VANISHING VISCOSITY VERSUS ROSENAU APPROXIMATION FOR SCALAR CONSERVATION LAWS: THE FRACTIONAL CASE

Nathaël Alibaud, Giuseppe Maria Coclite, Maxime Dalery, Carlotta Donadello

VANISHING VISCOSITY VERSUS ROSENAU APPROXIMATION FOR SCALAR CONSERVATION LAWS: THE FRACTIONAL CASE

N. ALIBAUD, G. M. COCLITE, M. DALERY, AND C. DONADELLO

Abstract. We consider approximations of scalar conservation laws by adding nonlocal diffusive operators. In particular, we consider solutions associated to fractional Laplacian and fractional Rosenau perturbations and show that, for any $t > 0$, the mutual L^1 distance of their profiles is negligible as compared to their common distance to the underlying inviscid entropy solution. We provide explicit examples showing that our rates are optimal in the supercritical and critical cases, in one space dimension and for strictly convex fluxes. For subcritical equations, our rates are not optimal but they remain explicit.

1. Introduction

The classical approach in the analysis of the Cauchy problem for multi-dimensional scalar conservation law,

$$\begin{cases} \partial_t w + \operatorname{div} f(w) = 0, & t \in \mathbb{R}^+,\ x \in \mathbb{R}^d, \\ w(0,x) = w_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where the divergence is taken with respect to the space variable x, the flux $f \in W^{1,\infty}_{loc}(\mathbb{R},\mathbb{R}^d)$ and the initial condition $w_0 \in L^\infty(\mathbb{R}^d)$, relies on singular approximations as the vanishing viscosity and the relaxation methods, see [18] and references therein.

The well-posedness of (1.1) was established in [31] in the framework of entropy solutions, that are weak solutions which satisfy additional selection criteria called entropy inequalities. The unique entropy solution coincides with the strong L^p limit as $\varepsilon \to 0$ of vanishing viscosity regularized solutions, which solve

$$\begin{cases} \partial_t w_\varepsilon + \operatorname{div} f(w_\varepsilon) = \varepsilon \Delta w_\varepsilon, & t \in \mathbb{R}^+,\ x \in \mathbb{R}^d, \\ w(0,x) = w_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

with the Laplacian in space. This regularization procedure is the most classical, but other ones have been considered. In [36], Rosenau proposed a regularized version of the Chapman-Enskog expansion of hydrodynamics, which can be rewritten as the following perturbation of (1.1):

$$\partial_t \mathfrak{v}_\varepsilon + \operatorname{div} f(\mathfrak{v}_\varepsilon) = -\varepsilon \mathcal{F}^{-1}\left(\frac{|\cdot|^2}{1+\varepsilon^2|\cdot|^2}\, \mathcal{F}(\mathfrak{v}_\varepsilon) \right). \tag{1.3}$$

Here and throughout, the space Fourier transform and its inverse are respectively denoted by

$$\mathcal{F}(\phi)(\xi) = \int_{\mathbb{R}^d} \phi(x) e^{-ix\cdot\xi}\, dx \qquad \text{and} \qquad \mathcal{F}^{-1}(\phi)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \phi(\xi) e^{ix\cdot\xi}\, d\xi. \tag{1.4}$$

Since the diffusion in (1.2) can be rewritten as

$$\Delta \phi = -\mathcal{F}^{-1}\left(|\cdot|^2 \mathcal{F}(\phi) \right),$$

Date: September 19, 2024.

2010 *Mathematics Subject Classification.* 35L65, 35R11, 35B25.

Key words and phrases. Conservation laws, vanishing viscosity, Lévy diffusion operators, fractional Laplacian operator, fractional Rosenau operator.

the right hand side of (1.3) resembles the one in (1.2) at low wave numbers ξ, but is intended to model a bounded approximation of a linearized collision operator for higher ξ. As explained in [36, 39], this avoids artificial instabilities occurring when the Chapman-Enskog expansion for such an operator is truncated after a finite number of terms. The model retains the essential properties of the usual viscosity approximation, e.g., existence of travelling waves, monotonicity, upper-Lipschitz continuity, etc., and it sharpens the standard viscous shock layers. However, discontinuities of w may persist in $\mathfrak{v}_\epsilon$.

In [39], it is proved in one space dimension that this regularization converges to the underlying inviscid entropy solution as $\varepsilon \to 0$, and estimated the convergence rate in all L^p spaces for $1 \leq p < \infty$. In particular, the convergence rate in L^1 coincides with the convergence rate of the vanishing viscosity approximation (1.2) established in [32]. The extension to several space dimensions (with the same rates) is due to [28]. If $w_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R}^d)$ then for any $T > 0$,

$$\|w_\varepsilon - w\|_{C([0,T];L^1)}, \|\mathfrak{v}_\varepsilon - w\|_{C([0,T];L^1)} = O(\sqrt{\varepsilon}). \tag{1.5}$$

These rates are expected to be optimal even for nonlinear equations as suggested by the analysis of [17] for numerical schemes, see also Theorem 2.5 for a proof in our setting for both (1.2) and (1.3).

It was expected from the modelization that the solution of the Rosenau approximation is closer to the viscous approximation than to the inviscid limit. The fact that $w_\varepsilon - w$ and $\mathfrak{v}_\epsilon - w$ decay with the same rate despite the persistence of discontinuities in $\mathfrak{v}_\epsilon$ goes in that sense. But in the literature we did not find any proof that $w_\varepsilon - \mathfrak{v}_\varepsilon$ indeed decays faster, and this is the motivation of our interest in the problem. Unfortunately we cannot handle standard Laplace and Rosenau operators, but we will give a rather complete analysis for fractional versions of (1.2) and (1.3). We are talking about (1.6) and (1.7) below, probably respectively studied for the first time by [10] and [25] as concerning the mathematical community. More references will be given later, after having introduced the equations on which we shall focus from now on.

Throughout, $\alpha \in (0, 2]$ is fixed, $\varepsilon > 0$ is the perturbation parameter and we consider the Cauchy problems

$$\begin{cases} \partial_t u + \operatorname{div} f(u) = \varepsilon \Delta^{\alpha/2} u, & t \in \mathbb{R}^+,\ x \in \mathbb{R}^d, \\ u(0,x) = w_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.6}$$

and

$$\begin{cases} \partial_t v + \operatorname{div} f(v) = \varepsilon R_\varepsilon^\alpha(v), & t \in \mathbb{R}^+,\ x \in \mathbb{R}^d, \\ v(0,x) = w_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.7}$$

with the space fractional Laplacian

$$\Delta^{\alpha/2}\phi = -\mathcal{F}^{-1}\left(|\cdot|^\alpha \mathcal{F}(\phi)\right), \tag{1.8}$$

and its corresponding Rosenau approximation

$$R_\varepsilon^\alpha(\phi) = -\mathcal{F}^{-1}\left(\frac{|\cdot|^\alpha}{1+\varepsilon^\alpha|\cdot|^\alpha}\,\mathcal{F}(\phi)\right). \tag{1.9}$$

We notably recover (1.2) and (1.3) if $\alpha = 2$. Equation (1.6) is respectively said to be

- subcritical if $\alpha > 1$,
- critical if $\alpha = 1$,
- supercritical if $\alpha < 1$,

in reference to the cases where the diffusion's order is less or above the nonlinear convection's order, see e.g. [12, 13, 29, 30] for the terminology.

Remark 1.1. *In* (1.6) *and* (1.7), $u = u_\varepsilon$ *and* $v = v_\varepsilon$ *depend on* ε *but we simply denote them* u *and* v *all along the paper.*

Scalar conservation laws with fractional diffusions and variants appear in many applications such as over-driven gas detonations [15], radiation hydrodynamics [25, 35, 36, 37, 39], anomalous diffusion in semiconductor growth [42], fractional porous media [19, 20], or fluid dynamics [29, 30]. Equation (1.7) is a very natural counterpart to (1.3) for fractional diffusions, see [25] and references therein. The Fourier symbol $\frac{|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}$ is a bounded approximation of $|\xi|^\alpha$ resembling the former Rosenau approximation for $\alpha = 2$. We recognize as well a standard approximation procedure from subordinate semigroup theory [7, 9, 27, 38], where the semigroup generated by R_ε^α is obtained by subordinating the semigroup generated by $\Delta^{\alpha/2}$ via the usual Bernstein function $r \mapsto \frac{r}{1+\varepsilon^\alpha r}$, see [27, Examples 3.9.23 & 4.3.3].

As concerning well-posedness, (1.6) admits a unique smooth solution in the subcritical and critical cases [12, 22], and a unique possibly discontinuous entropy solution in the supercritical one [1]. The notion of entropy solutions was then extended in [14] to scalar conservation laws with general pure jump Lévy diffusions, whose fractional Laplacian is a prototype. Lévy operators are generators of Lévy processes or equivalently linear and translation invariant operators $\mathcal{L} : \mathcal{C}_c^\infty(\mathbb{R}^d) \to C(\mathbb{R}^d)$ satisfying the maximum principle:

$$\mathcal{L}(\phi)(x) \le 0 \text{ at any global maximum point } x \text{ of } \phi, \tag{1.10}$$

see [7, 9, 16, 27, 38].

Lévy processes modelize diffusion via random motions of particles. Contrarily to the classical Laplacian, these particles may jump for the fractional Laplacian which roughly speaking makes it nonlocal, see [7, 9, 27, 38, 42]. The classical and fractional Rosenau operators are bounded nonlocal Lévy operators, see [7, 9, 27, 38, 39] or Lemma 2.1. The well-posedness of [14] then applies to both (1.6) and (1.7).

Now considering the limits as $\varepsilon \to 0$, we recover the unique entropy solution w of (1.1) from both (1.6) and (1.7). Moreover, for all $\alpha \in (0, 2]$, the solutions of (1.6) and (1.7) converge to the inviscid limit w with the same rate

$$\|u - w\|_{C([0,T];L^1)}, \|v - w\|_{C([0,T];L^1)} = \begin{cases} O(\varepsilon^{1/\alpha}), & \alpha \in (1, 2], \\ O(\varepsilon \ln(1/\varepsilon)), & \alpha = 1, \\ O(\varepsilon), & \alpha \in (0, 1), \end{cases} \tag{1.11}$$

see Corollary 2.1. This result is due to [1, 21] for (1.6). As concerning (1.7), this follows from general continuous dependence estimates for scalar conservation laws with Lévy diffusions [2].

Our contribution is to first rigorously verify that the above rates are optimal even for nonlinear f, and then show that $u - v$ decays strictly faster at least in the purely fractional case $\alpha \in (0, 2)$. We identify in particular the optimal rates in the supercritical and critical cases, and we prove non optimal but explicit rates in the subcritical case. More precisely we establish in Theorem 2.4 that

$$\|u - v\|_{C([0,T];L^1)} = \begin{cases} O(\varepsilon^{2-\alpha}), & \alpha \in (1/2, 1], \\ O(\varepsilon^{3/2} \ln(1/\varepsilon)), & \alpha = 1/2, \\ O(\varepsilon^{\alpha+1}), & \alpha \in (0, 1/2), \end{cases} \tag{1.12}$$

and we show that these general estimates are optimal even for nonlinear f as concerning the asymptotic behavior of the above global L^1 distance for small ε; see Theorem 2.5 and Remark 2.4. For subcritical equations, see Theorem 2.3, we get that

$$\|u - v\|_{C([0,T];L^1)} = O\left(\varepsilon^{\frac{3-\alpha}{2}}\right), \quad \alpha \in (1, 2). \tag{1.13}$$

This suffices to show that $u - v$ vanishes faster than $u - w$ and $v - w$.

The proofs of (1.12) and (1.13) are based on the error estimation technique of Kuznestov [32]; see [1, 2, 3, 21, 23, 28, 35, 39] for nonlocal PDEs. More precisely (1.13) follows from the continuous dependence result of [2] combined with a fine estimate on the difference between the respective Lévy measures of $\Delta^{\alpha/2}$ and R_ε^α (Lemma 3.1). To prove the optimal rates (1.12), we go back inside the Kuznetsov's argumentation while taking more advantage of properties specific to our equations. We notably show that if $\alpha \in (0,1]$, then $\varepsilon(\Delta^{\alpha/2} - R_\varepsilon^\alpha)$ satisfies the maximum principle (1.10). This is particularly important for $\alpha = 1/2$ which is a critical value in (1.12). The monotonicity of this error operator relies on a finer estimate on the Lévy measures (Lemma 3.2). It fails for $\alpha > 1$ which might explain why our Kuznetsov's argumentation does not give optimal rates in that case.

Another ingredient to get (1.12) is a fractional Sobolev estimate on the resolvent's kernel of the fractional Laplacian (Lemma A.6). It roughly speaking gives enough regularizing effects from the error term itself.

We prove Lemmas 3.1 and 3.2 via sophisticated tools on Bessel functions inspired from classical far field asymptotic results on the fractional heat kernel [11, 33]. We also have simpler and self-contained proofs in one space dimension, relying on PDEs arguments.

Our optimal examples are inspired from [17] as previously noticed, and also from [1, 3, 21, 23, 26]. The paper [3] estimated the continuous dependence in both α and porous medium nonlinearities. The optimality of the obtained results was established for the linearized equations via Fourier methods; see [1, 21, 23, 26] for the vanishing viscosity case. In [17], the author proved the optimality of error estimates for numerical approximations of (1.1) with nonlinear strictly convex fluxes. The idea is that, if considering the global L^1 distance and appropriate initial datas generating infinitely many shocks, then we cannot get better rates for nonlinear fluxes than for linear fluxes. Here we combine Fourier arguments and the latter idea to get optimal examples for (1.11) and (1.12).

The most difficult is for (1.12). It necessitates sufficiently larger and larger distances between shock waves, at infinity. The key to handle the nonlocal effects is a "finite-infinite" propagation speed estimate from [1, 24]. It is a modified version of the finite propagation speed estimate for scalar conservation laws involving the convolution semigroups of the diffusions.

We are finally convinced that (1.13) is suboptimal, because we can get better rates for linear fluxes. Remaining open questions are to strictly improve (1.13), and show that $u - v$ vanishes faster than $u - w$ and $v - w$ when $\alpha = 2$. The latter is indeed not a consequence of (1.13), whose limit as $\alpha \to 2$ is robust but just gives the same rate than in (1.5).

The rest of this paper is organized as follows. In Section 2 we recall the representation of the fractional Laplace and Rosenau operators as Lévy operators, the notion of entropy solution for conservation laws with such diffusion terms, and the continuous dependence result of [2]. We conclude the section stating our main results (Theorems 2.2, 2.3, 2.4 and 2.5). Section 3 contains the key estimates on the Lévy measures (Lemmas 3.1 and 3.2). They may be of independent interest, particularly Lemma 3.2. We establish our convergence rates in Section 4, while their optimality is the topic of Section 5. Two technical results in the latter have rather long proofs, postponed to Section 6. Some classical results in fractional calculus and for Bessel functions are collected in Appendix A, together with technical lemmas. Appendix B recalls the finite-infinite propagation speed estimate of [1, 24], and Appendix C provides the alternative proofs of Lemmas 3.1 and 3.2, in one space dimension.

2. Preliminaries and main results

In this section we first recall known facts on the well-posedness of (1.6) and (1.7), and then state our main results.

2.1. **Reminders on Lévy operators and the fractional Laplacian.** Given a Borel measure μ on $\mathbb{R}^d$ such that

$$\mu \geq 0, \qquad \mu(\{0\}) = 0, \qquad \mu(z) = \mu(-z) \qquad \text{and} \qquad \int_{\mathbb{R}^d} \min\left\{|z|^2, 1\right\} d\mu(z) < \infty, \tag{2.1}$$

we define the operator

$$\mathcal{L}^\mu(\phi)(x) = \int_{\mathbb{R}^d} \left(\phi(x+z) - \phi(x) - \nabla\phi(x)\cdot z\chi_{\{|z|\leq 1\}}\right) d\mu(z) \tag{2.2}$$

where χ denotes indicator function of sets. We recognize a symmetric pure jump Lévy operator with Lévy measure μ. These are nonlocal diffusion operators whose above general form is related to the Lévy-Khintchine Formula [7, 9, 27, 38]. Because of the symmetry

$$\mathcal{L}^\mu = \mathcal{L}^{\mu|_{\{|z|\leq r\}}} + \mathcal{L}^{\mu|_{\{|z|>r\}}}$$

for any $r > 0$, where $\mathcal{L}^{\mu|_{\{|z|\leq r\}}} : W^{2,p} \to L^p$ and $\mathcal{L}^{\mu|_{\{|z|>r\}}} : L^p \to L^p$ are linear and bounded for any $p \in [1, \infty]$.

The fractional Laplacian in (1.8) is of the form (2.1)–(2.2) when $\alpha \in (0, 2)$. Its Lévy measure is given by

$$d\mu(z) = \frac{c_\alpha}{|z|^{d+\alpha}}\, dz, \qquad \text{where} \qquad c_\alpha = \frac{\alpha 2^{\alpha-1}\Gamma\left(\frac{d+\alpha}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{2-\alpha}{2}\right)} > 0 \tag{2.3}$$

and Γ is the usual Gamma function. Therefore

$$\begin{aligned}\Delta^{\alpha/2}\phi(x) = c_\alpha &\int_{|z|<r} \left(\phi(x+z) - \phi(x) - \nabla\phi(x)\cdot z\right) \frac{dz}{|z|^{d+\alpha}} \\ &+ c_\alpha \int_{|z|>r} \left(\phi(x+z) - \phi(x)\right) \frac{dz}{|z|^{d+\alpha}},\end{aligned} \tag{2.4}$$

for all $\alpha \in (0, 2)$, $\phi \in C_c^\infty(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $r > 0$. We refer to e.g. [23, Theorem 1] for the computation of c_α whose exact value is important for us.[1] For shortness we also use the following notation:

$$\Delta^{\alpha/2}\phi(x) = c_\alpha\, \mathrm{PV} \int \left(\phi(x+z) - \phi(x)\right) \frac{dz}{|z|^{d+\alpha}}.$$

2.2. **Lévy-Khintchine Formula for Rosenau operators.** The classical and fractional Rosenau operators are bounded Lévy operators.

Lemma 2.1. *Let $\alpha \in (0, 2]$ and $g_\alpha = \mathcal{F}^{-1}\left(\frac{1}{1+|\cdot|^\alpha}\right)$. Then*

$$0 \leq g_\alpha \in L^1(\mathbb{R}^d) \text{ and is radially symmetric with } \|g_\alpha\|_{L^1} = 1. \tag{2.5}$$

Moreover for any $\phi \in C_c^\infty(\mathbb{R}^d)$, the operator R_ε^α in (1.9) satisfies

$$R_\varepsilon^\alpha(\phi)(x) = \int_{\mathbb{R}^d} \left(\phi(x+z) - \phi(x)\right) \frac{g_\alpha(z/\varepsilon)}{\varepsilon^{d+\alpha}}\, dz.$$

In particular $R_\varepsilon^\alpha(\phi)$ is a Lévy operator of the form (2.1)–(2.2) with the measure

$$d\nu(z) = \varepsilon^{-d-\alpha} g_\alpha(z/\varepsilon)\, dz. \tag{2.6}$$

[1] Notably $\mathcal{F}(\phi)(\xi) = \int e^{-2i\pi x\cdot\xi}\phi(x)\, dx$ in [23] and we get the constant in (2.3) with our choice (1.4) of Fourier transform.

This follows from fundamental results on subordinate semigroups since $|\xi|^\alpha$ is continuous and negative definite as opposite of the symbol of the Lévy operator $\Delta^{\alpha/2}$, and $r \mapsto \frac{r}{1+\varepsilon^\alpha r}$ is a Bernstein function. For details, see [27, Theorem 3.7.8] and [27, Examples 3.9.23 & 4.3.3], as well as [39] for $\alpha = 2$. We will notably compare μ and ν, see (2.3) and (2.6), and the exact computation of ν is also important. We thus provide for completeness a self-contained proof of Lemma 2.1; see Appendix A.2.

2.3. **Reminders on scalar conservation laws with Lévy diffusions.** Let us consider a general Cauchy problem of the form

$$\begin{cases} \partial_t w^\mu + \operatorname{div} f(w^\mu) = \mathcal{L}^\mu(w^\mu), & t \in \mathbb{R}^+,\ x \in \mathbb{R}^d, \\ w^\mu(t=0,x) = w_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.7}$$

where $f \in W^{1,\infty}_{loc}(\mathbb{R},\mathbb{R}^d)$, $w_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$ and $\mathcal{L}^\mu$ is as in (2.1)–(2.2). Let us recall the well-posedness and continuous dependence results of [2, 14].

Definition 2.1. *In the setting above, we say that $w^\mu \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d) \cap C([0,\infty); L^1)$ is an entropy solution of (2.7) if and only if for every $r > 0$, convex entropy $\eta \in C^\infty(\mathbb{R})$, entropy flux $q \in C^\infty(\mathbb{R},\mathbb{R}^d)$ such that $q' = \eta' f'$, and $0 \le \phi \in C^\infty_c([0,\infty)\times\mathbb{R}^d)$,*

$$\iint_{\mathbb{R}^+\times\mathbb{R}^d} \left(\eta(w^\mu)\partial_t\phi + q(w^\mu)\cdot\nabla\phi + \eta(w^\mu)\mathcal{L}^{\mu|_{\{|z|\le r\}}}(\phi) + \eta'(w^\mu)\mathcal{L}^{\mu|_{\{|z|>r\}}}(u)\phi\right) dx\,dt$$
$$+ \int_{\mathbb{R}^d} \eta(w^\mu(0,x))\phi(0,x)\,dx \ge 0.$$

Theorem 2.1. *In the hypothesis above, the Cauchy problem (2.7) admits a unique entropy solution w^μ. As concerning moreover the continuous dependence in the Lévy measure, we have the following estimate: For any $T > 0$ there is a constant C_T, only depending on T and d, such that*

$$\begin{aligned} \|w^\mu - w^\nu\|_{C([0,T];L^1)} \le C_T\,\mathrm{TV}(w_0) &\left(\int_{|z|\le r} |z|^2\, d|\mu-\nu|(z)\right)^{1/2} \\ &+ C_T \int_{|z|>r} \|w_0(\cdot+z) - w_0(\cdot)\|_{L^1}\, d|\mu-\nu|(z), \end{aligned} \tag{2.8}$$

for all $w_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R}^d)$, measures μ and ν satisfying (2.1), respective corresponding entropy solutions w^μ and w^ν of (2.7), and $r > 0$.

Remark 2.1. *We also have the following maximum and L^1 contraction principles [14]:*

$$\|w^\mu(t,\cdot)\|_{L^\infty} \le \|w_0\|_{L^\infty} \qquad \text{and} \qquad \|w^\mu(t,\cdot) - \widetilde{w}^\mu(t,\cdot)\|_{L^1} \le \|w_0 - \widetilde{w}_0\|_{L^1}$$

for all $t > 0$ and entropy solutions of (2.7) with respective initial datas w_0 and $\widetilde{w}_0$.

If $\nu \equiv 0$ and μ is successively taken in (2.8) as the Lévy measures of $\varepsilon\Delta^{\alpha/2}$ and $\varepsilon R^\alpha_\varepsilon$, then we recover the rates of convergence below, for the respective corresponding fractional approximations (1.6) and (1.7) of the scalar conservation law (1.1).

Corollary 2.1. *Assume $\alpha \in (0,2]$, $f \in W^{1,\infty}_{loc}(\mathbb{R},\mathbb{R}^d)$, $w_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R}^d)$, and $T > 0$. There then exists $C_T > 0$ such that for any $\varepsilon \in (0,1/2)$,*

$$\|u - w\|_{C([0,T];L^1)}, \|v - w\|_{C([0,T];L^1)} \le C_T\sigma_\varepsilon,$$

where w, u and v respectively solve (1.1), (1.6) and (1.7), and

$$\sigma_\varepsilon = \begin{cases} \varepsilon^{1/\alpha}, & \alpha > 1, \\ \varepsilon\ln(1/\varepsilon), & \alpha = 1, \\ \varepsilon, & \alpha < 1. \end{cases} \tag{2.9}$$

See [1, 21, 32, 39] for the original proofs as concerning (1.2), (1.3) and (1.6). The dependences of the constant C_T is precised in Remark 2.3. These rates are known to be optimal for linear f. This might be less known for the fractional Rosenau operator, but in any way we will give optimal examples even for nonlinear f in Theorem 2.5.

2.4. **Main results.** We are ready to state our main results. The proofs are given in the next sections. Our first contribution is the following:

Theorem 2.2. *Assume $\alpha \in (0,2)$, $f \in W^{1,\infty}_{loc}(\mathbb{R},\mathbb{R}^d)$, $w_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R}^d)$, and $T>0$. Then*

$$\lim_{\varepsilon\to 0} \frac{\|u-v\|_{C([0,T];L^1)}}{\sigma_\varepsilon} = 0,$$

where u and v respectively solve (1.6) *and* (1.7)*, and σ_ε is defined as in Corollary 2.1.*

This is a consequence of our next two results, where we provide sharp estimates of the decay rate of the distance between u and v.

Theorem 2.3 (Explicit rates for subcritical PDEs)**.** *Assume $\alpha \in (1,2)$, $f \in W^{1,\infty}_{loc}(\mathbb{R},\mathbb{R}^d)$, $w_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R}^d)$, and $T>0$. There then exists $C_T>0$ such that for any $\varepsilon \in (0,1/2)$,*

$$\|u-v\|_{C([0,T];L^1)} \le C_T \varepsilon^{\frac{3-\alpha}{2}},$$

where u and v respectively solve (1.6) *and* (1.7)*.*

Remark 2.2. *These rates are strictly better than in* (2.9) *excepted for the limit case $\alpha = 2$. Let us recall that it remains open to show a result as in Theorem 2.2 in that case.*

As concerning $\alpha \in (0,1]$, we have optimal estimates given by the next result.

Theorem 2.4 (Optimal rates for critical and supercritical PDEs)**.** *Let now $\alpha \in (0,1]$, $f \in W^{1,\infty}_{loc}(\mathbb{R},\mathbb{R}^d)$, $w_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R}^d)$, and $T>0$. There then exists $C_T>0$ such that for any $\varepsilon \in (0,1/2)$,*

$$\|u-v\|_{C([0,T];L^1)} \le C_T \mathfrak{s}_\varepsilon,$$

where u and v respectively solve (1.6) *and* (1.7)*, and*

$$\mathfrak{s}_\varepsilon = \begin{cases} \varepsilon^{2-\alpha}, & 1/2 < \alpha \le 1, \\ \varepsilon^{\frac{3}{2}} \ln(1/\varepsilon), & \alpha = 1/2, \\ \varepsilon^{\alpha+1}, & \alpha < 1/2. \end{cases}$$

Remark 2.3. *The constants C_T in Corollary 2.1 and Theorems 2.3 and 2.4 only depend on T, d, α, $\mathrm{TV}(w_0)$, and eventually $\|w_0\|_{L^1}$. More precisely they depend on the latter norm if $\alpha \le 1$ in Corollary 2.1, and if respectively $\alpha \le 1/2$ in Theorem 2.4.*

Let us now give optimal examples for Corollary 2.1 and Theorem 2.4. For simplicity we consider one-dimensional PDEs, but we allow for nonlinear f.[2] We say that $\omega : (0,\infty) \to (0,\infty)$ is a *modulus* if it is nondecreasing and $\lim_{\varepsilon\to 0} \omega(\varepsilon) = 0$.

Theorem 2.5 (Optimal examples)**.** *Let $d=1$, $f \in W^{1,\infty}_{loc}(\mathbb{R})$ be strictly convex, and $T>0$.*

(i) *Assume that $\alpha \in (0,2]$ and define σ_ε as in Corollary 2.1 for any $\varepsilon>0$. Then for any modulus ω such that $\lim_{\varepsilon\to 0} \frac{\omega(\varepsilon)}{\sigma_\varepsilon} = 0$, there is $w_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R})$ whose corresponding solutions of* (1.1)*,* (1.6) *and* (1.7) *satisfy*

$$\lim_{\varepsilon\to 0} \frac{\|u(t,\cdot)-w(t,\cdot)\|_{L^1}}{\omega(\varepsilon)} = \lim_{\varepsilon\to 0} \frac{\|v(t,\cdot)-w(t,\cdot)\|_{L^1}}{\omega(\varepsilon)} = \infty, \quad \forall t \in (0,T].$$

[2] See Propositions 5.1 and 5.2 for $f' \equiv 0$, or equivalently linear f up to changing variables.

(ii) *Assume that* $\alpha \in (0,1]$ *and define* $\mathfrak{s}_\varepsilon$ *as in Theorem 2.4. Then for any modulus* ω *such that* $\lim_{\varepsilon\to 0} \frac{\omega(\varepsilon)}{\mathfrak{s}_\varepsilon} = 0$, *there is* $w_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R})$ *whose corresponding solutions of* (1.6) *and* (1.7) *satisfy*

$$\lim_{\varepsilon\to 0} \frac{\|u(t,\cdot) - v(t,\cdot)\|_{L^1}}{\omega(\varepsilon)} = \infty, \quad \forall t \in (0,T]. \tag{2.10}$$

Remark 2.4. *Being fixed* f, α, T, *note that for any modulus* ω *such that* $\mathfrak{s}_\varepsilon \neq O(\omega(\varepsilon))$ *as* $\varepsilon \to 0$, *there is always an initial data* w_0 *such that* $\|(u-v)(t)\|_{L^1} \neq O(\omega(\varepsilon))$ *for any* $t \in (0,T]$.[3] *Our convergence rates in Theorem 2.4 are optimal in that sense, and so are those in Corollary 2.1.*

3. Preliminary estimates on $|x|^{d+\alpha} g_\alpha(x)$

This section provides two key estimates on the difference between the Lévy measures in (2.3) and (2.6). They are in terms of far field asymptotic results on the convolution kernel g_α of the fractional Laplacian's resolvent appearing in Lemma 2.1. It is more or less standard that

$$\begin{cases} 0 \leq g_\alpha \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d \setminus \{0\}), \text{ and} \\ g_\alpha(x) \sim c_\alpha |x|^{-d-\alpha} \text{ as } |x| \to \infty, \end{cases}$$

for some positive constant c_α only depending on d and α; see Appendices A.2 and A.3. It is particularly important that the above c_α is exactly as in (2.3), and this follows from [11, 33]; cf. Lemmas A.1 and A.3.

The first result below estimates the rate at which $|x|^{d+\alpha} g_\alpha(x) \xrightarrow{|x|\to\infty} c_\alpha$.

Lemma 3.1. *Let* $\alpha \in (0,2)$. *There exists* $N_\alpha > 0$, *only depending on* d *and* α, *such that for any* $x \neq 0$,

$$\left| |x|^{d+\alpha} g_\alpha(x) - c_\alpha \right| \leq \frac{N_\alpha}{|x|^\alpha}$$

where $g_\alpha = \mathcal{F}^{-1}\left(\frac{1}{1+|\cdot|^\alpha}\right)$ *and* c_α *is as in* (2.3).

To show it, we will go back inside the computations of [11, 33] and call for a more general version of these results from [34]. From now on, we use the notation from Appendix A.1 for the fractional heat kernel $G_\alpha(t,x) = t^{-d/\alpha} P_\alpha\left(x/t^{1/\alpha}\right)$ and Bessel functions $\mathcal{J}_\gamma(z)$.

Proof of Lemma A.1. Since $P_\alpha(x) = \mathcal{F}^{-1}(e^{-|\cdot|^\alpha})$, from Theorem A.1, we have

$$\begin{aligned} |x|^{d+\alpha} P_\alpha(x) &= \frac{|x|^{d+\alpha}}{(2\pi)^{\frac{d}{2}} |x|^{\frac{d}{2}-1}} \int_{\mathbb{R}^+} e^{-r^\alpha} r^{\frac{d}{2}} \mathcal{J}_{\frac{d}{2}-1}(r|x|)\, dr \\ &= \frac{|x|^\alpha + 1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-r^\alpha} \underbrace{(r|x|)^{\frac{d}{2}} \mathcal{J}_{\frac{d}{2}-1}(r|x|)}_{= \frac{1}{|x|}\frac{d}{dr}\left((r|x|)^{\frac{d}{2}} \mathcal{J}_{\frac{d}{2}}(r|x|)\right)} dr \\ &\overset{\text{(integrating by parts)}}{=} \frac{\alpha |x|^\alpha}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} r^{\alpha-1} e^{-r^\alpha} (r|x|)^{\frac{d}{2}} \mathcal{J}_{\frac{d}{2}}(r|x|)\, dr \\ &\overset{\text{(changing the variable to } s = r|x|)}{=} \frac{\alpha}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-\frac{s^\alpha}{|x|^\alpha}} s^{\frac{d}{2}+\alpha-1} \mathcal{J}_{\frac{d}{2}}(s)\, ds. \end{aligned}$$

[3] Indeed there is then $\varepsilon_n \to 0$ such that $\frac{\omega(\varepsilon_n)}{\mathfrak{s}_{\varepsilon_n}} \to 0$ and we can construct another modulus $\widetilde{\omega}$ such that $\lim_{\varepsilon\to 0} \frac{\widetilde{\omega}(\varepsilon)}{\mathfrak{s}_\varepsilon} = 0$ while $\widetilde{\omega}(\varepsilon_n) = \omega(\varepsilon_n)$. By Theorem 2.5 there is w_0 such that $\lim_{\varepsilon\to 0} \|(u-v)(t)\|_{L^1} / \widetilde{\omega}(\varepsilon) = \infty$, for any $t \in (0,T]$. Recall that $u = u_\varepsilon$ and $v = v_\varepsilon$, so taking the latter limit along the sequence ε_n proves our claim.

These computations are justified by the properties of Bessel functions recalled in (A.4). The above function of x is radially symmetric. Let us take its derivative with respect to the norm of x. It is equaled to

$$\frac{\alpha^2}{(2\pi)^{\frac{d}{2}}|x|^{\alpha+1}} \underbrace{\int_{\mathbb{R}^+} e^{-\frac{s^\alpha}{|x|^\alpha}} s^{\frac{d}{2}+2\alpha-1} \mathcal{J}_{\frac{d}{2}}(s)\, ds}_{=:I(x)}. \tag{3.1}$$

The term $I(x)$ is as the integral in [34, Lemma 1].[4] This result implies that $I(x) = O(1)$ as $|x| \to \infty$. The whole term in (3.1) thus equals $O(|x|^{-\alpha-1})$ for large $|x|$. Using now Lemma A.1 and integrating the function $r \in (1,\infty) \mapsto |rx|^{d+\alpha} P_\alpha(rx)$ for an arbitrary fixed x,

$$-\frac{N_\alpha}{2|x|^\alpha} + c_\alpha \le |x|^{d+\alpha} P_\alpha(x) \le \frac{N_\alpha}{2|x|^\alpha} + c_\alpha \tag{3.2}$$

for some constant $N_\alpha > 0$ independent of x. Injecting this into (A.7) gives us the desired result. □

The following lemma improves the estimate above for $\alpha \in (0,1]$.

Lemma 3.2. *Assume $\alpha \in (0,1]$, then $|x|^{d+\alpha} g_\alpha(x) \le c_\alpha$ for any $x \neq 0$ (where g_α and c_α are as previously).*

Remark 3.1. *In that case $\mu - \nu \ge 0$, see (2.3) and (2.6), and $\Delta^{\alpha/2} - R_\varepsilon^\alpha$ is a Lévy operator.*

Proof. It is enough to show that $x \mapsto |x|^{d+\alpha} P_\alpha(x)$ is radially nondecreasing. Recalling (3.2), its derivative with respect to the norm of x is

$$\begin{aligned}
\frac{\alpha^2}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-\frac{s^\alpha}{|x|^\alpha}} \frac{s^{\frac{d}{2}+2\alpha-1}}{|x|^{\alpha+1}} \mathcal{J}_{\frac{d}{2}}(s)\, ds &= \frac{\alpha^2}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-\frac{s^\alpha}{|x|^\alpha}} \frac{s^{2\alpha-2}}{|x|^{\alpha+1}} s^{\frac{d}{2}+1} \mathcal{J}_{\frac{d}{2}}(s)\, ds \\
\text{(setting } r|y| = s \text{ for a fixed } y \in \mathbb{R}^{d+2} \text{ s.t. } |y| = |x|) \quad &= \frac{\alpha^2 |y|^{\frac{d}{2}+\alpha-1}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-r^\alpha} r^{2\alpha-2} r^{\frac{d}{2}+1} \mathcal{J}_{\frac{d}{2}}(r|y|)\, dr \\
\text{(by Theorem A.1)} \quad &= 2\pi\alpha^2 |y|^{d+\alpha-1} \mathcal{F}_{d+2}^{-1}\left(e^{-|\cdot|^\alpha} |\cdot|^{2\alpha-2}\right)(y),
\end{aligned}$$

with the Fourier transform $\mathcal{F}_{d+2}$ in space dimension $d+2$. If $\alpha \in (0,1)$ then we have $\mathcal{F}_{d+2}^{-1}\left(|\cdot|^{2\alpha-2}\right) = C|\cdot|^{-d-2\alpha}$ for some $C = C(d,\alpha) > 0$; see e.g. [40, Chap. IV, Sec. 4, Thm. 4.1]. Hence

$$\mathcal{F}_{d+2}^{-1}\left(e^{-|\cdot|^\alpha} |\cdot|^{2\alpha-2}\right)(y) = C\left((P_\alpha)_{d+2} *_{d+2} |\cdot|^{-d-2\alpha}\right)(y) \ge 0,$$

with the convolution in $\mathbb{R}^{d+2}$ and where we used properties of $(P_\alpha)_{d+2} := \mathcal{F}_{d+2}^{-1}\left(e^{-|\cdot|^\alpha}\right)$ recalled in (A.3). Otherwise, if $\alpha = 1$,

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-\frac{s}{|x|}} \frac{s^{\frac{d}{2}+1}}{|x|^2} \mathcal{J}_{\frac{d}{2}}(s)\, ds = 2\pi |y|^d (P_1)_{d+2}(y) \ge 0,$$

and we deduce that $x \mapsto |x|^{d+\alpha} P_\alpha(x)$ is radially nondecreasing in all cases. □

[4] See [34, page 302] for the definition of D in this reference.

4. Proofs of Theorems 2.3 and 2.4

We are ready to prove Theorems 2.3 and 2.4. Let us recall from Sections 2.1 and 2.2 that the Lévy operators $\Delta^{\alpha/2}$ and R_ε^α are associated respectively to the measures

$$d\mu(z) = \frac{c_\alpha}{|z|^{d+\alpha}}\,dz \qquad \text{and} \qquad d\nu(z) = \frac{g_\alpha\left(z/\varepsilon\right)}{\varepsilon^{d+\alpha}}\,dz.$$

We also use the shorthand notation

$$g_\alpha^\varepsilon(z) = \frac{g_\alpha(z/\varepsilon)}{\varepsilon^d} \qquad \text{so that} \qquad d\nu(z) = \frac{g_\alpha^\varepsilon(z)}{\varepsilon^\alpha}\,dz. \tag{4.1}$$

Proof of Theorem 2.3. Let u and v respectively solve (1.6) and (1.7). The general continuous dependence estimate (2.8) implies that

$$\begin{aligned}\|u-v\|_{C([0,T];L^1)} \le C_T\varepsilon^{\frac12}\,\mathrm{TV}(w_0)\left(\int_{|z|<r}|z|^2\left|\frac{1}{\varepsilon^{d+\alpha}}\,g_\alpha\left(\frac{z}{\varepsilon}\right)-\frac{c_\alpha}{|z|^{d+\alpha}}\right|dz\right)^{1/2}\\ + C_T\varepsilon\int_{|z|>r}\|w_0(\cdot-z)-w_0(\cdot)\|_{L^1}\left|\frac{1}{\varepsilon^{d+\alpha}}\,g_\alpha\left(\frac{z}{\varepsilon}\right)-\frac{c_\alpha}{|z|^{d+\alpha}}\right|dz,\end{aligned} \tag{4.2}$$

for all positive T, ε and r. The estimate in Lemma 3.1 together with Lemma A.3 allow us to write

$$\left|\frac{1}{\varepsilon^{d+\alpha}}\,g_\alpha\left(\frac{z}{\varepsilon}\right)-\frac{c_\alpha}{|z|^{d+\alpha}}\right| \le \max\left\{M_\alpha+c_\alpha,N_\alpha\right\}\cdot\begin{cases}\frac{1}{|z|^{d+\alpha}}, & \text{if } |z|\le\varepsilon,\\ \frac{\varepsilon^\alpha}{|z|^{d+2\alpha}}, & \text{if } |z|>\varepsilon.\end{cases} \tag{4.3}$$

Recall that $\alpha\in(1,2)$ in Theorem 2.3. We can then consider $r=\infty$ in (4.2) and cut the remaining integral in two pieces to obtain

$$\|u-v\|_{C([0,T];L^1)} \le C_T\varepsilon^{\frac12}\left(\int_{|z|<\varepsilon}|z|^2\,\frac{dz}{|z|^{d+\alpha}}+\varepsilon^\alpha\int_{|z|>\varepsilon}|z|^2\,\frac{dz}{|z|^{d+2\alpha}}\right)^{1/2} \le C_T\varepsilon^{\frac{3-\alpha}{2}},$$

with a new constant C_T only depending on T, d, α, $\mathrm{TV}(w_0)$.

This completes the proof of Theorem 2.3. □

Proof of Theorem 2.4. For $\alpha=1$ we argue as previously by taking $r=\varepsilon$ in (4.2) while using that $\|w_0(\cdot-z)-w_0(\cdot)\|_{L^1}\le|z|\,\mathrm{TV}(w_0)$ and (4.3). It only remains to consider $\alpha\in(0,1)$. Let us go back inside the Kuznetsov's argumentation [1, 2, 3, 21, 23, 28, 32, 35, 39] to get optimal estimates.[5]

Define $\rho_\kappa(x)=\frac{1}{\kappa^d}\,\rho(\frac{x}{\kappa})$ for any $\kappa>0$ and some fixed

$$0\le\rho\in C_c^\infty(\mathbb{R}^d) \qquad \text{such that} \qquad \|\rho\|_{L^1}=1.$$

Restarting from [2, Equation (5.22)], there is $C=C(d)>0$ such that

$$\begin{aligned}&\|u(T,\cdot)-v(T,\cdot)\|_{L^1}\\ &\le C\,\mathrm{TV}(w_0)\kappa+\varepsilon\,\mathrm{PV}\iiiint_{(0,T)\times\mathbb{R}^{3d}}\rho_\kappa(x-y)\,\mathrm{sign}(u(t,x)-v(t,y))\\ &\qquad\cdot\left[\left(u(t,x+z)-u(t,x)\right)d\mu(z)-\left(v(t,y+z)-v(t,y)\right)d\nu(z)\right]dx\,dy\,dt\end{aligned} \tag{4.4}$$

for all positive T, ε and κ.[6] Recall that principal values are computed by first integrating with $|z|>r>0$ before letting $r\to0$. Recall also that $\nu\ge0$, and $\mu-\nu\ge0$ by Remark 3.1.

[5] In our setting (2.8) provides optimal results only if $\alpha\le1/3$ or $\alpha=1$.

[6] There were two time variables t and s in the integral term of [2, Equation (5.22)], and the test function was $\theta_\delta(t-s)\rho_\kappa(x-y)$ for some time approximate unit θ_δ as $\delta\to0$. Letting $\delta\to0$ led us to (4.4). Moreover C is $C_{\overline{\theta}}$ from [2, Lemma 3.1] where $\overline{\theta}$ corresponds to our ρ. This constant only depends on d via ρ.

Therefore

$$\begin{aligned}&\operatorname{sign}(a-c)\left[(b-a)\,d\mu(z)-(d-c)\,d\nu(z)\right]\\&\leq\left(|b-d|-|a-c|\,d\nu(z)+\left(|b-c|-|a-c|\right)d(\mu-\nu)(z)\right.\end{aligned}$$

for all reals a,b,c,d. Hence, denoting I the integral of the right hand side of (4.4),

$$\begin{aligned}I\leq\operatorname{PV}&\iiiint_{(0,T)\times\mathbb{R}^{3d}}\rho_\kappa(x-y)\\&\cdot\left(|u(t,x+z)-v(t,y+z)|-|u(t,x)-v(t,y)|\right)d\nu(z)\,dx\,dy\,dt\\+\operatorname{PV}&\iiiint_{(0,T)\times\mathbb{R}^{3d}}\rho_\kappa(x-y)\\&\cdot\left(|u(t,x+z)-v(t,y)|-|u(t,x)-v(t,y)|\right)d(\mu-\nu)(z)\,dx\,dy\,dt\\=:I_1+I_2.\end{aligned}$$

Integrating by parts,[7]

$$I_1\leq\operatorname{PV}\iiiint_{(0,T)\times\mathbb{R}^{3d}}|u(t,x)-v(t,y)|\underbrace{\left(\rho_\kappa((x+z)-(y+z))-\rho_\kappa(x-y)\right)}_{=0}d\nu(z)\,dx\,dy\,dt$$

and similarly

$$I_2\leq\operatorname{PV}\iiiint_{(0,T)\times\mathbb{R}^{3d}}|u(t,x)-v(t,y)|\underbrace{\left(\rho_\kappa(x-y+z)-\rho_\kappa(x-y)\right)d(\mu-\nu)(z)}_{=\Delta^{\alpha/2}\rho_\kappa(x-y)-R_\varepsilon^\alpha(\rho_\kappa)(x-y)\text{ after integrating in }z}dx\,dy\,dt.$$

Injecting this into (4.4) while using Lemma A.4 in appendix,

$$\begin{aligned}&\|u(T,\cdot)-v(T,\cdot)\|_{L^1}\\&\leq C\operatorname{TV}(w_0)\kappa+\varepsilon^{1+\alpha}\iiint_{(0,T)\times\mathbb{R}^{2d}}|u(t,x)-v(t,y)|\left(g_\alpha^\varepsilon*\Delta^\alpha\rho_\kappa\right)(x-y)\,dx\,dy\,dt\end{aligned}\tag{4.5}$$

for all positive T, ε and κ, and where we recall that $C=C(d)$. In the remaining of the proof we consider several cases according as the value of α.

CASE 1: $\alpha\in(1/2,1)$. Let us notice that $\nabla^{2\alpha-1}g_\alpha\in L^1(\mathbb{R}^d,\mathbb{R}^d)$ by Lemma A.6, see also Lemma A.5. By (4.1) we infer that $\rho_\kappa*\nabla^{2\alpha-1}g_\alpha^\varepsilon\in C_b^\infty\cap W^{\infty,1}(\mathbb{R}^d)$ and

$$g_\alpha^\varepsilon*\Delta^\alpha\rho_\kappa=\operatorname{div}\left(\rho_\kappa*\nabla^{2\alpha-1}g_\alpha^\varepsilon\right),$$

thanks to the formula $\Delta^\alpha=\operatorname{div}(\nabla^{2\alpha-1})$ which readily follows from (1.8) and (A.8). With that regularity in hands, we can integrate the last term in (4.5) by parts in x. For any fixed $(t,y)\in\mathbb{R}^+\times\mathbb{R}^d$ we get

$$\begin{aligned}&\int_{\mathbb{R}^d}|u(t,x)-v(t,y)|\left(g_\alpha^\varepsilon*\Delta^\alpha\rho_\delta\right)(x-y)\,dx\\&=-\int_{\mathbb{R}^d}\left(g_\alpha^\varepsilon*\nabla^{2\alpha-1}\rho_\kappa\right)(x-y)\cdot d\left(\nabla_x|u(t,\cdot)-v(t,y)|\right)(x)\\&\leq\int_{\mathbb{R}^d}\left|g_\alpha^\varepsilon*\nabla^{2\alpha-1}\rho_\kappa\right|(x-y)\,d|\nabla u(t,\cdot)|(x).\end{aligned}\tag{4.6}$$

We notably used the BV regularity of $x\mapsto u(t,x)$ due to the L^1 contraction principle, see Remark 2.1, and in particular the fact that

$$|\nabla_x|u(t,\cdot)-v(t,y)||\leq|\nabla u(t,\cdot)|$$

[7]That is we integrate with $|z|>r>0$, change variables to put the translations in z on ρ_κ, and let $r\to0$.

in the sense of measures.[8] Injecting (4.6) into (4.5), and integrating in y first, then x, and finally t,

$$\|u(T,\cdot)-v(T,\cdot)\|_{L^1} \le C\,\mathrm{TV}(w_0)\kappa + \varepsilon^{1+\alpha}\underbrace{\|\rho_\kappa\|_{L^1}}_{=1}\left\|\nabla^{2\alpha-1}g^\varepsilon_\alpha\right\|_{L^1}\underbrace{\int_0^T \mathrm{TV}(u(t,\cdot))\,dt}_{\le T\,\mathrm{TV}(w_0)\text{ by Remark 2.1}}.$$

We could notably call for the Fubini-Tonelli Theorem because t was fixed since (4.6), and $d|\nabla u(t,\cdot)|(x)\,dy$ is a product of σ-finite measures. We conclude the case $\alpha\in(1/2,1)$ by letting $\kappa\to 0$ and using that $\|\nabla^{2\alpha-1}g^\varepsilon_\alpha\|_{L^1}=\varepsilon^{1-2\alpha}\left\|\nabla^{2\alpha-1}g_\alpha\right\|_{L^1}$ by (4.1) and (A.8).

CASE 2: $\alpha=1/2$. Let $r\in(0,1)$ and let us split the integral in the right hand side of (4.5) into three terms. We omit the integration in time for a while and we get

$$\begin{aligned}
&\iint_{\mathbb{R}^{2d}}|u(t,x)-v(t,y)|\Delta^{1/2}(g^\varepsilon_{1/2}*\rho_\kappa)(x-y)\,dx\,dy\\
&=\iiint_{\mathbb{R}^{2d}\times\{|z|<r\}}|u(t,x)-v(t,y)|\\
&\qquad\cdot\left(g^\varepsilon_{1/2}*\rho_\kappa(x-y+z)-g^\varepsilon_{1/2}*\rho_\kappa(x-y)-z\cdot\nabla(g^\varepsilon_{1/2}*\rho_\kappa)(x-y)\right)\frac{c_1\,dz\,dx\,dy}{|z|^{d+1}}\\
&\quad+\iiint_{\mathbb{R}^{2d}\times\{r<|z|<1\}}g^\varepsilon_{1/2}*\rho_\kappa(x-y)\left(|u(t,x+z)-v(t,y)|-|u(t,x)-v(t,y)|\right)\frac{c_1\,dz\,dx\,dy}{|z|^{d+1}}\\
&\quad+\iiint_{\mathbb{R}^{2d}\times\{|z|>1\}}g^\varepsilon_{1/2}*\rho_\kappa(x-y)\left(|u(t,x+z)-v(t,y)|-|u(t,x)-v(t,y)|\right)\frac{c_1\,dz\,dx\,dy}{|z|^{d+1}}\\
&=:J_1+J_2+J_3.
\end{aligned}\tag{4.7}$$

Here we used (2.2) while putting back the translations in z on entropy solutions in J_2 and J_3.[9] Let us use Taylor's formula to compute J_1. We denote the Hessian by D^2 and the inner product by $\langle,\rangle$. Using as previously the BV regularity of $x\mapsto u(t,x)$ to integrate by parts, etc., we get

$$\begin{aligned}
J_1&=\iiint_{\mathbb{R}^{2d}\times\{|z|<r\}}|u(t,x)-v(t,y)|\\
&\qquad\cdot\int_0^1\left\langle\left(D^2(g^\varepsilon_{1/2}*\rho_\kappa)(x-y+sz)\right)z,z\right\rangle(1-s)\,ds\,\frac{c_1\,dz\,dx\,dy}{|z|^{d+1}}\\
&=\iiint_{\mathbb{R}^d\times\{|z|<r\}\times(0,1)}(1-s)\\
&\qquad\cdot\int_{\mathbb{R}^d}\underbrace{\langle-\nabla(g^\varepsilon_{1/2}*\rho_\kappa)(x-y+sz),z\rangle\langle z,d\nabla_x|u(t,\cdot)-v(t,y)|(x)\rangle}_{\le|z|^2\left|g^\varepsilon_{1/2}*\nabla\rho_\kappa(x-y+sz)\right|d|\nabla u(t,\cdot)|(x)}\frac{c_1\,ds\,dz\,dy}{|z|^{d+1}}\\
&\le\left\|g^\varepsilon_{1/2}\right\|_{L^1}\|\nabla\rho_\kappa\|_{L^1}\,\mathrm{TV}(w_0)\,\frac{1}{2}\int_{|z|<r}\frac{dz}{|z|^{d-1}}\\
&=\frac{Cr}{\kappa}\,\mathrm{TV}(w_0),
\end{aligned}$$

for some constant $C=C(d)$ thanks to (4.1) and (2.5). Notably C may differ from the constant in (4.5), but we use the same letter for simplicity and we may proceed similarly below as well

[8] Recalling that (t,y) is fixed, this is standard since $\eta(\cdot):=|\cdot-v(t,y)|\in W^{1,\infty}_{loc}(\mathbb{R})$ with $|\eta'|\le 1$.

[9] As before this follows from using that $\iiint(\cdots-\dots)=\iiint\cdots-\iiint\dots$ and changing variables.

as in other proofs. Integrating also J_2 first in y, then x, and z, we have

$$J_2 \le -C\ln(r)\left\|g^\varepsilon_{1/2}\right\|_{L^1}\|\rho_\kappa\|_{L^1}\operatorname{TV}(w_0) = -C\ln(r)\operatorname{TV}(w_0)$$

and similarly $J_3 \le C\|u(t,\cdot)\|_{L^1} \le C\|w_0\|_{L^1}$, where $C = C(d)$ thanks again to Remark 2.1. We can put all our estimates together in (4.7) and then (4.5) to get

$$\|u(T,\cdot) - v(T,\cdot)\|_{L^1} \le C\operatorname{TV}(w_0)\kappa + CT\varepsilon^{3/2}\left[\operatorname{TV}(w_0)\left(\frac{r}{\kappa} - \ln(r)\right) + \|w_0\|_{L^1}\right],$$

so taking $r = \kappa = \varepsilon^{3/2}$ completes the proof.

CASE 3: $\alpha \in (0, 1/2)$. Now $\Delta^\alpha : L^1 \cap BV \to L^1$ is bounded from (2.2) and Taylor's formula. Putting back the fractional Laplacian on entropy solutions in the last term of (4.5) then gives the desired result. The proof of Theorem 2.4 is complete. □

5. Proof of Theorem 2.5

Let us now construct examples for which our rates are optimal. We follow the approach of Şabac in [17], which first requires optimal examples for linear fluxes or equivalently for $f' \equiv 0$ up to changing variables. We start with the latter case in Section 5.1, whose proofs are rather long and postponed in Section 6. Nonlinear f are consider in Section 5.2, and all our examples are for $d = 1$.

5.1. Examples of optimal convergence rates for the zero convection case.

Let us give two examples, respectively for the rates in $u - w$ and $u - v$.

5.1.1. *Vanishing fractional Laplacian approximation of a stationary solution.* Consider the initial condition $w_0 = \chi_{E_J}$ as the characteristic function of the set

$$E_J = \bigcup_{j=0}^{J}\left[\left(-2j - \frac{3}{2}, -2j - \frac{1}{2}\right) \cup \left(2j + \frac{1}{2}, 2j + \frac{3}{2}\right)\right]. \tag{5.1}$$

and the initial value problems

$$\begin{cases} \partial_t u = \varepsilon\partial^\alpha_{xx} u, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = w_0(x), & x \in \mathbb{R}, \end{cases} \qquad \begin{cases} \partial_t w = 0, & t > 0,\ x \in \mathbb{R}, \\ w(0,x) = w_0(x), & x \in \mathbb{R}. \end{cases} \tag{5.2}$$

Hereafter we use the shorthand notation $\partial^\alpha_{xx} = (\partial^2_{xx})^{\alpha/2}$ for the one-dimensional fractional Laplacian (cf. (1.8)). To distinguish it with the usual derivatives $\partial^n_{x^n}$, $n \in \mathbb{N}$, we use the subscript "$_{xx}$" in ∂^α_{xx} referring to ∂^2_{xx} from which we take the fractional power.

Proposition 5.1. *Let $\alpha \in (0, 2]$. Then*

$$\inf_{0<t\varepsilon\le 1,\, J\ge 1} \frac{\int_{E_J}(w(t,x) - u(t,x))\,dx}{J\sigma_{t\varepsilon}} > 0, \tag{5.3}$$

where for every $\varepsilon > 0$ and integer J, we let u and w denote the solutions of (5.2) with w_0 as in (5.1). Recall also that σ_r is the rate from Corollary 2.1, i.e.

$$\sigma_r = \begin{cases} r^{1/\alpha}, & \alpha \in (1,2], \\ r\ln(1/r), & \alpha = 1, \\ r, & \alpha \in (0,1), \end{cases} \qquad r \ge 0.$$

Remark 5.1. *As concerning our notation, $u = u_{\varepsilon,J}$ and $w = w_J$ but we denote them u and w for short. We proceed similalry for u and v in (5.5), etc. It is also understood that t and ε are positive in (5.3).*

This result gives in particular an optimal example for the rates in Corollary 2.1 when $f' \equiv 0$. Fixing indeed $t > 0$ and taking $\varepsilon > 0$ small enough,

$$\|u(t,\cdot) - w(t,\cdot)\|_{L^1} \geq \int_{E_J} (w-u)(t,x)\,dx \geq cJ\sigma_{t\varepsilon} \tag{5.4}$$

where $c > 0$ is the infimum in (5.3) and only depends on α.

Remark 5.2. *We could also take E_J as the union of two intervals, i.e. $J = 0$, or even one interval. This would provide optimal examples even though this cannot be seen from* (5.4)*. A more accurate estimate in fact holds with J replaced by the number of intervals constituting E_J, a number that could be any integer. But taking such E_J and showing only* (5.3) *will simplify the computation, and this will be enough for later.*

5.1.2. *On the L^1 distance between the vanishing fractional Laplacian and vanishing Rosenau approximations.* Consider now the initial value problems

$$\begin{cases} \partial_t u = \varepsilon \partial_{xx}^{\alpha} u, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = w_0(x) = \chi_{E_J}(x), & x \in \mathbb{R}, \end{cases} \qquad \begin{cases} \partial_t v = \varepsilon R_{\varepsilon}^{\alpha}(v), & t > 0,\ x \in \mathbb{R}, \\ v(0,x) = w_0(x), & x \in \mathbb{R}, \end{cases} \tag{5.5}$$

with common initial condition w_0 in (5.1).

Proposition 5.2. *Let $\alpha \in (0,1]$. For any fixed $T > 0$ we have that*

$$\inf_{\substack{0<t\leq T,\\ 0<\varepsilon\leq 1,\, J\geq 1}} \frac{\int_{E_J} (v(t,x) - u(t,x))\,dx}{Jt\mathfrak{s}_\varepsilon} > 0, \tag{5.6}$$

where for every $\varepsilon > 0$ and integer J, we let u and v denote the solutions of (5.5) *with w_0 as in* (5.1)*. Recall also that $\mathfrak{s}_\varepsilon$ is the rate from Theorem 2.5, i.e.*

$$\mathfrak{s}_\varepsilon = \begin{cases} \varepsilon^{2-\alpha}, & 1/2 < \alpha \leq 1, \\ \varepsilon^{\frac{3}{2}} \ln(1/\varepsilon), & \alpha = 1/2, \\ \varepsilon^{\alpha+1}, & \alpha < 1/2. \end{cases} \tag{5.7}$$

Arguing as previously, this result gives an optimal example for the rates in Theorem 2.5 when $f' \equiv 0$.

5.2. **Optimal convergence rates in the nonlinear, strictly convex case.** We are ready to prove Theorem 2.5. To make the computations as readable as possible we focus on the Burgers' equation, in which $f(\xi) = \xi^2/2$, but the proof applies with very minor modifications to the case of a general strictly convex flux.

Proof of Theorem 2.5(ii). Consider the initial value problems

$$\begin{cases} \partial_t u + \partial_x \left(\frac{u^2}{2}\right) = \varepsilon \partial_{xx}^{\alpha} u, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = w_0(x), & x \in \mathbb{R}, \end{cases} \qquad \begin{cases} \partial_t v + \partial_x \left(\frac{v^2}{2}\right) = \varepsilon R_{\varepsilon}^{\alpha}(v), & t > 0,\ x \in \mathbb{R}, \\ v(0,x) = w_0(x), & x \in \mathbb{R}, \end{cases} \tag{5.8}$$

where w_0 is defined below. It depends on given $\ell > 0$, $\{\ell_i\}_{i\geq 1} \subset (\ell, \infty)$ and $\{J_i\}_{i\geq 1} \subset 4 + 2\mathbb{N}$. We assume that $\ell_{i+1} \geq \ell_i$. For every $i \geq 1$ and $1 \leq j \leq J_i$ define the real intervals I_{ij} by

$$\inf I_{ij} = \left(2\left(j + \sum_{m=1}^{i-1} J_m\right) - i\right)\ell + \sum_{m=1}^{i-1} \ell_m, \qquad \sup I_{ij} = \inf I_{ij} + \ell,$$

with the convention $\sum_{m=1}^{0}=0$, i.e. $I_{1j}=\big((2j-1)\ell,2j\ell\big)$.[10] Let $c_i>0$ be such that

$$\sum_{i=1}^{\infty} J_i c_i<\infty, \tag{5.9}$$

and define $w_0=\sum_{i=1}^{\infty}\sum_{j=1}^{J_i} c_i\chi_{I_{ij}}$.

Being the sets I_{ij} pairwise disjoint we have $w_0\in L^1\cap L^\infty\cap BV(\mathbb{R})$. We moreover choose ℓ large enough so that when the dynamics is governed by the inviscid conservation law, then the waves associated to the evolution of each of the non-zero components of w_0, $c_i\chi_{I_{ij}}$, do not interact before time T, i.e.

$$\ell\geq 2T\left\|f'(w_0)\right\|_{L^\infty}=2T\max_{i\geq 1}c_i. \tag{5.10}$$

Let us more precisely impose that

$$\max_{i\geq 1}c_i\leq 1 \qquad \text{and} \qquad \ell=2T. \tag{5.11}$$

We will require more conditions on $\{\ell_i\}_i$, $\{J_i\}_i$ and $\{c_i\}_i$, as we go along.

Consider now, for any $i\geq 1$, the Cauchy problems given by the linearization of the equations[11] about the state c_i and the truncated initial conditions $w_0^i=c_i\sum_{j=1}^{J_i}\chi_{I_{ij}}$, i.e.

$$\begin{cases}\partial_t q_i+c_i\partial_x q_i=\varepsilon\partial_{xx}^{\alpha}q_i, & t>0,\,x\in\mathbb{R},\\ q_i(0,x)=w_0^i(x), & x\in\mathbb{R},\end{cases}\qquad \begin{cases}\partial_t z_i+c_i\partial_x z_i=\varepsilon R_\varepsilon^{\alpha}(v), & t>0,\,x\in\mathbb{R},\\ z_i(0,x)=w_0^i(x), & x\in\mathbb{R},\end{cases} \tag{5.12}$$

and the sets

$$I_i=\left(\inf I_{i1},\sup I_{iJ_i}\right),\qquad I_i^t=I_i+c_it. \tag{5.13}$$

We call u_i and v_i the solutions of the equations in (5.8) corresponding to the initial condition w_0^i, then we observe that

$$\begin{aligned}\|u(t,\cdot)-v(t,\cdot)\|_{L^1}&\geq\|u(t,\cdot)-v(t,\cdot)\|_{L^1(I_i^t)}\\ &\geq\|u_i(t,\cdot)-v_i(t,\cdot)\|_{L^1(I_i^t)}-\|u(t,\cdot)-u_i(t,\cdot)\|_{L^1(I_i^t)}-\|v(t,\cdot)-v_i(t,\cdot)\|_{L^1(I_i^t)}.\end{aligned} \tag{5.14}$$

By Lemma B.1 in appendix and (5.10),

$$\begin{aligned}&\|u(t,\cdot)-u_i(t,\cdot)\|_{L^1(I_i^t)}\\ &\leq\int_{I_i^t+\left(-\|f'(w_0)\|_{L^\infty},\|f'(w_0)\|_{L^\infty}\right)t}(G_\alpha(\varepsilon t,\cdot)*|u(0,\cdot)-u_i(0,\cdot)|)(x)\,dx\\ &\leq\int_{I_i^t+(-\frac{\ell}{2},\frac{\ell}{2})}(G_\alpha(\varepsilon t,\cdot)*|u(0,\cdot)-u_i(0,\cdot)|)(x)\,dx\\ &=\sum_{k\neq i,\,k\geq 1}\sum_{j=1}^{J_k}c_k\underbrace{\int_{I_i^t+(-\frac{\ell}{2},\frac{\ell}{2})}(G_\alpha(\varepsilon t,\cdot)*\chi_{I_{kj}})(x)\,dx}_{=:E_{kj}},\end{aligned} \tag{5.15}$$

[10] If denoting $E\leq F$ iff $\sup E\leq\inf F$ for subsets of the real line, each $E_i:=\cup_{j=1}^{J_i}I_{ij}$ is such that ℓ is the common lenght of the I_{ij} as well as the common distance between two consecutive $I_{ij}\leq I_{i,j+1}$. The distance between two consecutive $E_i\leq E_{i+1}$ is $\ell_i>\ell$.

[11] If f is a general strictly convex function we have to modify this part, but the only difference would be to consider $f'(c_i)$ as coefficients instead of c_i.

while $t \le T$. Similar computations show that

$$\begin{aligned}&\|v(t,\cdot)-v_i(t,\cdot)\|_{L^1(I_i^t)}\\ &\le \sum_{k\neq i,\,k\ge 1}\sum_{j=1}^{J_k} c_k \underbrace{\int_{I_i^t+\left(-\frac{\ell}{2},\frac{\ell}{2}\right)} \left(\mathcal{F}^{-1}\left(e^{-\frac{\varepsilon t|\cdot|^\alpha}{1+\varepsilon^\alpha|\cdot|^\alpha}}\right) * \chi_{I_{kj}}\right)(x)\,dx}_{=:\widetilde{E}_{kj}}.\end{aligned} \tag{5.16}$$

Putting these bounds in (5.14), we get

$$\begin{aligned}&\|u(t,\cdot)-v(t,\cdot)\|_{L^1} \ge \|q_i(t,\cdot)-z_i(t,\cdot)\|_{L^1(I_i^t)}\\ &\quad-\sum_{k\neq i,\,k\ge 1}\sum_{j=1}^{J_k} c_k\left(E_{kj}+\widetilde{E}_{kj}\right) - \|u_i(t,\cdot)-q_i(t,\cdot)\|_{L^1(I_i^t)} - \|w_i(t,\cdot)-z_i(t,\cdot)\|_{L^1(I_i^t)}.\end{aligned} \tag{5.17}$$

For the last two terms, we use the continuous dependence results with respect to the flux (see [2, Theorem 3.1] for the nonlocal case). We have

$$\|u_i(t,\cdot)-q_i(t,\cdot)\|_{L^1(I_i^t)} \le \underbrace{\|u_i(0,\cdot)-q_i(0,\cdot)\|_{L^1}}_{=0} + \mathrm{TV}\left(\sum_{j=1}^{J_i} c_i\chi_{I_{ij}}\right)\left\|f'-c_i\right\|_{L^\infty(0,c_i)} t \le 2J_i c_i^2 t,$$

and we argue similary for $\|w_i(t,\cdot)-z_i(t,\cdot)\|_{L^1(I_i^t)}$.

Let us now focus on the first term of the right hand side of (5.17). We claim that there are positive C_T and ε_T, only depending on T and α, such that for all $t\in(0,T]$, $\varepsilon\in(0,\varepsilon_T]$ and $i\ge 1$,

$$\|q_i(t,\cdot)-z_i(t,\cdot)\|_{L^1(I_i^t)} \ge C_T J_i c_i t\mathfrak{s}_\varepsilon. \tag{5.18}$$

Let us show it via Proposition 5.2. Note that $w_0^i = c_i\sum_{j=1}^{J_i}\chi_{I_{ij}} = c_i\chi_{E_i}$ where $E_i = \cup_{i=1}^{J_i} I_{ij}$ is almost as in (5.1). The main difference is that the common length and distance between the intervals constituting E_i is not 1 but ℓ. The flux in (5.12) is also not zero as it was in (5.5). We then consider the rescaled solutions $\widetilde{q}_i(t,x) = q_i(\ell^{\alpha-1}t, \ell x + c_i\ell^{\alpha-1}t)$ and $\widetilde{z}_i$ similarly defined. They solve $\partial_t\widetilde{q}_i = \widetilde{\varepsilon}\partial_{xx}^\alpha\widetilde{q}_i$ and $\partial_t\widetilde{z}_i = R_{\widetilde{\varepsilon}}^\alpha(\widetilde{z}_i)$ with $\widetilde{\varepsilon}=\varepsilon/\ell$, while

$$\|q_i(t,\cdot)-z_i(t,\cdot)\|_{L^1(I_i^t)} \ge \ell\int_{E_i/\ell}\left(\widetilde{z}_i(\ell^{1-\alpha}t,x)-\widetilde{q}_i(\ell^{1-\alpha}t,x)\right)dx.$$

We can apply Proposition 5.2 to the latter integral. Indeed $\widetilde{q}_i(t=0) = \widetilde{z}_i(t=0) = c_i\chi_{E_i/\ell}$ where E_i/ℓ is as in (5.1), up to a last space translation to recenter it. With these observations in hands, we can easily obtain (5.18) from (5.6).[12] Notably C_T and ε_T only depend on T and α because $\ell = 2T$, see (5.11). Putting all these estimates in (5.17),

$$\|u(t,\cdot)-v(t,\cdot)\|_{L^1} \ge C_T J_i c_i t\mathfrak{s}_\varepsilon - 4J_i c_i^2 t - \sum_{k\neq i,\,k\ge 1}\sum_{j=1}^{J_k} c_k\left(E_{kj}+\widetilde{E}_{kj}\right). \tag{5.19}$$

We need a more explicit estimate on the nonlocal terms to continue.

Lemma 5.1. *Assume $\alpha\in(0,2]$ and the previous hypotheses on ℓ, $\{\ell_i\}_i$ and $\{J_i\}_i$. There is an increasing modulus $\omega_T(\cdot)$ only depending on T and α such that for all $t\in(0,T]$, $\varepsilon\in(0,1]$, $i\ge 2$, $k\neq i$, $k\ge 1$, $1\le j\le J_k$,*

$$E_{kj}, \widetilde{E}_{kj} \le \varepsilon^{1-\alpha} t\omega_T\left(1/\ell_{i-1}\right). \tag{5.20}$$

[12] To be more precise $2J+2$ was the number of intervals constituting E_J in (5.1). It was larger than 4 in (5.6). Here E_i has J_i intervals. To be in accordance with Proposition 5.2, J_i will then be taken even and larger than 4 in (5.23). In particular $\frac{J_i-2}{2} \ge \frac{J_i}{4}$ and we used it to get (5.18) from (5.6).

Let us admit it for a while. We will do the proof once having completed the one of Theorem 2.5(ii). Using (5.19)

$$\|u(t,\cdot)-v(t,\cdot)\|_{L^1} \geq J_i c_i t \mathfrak{s}_\varepsilon \left(C_T - \frac{4c_i}{\mathfrak{s}_\varepsilon} - \frac{2\varepsilon^{1-\alpha}\sum_{k\geq 1} J_k c_k}{J_i c_i \mathfrak{s}_\varepsilon}\, \omega_T\left(1/\ell_{i-1}\right)\right). \tag{5.21}$$

Recalling (5.18) the above estimate holds for all $t \in (0,T]$, $\varepsilon \in (0,\varepsilon_T]$ and $i \geq 2$, as well as all $\{\ell_i\}_i \subset (\ell,\infty)$, $\{J_i\}_i \subset 4+2\mathbb{N}$ and$\{c_i\}_i$, satisfying (5.9), (5.11) and $\ell_{i+1} \geq \ell_i$. Moreover the positive constants C_T and ε_T only depend on T and α. We notably assumed $\varepsilon_T \leq 1$ without loss of generality when injecting (5.20) into (5.19).

Consider now $\omega : (0,\infty) \to (0,\infty)$ nondecreasing such that $\lim_{\varepsilon\to 0} \omega(\varepsilon)/\mathfrak{s}_\varepsilon = 0$, and let us specify more precisely how to choose the latter sequences to get (2.10). It is useful to have in mind that $\varepsilon \mapsto \mathfrak{s}_\varepsilon$ is increasing on $[0,\varepsilon_0]$ for some $\varepsilon_0 > 0$ only depending on α, see (5.7). Define

$$\widetilde{\omega}(\varepsilon) = \frac{1}{\varepsilon}\int_\varepsilon^{2\varepsilon} \sqrt{\mathfrak{s}_\xi \omega(\xi)}\, d\xi.$$

Being $\widetilde{\omega}$ continuous, increasing and positive for $\varepsilon \leq \varepsilon_0$, we have

$$\lim_{\varepsilon\to 0} \frac{\widetilde{\omega}(\varepsilon)}{\omega(\varepsilon)} = \infty, \qquad \lim_{\varepsilon\to 0} \frac{\widetilde{\omega}(\varepsilon)}{\mathfrak{s}_\varepsilon} = 0 \tag{5.22}$$

([13]). Denoting $\mathfrak{s} \mapsto \varepsilon_\mathfrak{s}$ the inverse of $\varepsilon \mapsto \mathfrak{s}_\varepsilon$ (for $\varepsilon \leq \varepsilon_0$), define for any $i \geq 1$,

$$\begin{cases} i(\varepsilon) = \left\lfloor \sqrt{\mathfrak{s}_\varepsilon/\widetilde{\omega}(\varepsilon)} \right\rfloor, \\ S(i) = \left\{\mathfrak{s} \in (0,\mathfrak{s}_{\varepsilon_0}]\,;\, i = i(\varepsilon_\mathfrak{s})\right\}, \\ c_i = \min\left\{ \frac{C_T}{16}\min S(i), i^{-2}\right\}, \\ J_i = 4\left\lceil c_i^{-1} i^{-2}\right\rceil, \end{cases} \tag{5.23}$$

where the third line is replaced by $c_i = i^{-2}$ if $S(i) = \emptyset$. Denoting moreover $c = 16\sum_{k\geq 1} k^{-2}$ and ω_T^{-1} the reciprocal function to the increasing modulus from Lemma 5.1, define finally

$$\ell_i = \frac{1}{\omega_T^{-1}\left(\frac{C_T}{c(i+1)^2}\min\left\{\frac{\mathfrak{s}_\varepsilon}{\varepsilon^{1-\alpha}}; \varepsilon \in (0,\varepsilon_0] \text{ and } i(\varepsilon) \leq i+1\right\}\right)} \tag{5.24}$$

if $\{\varepsilon \in (0,\varepsilon_0]; i(\varepsilon) \leq i+1\} \neq \emptyset$ and the above right hand side is larger than $2T+1$. If at least one of the two latter conditions fails, set $\ell_i = 2T+1$. We are planning to take $i = i(\varepsilon)$ in (5.21), but we first need to check that the above ℓ_i, J_i and c_i satisfy the conditions required from the beginning.

Note that $\mathfrak{s} \mapsto \mathfrak{s}/\widetilde{\omega}(\varepsilon_\mathfrak{s})$ is continuous and positive on $(0,\mathfrak{s}_{\varepsilon_0}]$ while $\lim_{s\to 0} s/\widetilde{\omega}(\varepsilon_s) = \infty$, thanks to (5.22). Hence $S(i) \neq \emptyset$ provided that i is large enough, and it is moreover positively lower bounded for such i. In particular $0 < c_i \leq i^{-2}$ for any $i \geq 1$, and clearly

$$4i^{-2} \leq J_i c_i \leq 8i^{-2}. \tag{5.25}$$

This shows that (5.9) holds. Moreover $\max_{i\geq 1} c_i \leq 1$ as desired in (5.11), so we have (5.10) with $\ell = 2T$. Finally (5.24) implies that $\infty > \ell_{i+1} \geq \ell_i > \ell$, for any $i \geq 1$, because $\varepsilon \in (0,\varepsilon_0] \mapsto \mathfrak{s}_\varepsilon/\varepsilon^{1-\alpha}$ is continuous and positive while $\lim_{\varepsilon\to 0} i(\varepsilon) = \infty$. This verifies all the conditions previously required.

To continue, use (5.23) and (5.25) to show that for any $\varepsilon \in (0,\varepsilon_0]$,

$$J_{i(\varepsilon)} c_{i(\varepsilon)} \mathfrak{s}_\varepsilon \geq 4i(\varepsilon)^{-2}\mathfrak{s}_\varepsilon = \frac{4\mathfrak{s}_\varepsilon}{\left\lfloor \sqrt{\mathfrak{s}_\varepsilon/\widetilde{\omega}(\varepsilon)}\right\rfloor^2} \geq \frac{4\mathfrak{s}_\varepsilon}{\mathfrak{s}_\varepsilon/\widetilde{\omega}(\varepsilon)} = 4\widetilde{\omega}(\varepsilon).$$

[13] Indeed $\frac{\widetilde{\omega}(\varepsilon)}{\omega(\varepsilon)} \geq \frac{\sqrt{\mathfrak{s}_\varepsilon\,\omega(\varepsilon)}}{\omega(\varepsilon)} = \sqrt{\frac{\mathfrak{s}_\varepsilon}{\omega(\varepsilon)}} \to \infty$ and $\frac{\widetilde{\omega}(\varepsilon)}{\mathfrak{s}_\varepsilon} = \frac{1}{\varepsilon}\int_\varepsilon^{2\varepsilon} \frac{\mathfrak{s}_\xi}{\mathfrak{s}_\varepsilon}\sqrt{\frac{\omega(\xi)}{\mathfrak{s}_\xi}}\, d\xi \to 0$ as $\varepsilon \to 0$.

Taking $i = i(\varepsilon)$ in (5.21) then gives

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1} \geq 4t\widetilde{\omega}(\varepsilon)\left(C_T - \frac{4c_{i(\varepsilon)}}{\mathfrak{s}_\varepsilon} - \frac{c\varepsilon^{1-\alpha}i(\varepsilon)^2}{4\mathfrak{s}_\varepsilon}\,\omega_T\left(1/\ell_{i(\varepsilon)-1}\right)\right), \tag{5.26}$$

where for the last term we used that $\frac{i(\varepsilon)^2}{4} \geq \frac{1}{J_{i(\varepsilon)}c_{i(\varepsilon)}}$ and $c = 16\sum_{k\geq 1}k^{-2} \geq 2\sum_{k\geq 1}J_k c_k$ by (5.25). It remains to estimate the last two terms in parenthesis of (5.26), and it is sufficient to consider small ε because we will take the limit as $\varepsilon \to 0$. Consider more precisely $0 < \varepsilon_\omega \leq \varepsilon_0$ such that $S(i(\varepsilon)) \neq \emptyset$ for any $\varepsilon \in (0, \varepsilon_\omega]$. Such an ε_ω exists and only depends on α and $\omega(\cdot)$ by (5.22). Moreover $\mathfrak{s}_\varepsilon \in S(i(\varepsilon))$ for $\varepsilon \leq \varepsilon_\omega$, by construction. Recalling also the choice of c_i in (5.23), we infer that

$$-4\frac{c_{i(\varepsilon)}}{\mathfrak{s}_\varepsilon} \geq -\frac{C_T}{4\mathfrak{s}_\varepsilon}\min\underbrace{S(i(\varepsilon))}_{\text{containing }\mathfrak{s}_\varepsilon} \geq -\frac{C_T}{4}.$$

Finally using in (5.24) that each ε is such that $i(\varepsilon) \leq (i(\varepsilon)-1)+1$,

$$\ell_{i(\varepsilon)-1} \geq \frac{1}{\omega_T^{-1}\left(\frac{C_T\mathfrak{s}_\varepsilon}{c\varepsilon^{1-\alpha}i(\varepsilon)^2}\right)},$$

and the last term in parenthesis in (5.26) is also larger than $-\frac{C_T}{4}$. The latter equation then rewrites

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1} \geq 2C_T t\widetilde{\omega}(\varepsilon),$$

for all $t \in (0,T]$ and $\varepsilon \in (0, \min\{\varepsilon_0, \varepsilon_\omega\}]$. Recalling (5.22) the proof of Theorem 2.5(ii) is complete. □

Let us now show the lemma which we admitted during the previous proof.

Proof of Lemma 5.1. Let us estimate E_{kj} from (5.15). Consider the case $i < k$. Recalling (5.13) and using the notation from the footnote 10 at page 15, we have $I_i \leq I_{kj}$ and $\operatorname{dist}(I_i, I_{kj}) \geq \ell_i > \ell$. In particular $I_i^t + \left(-\frac{\ell}{2}, \frac{\ell}{2}\right) \leq I_{kj}$ by (5.10). Now by (A.1)–(A.3) and Lemma A.1, there is $c = c(\alpha)$ such that $P_\alpha(x) \leq \frac{c}{|x|^{1+\alpha}}$ and thus $G_\alpha(\varepsilon t, x) \leq \frac{c\varepsilon t}{|x|^{1+\alpha}}$. It follows that

$$\begin{aligned}
E_{kj} &\leq c\varepsilon t\int_{I_{kj}}\left(\int_{I_i^t+\left(-\frac{\ell}{2},\frac{\ell}{2}\right)}\frac{1}{|x-y|^{1+\alpha}}\,dx\right)dy \\
&= \frac{c\varepsilon t}{\alpha}\int_{I_{kj}}\left(\frac{1}{|y-\sup I_i - c_i t - \frac{\ell}{2}|^\alpha} - \frac{1}{|y-\inf I_i - c_i t + \frac{\ell}{2}|^\alpha}\right)dy \\
&\leq \frac{c\varepsilon t\ell}{\alpha}\left(\inf I_{kj} - \sup I_i - \ell\right)^{-\alpha}.
\end{aligned}$$

Since $\ell = 2T$ and $\inf I_{kj} - \sup I_i \geq \ell_i$, we get $E_{kj} \leq \varepsilon t\omega_T(1/\ell_i)$ with $\omega_T(\kappa) = \frac{2cT}{\alpha}\left(1/\kappa - 2T\right)^{-\alpha}$ for any $\kappa > 0$. We leave it to the reader to verify that we can argue similarly for $i > k$. Note just that $\operatorname{dist}(I_{kj}, I_i) \geq \ell_{i-1}$ since then $I_{kj} \leq I_i$. In particular we will get $E_{kj} \leq \varepsilon t\omega_T(1/\ell_{i-1})$. In anyway $\ell_{i-1} \leq \ell_i$ and the latter estimate holds in all cases. It is moreover as desired in (5.20) since $\varepsilon^\alpha \leq 1$ when $\varepsilon \leq 1$.

Let us now estimate $\widetilde{E}_{kj}$ from (5.16). Arguing in Fourier variables,

$$\begin{aligned}
\widetilde{E}_{kj} &= \int_{I_i^t+\left(-\frac{\ell}{2},\frac{\ell}{2}\right)}\mathcal{F}^{-1}\left(e^{-\frac{\varepsilon t|\cdot|^\alpha}{1+\varepsilon^\alpha|\cdot|^\alpha}}\mathcal{F}\left(\chi_{I_{kj}}\right)\right)(x)\,dx \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{\varepsilon t|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}}\mathcal{F}\left(\chi_{I_{kj}}\right)(\xi)\overline{\mathcal{F}\left(\chi_{I_i^t+\left(-\frac{\ell}{2},\frac{\ell}{2}\right)}\right)(\xi)}\,d\xi.
\end{aligned}$$

Recall that $\mathcal{F}(\chi_{(a,b)})(\xi) = 2\frac{\sin\left(\frac{b-a}{2}\xi\right)}{\xi}e^{-i\frac{a+b}{2}\xi}$ for all $a<b$, and let us continue with the shorthand notation $I = I_i^t + \left(-\frac{\ell}{2},\frac{\ell}{2}\right)$. Simple computations lead to

$$\widetilde{E}_{kj} = \frac{2}{\pi}\int_{\mathbb{R}} e^{-\frac{\varepsilon t|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}}\,\frac{\sin\left(\ell\xi/2\right)}{\xi}\,\frac{\sin(|I|\xi/2)}{\xi}\cos\left((m_I - m_{I_{kj}})\xi\right)d\xi \tag{5.27}$$

since $e^{-im_{I_{kj}}\xi}\overline{e^{-im_I\xi}} = \cos\left((m_I-m_{I_{kj}})\xi\right)+i\sin\left((m_I-m_{I_{kj}})\xi\right)$ with the respective intervals' middle points $m_{I_{kj}}$ and m_I. The imaginary part canceled in (5.27) as integral of an odd function. We also denoted $|I|$ the lenght of I and used that $|I_{kj}|=\ell$. Note that if having 1 instead of $e^{-\frac{\varepsilon t|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}}$, the right hand side of (5.27) would cancel. Doing indeed the reverse computations, it would equal $\int_I \chi_{I_{kj}} = 0$ since $I_{kj}\cap I=\emptyset$. Applying then Taylor's formula to rewrite the exponential,

$$\widetilde{E}_{kj} = -\frac{2}{\pi}\int_{\mathbb{R}}\int_0^1 \frac{\varepsilon t|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}\,e^{-\frac{s\varepsilon t|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}}\,\frac{\sin\left(\ell\xi/2\right)}{\xi}\,\frac{\sin(|I|\xi/2)}{\xi}\cos\left((m_I - m_{I_{kj}})\xi\right)ds\,d\xi.$$

Recall now that $\mathcal{F}^{-1}\left(e^{-\frac{t|\cdot|^\alpha}{1+\varepsilon^\alpha|\cdot|^\alpha}}\right)\geq 0$ because $\frac{|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}$ is the (opposite of the) symbol of the Lévy operator R_ε^α, see [27, Thm. 3.7.8 & Def. 3.6.1]. In particular

$$I\mapsto \widetilde{E}_{kj} = \int_I\left(\mathcal{F}^{-1}\left(e^{-\frac{\varepsilon t|\cdot|^\alpha}{1+\varepsilon^\alpha|\cdot|^\alpha}}\right)*\left(\chi_{I_{kj}}\right)\right)(x)\,dx$$

is nondecreasing for the inclusion. Let us consider the case $i>k$ to continue. Defining $I_R = \left(\inf I_i - \frac{\ell}{2}, R\right)\supset I$ for large R, the previous monotonicity shows that

$$\begin{aligned}\widetilde{E}_{kj} \leq -\frac{2}{\pi}\int_{\mathbb{R}}\int_0^1 &\frac{\varepsilon t|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}\,e^{-\frac{s\varepsilon t|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}}\\ &\cdot\frac{\sin\left(\ell\xi/2\right)}{\xi}\,\frac{\sin(|I_R|\xi/2)}{\xi}\cos\left((m_{I_R}-m_{I_{kj}})\xi\right)ds\,d\xi.\end{aligned} \tag{5.28}$$

Moreover $m_{I_R}-m_{I_{kj}} = \left(m_{I_R}-\inf I_i+\frac{\ell}{2}\right)+\left(\inf I_i-\frac{\ell}{2}-m_{I_{kj}}\right) = |I_R|/2+\mathrm{dist}(I_{kj},I_i)$, so

$$\cos\left((m_{I_R}-m_{I_{kj}})\xi\right) = \cos\left(|I_R|\xi/2\right)\cos\left(\xi\mathrm{dist}(I_{kj},I_i)\right)-\sin\left(|I_R|\xi/2\right)\sin\left(\xi\mathrm{dist}(I_{kj},I_i)\right)$$

and injecting this into (5.28) while letting $R\to\infty$ leads to

$$\widetilde{E}_{kj}\leq 2\int_{\mathbb{R}}\int_0^1 \frac{\varepsilon t|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}\,e^{-\frac{s\varepsilon t|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}}\,\frac{\sin\left(\ell\xi/2\right)}{\xi^2}\sin\left(\xi\mathrm{dist}(I_{kj},I_i)\right)ds\,d\xi. \tag{5.29}$$

During this passage to the limit, we used that $\cos\left(|I_R|\xi/2\right)\sin\left(|I_R|\xi/2\right)$ and $\sin^2\left(|I_R|\xi/2\right)$ weakly-$*$ converge in $L^\infty(\mathbb{R})$ to their respective mean values, and noticed that the first limit cancels as integral of an odd function. Now define

$$f_T(\kappa) = 2\sup_{t\in(0,T],\varepsilon\in(0,1]}\int_{\mathbb{R}}\int_0^1 \frac{\varepsilon^\alpha|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}\,e^{-\frac{s\varepsilon t|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}}\,\frac{\sin\left(T\xi\right)}{\xi^2}\sin\left(\xi/\kappa\right)ds\,d\xi,$$

for $\kappa>0$. Note that $\sin\left(\xi/\kappa\right)$ weakly-$*$ vanishes as $\kappa\to 0$, while $\frac{\varepsilon^\alpha|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}\leq|\xi|^\beta$ for all $\varepsilon\in(0,1]$, $\xi\in\mathbb{R}$ and $\beta\in(0,\alpha)$, and $\frac{\sin(T\xi)}{\xi^{2-\beta}}$ is integrable if fixing moreover $0<\beta<\min\{\alpha,1\}$. Hence, elementary arguments using sequences show that $\lim_{\kappa\to0}f_T(\kappa)=0$. We then easily get (5.20) from (5.29) for any increasing modulus above f_T; recall indeed that $\ell=2T$ and $\mathrm{dist}(I_{kj},I_i)\geq\ell_{i-1}$. For $i<k$, we leave it to the reader to verify that taking $I_R = (-R,\sup I_i+\ell)\supset I$ allows to argue similarly. The proof of Lemma 5.1 is complete. □

It remains to show the first part of Theorem 2.5.

Sketch of the proof of Theorem 2.5(i). We can argue as for (ii). Let us give insights for $u-w$. Take w in place of v, σ_ε in place of $\mathfrak{s}_\varepsilon$, and the second equation in (5.12) with zero right hand side. In (5.14) and thanks to (5.10), $\|w(t,\cdot)-w_i(t,\cdot)\|_{L^1(I_i^t)}=0$ while $t\le T$. There is thus no $\widetilde{E}_{kj}$ in (5.19) and we can reach (5.21) more easily. Notably we used (5.18) to show (5.21), and this followed from Proposition 5.2. Now this will follow from Proposition 5.1 but we have to take care of the different forms of $\sigma_{t\varepsilon}$ and $t\mathfrak{s}_\varepsilon$ in (5.3) and (5.6). Note then that $\sigma_{t\varepsilon}\ge t\sigma_\varepsilon$ for all $t\in(0,T]$ and ε small enough, up to a positive multiplicative constant only depending on T and α. We can thus argue with the term $t\sigma_\varepsilon$ being of the same form than $t\mathfrak{s}_\varepsilon$, and the same computations work to get (5.18). The rest of the proof will be the same as well. We could similarly consider $v-w$, for which there would remain $\widetilde{E}_{kj}$ in (5.19) instead of E_{kj}. □

6. Proofs of Propositions 5.1 and 5.2

Let us now establish the results admitted in Section 5.1.

First step: reduction of the proofs to two lemmas. Consider $w_0=\chi_{E_J}$ as in (5.1), and let us first provide some computations on its Fourier transform. We have

$$\begin{aligned}
\mathcal{F}\left(\chi_{E_J}\right)(\xi)&=\int_{\mathbb{R}}\chi_{E_J}(x)e^{-ix\xi}\,dx=\sum_{j=0}^{J}\left(\int_{-2j-\frac{3}{2}}^{-2j-\frac{1}{2}}e^{-ix\xi}\,dx+\int_{2j+\frac{1}{2}}^{2j+\frac{3}{2}}e^{-ix\xi}\,dx\right)\\
&=\sum_{j=0}^{J}\left(e^{i(2j+1)\xi}\int_{-2j-\frac{3}{2}}^{-2j-\frac{1}{2}}e^{-i(x+2j+1)\xi}\,dx+e^{-i(2j+1)\xi}\int_{2j+\frac{1}{2}}^{2j+\frac{3}{2}}e^{-i(x-2j-1)\xi}\,dx\right)\\
&=\sum_{j=0}^{J}\left(e^{i(2j+1)\xi}+e^{-i(2j+1)\xi}\right)\int_{-\frac{1}{2}}^{\frac{1}{2}}e^{-ix\xi}\,dx\\
&=\sum_{j=0}^{J}\left(e^{i(2j+1)\xi}+e^{-i(2j+1)\xi}\right)\frac{\sin\left(\xi/2\right)}{\xi/2}\\
&=\frac{4}{\xi}\sin\left(\frac{\xi}{2}\right)\mathfrak{Re}\left(\sum_{j=0}^{J}e^{i(2j+1)\xi}\right)=\frac{4}{\xi}\sin\left(\frac{\xi}{2}\right)\mathfrak{Re}\left(e^{i\xi}\sum_{j=0}^{J}e^{2ij\xi}\right)\\
&=\frac{4}{\xi}\sin\left(\frac{\xi}{2}\right)\mathfrak{Re}\left(e^{i\xi}\,\frac{e^{2i(J+1)\xi}-1}{e^{2i\xi}-1}\right)=\frac{4}{\xi}\sin\left(\frac{\xi}{2}\right)\mathfrak{Re}\left(\frac{e^{2i(J+1)\xi}-1}{e^{i\xi}-e^{-i\xi}}\right)\\
&=\frac{4}{\xi}\sin\left(\frac{\xi}{2}\right)\mathfrak{Re}\left(\frac{e^{2i(J+1)\xi}-1}{2i\sin(\xi)}\right)=\frac{2}{\xi\sin(\xi)}\sin\left(\frac{\xi}{2}\right)\mathfrak{Re}\left(i-ie^{2i(J+1)\xi}\right),
\end{aligned}$$

which finally gives

$$\mathcal{F}\left(\chi_{E_J}\right)(\xi)=\frac{2\sin(2(J+1)\xi)}{\xi\sin(\xi)}\sin\left(\frac{\xi}{2}\right). \tag{6.1}$$

Consider now entropy solutions u and w of (5.2), as in Proposition 5.1. Since $f'\equiv 0$ they correspond to the classical solutions

$$w(t,x)=\chi_{E_J}(x),\qquad u(t,x)=\left(\mathcal{F}^{-1}\left(e^{-\varepsilon t|\cdot|^\alpha}\right)*\chi_{E_J}\right)(x), \tag{6.2}$$

where we recall that $G_\alpha(t,x)=\mathcal{F}^{-1}\left(e^{-|\cdot|^\alpha}\right)(x)$; see [1, 14] and Appendix A.1. Now by Parseval Identity, the formula $\int\mathcal{F}(f)\,\overline{g}=2\pi\int f\overline{\mathcal{F}^{-1}(g)}$,[14] and since $\chi_{E_J}(\cdot)$ and $e^{-\varepsilon t|\cdot|^\alpha}$ are

[14]The factor 2π comes from our choice of Fourier transform in (1.4).

real-valued and even, we have

$$\begin{aligned}\int_{E_J}(w-u)\,dx &= \int_{E_J}\chi_{E_J}\,dx - \int_{E_J}\left(\mathcal{F}^{-1}\left(e^{-\varepsilon t|\cdot|^\alpha}\right)*\chi_{E_J}\right)dx\\ &= \int_{\mathbb{R}}|\chi_{E_J}|^2\,dx - \int_{\mathbb{R}}\mathcal{F}^{-1}\left(e^{-\varepsilon t|\cdot|^\alpha}\right)(\chi_{E_J}*\chi_{E_J})\,dx\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}|\mathcal{F}(\chi_{E_J})|^2\,d\xi - \frac{1}{2\pi}\int_{\mathbb{R}}e^{-\varepsilon t|\xi|^\alpha}\mathcal{F}\underbrace{(\chi_{E_J}*\chi_{E_J})}_{=|\mathcal{F}(\chi_{E_J})|^2}d\xi\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\left(1-e^{-\varepsilon t|\xi|^\alpha}\right)|\mathcal{F}(\chi_{E_J})|^2\,d\xi.\end{aligned}$$

Therefore, using (6.1),

$$\int_{E_J}(w-u)\,dx = \frac{2}{\pi}\int_{\mathbb{R}}\left(1-e^{-\varepsilon t|\xi|^\alpha}\right)\frac{\sin^2(2(J+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi. \tag{6.3}$$

Denoting $t\varepsilon = r$ and defining

$$\mathcal{E}(r,J) = \frac{1}{J\sigma_r}\int_{\mathbb{R}}\left(1-e^{-r|\xi|^\alpha}\right)\frac{\sin^2(2(J+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi,$$

we deduce from (6.3) that

$$\frac{1}{J\sigma_{t\varepsilon}}\int_{E_J}(w-u)(t,x)\,dx = \frac{2}{\pi}\,\mathcal{E}(t\varepsilon,J).$$

This reduces the proof of (5.3) to the one of the following lemma.

Lemma 6.1. *The above quantity satisfies* $\inf_{0<r\le 1,\,J\ge 1}\mathcal{E}(r,J)>0$.

The proof consists of a long case by case study. Before doing it, let us repeat the same procedure for Proposition 5.2.

Starting again from (6.1), the difference is that w is replaced by v solving (5.5). Recalling (1.9), a standard Fourier procedure now leads to

$$v(t,x) = \left(\mathcal{F}^{-1}\left(e^{-\varepsilon t\frac{|\cdot|^\alpha}{1+\varepsilon^\alpha|\cdot|^\alpha}}\right)*\chi_{E_J}\right)(x),$$

in place of w in (6.2), and Equation (6.3) becomes

$$\int_{E_J}(v-u)\,dx = \frac{2}{\pi}\int_{\mathbb{R}}\left(e^{-\varepsilon t\frac{|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}}-e^{-\varepsilon t|\xi|^\alpha}\right)\frac{\sin^2(2(J+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi.$$

Defining

$$\mathcal{E}(t,\varepsilon,J) = \frac{1}{Jt\mathfrak{s}_\varepsilon}\int_{\mathbb{R}}\left(e^{-\varepsilon t\frac{|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}}-e^{-t\varepsilon|\xi|^\alpha}\right)\frac{\sin^2(2(J+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi,$$

the proof of (5.6) reduces to the one of the following lemma.

Lemma 6.2. *For any* $T>0$*, the above quantity is such that* $\inf\limits_{\substack{0<t\le T,\\ 0<\varepsilon\le 1,\,J\ge 1}}\mathcal{E}(t,\varepsilon,J)>0$.

Second step: proofs of Lemmas 6.1 and 6.2. We need a preliminary result.

Lemma 6.3. *For any* $\varphi\in C\left(\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\right)$,

$$\lim_{J\to\infty}\frac{1}{J}\int_{\pi/2}^{3\pi/2}\varphi(\xi)\,\frac{\sin^2(2(J+1)\xi)}{\sin^2(\xi)}\,d\xi = 2\pi\varphi(\pi). \tag{6.4}$$

Proof. Using that $\xi \mapsto \frac{\sin^2(2(J+1)\xi)}{\sin^2(\xi)}$ is π-periodic, and denoting $z = (\xi - \pi)(J+1)$,

$$\frac{1}{J}\int_{\pi/2}^{3\pi/2} \varphi(\xi)\,\frac{\sin^2(2(J+1)\xi)}{\sin^2(\xi)}\,d\xi = 4\,\frac{J+1}{J}\int_{-\pi(J+1)/2}^{\pi(J+1)/2} \varphi\left(\pi + \frac{z}{J+1}\right)\frac{\sin^2(2z)}{(2z)^2}\,\frac{\left(\frac{z}{J+1}\right)^2}{\sin^2\left(\frac{z}{J+1}\right)}\,dz.$$

Since

$$\int_{\mathbb{R}} \frac{\sin^2(2z)}{(2z)^2}\,dz = \frac{\pi}{2} \qquad \text{and} \qquad \max_{z\in\left[-\frac{\pi(J+1)}{2},\frac{\pi(J+1)}{2}\right]} \frac{\left(\frac{z}{J+1}\right)^2}{\sin^2\left(\frac{z}{J+1}\right)} = \frac{\pi^2}{4},$$

([15]) the Dominated Convergence Theorem shows that

$$\begin{aligned}&\lim_{J\to\infty} 4\,\frac{J+1}{J}\int_{-\pi(J+1)/2}^{\pi(J+1)/2} \varphi\left(\pi + \frac{z}{J+1}\right)\frac{\sin^2(2z)}{(2z)^2}\,\frac{\left(\frac{z}{J+1}\right)^2}{\sin^2\left(\frac{z}{J+1}\right)}\,dz\\ &= 4\varphi(\pi)\int_{\mathbb{R}} \frac{\sin^2(2z)}{(2z)^2}\,dz = 2\pi\varphi(\pi).\end{aligned}$$

□

We are ready to prove Lemmas 6.1 and 6.2.

Proof of Lemma 6.1. Let $\{(r_n, J_n)\}_n \subset (0,1]\times[1,\infty)$ be a minimizing sequence, namely

$$\inf_{0<r\le 1,\, J\ge 1} \mathcal{E}(r,J) = \lim_n \mathcal{E}(r_n, J_n). \tag{6.5}$$

Passing to a subsequence, we can assume that

$$\exists \lim_n r_n = r_\infty \in [0,1], \qquad \exists \lim_n J_n = J_\infty \in [1,\infty].$$

We distinguish eigth cases. Let us recall and notice that

$$\begin{aligned}\mathcal{E}(r_n,J_n) &= \frac{1}{J_n\sigma_{r_n}}\int_{\mathbb{R}} \left(1 - e^{-r_n|\xi|^\alpha}\right)\frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi\\ &= \frac{r_n}{J_n\sigma_{r_n}}\int_{\mathbb{R}}\int_0^1 |\xi|^\alpha e^{-sr_n|\xi|^\alpha}\,\frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)ds\,d\xi.\end{aligned} \tag{6.6}$$

CASE 1. Assume

$$r_\infty = 0, \qquad J_\infty = \infty, \qquad \alpha < 1.$$

Since $r_n \le 1$, (6.6) and (6.4) imply that

$$\mathcal{E}(r_n,J_n) \ge \frac{1}{J_n}\int_{\pi/2}^{3\pi/2} \xi^\alpha e^{-\xi^\alpha}\,\frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi \xrightarrow{n\to\infty} 2\pi^{\alpha-1}e^{-\pi^\alpha} > 0.$$

CASE 2. Assume

$$r_\infty = 0, \qquad J_\infty = \infty, \qquad \alpha > 1.$$

Denoting $c_0 = \min_{z\in[\pi/2,3\pi/2]} \frac{1-e^{-z^\alpha}}{z^2} > 0$ and using that $\xi \mapsto \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)$ is 2π-periodic, we have

$$\begin{aligned}\mathcal{E}(r_n,J_n) &= \frac{1}{J_n r_n^{1/\alpha}}\int_{\mathbb{R}} \left(1 - e^{-r_n|\xi|^\alpha}\right)\frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi\\ &= \frac{1}{J_n}\int_{\mathbb{R}} \left(1 - e^{-|z|^\alpha}\right)\frac{\sin^2(2(J_n+1)r_n^{-1/\alpha}z)}{z^2\sin^2(r_n^{-1/\alpha}z)}\sin^2\left(\frac{r_n^{-1/\alpha}z}{2}\right)dz\end{aligned}$$

[15] For the first identity use that $1 = \int|\chi_{(-1/2,1/2)}|^2 = \frac{1}{2\pi}\int|\mathcal{F}(\chi_{(-1/2,1/2)})|^2 = \frac{1}{2\pi}\int_{\mathbb{R}}\frac{\sin(\xi/2)}{\xi/2}\,d\xi$. For the second one, use the symmetry and monotonicity of the function whose we take the maximum.

$$
\begin{aligned}
&\geq \frac{c_0}{J_n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sin^2(2(J_n+1)r_n^{-1/\alpha}z)}{\sin^2(r_n^{-1/\alpha}z)}\sin^2\left(\frac{r_n^{-1/\alpha}z}{2}\right)dz\\
&= \frac{c_0 r_n^{1/\alpha}}{J_n}\int_{\frac{\pi r_n^{-1/\alpha}}{2}}^{\frac{3\pi r_n^{-1/\alpha}}{2}} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi\\
&\geq \frac{c_0 r_n^{1/\alpha}}{J_n}\int_{\frac{\pi r_n^{-1/\alpha}}{2}}^{\frac{\pi r_n^{-1/\alpha}}{2}+2\pi\left\lfloor\frac{1}{2r_n^{1/\alpha}}\right\rfloor} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi\\
&= \frac{c_0 r_n^{1/\alpha}}{J_n}\left\lfloor\frac{1}{2r_n^{1/\alpha}}\right\rfloor\int_{\frac{\pi r_n^{-1/\alpha}}{2}}^{\frac{\pi r_n^{-1/\alpha}}{2}+2\pi} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi\\
&= \frac{c_0 r_n^{1/\alpha}}{J_n}\left\lfloor\frac{1}{2r_n^{1/\alpha}}\right\rfloor\int_{\pi/2}^{3\pi/2} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi.
\end{aligned}
$$

Hence

$$
\liminf_n \mathcal{E}(r_n,J_n) \geq \liminf_n \frac{c_0 r_n^{1/\alpha}}{J_n}\left\lfloor\frac{1}{2r_n^{1/\alpha}}\right\rfloor\int_{\pi/2}^{3\pi/2} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi, \tag{6.7}
$$

and this limit is $c_0\pi > 0$ by (6.4).

CASE 3. Assume

$$
r_\infty = 0, \qquad J_\infty = \infty, \qquad \alpha = 1.
$$

For n so large that $\pi/2$ is less than $1/r_n$, we infer from (6.6) that

$$
\begin{aligned}
\mathcal{E}(r_n,J_n) &\geq \frac{1}{J_n e\ln(1/r_n)}\int_{\pi/2}^{1/r_n} \frac{\sin^2(2(J_n+1)\xi)}{\xi\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi\\
&\geq \frac{1}{J_n e\ln(1/r_n)}\sum_{k\geq 0,\,(2k+\frac{3}{2})\pi\leq\frac{1}{r_n}}\int_{(2k+\frac{1}{2})\pi}^{(2k+\frac{3}{2})\pi} \frac{\sin^2(2(J_n+1)\xi)}{\xi\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi\\
&\geq \frac{1}{2J_n e\ln(1/r_n)}\sum_{k\geq 0,\,(2k+\frac{3}{2})\pi\leq\frac{1}{r_n}}\int_{(2k+\frac{1}{2})\pi}^{(2k+\frac{3}{2})\pi} \frac{\sin^2(2(J_n+1)\xi)}{\xi\sin^2(\xi)}d\xi\\
&\geq \frac{1}{2J_n e\ln(1/r_n)}\sum_{k\geq 0,\,(2k+\frac{3}{2})\pi\leq\frac{1}{r_n}}\frac{1}{(2k+\frac{3}{2})\pi}\int_{(2k+\frac{1}{2})\pi}^{(2k+\frac{3}{2})\pi} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}d\xi\\
&= \frac{1}{2J_n e\ln(1/r_n)}\int_{\pi/2}^{3\pi/2} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}d\xi\sum_{k\geq 0,\,(2k+\frac{3}{2})\pi\leq\frac{1}{r_n}}\frac{1}{(2k+\frac{3}{2})\pi}.
\end{aligned}
$$

We notably used that $\sin^2(\xi/2) \geq 1/2$ for $\xi \in \left(\frac{\pi}{2},\frac{3\pi}{2}\right)+2\pi\mathbb{N}$. Using now that $\sum_{m=1}^M m^{-1} \overset{M\to\infty}{\sim} \ln(M)$, we have $c_1 := \lim_n \frac{1}{\ln(1/r_n)}\sum_{k\geq 0,\,(2k+\frac{3}{2})\pi\leq\frac{1}{r_n}}\frac{1}{(2k+\frac{3}{2})\pi} > 0$. In particular

$$
\liminf_n \mathcal{E}(r_n,J_n) \geq \frac{c_1}{2e}\liminf_n \frac{1}{J_n}\int_{\pi/2}^{3\pi/2} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}d\xi, \tag{6.8}
$$

and this limit is $\frac{c_1\pi}{e} > 0$ by (6.4).

CASE 4. Assume

$$
r_\infty > 0, \qquad J_\infty = \infty.
$$

Since $\sigma_{r_n} \le 1$ and $r_n \ge \frac{r_\infty}{2}$ for large n, (6.6) implies that

$$\mathcal{E}(r_n, J_n) \ge \frac{1}{J_n} \int_{\pi/2}^{3\pi/2} \left(1 - e^{-\frac{r_\infty}{2}\xi^\alpha}\right) \frac{\sin^2(2(J_n+1)\xi)}{\xi^2 \sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi \xrightarrow{n\to\infty} \frac{2}{\pi}\left(1 - e^{-\frac{r_\infty}{2}\pi^\alpha}\right) > 0.$$

CASE 5. Assume

$$r_\infty = 0, \qquad J_\infty < \infty, \qquad \alpha < 1.$$

By (6.6), we have $\mathcal{E}(r_n, J_n) \ge \frac{1}{J_\infty} \int_{\mathbb{R}} |\xi|^\alpha e^{-|\xi|^\alpha} \frac{\sin^2(2(J_\infty+1)\xi)}{\xi^2 \sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi > 0$ for any n large enough so that $J_n = J_\infty$.

CASE 6. Assume

$$r_\infty = 0, \qquad J_\infty < \infty, \qquad \alpha > 1.$$

By (6.7), $\liminf_n \mathcal{E}(r_n, J_n) \ge \frac{c_0}{2J_\infty} \int_{\pi/2}^{3\pi/2} \frac{\sin^2(2(J_\infty+1)s)}{\sin^2(s)} \sin^2\left(\frac{s}{2}\right) ds > 0$.

CASE 7. Assume

$$r_\infty = 0, \qquad J_\infty < \infty, \qquad \alpha = 1.$$

By (6.8), $\liminf_n \mathcal{E}(r_n, J_n) \ge \frac{c_1}{2eJ_\infty} \int_{\pi/2}^{3\pi/2} \frac{\sin^2(2(J_\infty+1)\xi)}{\sin^2(\xi)} d\xi > 0$.

CASE 8. Assume

$$r_\infty > 0, \qquad J_\infty < \infty.$$

Using again that $\sigma_{r_n} \le 1$, $r_n \ge \frac{r_\infty}{2}$ and $J_n = J_\infty$ for large n, we have

$$\mathcal{E}(r_n, J_n) \ge \frac{1}{J_\infty} \int_{\mathbb{R}} \left(1 - e^{-\frac{r_\infty}{2}|\xi|^\alpha}\right) \frac{\sin^2(2(J_\infty+1)\xi)}{\xi^2 \sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi > 0.$$

In this way, thanks to (6.5), Lemma 6.1 is completely proved. □

Proof of Lemma 6.2. Let $\{(t_n, \varepsilon_n, J_n)\}_n \subset (0,T] \times (0,1] \times [1,\infty)$ be a minimizing sequence, namely

$$\inf_{\substack{0<t\le T,\\ 0<\varepsilon\le 1,\, J\ge 1}} \mathcal{E}(t,\varepsilon,J) = \lim_n \mathcal{E}(t_n, \varepsilon_n, J_n).$$

Passing to a subsequence, we can assume that

$$\exists \lim_n \varepsilon_n = \varepsilon_\infty \in [0,1], \qquad \exists \lim_n J_n = J_\infty \in [1,\infty].$$

We distinguish eigth cases. Recall and notice that

$$\begin{aligned} \mathcal{E}(t_n,\varepsilon_n,J_n) &= \frac{1}{J_n t_n \mathfrak{s}_{\varepsilon_n}} \int_{\mathbb{R}} \left(e^{-t_n\varepsilon_n \frac{|\xi|^\alpha}{1+\varepsilon_n^\alpha|\xi|^\alpha}} - e^{-t_n\varepsilon_n|\xi|^\alpha}\right) \frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi \\ &= \frac{\varepsilon_n^{1+\alpha}}{J_n \mathfrak{s}_{\varepsilon_n}} \int_{\mathbb{R}} \int_0^1 \frac{|\xi|^{2\alpha}}{1+\varepsilon_n^\alpha|\xi|^\alpha} \exp\left(-t_n\varepsilon_n|\xi|^\alpha \frac{1+s\varepsilon_n^\alpha|\xi|^\alpha}{1+\varepsilon_n^\alpha|\xi|^\alpha}\right) \\ &\qquad \cdot \frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) ds\, d\xi. \end{aligned} \tag{6.9}$$

CASE 1. Assume

$$\varepsilon_\infty = 0, \qquad J_\infty = \infty, \qquad \alpha < 1/2.$$

By (6.9) and (6.4),

$$\mathcal{E}(t_n,\varepsilon_n,J_n) \ge \frac{1}{J_n} \int_{\pi/2}^{3\pi/2} \int_0^1 \frac{\xi^{2(\alpha-1)}}{1+\xi^\alpha} e^{-T\xi^\alpha} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) ds\, d\xi \xrightarrow{n\to\infty} \frac{2\pi^{2\alpha-1}}{1+\pi^\alpha} e^{-T\pi^\alpha} > 0.$$

CASE 2. Assume

$$\varepsilon_\infty = 0, \qquad J_\infty = \infty, \qquad 1/2 < \alpha \leq 1.$$

Define $c_0 = \min_{z\in[\pi/2,3\pi/2]} \frac{z^{2(\alpha-1)}}{1+z^\alpha}\, e^{-Tz^\alpha} > 0$. By (6.9) and 2π-periodicity of the function $\xi \mapsto \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right)$, we have

$$\begin{aligned}
&\mathcal{E}(t_n,\varepsilon_n,J_n)\\
&= \frac{1}{J_n\varepsilon_n^{1-2\alpha}} \int_{\mathbb{R}}\int_0^1 \frac{|\xi|^{2(\alpha-1)}}{1+\varepsilon_n^\alpha|\xi|^\alpha} \exp\left(-t_n\varepsilon_n|\xi|^\alpha \frac{1+s\varepsilon_n^\alpha|\xi|^\alpha}{1+\varepsilon_n^\alpha|\xi|^\alpha}\right) \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) ds\, d\xi\\
&= \frac{1}{J_n} \int_{\mathbb{R}}\int_0^1 \frac{|z|^{2(\alpha-1)}}{1+|z|^\alpha} \exp\left(-t_n\varepsilon_n^{1-\alpha}|z|^\alpha \frac{1+s|z|^\alpha}{1+|z|^\alpha}\right) \frac{\sin^2(2(J_n+1)z\varepsilon_n^{-1})}{\sin^2(z\varepsilon_n^{-1})} \sin^2\left(\frac{z}{2\varepsilon_n}\right) ds\, dz\\
&\geq \frac{1}{J_n} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{z^{2(\alpha-1)}}{1+z^\alpha} e^{-Tz^\alpha} \frac{\sin^2(2(J_n+1)z\varepsilon_n^{-1})}{\sin^2(z\varepsilon_n^{-1})} \sin^2\left(\frac{z}{2\varepsilon_n}\right) dz\\
&\geq \frac{c_0}{J_n} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sin^2(2(J_n+1)z\varepsilon_n^{-1})}{\sin^2(z\varepsilon_n^{-1})} \sin^2\left(\frac{z}{2\varepsilon_n}\right) dz = \frac{c_0\varepsilon_n}{J_n} \int_{\frac{\pi}{2\varepsilon_n}}^{\frac{3\pi}{2\varepsilon_n}} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi\\
&\geq \frac{c_0\varepsilon_n}{J_n} \int_{\frac{\pi}{2\varepsilon_n}}^{\frac{\pi}{2\varepsilon_n}+2\pi\left\lfloor\frac{1}{2\varepsilon_n}\right\rfloor} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi\\
&= \frac{c_0\varepsilon_n}{J_n} \left\lfloor\frac{1}{2\varepsilon_n}\right\rfloor \int_0^{2\pi} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi.
\end{aligned}$$

Hence

$$\lim_n \mathcal{E}(t_n,\varepsilon_n,J_n) \geq \lim_n \frac{c_0\varepsilon_n}{J_n} \left\lfloor\frac{1}{2\varepsilon_n}\right\rfloor \int_{\pi/2}^{3\pi/2} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi, \tag{6.10}$$

and this limit is $c_0\pi > 0$ by (6.4).

CASE 3. Assume

$$\varepsilon_\infty = 0, \qquad J_\infty = \infty, \qquad \alpha = 1/2.$$

Let us start again from (6.9) and argue as in CASE 3 of the previous lemma. We get

$$\begin{aligned}
&\mathcal{E}(t_n,\varepsilon_n,J_n)\\
&\geq \frac{1}{2J_ne^T\ln(1/\varepsilon_n)} \int_{\pi/2}^{\varepsilon_n^{-1}} \frac{\sin^2(2(J_n+1)\xi)}{\xi\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi\\
&\geq \frac{1}{4J_ne^T\ln(1/\varepsilon_n)} \sum_{k\geq 0, \left(2k+\frac{3}{2}\right)\pi\leq\varepsilon_n^{-1}} \int_{\left(2k+\frac{1}{2}\right)\pi}^{\left(2k+\frac{3}{2}\right)\pi} \frac{\sin^2(2(J_n+1)\xi)}{\xi\sin^2(\xi)} d\xi\\
&\geq \frac{1}{4J_ne^T\ln(1/\varepsilon_n)} \sum_{k\geq 0, \left(2k+\frac{3}{2}\right)\pi\leq\varepsilon_n^{-1}} \frac{1}{\left(2k+\frac{3}{2}\right)\pi} \int_{\left(2k+\frac{1}{2}\right)\pi}^{\left(2k+\frac{3}{2}\right)\pi} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} d\xi\\
&= \frac{1}{4J_ne^T\ln(1/\varepsilon_n)} \int_{\pi/2}^{3\pi/2} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} d\xi \sum_{k\geq 0, \left(2k+\frac{3}{2}\right)\pi\leq\varepsilon_n^{-1}} \frac{1}{\left(2k+\frac{3}{2}\right)\pi}.
\end{aligned}$$

Denoting $c_1 = \lim_n \frac{1}{\ln(1/\varepsilon_n)} \sum_{k\geq 0,\, (2k+\frac{3}{2})\pi\leq\varepsilon_n^{-1}} \frac{1}{(2k+\frac{3}{2})\pi} > 0$, we have

$$\lim_n \mathcal{E}(t_n,\varepsilon_n,J_n) \geq \frac{c_1}{4e^T} \lim_n \frac{1}{J_n} \int_{\pi/2}^{3\pi/2} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} d\xi, \tag{6.11}$$

and this limit is $c_1\pi e^{-T}/2 > 0$ by (6.4).

CASE 4. Assume

$$\varepsilon_\infty > 0, \qquad J_\infty = \infty.$$

Since $\mathfrak{s}_\varepsilon \le 1$ and $\varepsilon_n \in (\frac{\varepsilon_\infty}{2}, 1]$ for large n, (6.9) implies that

$$\mathcal{E}(t_n, \varepsilon_n, J_n) \ge \frac{(\frac{\varepsilon_\infty}{2})^{1+\alpha}}{J_n} \int_{\pi/2}^{3\pi/2} \frac{\xi^{2(\alpha-1)}}{1+\xi^\alpha} e^{-T\xi^\alpha} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi,$$

whose limit as $n \to \infty$ is $\frac{2\pi^{2\alpha-1}}{1+\pi^\alpha} e^{-T\pi^\alpha} (\frac{\varepsilon_\infty}{2})^{1+\alpha} > 0$ by (6.4).

CASE 5. Assume

$$\varepsilon_\infty = 0, \qquad J_\infty < \infty, \qquad \alpha < 1/2.$$

Recall that $J_n = J_\infty$ for large n, so by (6.9) we have

$$\mathcal{E}(t_n, \varepsilon_n, J_n) \ge \frac{1}{J_\infty} \int_{\mathbb{R}} \frac{|\xi|^{2\alpha}}{1+|\xi|^\alpha} e^{-T|\xi|^\alpha} \frac{\sin^2(2(J_\infty+1)\xi)}{\xi^2 \sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi > 0.$$

CASE 6. Assume

$$\varepsilon_\infty = 0, \qquad J_\infty < \infty, \qquad 1/2 < \alpha \le 1.$$

By (6.10), $\lim_n \mathcal{E}(t_n, \varepsilon_n, J_n) \ge \frac{c_0}{2J_\infty} \int_{\pi/2}^{3\pi/2} \frac{\sin^2(2(J_\infty+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi > 0.$

CASE 7. Assume

$$\varepsilon_\infty = 0, \qquad J_\infty < \infty, \qquad \alpha = 1/2.$$

By (6.11), $\lim_n \mathcal{E}(t_n, \varepsilon_n, J_n) \ge \frac{c_1 e^{-T}}{4J_\infty} \int_{\pi/2}^{3\pi/2} \frac{\sin^2(2(J_\infty+1)\xi)}{\sin^2(\xi)} d\xi > 0.$

CASE 8. Assume

$$\varepsilon_\infty > 0, \qquad J_\infty < \infty.$$

Using again that $\mathfrak{s}_{\varepsilon_n} \le 1$, $\varepsilon_n \in (\frac{\varepsilon_\infty}{2}, 1]$ and $J_n = J_\infty$ for large n, (6.9) implies that

$$\mathcal{E}(t_n, \varepsilon_n, J_n) \ge \frac{(\frac{\varepsilon_\infty}{2})^{1+\alpha}}{J_\infty} \int_{\mathbb{R}} \frac{|\xi|^{2(\alpha-1)}}{1+|\xi|^\alpha} e^{-T|\xi|^\alpha} \frac{\sin^2(2(J_\infty+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi > 0. \qquad \square$$

Conclusion. Having established Lemmas 6.1 and 6.2 and recalling that the proofs of Propositions 5.1 and 5.2 were reduced to these two lemmas, we have completed the proofs of these two propositions. $\square$

Appendix A. Some tools of fractional calculus

This appendix recalls basic facts on the fractional heat kernel and Bessel functions. It moreover contains a proof of Lemma 2.1, and technical results on g_α. Most of the material is well known, see e.g. the books [7, 8, 9, 27, 38, 40, 41] and references therein.

A.1. **The fractional heat kernel and Bessel functions.** Given $\alpha \in (0, 2]$, define for any $t > 0$ and $x \in \mathbb{R}^d$,

$$G_\alpha(t,x) = \mathcal{F}^{-1}\left(e^{-t|\cdot|^\alpha}\right)(x) \qquad \text{and} \qquad P_\alpha(x) = G_\alpha(1,x). \tag{A.1}$$

Recall that

$$G_\alpha(t,x) = G_\alpha(1, t^{-\frac{1}{\alpha}}x) =: t^{-\frac{d}{\alpha}} P_\alpha(t^{-\frac{1}{\alpha}}x), \tag{A.2}$$

where

$$\begin{cases} 0 \le P_\alpha \in C_b^\infty \cap W^{\infty,1}(\mathbb{R}^d), \text{ and} \\ P_\alpha \text{ is radially symmetric with } \|P_\alpha\|_{L^1} = 1. \end{cases} \tag{A.3}$$

For proofs, see e.g. [11, 22, 23, 26, 33]. Here is another property due to [11, 33].

Lemma A.1. *Let $\alpha \in (0,2)$. Then $\lim_{|x|\to\infty} |x|^{d+\alpha} P_\alpha(x) = c_\alpha$ with c_α as in (2.3).*

The exact value of this limit is important for us. Here are details for completeness.

Proof. By [11, Theorem 1], $\lim_{|x|\to\infty} |x|^{d+\alpha} P_\alpha(x) = \frac{\alpha 2^{\alpha-1}}{\pi^{\frac{d}{2}+1}} \sin\left(\frac{\alpha\pi}{2}\right) \Gamma\left(\frac{d+\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)$. Indeed the Fourier transform used in [11, Equation (1.1)] is as in (1.4) up to the choice of the sign of $\pm ix\cdot\xi$, but this sign does not change the above limit since $e^{-|\cdot|^\alpha}$ and $P_\alpha = \mathcal{F}^{-1}(e^{-|\cdot|^\alpha})$ are even. Conclude by Euler Reflection Formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, valid for $z \notin \mathbb{Z}$. □

Consider now the Bessel function

$$\mathcal{J}_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\gamma+1)} \left(\frac{z}{2}\right)^{2n+\gamma},$$

which we use for $\gamma = -1/2$, $1/2$, $3/2$, etc. For any $\gamma > -1/2$,

$$\text{(A.4)} \qquad \begin{cases} \mathcal{J}_\gamma \in C^\infty(\mathbb{R}) \text{ (meaning it is real-valued if considering it on } \mathbb{R}), \\ \mathcal{J}_\gamma(z) = O\left(1/\sqrt{|z|}\right) \text{ as } |z| \to \infty, \text{ and} \\ (z^\gamma \mathcal{J}_\gamma)' = z^\gamma \mathcal{J}_{\gamma-1}, \end{cases}$$

while $\mathcal{J}_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos(z)$. Moreover:

Theorem A.1. *Let $f \in L^1(\mathbb{R}^d)$ be radially symmetric of the form $f(|\xi|) = f_0(|\xi|)$. Then*[16]

$$\text{(A.5)} \qquad \mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{\frac{d}{2}} |x|^{\frac{d}{2}-1}} \int_{\mathbb{R}^+} f_0(r) r^{\frac{d}{2}} \mathcal{J}_{\frac{d}{2}-1}(r|x|)\, dr.$$

For proofs, see e.g. [40, Chap. IV, Sect. 3, Thm. 3.3 & Lem. 3.11].[17] See also [8, 41] for standard references on Bessel functions.

A.2. **Calculation of the fractional Rosenau Lévy measure.** In this section we prove Lemma 2.1. Let us recall that it is standard in subordinate semigroup theory [7, 9, 27, 38], but the exact computation of the measure ν is fundamental for us, so we give details for completeness. The proof is very short from the knowledge of G_α from (A.1) and the result below.

Lemma A.2. *Let $\alpha \in (0,2]$. Then $g_\alpha = \mathcal{F}^{-1}\left(\frac{1}{1+|\cdot|^\alpha}\right) \in L^1(\mathbb{R}^d)$ and*

$$\text{(A.6)} \qquad g_\alpha(x) = \int_0^\infty e^{-t} G_\alpha(t,x)\, dt = \int_0^\infty e^{-t} t^{-\frac{d}{\alpha}} P_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dt.$$

This is a classical formula on semigroups and their resolvents, see e.g. [27]. Let us give the proof for completeness.

Proof. Define $f(x)$ to be the right hand side of (A.6). It is measurable with values in $[0,\infty]$, thanks to the nonnegativity of P_α, see (A.2)–(A.3). Moreover

$$\|f\|_{L^1} = \int_0^\infty e^{-t} \|G_\alpha(t,\cdot)\|_{L^1}\, dt = \int_0^\infty e^{-t}\, dt = 1 < \infty.$$

This shows that $f \in L^1(\mathbb{R}^d)$ thus $\mathcal{F}(f)$ is continuous, and it suffices to show that $\mathcal{F}(f)(\xi) = \frac{1}{1+|\xi|^\alpha}$ for any $\xi \in \mathbb{R}^d$. By the Fubini-Tonelli Theorem,

$$\begin{aligned} \mathcal{F}(f)(\xi) &= \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \int_0^\infty e^{-t} G_\alpha(t,x)\, dt\, dx = \int_0^\infty e^{-t} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} G_\alpha(t,x)\, dx\, dt \\ &= \int_0^\infty e^{-t(1+|\xi|^\alpha)}\, dt = \frac{1}{1+|\xi|^\alpha}. \end{aligned}$$

□

[16] The right hand side of (A.5) makes sense by (A.4) and since $\int_0^\infty |f_0(r)| r^{d-1}\, dr < \infty$, thanks to recomputing $\|f\|_{L^1}$ in spherical coordinates.

[17] In [40] the above theorem is stated for another choice of Fourier transform, see [40, Page 2]. This is why we have different constants. The proof is given for $d \geq 2$, but the result for $d = 1$ easily follows from $\mathcal{F}^{-1}(f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \cos(x\xi) f(\xi)\, d\xi$ rewritting ξ as r.

Proof of Lemma 2.1. The properties in (2.5) are immediate from Lemma A.2 and (A.3). Recalling our notation $g_\alpha^\varepsilon(x) = \varepsilon^{-d} g_\alpha(x/\varepsilon)$ we have for all $\phi \in C_c^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} R_\varepsilon^\alpha(\phi)(x) &= -\mathcal{F}^{-1}\left(\frac{|\cdot|^\alpha}{1+\varepsilon^\alpha|\cdot|^\alpha}\,\mathcal{F}(\phi)\right)(x) \\ &= -\frac{\phi(x)}{\varepsilon^\alpha} + \frac{1}{\varepsilon^\alpha}\left(\mathcal{F}^{-1}\left(\frac{1}{1+\varepsilon^\alpha|\cdot|^\alpha}\right) * \phi\right)(x) \\ &= -\frac{\phi(x)}{\varepsilon^\alpha} + \frac{1}{\varepsilon^\alpha}\left(g_\alpha^\varepsilon * \phi\right)(x) \\ &= \int_{\mathbb{R}^d} \left(\phi(x+z)-\phi(x)\right)\frac{g_\alpha^\varepsilon(z)}{\varepsilon^\alpha}\,dz, \end{aligned}$$

since $\int g_\alpha^\varepsilon = 1$ by (2.5) which we already proved. Using now that g_α^ε is radially symmetric (thus even), we have $\int_{|z|\le 1} \nabla\phi(x)\cdot z g_\alpha^\varepsilon(z)\,dz = 0$ and

$$R_\varepsilon^\alpha(\phi)(x) = \int_{\mathbb{R}^d} \left(\phi(x+z)-\phi(x)-\nabla\phi(x)\cdot z\chi_{\{|z|\le 1\}}\right)\frac{g_\alpha^\varepsilon(z)}{\varepsilon^\alpha}\,dz. \qquad \square$$

A.3. **Further technical results.** Let us end this appendix with several lemmas on R_ε^α (cf. (1.9)) and $g_\alpha = \mathcal{F}^{-1}\left(\frac{1}{1+|\cdot|^\alpha}\right)$.

Lemma A.3. *Let $\alpha \in (0,2)$. Then g_α is continuous in $\mathbb{R}^d \setminus \{0\}$ with*

$$\lim_{|x|\to\infty} |x|^{d+\alpha} g_\alpha(x) = c_\alpha, \qquad M_\alpha := \sup_{x\neq 0} |x|^{d+\alpha} g_\alpha(x) < \infty,$$

where c_α is as in (2.3).

Remark A.1. *Note that $\lim_{x\to 0} g_\alpha(x) = \infty$ since $g_\alpha(x) \ge c_0 \int_{|x|^\alpha}^\infty e^{-t} t^{-\frac{d}{\alpha}}\,dt \sim_{x\to 0} c_1|x|^{-d+\alpha}$, for some positive constants c_i, thanks to* (A.6) *and radial monotonicity of P_α (cf.* [4]*).*

Proof. We remark first by (A.6) that

$$|x|^{d+\alpha} g_\alpha(x) = \int_0^\infty e^{-t}\left|\frac{x}{t^{1/\alpha}}\right|^{d+\alpha} P_\alpha\left(\frac{x}{t^{1/\alpha}}\right)dt. \tag{A.7}$$

then Lemma A.1 and the Dominated Convergence Theorem allow us to conclude. $\square$

Lemma A.4. *For any $\alpha \in (0,2]$, $\varepsilon > 0$ and $\phi \in C_c^\infty(\mathbb{R}^d)$, $\Delta^{\alpha/2}\phi - R_\varepsilon^\alpha(\phi) = \varepsilon^\alpha g_\alpha^\varepsilon * \Delta^\alpha(\phi)$ where $g_\alpha^\varepsilon(\cdot) = \varepsilon^{-d} g_\alpha(\cdot/\varepsilon)$.*

Proof. We have

$$\begin{aligned} \Delta^{\alpha/2}\phi - R_\varepsilon^\alpha(\phi) &= -\mathcal{F}^{-1}\left(\left(|\cdot|^\alpha - \frac{|\cdot|^\alpha}{1+\varepsilon^\alpha|\cdot|^\alpha}\right)\mathcal{F}(\phi)\right) \\ &= -\mathcal{F}^{-1}\left(\frac{\varepsilon^\alpha|\cdot|^{2\alpha}}{1+\varepsilon^\alpha|\cdot|^\alpha}\,\mathcal{F}(\phi)\right) \\ &= \varepsilon^\alpha \underbrace{\mathcal{F}^{-1}\left(\frac{1}{1+\varepsilon^\alpha|\cdot|^\alpha}\right)}_{=g_\alpha^\varepsilon} * \mathcal{F}^{-1}\left(-|\cdot|^{2\alpha}\mathcal{F}(\phi)\right). \end{aligned}$$

$\square$

Lemma A.5 (Fractional gradient)**.** *For any $\beta \in (0,1)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$, define*

$$\nabla^\beta \phi = \mathcal{F}^{-1}\left(i h_\beta \mathcal{F}(\phi)\right) \tag{A.8}$$

where $h_\beta(\xi) = \xi|\xi|^{\beta-1}$. Then for any x in $\mathbb{R}^d$,

$$\nabla^\beta\phi(x) = C\int_{\mathbb{R}^d} \left(\phi(x+z)-\phi(x)\right)\frac{z}{|z|^{d+\beta+1}}\,dz, \tag{A.9}$$

for some constant $C = C(d, \beta)$.

The proof is probably classical but we did not find any reference. Let us give it.

Proof. The Fourier transform of the right hand side of (A.9) equals

$$\mathcal{F}(\phi)(\xi) \int_{\mathbb{R}^d} \frac{e^{iz\cdot\xi} - 1}{|z|^{d+\beta+1}}\, z\, dz = \mathcal{F}(\phi)(\xi) i \int_{\mathbb{R}^d} \frac{\sin(z\cdot\xi) z}{|z|^{d+\beta+1}}\, dz.$$

For $\xi \neq 0$ fixed, write $z = \frac{z\cdot\xi}{|\xi|^2}\,\xi + z^\perp$ where $z^\perp \in (\mathrm{span}\{\xi\})^\perp$. Then

$$\int_{\mathbb{R}^d} \frac{\sin(z\cdot\xi) z}{|z|^{d+\beta+1}}\, dz = \xi \underbrace{\frac{1}{|\xi|^2} \int_{\mathbb{R}^d} \frac{\sin(z\cdot\xi) z\cdot\xi}{|z|^{d+\beta+1}}\, dz}_{=: I_\beta(\xi)},$$

since $\int_{\mathbb{R}^d} \frac{\sin(z\cdot\xi) z^\perp}{|z|^{d+\beta+1}}\, dz = 0$. The latter integral can indeed be computed by fixing the coordinate in ξ, while noting that $\sin(z\cdot\xi)$ and $z^\perp/|z|^{d+\beta+1}$ will then be respectively constant and odd during the integration in $z^\perp$. Given moreover any orthogonal transformation O, $I_\beta(O\xi) = I_\beta(\xi)$ thanks to the change of variable $z \mapsto Oz$. In particular $I_\beta(\cdot)$ is radial and noting that $I_\beta(\lambda\xi) = \lambda^{\beta-1} I_\beta(\xi)$, for any $\lambda > 0$, $I_\beta(\xi) = |\xi|^{\beta-1}$ up to a multiplicative constant. □

Lemma A.6. *For any $0 < \beta < \alpha \leq 1$, $\nabla^\beta g_\alpha \in L^1(\mathbb{R}^d, \mathbb{R}^d)$.*

Proof. We will use the Fubini-Tonelli Theorem applied to the right hand side of (A.9), while computing g_α via (A.6). To justify the computations, let us first show that

$$I := \iiint_{\mathbb{R}^+\times\mathbb{R}^{2d}} e^{-t} t^{-\frac{d}{\alpha}} \frac{\left| P_\alpha\left(\frac{x+z}{t^{1/\alpha}}\right) - P_\alpha\left(\frac{x}{t^{1/\alpha}}\right)\right|}{|z|^{d+\beta}}\, dz\, dx\, dt < \infty.$$

Changing the variables by $\frac{x}{t^{1/\alpha}} \mapsto x$, and similarly for z, we get

$$I = \iiint_{\mathbb{R}^+\times\mathbb{R}^{2d}} e^{-t} t^{-\frac{\beta}{\alpha}} \frac{|P_\alpha(x+z) - P_\alpha(x)|}{|z|^{d+\beta}}\, dz\, dx\, dt.$$

Cut I in two pieces $I_1 + I_2$, according as $|z| < 1$ or not. Writing P_α as integral of its derivative,

$$\begin{aligned} I_1 &\leq \iiint_{\mathbb{R}^+\times\mathbb{R}^d\times\{|z|<1\}} \int_0^1 e^{-t} t^{-\frac{\beta}{\alpha}} \frac{|\nabla P_\alpha(x+sz)|}{|z|^{d+\beta-1}}\, ds\, dz\, dx\, dt \\ &= \int_0^\infty e^{-t} t^{-\frac{\beta}{\alpha}}\, dt\, \|\nabla P_\alpha\|_{L^1} \int_{|z|<1} |z|^{-d+(1-\beta)}\, dz < \infty, \end{aligned}$$

thanks to integrating first in x. Using that $|P_\alpha(x+z) - P_\alpha(x)| \leq |P_\alpha(x+z)| + |P_\alpha(x+z)|$, a similar reasoning gives

$$I_2 \leq 2 \int_0^\infty e^{-t} t^{-\frac{\beta}{\alpha}}\, dt\, \|P_\alpha\|_{L^1} \int_{|z|>1} |z|^{-d-\beta}\, dz < \infty.$$

The Fubini-Tonelli Theorem then implies that

$$x \mapsto \int_{\mathbb{R}^d} \frac{\int_0^\infty e^{-t} t^{-\frac{d}{\alpha}} P_\alpha\left(\frac{x+z}{t^{1/\alpha}}\right) dt - \int_0^\infty e^{-t} t^{-\frac{d}{\alpha}} P_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dt}{|z|^{d+\beta+1}} z\, dz$$

is a well-defined L^1 function. We recognize $\nabla^\beta g_\alpha$ as in the right hand side of (A.9), up to the multiplicative constant C. We denote this operator (with C) by $\widetilde{\nabla}^\beta g_\alpha$ for a while. It remains to show that it equals $\nabla^\beta g_\alpha$ in (A.8). The latter makes sense as a tempered distribution. Considering a test function ϕ,

$$\langle \nabla^\beta g_\alpha, \phi\rangle = -\int_{\mathbb{R}^d} g_\alpha \nabla^\beta \phi\, dx = -\int_{\mathbb{R}^d} g_\alpha \widetilde{\nabla}^\beta \phi\, dx = \int_{\mathbb{R}^d} \phi \widetilde{\nabla}^\beta g_\alpha\, dx,$$

where the latter integration by parts follows from elementary computations.[18] This proves that $\nabla^\beta g_\alpha = \widetilde{\nabla}^\beta g_\alpha \in L^1(\mathbb{R}^d)$. □

Appendix B. Finite-infinite propagation speed estimate

We adapt here [1, Theorem 3.2] to (2.7), see [24] and also [5, 6] for nonlinear diffusions. Let us first recall standard results on Lévy operators and convolution semigroups. Assuming (2.1)–(2.2) and denoting

$$\psi(\xi) = \int_{\mathbb{R}^d} \left(1 - e^{iz\cdot\xi} + iz\cdot\xi\chi_{\{|z|\le 1\}}\right) d\mu(z),$$

we have $\mathcal{L}^\mu(\phi) = -\mathcal{F}^{-1}\left(\psi\mathcal{F}(\phi)\right)$ for any $\phi \in C_c^\infty(\mathbb{R}^d)$, where ψ is continuous and negative definite, see [27, Theorem 3.7.8].[19] In particular $G_\mu(t,\cdot) := \mathcal{F}^{-1}\left(e^{-t\psi}\right)$ is a nonnegative Borel measure of mass 1, defining a convolution semigroup as in [27, Thm. 3.6.16 & Def. 3.6.1].

Proposition B.1. *Assume* (2.1)–(2.2) *and* $f \in W^{1,\infty}_{loc}(\mathbb{R},\mathbb{R}^d)$. *Let* w *and* $\widetilde{w}$ *be the respective entropy solutions of* (1.7) *associated to given* w_0 *and* $\widetilde{w}_0$ *in* $L^1\cap L^\infty(\mathbb{R}^d)$. *Then for all* $t>0$, $x_0\in\mathbb{R}^d$, *and* $R>0$,

$$\int_{|x-x_0|<R} |w-\widetilde{w}|(x,t)\,dx \le \int_{|x-x_0|<R+Lt} \left(G_\mu(t,\cdot) * |w_0-\widetilde{w}_0|\right)(x)\,dx$$

where $L = \|f'\|_{L^\infty((-M,M),\mathbb{R}^d)}$, $M = \max\{\|w_0\|_{L^\infty}, \|\widetilde{w}_0\|_{L^\infty}\}$, *and* G_μ *is as above.*

Proof. Use [24, Proposition 4.2] for (2.7) to get

$$\begin{aligned}(\text{B.1})\qquad &\int_0^T\int_{\mathbb{R}^d} |w-\widetilde{w}|\partial_t\phi + \operatorname{sgn}(w-\widetilde{w})\left(f(w)-f(\widetilde{w})\right)\nabla\phi + |w-\widetilde{w}|\mathcal{L}^\mu(\phi)\,dx\,dt\\ &+\int_{\mathbb{R}^d}|w_0(x)-\widetilde{w}_0(x)|\phi(0,x)\,dx \ge \int_{\mathbb{R}^d}|w(T,x)-\widetilde{w}(T,x)|\phi(T,x)\,dx\end{aligned}$$

for all $T>0$ and $0\le\phi\in C_b^\infty([0,T]\times\mathbb{R}^d)$.[20] As usually

$$|\operatorname{sgn}(w-\widetilde{w})\left(f(w)-f(\widetilde{w})\right)| \le L|w-\widetilde{w}|,$$

since w and $\widetilde{w}$ are bounded by M by the maximum principle, see Remark 2.1. Choose

$$\phi_n(x,t) = \left(G_\mu(T-t,\cdot) * \varphi_n(t,\cdot)\right)(x)$$

where $\varphi_n(t,x) = \gamma\left(\sqrt{|x-x_0|^2+1/n^2} - R - L(T-t)\right)$ for some $0\le\gamma\in C^\infty(\mathbb{R})$ fixed, nonincreasing, and equaled to 1 on $\mathbb{R}^-$. Note that $\varphi_n\in C_c^\infty([0,T]\times\mathbb{R}^2)$ with $\partial_t\varphi_n \le -L|\nabla\varphi_n|$. Recalling moreover that for any fixed $\Phi\in C_c^\infty(\mathbb{R}^d)$, $(x,t)\mapsto\left(G_\mu(T-t,\cdot)*\Phi\right)(x)$ is smooth with

$$\partial_t\left(G_\mu(T-t,\cdot)*\Phi\right)(x) = -\mathcal{L}^\mu\left(G_\mu(T-t,\cdot)*\Phi\right)(x)$$

([21]), easy computations show that $\phi_n\in C_b^\infty([0,T]\times\mathbb{R}^d)$ with

$$\partial_t\phi_n + L|\nabla\phi_n| + \mathcal{L}^\mu(\phi_n) \le 0.$$

Putting finally ϕ_n in (B.1), letting $n\to\infty$, and then γ approximate $\chi_{\mathbb{R}^-}$ gives the desired result. □

[18] $\iint_{|z|>r} g_\alpha(x)\frac{\phi(x+z)-\phi(x)}{|z|^{d+\beta+1}}\,z\,dz\,dx = \iint_{|z|>r}\frac{g_\alpha(x)\phi(x+z)}{|z|^{d+\beta+1}}\,z\,dz\,dx - \iint_{|z|>r}\frac{g_\alpha(x)\phi(x)}{|z|^{d+\beta+1}}\,z\,dz\,dx$, do $x+z\mapsto x$ and $z\mapsto -z$ in the first integral, let $r\to 0^+$, and get the result.

[19] It is standard that $\mathcal{L}^\mu(\phi) = -\mathcal{F}^{-1}\left(\psi\mathcal{F}(\phi)\right)$ by computing $\mathcal{F}\left(\mathcal{L}^\mu(\phi)\right)$ and identifying the multiplier.

[20] In [24], $T=\infty$ and $\phi\in C_c^\infty(\{t\ge 0\})$. But a standard procedure gives (B.1) since entropy solutions are $C_t(L^1_x)$. It is moreover easy to pass to the limit in the Rosenau term via Lemma 2.1, when approximating $\phi\in C_b^\infty$ by functions with compact supports.

[21] See [27, Example 4.3.1], or derivate $\mathcal{F}^{-1}(e^{-(T-t)\psi}\mathcal{F}(\Phi))(x)$ under the integral sign using that $\mathfrak{Re}(\psi)\ge 0$ and $|\psi(\xi)|\le c_\psi(1+|\xi|^2)$ to justify the computations; see respectively (3.123) and Lemma 3.6.22 in [27].

Appendix C. Alternative proofs of Lemmas 3.1 and 3.2 in 1-D

This final appendix contains more simple an purely PDEs proofs of Lemmas 3.1 and 3.2 when $d = 1$. It is convenient to start with the second lemma.

Proof of Lemma 3.2. Define $H_\alpha(t,x) = \int_{-\infty}^x G_\alpha(t,y)\,dy$ and recall that

$$\text{(C.1)}\qquad \begin{cases} \partial_t H_\alpha = \partial_{xx}^\alpha H_\alpha, & t \in \mathbb{R}^+,\ x \in \mathbb{R}^d, \\ H_\alpha(0,x) = \mathrm{Heav}(x) := \begin{cases} 1, & \text{for } x > 0, \\ 0, & \text{otherwise,} \end{cases} & x \in \mathbb{R}, \end{cases}$$

thanks to (A.1) and (A.3). Recall also our shorthand notation $\partial_{xx}^\alpha = (\partial_{xx}^2)^{\alpha/2}$ defined as in (1.8). Injecting the identity (A.2) into the above equation, straightforward computations give

$$\text{(C.2)}\qquad \left(\frac{x}{t^{1/\alpha}}\right)^{1+\alpha} P_\alpha\left(\frac{x}{t^{1/\alpha}}\right) = -\alpha x^\alpha \partial_{xx}^\alpha H_\alpha(t,x) =: f(t,x),$$

for all positive t and x. Using again the equation in (C.1),

$$\text{(C.3)}\qquad \partial_t f(t,x) = -\alpha x^\alpha \underbrace{\partial_{xx}^\alpha\left(\partial_{xx}^\alpha H_\alpha\right)}_{=-\partial_{xx}^{2\alpha} H_\alpha \text{ by (1.8)}}(t,x).$$

When $\alpha \in (0,1]$, $\partial_{xx}^{2\alpha}$ is a Lévy operator. If in particular $\alpha \neq 1$ we can apply (2.4). Hence for all positive t and x,

$$\text{(C.4)}\qquad \partial_{xx}^{2\alpha} H_\alpha(t,x) = c_{2\alpha} \lim_{r\to\infty} \int_{|z|<r} \left(H_\alpha(t,x+z) - H_\alpha(t,x) - \partial_x H_\alpha(t,x) z\right) \frac{dz}{|z|^{1+2\alpha}} \le 0.$$

The latter inequality holds since $x \mapsto H_\alpha(t,x)$ is odd, nondecreasing, and concave on $\mathbb{R}^+$; see [4].[22] Indeed the integrand in (C.4) is nonpositive for positive or small z, while these regions compensate those with the wrong sign by symmetry. If $\alpha = 1$ we still have $\partial_{xx}^{2\alpha=2} H_\alpha(t,x) \le 0$ for positive t and x by concavity of $H_\alpha(t,\cdot)$ on $\mathbb{R}^+$. From (C.2), (C.3), (C.4), and Lemma A.1, we get that $f(t,x) \le \lim_{s\to 0} f(s,x) = c_\alpha$ for positive t and x. This proves Lemma 3.2 via (A.7), first for positive x and then all x since g_α is even. □

Remark C.1. *We can also recover Lemma A.1 with our arguments. Indeed* (2.4) *shows that*

$$\lim_{t\to 0} -\alpha x^\alpha \partial_{xx}^\alpha H_\alpha(t,x) = -\alpha x^\alpha c_\alpha \underbrace{\int_{\mathbb{R}} \left(\mathrm{Heav}(x+z) - \mathrm{Heav}(x)\right) \frac{dz}{|z|^{1+\alpha}}}_{=-\frac{1}{\alpha x^\alpha}} = c_\alpha$$

for any $x > 0$, and this proves Lemma A.1 via (C.2). *The computation of the abvove limit via* (2.4) *is notably justified by Lemma C.1.*

Proof of Lemma 3.1. Now $\alpha \in (0,2)$, ∂_{xx}^α is well-defined via (1.8), but it is not a Lévy operator if $\alpha \in (1,2)$. Assume the latter case, since Lemma 3.1 is a consequence of Lemma 3.2 if $\alpha \in (0,1]$. We need the result below.

Lemma C.1. *Assume $\alpha > 0$ and H_α solve* (C.1) *as previously. Then $H_\alpha(t,\cdot) \xrightarrow{t\to 0} \mathrm{Heav}(\cdot)$ in $C^\infty(\mathbb{R}\setminus\{0\})$ (meaning all space derivatives converge locally uniformly).*

Proof. For any $n \in \mathbb{N}$, $|\partial_{x^n}^n P_\alpha(x)| = O(|x|^{-n-1})$ as $|x| \to \infty$,[23] because $\partial_{\xi^{n+1}}^{n+1}(\xi|\xi|^{n-1}e^{-|\xi|^\alpha}) = O(|\xi|^{\alpha-1})$ as $\xi \to 0$ and thus remains integrable on $\mathbb{R}$. For any $0 < a < b$,

$$\|\partial_{x^n}^n G_\alpha(t,\cdot)\|_{L^1(a,b)} = \frac{1}{t^{n/\alpha}} \int_{at^{-1/\alpha}}^{bt^{-1/\alpha}} |\partial_{x^n}^n P_\alpha(x)|\,dx \le \frac{Ca}{t^{n/\alpha}} \frac{(b-a)t^{\frac{n+1}{\alpha}}}{t^{1/\alpha}} \le Ca(b-a),$$

[22] These properties are also straightforward from the maximum and comparison principles for (C.1).

[23] In fact $O(|x|^{-n-1-\alpha})$, see [34], but we will not need it.

thanks to (A.2) and to what precedes to ensure that the above constant C depends on $a > 0$ but not on small t. This gives enough compacity estimates on $G_\alpha(t,\cdot)$ as $t \to 0$ to conclude. □

Going back to the proof of Lemma 3.1, note that (C.2) and (C.3) remain valid under our assumption that $\alpha \in (1,2)$. By (C.3) and Lemma C.1, $\lim_{t\to 0} \partial_t f(t,x) = \alpha x^\alpha \partial_{xx}^{2\alpha} \mathrm{Heav}(x)$ for any $x > 0$. But $\partial_{xx}^{2\alpha} \mathrm{Heav}(x) = -\partial_{xx}^2(\partial_{xx}^{2\alpha-2}\mathrm{Heav})(x)$ where $2\alpha - 2 \in (0,2)$. Hence

$$\partial_{xx}^{2\alpha-2}\mathrm{Heav}(x) = c_{2\alpha-2}\int_{\mathbb{R}} (\mathrm{Heav}(x+z) - \mathrm{Heav}(x))\,\frac{dz}{|z|^{2\alpha-1}} = -\frac{c_{2\alpha-2}}{2\alpha-2}\,\frac{1}{x^{2\alpha-2}}$$

by (2.4), and $\lim_{t\to 0} \partial_t f(t,x) = \alpha x^\alpha c_{2\alpha-2}\,\frac{2\alpha-1}{x^{2\alpha}} = \frac{\alpha(2\alpha-1)c_{2\alpha-2}}{x^\alpha}$. Lemma 3.1 then follows from (C.2), Lemma A.1, and (A.7). □

Acknowledgments

The authors warmly thank Denis Serre for suggesting the problem.

G. M. Coclite has been partially supported by the Project funded under the National Recovery and Resilience Plan (NRRP), Mission 4, Component 2, Investment 1.4 (Call for tender No. 3138 of 16/12/2021), of Italian Ministry of University and Research funded by the European Union (NextGenerationEU Award, No. CN000023, Concession Decree No. 1033 of 17/06/2022) adopted by the Italian Ministry of University and Research (CUP D93C22000410001), Centro Nazionale per la Mobilità Sostenibile. He has also been supported by the Italian Ministry of Education, University and Research under the Programme "Department of Excellence" Legge 232/2016 (CUP D93C23000100001), and by the Research Project of National Relevance "Evolution problems involving interacting scales" granted by the Italian Ministry of Education, University and Research (MIUR PRIN 2022, project code 2022M9BKBC, CUP D53D23005880006). He warmly thanks the Laboratoire de Mathématiques de Besançon for the hospitality.

The work of C. Donadello has been partially supported by EIPHI Graduate School (contract ANR-17-EURE-0002) and the Region Bourgogne-Franche-Comté, and by the International Emerging Action CNRS "Conpanhye".

References

[1] N. Alibaud. Entropy formulation for fractal conservation laws. *J. Evol. Equ.*, 7(1):145–175, 2007.
[2] N. Alibaud, S. Cifani, and E. R. Jakobsen. Continuous dependence estimates for nonlinear fractional convection-diffusion equations. *SIAM J. Math. Anal.*, 44(2):603–632, 2012.
[3] N. Alibaud, S. Cifani, and E. R. Jakobsen. Optimal continuous dependence estimates for fractional degenerate parabolic equations. *Arch. Ration. Mech. Anal.*, 213(3):705–762, 2014.
[4] N. Alibaud, J. Droniou, and J. Vovelle. Occurrence and non-appearance of shocks in fractal Burgers equations. *J. Hyperbolic Differ. Equ.*, 4(3):479–499, 2007.
[5] N. Alibaud, J. Endal, and E. R. Jakobsen. Optimal stability results and nonlinear duality for L^∞ entropy and L^1 viscosity solutions. *J. Math. Pures Appl. (9)*, 188:26–72, 2024.
[6] B. Andreianov and M. Brassart. Uniqueness of entropy solutions to fractional conservation laws with "fully infinite" speed of propagation. *J. Differential Equations*, 268(7):3903–3935, 2020.
[7] D. Applebaum. *Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2009.
[8] H. Bateman and B. M. Project. *Higher Transcendental Functions [Volume 2]*. McGraw-Hill Book Company, 1953.
[9] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.
[10] P. Biler, T. Funaki, and W. A. Woyczynski. Fractal Burgers equations. *J. Differential Equations*, 148(1):9–46, 1998.
[11] R. M. Blumenthal and R. K. Getoor. Some theorems on stable processes. *Trans. Amer. Math. Soc.*, 95:263–273, 1960.
[12] C. H. Chan and M. Czubak. Regularity of solutions for the critical N-dimensional Burgers' equation. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 27(2):471–501, 2010.

[13] C. H. Chan, M. Czubak, and L. Silvestre. Eventual regularization of the slightly supercritical fractional Burgers equation. *Discrete Contin. Dyn. Syst.*, 27(2):847–861, 2010.
[14] S. Cifani and E. R. Jakobsen. Entropy solution theory for fractional degenerate convection-diffusion equations. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 28(3):413–441, 2011.
[15] P. Clavin. *Instabilities and Nonlinear Patterns of Overdriven Detonations in Gases*, pages 49–97. Springer Netherlands, Dordrecht, 2002.
[16] P. Courrège. Générateur infinitésimal d'un semi-groupe de convolution sur R^n, et formule de Lévy-Khinchine. *Bull. Sci. Math. (2)*, 88:3–30, 1964.
[17] F. Şabac. The optimal convergence rate of monotone finite difference methods for hyperbolic conservation laws. *SIAM J. Numer. Anal.*, 34(6):2306–2318, 1997.
[18] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 2000.
[19] A. de Pablo, F. Quirós, A. Rodríguez, and J. L. Vázquez. A fractional porous medium equation. *Advances in Mathematics*, 226(2):1378–1409, 2011.
[20] A. de Pablo, F. Quirós, A. Rodríguez, and J. L. Vázquez. A general fractional porous medium equation. *Communications on Pure and Applied Mathematics*, 65(9):1242–1284, 2012.
[21] J. Droniou. Vanishing non-local regularization of a scalar conservation law. *Electron. J. Differential Equations*, pages No. 117, 20, 2003.
[22] J. Droniou, T. Gallouet, and J. Vovelle. Global solution and smoothing effect for a non-local regularization of a hyperbolic equation. *J. Evol. Equ.*, 3(3):499–521, 2003. Dedicated to Philippe Bénilan.
[23] J. Droniou and C. Imbert. Fractal first-order partial differential equations. *Arch. Ration. Mech. Anal.*, 182(2):299–331, 2006.
[24] J. Endal and E. R. Jakobsen. L^1 contraction for bounded (nonintegrable) solutions of degenerate parabolic equations. *SIAM J. Math. Anal.*, 46(6):3957–3982, 2014.
[25] G. Furioli, A. Pulvirenti, E. Terraneo, and G. Toscani. On Rosenau-type approximations to fractional diffusion equations. *Commun. Math. Sci.*, 13(5):1163–1191, 2015.
[26] C. Imbert. A non-local regularization of first order Hamilton–Jacobi equations. *Journal of Differential Equations*, 211(1):218–246, 2005.
[27] N. Jacob. *Pseudo differential operators and Markov processes. Vol. I.* Imperial College Press, London, 2001. Fourier analysis and semigroups.
[28] M. A. Katsoulakis and A. E. Tzavaras. Contractive relaxation systems and the scalar multidimensional conservation law. *Comm. Partial Differential Equations*, 22(1-2):195–233, 1997.
[29] A. Kiselev, F. Nazarov, and R. Shterenberg. Blow up and regularity for fractal Burgers equation. *Dyn. Partial Differ. Equ.*, 5(3):211–240, 2008.
[30] A. Kiselev, F. Nazarov, and A. Volberg. Global well-posedness for the critical 2D dissipative quasi-geostrophic equation. *Invent. Math.*, 167(3):445–453, 2007.
[31] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.
[32] N. N. Kuznetsov. Accuracy of some approximate methods for computing the weak solutions of a first-order quasi-linear equation. *Ussr Computational Mathematics and Mathematical Physics*, 16:105–119, 1976.
[33] G. Pólya. On the zeros of an integral function represented by Fourier's integral. *Messenger of Math.*, 52:185–188, 1923.
[34] W. E. Pruitt and S. J. Taylor. The potential kernel and hitting probabilities for the general stable process in R^N. *Trans. Amer. Math. Soc.*, 146:299–321, 1969.
[35] C. Rohde and W.-A. Yong. The nonrelativistic limit in radiation hydrodynamics: I. weak entropy solutions for a model problem. *Journal of Differential Equations*, 234(1):91–109, 2007.
[36] P. Rosenau. Extending hydrodynamics via the regularization of the Chapman-Enskog expansion. *Phys. Rev. A*, 40:7193–7196, Dec 1989.
[37] P. Rosenau. Tempered diffusion: A transport process with propagating fronts and inertial delay. *Phys. Rev. A*, 46:R7371–R7374, Dec 1992.
[38] K.-i. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original, Revised by the author.
[39] S. Schochet and E. Tadmor. The regularized Chapman-Enskog expansion for scalar conservation laws. *Archive for Rational Mechanics and Analysis*, 119(2):95–107, June 1992.
[40] E. M. Stein and G. Weiss. *Introduction to Fourier analysis on Euclidean spaces*, volume No. 32 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1971.
[41] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, Cambridge; The Macmillan Company, New York, 1944.

[42] W. A. Woyczyński. Lévy processes in the physical sciences. In *Lévy processes. Theory and applications*, pages 241–266. Boston: Birkhäuser, 2001.

(Nathaël Alibaud)
SUPMICROTECH-ENSMM, 26 Chemin de l'Epitaphe, 25030 Besançon cedex, France, and Université de Franche-Comté, CNRS, LmB (UMR 6623), F-25000 Besançon, France.
Email address: nathael.alibaud@ens2m.fr
URL: http://lmb.univ-fcomte.fr/Alibaud-Nathael

(Giuseppe Maria Coclite)
Department of Mechanics, Mathematics and Management, Polytechnic University of Bari, Via E. Orabona 4, 70125 Bari, Italy.
Email address: giuseppemaria.coclite@poliba.it
URL: http://www.dmmm.poliba.it/index.php/it/profile/gmcoclite

(Maxime Dalery)
Université de Franche-Comté, CNRS, LmB (UMR 6623), F-25000 Besançon, France.
Email address: maxime.dalery@univ-fcomte.fr
URL: https://lmb.univ-fcomte.fr/Dalery-Maxime-1783

(Carlotta Donadello)
Université de Franche-Comté, CNRS, LmB (UMR 6623), F-25000 Besançon, France.
Email address: carlotta.donadello@univ-fcomte.fr
URL: http://lmb.univ-fcomte.fr/Donadello-Carlotta