\sqrt{n+M} \leq \sqrt{n} + \sqrt{M}$, we have that

$$\sum_{n=M+1}^{\infty} p(n)q^n \leq e^{M \log q + c\sqrt{M}} \sum_{n \geq 1} e^{n \log q + c\sqrt{n}}$$

We denote by $S_q := \sum_{n \geq 1} e^{n \log q + c\sqrt{n}}$, the sum being finite when $|q| < 1$. We conclude by observing that there exists T_q such that for any $M \geq 1$,

$$e^{M \log q + c\sqrt{M}} \leq T_q e^{\frac{1}{2} M \log q},$$

and the lemma holds with $C_q = S_q T_q$. □

As a direct consequence, let us start with some simple deviation inequalities for partitions with distribution $\mathcal{U}(q)$:

Proposition 2.5. *Let $q \in (0, 1)$ be fixed, and $\alpha \sim \mathcal{U}(q)$ be a random partition. Then there exists a constant C_q such that*

$$\mathbb{P}(\ell(\alpha) > \lfloor (N-1)/2 \rfloor) \leq C_q e^{\frac{N}{2} \log q}$$

and

$$\mathbb{P}(|\alpha| > N) \leq C_q e^{\frac{N}{2} \log q}.$$

Consequently, for any $p \in \mathbb{N}$, there are constants $C_1(p, q), C_2(p, q) > 0$ such that

$$\mathbb{P}(\ell(\alpha) > \lfloor (N-1)/2 \rfloor) \leq C_1(p, q) N^{-p}, \quad (16)$$

and

$$\mathbb{P}(|\alpha| > N) \leq C_2(p, q) N^{-p}. \quad (17)$$

Proof. We start by writing the expression of the probability that $2\ell(\alpha) > N$.

$$\mathbb{P}(2\ell(\alpha) > N) = \phi(q) \sum_{\substack{\alpha \in \mathfrak{P} \\ 2\ell(\alpha) > N}} q^{|\alpha|} = \phi(q) \sum_{n \geq 0} q^n \sum_{\substack{\alpha \vdash n \\ 2\ell(\alpha) > N}} 1.$$

The sum over α is nothing else than $p(n) - p_{\leq \lfloor (N-1)/2 \rfloor}(n)$. We can apply Lemma 2.3 to get that

$$\mathbb{P}(2\ell(\alpha) > N) \leq \phi(q) 5.44 q^{\lfloor (N-1)/2 \rfloor + 1} \sum_{m \geq 0} (1 + e^{\pi \sqrt{\frac{2m}{3}}}) e^{m \log q}.$$

As the sum is finite and $\lfloor (N-1)/2 \rfloor + 1 \geq \frac{N}{2}$, we get the first inequality.

The second one is very similar :

$$\mathbb{P}(|\alpha| > N) \leq \phi(q) \sum_{n \geq N+1} q^n p(n)$$

and we proceed similarly, by using Lemma 2.4 instead of Lemma 2.3. \square

3. THE GAUSSIAN MEASURE ON $\widehat{U(N)}$

In this section, we will study the measure $\mathcal{G}(N, q)$ on two different aspects. First, we shall describe the specific case of $N = 1$ where all moments can be computed explicitly, then we will prove that in the general case a highest weight of law $\mathcal{G}(N, q)$ is a coupling of two random q -uniform partitions and a highest weight of $U(1)$ with law $\mathcal{G}(1, q)$. We will also prove Theorem 1.1 which can be interpreted as an asymptotic decoupling.

3.1. Definition of $\mathcal{G}(N, q)$. One of the most important results in the representation theory of compact Lie groups is the celebrated Peter–Weyl theorem, which states that the characters of irreducible representations of such groups form a Hilbert basis of the space of square integrable central functions on G . Moreover, if we denote by Δ_G the Laplace–Beltrami operator on G and by \widehat{G} the set of equivalence classes of irreducible representations of G (called *dual* of G), we have the following:

Theorem 3.1. *Let G be a compact Lie group. The set of irreducible characters $\{\chi_\lambda, \lambda \in \widehat{G}\}$ is a Hilbert basis of eigenfunctions of Δ_G , and there is a family of nonnegative numbers, $(c_2(\lambda), \lambda \in \widehat{G})$, called the Casimir numbers, such that*

$$\Delta_G \chi_\lambda = -c_2(\lambda) \chi_\lambda, \quad \forall \lambda \in \widehat{G}.$$

In the case of $G = \mathrm{U}(N)$, the irreducible representations can be labelled by tuples of integers $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N$ such that $\lambda_1 \geq \dots \geq \lambda_N$, called *highest weights*. If we equip the Lie algebra $\mathfrak{u}(N)$ by the metric given by the Killing form

$$\langle X, Y \rangle = N \mathrm{Tr}(XY^*), \quad \forall X, Y \in \mathfrak{u}(N),$$

then the Casimir numbers are given as follows:

$$c_2(\lambda) = \frac{1}{N} \sum_{i=1}^N \lambda_i (\lambda_i + N + 1 - 2i), \quad \forall \lambda \in \widehat{\mathrm{U}(N)}. \quad (18)$$

We will see that it is a natural scaling for asymptotic analysis.

As explained in the introduction, we are interested in the measure $\mathcal{G}(N, q)$ defined as follows: a random highest weight $\lambda \in \widehat{\mathrm{U}(N)}$ is distributed according to $\mathcal{G}(N, q)$ if

$$\mathbb{P}(\lambda \in \Lambda) = \frac{1}{Z_N(q)} \sum_{\mu \in \Lambda} q^{c_2(\mu)}, \quad \forall \Lambda \subset \widehat{\mathrm{U}(N)}. \quad (19)$$

If we set $q = e^{-\frac{t}{2}}$, we see that $\mathcal{G}(N, q_t)$ is actually a discrete Gaussian measure on $\widehat{\mathrm{U}(N)}$. Indeed, seeing the set of highest weights as a discrete subset of the Weyl chamber

$$\mathcal{C}_N = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_1 \geq \dots \geq x_N\},$$

the measure $\mathcal{G}(N, q)$ can be seen as a measure on \mathcal{C}_N with discrete support. If we denote by ρ the element of \mathcal{C}_N such that, for any $1 \leq i \leq N$, $\rho_i = i - \frac{N+1}{2}$, then

$$\mathbb{P}(\lambda \in \Lambda) \propto \sum_{\mu \in \Lambda} e^{-\frac{t}{2} \|\mu - \rho\|^2},$$

hence it is indeed a kind of *discrete Gaussian measure* in the Weyl chamber.

3.2. The case $N = 1$. The measure $\mathcal{G}(1, q)$, as explained in the next section, is involved in the study of the general measure $\mathcal{G}(N, q)$ by a coupling with random partitions. The highest weights of $\mathrm{U}(1)$ are just integers, and for any $n \in \mathbb{Z}$, its Casimir number is equal to n^2 . It means that $n \sim \mathcal{G}(1, q)$ if and only if

$$\mathbb{P}(n \in A) = \frac{1}{Z_1(q)} \sum_{m \in A} q^{m^2}, \quad \forall A \subset \mathbb{Z}. \quad (20)$$

The partition function is the Jacobi theta function

$$Z_1(q) = \theta(q) := \sum_{n \in \mathbb{Z}} q^{n^2},$$

which is analytic for $|q| < 1$. The measure $\mathcal{G}(1, q)$ has been used by Borodin [Bor07] as a way to modify the periodic Schur process in order to obtain a

determinantal point process. We will only consider the case where $q \in (0, 1)$ in order to keep positive measures. The moments and cumulants of this distribution have already been described by Romik [Rom20] in the case $q = e^{-\pi}$ and by Wakhare and Vignat [WV20] in the general case, yielding deep formulas in terms of elliptic integrals, and the reader interested in the interplay between probabilistic and number-theoretic aspects of the theta function can have an eye on [BPY01, SV23]. We shall provide a simple alternative expression of the moments obtained by Wakhare and Vignat. It involves the differential operators $D^n = (x \frac{d}{dx})^n$ for all $n \geq 0$, which are related to the *infinite wedge representation* of the Lie algebra \mathcal{D} of differential operators on the unit circle [BO00].

Proposition 3.2. *For any $q \in (0, 1)$ and $k \geq 0$, if $n \sim \mathcal{G}(1, q)$ we have*

$$\mathbb{E}[n^{2k}] = \frac{D^k \theta(q)}{\theta(q)} \text{ and } \mathbb{E}[n^{2k+1}] = 0. \quad (21)$$

It would be satisfactory to interpret Proposition 3.2 as a discrete analog of the Wick formula for the Gaussian distribution, and to interpret its even moments as counting functions of combinatorial objects. However we were unable to do so, and leave it as an open question. Surprisingly, we will find similar expressions of moments of another measure in Proposition 5.2, where the theta function is replaced by some generating series.

We now go to the study of the measure $\mathcal{G}(N, q)$ in the large N limit.

3.3. The general case: from highest weights to partitions and back.

The main issue with the current expression (19) of $\mathcal{G}(N, q)$ is that not only the summand depends on N , but also does the set $\widehat{U}(N)$, which is not convenient for letting $N \rightarrow \infty$. A better way to deal with this issue is to rewrite $\widehat{U}(N)$ in terms of partitions, using a decomposition of highest weights that has been widely used in the last decades [Ste87, Sta84, Koi89, GT93b, GT93a, Lem22, Mag22, Lem23], but seems to be already known since the works of Littlewood [Lit44a, Lit44b]. We recall the following result, that already appears in [Lem22] in a different form.

Proposition 3.3. *For any $N \geq 1$, there is a bijection*

$$\Phi_N : \widehat{U}(N) \xrightarrow{\sim} \Lambda_N,$$

where

$$\Lambda_N = \{(\alpha, \beta, n) \in \mathfrak{Y}^2 \times \mathbb{Z} : \ell(\alpha) \leq A_N, \ell(\beta) \leq B_N\}.$$

has been defined in (1).

Proof. The bijection consists in fixing $n = \lambda_{\lfloor (N+1)/2 \rfloor} \in \mathbb{Z}$ and seeing the left part of λ as $\alpha + n = (\alpha_1 + n, \dots, \alpha_r + n)$ and the right part as $n - \beta$, as shown in Figure 1 from the introduction. We shall describe this bijection in more details: let $k = \lfloor (N+1)/2 \rfloor$ and for any $\lambda = (\lambda_1, \dots, \lambda_N) \in \widehat{U}(N)$ denote by ℓ_1 the greatest integer such that $\lambda_{\ell_1} > \lambda_k$ and ℓ_2 the smallest integer such that $\lambda_{\ell_2} < \lambda_k$ ². Then, we set $\alpha_i = \lambda_i - \lambda_k$ for $1 \leq i \leq \ell_1$ and

²It is important that $\ell(\alpha) < k$ and $\ell(\beta) \leq N - k$, otherwise there would be several possible decompositions, by shifting n and either α or β .

$\beta_j = \lambda_k - \lambda_{N-j-1}$, for any $1 \leq j \leq \ell_2$. We obtain

$$\lambda = (\alpha_1 + \lambda_k, \dots, \alpha_{\ell_1} + \lambda_k, \underbrace{\lambda_k, \dots, \lambda_k}_{N-\ell_1-\ell_2}, \lambda_k - \beta_{\ell_2}, \dots, \lambda_k - \beta_1), \quad (22)$$

where $\alpha = (\alpha_1, \dots, \alpha_{\ell_1})$ and $\beta = (\beta_1, \dots, \beta_{\ell_2})$ are indeed integer partitions. Furthermore, we also have by construction

$$\ell_1 \leq k - 1 = \lfloor (N + 1)/2 \rfloor - 1, \text{ and } \ell_2 \leq N - k = N - \lfloor (N + 1)/2 \rfloor.$$

We can set $\Phi_N(\lambda) = (\alpha, \beta, \lambda_k)$ where $\alpha = (\alpha_1, \dots, \alpha_{\ell_1})$ and $\beta = (\beta_1, \dots, \beta_{\ell_2})$. The expression of α_i and β_i yields the injectivity of Φ_N . Reciprocally, from $(\alpha, \beta, n) \in \Lambda_N$ we can construct $\lambda \in \widehat{\mathbb{U}(N)}$ with (22), which yields the surjectivity of Φ_N , hence Φ_N is indeed a bijection. \square

In the sequel, given a pair of partitions (α, β) and an integer n , we will denote by $\lambda_N(\alpha, \beta, n) = \Phi_N^{-1}(\alpha, \beta, n)$ the highest weight constructed through Φ_N^{-1} for $N \geq \max(2\ell(\alpha), 2\ell(\beta) - 1)$. A crucial result is that $c_2(\lambda)$ has then an explicit expression as a function of α, β, n and N that will be convenient for asymptotic analysis:

Proposition 3.4 ([Lem22]). *For any $N \geq 1$ and $(\alpha, \beta, n) \in \Lambda_N$, we have*

$$c_2(\lambda_N(\alpha, \beta, n)) = |\alpha| + |\beta| + n^2 + \frac{2}{N}F(\alpha, \beta, n), \quad (23)$$

where

$$F(\alpha, \beta, n) = K(\alpha) + K(\beta) + n(|\alpha| - |\beta|).$$

We are now ready to exhibit the relationship between $\mathcal{G}(N, q)$, $\mathcal{G}(1, q)$ and $\mathcal{U}(q)$. It can be roughly stated as follows: the pushforward of $\mathcal{G}(N, q)$ by the map Φ_N is absolutely continuous with respect to the product measure $\mathcal{U}(q)^{\otimes 2} \otimes \mathcal{G}(1, q)$, with an explicit density. Here is the exact result:

Proposition 3.5. *Let $N \geq 1$ be an integer and $q \in (0, 1)$ be a real number. For any bounded measurable $f : \Lambda_N \rightarrow \mathbb{R}$, if $\lambda \sim \mathcal{G}(N, q)$, we have*

$$\mathbb{E}[f \circ \Phi_N(\lambda)] = \frac{\theta(q)}{Z_N(q)\phi(q)^2} \mathbb{E} \left[f(\alpha, \beta, n) q^{\frac{2}{N}F(\alpha, \beta, n)} \mathbf{1}_{\Lambda_N}(\alpha, \beta, n) \right], \quad (24)$$

where α, β, n are independent, $\alpha, \beta \sim \mathcal{U}(q)$ and $n \sim \mathcal{G}(1, q)$.

Proof. It follows straightforwardly from the definitions of the measures and Proposition 3.4. \square

4. PROOFS OF THE MAIN THEOREMS

4.1. The large N limit of $\mathcal{G}(N, q)$. The behavior of $\mathcal{G}(N, q)$ (or more precisely of the pushforward of $\mathcal{G}(N, q)$ by the map Φ_N) in the large N limit is given by Theorem 1.1, that we will prove in the present subsection.

Before that, we need some preliminary control on the Casimir numbers. The main interest of this result is to give a minoration of c_2 which involves only the integers n and N and the sizes of the two partitions α and β .

Lemma 4.1 (Domination lemma). *For any $(\alpha, \beta, n) \in \Lambda_N$, for $N \geq 1$,*

$$c_2(\lambda_N(\alpha, \beta, n)) \geq \frac{1}{2}(|\alpha| + |\beta|) + \left(n + \frac{|\alpha| - |\beta|}{N}\right)^2 =: C_N(\alpha, \beta, n). \quad (25)$$

Furthermore, for any $q \in (0, 1)$,

$$\sup_{N \geq 1} \sum_{(\alpha, \beta, n) \in \Lambda_N} q^{C_N(\alpha, \beta, n)} < \infty.$$

Proof. Let $(\alpha, \beta, n) \in \Lambda_N$. We have

$$\begin{aligned} c_2(\lambda_N(\alpha, \beta, n)) &= |\alpha| + |\beta| + n^2 + \frac{2}{N}(K(\alpha) + K(\beta) + n(|\alpha| - |\beta|)) \\ &= |\alpha| + |\beta| + \frac{2}{N}(K(\alpha) + K(\beta)) + \left(n + \frac{|\alpha| - |\beta|}{N}\right)^2 - \frac{1}{N^2}(|\alpha| - |\beta|)^2 \\ &\geq |\alpha| + |\beta| + \frac{2}{N}(K(\alpha) + K(\beta)) + \left(n + \frac{|\alpha| - |\beta|}{N}\right)^2 - \frac{1}{N^2}(|\alpha|^2 + |\beta|^2) \end{aligned}$$

Using the Cauchy–Schwarz inequality, we have

$$\frac{1}{N^2}|\alpha|^2 \leq \left(\sum_{k=1}^{\ell(\alpha)} \frac{1}{N^2}\right) \left(\sum_{k=1}^{\ell(\alpha)} \alpha_k^2\right) = \frac{\ell(\alpha)}{N^2} \sum_{k=1}^{\ell(\alpha)} \alpha_k^2 \leq \frac{1}{2N} \sum_k \alpha_k^2,$$

where, for the last inequality, we have used that $\ell(\alpha) \leq N/2$. It yields

$$\begin{aligned} \frac{2}{N}K(\alpha) - \frac{1}{N^2}|\alpha|^2 &\geq \frac{1}{N} \left(\sum_{i=1}^{\ell(\alpha)} \alpha_i(\alpha_i + 1 - 2i) - \frac{1}{2} \sum_{i=1}^{\ell(\alpha)} \alpha_i^2\right) \\ &\geq \frac{1}{N} \sum_{i=1}^{\ell(\alpha)} \alpha_i(1 - 2i) \\ &\geq -\frac{|\alpha|\ell(\alpha)}{N}. \end{aligned}$$

Using the same bound for β leads to the following inequality:

$$c_2(\lambda_N(\alpha, \beta, n)) \geq |\alpha| \left(1 - \frac{\ell(\alpha)}{N}\right) + |\beta| \left(1 - \frac{\ell(\beta)}{N}\right) + \left(n + \frac{|\alpha| - |\beta|}{N}\right)^2.$$

However, by assumption we have $\ell(\alpha) \leq A_N, \ell(\beta) \leq B_N$, and for $N \geq 1$, we have

$$\frac{\max(A_N, B_N)}{N} \leq \frac{1}{2}.$$

We thus obtain (25). Now, let us bound

$$\sum_{(\alpha, \beta, n) \in \Lambda_N} q^{C_N(\alpha, \beta, n)}.$$

We have

$$\sum_{(\alpha, \beta, n) \in \Lambda_N} q^{C_N(\alpha, \beta, n)} = \sum_{\substack{\ell(\alpha) \leq A_N \\ \ell(\beta) \leq B_N}} q^{\frac{1}{2}(|\alpha| + |\beta|)} \left(\sum_{n \in \mathbb{Z}} q^{\left(n + \frac{|\alpha| - |\beta|}{N}\right)^2}\right).$$

The sum between the brackets can be bounded independently from N, α, β by

$$C := \sup_{x \in [0,1]} \sum_{n \in \mathbb{Z}} q^{(n+x)^2} < \infty. \quad (26)$$

and therefore

$$\sup_{N \geq 1} \sum_{(\alpha, \beta, n) \in \Lambda_N} q^{C_N(\alpha, \beta, n)} \leq \frac{C}{\phi(q^{1/2})^2}.$$

□

Remark 4.2. The quasi-periodicity (26) of the theta function is critical in the proof of the domination lemma. There might be an interpretation in terms of quasimodular forms.

We are now in position to prove the asymptotic decorrelation. As a matter of fact, we will simultaneously prove Theorem 1.3 and 1.1.

Proof of Theorem 1.1 and Theorem 1.3. Let $f : \mathfrak{Y}^2 \times \mathbb{Z} \rightarrow \mathbb{R}$ be measurable bounded. We have to study the convergence of the following sum:

$$\sum_{(\alpha, \beta, n) \in \mathfrak{Y}^2 \times \mathbb{Z}} G_N(\alpha, \beta, n)$$

with

$$G_N(\alpha, \beta, n) := f(\alpha, \beta, n) q^{c_2(\lambda_N(\alpha, \beta, n))} \mathbf{1}_{\Lambda_N}(\alpha, \beta, n)$$

It is clear that, for any $(\alpha, \beta, n) \in \mathfrak{Y}^2 \times \mathbb{Z}$, we have the pointwise convergence

$$G_N(\alpha, \beta, n) \xrightarrow{N \rightarrow \infty} f(\alpha, \beta, n) q^{|\alpha|} q^{|\beta|} q^{n^2}.$$

As we are working with measures of infinite mass, the dominated convergence criterion that we will apply is not standard. For the sake of completeness, we state and prove it in Appendix A. The proof therefore boils down to show that the family of functions (G_N) satisfies (46), thanks to Lemma A.1. For any positive integer M , set

$$X_M = \{(\alpha, \beta, n) \in \mathfrak{Y}^2 \times \mathbb{Z} : |\alpha| \leq M, |\beta| \leq M, |n| \leq M\}.$$

It is always a finite set so it has finite counting measure. We first have :

$$\sum_{(\alpha, \beta, n) \in X_M^c} G_N(\alpha, \beta, n) \leq \|f\|_\infty \sum_{(\alpha, \beta, n) \in X_M^c \cap \Lambda_N} q^{c_2(\lambda_N(\alpha, \beta, n))},$$

we now use Lemma 4.1 to get

$$\sum_{(\alpha, \beta, n) \in X_M^c} G_N(\alpha, \beta, n) \leq \|f\|_\infty \sum_{(\alpha, \beta, n) \in X_M^c \cap \Lambda_N} q^{C_N(\alpha, \beta, n)}.$$

Let us show that the sum in the RHS can be arbitrarily small for M large enough. By the same arguments as in the proof of Lemma 4.1,

$$\begin{aligned} & \sum_{(\alpha, \beta, n) \in \Lambda_N} \mathbf{1}_{X_M}(\alpha, \beta, n) q^{C_N(\alpha, \beta, n)} \\ &= \sum_{\substack{\ell(\alpha) \leq A_N \\ \ell(\beta) \leq B_N}} \mathbf{1}_{\{|\alpha| > M, |\beta| > M\}} q^{\frac{1}{2}(|\alpha| + |\beta|)} \left(\sum_{n \in \mathbb{Z}} \mathbf{1}_{\{|n| > M\}} q^{\left(n + \frac{|\alpha| - |\beta|}{N}\right)^2} \right). \end{aligned}$$

As all terms are nonnegative, it is clear that the right handside is bounded by

$$\begin{aligned} \sum_{|\alpha|, |\beta| > M} q^{\frac{1}{2}(|\alpha|+|\beta|)} \left(\sum_{n \in \mathbb{Z}} q^{\left(n + \frac{|\alpha| - |\beta|}{N}\right)^2} \right) &\leq C \sum_{|\alpha|, |\beta| > M} q^{\frac{1}{2}(|\alpha|+|\beta|)} \\ &\leq C \phi(q^{\frac{1}{2}})^2 \mathbb{P}(|\alpha| > M)^2, \end{aligned}$$

where $\alpha \sim \mathcal{U}(q)$ and C is defined in (26). Using Lemma 2.4, we get that, for M large enough, it is smaller than

$$\frac{\varepsilon}{C \|f\|_\infty},$$

and it follows that

$$\sup_{N \geq 1} \sum_{(\alpha, \beta, n) \in X_M^c} G_N(\alpha, \beta, n) \leq \varepsilon,$$

as required.

Note that we have actually proved the convergence for the non-normalized measure rather than $\mathcal{G}(N, q)$; if we apply the result to $f(\alpha, \beta, n) = 1$ we obtain the convergence of the partition function, which is Theorem 1.3, and the proof of Theorem 1.1 follows directly. \square

4.2. Asymptotic expansion of the partition function. Our goal is now to prove Theorem 1.2. Before going to its proof, we will again need some preliminary lemmas. For any $k, \ell \geq 0$ and any $q \in (0, 1)$, we set

$$A_{k, \ell} = \mathbb{E} \left[K(\alpha)^k |\alpha|^\ell \right] \quad \text{and} \quad B_{k, \ell} = \mathbb{E} \left[n^{k+\ell} \right],$$

for $\alpha \sim \mathcal{U}(q)$ and $n \sim \mathcal{G}(1, q)$.

Lemma 4.3. *Let $q \in (0, 1)$, and $\alpha, \beta \sim \mathcal{U}(q)$ and $n \sim \mathcal{G}(1, q)$ be independent random variables. For any $k \geq 0$, we have*

$$\mathbb{E} \left[F(\alpha, \beta, n)^{2k} \right] = \sum \binom{2k}{2k_1 \ 2k_2 \ k_3 \ k_4} (-1)^{k_4} A_{2k_1, k_3} A_{2k_2, k_4} B_{k_3, k_4}, \quad (27)$$

where the sum is over $k_1, k_2, k_3, k_4 \geq 0$ such that $2k_1 + 2k_2 + k_3 + k_4 = 2k$, and

$$\mathbb{E} \left[F(\alpha, \beta, n)^{2k+1} \right] = 0. \quad (28)$$

Proof. The proof will be organized as follows. First, we expand $F(\alpha, \beta, n)^k$ for any k using the multinomial formula, then we prove that the expectation for odd k is zero, which will simplify the general formula.

Step 1: expansion. From the definition of $F(\alpha, \beta, n)$ and the multinomial formula, we have

$$\mathbb{E}[F(\alpha, \beta, n)^k] = \sum \binom{k}{k_1 \ k_2 \ k_3 \ k_4} \mathbb{E} \left[K(\alpha)^{k_1} K(\beta)^{k_2} n^{k_3+k_4} |\alpha|^{k_3} |\beta|^{k_4} (-1)^{k_4} \right].$$

where the sum is over $k_1, k_2, k_3, k_4 \geq 0$ such that $k_1 + k_2 + k_3 + k_4 = k$. By independence and linearity, it rewrites

$$\mathbb{E}[F(\alpha, \beta, n)^k] = \sum_{\substack{k_1, k_2, k_3, k_4 \geq 0 \\ k_1 + k_2 + k_3 + k_4 = k}} \binom{k}{k_1 \ k_2 \ k_3 \ k_4} (-1)^{k_4} A_{k_1, k_3} A_{k_2, k_4} B_{k_3, k_4}, \quad (29)$$

Step 2: cancellations. If λ is a partition, we denote by λ' its conjugate, in the sense that $\lambda'_i = \#\{j, \lambda_j \geq i\}$. As $\lambda \mapsto \lambda'$ is a bijection on \mathfrak{P} and $|\lambda| = |\lambda'|$, we have that, if $\alpha \sim \mathcal{U}(q)$ then $\alpha' \sim \mathcal{U}(q)$. Moreover, $K(\lambda') = -K(\lambda)$ so that

$$\mathbb{E} \left[K(\alpha)^{k_1} |\alpha|^{k_3} \mathbf{1}_{\{K(\alpha) > 0\}} \right] = (-1)^{k_1} \mathbb{E} \left[K(\alpha)^{k_1} |\alpha|^{k_3} \mathbf{1}_{\{K(\alpha) < 0\}} \right]$$

and

$$\mathbb{E} \left[K(\alpha)^{k_1} |\alpha|^{k_3} \right] = (1 + (-1)^{k_1}) \mathbb{E} \left[K(\alpha)^{k_1} |\alpha|^{k_3} \mathbf{1}_{\{K(\alpha) > 0\}} \right],$$

which cancels when k_1 is odd. By a similar argument for k_2 , we see that we can restrict the sum in (29) to

$$\{k_1, k_2, k_3, k_4 \geq 0 \mid k_1 + k_2 + k_3 + k_4 = k, k_1 \text{ and } k_2 \text{ are even}\}.$$

Now, observe that for any k_1, k_2, k_3, k_4 , the terms

$$(-1)^{k_4} A_{k_1, k_3} A_{k_2, k_4} B_{k_3, k_4}$$

and

$$(-1)^{k_3} A_{k_2, k_4} A_{k_1, k_3} B_{k_4, k_3}$$

cancel each other if k_3 and k_4 do not have the same parity. Therefore k has to be even and we obtain both (27) and (28). \square

Lemma 4.4. *Let $q \in (0, 1)$, and $\alpha, \beta \sim \mathcal{U}(q)$ and $n \sim \mathcal{G}(1, q)$ be independent random variables. For any $k \geq 1$, there exists a constant $C_{k,q}$ such that*

$$\mathbb{E} \left[F(\alpha, \beta, n)^k \mathbf{1}_{\Lambda_N^c}(\alpha, \beta, n) \right] \leq C_{k,q} e^{\frac{N}{2} \log q}.$$

Consequently, for any $p \in \mathbb{N}$ and $k \geq 1$, there is a constant $C(k, p, q) > 0$ such that, for any $N \geq 1$,

$$\mathbb{E} \left[F(\alpha, \beta, n)^k \mathbf{1}_{\Lambda_N^c}(\alpha, \beta, n) \right] \leq C(k, p, q) N^{-p}. \quad (30)$$

Proof. As α and β are playing a similar role, we have

$$\mathbb{E} \left[F(\alpha, \beta, n)^k \mathbf{1}_{\Lambda_N^c}(\alpha, \beta, n) \right] \leq 2 \mathbb{E} \left[F(\alpha, \beta, n)^k \mathbf{1}_{\{\ell(\alpha) > \lfloor (N-1)/2 \rfloor\}} \right].$$

By Cauchy-Schwarz inequality, we have

$$\mathbb{E} \left[F(\alpha, \beta, n)^k \mathbf{1}_{\{\ell(\alpha) > \lfloor (N-1)/2 \rfloor\}} \right]^2 \leq \mathbb{P}(\ell(\alpha) > \lfloor (N-1)/2 \rfloor) \mathbb{E} \left[F(\alpha, \beta, n)^{2k} \right],$$

which is $O(N^{-p})$ with a bound that only depends on k, p and q , according to Proposition 2.5 and Lemma 4.3. \square

We are now in position to prove Theorem 1.2, whose statement we detail in the following:

Theorem 4.5. *Let $t > 0$ be a positive real number, and set $q_t = e^{-\frac{t}{2}}$. For any $p \geq 0$ and any integer N large enough, the partition function $Z_N(q_t)$ admits the asymptotic expansion*

$$Z_N(q_t) = \sum_{k=0}^p \frac{a_{2k}(t)}{N^{2k}} + O(N^{-2(p+1)}), \quad (31)$$

where $a_{2k}(t)$ is given by

$$a_{2k}(t) = \frac{t^{2k} \theta(q_t)}{(2k)! \phi(q_t)^2} \mathbb{E}[F(\alpha, \beta, n)^{2k}].$$

Remark 4.6. This asymptotic expansion is also a topological expansion in the sense of Section 1.4. We do not address this aspect in the following proof but we will give all the details on this aspect in Section 6.

Proof. Let us start by writing the partition function with the help of Proposition 3.5:

$$Z_N(q_t) = \frac{\theta(q_t)}{\phi(q_t)^2} \mathbb{E} \left[e^{-\frac{t}{N} F(\alpha, \beta, n)} \mathbf{1}_{\Lambda_N}(\alpha, \beta, n) \right],$$

where we recall that α, β and n are independent, $\alpha, \beta \sim \mathcal{U}(q_t)$ and $n \sim \mathcal{G}(1, q_t)$.

We perform a Taylor expansion of the exponential up to order $2p + 1$:

$$e^{-\frac{t}{N} F(\alpha, \beta, n)} = \sum_{k=0}^{2p+1} \frac{(-t)^k}{k! N^k} F(\alpha, \beta, n)^k + \frac{F(\alpha, \beta, n)^{2p+2}}{(2p+1)! N^{2p+2}} I_{N,t}(\alpha, \beta, n),$$

where

$$I_{N,t}(\alpha, \beta, n) = \int_0^t (t-s)^{2p+1} e^{-\frac{s}{N} F(\alpha, \beta, n)} ds.$$

By linearity, it follows that

$$\begin{aligned} Z_N(q_t) &= \frac{\theta(q_t)}{\phi(q_t)^2} \left(\sum_{k=0}^{2p+1} \frac{(-t)^k}{k! N^k} \mathbb{E}[F(\alpha, \beta, n)^k] - \sum_{k=0}^{2p+1} \frac{(-t)^k}{k! N^k} \mathbb{E}[F(\alpha, \beta, n)^k \mathbf{1}_{\Lambda_N^c}(\alpha, \beta, n)] \right) \\ &\quad + \frac{\theta(q_t)}{\phi(q_t)^2} \frac{1}{(2p+1)! N^{2p+2}} \mathbb{E}[F(\alpha, \beta, n)^{2p+2} I_{N,t}(\alpha, \beta, n) \mathbf{1}_{\Lambda_N}(\alpha, \beta, n)]. \end{aligned}$$

As a consequence of Lemma 4.4, we have

$$\sum_{k=0}^{2p+1} \frac{(-t)^k}{k! N^k} \mathbb{E}[F(\alpha, \beta, n)^k \mathbf{1}_{\Lambda_N^c}(\alpha, \beta, n)] = O(N^{-2p-2}). \quad (32)$$

Now we conclude the control of the remainder by claiming that

$$\sup_n \mathbb{E}[F(\alpha, \beta, n)^{2p+2} I_{N,t}(\alpha, \beta, n) \mathbf{1}_{\Lambda_N}(\alpha, \beta, n)] < \infty. \quad (33)$$

Indeed, for any $s \leq t$,

$$\begin{aligned} &\mathbb{E}[F(\alpha, \beta, n)^{2p+2} e^{-\frac{s}{N} F(\alpha, \beta, n)} \mathbf{1}_{\Lambda_N}(\alpha, \beta, n)] \\ &\leq \mathbb{E}[F(\alpha, \beta, n)^{2p+2} (1 + e^{-\frac{t}{N} F(\alpha, \beta, n)}) \mathbf{1}_{\Lambda_N}(\alpha, \beta, n)] \\ &\leq \mathbb{E}[F(\alpha, \beta, n)^{2p+2}] + \mathbb{E}[F(\alpha, \beta, n)^{6p+6}]^{1/3} \mathbb{E}[e^{-\frac{3t}{2N} F(\alpha, \beta, n)} \mathbf{1}_{\Lambda_N}(\alpha, \beta, n)]^{2/3}, \end{aligned}$$

where we have used Hölder inequality with conjugate coefficients 3 and 3/2. Now, by the same argument as in the proof of Lemma 4.1, the last term can be bounded as follows:

$$\sum_{(\alpha, \beta, n) \in \Lambda_N} q_t^{\alpha + \beta + n^2 + \frac{3}{N} F(\alpha, \beta, n)} \leq \sum_{\substack{\ell(\alpha) \leq A_N \\ \ell(\beta) \leq B_N}} q_t^{\frac{1}{4}(|\alpha| + |\beta|)} \left(\sum_{n \in \mathbb{Z}} q_t^{\left(n + \frac{3|\alpha| - |\beta|}{2N}\right)^2} \right),$$

which is finite as explained in the proof of Lemma 4.1. We therefore get that

$$Z_N(q_t) = \frac{\theta(q_t)}{\phi(q_t)^2} \sum_{k=0}^{2p+1} \frac{(-t)^k}{k! N^k} \mathbb{E}[F(\alpha, \beta, n)^k] + O(N^{-2p-2})$$

and we conclude by using (28). \square

5. RANDOM PARTITIONS AND ENUMERATION OF RAMIFIED COVERINGS

In this section, we develop a new approach to deal with the enumeration of ramified coverings of compact Riemann surfaces, which is inspired by Okounkov's works [Ok00, BO00, EO01]. It is based on the idea that one can turn generating series of the generalized Hurwitz numbers into probability measures on partitions, making an effective link between models of random partitions and enumerative geometry. We start by recalling the relationship between partitions and enumeration of ramified coverings, then we introduce and study new measures on partitions, that we call respectively the *Frobenius* and the *Plancherel–Hurwitz* measures. The main point is that these measure are related to the q -uniform measure and can benefit from the results we obtained in Section 2. We finish this section by a topological interpretation of the expectations involved in the coefficients of the asymptotic expansion (31).

5.1. Enumeration of ramified coverings. If X, Y are two topological spaces, a continuous surjective map $p : Y \rightarrow X$ is a *covering* of X if for any $x \in X$, there is an open neighbourhood U of x such that preimage $p^{-1}(U)$ is the disjoint union of open sets V_1, \dots, V_n , and $p|_{V_i} : V_i \rightarrow U$ is a homeomorphism for all $1 \leq i \leq n$. If X is a smooth path-connected surface and $R \subset X$ is a finite subset of X , a *ramified covering* of X with *ramification locus* R is the datum of a topological space Y and a continuous mapping $p : Y \rightarrow X$ such that:

- (i) the mapping $p|_{p^{-1}(X \setminus R)} : p^{-1}(X \setminus R) \rightarrow X \setminus R$ is a covering;
- (ii) for all $x \in X$ and $y \in p^{-1}(\{x\})$, there exists an open neighbourhood V of y and an integer $n \geq 1$ such that the mapping $p|_V$ is (topologically) conjugated to the mapping

$$\begin{cases} \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & z^k \end{cases} .$$

In this case, k is called the *ramification index*, or the *ramification order*, of X . We also denote by B , and call *branch locus* of X , the set $p(R)$. Its elements are called *branch points*. In what follows, we will only consider single branch points, i.e. the ramification index of X will be 2 (the ramification is

also called *generic* in this case).

A powerful result is that if the base space is a Riemann surface, then the ramified covering can be equipped with a complex structure such that p is holomorphic³. In particular it is orientable and it has a well-defined genus. The set of n -sheeted coverings of a closed orientable surface X with ramification locus R is in bijection with the representation space of the fundamental group $\pi_1(X \setminus R)$ into the symmetric group \mathfrak{S}_n [EO01]:

$$\{n\text{-sheeted ramified coverings of } X \setminus R\} \simeq \text{Hom}(\pi_1(X \setminus R), \mathfrak{S}_n) / \mathfrak{S}_n. \quad (34)$$

The standard way to count ramified coverings is therefore by counting weighted isomorphism classes of the RHS of (34), the weight being given by $\frac{1}{|\mathfrak{S}_n|} = \frac{1}{n!}$. Assuming that we only consider generic ramifications, the corresponding number of ramified coverings is given by

$$H_g(n, k) = \sum_{\alpha \vdash n} \left(\frac{n!}{\chi_\alpha(1)} \right)^{2g-2} K(\alpha)^k. \quad (35)$$

A detailed derivation of these numbers can be found in [LZ04, Appendix A]. We shall call them *generalized Hurwitz number*, because for $g = 0$ we recover the standard Hurwitz numbers that count ramified coverings of the sphere. By the Riemann–Hurwitz formula, if Y is a n -sheeted ramified covering over a surface X of genus g with ramification locus $R = \{x_1, \dots, x_k\}$, then the genus g' of Y is given by

$$g' = \frac{k}{2} + 1 + n(g - 1). \quad (36)$$

Here, we are only interested in the particular case of $g = 1$ (when the base is a torus). We obtain a nice simple relationship between the number of ramification points and the genus of the ramified covering: $k = 2g' - 2$. Besides, in order to ensure that the ramified covering corresponds to an orientable surface, we see that k has to be even. In order to stress this fact, we will enumerate the generalized Hurwitz numbers as $H_g(n, 2g' - 2)$ with $g' \geq 1$ in the sequel. Their generating function is the formal power series

$$\mathcal{F}_{g,g'}(q) = \sum_{n \geq 1} H_g(n, 2g' - 2) q^n. \quad (37)$$

Let us make a few comments on how to compute these generating functions when the base space is a torus.

- The case $g' = 1$ is trivial:

$$\mathcal{F}_{1,1}(q) = \sum_{n \in \mathbb{N}} p(n) q^n = \phi(q)^{-1}.$$

- For $g' \geq 2$, Kaneko and Zagier [KZ95] proved that $\mathcal{F}_{g,g'}$ is a quasimodular form of weight $6k$, inspired by works of Douglas [Dou95] and Dijkgraaf [Dij95] on mirror symmetry and conformal field theory. See [LZ04, Corollary A.2.17] for a precise statement and a few explicit computations.

³This fact was one of the initial motivations from Eskin and Okounkov, because it enables to link the enumeration of ramified coverings to the computation of Masur–Veech volumes, see [Agg20] and references therein for more recent developments.

5.2. New measures on partitions. Using formulas (37) and (35), we can define a measure on \mathfrak{P} with parameters $g \in \mathbb{N}$, $g' \in \mathbb{N}^*$ and $q \in (0, 1)$ by

$$\mathbb{P}(\alpha) \propto \left(\frac{\chi_\alpha(1)}{|\alpha|!} \right)^{2-2g} K(\alpha)^{2g'-2} q^{|\alpha|}.$$

We call it the *Frobenius measure* of genera g and g' and parameter q and denote it by $\mathfrak{F}_{g,g'}(q)$. Its partition function is simply the generating function $\mathcal{F}_{g,g'}(q)$. We made the choice of only considering even powers of the total content of partitions in order to avoid signed measures, but a similar construction is possible with any power of $K(\alpha)$. The Frobenius measure is related to another measure $\mathfrak{M}_{g,g'}(n)$ on \mathfrak{P}_n , that generalizes the Plancherel–Hurwitz measure introduced by Chapuy, Louf and Walsh [CLW23], corresponding to the case $g = 0$. It is defined by

$$\mathbb{P}(\alpha) \propto \left(\frac{\chi_\alpha(1)}{|\alpha|!} \right)^{2-2g} K(\alpha)^{2g'-2},$$

and its partition function is just the Hurwitz number $H_g(n, 2g' - 2)$. We have the following relationship between $\mathfrak{F}_{g,g'}(q)$ and $\mathfrak{M}_{g,g'}(n)$, reminiscent of the ones for the Poissonized Plancherel measure ($g = 0, g' = 1$) and the q -uniform measure ($g = g' = 1$).

Proposition 5.1. *For any $g \geq 0$, $g' \geq 1$ and any $q \in (0, 1)$, if $\alpha \sim \mathfrak{F}_{g,g'}(q)$, then the distribution of $|\alpha|$ is given by*

$$\mathbb{P}(|\alpha| = n) \propto H_g(n, 2g' - 2) q^n. \quad (38)$$

Besides, conditionally to $|\alpha| = n$, the distribution of α on \mathfrak{P}_n is given by $\mathfrak{M}_{g,g'}(n)$.

Proof. It is the same proof as in Proposition 2.2: the first assertion comes from the averaging over all partitions with the same size, and the second one from the definition of the conditional probability. \square

Here is an elementary result, whose proof is identical to Proposition 3.2.

Proposition 5.2. *For any $g \geq 0$, $g' \geq 1$ and $q \in (0, 1)$, if $\alpha \sim \mathfrak{F}_{g,g'}(q)$ is a random partition and $\ell \geq 0$ is an integer, then*

$$\mathbb{E} \left[|\alpha|^\ell \right] = \frac{D^\ell \mathcal{F}_{g,g'}(q)}{\mathcal{F}_{g,g'}(q)}. \quad (39)$$

Let us comment (39) in a few simple cases.

- If $\ell = 1$, we find the average size of a $\mathfrak{F}_{g,g'}(q)$ -distributed random partition:

$$\mathbb{E} [|\alpha|] = \frac{q \frac{d}{dq} \mathcal{F}_{2k,g}(q)}{\mathcal{F}_{2k,g}(q)} = -q \frac{d}{dq} \log \mathcal{F}_{2k,g}(q).$$

- If $\ell = g = g' = 1$, we retrieve a result by Okounkov [Oko01]:

$$\phi(q) \sum_{\alpha \in \mathfrak{P}} |\alpha| q^{|\alpha|} = q \frac{d}{dq} \log \phi(q) = \frac{1}{24} + G_2(q), \quad (40)$$

where $G_2(q) = \frac{1}{2} \zeta(-1) + \sum_n q^n \sum_{d|n} d$ is an Eisenstein series.

Although it might be of independent interest to study in detail the Plancherel–Hurwitz and the Frobenius measures, we will not go further in this direction. We will simply link the expectations from (39) to the expectations $A_{k,\ell}$ that appear in Lemma 4.3: recall that if α is a q -uniform random partition,

$$A_{k,\ell} = \mathbb{E} \left[K(\alpha)^k |\alpha|^\ell \right].$$

It follows that for any integer $g \geq 1$,

$$A_{2g-2,\ell} = \phi(q) \sum_{\alpha \in \mathfrak{Q}} K(\alpha)^{2g-2} |\alpha|^\ell = \phi(q) \mathcal{F}_{1,1}(q) \mathbb{E} \left[|\beta|^\ell \right],$$

where $\beta \sim \mathfrak{F}_{1,g}(q)$. Finally,

$$A_{2g-2,\ell} = \phi(q) D^\ell \mathcal{F}_{1,g}(q). \quad (41)$$

Now we shall state the main result of this subsection.

Theorem 5.3. *For any $k \geq 0$ and $q \in (0, 1)$, if (α, β, n) is a triple of independent random variables with $\alpha, \beta \sim \mathcal{U}(q)$ and $n \sim \mathcal{G}(1, q)$, we have*

$$\begin{aligned} \mathbb{E} \left[F(\alpha, \beta, n)^{2k} \right] = \\ \sum \frac{(-1)^{k_4} (2k)! \phi(q)^2}{(2k_1)! (2k_2)! k_3! k_4! \theta(q)} D^{k_3} \mathcal{F}_{1,k_1+1}(q) D^{k_4} \mathcal{F}_{1,k_2+1}(q) D^{\frac{k_3+k_4}{2}} \theta(q), \end{aligned} \quad (42)$$

where the sum is over $k_1, k_2, k_3, k_4 \geq 0$ such that $2k_1 + 2k_2 + k_3 + k_4 = 2k$.

Proof. This is a direct consequence of (27), (39), (21) and (41). \square

6. TOPOLOGICAL EXPANSIONS

In this section, we combine the asymptotic expansions from Section 4 and the results from Section 5 in order to get the topological expansion of the Yang–Mills partition function on a torus. We will first focus on a special case, the so-called *chiral setting*, introduced by Gross and Taylor in [GT93b, GT93a]. It consists in replacing, in (5), the set of summation $\widehat{U(N)}$ by $\{\alpha \in \mathfrak{Q}, \ell(\alpha) \leq N\}$, and apply the dimension and Casimir formulas to the highest weight obtained from α by adding the right amount of zeros: $\tilde{\alpha} = (\alpha_1, \dots, \alpha_{\ell(\alpha)}, 0, \dots, 0) \in \widehat{U(N)}$. More precisely,

$$Z_N^{\text{chir}}(g, t) = \phi(q)^{-1} \mathbb{E} \left[d_{\tilde{\alpha}}^{2-2g} q_t^{\frac{2}{N} K(\alpha)} \mathbf{1}_{\{\ell(\alpha) \leq N\}} \right],$$

where $d_{\tilde{\alpha}}$ is the dimension of the associated representation. As we treat here the case $g = 1$, we do not need to provide an exact formula for this dimension.

Let us now prove the analog of Theorem 1.2 in the chiral setting.

Proof of Theorem 1.4. Using the exact same estimates as in the proof of Theorem 4.5, we have

$$Z_N^{\text{chir}}(1, t) = \sum_{k=0}^{2p+1} \frac{(-t)^k}{k! N^k} \mathbb{E}[K(\alpha)^k] + O(N^{-2p-2}).$$

The odd coefficients vanish because of the identity $K(\alpha)^{2k+1} = -K(\alpha')^{2k+1}$, and the even coefficients can be computed using Proposition 5.2. The result follows. \square

Theorem 1.4 has also been proved independently by Novak [Nov24], who recovered the power series expansion from Gross and Taylor and proved that it is absolutely convergent. The fact that it is a topological expansion follows from the Riemann–Hurwitz formula (36). We claim that one can go even further, and recover a beautiful formula, due to Gross and Taylor, written as an integral over the space of ramified coverings. In order to do so, we shall borrow⁴ some constructions from [Lév08]. Let X be a compact complex torus, i.e. a compact connected Riemann surface of genus 1. Denote by \mathcal{R} (resp. \mathcal{R}_n) the set of ramified coverings (resp. n -sheeted ramified coverings) $Y \rightarrow X$ such that:

- (i) Y is also a compact⁵ Riemann surface,
- (ii) All ramifications are simple.

If $R = \{x_1, \dots, x_{2k}\}$ is a finite subset of X , denote by $\mathcal{R}_{n,R}$ the set of n -sheeted ramified coverings $Y \rightarrow X$ with ramification locus R . In this case, we say that n is the degree of the ramified covering Y , denoted by $\deg(Y)$ and the Euler characteristics of Y is denoted by $\chi(Y)$.

Definition 6.1. Let $t > 0$ be a fixed positive real number.

- (i) For any $R = \{x_1, \dots, x_{2k}\} \subset X$, we define the measure $\rho_{n,R}$ on $\mathcal{R}_{n,R}$ by

$$\rho_{n,R} = \sum_{Y \in \mathcal{R}_n} \frac{n!}{|\text{Aut}(Y)|} \delta_Y.$$

- (ii) Assume that X is endowed with a smooth volume measure vol such that $\text{vol}(X) = t$, and that Ξ_t is the distribution of a Poisson point process on X with intensity vol , conditioned to only take even values. We define the measure $\rho_{n,t}$ on \mathcal{R}_n by

$$\rho_{n,t}(dY) = \int_{\text{Conf}(X)} \rho_{n,R}(dY) \Xi_t(dR),$$

where $\text{Conf}(X)$ is the set of configurations, i.e. locally finite subsets of X .

- (iii) We define the measure ρ_t on \mathcal{R} by

$$\rho_t(dY) = \sum_{n \geq 0} \mathbf{1}_{\mathcal{R}_n}(Y) \rho_{n,t}(dY).$$

Roughly speaking, ρ_t is the uniform counting measure of isomorphism classes of ramified coverings $Y \rightarrow X$ with simple ramifications, and with a random ramification locus R whose distribution is given by Ξ_t .

⁴We actually rather adapt them, in the sense that Lévy defined the objects when the base space is the unit disk, but everything remains well defined if we replace it by the torus.

⁵Not necessarily connected!

We can now rewrite the chiral partition function as an integral over the space of ramified coverings of the torus, recovering an analog⁶ of the equation (2.27) from [GT93a].

Corollary 6.2. *The chiral partition function on a compact torus of area $t > 0$ admits the following expression.*

$$Z_N^{\text{chir}}(1, t) = \int_{\mathcal{R}} q_t^{\deg(Y)} N^{\chi(Y)} \rho_t(dY). \quad (43)$$

Proof. Let us start by taking the limit $p \rightarrow \infty$ in Theorem 1.4. By Fubini's theorem (all terms are nonnegative),

$$Z_N^{\text{chir}}(1, t) = \sum_{n \geq 0} q_t^n \sum_{k \geq 0} \frac{t^{2k}}{(2k)! N^{2k}} H_1(n, 2k).$$

By the Riemann–Hurwitz formula (36), a ramified covering Y of the torus has genus g if and only if its Euler characteristic equals minus the number of branch points: $2k = 2g - 2 = -\chi(Y)$. Furthermore, from the definition of the Hurwitz numbers, we have for all $k \geq 0$

$$H_1(n, 2k) = \sum_{\substack{Y \in \mathcal{R}_n \\ \chi(Y) = -2k}} \frac{n!}{|\text{Aut}(Y)|},$$

so that

$$Z_N^{\text{chir}}(1, t) = \sum_{n \geq 0} q_t^n \sum_{Y \in \mathcal{R}_n} N^{\chi(Y)} \int_{\text{Conf}(X)} \rho_{n,R}(dY) \Xi_t(dR).$$

The result follows from the fact that $\mathcal{R} = \bigsqcup_n \mathcal{R}_n$. \square

If we combine Theorem 1.2 with Theorem 5.3 and replace the indices $2k$ for $k \geq 0$ by $2g - 2$ for $g \geq 1$, we can also somehow interpret the asymptotic expansions of the full partition function on the torus as a topological expansion. However, the actual coefficients are much more involved, and we are unable to express the partition function as an explicit integral over random ramified coverings as in the chiral case.

Theorem 6.3. *For any $t > 0$, the partition function $Z_N^{\text{YM}}(1, t)$ admits the following expansion for any $p \geq 1$:*

$$Z_N^{\text{YM}}(1, t) = \sum_{g=1}^{p+1} \frac{a_{2g-2}(t)}{N^{2g-2}} + O(N^{-2(p+1)}). \quad (44)$$

The coefficients $a_{2g-2}(t)$ are given by

$$a_{2g-2}(t) = \sum \frac{t^{2g-2} (-1)^{k_2} D^{k_1} \mathcal{F}_{1,g_1}(q_t) D^{k_2} \mathcal{F}_{1,g_2}(q_t) D^{\frac{k_1+k_2}{2}} \theta(q_t)}{(2g_1 - 2)! (2g_2 - 2)! k_1! k_2!}, \quad (45)$$

where the sum is over $g_1, g_2, \geq 1$, $k_1, k_2 \geq 0$ such that $2g_1 - 2 + 2g_2 - 2 + k_1 + k_2 = 2g - 2$.

⁶In their expression, Gross and Taylor actually integrate over a larger set of coverings, which are allowed to have what they call *twists and loops* in addition to the branch points. They also have an additional factor $(-1)^i$ where i denotes the number of branch points of the covering. However, in our case the number of branch points is always even, which make the factor $(-1)^i$ disappear.

Remark 6.4. The leading term in the large N limit is given by $g = 1$, which corresponds to $g_1 = g_2 = 1$ and $k_1 = k_2 = 0$, so that

$$a_0(t) = \mathcal{F}_{1,1}(q_t)^2 \theta(q_t) = \frac{\theta(q_t)}{\phi(q_t)^2}.$$

The limit (7) is again recovered. The next order term, corresponding to $g = 2$, is already much more involved, and is obtained by adding the terms corresponding to the following values of (g_1, g_2, k_1, k_2) : $(2, 1, 0, 0)$, $(1, 2, 0, 0)$, $(1, 1, 2, 0)$, $(1, 1, 0, 2)$ and $(1, 1, 1, 1)$. We get

$$a_2(t) = t^2 \mathcal{F}_{1,2}(q_t) \mathcal{F}_{1,1}(q_t) \theta(q_t) + t^2 \mathcal{F}_{1,1}(q_t) D^2 \mathcal{F}_{1,1}(q_t) D\theta(q_t) - t^2 (D\mathcal{F}_{1,1}(q_t))^2 D\theta(q_t).$$

There might be an algorithmic procedure for iteratively compute the coefficients a_{2g-2} for higher g , using topological recursion techniques in the spirit of [LMS18].

APPENDIX A. DOMINATED CONVERGENCE ON COUNTABLE MEASURED SPACES

Lemma A.1. *Let X be a countable set, endowed with the σ -algebra $\mathcal{P}(X)$ of all its subsets, and let $\mu : \mathcal{P}(X) \rightarrow [0, \infty]$ be its counting measure. Let $(f_n)_{n \geq 1}$ be a sequence of integrable functions on X that converges pointwise on X . Then, it converges in $L^1(\mu)$ if and only if for any $\varepsilon > 0$, there exists a measurable subset X_ε such that $\mu(X_\varepsilon) < \infty$ and*

$$\sup_n \int_{X \setminus X_\varepsilon} |f_n| d\mu < \varepsilon. \quad (46)$$

Proof. According to [Bog07, Corollary 4.5.5], which applies on any measured space with a positive measure, the convergence of (f_n) in $L^1(\mu)$ holds if and only if (46) holds true and

$$\lim_{\mu(E) \rightarrow 0} \sup_n \int_E |f_n| d\mu = 0.$$

If we prove that the latter holds for any countable measured space with its counting measure, then the lemma will follow.

For any $\varepsilon > 0$, let $\delta = \frac{1}{2}$. For any $E \subset X$ such that $\mu(E) \leq \delta$, we have $E = \emptyset$ and it is clear that

$$\sup_n \int_{\emptyset} |f_n| d\mu = 0 < \varepsilon.$$

Hence, the condition is satisfied. \square

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