



HAL
open science

p -adic Fourier theory for \mathbb{Q}_p^2 and the Monna map

Konstantin Ardakov, Laurent Berger

► **To cite this version:**

Konstantin Ardakov, Laurent Berger. p -adic Fourier theory for \mathbb{Q}_p^2 and the Monna map. 2024.
hal-04572755

HAL Id: hal-04572755

<https://hal.science/hal-04572755>

Preprint submitted on 12 May 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

p -ADIC FOURIER THEORY FOR \mathbf{Q}_{p^2} AND THE MONNA MAP

KONSTANTIN ARDAKOV AND LAURENT BERGER

ABSTRACT. We show that the coefficients of a power series occurring in p -adic Fourier theory for \mathbf{Q}_{p^2} have valuations that are given by an intriguing formula.

INTRODUCTION

Let L be a finite extension of \mathbf{Q}_p , let π be a uniformizer of o_L and let LT be the Lubin-Tate formal o_L -module attached to π . The formal group maps over $o_{\mathbf{C}_p}$ from LT to \mathbf{G}_m play an important role in p -adic Fourier theory (see [ST01]). Choose a coordinate Z on LT , and let $G(Z) \in o_{\mathbf{C}_p}[[Z]]$ be a generator of $\text{Hom}_{o_{\mathbf{C}_p}}(\text{LT}, \mathbf{G}_m)$, so that

$$G(Z) = \sum_{k \geq 1} P_k(\Omega) \cdot Z^k = \exp(\Omega \cdot \log_{\text{LT}}(Z)) - 1$$

for a certain element $\Omega \in o_{\mathbf{C}_p}$ and polynomials $P_k(Y) \in L[Y]$. We have (§3 of [ST01]) $\text{val}_p(\Omega) = 1/(p-1) - 1/e(q-1)$ where e is the ramification index of L and $q = |o_L/\pi o_L|$. The power series $G(Z)$ gives rise to a function on $\mathfrak{m}_{\mathbf{C}_p}$ and the theory of Newton polygons then allows us to compute the valuation of $P_k(\Omega)$ for $k = q^j/p^{\lfloor (j-1)/e \rfloor + 1}$ with $j \geq 0$ (Theorem 1.5.2 of [AB24]). However, the valuation of $P_k(\Omega)$ for most $k \geq 2$ has no geometric significance and depends on the choice of the coordinate Z .

During our work on the character variety, we computed the valuation of $P_k(\Omega)$ for many small values of k in a special case: we took $L = \mathbf{Q}_{p^2}$ and $\pi = p$ and chose a coordinate Z on LT for which $\log_{\text{LT}}(Z) = \sum_{m \geq 0} Z^{q^m}/p^m$ (this is possible by §8.3 of [Haz12]). Note that in this setting, the theory of Newton polygons gives $\text{val}_p(P_k(\Omega))$ precisely when k is a power of p . Let $w : \mathbf{Z}_{\geq 0} \rightarrow \mathbf{Q}$ be the map defined by

$$w(k) = \frac{p}{q-1} \cdot (k_0 + p^{-1}k_1 + \cdots + p^{-h} \cdot k_h) \text{ if } k = (k_h \cdots k_0)_p \text{ in base } p.$$

For all k for which we were able to compute $\text{val}_p(P_k(\Omega))$, we found that $\text{val}_p(P_k(\Omega)) = w(k)$. The main result of this note is that this formula holds for all k .

Theorem A. *For all $k \geq 1$, we have $\text{val}_p(P_k(\Omega)) = w(k)$.*

The proof involves a careful study of the functional equation that $G(Z)$ satisfies, and a direct computation of $\text{val}_p(P_k(\Omega))$ for small values of k . The function w is related to the Monna map, defined in [Mon52].

1. THE POLYNOMIALS $P_m(Y)$

Let $L = \mathbf{Q}_{p^2}$ and $\pi = p$, so that $q = p^2$, and choose a coordinate Z on LT for which $\log_{\text{LT}}(Z) = \sum_{k \geq 0} Z^{q^k} / p^k$. The polynomials $P_m(Y) \in L[Y]$ are given by

$$\exp(Y \cdot \log_{\text{LT}}(Z)) = \sum_{m=0}^{+\infty} P_m(Y) \cdot Z^m.$$

Proposition 1.1. *We have*

$$P_m(Y) = \sum_{m_0 + qm_1 + \dots + q^d m_d = m} \frac{Y^{m_0 + \dots + m_d}}{m_0! \cdots m_d! \cdot p^{1 \cdot m_1 + 2 \cdot m_2 + \dots + d \cdot m_d}}$$

Proof. Since $\log_{\text{LT}}(Z) = \sum_{k \geq 0} Z^{q^k} / p^k$ and \exp is the usual exponential,

$$\sum_{m=0}^{+\infty} P_m(Y) Z^m = \exp(Y \cdot \log_{\text{LT}}(Z)) = \prod_{k \geq 0} \exp(Y \cdot Z^{q^k} / p^k) = \prod_{k \geq 0} \sum_{j \geq 0} (Y \cdot Z^{q^k} / p^k)^j / j!$$

The coefficient of Z^m is the sum of $Y^{m_0 + \dots + m_d} / m_0! \cdots m_d! \cdot p^{1 \cdot m_1 + 2 \cdot m_2 + \dots + d \cdot m_d}$ over all $d \geq 0$ and $(m_0, \dots, m_d) \in \mathbf{Z}_{\geq 0}^{d+1}$ such that $m_0 + qm_1 + \dots + q^d m_d = m$. \square

For example, if $i \leq q - 1$, then

$$\begin{aligned} P_i(Y) &= Y^i / i! \\ P_{q+i}(Y) &= \frac{Y^{q+i}}{(q+i)!} + \frac{Y^{i+1}}{p \cdot i!} \\ P_{2q+i}(Y) &= \frac{Y^{2q+i}}{(2q+i)!} + \frac{Y^{q+i+1}}{p \cdot (q+i)!} + \frac{Y^{i+2}}{2p^2 \cdot i!}. \end{aligned}$$

Because $L = \mathbf{Q}_{p^2}$, it follows from Lemma 3.4.b of [ST01] that

$$\text{val}_p(\Omega) = \frac{1}{p-1} - \frac{1}{e(q-1)} = \frac{p}{q-1}.$$

Lemma 1.2. *If $i \leq q - 1$ and $i = (ab)_p$ in base p , then $\text{val}_p(P_i(\Omega)) = \frac{a+bp}{q-1} = w(i)$.*

Proof. If $i \leq q - 1$, then $P_i(\Omega) = \Omega^i / i!$ by Proposition 1.1, so that

$$\text{val}_p(P_i(\Omega)) = i \cdot \left(\frac{1}{p-1} - \frac{1}{q-1} \right) - \frac{i - s_p(i)}{p-1} = \frac{s_p(i)}{p-1} - \frac{i}{q-1} = \frac{a+bp}{q-1}. \quad \square$$

2. THE MAP w

Recall that $w : \mathbf{Z}_{\geq 0} \rightarrow \mathbf{Q}$ is the map defined by

$$w(k) = \frac{p}{q-1} \cdot (k_0 + p^{-1}k_1 + \dots + p^{-h} \cdot k_h) \text{ if } k = (k_h \cdots k_0)_p \text{ in base } p.$$

Proposition 2.1. *The function $w : \mathbf{Z}_{\geq 0} \rightarrow \mathbf{Q}_{\geq 0}$ has the following properties:*

- (1) $w(k) < 1 + 1/(q-1)$;
- (2) $w(k) \geq 1$ if and only if $k \equiv -1 \pmod{q}$, and then $w(k) > 1$ unless $k = q - 1$;

- (3) if $\ell > k$, then $w(\ell) - w(k) \in \mathbf{Z}$ if and only if $k = qj$ and $\ell = qj + (q - 1)$;
- (4) $w(pk) = 1/p \cdot w(k)$;
- (5) $w(p^n k + i) = w(p^n k) + w(i)$ if $0 \leq i \leq p^n - 1$;
- (6) For all $a, b \geq 0$ we have $w(a + b) \leq w(a) + w(b)$.

Proof. Item (1) results from the fact that

$$w(k) = (k_0 + p^{-1}k_1 + \cdots + p^{-h} \cdot k_h) \cdot \frac{p}{q-1} < \frac{p^2}{q-1} = 1 + \frac{1}{q-1}.$$

If $k_0 \leq p-2$, or if $k_0 = p-1$ and $k_1 \leq p-2$, then $w(k) \leq (p^{h+1} - 1 - p^{h-1})/p^{h-1}(q-1) < 1$, so if $w(k) \geq 1$, then $k_0 = p-1$ and $k_1 = p-1$, and $k \equiv -1 \pmod{q}$. Conversely, if $k \equiv -1 \pmod{q}$, then $k_0 = p-1$ and $k_1 = p-1$, and $w(k) \geq 1$. Finally, if we have equality, then $k_i = 0$ for all $i \geq 2$. This proves (2).

Write $k = (k_h \cdots k_0)_p$ and $\ell = (\ell_i \cdots \ell_0)_p$. Since $w(k) < 1 + 1/(q-1)$, if $w(\ell) - w(k) \in \mathbf{Z}_{\geq 0}$, then $w(\ell) = w(k)$ or $w(\ell) = w(k) + 1$. If $w(\ell) = w(k)$, then $k_0 + p^{-1}k_1 + \cdots + p^{-h} \cdot k_h = \ell_0 + p^{-1}\ell_1 + \cdots + p^{-i} \cdot \ell_i$. By comparing p -adic valuations, we get $h = i$, and then $k_h \equiv \ell_i \pmod{p}$ so that $k_h = \ell_i$. By descending induction, $k_j = \ell_j$ for all j , and $k = \ell$. If $w(\ell) = w(k) + 1$, then $w(\ell) \geq 1$, and hence $\ell = (\ell_i \cdots \ell_2(p-1)_1(p-1)_0)_p$ by item (2). We then have $w((\ell_i \cdots \ell_2 0_1 0_0)_p) = w(k)$ and hence $k = (\ell_i \cdots \ell_2 0_1 0_0)_p$. This implies (3).

Items (4) and (5) are straightforward. For item (6), let $\{a_i\}$, $\{b_i\}$ and $\{c_i\}$ be the digits of a , b and c in base p . Let $r_0 = 0$ and let $r_i \in \{0, 1\}$ be the i th carry when adding a and b , so that $c_i = a_i + b_i + r_i - pr_{i+1}$. The result follows from the following computation.

$$\sum_{i \geq 0} \frac{c_i}{p^i} = \sum_{i \geq 0} \frac{a_i + b_i}{p^i} + \frac{r_i}{p^i} - \frac{pr_{i+1}}{p^i} = \sum_{i \geq 0} \frac{a_i + b_i}{p^i} - (p^2 - 1) \sum_{i \geq 1} \frac{r_i}{p^i} \leq \sum_{i \geq 0} \frac{a_i + b_i}{p^i}. \quad \square$$

3. CONGRUENCES FOR THE $P_k(\Omega)$

From now on, we write u_k for $P_k(\Omega)$ to lighten the notation. Recall that $q = p^2$. The power series $G(Z)$ is a map between LT and \mathbf{G}_m , so that $G([p]_{\text{LT}}(Z)) = [p]_{\mathbf{G}_m}(G(Z))$.

Proposition 3.1. *We have $\sum_{m=1}^{+\infty} u_m Z^{qm} \equiv \sum_{k=1}^{+\infty} u_k^p Z^{kp} \pmod{p \cdot \mathbf{m}_{\mathbf{C}_p}}$.*

Proof. We have $G(Z) \in \mathbf{m}_{\mathbf{C}_p}[[Z]]$ and $[p]_{\text{LT}}(Z) \equiv Z^q \pmod{p}$ and $[p]_{\mathbf{G}_m}(Z) = Z^p \pmod{p}$.

Since $G([p]_{\text{LT}}(Z)) = [p]_{\mathbf{G}_m}(G(Z))$, we get $G(Z^q) \equiv G(Z)^p \pmod{p \cdot \mathbf{m}_{\mathbf{C}_p}}$. \square

Corollary 3.2. *If k is not divisible by p , then $\text{val}_p(u_k) > 1/p$.*

Corollary 3.3. *We have $u_{pm}^p \equiv u_m \pmod{p \cdot \mathbf{m}_{\mathbf{C}_p}}$.*

Proof. Take $k = pm$ in Proposition 3.1. \square

Corollary 3.4. *Take $m \geq 0$.*

- (1) *Suppose that $\text{val}_p(u_m) \leq 1$. Then $\text{val}_p(u_{pm}) = 1/p \cdot \text{val}_p(u_m)$.*

(2) Suppose that $\text{val}_p(u_m) > 1$. Then $\text{val}_p(u_{pm}) > 1/p$.

Proof. Both cases follow easily from Corollary 3.3. \square

We now compare $[p]_{\text{LT}}(Z)$ and $Z^q + pZ$ (compare with (iv) of §2.2 of [Haz12]).

Lemma 3.5. *We have $[p]_{\text{LT}}(Z) = Z^q + pZ + p^2 \cdot s(Z)$ for some $s(Z) \in Z^2 \cdot \mathbf{Z}_p[[Z]]$.*

Proof. There exists $r(Z) \in Z^2 \cdot \mathbf{Z}_p[[Z]]$ such that $[p]_{\text{LT}}(Z) = Z^q + pZ + pr(Z)$. By the properties of \log_{LT} , we have $\log_{\text{LT}}([p]_{\text{LT}}(Z)) = p \log_{\text{LT}}(Z)$. Expanding around Z^q , we get $\log_{\text{LT}}(Z^q + pZ + pr(Z)) = \log_{\text{LT}}(Z^q) + (pZ + pr(Z)) \log'_{\text{LT}}(Z^q) + \sum_{i \geq 2} \frac{(pZ + pr(Z))^i}{i!} \log_{\text{LT}}^{(i)}(Z^q)$

Our choice of \log_{LT} is such that $\log_{\text{LT}}(Z^q) = p \log_{\text{LT}}(Z) - pZ$ and $\log'_{\text{LT}}(Z) \in 1 + pZ \cdot \mathbf{Z}_p[[Z]]$ and $\log_{\text{LT}}^{(i)}(Z) \in p\mathbf{Z}_p[[Z]]$ for all $i \geq 2$. Note also that $p^{i+1}/i! \in p^2\mathbf{Z}_p$ for all $i \geq 2$.

The above equation now implies that $pr(Z) \equiv 0 \pmod{p^2}$ so that $r(Z) = ps(Z)$. \square

Corollary 3.6. *The coefficient of Z^{qn} in $G([p]_{\text{LT}}(Z))$ is congruent to $u_n \pmod{p^2}$.*

Proof. Since $[p]_{\text{LT}}(Z) \equiv Z^q + pZ \pmod{p^2}$, Lemma 3.5 tells us that

$$\begin{aligned} G([p]_{\text{LT}}(Z)) &\equiv G(Z^q) + pZ \cdot G'(Z^q) \pmod{p^2} \\ &\equiv \sum_{k \geq 1} u_k Z^{qk} + \sum_{m \geq 1} pm \cdot u_m Z^{q(m-1)+1} \pmod{p^2}. \end{aligned}$$

Hence $pZ \cdot G'(Z^q)$ doesn't contribute to the coefficient of Z^{qn} modulo p^2 . \square

Proposition 3.7. *For all $k \geq 1$, we have $k \cdot u_k = u_1 \cdot \sum_{r=0}^{\lfloor \log_q(k) \rfloor} p^r u_{k-q^r}$.*

Proof. We have $\sum_{k \geq 0} u_k Z^k = \exp(u_1 \cdot \log_{\text{LT}}(Z))$. Applying d/dZ , we get

$$\begin{aligned} \sum_{k \geq 1} k u_k Z^{k-1} &= \exp(u_1 \cdot \log_{\text{LT}}(Z)) \cdot u_1 \cdot \log'_{\text{LT}}(Z) \\ &= u_1 \cdot \left(\sum_{i \geq 0} u_i Z^i \right) \cdot \left(\sum_{r \geq 0} (q/p)^r Z^{q^r-1} \right). \end{aligned}$$

The result follows from looking at the coefficient of Z^{k-1} on both sides. \square

Corollary 3.8. *We have $u_1 \cdot u_{k-1} \equiv k u_k \pmod{p}$ for all $k \geq 1$.*

Proposition 3.9. *If $0 \leq i \leq p-1$ and $m \geq p$, then there exists $\zeta_{i,m} \in o_L$ such that*

$$u_{mp+i} \equiv \binom{mp+i}{i}^{-1} \cdot u_{mp} \cdot u_i + p \cdot \zeta_{i,m} \cdot u_{p(m-p)+i+1} \pmod{p^2}.$$

Proof. We proceed by induction on i . When $i = 0$, we can even achieve equality by setting $\zeta_{0,m} := 0$, because $u_0 = 1$. Write $k := mp + i$ for brevity. For $i \geq 1$ we have

$$u_k \equiv \frac{1}{k} u_1 \cdot u_{k-1} + \frac{p}{k} u_1 \cdot u_{k-q} \pmod{p^2}$$

by Proposition 3.7, because here $k \in o_L^\times$. By the inductive hypothesis, we have

$$u_{k-1} \equiv \binom{k-1}{i-1}^{-1} u_{mp} \cdot u_{i-1} + p\zeta_{i-1,m} \cdot u_{k-q} \pmod{p^2}.$$

Note that since $i \leq p-1$, we have $u_i = u_1^i/i!$ by Proposition 1.1, so $u_1 u_{i-1} = \frac{u_1^i}{(i-1)!} = i u_i$. Substituting this information, we obtain

$$\begin{aligned} u_k &\equiv \frac{u_1}{k} \cdot \left(\binom{k-1}{i-1}^{-1} u_{mp} \cdot u_{i-1} + p\zeta_{i-1,m} u_{k-q} \right) + \frac{p}{k} u_1 \cdot u_{k-q} \\ &\equiv \frac{i}{k} \binom{k-1}{i-1}^{-1} u_{mp} \cdot u_i + \frac{p}{k} (\zeta_{i-1,m} + 1) u_1 \cdot u_{k-q} \pmod{p^2}. \end{aligned}$$

On the other hand, by Corollary 3.8, we have

$$p u_1 \cdot u_{k-q} \equiv p(k-q+1) u_{k-q+1} \pmod{p^2}.$$

Hence we can rewrite the congruence as follows:

$$u_k \equiv \binom{k}{i}^{-1} u_{mp} \cdot u_i + p \frac{k-q+1}{k} (\zeta_{i-1,m} + 1) u_{k-q+1} \pmod{p^2}.$$

Define $\zeta_{i,m} := \frac{k-q+1}{k} (\zeta_{i-1,m} + 1)$ and observe that this lies in o_L because $p \nmid k$. □

We need to know what $\zeta_{p-1,m}$ is modulo p .

Lemma 3.10. *Take $1 \leq i \leq p-1$ and $m \geq 0$ and let $k = mp + i$.*

If $\zeta_{0,m} = 0$ and $\zeta_{i,m} = \frac{k-q+1}{k} (\zeta_{i-1,m} + 1)$ whenever $1 \leq i \leq p-1$, then $\zeta_{p-1,m} \equiv 0 \pmod{p}$.

Proof. Note that modulo p , the recurrence relation satisfied by $\zeta_{i,m}$ is simply

$$\zeta_{i,m} \equiv \frac{i+1}{i} (\zeta_{i-1,m} + 1) \pmod{p}.$$

Now set $i = p-1$ to see that $\zeta_{p-1,m} \equiv 0 \pmod{p}$. □

4. PROOF OF THEOREM A

We now use the functional equation of $G(Z)$ modulo p^2 in order to prove Theorem A.

Definition 4.1. *For each $n \geq 0$, let C_n be the coefficient of Z^{qn} in*

$$(1 + G(Z))^p = \left(\sum_{k=0}^{\infty} u_k Z^k \right)^p.$$

We develop some notation to compute C_n .

Definition 4.2.

- (1) Let $|\mathbf{k}| := k_1 + \cdots + k_p$ for all $\mathbf{k} \in \mathbf{N}^p$.
- (2) For each $\mathbf{k} \in \mathbf{N}^p$, define $u_{\mathbf{k}} := u_{k_1} \cdot u_{k_2} \cdots u_{k_p}$.
- (3) For each $n \geq 0$, let $X_n \subset \mathbf{N}^p$ be a complete set of representatives for the orbits of the natural action of S_p on $\{\mathbf{k} \in \mathbf{N}^p : |\mathbf{k}| = n\}$.

In this language, expanding $(\sum_{k=0}^{\infty} u_k Z^k)^p$ gives the following

Lemma 4.3. *We have $C_n = \sum_{\mathbf{k} \in X_{qn}} |S_p \cdot \mathbf{k}| u_{\mathbf{k}}$.*

Lemma 4.4. *We have $\text{val}_p(|S_p \cdot \mathbf{k}|) = 1$ whenever $k_i \neq k_j$ for some $i \neq j$.*

Proof. Let H be the stabiliser of \mathbf{k} in S_p , so that $|S_p \cdot \mathbf{k}| = |S_p|/|H|$. If $k_i \neq k_j$ for some $i \neq j$, then H cannot contain any p -cycle. The only elements of S_p of order p are p -cycles, so by Cauchy's Theorem, $\text{val}_p(|H|) = 0$. Hence $\text{val}_p(|S_p|/|H|) = \text{val}_p(|S_p|) = 1$. \square

Lemma 4.5. *If $\mathbf{k} \in X_{qn} \setminus q\mathbf{N}^p$, then $\text{val}_p(u_{\mathbf{k}}) > w(n) - 1$.*

Proof. Since $\frac{1}{q-1} > w(n) - 1$ by Proposition 2.1(1), it is enough to show that

$$\text{val}_p(u_{\mathbf{k}}) > \frac{1}{q-1}.$$

If some k_i is not divisible by p , then by Corollary 3.2,

$$\text{val}_p(u_{\mathbf{k}}) \geq \text{val}_p(u_{k_i}) > \frac{1}{p} > \frac{1}{q-1}.$$

Assume now that for each $i = 1, \dots, p$, we can write $k_i = pm_i$ for some $m_i \geq 0$ so that $|\mathbf{m}| = \frac{1}{p}|\mathbf{k}| = pn$. Since $\mathbf{k} \notin q\mathbf{N}^p$ by assumption, we must have $m_i \not\equiv 0 \pmod{p}$ for some i . Because $|\mathbf{m}| = np \equiv 0 \pmod{p}$, in this case there must be at least two distinct indices i, j such that $m_i \not\equiv 0 \pmod{p}$ and $m_j \not\equiv 0 \pmod{p}$. Using Corollary 3.2 again, we obtain

$$\text{val}_p(u_{\mathbf{m}}) \geq \text{val}_p(u_{m_i}) + \text{val}_p(u_{m_j}) \geq \frac{2}{p} > \frac{p}{q-1}.$$

Suppose now that $\text{val}_p(u_{m_i}) \leq 1$ for all i . Then Corollary 3.4(1) implies that

$$\text{val}_p(u_{\mathbf{k}}) = \frac{1}{p} \text{val}_p(u_{\mathbf{m}}) > \frac{1}{p} \cdot \frac{p}{q-1} = \frac{1}{q-1}.$$

Otherwise, for at least one index i we have $\text{val}_p(u_{m_i}) > 1$, and then Corollary 3.4(2) gives

$$\text{val}_p(u_{\mathbf{k}}) \geq \text{val}_p(u_{k_i}) > \frac{1}{p} > \frac{1}{q-1}. \quad \square$$

We can now prove Theorem A.

Theorem 4.6. *We have $\text{val}_p(u_n) = w(n)$ for all $n \geq 0$.*

Proof. We prove the stronger statement $\text{val}_p(u_n) = w(n) = p \cdot \text{val}_p(u_{pn})$ by induction on n . The base case $n = 0$ is clear, so assume $n \geq 1$. We first show that $\text{val}_p(u_n) = w(n)$.

Write $n = mp + i$ with $0 \leq i \leq p-1$. Then $\text{val}_p(u_i) = w(i)$ holds by Lemma 1.2. Since $n \neq 0$, we must have $m < n$ so $\text{val}_p(u_{mp}) = \frac{1}{p}w(m)$ by the inductive hypothesis. Using (4) and (5) of Proposition 2.1, we see that

$$\text{val}_p(u_i u_{mp}) = \text{val}_p(u_i) + \text{val}_p(u_{mp}) = w(i) + \frac{1}{p}w(m) = w(pm + i) = w(n).$$

Suppose first that $n \not\equiv -1 \pmod{q}$. Then $w(n) < 1$ by Proposition 2.1(2), which means that $\text{val}_p(u_i u_{mp}) = w(n) < 1$. By Proposition 3.9, we have

$$u_n \equiv \binom{mp+i}{i}^{-1} u_i u_{mp} \pmod{p}.$$

We have $\binom{mp+i}{i} \equiv 1 \pmod{p}$ by Lucas' theorem, and therefore $\text{val}_p(u_n) = w(n)$.

Suppose now that $n \equiv -1 \pmod{q}$. Then $i = p-1$, and Proposition 3.9 tells us that

$$u_n \equiv \binom{n}{p-1}^{-1} u_{mp} \cdot u_{p-1} + p\zeta_{p-1,m} \cdot u_{n-q+1} \pmod{p^2}.$$

We have $\zeta_{p-1,m} \equiv 0 \pmod{p}$ by Lemma 3.10. Hence in fact $u_n \equiv \binom{n}{p-1}^{-1} u_{mp} u_{p-1} \pmod{p^2}$. Since $\text{val}_p(u_{mp} u_{p-1}) = w(n) < 2$ by Proposition 2.1(1), we again conclude that

$$\text{val}_p(u_n) = \text{val}_p(u_{mp}) + \text{val}_p(u_{p-1}) = w(n).$$

To complete the induction step, we must show that $w(n) = p \text{val}_p(u_{pn}) = \text{val}_p(u_{pn}^p)$. In order to do this, we compare the coefficients of Z^{qn} in the functional equation for $G(Z)$

$$G([p]_{\text{LT}}(Z)) = [p]_{\mathbf{G}_m}(G(Z)) = (1 + G(Z))^p - 1$$

modulo p^2 . Using Corollary 3.6 and Lemma 4.3, we see that

$$(\diamond) \quad u_n \equiv C_n = \sum_{\mathbf{k} \in X_{qn}} |S_p \cdot \mathbf{k}| u_{\mathbf{k}} \pmod{p^2}.$$

Define $\mathbf{k}_0 := (pn, pn, \dots, pn)$. We will now proceed to show that in fact

$$(\star) \quad \text{val}_p(|S_p \cdot \mathbf{k}| u_{\mathbf{k}}) > w(n) \quad \text{for all } \mathbf{k} \in X_{qn} \setminus \{\mathbf{k}_0\}.$$

Note that $w(n) < 2$ by Proposition 2.1(1) and that $u_{\mathbf{k}_0} = u_{pn}^p$. Hence congruence (\diamond) together with (\star) imply that $\text{val}_p(u_n - u_{pn}^p) > w(n)$. Since we already know that $\text{val}_p(u_n) = w(n)$ this shows that $\text{val}_p(u_{pn}^p) = \text{val}_p(u_n) = w(n)$ and completes the proof.

Since at least two entries of \mathbf{k} must be distinct when $\mathbf{k} \neq \mathbf{k}_0$, we have $\text{val}_p(|S_p \cdot \mathbf{k}|) = 1$ by Lemma 4.4, so we're reduced to showing that

$$(\star\star) \quad \text{val}_p(u_{\mathbf{k}}) > w(n) - 1 \quad \text{for all } \mathbf{k} \in X_{qn} \setminus \{\mathbf{k}_0\}.$$

Fix $\mathbf{k} \in X_{qn} \setminus \{\mathbf{k}_0\}$. When $\mathbf{k} \notin q\mathbf{N}^p$, $(\star\star)$ is precisely the conclusion of Lemma 4.5, so we may assume that $\mathbf{k} \in q\mathbf{N}^p$. Write $\mathbf{k} = q\mathbf{m}$ for some $\mathbf{m} \in \mathbf{N}^p$, so that $|\mathbf{m}| = \frac{1}{q}|\mathbf{k}| = \frac{qn}{q} = n$. We first consider the case where $m_i < n$ for all i , so that by the inductive hypothesis we have $\text{val}_p(u_{pm_i}) = w(m_i)/p$. Suppose that $\text{val}_p(u_{pm_i}) > 1$ for some i . Then by Corollary 3.4(2) and Proposition 2.1(1),

$$\text{val}_p(u_{\mathbf{k}}) \geq \text{val}_p(u_{k_i}) = \text{val}_p(u_{qm_i}) > \frac{1}{p} > \frac{1}{q-1} > w(n) - 1$$

and $(\star\star)$ holds. Otherwise, $\text{val}_p(u_{pm_i}) \leq 1$ for all i and then by Corollary 3.4(1) we have

$$\text{val}_p(u_{k_i}) = \text{val}_p(u_{qm_i}) = \frac{1}{p} \text{val}_p(u_{pm_i}) = \frac{1}{q} w(m_i).$$

Since $|\mathbf{m}| = n$, Proposition 2.1(6) gives

$$\mathrm{val}_p(u_{\mathbf{k}}) \geq \frac{1}{q} \sum w(m_i) \geq \frac{1}{q} \cdot w(n) > w(n) - 1$$

because $w(n) < 1 + 1/(q-1)$ by Proposition 2.1(1). Hence $(\star\star)$ follows.

We're left with the case where at least one m_i is equal to n . But then since $|\mathbf{m}| = n$, all other m_j 's are zero and such \mathbf{m} 's form a single S_p -orbit of size p . Hence we have to show $(\star\star)$ holds when $\mathbf{k} = (0, 0, \dots, qn)$.

The congruence (\diamond) together with our estimates above implies

$$\mathrm{val}_p(u_n - (u_{np}^p + pu_{nq})) > w(n).$$

Now, $u_{np} \equiv u_{nq}^p \pmod{p}$ by Corollary 3.3 so that $u_{np}^p \equiv u_{nq}^q \pmod{p^2}$. Therefore

$$\mathrm{val}_p(u_n - (u_{nq}^q + pu_{nq})) > w(n).$$

Since we already know that $\mathrm{val}_p(u_n) = w(n)$, we get that

$$\mathrm{val}_p(u_{nq}^q + pu_{nq}) = w(n).$$

We will now see that $\mathrm{val}_p(pu_{nq}) \leq w(n)$ is not possible. Indeed, if $\mathrm{val}_p(pu_{nq}) = w(n)$, then $\mathrm{val}_p(u_{nq}^q) \geq w(n)$ so that $\mathrm{val}_p(u_{nq}) \geq w(n)/q$ and $\mathrm{val}_p(pu_{nq}) \geq 1 + w(n)/q > w(n)$. And if $\mathrm{val}_p(pu_{nq}) < w(n)$ then $\mathrm{val}_p(pu_{nq}) = \mathrm{val}_p(u_{nq}^q)$, so $\mathrm{val}_p(u_{nq}) = 1/(q-1)$. But then $\mathrm{val}_p(pu_{nq}) > 1 + 1/(q-1) > w(n)$ by Proposition 2.1(1).

Hence $\mathrm{val}_p(pu_{nq}) > w(n)$ after all, which is $(\star\star)$ for $\mathbf{k} = (0, 0, \dots, 0, qn)$. \square

REFERENCES

- [AB24] K. Ardakov and L. Berger, *Bounded functions on the character variety*, Münster J. Math. (2024), to appear.
- [Haz12] M. Hazewinkel, *Formal groups and applications*, AMS Chelsea Publishing, Providence, RI, 2012, Corrected reprint of the 1978 original.
- [Mon52] A. F. Monna, *Sur une transformation simple des nombres p -adiques en nombres réels*, Indag. Math. **14** (1952), 1–9, Nederl. Akad. Wetensch. Proc. Ser. A **55**.
- [ST01] P. Schneider and J. Teitelbaum, *p -adic Fourier theory*, Doc. Math. **6** (2001), 447–481.

KONSTANTIN ARDAKOV, MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD

Email address: `ardakov@maths.ox.ac.uk`

URL: `http://people.maths.ox.ac.uk/ardakov/`

LAURENT BERGER, UMPA, ENS DE LYON, UMR 5669 DU CNRS

Email address: `laurent.berger@ens-lyon.fr`

URL: `https://perso.ens-lyon.fr/laurent.berger/`