VANISHING VISCOSITY VERSUS ROSENAU APPROXIMATION FOR SCALAR CONSERVATION LAWS: THE FRACTIONAL CASE

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Abstract. We consider approximations of scalar conservation laws obtained by adding nonlocal diffusive operators. In particular, we compare solutions associated to fractional Laplacian and fractional Rosenau perturbations and show that for any $t > 0$ the mutual L^1-distance of their profiles is lower than their common distance to the underlying inviscid entropy solution. We provide explicit examples showing that our rates are optimal in the subcritical case, in one space dimension and for convex fluxes.

1. Introduction

The classical approach in the analysis of the Cauchy problem for multi-dimensional scalar conservation law,

$$\begin{cases} \partial_t w + \operatorname{div} f(w) = 0, & t \in \mathbb{R}^+,\ x \in \mathbb{R}^d, \\ w(0,x) = w_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where the flux f is in $W^{1,\infty}_{loc}(\mathbb{R},\mathbb{R}^d)$ and the initial condition w_0 is in $L^\infty \cap \mathrm{BV}(\mathbb{R}^d)$, relies on singular approximations as the vanishing viscosity and the relaxation methods, see [9] and references therein.

The well-posedness of (1.1) was established in [18] in the framework of entropy solutions, that are weak solutions which satisfy additional selection criteria called entropy inequalities. The unique entropy solution coincides with the strong L^p-limit as $\varepsilon \to 0$ of vanishing viscosity regularized solutions, which solve

$$\begin{cases} \partial_t w_\varepsilon + \operatorname{div} f(w_\varepsilon) = \varepsilon \Delta w_\varepsilon, & t \in \mathbb{R}^+,\ x \in \mathbb{R}^d, \\ w(0,x) = w_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{1.2}$$

This regularization procedure is the most classical, but other ones have been considered. In [22], Rosenau proposed a regularized version of the Chapman-Enskog expansion of hydrodynamics, which can be rewritten as the following nonlocal perturbation of (1.1)

$$\partial_t \mathfrak{v}_\varepsilon + \operatorname{div} f(\mathfrak{v}_\varepsilon) = -\varepsilon \mathcal{F}^{-1}\left(\frac{|\cdot|^2}{1+\varepsilon^2|\cdot|^2} \mathcal{F}(\mathfrak{v}_\varepsilon) \right). \tag{1.3}$$

Here and throughout,

$$\mathcal{F}(\phi)(\xi) = \int_{\mathbb{R}^d} \phi(x) e^{-ix\cdot\xi}\, dx \quad \text{and} \quad \mathcal{F}^{-1}(\phi)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \phi(\xi) e^{ix\cdot\xi}\, d\xi,$$

stand for the Fourier transform and its inverse, respectively. Since the diffusion in (1.2) can be rewritten as

$$\Delta \phi = -\mathcal{F}^{-1}\left(|\cdot|^2 \mathcal{F}(\phi) \right),$$

the right-hand side of (1.3) resembles the one in (1.2) at low wave numbers ξ, but is intended to model a bounded approximation of a linearized collision operator for higher ξ. As explained

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in [22, 24], this avoids artificial instabilities occurring when the Chapman-Enskog expansion for such an operator is truncated after a finite number of terms. The model retains the essential properties of the usual viscosity approximation, e.g., existence of travelling waves, monotonicity, upper-Lipschitz continuity, etc., and it sharpens the standard viscous shock layers. However, $\mathfrak{v}_\epsilon$ may remain discontinuous. Schochet and Tadmor, in [24] proved, in one space dimension, that this regularization converges to the underlying inviscid entropy solution as ε goes to zero, and estimated the convergence rate in all L^p spaces for $1 \leq p < \infty$. In particular, the convergence rate in L^1 coincides with the convergence rate of the vanishing viscosity approximation (1.2) established in [19]. The extension to several space dimensions (with the same convergence rates) is due to Katsoulakis and Tzavaras in [17]. For any $T > 0$, we have

$$\|w_\varepsilon - w\|_{C([0,T],L^1)}, \|\mathfrak{v}_\varepsilon - w\|_{C([0,T],L^1)} = O(\sqrt{\varepsilon T}). \tag{1.4}$$

The rates in (1.4) are expected to be optimal even for nonlinear equations as suggested by the analysis of [8] for numerical schemes, see also Theorem 2.5 for a proof in our setting for both (1.2) and (1.3). It is thus remarkable that $w_\varepsilon - w$ and $\mathfrak{v}_\epsilon - w$ converge to 0 at the same speed, since discontinuities from w may persist in $\mathfrak{v}_\epsilon$ whereas w_ε is smooth.

Formally, however, the solutions to the Rosenau approximate problem should be closer to the viscous approximations than to the inviscid limit. In the literature we did not find a rigorous proof of this fact, and this is the motivation of our interest in the problem. In this paper we focus our attention to the comparison between the fractional Laplacian and their corresponding Rosenau approximations as defined in [14], see respectively (1.5) and (1.6) below, for the whole range $\alpha \in (0, 2]$. Unfortunately, our results are not optimal in the case $\alpha \in (1, 2]$, which includes the classical approximations (1.2) and (1.3).

Let us now introduce the fractional vanishing viscosity and Rosenau approximations of (1.1). Throughout, $\alpha \in (0, 2]$ is fixed, $\varepsilon > 0$ is the perturbation parameter and we consider the following Cauchy problems

$$\begin{cases} \partial_t u + \operatorname{div} f(u) = \varepsilon \Delta^{\alpha/2} u, & t \in \mathbb{R}^+,\ x \in \mathbb{R}^d, \\ u(0,x) = w_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.5}$$

and

$$\begin{cases} \partial_t v + \operatorname{div} f(v) = \varepsilon R_\varepsilon^\alpha(v), & t \in \mathbb{R}^+,\ x \in \mathbb{R}^d, \\ v(0,x) = w_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.6}$$

with the fractional Laplacian

$$\Delta^{\alpha/2}\phi = -\mathcal{F}^{-1}\left(|\cdot|^\alpha \mathcal{F}(\phi)\right), \tag{1.7}$$

and its corresponding Rosenau approximation

$$R_\varepsilon^\alpha(\phi) = -\mathcal{F}^{-1}\left(\frac{|\cdot|^\alpha}{1 + \varepsilon^\alpha |\cdot|^\alpha} \mathcal{F}(\phi)\right). \tag{1.8}$$

We notably recover (1.2) and (1.3) if $\alpha = 2$. Equation (1.5) is respectively said to be

- subcritical if $\alpha > 1$,
- critical if $\alpha = 1$,
- supercritical if $\alpha < 1$,

in reference to the cases where the diffusion's order is less or above the nonlinear convection's order. Such fractional conservation laws appear, of course, in radiation hydrodynamics [21, 22, 23, 24], but also in the description of over-driven gas detonations [7], anomalous diffusion in semiconductor growth [25], flow in porous media [10, 11]. Problem (1.5) admits a unique smooth solution in the subcritical and critical cases [13], and a unique possibly discontinuous entropy solution in the supercritical one [1]. Equation (1.6) is a very natural counterpart to

(1.3) for fractional diffusions, see [14] and references therein. As concerning well-posedness, the notion of entropy solutions from [1] has been extended in [6] to scalar conservation laws with general Lévy diffusions, whose fractional Laplacian is a prototype. This theory applies in particular to (1.6), and actually also to (1.3), see Section 2 for details. Now considering the limits as $\varepsilon \to 0^+$, we recover the unique entropy solution w of (1.1) from both (1.5) and (1.6). Moreover, for all $\alpha \in (0,2]$, the solutions of (1.5) and (1.6) converge to the inviscid limit w with the same rate

$$\|u-w\|_{C([0,T],L^1)}, \|v-w\|_{C([0,T],L^1)} = \begin{cases} O(\varepsilon^{1/\alpha}), & \alpha \in (1,2], \\ O(\varepsilon \ln(1/\varepsilon)), & \alpha = 1, \\ O(\varepsilon), & \alpha \in (0,1). \end{cases} \tag{1.9}$$

This result is due to [1, 12] for (1.5). It also follows from general continuous dependence estimates for scalar conservation laws with Lévy diffusions, see [2], which apply to both (1.5) and (1.6).

In this paper we identify the optimal rates of decay of the distance between u and v in the supercritical and critical cases, and we prove non optimal but explicit rates in the subcritical case. More precisely, for $\alpha \in (0,1)$, we establish in Theorem 2.4 that

$$\|u-v\|_{C([0,T],L^1)} = \begin{cases} O(\varepsilon^{2-\alpha}), & \alpha \in (1/2,1], \\ O(\varepsilon^{3/2} \ln(1/\varepsilon)), & \alpha = 1/2, \\ O(\varepsilon^{\alpha+1}), & \alpha \in (0,1/2), \end{cases} \tag{1.10}$$

and we exhibit optimal examples for (1.9) and (1.10) in Theorem 2.5. These examples, given for simplicity in one space dimension, are inspired from the approach used in [8] combined with Fourier arguments, and they work for nonlinear strictly convex fluxes f. As concerning subcritical equations, see Theorem 2.3, we get that

$$\|u-v\|_{C([0,T],L^1)} = O\left(\varepsilon^{\frac{3-\alpha}{2}}\right), \quad \alpha \in (1,2]. \tag{1.11}$$

This suffices to show that $u-v$ vanishes faster than $u-w$ and $v-w$, but these rates are probably not optimal. Indeed, as we show in Proposition 5.1, in the zero convection case (1.9) and (1.10) are optimal, while the optimal rates for subcritical equations turn out to be

$$\|u-v\|_{C([0,T],L^1)} = O(\varepsilon^{-\alpha}), \quad \alpha > 1 \text{ and } f' \equiv 0. \tag{1.12}$$

Hence, we conjecture that (1.12) hold for nonlinear f as well, and this would be the best possible rates in the subcritical case, including for the classical one $\alpha = 2$.

As concerning the proofs, Theorems 2.3, 2.4 and 2.5 and are based on the Kuznetsov's method [19], nonlocal arguments from [2], and fine estimates on the fractional heat and Rosenau kernels. The general estimates from [2] compare solutions of different scalar conservation laws with Lévy diffusions, as we have. To get Corollary 2.1, we combine these now standard results with a fine estimate on the asymptotic behavior of the kernels for large $|x|$, see Lemma 3.1. But for the optimal estimates (1.10), we roughly speaking need to restart the Kuznetsov's argumentation, while essentially taking advantage of a key monotonicity result on the difference operator $\Delta^{\alpha/2} - R_\varepsilon^\alpha$. The latter estimate on the kernels is even more precise, see Lemma 3.2.

The rest of this paper is organized as follows. In Section 2 we recall the representation of the fractional Laplacian and the Rosenau operators as Lévy operators, together with the definition of entropy solution for fractal conservation laws, and a result on their continuous dependence. We conclude the section stating our main results. Section 3 contains the proof of preliminary technical lemmas. We establish our convergence rates in Section 4, while their optimality in the topic of Section 5. Two lemmas in the latter have rather long proofs, postponed to Section

6. Some classical results in fractional calculus are collected in the Appendix A. Finally, in Appendix B we present alternative proofs in one space dimension for two technical lemmas.

2. Preliminaries and main results

In this section we first recall known facts on the well-posedness of (1.5) and (1.6), and then state our main results.

2.1. **Lévy diffusion operators.** Given a Borel measure μ on $\mathbb{R}^d$ such that

$$\mu \geq 0, \quad \mu(z) = \mu(-z) \quad \text{and} \quad \int_{\mathbb{R}^d} \min\left\{|z|^2, 1\right\} d\mu(z) < \infty, \tag{2.1}$$

we define the (symmetric) Lévy diffusion operator $\mathcal{L}^\mu$ associated to the Lévy measure μ as

$$\mathcal{L}^\mu[\phi](x) = \int_{\mathbb{R}^d} (\phi(x+z) - \phi(x) - \nabla\phi(x)\cdot z)\, d\mu(z). \tag{2.2}$$

Any Lévy diffusion operator can be written as the sum of two operators associated respectively to $\mu_{|\{|z|\leq r\}}$ and $\mu_{|\{|z|>r\}}$, for any $r>0$

$$\mathcal{L}^\mu = \mathcal{L}^{\mu_{|\{|z|\leq r\}}} + \mathcal{L}^{\mu_{|\{|z|>r\}}}.$$

Note that $\mathcal{L}^{\mu_{|\{|z|>r\}}}(\phi) \in L^\infty(\mathbb{R}^d)$ when $\phi \in L^\infty(\mathbb{R}^d)$, since the measure $\mu_{|\{|z|>r\}}$ is finite on $\mathbb{R}^d$. Following the classical results in [13, 16] the fractional Laplacian, defined in (1.7), is a Lévy diffusion operator associated to the measure

$$\mu(z) = \frac{c_d(\alpha)}{|z|^{d+\alpha}} dz, \qquad \text{where} \quad c_d(\alpha) := \frac{\alpha 2^{\alpha-1}\Gamma\left(\frac{d+\alpha}{2}\right)}{\pi^{\frac{d}{2}}\Gamma\left(\frac{2-\alpha}{2}\right)} > 0, \tag{2.3}$$

and Γ is the usual Gamma function. Therefore, the fractional Laplacian admits at each point $x \in \mathbb{R}^d$ the following integral representation for $\alpha \in (0,2)$, $\phi \in C_c^\infty(\mathbb{R}^d)$, $r>0$,

$$\begin{aligned} \Delta^{\alpha/2}\phi(x) =& c_d(\alpha) \int_{|z|<r} (\phi(x+z) - \phi(x) - \nabla\phi(x)\cdot z) \frac{dz}{|z|^{d+\alpha}} \\ &+ c_d(\alpha) \int_{|z|>r} (\phi(x+z) - \phi(x)) \frac{dz}{|z|^{d+\alpha}} \\ =& \mathcal{L}_{\alpha,r}[\phi](x) + \mathcal{L}^{\alpha,r}[\phi](x). \end{aligned} \tag{2.4}$$

For shortness, in the next Sections we will also use the following notation

$$\Delta^{\alpha/2}\phi(x) = c_d(\alpha)\mathrm{PV}\int (\phi(x+z) - \phi(x)) \frac{dz}{|z|^{d+\alpha}}.$$

The fractional Rosenau operator defined in (1.8) is also a Lévy diffusion operator for any $\alpha \in (0,2)$ and $\varepsilon > 0$, associated to the measure

$$\nu(z) = \frac{g_\alpha(z/\varepsilon)}{\varepsilon^{d+\alpha}} dz, \qquad \text{where} \quad g_\alpha := \mathcal{F}^{-1}\left(\frac{1}{1+|\cdot|^\alpha}\right). \tag{2.5}$$

A detailed proof of this (known) fact is deferred to the Appendix A.

2.2. **Entropy solutions for fractal conservation laws with Lévy diffusion.** Let us consider a general Cauchy problem of the form

$$\begin{cases} \partial_t w^\mu + \operatorname{div} f(w^\mu) = \mathcal{L}^\mu(w^\mu), & t \in \mathbb{R}^+,\ x \in \mathbb{R}^d, \\ w^\mu(t=0,x) = w_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.6}$$

where $f \in W^{1,\infty}_{loc}(\mathbb{R},\mathbb{R}^d)$, $w_0 \in L^\infty \cap \mathrm{BV}(\mathbb{R}^d)$ and $\mathcal{L}^\mu$ is a symmetric Lévy diffusion operator. This problem is well-posed in L^∞ within the class of entropy solutions.

Definition 2.1. *In the setting above, we say that $w^\mu \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ is an entropy solution of (2.6) if and only if for all $r > 0$, for all non-negative $\phi \in C_c^\infty([0,\infty) \times \mathbb{R}^d)$ and for every convex entropy $\eta \in C^\infty(\mathbb{R})$ and entropy flux $q \in C^\infty(\mathbb{R}, \mathbb{R}^d)$ such that $q_i'(\xi) = \eta'(\xi) f_i'(\xi)$, for any $\xi \in \mathbb{R}$ and $i = 1, \dots, d$,*

$$\int_0^\infty \int_{\mathbb{R}^d} \Big(\eta(w^\mu)\partial_t \phi + q(w^\mu) \cdot \nabla \phi + \eta(w^\mu) \mathcal{L}_{\mu|_{\{|z| \le r\}}}(\phi) - \eta'(w^\mu) \mathcal{L}^{\mu|_{\{|z|>r\}}}(u)\phi \Big) \, dx \, dt$$
$$+ \int_{\mathbb{R}^d} \eta(w^\mu(0,x))\phi(0,x) \, dx \ge 0.$$

An important ingredient in our analysis are the general continuous dependence estimates of [2]. They mesure the distance between two entropy solutions in terms of their respective data. Here we are only interested in the dependence with respect to the Lévy measure. We summarize in the next theorem the well-posedness results from [1, 2, 13] needed for our analysis

Theorem 2.1. *In the hypothesis above, the Cauchy problem (2.6) admits a unique entropy solution $w^\mu \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$. Additionally, $w^\mu \in C([0,\infty); L^1_{loc})$ with $w^\mu(t = 0, \cdot) = w_0(\cdot)$.*

If in addition $w_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R}^d)$, μ and ν are two measures satisfying (2.1), and w^μ and w^ν are the respective corresponding entropy solutions of (2.6). Then for all $T > 0$ and $r > 0$,

$$\|w^\mu - w^\nu\|_{C([0,T];L^1)} \le T^{1/2} \frac{2d}{\sqrt{d+1}} TV(w_0) \left(\int_{|z| \le r} |z|^2 d|\mu - \nu|(z) \right)^{1/2}$$
$$+ T \int_{|z|>r} \|w_0(\cdot + z) - w_0(\cdot)\|_{L^1} d|\mu - \nu|(z).$$

If $\nu \equiv 0$ and μ is successively taken as the Lévy measures of $\varepsilon \Delta^{\alpha/2}$ and $\varepsilon R_\varepsilon^\alpha$, then we recover existence for the respective corresponding fractional approximations (1.5) and (1.6) of the scalar conservation law (1.1) and the known rates of convergence.

Corollary 2.1. *Assume $f \in W^{1,\infty}_{loc}(\mathbb{R}, \mathbb{R}^d)$, $\alpha \in (0,2)$, $w_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R}^d)$, and $T > 0$. There then exists $C_T > 0$ such that for any $\varepsilon \in (0, 1/2)$,*

$$\|u - w\|_{C([0,T],L^1)}, \|v - w\|_{C([0,T],L^1)} \le C_T \sigma_\varepsilon,$$

where w, u and v respectively solve (1.1), (1.5) and (1.6), and

$$\sigma_\varepsilon := \begin{cases} \varepsilon^{1/\alpha}, & \alpha > 1, \\ \varepsilon \ln(1/\varepsilon), & \alpha = 1, \\ \varepsilon, & \alpha < 1. \end{cases} \tag{2.7}$$

The convergence rates of Corollary 2.1 are optimal for zero convection PDEs ($f' \equiv 0$); see the discussion in [8] and Proposition 5.1. We will also give optimal examples for nonlinear strictly convex f in Theorem 2.5.

We refer to [1, 12] for the original proofs of Corollary 2.1 and to [2] for alternative proofs using Theorem 2.1.

2.3. **Main results.** We are ready to state our main results. The proofs are given in the next sections. Our first contribution is the following

Theorem 2.2. *Assume $f \in W^{1,\infty}_{loc}(\mathbb{R}, \mathbb{R}^d)$, $\alpha \in (0,2)$, $w_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R}^d)$, and $T > 0$. Then*

$$\lim_{\varepsilon \to 0^+} \frac{\|u - v\|_{C([0,T];L^1)}}{\sigma_\varepsilon} = 0,$$

where u and v respectively solve (1.5) and (1.6), and σ_ε is defined as in Corollary 2.1.

Let us now provide sharp estimates of the decay rate of the distance between u and v.

Theorem 2.3 (Explicit rates for subcritical PDEs)**.** *Let $\alpha \in (1,2]$, $f \in W^{1,\infty}_{loc}(\mathbb{R},\mathbb{R}^d)$, $w_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R}^d)$, and $T>0$. There then exists $C_T>0$ such that for any $\varepsilon \in (0,1/2)$,*

$$\|u-v\|_{C([0,T],L^1)} \leq C_T \varepsilon^{\frac{3-\alpha}{2}},$$

where u and v respectively solve (1.5) *and* (1.6)*.*

As concerning $\alpha \in (0,1]$, we have optimal estimates given by the next result.

Theorem 2.4 (Optimal rates for critical and supercritical PDEs)**.** *Let now $\alpha \in (0,1]$, $f \in W^{1,\infty}_{loc}(\mathbb{R},\mathbb{R}^d)$, $w_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R}^d)$, and $T>0$. There then exists $C_T>0$ such that for any $\varepsilon \in (0,1/2)$,*

$$\|u-v\|_{C([0,T],L^1)} \leq C_T \mathfrak{s}_\varepsilon, \tag{2.8}$$

where u and v respectively solve (1.5) *and* (1.6)*, and*

$$\mathfrak{s}_\varepsilon := \begin{cases} \varepsilon^{2-\alpha}, & 1/2 < \alpha \leq 1, \\ \varepsilon^{\frac{3}{2}} \ln(1/\varepsilon), & \alpha = 1/2, \\ \varepsilon^{\alpha+1}, & \alpha < 1/2. \end{cases}$$

Remark 2.1. *The constants C_T appearing in the statements of Corollary 2.1 and Theorems 2.3 and 2.4 only depend on T, d, α, $\|w_0\|_{L^\infty}$, $TV(w_0)$ and, if $\alpha \leq 1$, on $\|w_0\|_{L^1}$.*

Let us now give optimal examples for Corollary 2.1 and Theorem 2.4. For simplicity we consider one-dimensional PDEs, but we allow for nonlinear f. We say that $\omega : (0,\infty) \to (0,\infty)$ is a *modulus* if it is nondecreasing and $\lim_{\varepsilon \to 0} \omega(\varepsilon) = 0$.

Theorem 2.5 (Optimal examples)**.** *Let $d=1$, $f \in W^{1,\infty}_{loc}(\mathbb{R})$ be strictly convex, and $T>0$.*

(i) *Assume that $\alpha \in (0,2)$ and define σ_ε as in Corollary 2.1 for any $\varepsilon>0$. Then for any modulus ω such that*

$$\lim_{\varepsilon \to 0} \frac{\omega(\varepsilon)}{\sigma_\varepsilon} = 0, \tag{2.9}$$

there is $w_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R})$ whose corresponding solutions of (1.1)*,* (1.5) *and* (1.6) *satisfy*

$$\lim_{\varepsilon \to 0} \frac{\|u(t,\cdot)-w(t,\cdot)\|_{L^1}}{\omega(\varepsilon)} = \lim_{\varepsilon \to 0} \frac{\|v(t,\cdot)-w(t,\cdot)\|_{L^1}}{\omega(\varepsilon)} = \infty, \quad \forall t \in (0,T].$$

(ii) *Assume that $\alpha \in (0,1]$ and define $\mathfrak{s}_\varepsilon$ as in Theorem 2.4. Then for any modulus ω such that*

$$\lim_{\varepsilon \to 0} \frac{\omega(\varepsilon)}{\mathfrak{s}_\varepsilon} = 0, \tag{2.10}$$

there is $w_0 \in L^1 \cap L^\infty \cap BV(\mathbb{R})$ whose corresponding solutions of (1.5) *and* (1.6) *satisfy*

$$\lim_{\varepsilon \to 0} \frac{\|u(t,\cdot)-v(t,\cdot)\|_{L^1}}{\omega(\varepsilon)} = \infty, \quad \forall t \in (0,T]. \tag{2.11}$$

Remark 2.2. *If* (2.9) *and* (2.10) *simultaneously hold, there is a common w_0 which works for both* (i) *and* (ii)*.*

3. Preliminary estimates on $|x|^{d+\alpha} g_\alpha(x)$

The two following lemmas are crucial ingredients in our estimates of convergence rates.

Lemma 3.1. *There exists $N_\alpha > 0$ such that for any $x \in \mathbb{R}^d$,*

$$\left| |x|^{d+\alpha} g_\alpha(x) - c_d(\alpha) \right| \leq \frac{N_\alpha}{|x|^\alpha},$$

where g_α and $c_d(\alpha)$ are respectively defined in (2.5) *and* (2.3).

Proof. Thanks to (A.12) the function $x \mapsto |x|^{d+\alpha} P_\alpha(x)$ admits the limit value $c_d(\alpha)$ as $|x| \to \infty$. Using the identity (see the proof of Lemma A.1)

$$|x|^{d+\alpha} P_\alpha(x) = \frac{\alpha}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-\frac{s^\alpha}{|x|^\alpha}} s^{\frac{d}{2}+\alpha-1} \mathcal{J}_{\frac{d}{2}}(s)\, ds, \tag{3.1}$$

it is also clear that

$$\lim_{|x|\to 0^+} |x|^{d+\alpha} P_\alpha(x) = 0.$$

Since the function is radially symmetric and its derivative with respect to the norm of x, that is

$$\frac{\alpha^2}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-\frac{s^\alpha}{|x|^\alpha}} \frac{s^{\frac{d}{2}+2\alpha-1}}{|x|^{\alpha+1}} \mathcal{J}_{\frac{d}{2}}(s)\, ds, \tag{3.2}$$

is bounded as $|x| \to \infty$. Then $x \mapsto |x|^{d+\alpha} P_\alpha(x)$ is bounded on $\mathbb{R}^d$ and there exist $N_\alpha/2 > 0$ such that

$$-\frac{N_\alpha}{2|x|^\alpha} + c_d(\alpha) \leq |x|^{d+\alpha} P_\alpha(x) \leq \frac{N_\alpha}{2|x|^\alpha} + c_d(\alpha). \tag{3.3}$$

From the equation (A.5) we have

$$|x|^{d+\alpha} g_\alpha(x) = \int_0^\infty e^{-t} t \left| \frac{x}{t^{1/\alpha}} \right|^{d+\alpha} P_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dt, \tag{3.4}$$

so that (3.3) gives us

$$-\frac{N_\alpha}{|x|^\alpha} + c_d(\alpha) \leq |x|^{d+\alpha} g_\alpha(x) \leq \frac{N_\alpha}{|x|^\alpha} + c_d(\alpha).$$

□

The following lemma improves the estimate above for $\alpha \in (0, 1]$. The convergence rates we obtain for this range are optimal, see Section 5.

Lemma 3.2. *Assume $\alpha \in (0, 1]$, then for any $x \in \mathbb{R}^d \setminus \{0\}$,*

$$|x|^{1+\alpha} g_\alpha(x) \leq c_d(\alpha). \tag{3.5}$$

Proof. It is enough to show that the function $x \mapsto |x|^{d+\alpha} P_\alpha(x)$ is radially monotone increasing. Its derivative with respect to the norm of x is

$$
\begin{aligned}
&\frac{\alpha^2}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-\frac{s^\alpha}{|x|^\alpha}} \frac{s^{\frac{d}{2}+2\alpha-1}}{|x|^{\alpha+1}} \mathcal{J}_{\frac{d}{2}}(s)\, ds \\
&\qquad = \frac{\alpha^2}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-\frac{s^\alpha}{|x|^\alpha}} \frac{s^{2\alpha-2}}{|x|^{\alpha+1}} s^{\frac{d}{2}+1} \mathcal{J}_{\frac{d}{2}}(s)\, ds \\
&\qquad \overset{\text{Change variable to } r|y|=s,\ y\in\mathbb{R}^{d+2}}{=} \frac{\alpha^2 |y|^{\frac{d}{2}+\alpha-1}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-r^\alpha} r^{2\alpha-2} r^{\frac{d}{2}+1} \mathcal{J}_{\frac{d}{2}}(r|y|)\, dr \\
&\qquad \overset{\text{By Th. A.1}}{=} 2\pi\alpha^2 |y|^{d+\alpha-1} \mathcal{F}^{-1}\left(e^{-|\cdot|^\alpha} |\cdot|^{2\alpha-2}\right)(y) \\
&\qquad = 2\pi\alpha^2 |y|^{d+\alpha-1} \left(P_\alpha * \mathcal{F}^{-1}\left(|\cdot|^{2\alpha-2}\right)\right)(y).
\end{aligned}
\tag{3.6}
$$

If $\alpha \in (0,1)$ then

$$
\frac{\alpha^2}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-\frac{s^\alpha}{|x|^\alpha}} \frac{s^{\frac{d}{2}+2\alpha-1}}{|x|^{\alpha+1}} \mathcal{J}_{\frac{d}{2}}(s)\, ds = \alpha^2 |y|^{d+\alpha-1} \frac{2^{\alpha-1}\Gamma\left(\frac{d}{2}+\alpha\right)}{\pi^{\frac{d}{2}}\Gamma(1-\alpha)} \left(P_\alpha * |\cdot|^{-d-2\alpha}\right)(y) \geq 0.
$$

Otherwise, if $\alpha = 1$,

$$
\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-\frac{s}{|x|}} \frac{s^{\frac{d}{2}+1}}{|x|^2} \mathcal{J}_{\frac{d}{2}}(s)\, ds = 2\pi |y|^d P_1(y) \geq 0.
$$

□

4. Proof of Theorems 2.3 and 2.4

As recalled in the Introduction, the Lévy operators $\Delta^{\alpha/2}$ and R_ε^α are associated respectively to the measures μ and ν

$$
\mu(z) = \frac{c_d(\alpha)}{|z|^{d+\alpha}} dz, \qquad \nu(z) = \frac{1}{\varepsilon^{d+\alpha}} g_\alpha\left(\frac{z}{\varepsilon}\right) dz = \frac{1}{\varepsilon^\alpha} g_\alpha^\varepsilon(z)\, dz.
$$

The following preliminary lemma is essential in the proof of our main results.

Lemma 4.1. *For any $\phi \in C_c^\infty(\mathbb{R}^d)$ and $\alpha \in (0,1]$, we have*

$$
\Delta^{\alpha/2}\phi - R_\varepsilon^\alpha(\phi) = \varepsilon^\alpha g_\alpha^\varepsilon * \Delta^\alpha(\phi).
$$

Proof. We can verify the equality by a direct computation

$$
\begin{aligned}
\Delta^{\alpha/2}\phi - R_\varepsilon^\alpha(\phi) =& \mathcal{F}^{-1}\left(\left(|\cdot|^\alpha - \frac{|\cdot|^\alpha}{1+\varepsilon^\alpha|\cdot|^\alpha}\right)\mathcal{F}(\phi)\right) \\
=& \mathcal{F}^{-1}\left(\frac{\varepsilon^\alpha|\cdot|^{2\alpha}}{1+\varepsilon^\alpha|\cdot|^\alpha}\mathcal{F}(\phi)\right) \\
=& \varepsilon^\alpha \underbrace{\mathcal{F}^{-1}\left(\frac{1}{1+\varepsilon^\alpha|\cdot|^\alpha}\right)}_{=g_\alpha^\varepsilon} * \mathcal{F}^{-1}\left(|\cdot|^{2\alpha}\mathcal{F}(\phi)\right).
\end{aligned}
$$

□

Let u and v be the solutions of (1.5) and (1.6). For any $r>0$, Theorem 2.1 implies

$$\begin{aligned}\|v-u\|_{C([0,T];L^1)} \leq & C\,(\varepsilon T)^{1/2}\,\mathrm{TV}(w_0)\left(\int_{|z|<r}|z|^2\left|\frac{1}{\varepsilon^{d+\alpha}}g_\alpha\left(\frac{z}{\varepsilon}\right)-\frac{c_d(\alpha)}{|z|^{d+\alpha}}\right|dz\right)^{1/2}\\ &+\varepsilon T\int_{|z|>r}\|w_0(\cdot-z)-w_0(\cdot)\|_{L^1}\left|\frac{1}{\varepsilon^{d+\alpha}}g_\alpha\left(\frac{z}{\varepsilon}\right)-\frac{c_d(\alpha)}{|z|^{d+\alpha}}\right|dz.\end{aligned}\tag{4.1}$$

The estimate in Lemma 3.1 together with (A.4) allows us to write

$$\left|\frac{1}{\varepsilon^{d+\alpha}}g_\alpha\left(\frac{z}{\varepsilon}\right)-\frac{c_d(\alpha)}{|z|^{d+\alpha}}\right|\leq\max\left\{M_\alpha+c_d(\alpha),N_\alpha\right\}\cdot\begin{cases}\frac{1}{|z|^{d+\alpha}}, & \text{if } |z|\leq\varepsilon,\\ \frac{\varepsilon^\alpha}{|z|^{d+2\alpha}}, & \text{if } |z|>\varepsilon.\end{cases}$$

If $\alpha\in(1,2)$, we consider $r=+\infty$ in (4.1) and we obtain

$$\|v-u\|_{C([0,T];L^1)}\leq C_T\varepsilon^{\frac{1}{2}}\left(\int_{|z|<\varepsilon}|z|^2\frac{dz}{|z|^{d+\alpha}}+\varepsilon^\alpha\int_{|z|\geq\varepsilon}|z|^2\frac{dz}{|z|^{d+2\alpha}}\right)^{1/2}\leq C_T\varepsilon^{\frac{3-\alpha}{2}},$$

where $C_T=C_T(T,\alpha,\mathrm{TV}(w_0))$.

If $\alpha\in(0,1]$, we apply the same argument as in the proof of Theorem 2.1 as it appears in [2] to get a more precise result. The argument relies on a doubling of variables, then in the following $(t,x)\in\mathbb{R}^+\times\mathbb{R}^d$ are the variables of u and $(t,y)\in\mathbb{R}^+\times\mathbb{R}^d$ the ones of v. For any given $t>0$ we have

$$\begin{aligned}&\partial_t|u(t,x)-v(t,y)|+(\mathrm{div}_x+\mathrm{div}_y)\left[\mathrm{sign}(u(t,x)-v(t,y))(f(u(t,x))-f(v(t,y)))\right]\\ &\qquad=\varepsilon\,\mathrm{sign}(u(t,x)-v(t,y))\left(\Delta^{\alpha/2}u(t,x)-R_\varepsilon^\alpha v(t,y)\right),\end{aligned}\tag{4.2}$$

and we can use the measures μ and ν to rewrite the right hand side as follows

$$\begin{aligned}&\mathrm{sign}(u(t,x)-v(t,y))\left(\Delta^{\alpha/2}u(t,x)-R_\varepsilon^\alpha v(t,y)\right)\\ &=\mathrm{sign}(u-v)\left(\int_{E^+}u(t,x+z)-u(t,x)\,d\mu(z)-\int_{E^+}v(t,y+z)-v(t,y)\,d\nu(z)\right)\\ &\quad+\mathrm{sign}(u-v)\left(\int_{E^-}u(t,x+z)-u(t,x)\,d\mu(z)-\int_{E^-}v(t,y+z)-v(t,y)\,d\nu(z)\right)\\ &=I_1+I_2,\end{aligned}$$

where $E^+=\{z\in\mathbb{R}^d|(\nu-\mu)(z)\geq0\}$ and $E^-=\{z\in\mathbb{R}^d|(\nu-\mu)(z)<0\}$. We have

$$\begin{aligned}I_1=&\mathrm{sign}(u-v)\left(\int_{E^+}u(t,x+z)-u(t,x)-v(t,y+z)+v(t,y)\,d\mu(z)\right.\\ &\left.-\int_{E^+}v(t,y+z)-v(t,y)+u(t,x)-u(t,x)\,d(\nu-\mu)(z)\right)\\ \leq&\int_{E^+}|u(t,x+z)-v(t,y+z)|-|u(t,x)-v(t,y)|\,d\mu(z)\\ &+\int_{E^+}|v(t,y+z)-u(t,x)|-|v(t,y)-u(t,x)|\,d(\nu-\mu)(z).\end{aligned}$$

A similar computation gives us

$$\begin{aligned}I_2\leq&\int_{E^-}|u(t,x+z)-v(t,y+z)|-|u(t,x)-v(t,y)|\,d\nu(z)\\ &+\int_{E^-}|u(t,x+z)-v(t,y)|-|u(t,x)-v(t,y)|\,d(\mu-\nu)(z).\end{aligned}$$

Let $\xi \mapsto \rho_\delta(\xi)$, for $\xi \in \mathbb{R}^d$, be defined by

$$\rho_\delta(\xi) = \frac{1}{\delta}\rho\left(\frac{|\xi|}{\delta}\right), \quad \rho \in C_b^\infty \cap L^1(\mathbb{R}), \quad \rho > 0, \quad \int_{\mathbb{R}} \rho(\xi)\, d\xi = 1.$$

We multiply the two sides of the equation (4.2) by $\rho_\delta(x-y)$, then we integrate with respect to both variables x and y. To shorten our notation, in the following we call γ the Borel measure $\gamma = \mu_{|E^+} + \nu_{|E^-}$.

$$\begin{aligned}
&\frac{1}{\varepsilon}\frac{d}{dt}\left\|(u(t,\cdot_x) - v(t,\cdot_y))\rho_\delta(\cdot_x - \cdot_y)\right\|_{L^1} \\
&\quad \le \iiint_{\mathbb{R}^{3d}} \left(|u(t,x+z) - v(t,y+z)| - |u(t,x) - v(t,y)|\right) \rho_\delta(x-y)\, d\gamma(z)\, dx\, dy \\
&\qquad + \iiint_{\mathbb{R}^{2d}\times E^+} \left(|v(t,y+z) - u(t,x)| - |v(t,y) - u(t,x)|\right) \rho_\delta(x-y)\, d(\nu-\mu)(z)\, dx\, dy \\
&\qquad + \iiint_{\mathbb{R}^{2d}\times E^-} \left(|u(t,x+z) - v(t,y)| - |u(t,x) - v(t,y)|\right) \rho_\delta(x-y)\, d(\mu-\nu)(z)\, dx\, dy \\
&\quad = \iiint_{\mathbb{R}^{3d}} |u(t,x) - v(t,y)| \underbrace{\left(\rho_\delta((x-z)-(y-z)) - \rho_\delta(x-y)\right)}_{=0} d\gamma(z)\, dx\, dy \\
&\qquad + \iiint_{\mathbb{R}^{2d}\times E^+} |v(t,y) - u(t,x)| \underbrace{\left(\rho_\delta(x-y+z) - \rho_\delta(x-y)\right) d(\nu-\mu)(z)}_{\mathcal{L}^{(\nu-\mu)^+}(\rho_\delta)(x-y)}\, dx\, dy \\
&\qquad + \iiint_{\mathbb{R}^{2d}\times E^-} |u(t,x) - v(t,y)| \underbrace{\left(\rho_\delta(x-y-z) - \rho_\delta(x-y)\right) d(\mu-\nu)(z)}_{\mathcal{L}^{(\mu-\nu)^+}(\rho_\delta)(x-y)}\, dx\, dy \\
&\quad = \iint_{\mathbb{R}^{2d}} |u(t,x) - v(t,y)| \mathcal{L}^{|\mu-\nu|}(\rho_\delta)(x-y)\, dx\, dy.
\end{aligned} \tag{4.3}$$

Under the assumption $\alpha \in (0,1]$, Lemma 3.2 implies that

$$(\mu-\nu)(z) = \left(\frac{c_d(\alpha)}{|z|^{d+\alpha}} - \frac{1}{\varepsilon^{d+\alpha}} g_\alpha\left(\frac{z}{\varepsilon}\right)\right) = \frac{1}{\varepsilon^{d+\alpha}}\left(\frac{c_d(\alpha)}{|z/\varepsilon|^{d+\alpha}} - g_\alpha\left(\frac{z}{\varepsilon}\right)\right) \ge 0.$$

Hence

$$\mathcal{L}^{|\mu-\nu|} = \mathcal{L}^{(\mu-\nu)} = \left(\Delta^{\alpha/2} - R_\varepsilon^\alpha\right),$$

and, by Lemma 4.1, the inequality in (4.3) becomes

$$\begin{aligned}
&\frac{d}{dt}\left\|(u(t,\cdot_x) - v(t,\cdot_y))\rho_\delta(\cdot_x - \cdot_y)\right\|_{L^1} \\
&\qquad \le \varepsilon^{\alpha+1} \iint_{\mathbb{R}^{2d}} |u(t,x) - v(t,y)| \left(g_\alpha^\varepsilon * \Delta^\alpha \rho_\delta\right)(x-y)\, dx\, dy.
\end{aligned} \tag{4.4}$$

In the remaining of the proof we consider separately the three cases $\alpha \in (1/2, 1]$, $\alpha = 1/2$ and $\alpha \in (0, 1/2)$.

4.1. **Case** $\alpha \in (1/2, 1]$**.** For any fixed $y \in \mathbb{R}^d$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} |u(t,x) - v(t,y)| \left(g_\alpha^\varepsilon * \Delta^\alpha \rho_\delta\right)(x-y)\, dx \\
&= \int_{\mathbb{R}^d} |u(t,x) - v(t,y)| \Delta^\alpha \left(g_\alpha^\varepsilon * \rho_\delta\right)(x-y)\, dx \\
&= \int_{\mathbb{R}^d} |\xi|^{2\alpha} \mathcal{F}\left(|u(t,\cdot) - v(t,y)|\right) * \mathcal{F}\left(g_\alpha^\varepsilon * \rho_\delta(\cdot - y)\right)(\xi)\, d\xi
\end{aligned}$$

$$= \int_{\mathbb{R}^d} \mathcal{F}^{-1}\left(-i\xi \mathcal{F}\left(|u(t,\cdot) - v(t,y)|\right)\right) \mathcal{F}^{-1}\left(i\xi|\xi|^{2\alpha-2} \mathcal{F}\left(g_\alpha^\varepsilon * \rho_\delta(\cdot - y)\right)\right) dx$$
$$= -\int_{\mathbb{R}^d} \operatorname{sign}(u(t,x) - v(t,y)) \nabla_x u(x) \nabla_x^{2\alpha-1} \left(g_\alpha^\varepsilon * \rho_\delta\right)(x-y)\, dx,$$

where the operator $\nabla_x^{2\alpha-1}$ is defined as in (A.13) and we use that

$$\mathcal{F}\left(g_\alpha^\varepsilon * \rho_\delta(\cdot - y)\right)(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \left(g_\alpha^\varepsilon * \rho_\delta(x-y)\right) dx$$
$$= e^{-iy\cdot\xi} \int_{\mathbb{R}^d} e^{-i(x-y)\cdot\xi} \left(g_\alpha^\varepsilon * \rho_\delta(x-y)\right) dx = e^{-iy\cdot\xi} \mathcal{F}\left(g_\alpha^\varepsilon * \rho_\delta\right)(\xi),$$

which means

$$\nabla_x^{2\alpha-1}\left(\left(g_\alpha^\varepsilon * \rho_\delta\right)(\cdot - y)\right)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} i\xi|\xi|^{2\alpha-2} e^{ix\cdot\xi} e^{-iy\cdot\xi} \mathcal{F}\left(g_\alpha^\varepsilon * \rho_\delta(\cdot)\right)(\xi)\, d\xi$$
$$= \nabla_x^{2\alpha-1}\left(g_\alpha^\varepsilon * \rho_\delta\right)(x-y).$$

Finally, using Lemma A.5, the estimate (4.4) becomes

$$\begin{aligned}
&\frac{d}{dt} \left\|(u(t,\cdot_x) - v(t,\cdot_y))\rho_\delta(\cdot_x - \cdot_y)\right\|_{L^1} \\
&\quad \le \varepsilon^{\alpha+1} \iint_{\mathbb{R}^{2d}} |u(t,x) - v(t,y)| \left(g_\alpha^\varepsilon * \Delta^\alpha \rho_\delta\right)(x-y)\, dx\, dy \\
&\quad = -\varepsilon^{\alpha+1} \iint_{\mathbb{R}^{2d}} \operatorname{sign}(u(t,x) - v(t,y)) \nabla_x u(x) \nabla_x^{2\alpha-1} \left(g_\alpha^\varepsilon * \rho_\delta\right)(x-y)\, dx\, dy \qquad (4.5) \\
&\quad \le \varepsilon^{\alpha+1} \mathrm{TV}(w_0) \left\|\nabla_x^{2\alpha-1}\left(g_\alpha^\varepsilon * \rho_\delta\right)\right\|_{L^1} \le \varepsilon^{\alpha+1} \mathrm{TV}(w_0) \left\|\nabla_x^{2\alpha-1} g_\alpha^\varepsilon\right\|_{L^1} \underbrace{\|\rho_\delta\|_{L^1}}_{=1} \\
&\quad = \varepsilon^{\alpha+1} \mathrm{TV}(w_0) \frac{1}{\varepsilon^{2\alpha-1}} \left\|\nabla_x^{2\alpha-1} g_\alpha\right\|_{L^1} = C\varepsilon^{2-\alpha} \mathrm{TV}(w_0),
\end{aligned}$$

for a constant C depending on α. We notice that

$$\begin{aligned}
\left\|u(t,\cdot_x) - v(t,\cdot_x)\right\|_{L^1} &= \int_{\mathbb{R}^d} |u(t,x) - v(t,x)|\, dx \qquad (4.6) \\
&\le \left\|(u(t,\cdot_x) - v(t,\cdot_y))\rho_\delta(\cdot_x - \cdot_y)\right\|_{L^1} + \delta \mathrm{TV}(w_0).
\end{aligned}$$

Then integrating with respect to time the inequality (4.5) we get

$$\left\|u(t,\cdot_x) - v(t,\cdot_x)\right\|_{L^1} \le \left(\delta + Ct\varepsilon^{2-\alpha}\right) \mathrm{TV}(w_0). \qquad (4.7)$$

Then we conclude

$$\left\|u(t,\cdot_x) - v(t,\cdot_x)\right\|_{L^1} \le C\mathrm{TV}(w_0) t\varepsilon^{2-\alpha}. \qquad (4.8)$$

4.2. **Case $\alpha = 1/2$.** We fix $r \in (0,1)$ and we split the integral in the right hand side of (4.4) into three terms as follows

$$\begin{aligned}
&\iint_{\mathbb{R}^{2d}} |u(t,x) - v(t,y)| \Delta^{1/2} \left(g_{1/2}^\varepsilon * \rho_\delta\right)(x-y)\, dx\, dy \\
&= \iiint_{\mathbb{R}^{2d}\times\{|z|<r\}} |u(t,x) - v(t,y)| \times \\
&\qquad \times \left(g_{1/2}^\varepsilon * \rho_\delta(x-y+z) - g_{1/2}^\varepsilon * \rho_\delta(x-y) - z\cdot\nabla_x \left(g_{1/2}^\varepsilon * \rho_\delta\right)(x-y)\right) \frac{c_d(1)}{|z|^{d+1}}\, dz\, dx\, dy \\
&+ \iiint_{\mathbb{R}^{2d}\times\{r<|z|<1\}} |u(t,x) - v(t,y)| \left(g_{1/2}^\varepsilon * \rho_\delta(x-y+z) - g_{1/2}^\varepsilon * \rho_\delta(x-y)\right) \frac{c_d(1)}{|z|^{d+1}}\, dz\, dx\, dy
\end{aligned}$$

$$+\iiint_{\mathbb{R}^{2d}\times\{|z|>1\}} |u(t,x)-v(t,y)|\left(g^\varepsilon_{1/2}*\rho_\delta(x-y+z)-g^\varepsilon_{1/2}*\rho_\delta(x-y)\right)\frac{c_d(1)}{|z|^{d+1}}\,dz\,dx\,dy$$
$$=I_A+I_B+I_C.$$

We consider first the integral I_A

$$\begin{aligned}I_A&=\iiint_{\mathbb{R}^{2d}\times\{|z|<r\}}|u(t,x)-v(t,y)|\times\\&\qquad\times\left(\int_0^1\Delta\left(g^\varepsilon_{1/2}*\rho_\delta\right)(x-y+sz)(1-s)\,ds\right)\frac{|z|^2c_d(1)}{|z|^{d+1}}\,dz\,dx\,dy\\&\le Cr\mathrm{TV}(w_0)\underbrace{\left\|g^\varepsilon_{1/2}\right\|_{L^1}}_{=1}\|\partial_x\rho_\delta\|_{L^1}\le\frac{Cr}{\delta}\mathrm{TV}(w_0).\end{aligned}$$

Then, we focus on I_B and thanks to a change of variable we get

$$\begin{aligned}I_B&\le\iiint_{\mathbb{R}^{2d}\times\{r<|z|<1\}}|u(t,x-z)-u(t,x)|\left(g^\varepsilon_{1/2}*\rho_\delta\right)(x-y)\frac{c_d(1)}{|z|^{d+1}}\,dz\,dx\,dy\\&\le-C\ln(r)\mathrm{TV}(w_0)\left\|g^\varepsilon_{1/2}\right\|_{L^1}\underbrace{\|\rho_\delta\|_{L^1}}_{=1}=-C\ln(r)\mathrm{TV}(w_0).\end{aligned}\tag{4.9}$$

Similar computations, using the classical inequality $\|u(\cdot-z)-u(\cdot)\|_{L^1}\le 2\|w_0\|_{L^1}$, lead to establish an upper bound for I_C

$$I_C\le C\|w_0\|_{L^1}.$$

Finally, we can put all our estimates together to get

$$\frac{d}{dt}\|(u(t,\cdot_x)-v(t,\cdot_y))\rho_\delta(\cdot_x-\cdot_y)\|_{L^1}\le C\varepsilon^{3/2}\left[\mathrm{TV}(w_0)\left(\frac{r}{\delta}-\ln(r)\right)+\|w_0\|_{L^1}\right].$$

Then we integrate with respect to time and apply the inequality in (4.6) to obtain

$$\|u(t,\cdot)-v(t,\cdot)\|_{L^1}\le\mathrm{TV}(w_0)\delta+C\varepsilon^{3/2}t\left[\mathrm{TV}(w_0)\left(\frac{r}{\delta}-\ln(r)\right)+\|w_0\|_{L^1}\right].$$

In order to optimize this estimate we give to r and δ the common optimal value $t\varepsilon^{3/2}$, and we obtain

$$\|u(t,\cdot)-v(t,\cdot)\|_{L^1}\le Ct\varepsilon^{3/2}|\ln(\varepsilon)|.$$

4.3. **Case** $\alpha\in(0,1/2)$**.** We fix $r\in(0,1)$ and we split the integral in the right hand side of (4.4) into two terms as follows

$$\begin{aligned}&\iint_{\mathbb{R}^{2d}}|u(t,x)-v(t,y)|\Delta^\alpha\left(g^\varepsilon_\alpha*\rho_\delta\right)(x-y)\,dx\,dy\\&=\iiint_{\mathbb{R}^{2d}\times\{|z|<r\}}|u(t,x)-v(t,y)|\left(\left(g^\varepsilon_\alpha*\rho_\delta\right)(x-y+z)-\left(g^\varepsilon_\alpha*\rho_\delta\right)(x-y)\right)\frac{c_d(\alpha)}{|z|^{d+2\alpha}}\,dz\,dx\,dy\\&+\iiint_{\mathbb{R}^{2d}\times\{|z|>r\}}|u(t,x)-v(t,y)|\left(\left(g^\varepsilon_\alpha*\rho_\delta\right)(x-y+z)-\left(g^\varepsilon_\alpha*\rho_\delta\right)(x-y)\right)\frac{c_d(\alpha)}{|z|^{d+2\alpha}}\,dz\,dx\,dy\\&=I_<+I_>.\end{aligned}$$

We estimate $I_<$ and $I_>$ by computations similar to the ones we did for I_B and I_C in (4.9). Then we get

$$\iint_{\mathbb{R}^{2d}}|u(t,x)-v(t,y)|\Delta^\alpha\left(g^\varepsilon_\alpha*\rho_\delta\right)(x-y)\,dx\,dy\le C\left(\mathrm{TV}(w_0)r^{1-2\alpha}+\|w_0\|_{L^1}r^{-2\alpha}\right).$$

We observe that the inequality above does not depend on δ. As in analysis of the previous cases we integrate with respect to time and apply the inequality in (4.6), then after optimizing the choice of r, we get the desired estimate

$$\|u(t,\cdot)-v(t,\cdot)\|_{L^1}\leq Ct\varepsilon^{\alpha+1}.$$

5. Proof of Theorem 2.5

In this Section we follow the approach used by Şabac in [8], which requires optimal convergence rates for the case where flux f is linear, to prove Theorem 2.5.

For this reason in Section 5.1 we consider the zero convection case and we give examples in which, as ε vanishes, the L^1 distance between u and w and between u and v approaches zero at the rates announced respectively in Corollary 2.1 in for $\alpha\in(0,2)$, and in Theorem 2.3 for $\alpha\in(1,2)$. Please notice that for the reader convenience, the proofs of two lemmas in this part are postponed to Section 6 as they consists of rather long case by case studies.

Finally, Section 5.2 contains the proof of Theorem 2.5.

5.1. Examples of optimal convergence rates in the linear case (zero convection).

5.1.1. *Vanishing fractional Laplacian approximation of a stationary solution.* Define the initial condition $w_0=\chi_{E_J}$ as the characteristic function of the set

$$E_J=\bigcup_{j=0}^{J}\left[\left(-2j-\frac{3}{2},-2j-\frac{1}{2}\right)\cup\left(2j+\frac{1}{2},2j+\frac{3}{2}\right)\right], \tag{5.1}$$

and consider the initial value problems

$$\begin{cases}\partial_t u=\varepsilon\partial_{xx}^{\alpha/2}u, & t>0,\ x\in\mathbb{R},\\ u(0,x)=w_0(x), & x\in\mathbb{R},\end{cases}\qquad\begin{cases}\partial_t w=0, & t>0,\ x\in\mathbb{R},\\ w(0,x)=w_0(x), & x\in\mathbb{R}.\end{cases}$$

We observe that

$$\begin{aligned}\mathcal{F}^{-1}\left(\chi_{E_J}\right)(\xi)=&\int_{\mathbb{R}}\chi_{E_J}(x)e^{ix\xi}dx=\sum_{j=0}^{J}\left(\int_{-2j-\frac{3}{2}}^{-2j-\frac{1}{2}}e^{ix\xi}dx+\int_{2j+\frac{1}{2}}^{2j+\frac{3}{2}}e^{ix\xi}dx\right)\\ =&\sum_{j=0}^{J}\left(e^{-i(2j+1)\xi}\int_{-2j-\frac{3}{2}}^{-2j-\frac{1}{2}}e^{i(x+2j+1)\xi}dx+e^{i(2j+1)\xi}\int_{2j+\frac{1}{2}}^{2j+\frac{3}{2}}e^{i(x-2j-1)\xi}dx\right)\\ =&\sum_{j=0}^{J}\left(e^{-i(2j+1)\xi}+e^{i(2j+1)\xi}\right)\int_{-\frac{1}{2}}^{\frac{1}{2}}e^{ix\xi}dx\\ =&\sum_{j=0}^{J}\left(e^{-i(2j+1)\xi}+e^{i(2j+1)\xi}\right)\frac{e^{i\xi/2}-e^{-i\xi/2}}{i\xi}\\ =&\frac{4}{\xi}\sin\left(\frac{\xi}{2}\right)\mathfrak{Re}\left(\sum_{j=0}^{J}e^{i(2j+1)\xi}\right)=\frac{4}{\xi}\sin\left(\frac{\xi}{2}\right)\mathfrak{Re}\left(e^{i\xi}\sum_{j=0}^{J}e^{2ij\xi}\right)\\ =&\frac{4}{\xi}\sin\left(\frac{\xi}{2}\right)\mathfrak{Re}\left(e^{i\xi}\frac{e^{2i(J+1)\xi}-1}{e^{2i\xi}-1}\right)=\frac{4}{\xi}\sin\left(\frac{\xi}{2}\right)\mathfrak{Re}\left(\frac{e^{2i(J+1)\xi}-1}{e^{i\xi}-e^{-i\xi}}\right)\\ =&\frac{4}{\xi}\sin\left(\frac{\xi}{2}\right)\mathfrak{Re}\left(\frac{e^{2i(J+1)\xi}-1}{2i\sin(\xi)}\right)=\frac{2}{\xi\sin(\xi)}\sin\left(\frac{\xi}{2}\right)\mathfrak{Re}\left(i-ie^{2i(J+1)\xi}\right)\\ =&\frac{2\sin(2(J+1)\xi)}{\xi\sin(\xi)}\sin\left(\frac{\xi}{2}\right).\end{aligned} \tag{5.2}$$

We prove the following, which establishes the optimality of the estimate in (2.7)

Proposition 5.1. *We have that*

$$\inf_{0<t\varepsilon\leq 1,\, J\geq 1} \frac{\|u(t,\cdot)-w(t,\cdot)\|_{L^1}}{J\sigma_{t\varepsilon}} > 0, \tag{5.3}$$

where

$$\sigma_r = \begin{cases} r^{1/\alpha}, & \alpha\in(1,2], \\ r\ln(1/r), & \alpha = 1, \\ r, & \alpha\in(0,1), \end{cases} \qquad r\geq 0. \tag{5.4}$$

It is clear that

$$w(t,x) = \chi_{E_J}(x), \qquad u(t,x) = \left(\mathcal{F}^{-1}\left(e^{-\varepsilon t|\cdot|^\alpha}\right) * \chi_{E_J}\right)(x),$$

and

$$w(t,x)\geq u(t,x), \qquad x\in E_J,$$

then, thanks to the Parseval Identity

$$\begin{aligned} 0\leq \int_{E_J}(w-u)dx &= \int_{E_J}\chi_{E_J}dx - \int_{E_J}\left(\mathcal{F}^{-1}\left(e^{-\varepsilon t|\cdot|^\alpha}\right)*\chi_{E_J}\right)dx \\ &= \int_{E_J}\chi_{E_J}dx - \int_{\mathbb{R}}\left(\mathcal{F}^{-1}\left(e^{-\varepsilon t|\cdot|^\alpha}\right)*\chi_{E_J}\right)\chi_{E_J}dx \\ &= \int_{E_J}\chi_{E_J}dx - \int_{\mathbb{R}}\mathcal{F}^{-1}\left(e^{-\varepsilon t|\cdot|^\alpha}\right)\left(\chi_{E_J}*\chi_{E_J}\right)dx \\ &= \int_{\mathbb{R}}\left(\chi_{E_J}\right)^2 dx - \int_{\mathbb{R}} e^{-\varepsilon t|\xi|^\alpha}\mathcal{F}^{-1}\left(\chi_{E_J}*\chi_{E_J}\right)d\xi \\ &= 2\pi\int_{\mathbb{R}}\left(\mathcal{F}^{-1}\left(\chi_{E_J}\right)\right)^2 d\xi - 2\pi\int_{\mathbb{R}} e^{-\varepsilon t|\xi|^\alpha}\left(\mathcal{F}^{-1}\left(\chi_{E_J}\right)\right)^2 d\xi \\ &= 2\pi\int_{\mathbb{R}}\left(1-e^{-\varepsilon t|\xi|^\alpha}\right)\left(\mathcal{F}^{-1}\left(\chi_{E_J}\right)\right)^2 d\xi. \end{aligned}$$

Therefore, using (5.2)

$$\int_{E_J}(w-u)dx = 8\pi\int_{\mathbb{R}}\left(1-e^{-\varepsilon t|\xi|^\alpha}\right)\frac{\sin^2(2(J+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi. \tag{5.5}$$

Lemma 5.1. *For any given $\varphi\in C_c^\infty(\mathbb{R})$ there holds*

$$\lim_{J\to\infty}\frac{1}{J}\int_{\mathbb{R}}\varphi(\xi)\frac{\sin^2(2(J+1)\xi)}{\sin^2(\xi)}d\xi = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\varphi(k\pi). \tag{5.6}$$

Proof. Let $\varphi\in C_c^\infty(\mathbb{R})$ be given. Since φ has compact support and the function $\xi\mapsto\frac{\sin^2(2(J+1)\xi)}{\sin^2(\xi)}$ is π-periodic, an argument based on a partition of unity allows us to assume that

$$\mathrm{supp}(\varphi)\subset\left[-\frac{3\pi}{4},\frac{3\pi}{4}\right].$$

We rewrite the left hand side in (5.6) as follows

$$\begin{aligned} \frac{1}{J}\int_{\mathbb{R}}\varphi(\xi)\frac{\sin^2(2(J+1)\xi)}{\sin^2(\xi)}d\xi &= \frac{1}{J}\int_{-3\pi/4}^{3\pi/4}\varphi(\xi)\frac{\sin^2(2(J+1)\xi)}{\sin^2(\xi)}d\xi \\ &= 4\frac{J+1}{J}\int_{-3\pi(J+1)/4}^{3\pi(J+1)/4}\varphi\left(\frac{z}{J+1}\right)\frac{\sin^2(2z)}{(2z)^2}\frac{\left(\frac{z}{J+1}\right)^2}{\sin^2\left(\frac{z}{J+1}\right)}dz. \end{aligned}$$

Since

$$\int_{\mathbb{R}} \frac{\sin^2(2z)}{(2z)^2} dz = \frac{1}{8\pi}, \quad \text{and} \quad \sup_{z\in\left[-\frac{3\pi(J+1)}{4},\frac{3\pi(J+1)}{4}\right]} \frac{\left(\frac{z}{J+1}\right)^2}{\sin^2\left(\frac{z}{J+1}\right)} = \frac{9\pi^2}{8},$$

the Dominated Convergence Theorem allows us to prove (5.6)

$$\lim_{J\to\infty} 4\frac{J+1}{J}\int_{-3\pi(J+1)/4}^{3\pi(J+1)/4} \varphi\left(\frac{z}{J+1}\right)\frac{\sin^2(2z)}{(2z)^2}\frac{\left(\frac{z}{J+1}\right)^2}{\sin^2\left(\frac{z}{J+1}\right)}dz = 4\varphi(0)\int_{\mathbb{R}}\frac{\sin^2(2z)}{(2z)^2}dz = \frac{\varphi(0)}{2\pi}.$$

□

Define

$$\mathcal{E}(r,J) = \frac{1}{J\sigma_r}\int_{\mathbb{R}}\left(1-e^{-r|\xi|^\alpha}\right)\frac{\sin^2(2(J+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi, \tag{5.7}$$

where σ_r is defined in (5.4).

Thanks to (5.5) and (5.7), we have

$$\frac{\|u(t,\cdot)-w(t,\cdot)\|_{L^1}}{J\sigma_{t\varepsilon}} \geq \frac{1}{J\sigma_{t\varepsilon}}\int_{E_J}(w-u)dx = 8\pi\mathcal{E}(t\varepsilon,J),$$

then the proof of (5.3) relies on the following lemma.

Lemma 5.2. *The following estimate holds*

$$\inf_{0<r\leq 1,\, J\geq 1}\mathcal{E}(r,J) > 0. \tag{5.8}$$

The proof (consisting of a long case by case study) is postponed to Section 6.

5.1.2. *On the L^1 distance between the vanishing fractional Laplacian and vanishing Rosenau approximations.* Consider the initial value problems

$$\begin{cases}\partial_t u = \varepsilon\partial_{xx}^{\alpha/2}u, & t>0,\, x\in\mathbb{R},\\ u(0,x)=w_0(x)=\chi_{E_J}(x), & x\in\mathbb{R},\end{cases} \qquad \begin{cases}\partial_t v = \varepsilon R_\varepsilon^\alpha(v), & t>0,\, x\in\mathbb{R},\\ v(0,x)=w_0(x), & x\in\mathbb{R},\end{cases}$$

which common initial condition w_0 is again the characteristic function of the set E_J defined in (5.1). In this section we only consider α in the range $(0,1]$ and our goal is to prove that the convergence rates in Theorem 2.4 are optimal by showing the following

Proposition 5.2. *For any fixed $T>0$ we have that*

$$\inf_{\substack{0<t\leq T,\\ 0<\varepsilon\leq 1,\, J\geq 1}} \frac{\|u(t,\cdot)-v(t,\cdot)\|_{L^1}}{Jt\sigma_\varepsilon} > 0, \tag{5.9}$$

where

$$\sigma_\varepsilon = \begin{cases}\varepsilon^{\alpha+1}, & \alpha\in(0,1/2),\\ \varepsilon^{3/2}\ln(1/\varepsilon), & \alpha=1/2,\\ \varepsilon^{2-\alpha}, & \alpha\in(1/2,1].\end{cases} \tag{5.10}$$

We follow the same argument as in the proof of Lemma 5.1. In particular we have that

$$\int_{E_J}(v-u)dx = 8\pi\int_{\mathbb{R}}\left(e^{-\varepsilon t\frac{|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}}-e^{-\varepsilon t|\xi|^\alpha}\right)\frac{\sin^2(2(J+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi. \tag{5.11}$$

We define

$$\mathcal{E}(t,\varepsilon,J) = \frac{1}{Jt\sigma_\varepsilon}\int_{\mathbb{R}}\left(e^{-\varepsilon t\frac{|\xi|^\alpha}{1+\varepsilon^\alpha|\xi|^\alpha}}-e^{-t\varepsilon|\xi|^\alpha}\right)\frac{\sin^2(2(J+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi, \tag{5.12}$$

where σ_ε is defined in (5.10). As for any $(t,x)\in\mathbb{R}^+\times\mathbb{R}$ $v(t,x)\geq u(t,x)$ the estimate (5.9) is a direct consequence of (5.11), the definition of $\mathcal{E}$ in (5.12) and the inequality in the following lemma, whose proof is postponed to Section 6.

Lemma 5.3. *For any given $T>0$*

$$\inf_{\substack{0<t\leq T,\\ 0<\varepsilon\leq 1,\, J\geq 1}} \mathcal{E}(t,\varepsilon,J)>0. \tag{5.13}$$

5.2. **Optimal convergence rates in the nonlinear, strictly convex case.** This section is devoted to the proof of Theorem 2.5. To make the computations as readable as possible we focus on the Burgers' equation, in which $f(\xi)=\xi^2/2$, but the proof applies with very minor modifications to the case of a general strictly convex flux. Consider the initial value problems

$$\begin{cases}\partial_t u+\partial_x\left(\frac{u^2}{2}\right)=\varepsilon\partial_{xx}^{\alpha/2}u, & t>0,\, x\in\mathbb{R},\\ u(0,x)=w_0(x), & x\in\mathbb{R},\end{cases}\qquad\begin{cases}\partial_t w+\partial_x\left(\frac{w^2}{2}\right)=0, & t>0,\, x\in\mathbb{R},\\ w(0,x)=w_0(x), & x\in\mathbb{R},\end{cases} \tag{5.14}$$

where w_0 is a piecewise constant function defined as follows.

Let $\widetilde{\ell}>\ell>0$ and $\{J_n\}_{n\in\mathbb{N}}\subset\mathbb{N}$ be a monotone increasing diverging sequence. For every $i\geq 1$ and $1\leq j\leq J_i$ define the intervals I_{ij} by

$$\inf I_{ij}=\left(2\left(j+\sum_{m=1}^{i-1}J_m\right)-i\right)\ell+(i-1)\widetilde{\ell},\qquad\qquad \sup I_{ij}=\inf I_{ij}+\ell.$$

The initial condition w_0 takes the form $w_0=\displaystyle\sum_{i=1}^{\infty}\sum_{j=1}^{J_i}c_i\chi_{I_{ij}}$, where the positive numbers c_i satisfy

$$\sum_{i=1}^{\infty}c_iJ_i<\infty. \tag{5.15}$$

Being the set I_{ij} pairwise disjoint we have $w_0\in L^\infty\cap L^1\cap BV(\mathbb{R})$.

We choose $\ell>0$ large enough so that when the dynamics is governed by the inviscid conservation law the waves associated to the evolution of each of the non-zero components of w_0, $c_i\chi_{I_{ij}}$, do not interact before time T, i.e.,

$$\ell\geq 2T\left\|f'(w_0)\right\|_{L^\infty}=2T\max_{i\geq 1}c_i. \tag{5.16}$$

Consider the Cauchy problems given by the linearization of the equations[1] about the state c_i and the truncated initial conditions $w_0^i=c_i\displaystyle\sum_{j=1}^{J_i}\chi_{I_{ij}}$

$$\begin{cases}\partial_t q_i+c_i\partial_x q_i=\varepsilon\partial_{xx}^{\alpha/2}q_i, & t>0,\, x\in\mathbb{R},\\ q_i(0,x)=w_0^i(x), & x\in\mathbb{R},\end{cases}\qquad\begin{cases}\partial_t z_i+c_i\partial_x z_i=0, & t>0,\, x\in\mathbb{R},\\ z_i(0,x)=w_0^i(x), & x\in\mathbb{R},\end{cases}$$

and the sets

$$I_i=\left(\inf I_{i1},\sup I_{iJ_i}\right),\qquad I_i^t=I_i+c_it.$$

We call u_i and w_i the solutions of the equations in (5.14) corresponding to the initial condition w_0^i, then we observe that

$$\begin{aligned}\|u(t,\cdot)-w(t,\cdot)\|_{L^1}&\geq\|u(t,\cdot)-w(t,\cdot)\|_{L^1(I_i^t)}\\ &\geq\|u_i(t,\cdot)-w_i(t,\cdot)\|_{L^1(I_i^t)}-\|u(t,\cdot)-u_i(t,\cdot)\|_{L^1(I_i^t)}-\|w_i(t,\cdot)-w(t,\cdot)\|_{L^1(I_i^t)}.\end{aligned} \tag{5.17}$$

[1] If f is a general strictly convex function we have to modify this part, but the only difference would be to consider $f'(c_i)$ as coefficients instead of c_i.

Thanks to (5.16) we have

$$\|w_i(t,\cdot)-w(t,\cdot)\|_{L^1(I_i^t)}=0, \qquad t\in(0,T],$$

moreover, using the results of [1] and the choice of ℓ

$$\begin{aligned}
&\|u_i(t,\cdot)-u(t,\cdot)\|_{L^1(I_i^t)}\\
&\leq \int_{I_i^t+(-\|f'(w_0))\|_{L^\infty},\|f'(w_0))\|_{L^\infty})t}(G(\varepsilon t,\cdot)*|u(0,\cdot)-u_i(0,\cdot)|)(x)dx\\
&\leq \int_{I_i^t+(-\frac{\ell}{2},\frac{\ell}{2})}(G(\varepsilon t,\cdot)*|u(0,\cdot)-u_i(0,\cdot)|)(x)dx \qquad (5.18)\\
&\leq \sum_{k\neq i,\,k\geq 1}\sum_{j=1}^{J_k} c_k \underbrace{\int_{I_i^t+(-\frac{\ell}{2},\frac{\ell}{2})}(G(\varepsilon t,\cdot)*\chi_{I_{kj}})(x)dx}_{E_{kj}}.
\end{aligned}$$

Therefore, using Lemma 5.1 in (5.17) we get

$$\begin{aligned}
&\|u(t,\cdot)-w(t,\cdot)\|_{L^1}\geq \|u(t,\cdot)-w(t,\cdot)\|_{L^1(I_i^t)}\\
&\geq \|q_i(t,\cdot)-z_i(t,\cdot)\|_{L^1(I_i^t)}-\sum_{k\neq i,\,k\geq 1}\sum_{j=1}^{J_k}c_kE_{kj}\\
&\qquad -\|u_i(t,\cdot)-q_i(t,\cdot)\|_{L^1(I_i^t)}-\|w_i(t,\cdot)-z_i(t,\cdot)\|_{L^1(I_i^t)}\\
&\geq C_TJ_ic_i\varepsilon t-\sum_{k\neq i,\,k\geq 1}\sum_{j=1}^{J_k}c_kE_{kj}-\|u_i(t,\cdot)-q_i(t,\cdot)\|_{L^1(I_i^t)}-\|w_i(t,\cdot)-z_i(t,\cdot)\|_{L^1(I_i^t)}.
\end{aligned}$$

For the last two terms, we use the continuous dependence results with respect to the flux (see [2, Theorem 3.1] for the nonlocal case)

$$\|u_i(t,\cdot)-q_i(t,\cdot)\|_{L^1(I_i^t)}\leq \underbrace{\|u_i(0,\cdot)-q_i(0,\cdot)\|_{L^1}}_{=0}+TV\left(\sum_{j=1}^{J_i}c_i\chi_{I_{ij}}\right)\left\|f'-c_i\right\|_{L^\infty(0,c_i)}t\leq 2J_ic_i^2t.$$

Arguing in the same way on $\|w_i(t,\cdot)-z_i(t,\cdot)\|_{L^1(I_i^t)}$ we can conclude

$$\|u(t,\cdot)-w(t,\cdot)\|_{L^1}\geq J_ic_it(C_T\varepsilon-4c_i)-\sum_{k\neq i,\,k\geq 1}\sum_{j=1}^{J_k}c_kE_{kj}. \qquad (5.19)$$

Let us now estimate E_{kj} from (5.18)

$$\begin{aligned}
E_{kj}=&\frac{1}{(\varepsilon t)^{1/\alpha}}\int_{\mathbb{R}}\left(\int_{I_i^t+(-\frac{\ell}{2},\frac{\ell}{2})}P\left(\frac{x-y}{(\varepsilon t)^{1/\alpha}}\right)dx\right)\chi_{I_{kj}}(y)dy\\
\leq&\frac{c}{(\varepsilon t)^{1/\alpha}}\int_{I_{kj}}\left(\int_{I_i^t+(-\frac{\ell}{2},\frac{\ell}{2})}\frac{1}{1+\frac{|x-y|^{1+\alpha}}{(\varepsilon t)^{1+1/\alpha}}}dx\right)dy\\
\leq&c\varepsilon t\int_{I_{kj}}\left(\int_{I_i^t+(-\frac{\ell}{2},\frac{\ell}{2})}\frac{1}{|x-y|^{1+\alpha}}dx\right)dy\\
\leq&\frac{2c\varepsilon t}{\alpha}\operatorname{sign}(i-k)\int_{I_{kj}}\left(\frac{1}{|y-\inf I_i-c_it+\frac{\ell}{2}|^\alpha}-\frac{1}{|y-\sup I_i-c_it-\frac{\ell}{2}|^\alpha}\right)dy
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2c\varepsilon t}{\alpha}\begin{cases}(\inf I_i - \sup I_{kj} - \frac{\ell}{2})^{-\alpha}, & i>k,\\ (\inf I_{kj} - \sup I_i - \frac{\ell}{2})^{-\alpha}, & i<k,\end{cases}\\
&\leq \frac{2c\varepsilon t}{\alpha}\begin{cases}\left(\left(2\left(\sum_{m=k}^{i-1} J_m - j\right)+\frac{1}{2}\right)\ell + (i-k)(\widetilde{\ell}-\ell)\right)^{-\alpha}, & i>k,\\ \left(2\left(\sum_{m=i+1}^{k-1} J_m + j - 1\right)\ell + (k-i)(\widetilde{\ell}-\ell)\right)^{-\alpha}, & i<k.\end{cases}
\end{aligned}
$$

Hence, using (5.15),

$$
\begin{aligned}
&\sum_{k\neq i,\,k=1}^{\infty}\sum_{j=1}^{J_k} c_k E_{kj}\\
&\quad\leq \frac{2c\varepsilon t}{\alpha}\left(c_{i-1}\sum_{j=1}^{J_{i-1}}\frac{1}{(\widetilde{\ell}-\frac{\ell}{2})^{\alpha}} + c_{i+1}\sum_{j=1}^{J_{i+1}}\frac{1}{(\widetilde{\ell}-\ell)^{\alpha}}\right.\\
&\qquad + \sum_{k=1}^{i-2}\sum_{j=1}^{J_k}\frac{c_k}{\left(\left(2\sum_{m=k+1}^{i-1}J_m+\frac{1}{2}\right)\ell+(i-k)(\widetilde{\ell}-\ell)\right)^{\alpha}}\\
&\qquad \left. + \sum_{k=i+2}^{\infty}\sum_{j=1}^{J_k}\frac{c_k}{\left(\left(2\sum_{m=i+1}^{k-1}J_m\right)\ell+(k-i)(\widetilde{\ell}-\ell)\right)^{\alpha}}\right)\\
&\quad\leq \frac{2c\varepsilon t}{\alpha}\left(\frac{\max\{J_{i-1}c_{i-1},J_{i+1}c_{i+1}\}}{(\widetilde{\ell}-\ell)^{\alpha}} + \sum_{k\neq i,\,j-1,\,k=1}^{\infty} J_k c_k \max\left\{\frac{1}{(2J_{i-1}\ell)^{\alpha}},\frac{1}{(2J_{i+1}\ell)^{\alpha}}\right\}\right)\\
&\quad\leq c\varepsilon t\left(\frac{\max\{J_{i-1}c_{i-1},J_{i+1}c_{i+1}\}}{(\widetilde{\ell}-\ell)^{\alpha}} + \frac{1}{\ell^{\alpha}}\max\left\{\frac{1}{J_{i-1}^{\alpha}},\frac{1}{J_{i+1}^{\alpha}}\right\}\right).
\end{aligned}
$$

Using (5.19)

$$
\begin{aligned}
\|u(t,\cdot)-w(t,\cdot)\|_{L^1} \geq J_i c_i t\varepsilon\left(C_T - \frac{4c_i}{\varepsilon} - \frac{c}{(\widetilde{\ell}-\ell)^{\alpha}}\frac{\max\{J_{i-1}c_{i-1},J_{i+1}c_{i+1}\}}{J_i c_i}\right.\\
\left. - \frac{c}{\ell^{\alpha}}\max\left\{\frac{1}{J_i J_{i-1}^{\alpha}c_i},\frac{1}{J_i J_{i+1}^{\alpha}c_i}\right\}\right),
\end{aligned}
\tag{5.20}
$$

that holds for every $t\in(0,T)$, $i\geq 2$, $\varepsilon\in(0,1)$ and any choice of J_i, c_i and $\widetilde{\ell}$ satisfying (5.15) and (5.16).

Consider σ_ε as in (2.7) and let ω in $C((0,+\infty);(0,+\infty))$ be nondecreasing and such that $\omega(\varepsilon)=o(\sigma_\varepsilon)$, as in the statement of Theorem 2.5. We have to choose the indices i depending on ε in such a way that (2.11) holds. Define

$$
\widetilde{\omega}(\varepsilon) = \frac{1}{\varepsilon}\int_{\varepsilon}^{2\varepsilon}\sqrt{\sigma_\xi\omega(\xi)}d\xi.
$$

Being $\widetilde{\omega}$ continuous, increasing and positive we have

$$
\lim_{\varepsilon\to 0}\frac{\widetilde{\omega}(\varepsilon)}{\omega(\varepsilon)} = \infty, \qquad \lim_{\varepsilon\to 0}\frac{\widetilde{\omega}(\varepsilon)}{\sigma_\varepsilon} = 0.
\tag{5.21}
$$

Indeed

$$\frac{\widetilde{\omega}(\varepsilon)}{\omega(\varepsilon)} \geq \frac{\sqrt{\sigma_\varepsilon \omega(\varepsilon)}}{\omega(\varepsilon)} = \sqrt{\frac{\sigma_\varepsilon}{\omega(\varepsilon)}}, \qquad \frac{\widetilde{\omega}(\varepsilon)}{\sigma_\varepsilon} = \frac{1}{\varepsilon}\int_\varepsilon^{2\varepsilon} \frac{\sigma_\xi}{\sigma_\varepsilon}\sqrt{\frac{\omega(\xi)}{\sigma_\xi}}d\xi.$$

Define

$$i(\varepsilon) = \left\lfloor \sqrt{\frac{\sigma_\varepsilon}{\widetilde{\omega}(\varepsilon)}} \right\rfloor, \qquad S(i) = \{\varepsilon \in (0,1); i = i(\varepsilon_\sigma)\},$$

for $i \geq \min_{\varepsilon\in(0,1)} i(\varepsilon)$, where, being $\varepsilon \mapsto \sigma_\varepsilon$ increasing, $\sigma \mapsto \varepsilon_\sigma$ is the inverse of $\varepsilon \mapsto \sigma_\varepsilon$. Notice that thanks to (5.21), $S(i) \neq \emptyset$. Moreover, we define

$$c_i = \min\left\{C_{T,\alpha} \min S(i), i^{-\frac{2(1+\alpha)}{\alpha}}\right\}, \qquad J_i = \lceil c_i^{-1} i^{-2} \rceil, \tag{5.22}$$

where $C_{T,\alpha}$ will be chosen later. Clearly,

$$i^{-2} \leq J_i c_i \leq 2i^{-2}, \tag{5.23}$$

and since $\sigma_\varepsilon \in S(i(\varepsilon))$ we have

$$c_{i(\varepsilon)} \leq C_{T,\alpha} \min S(i(\varepsilon)) \leq C_{T,\alpha}\sigma_\varepsilon. \tag{5.24}$$

Clearly, we have that (5.15) holds. Moreover, thanks to (5.21)

$$J_{i(\varepsilon)}c_{i(\varepsilon)}\sigma_\varepsilon \geq i(\varepsilon)^{-2}\sigma_\varepsilon = \frac{\sigma_\varepsilon}{\left\lfloor \sqrt{\frac{\sigma_\varepsilon}{\widetilde{\omega}(\varepsilon)}} \right\rfloor^2} \geq \frac{\sigma_\varepsilon}{\frac{\sigma_\varepsilon}{\widetilde{\omega}(\varepsilon)}} = \widetilde{\omega}(\varepsilon). \tag{5.25}$$

The remaining part of the proof consists in the analysis of three cases, depending on the value of α.

Case $\alpha < 1$: Using (5.20) and choosing $C_{T,\alpha} = C_T/8$

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \frac{\|u(t,\cdot) - w(t,\cdot)\|_{L^1}}{\omega(\varepsilon)} \\
&\quad \geq t \liminf_{\varepsilon\to 0} \frac{J_{i(\varepsilon)}c_{i(\varepsilon)}\varepsilon}{\omega(\varepsilon)} \left(C_T - \frac{4c_{i(\varepsilon)}}{\varepsilon} - \frac{c}{(\widetilde{\ell} - \ell)^\alpha} \frac{\max\{J_{i(\varepsilon)-1}c_{i(\varepsilon)-1}, J_{i(\varepsilon)+1}c_{i(\varepsilon)+1}\}}{J_{i(\varepsilon)}c_{i(\varepsilon)}} \right. \\
&\qquad \left. - \frac{c}{\ell^\alpha} \max\left\{ \frac{1}{J_{i(\varepsilon)}J_{i(\varepsilon)-1}^\alpha c_{i(\varepsilon)}}, \frac{1}{J_{i(\varepsilon)}J_{i(\varepsilon)+1}^\alpha c_{i(\varepsilon)}} \right\} \right).
\end{aligned}$$

Thanks to (5.25) in order to prove (2.11) we have to show that

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \left(C_T - \frac{4c_{i(\varepsilon)}}{\varepsilon} - \frac{c}{(\widetilde{\ell} - \ell)^\alpha} \frac{\max\{J_{i(\varepsilon)-1}c_{i(\varepsilon)-1}, J_{i(\varepsilon)+1}c_{i(\varepsilon)+1}\}}{J_{i(\varepsilon)}c_{i(\varepsilon)}} \right. \\
\left. - \frac{c}{\ell^\alpha} \max\left\{ \frac{1}{J_{i(\varepsilon)}J_{i(\varepsilon)-1}^\alpha c_{i(\varepsilon)}}, \frac{1}{J_{i(\varepsilon)}J_{i(\varepsilon)+1}^\alpha c_{i(\varepsilon)}} \right\} \right) > 0.
\end{aligned} \tag{5.26}$$

From to (5.22) and (5.23) we deduce that

$$\frac{J_{i-1}c_{i-1}}{J_i c_i} \leq 2\left(\frac{i}{i-1}\right)^2 \leq 4,$$

and

$$\frac{1}{J_i J_{i-1}^\alpha c_i} \leq \frac{J_{i-1}c_{i-1}}{J_i c_i} \frac{1}{J_{i-1}^{1+\alpha} c_{i-1}} \leq \frac{4}{\left(\frac{1}{c_{i-1}(i-1)^2}\right)^{1+\alpha} c_{i-1}} \leq 4c_{i-1}^\alpha (i-1)^{2(1+\alpha)} \leq 4,$$

so that

$$C_T - \frac{4c_i}{\varepsilon} - \frac{c}{(\widetilde{\ell} - \ell)^\alpha} \frac{\max\{J_{i-1}c_{i-1}, J_{i+1}c_{i+1}\}}{J_i c_i} - \frac{c}{\ell^\alpha} \max\left\{ \frac{1}{J_i J_{i-1}^\alpha c_i}, \frac{1}{J_i J_{i+1}^\alpha c_i} \right\}$$

$$\leq C_T - \frac{4c_i}{\varepsilon} - \frac{4c}{(\widetilde{\ell}-\ell)^\alpha} - \frac{4c}{\ell^\alpha}.$$

Since, thanks to (5.22) $4c_i/\varepsilon \leq C_T/2$, choosing $\widetilde{\ell}$ and ℓ large enough we have (5.26).
Case $\alpha > 1$: Since $\sigma_{t\varepsilon} = t^{1/\alpha}\sigma_\varepsilon = (t\varepsilon)^{1/\alpha}$ using (5.20) and (5.23) for $t \leq T$ and ε small enough

$$\begin{aligned}
\|u(t,\cdot) - w(t,\cdot)\|_{L^1(I_i^t)} &\geq J_{i(\varepsilon)}c_{i(\varepsilon)}\sigma_\varepsilon \left(C_T t^{1/\alpha} - 4C_{T,\alpha}t\right) \\
&\quad - \frac{c}{(\widetilde{\ell}-\ell)^\alpha}\max\{J_{i(\varepsilon)-1}c_{i(\varepsilon)-1}, J_{i(\varepsilon)+1}c_{i(\varepsilon)+1}\}t\varepsilon - \frac{c}{\ell^\alpha}\max\{J_{i(\varepsilon)-1}^{-\alpha}, J_{i(\varepsilon)+1}^{-\alpha}\}t\varepsilon \\
&\geq (t\varepsilon)^{1/\alpha}J_{i(\varepsilon)}c_{i(\varepsilon)}\left(C_T - 4C_{T,\alpha}t^{\frac{\alpha-1}{\alpha}}\right. \\
&\quad - \frac{c}{(\widetilde{\ell}-\ell)^\alpha}\frac{\max\{J_{i(\varepsilon)-1}c_{i(\varepsilon)-1}, J_{i(\varepsilon)+1}c_{i(\varepsilon)+1}\}}{J_{i(\varepsilon)}c_{i(\varepsilon)}}(t\varepsilon)^{\frac{\alpha-1}{\alpha}} \\
&\quad \left. - \frac{c}{\ell^\alpha}\frac{\max\{J_{i(\varepsilon)-1}^{-\alpha}, J_{i(\varepsilon)+1}^{-\alpha}\}}{J_{i(\varepsilon)}c_{i(\varepsilon)}}(t\varepsilon)^{\frac{\alpha-1}{\alpha}}\right).
\end{aligned} \tag{5.27}$$

Choosing

$$C_{T,\alpha} = \frac{C_T}{8T^{\frac{\alpha-1}{\alpha}}},$$

we gain

$$\begin{aligned}
&\|u(t,\cdot) - w(t,\cdot)\|_{L^1(I_i^t)} \\
&\geq t^{1/\alpha}\widetilde{\omega}(\varepsilon)\left(\frac{C_T}{2} - \frac{c}{(\widetilde{\ell}-\ell)^\alpha}\frac{\max\{J_{i(\varepsilon)-1}c_{i(\varepsilon)-1}, J_{i(\varepsilon)+1}c_{i(\varepsilon)+1}\}}{J_{i(\varepsilon)}c_{i(\varepsilon)}}(T\varepsilon)^{\frac{\alpha-1}{\alpha}}\right. \\
&\qquad \left. - \frac{c}{\ell^\alpha}\frac{\max\{J_{i(\varepsilon)-1}^{-\alpha}, J_{i(\varepsilon)+1}^{-\alpha}\}}{J_{i(\varepsilon)}c_{i(\varepsilon)}}(T\varepsilon)^{\frac{\alpha-1}{\alpha}}\right).
\end{aligned} \tag{5.28}$$

Since

$$\frac{c}{(\widetilde{\ell}-\ell)^\alpha}\frac{\max\{J_{i-1}c_{i-1}, J_{i+1}c_{i+1}\}}{J_ic_i} + \frac{c}{\ell^\alpha}\frac{\max\{J_{i-1}^{-\alpha}, J_{i+1}^{-\alpha}\}}{J_ic_i} \leq \frac{4c}{(\widetilde{\ell}-\ell)^\alpha} + \frac{4c}{\ell^\alpha}.$$

Thanks to (5.25) and (5.28), we obtain the desired estimate provided that $\widetilde{\ell}$ and ℓ are large enough.
Case $\alpha = 1$: Since

$$\sigma_{t\varepsilon} = t\varepsilon\ln(1/\varepsilon) + t\varepsilon\ln(1/t) = t\sigma_\varepsilon - \varepsilon t\ln(t),$$

(5.27) gives

$$\begin{aligned}
&\|u(t,\cdot) - w(t,\cdot)\|_{L^1(I_i^t)} \\
&\quad \geq C_T t J_{i(\varepsilon)}c_{i(\varepsilon)}\sigma_\varepsilon - 4C_{T,1}tJ_{i(\varepsilon)}c_{i(\varepsilon)}\sigma_\varepsilon - C_{T,1}\varepsilon t|\ln(t)|J_{i(\varepsilon)}c_{i(\varepsilon)} \\
&\qquad - \frac{c}{\widetilde{\ell}-\ell}\max\{J_{i(\varepsilon)-1}c_{i(\varepsilon)-1}, J_{i(\varepsilon)+1}c_{i(\varepsilon)+1}\}t\varepsilon - \frac{c}{\ell}\max\{J_{i(\varepsilon)-1}^{-1}, J_{i(\varepsilon)+1}^{-1}\}t\varepsilon \\
&\quad \geq tJ_{i(\varepsilon)}c_{i(\varepsilon)}\sigma_\varepsilon\left(C_T - 4C_{T,1} - \frac{C_{T,1}|\ln(t)|}{\ln(1/\varepsilon)}\right. \\
&\qquad \left. - \frac{c}{\widetilde{\ell}-\ell}\frac{\max\{J_{i(\varepsilon)-1}c_{i(\varepsilon)-1}, J_{i(\varepsilon)+1}c_{i(\varepsilon)+1}\}}{\ln(1/\varepsilon)} - \frac{c}{\ell}\frac{\max\{J_{i(\varepsilon)-1}^{-\alpha}, J_{i(\varepsilon)+1}^{-\alpha}\}}{\ln(1/\varepsilon)}\right) \\
&\quad \geq t\widetilde{\omega}(\varepsilon)\left(C_T - 4C_{T,1} - \frac{C_{T,1}|\ln(t)|}{\ln(1/\varepsilon)}\right.
\end{aligned}$$

$$- \frac{c}{\widetilde{\ell}-\ell}\frac{\max\{J_{i(\varepsilon)-1}c_{i(\varepsilon)-1}, J_{i(\varepsilon)+1}c_{i(\varepsilon)+1}\}}{\ln(1/\varepsilon)} - \frac{c}{\ell}\frac{\max\{J_{i(\varepsilon)-1}^{-\alpha}, J_{i(\varepsilon)+1}^{-\alpha}\}}{\ln(1/\varepsilon)}\Bigg),$$

thanks to (5.21) we have (2.10).

6. Technical details

6.1. **Proof of Lemma 5.2.** Let $\{(r_n, J_n)\}_n \subset (0,1]\times[1,\infty)$ be a minimizing sequence, namely

$$\inf_{0<r\leq 1,\, J\geq 1} \mathcal{E}(r,J) = \lim_n \mathcal{E}(r_n, J_n). \tag{6.1}$$

Passing to a subsequence, we can assume that

$$\exists \lim_n r_n = r_\infty \in [0,1], \qquad \exists \lim_n J_n = J_\infty \in [1,\infty].$$

We distinguish eigth cases.

CASE 1. Assume

$$r_\infty = 0, \qquad J_\infty = \infty, \qquad \alpha < 1.$$

We have

$$\begin{aligned}
\mathcal{E}(r_n,J_n) =& \frac{1}{J_n r_n}\int_{\mathbb{R}} \left(1-e^{-r_n|\xi|^\alpha}\right)\frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi\\
=& \frac{1}{J_n}\int_{\mathbb{R}}\int_0^1 |\xi|^\alpha e^{-sr_n|\xi|^\alpha}\frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)ds\,d\xi\\
\geq& \frac{1}{J_n}\int_{-r_n^{-1/\alpha}}^{r_n^{-1/\alpha}}\int_0^1 |\xi|^\alpha e^{-sr_n|\xi|^\alpha}\frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)ds\,d\xi\\
\geq& \frac{1}{eJ_n}\int_{-r_n^{-1/\alpha}}^{r_n^{-1/\alpha}} |\xi|^{\alpha-2}\frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi.
\end{aligned}$$

Thanks to (5.6),

$$\lim_n \mathcal{E}(r_n,J_n) \geq \frac{1}{2e\pi}\sum_{k\in\mathbb{Z}}|k\pi|^{\alpha-2}\sin^2\left(\frac{k\pi}{2}\right) = \frac{1}{2e\pi^{3-\alpha}}\sum_{k\in\mathbb{Z}}|2k+1|^{\alpha-2} > 0,$$

using to (6.1), we have (5.8).

CASE 2. Assume

$$r_\infty = 0, \qquad J_\infty = \infty, \qquad \alpha > 1.$$

Since

$$c_0 = \inf_{z\in[\pi/2,3\pi/2]}\frac{1-e^{-|z|^\alpha}}{z^2} > 0,$$

and the function $s\mapsto \frac{\sin^2(2(J_n+1)s)}{\sin^2(s)}\sin^2\left(\frac{s}{2}\right)$ is $2\pi-$ periodic, we have

$$\begin{aligned}
\mathcal{E}(r_n,J_n) =& \frac{1}{J_n r_n^{1/\alpha}}\int_{\mathbb{R}} \left(1-e^{-r_n|\xi|^\alpha}\right)\frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi\\
=& \frac{1}{J_n}\int_{\mathbb{R}} \left(1-e^{-|z|^\alpha}\right)\frac{\sin^2(2(J_n+1)r_n^{-1/\alpha}z)}{z^2\sin^2(r_n^{-1/\alpha}z)}\sin^2\left(\frac{r_n^{-1/\alpha}z}{2}\right)dz\\
\geq& \frac{1}{J_n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(1-e^{-|z|^\alpha}\right)\frac{\sin^2(2(J_n+1)r_n^{-1/\alpha}z)}{z^2\sin^2(r_n^{-1/\alpha}z)}\sin^2\left(\frac{r_n^{-1/\alpha}z}{2}\right)dz\\
\geq& \frac{c_0}{J_n}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sin^2(2(J_n+1)r_n^{-1/\alpha}z)}{\sin^2(r_n^{-1/\alpha}z)}\sin^2\left(\frac{r_n^{-1/\alpha}z}{2}\right)dz
\end{aligned}$$

$$
\begin{aligned}
=&\frac{c_0 r_n^{1/\alpha}}{J_n}\int_{\frac{\pi r_n^{-1/\alpha}}{2}}^{\frac{3\pi r_n^{-1/\alpha}}{2}}\frac{\sin^2(2(J_n+1)s)}{\sin^2(s)}\sin^2\left(\frac{s}{2}\right)ds\\
\geq&\frac{c_0 r_n^{1/\alpha}}{J_n}\int_{\frac{\pi r_n^{-1/\alpha}}{2}}^{\frac{\pi r_n^{-1/\alpha}}{2}+2\pi\left\lfloor\frac{1}{2r_n^{1/\alpha}}\right\rfloor}\frac{\sin^2(2(J_n+1)s)}{\sin^2(s)}\sin^2\left(\frac{s}{2}\right)ds\\
=&\frac{c_0 r_n^{1/\alpha}}{J_n}\left\lfloor\frac{1}{2r_n^{1/\alpha}}\right\rfloor\int_{\frac{\pi r_n^{-1/\alpha}}{2}}^{\frac{\pi r_n^{-1/\alpha}}{2}+2\pi}\frac{\sin^2(2(J_n+1)s)}{\sin^2(s)}\sin^2\left(\frac{s}{2}\right)ds\\
=&\frac{c_0 r_n^{1/\alpha}}{J_n}\left\lfloor\frac{1}{2r_n^{1/\alpha}}\right\rfloor\int_0^{2\pi}\frac{\sin^2(2(J_n+1)s)}{\sin^2(s)}\sin^2\left(\frac{s}{2}\right)ds.
\end{aligned}
$$

Thanks to (5.6),

$$
\lim_n \mathcal{E}(r_n,J_n)\geq\lim_n\frac{c_0 r_n^{1/\alpha}}{J_n}\left\lfloor\frac{1}{2r_n^{1/\alpha}}\right\rfloor\int_0^{2\pi}\frac{\sin^2(2(J_n+1)s)}{\sin^2(s)}\sin^2\left(\frac{s}{2}\right)ds=\frac{c_0}{4\pi}>0.
$$

CASE 3. Assume

$$
r_\infty=0,\qquad J_\infty=\infty,\qquad \alpha=1.
$$

We have

$$
\begin{aligned}
\mathcal{E}(r_n,J_n)=&\frac{1}{J_n r_n\ln(1/r_n)}\int_{\mathbb{R}}\left(1-e^{-r_n|\xi|}\right)\frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi\\
\geq&\frac{1}{J_n\ln(1/r_n)}\int_{-1/r_n}^{1/r_n}\int_0^1 e^{-r_n|\xi|s}\frac{\sin^2(2(J_n+1)\xi)}{|\xi|\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)ds\,d\xi\\
\geq&\frac{1}{J_n e\ln(1/r_n)}\int_{-1/r_n}^{1/r_n}\frac{\sin^2(2(J_n+1)\xi)}{|\xi|\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi\\
\geq&\frac{1}{J_n e\ln(1/r_n)}\sum_{k\geq 0,\,(2k+\frac{3}{2})\pi\leq\frac{1}{r_n}}\int_{(2k+\frac{1}{2})\pi}^{(2k+\frac{3}{2})\pi}\frac{\sin^2(2(J_n+1)\xi)}{|\xi|\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi\\
\geq&\frac{1}{J_n e2\ln(1/r_n)}\sum_{k\geq 0,\,(2k+\frac{3}{2})\pi\leq\frac{1}{r_n}}\int_{(2k+\frac{1}{2})\pi}^{(2k+\frac{3}{2})\pi}\frac{\sin^2(2(J_n+1)\xi)}{|\xi|\sin^2(\xi)}d\xi\\
\geq&\frac{1}{J_n e2\ln(1/r_n)}\sum_{k\geq 0,\,(2k+\frac{3}{2})\pi\leq\frac{1}{r_n}}\frac{1}{(2k+\frac{3}{2})\pi}\int_{(2k+\frac{1}{2})\pi}^{(2k+\frac{3}{2})\pi}\frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}d\xi\\
=&\frac{1}{J_n e2\ln(1/r_n)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}d\xi\sum_{k\geq 0,\,(2k+\frac{3}{2})\pi\leq\frac{1}{r_n}}\frac{1}{(2k+\frac{3}{2})\pi}.
\end{aligned}
$$

Since

$$
c_0=\lim_n\frac{1}{\ln(1/r_n)}\sum_{k\geq 0,\,(2k+\frac{3}{2})\pi\leq\frac{1}{r_n}}\frac{1}{(2k+\frac{3}{2})\pi}>0.
$$

Thanks to (5.6),

$$
\lim_n\mathcal{E}(r_n,J_n)\geq\lim_n\frac{c_0}{J_n 2e}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}d\xi=\frac{c_0}{4e\pi}>0.
$$

CASE 4. Assume

$$
r_\infty>0,\qquad J_\infty=\infty.
$$

Since it is not restrictive to assume

$$r_0 = \inf_{n\in\mathbb{N}} r_n > 0,$$

we have

$$\mathcal{E}(r_n, J_n) \geq \frac{1}{J_n\lambda_\alpha}\int_{\mathbb{R}} \left(1 - e^{-r_0|\xi|^\alpha}\right) \frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi,$$

where

$$\lambda_\alpha = \begin{cases} 1, & \text{if } \alpha \neq 1, \\ 1/e, & \text{if } \alpha = 1. \end{cases}$$

Thanks to (5.6),

$$\lim_n \mathcal{E}(r_n, J_n) \geq \frac{1}{2\pi^3\lambda_\alpha}\sum_{k\in\mathbb{Z}} \frac{1 - e^{-r_0|k\pi|^\alpha}}{k^2} \sin^2\left(\frac{k\pi}{2}\right) = \frac{1}{2\pi^3\lambda_\alpha}\sum_{k\in\mathbb{Z}} \frac{1 - e^{-r_0|(2k+1)\pi|^\alpha}}{(2k+1)^2} > 0.$$

CASE 5. Assume

$$r_\infty = 0, \qquad J_\infty < \infty, \qquad \alpha < 1.$$

We have

$$\begin{aligned}\lim_n \mathcal{E}(r_n, J_n) =& \lim_n \frac{1}{J_n r_n}\int_{\mathbb{R}} \left(1 - e^{-r_n|\xi|^\alpha}\right) \frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi \\ =& \frac{1}{J_\infty}\int_{\mathbb{R}} |\xi|^\alpha \frac{\sin^2(2(J_\infty+1)\xi)}{\xi^2\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi > 0.\end{aligned}$$

CASE 6. Assume

$$r_\infty = 0, \qquad J_\infty < \infty, \qquad \alpha > 1.$$

Arguing as in CASE 2 we get

$$\begin{aligned}\lim_n \mathcal{E}(r_n, J_n) \geq& \lim_n \frac{c_0 r_n^{1/\alpha}}{J_n}\left[\frac{1}{2r_n^{1/\alpha}}\right]\int_0^{2\pi} \frac{\sin^2(2(J_n+1)s)}{\sin^2(s)} \sin^2\left(\frac{s}{2}\right) ds \\ =& \frac{c_0}{2J_\infty}\int_0^{2\pi} \frac{\sin^2(2(J_\infty+1)s)}{\sin^2(s)} \sin^2\left(\frac{s}{2}\right) ds > 0.\end{aligned}$$

CASE 7. Assume

$$r_\infty = 0, \qquad J_\infty < \infty, \qquad \alpha = 1.$$

Arguing as in CASE 3 we get

$$\lim_n \mathcal{E}(r_n, J_n) \geq \lim_n \frac{c_0}{J_n 2e}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} d\xi = \frac{c_0}{J_\infty 2e}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2(2(J_\infty+1)\xi)}{\sin^2(\xi)} d\xi > 0.$$

CASE 8. Assume

$$r_\infty > 0, \qquad J_\infty < \infty.$$

We have

$$\lim_n \mathcal{E}(r_n, J_n) = \frac{1}{J_\infty\sigma_{r_\infty}}\int_{\mathbb{R}} \left(1 - e^{-r_\infty|\xi|^\alpha}\right) \frac{\sin^2(2(J_\infty+1)\xi)}{\xi^2\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi > 0.$$

In this way, thanks to (6.1), (5.8) is completely proved.

6.2. **Proof of Lemma 5.3.** Let $\{(t_n,\varepsilon_n,J_n)\}_n \subset (0,T]\times(0,1]\times[1,\infty)$ be a minimizing sequence, namely

$$\inf_{\substack{0<t\le T,\\ 0<\varepsilon\le 1,\, J\ge 1}} \mathcal{E}(t,\varepsilon,J) = \lim_n \mathcal{E}(t_n,\varepsilon_n,J_n). \tag{6.2}$$

Passing to a subsequence, we can assume that

$$\exists \lim_n t_n = t_\infty \in [0,T], \qquad \exists \lim_n \varepsilon_n = \varepsilon_\infty \in [0,1], \qquad \exists \lim_n J_n = J_\infty \in [1,\infty].$$

We distinguish eigth cases.

CASE 1. Assume

$$\varepsilon_\infty = 0, \qquad J_\infty = \infty, \qquad \alpha < 1/2.$$

We have

$$\begin{aligned}
\mathcal{E}(t_n,\varepsilon_n,J_n) &= \frac{1}{J_n t_n \varepsilon_n^{1+\alpha}} \int_{\mathbb{R}} \left(e^{-t_n\varepsilon_n \frac{|\xi|^\alpha}{1+\varepsilon_n^\alpha|\xi|^\alpha}} - e^{-t_n\varepsilon_n|\xi|^\alpha} \right) \frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi \\
&= \frac{1}{J_n} \int_{\mathbb{R}} \int_0^1 \frac{|\xi|^{2\alpha}}{1+\varepsilon_n^\alpha|\xi|^\alpha} \exp\left(-t_n\varepsilon_n|\xi|^\alpha \frac{1+s\varepsilon_n^\alpha|\xi|^\alpha}{1+\varepsilon_n^\alpha|\xi|^\alpha} \right) \frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) ds\, d\xi \\
&\ge \frac{1}{J_n} \int_{\mathbb{R}} \int_0^1 \frac{|\xi|^{2(\alpha-1)}}{1+|\xi|^\alpha} e^{-T|\xi|^\alpha} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) ds\, d\xi.
\end{aligned}$$

Thanks to (5.6),

$$\begin{aligned}
\lim_n \mathcal{E}(t_n,\varepsilon_n,J_n) &\ge \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \frac{|k\pi|^{2(\alpha-1)}}{1+|k\pi|^\alpha} e^{-T|k\pi|^\alpha} \sin^2\left(\frac{k\pi}{2}\right) \\
&= \frac{1}{2\pi^{3-\alpha}} \sum_{k\in\mathbb{Z}} \frac{|2k+1|^{2(\alpha-1)}}{1+|(2k+1)\pi|^\alpha} e^{-T|(2k+1)\pi|^\alpha} > 0,
\end{aligned}$$

using to (6.2), we have (5.13).

CASE 2. Assume

$$\varepsilon_\infty = 0, \qquad J_\infty = \infty, \qquad 1/2 < \alpha \le 1.$$

Since

$$c_0 = \inf_{z\in[\pi/2,3\pi/2]} \frac{|z|^{2(\alpha-1)}}{1+|z|^\alpha} e^{-T|z|^\alpha} > 0, \tag{6.3}$$

and the function $s \mapsto \frac{\sin^2(2(J_n+1)s)}{\sin^2(s)} \sin^2\left(\frac{s}{2}\right)$ is $2\pi-$periodic, we have

$$\begin{aligned}
\mathcal{E}(t_n,\varepsilon_n,J_n) &= \frac{1}{J_n t_n \varepsilon_n^{2-\alpha}} \int_{\mathbb{R}} \left(e^{-t_n\varepsilon_n \frac{|\xi|^\alpha}{1+\varepsilon_n^\alpha|\xi|^\alpha}} - e^{-t_n\varepsilon_n|\xi|^\alpha} \right) \frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi \\
&= \frac{1}{J_n\varepsilon_n^{1-2\alpha}} \int_{\mathbb{R}} \int_0^1 \frac{|\xi|^{2\alpha}}{1+\varepsilon_n^\alpha|\xi|^\alpha} \exp\left(-t_n\varepsilon_n|\xi|^\alpha \frac{1+s\varepsilon_n^\alpha|\xi|^\alpha}{1+\varepsilon_n^\alpha|\xi|^\alpha} \right) \frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) ds\, d\xi \\
&= \frac{1}{J_n} \int_{\mathbb{R}} \int_0^1 \frac{|z|^{2(\alpha-1)}}{1+|z|^\alpha} \exp\left(-t_n\varepsilon_n^{1-\alpha}|z|^\alpha \frac{1+s|z|^\alpha}{1+|z|^\alpha} \right) \frac{\sin^2(2(J_n+1)z\varepsilon_n^{-1})}{\sin^2(z\varepsilon_n^{-1})} \sin^2\left(\frac{z}{2\varepsilon_n}\right) ds\, dz \\
&\ge \frac{1}{J_n} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{|z|^{2(\alpha-1)}}{1+|z|^\alpha} e^{-T|z|^\alpha} \frac{\sin^2(2(J_n+1)z\varepsilon_n^{-1})}{\sin^2(z\varepsilon_n^{-1})} \sin^2\left(\frac{z}{2\varepsilon_n}\right) dz \\
&\ge \frac{c_0}{J_n} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sin^2(2(J_n+1)z\varepsilon_n^{-1})}{\sin^2(z\varepsilon_n^{-1})} \sin^2\left(\frac{z}{2\varepsilon_n}\right) dz = \frac{c_0\varepsilon_n}{J_n} \int_{\frac{\pi}{2\varepsilon_n}}^{\frac{3\pi}{2\varepsilon_n}} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi \\
&\ge \frac{c_0\varepsilon_n}{J_n} \int_{\frac{\pi}{2\varepsilon_n}}^{\frac{\pi}{2\varepsilon_n}+2\pi\left\lfloor\frac{1}{2\varepsilon_n}\right\rfloor} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi
\end{aligned}$$

$$=\frac{c_0\varepsilon_n}{J_n}\left\lfloor\frac{1}{2\varepsilon_n}\right\rfloor\int_0^{2\pi}\frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)\,d\xi.$$

Thanks to (5.6),

$$\lim_n \mathcal{E}(t_n,\varepsilon_n,J_n)\geq \lim_n \frac{c_0\varepsilon_n}{J_n}\left\lfloor\frac{1}{2\varepsilon_n}\right\rfloor\int_0^{2\pi}\frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)\,d\xi=\frac{c_0}{4\pi}>0.$$

CASE 3. Assume

$$\varepsilon_\infty=0,\qquad J_\infty=\infty,\qquad \alpha=1/2.$$

Computations similar to the ones we used in the cases above give us

$$\begin{aligned}
&\mathcal{E}(t_n,\varepsilon_n,J_n)\\
=&\frac{1}{J_nt_n\varepsilon_n^{3/2}\ln(1/\varepsilon_n)}\int_{\mathbb{R}}\left(e^{-t_n\varepsilon_n\frac{|\xi|^{1/2}}{1+\sqrt{\varepsilon_n|\xi|}}}-e^{-t_n\varepsilon_n|\xi|^{1/2}}\right)\frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi\\
\geq&\frac{1}{2J_n\ln(1/\varepsilon_n)}\int_{-\varepsilon_n^{-1}}^{\varepsilon_n^{-1}}e^{-T}\frac{\sin^2(2(J_n+1)\xi)}{|\xi|\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)\,d\xi\\
\geq&\frac{1}{4J_ne^T\ln(1/\varepsilon_n)}\sum_{k\geq 0,\left(2k+\frac{3}{2}\right)\pi\leq\varepsilon_n^{-1}}\int_{\left(2k+\frac{1}{2}\right)\pi}^{\left(2k+\frac{3}{2}\right)\pi}\frac{\sin^2(2(J_n+1)\xi)}{|\xi|\sin^2(\xi)}\,d\xi\\
\geq&\frac{1}{4J_ne^T\ln(1/\varepsilon_n)}\sum_{k\geq 0,\left(2k+\frac{3}{2}\right)\pi\leq\varepsilon_n^{-1}}\frac{1}{\left(2k+\frac{3}{2}\right)\pi}\int_{\left(2k+\frac{1}{2}\right)\pi}^{\left(2k+\frac{3}{2}\right)\pi}\frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}\,d\xi\\
=&\frac{1}{4J_ne^T\ln(1/\varepsilon_n)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}\,d\xi\sum_{k\geq 0,\left(2k+\frac{3}{2}\right)\pi\leq\varepsilon_n^{-1}}\frac{1}{\left(2k+\frac{3}{2}\right)\pi}.
\end{aligned}$$

Since

$$c_0=\lim_n\frac{1}{\ln(1/\varepsilon_n)}\sum_{k\geq 0,\,(2k+\frac{3}{2})\pi\leq\varepsilon_n^{-1}}\frac{1}{(2k+\frac{3}{2})\pi}>0. \tag{6.4}$$

Thanks to (5.6),

$$\lim_n\mathcal{E}(t_n,\varepsilon_n,J_n)\geq\lim_n\frac{c_0}{4J_ne^T}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}d\xi=\frac{c_0}{8e^T\pi}>0.$$

CASE 4. Assume

$$\varepsilon_\infty>0,\qquad J_\infty=\infty,$$

so that it is not restrictive to assume

$$\inf_{n\in\mathbb{N}}\varepsilon_n=\varepsilon_0>0.$$

Since the function $\varepsilon\mapsto\sigma_\varepsilon$ is bounded and increasing on $[0,1]$ we have

$$\begin{aligned}
&\mathcal{E}(t_n,\varepsilon_n,J_n)=\frac{1}{J_nt_n\sigma_{\varepsilon_n}}\int_{\mathbb{R}}\left(e^{-t_n\varepsilon_n\frac{|\xi|^\alpha}{1+\varepsilon_n^\alpha|\xi|^\alpha}}-e^{-t_n\varepsilon_n|\xi|^\alpha}\right)\frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)d\xi\\
\geq&\frac{1}{J_n\|\sigma_*\|_{L^\infty}}\int_{\mathbb{R}}\int_0^1\varepsilon_n^{1+\alpha}\frac{|\xi|^{2(\alpha-1)}}{1+\varepsilon_n^\alpha|\xi|^\alpha}\exp\left(-t_n\varepsilon_n|\xi|^\alpha\frac{1+s\varepsilon_n^\alpha|\xi|^\alpha}{1+\varepsilon_n^\alpha|\xi|^\alpha}\right)\frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)ds\,d\xi\\
\geq&\frac{\varepsilon_0^{1+\alpha}}{J_n\|\sigma_*\|_{L^\infty}}\int_{\mathbb{R}}\frac{|\xi|^{2(\alpha-1)}}{1+|\xi|^\alpha}e^{-T|\xi|^\alpha}\frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)}\sin^2\left(\frac{\xi}{2}\right)\,d\xi.
\end{aligned}$$

Thanks to (5.6)

$$\begin{aligned}\lim_n \mathcal{E}(t_n,\varepsilon_n,J_n) &\geq \lim_n \frac{\varepsilon_0^{1+\alpha}}{2\pi\|\sigma_*\|_{L^\infty}} \sum_{k\in\mathbb{Z}} \frac{(k\pi)^{2(\alpha-1)}}{1+(k\pi)^\alpha} e^{-T(k\pi)^\alpha} \sin\left(\frac{k\pi}{2}\right) \\ &= \lim_n \frac{\varepsilon_0^{1+\alpha}}{2\pi\|\sigma_*\|_{L^\infty}} \sum_{k\in\mathbb{Z}} \frac{((2k+1)\pi)^{2(\alpha-1)}}{1+((2k+1)\pi)^\alpha} e^{-T((2k+1)\pi)^\alpha} > 0.\end{aligned}$$

CASE 5. Assume

$$\varepsilon_\infty = 0, \qquad J_\infty < +\infty, \qquad \alpha < 1/2.$$

We have

$$\begin{aligned}\lim_n \mathcal{E}(t_n,\varepsilon_n,J_n) =& \lim_n \frac{1}{J_n t_n \varepsilon_n^{1+\alpha}} \int_{\mathbb{R}} \left(e^{-t_n\varepsilon_n \frac{|\xi|^\alpha}{1+\varepsilon_n^\alpha|\xi|^\alpha}} - e^{-t_n\varepsilon_n|\xi|^\alpha}\right) \frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi \\ =& \frac{1}{J_\infty} \int_{\mathbb{R}} |\xi|^{2(\alpha-1)} \frac{\sin^2(2(J_\infty+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi > 0.\end{aligned}$$

CASE 6. Assume

$$\varepsilon_\infty = 0, \qquad J_n \leq J_\infty < +\infty, \qquad 1/2 < \alpha \leq 1.$$

Using the same approach as in CASE 2 we have that

$$\begin{aligned}\lim_n \mathcal{E}(t_n,\varepsilon_n,J_n) =& \lim_n \frac{1}{J_n t_n \varepsilon_n^{2-\alpha}} \int_{\mathbb{R}} \left(e^{-t_n\varepsilon_n \frac{|\xi|^\alpha}{1+\varepsilon_n^\alpha|\xi|^\alpha}} - e^{-t_n\varepsilon_n|\xi|^\alpha}\right) \frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi \\ \geq& \lim_n \frac{c_0\varepsilon_n}{J_n} \left\lfloor \frac{1}{2\varepsilon_n} \right\rfloor \int_0^{2\pi} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi \\ =& \frac{c_0}{2J_\infty} \int_0^{2\pi} \frac{\sin^2(2(J_\infty+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi > 0,\end{aligned}$$

where c_0 in the constant defined in (6.3).

CASE 7. Assume

$$\varepsilon_\infty = 0, \qquad J_n \leq J_\infty < +\infty, \qquad \alpha = 1/2.$$

Arguing as in CASE 3 we obtain

$$\begin{aligned}&\lim_n \mathcal{E}(t_n,\varepsilon_n,J_n) \\ &= \lim_n \frac{1}{J_n t_n \varepsilon_n^{3/2} \ln(1/\varepsilon_n)} \int_{\mathbb{R}} \left(e^{-t_n\varepsilon_n \frac{|\xi|^{1/2}}{1+\sqrt{\varepsilon_n|\xi|}}} - e^{-t_n\varepsilon_n|\xi|^{1/2}}\right) \frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi \\ &\geq \lim_n \frac{c_0}{4J_n e^T} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} d\xi = \frac{c_0}{4J_\infty e^T} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2(2(J_\infty+1)\xi)}{\sin^2(\xi)} d\xi > 0,\end{aligned}$$

where c_0 in the constant defined in (6.4).

CASE 8. Assume

$$\varepsilon_n \geq \varepsilon_0 > 0, \qquad J_n \leq J_\infty < +\infty.$$

Then we simply have

$$\begin{aligned}\lim_n \mathcal{E}(t_n,\varepsilon_n,J_n) =& \lim_n \frac{1}{J_n t_n \sigma_{\varepsilon_n}} \int_{\mathbb{R}} \left(e^{-t_n\varepsilon_n \frac{|\xi|^\alpha}{1+\varepsilon_n^\alpha|\xi|^\alpha}} - e^{-t_n\varepsilon_n|\xi|^\alpha}\right) \frac{\sin^2(2(J_n+1)\xi)}{\xi^2\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi \\ \geq& \lim_n \frac{\varepsilon_0^{1+\alpha}}{J_n\|\sigma_*\|_{L^\infty}} \int_{\mathbb{R}} \frac{|\xi|^{2(\alpha-1)}}{1+|\xi|^\alpha} e^{-T|\xi|^\alpha} \frac{\sin^2(2(J_n+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi \\ =& \frac{\varepsilon_0^{1+\alpha}}{J_\infty\|\sigma_*\|_{L^\infty}} \int_{\mathbb{R}} \frac{|\xi|^{2(\alpha-1)}}{1+|\xi|^\alpha} e^{-T|\xi|^\alpha} \frac{\sin^2(2(J_\infty+1)\xi)}{\sin^2(\xi)} \sin^2\left(\frac{\xi}{2}\right) d\xi > 0.\end{aligned}$$

The estimates above prove (6.2), thus (5.13).

Appendix A. Some tools of fractional calculus

Most of the material in this section is well known and appears in different research papers, the interested reader may refer to [5, 15, 20]. For this reason some of the proofs are omitted.

A.1. The fractional Rosenau operator is a Lévy operator.

The function $G_\alpha \in C^\infty((0,+\infty)\times\mathbb{R}^d)$, given by

$$G_\alpha(t,x) = \mathcal{F}^{-1}\left(e^{-t|\cdot|^\alpha}\right)(x),$$

is the fundamental solution of the fractional heat equation. This means that for any given initial condition $\phi_0 \in C_c^\infty(\mathbb{R}^d)$, the unique $C_b^\infty(\mathbb{R}^+\times\mathbb{R}^d)$ solution of the Cauchy problem

$$\begin{cases} \partial_t\phi - \Delta^{\alpha/2}\phi = 0, & (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\ \phi(0,x) = \phi_0(x), & x\in\mathbb{R}^d, \end{cases}$$

is given by $\phi(t,x) = G_\alpha(t,\cdot) * \phi_0(x)$.

It is easy to show that for any $(t,x)\in\mathbb{R}^+\times\mathbb{R}^d$, $G_\alpha(t,x) = t^{-\frac{d}{\alpha}}G_\alpha(1,t^{-\frac{1}{\alpha}}x)$. Exploiting this equality, we define $P_\alpha(x) = G_\alpha(1,x)$ and we immediately have that $G(t,x) = t^{\frac{d}{\alpha}}P_\alpha(t^{-\frac{1}{\alpha}}x)$. The function $G_\alpha(t,\cdot)$, hence P_α, is radially symmetric, positive and its L^1-norm equals 1.

Moreover, by explicit computations one can show that there exists $k_\alpha > 0$ such that for all $t>0$,

$$\|\nabla P_\alpha\|_{L^1} \le k_\alpha, \qquad \text{so that} \qquad \|\nabla G_\alpha(t,\cdot)\|_{L^1} \le t^{-\frac{1}{\alpha}}k_\alpha.$$

Finally, we also use the following lemma, whose proof is deferred to Section A.2.

Lemma A.1. *The following equality holds*

$$\lim_{|x|\to\infty} |x|^{d+\alpha}P_\alpha(x) = c_d(\alpha). \tag{A.1}$$

We call g_α the function given by

$$g_\alpha(x) = \mathcal{F}^{-1}\left(\frac{1}{1+|\cdot|^\alpha}\right)(x). \tag{A.2}$$

We recall that for any given $\psi\in C_b^\infty(\mathbb{R}^d)$, the unique $C_b^\infty(\mathbb{R}^d)$ solution of

$$\phi - \Delta^{\alpha/2}\phi = \psi,$$

is given by $\phi(x) = g_\alpha(\cdot) * \psi(x)$.

Lemma A.2. *The following statements hold*

(1) The function g_α defined in (A.2) *can be expressed in the form*

$$g_\alpha(x) = \int_0^\infty e^{-t}G_\alpha(t,x)\,dt = \int_0^\infty e^{-t}t^{-\frac{d}{\alpha}}P_\alpha\left(\frac{x}{t^{1/\alpha}}\right)\,dt. \tag{A.3}$$

(2) g_α is radially symmetric, strictly positive and $\|g_\alpha\|_{L^1} = 1$.

(3) We have

$$\lim_{|x|\to\infty}|x|^{d+\alpha}g_\alpha(x) = c_d(\alpha), \qquad M_\alpha := \sup_{x\in\mathbb{R}^d}|x|^{d+\alpha}g_\alpha(x) \le \sup_{x\in\mathbb{R}^d}|x|^{d+\alpha}P_\alpha(x). \tag{A.4}$$

Proof. (1) The equalities in (A.3) follow from the definition and properties of G_α and the Fubini-Tonelli Theorem, as

$$\begin{aligned}\int_0^\infty e^{-t}G_\alpha(t,x)\,dt &= \int_0^\infty e^{-t}t^{-\frac{d}{\alpha}}G_\alpha\left(1,\frac{x}{t^{1/\alpha}}\right)\,dt\\ &= \int_0^\infty \frac{1}{(2\pi)^d}\int_{\mathbb{R}} e^{-t(1+|\xi|^\alpha)}e^{ix\cdot\xi}\,d\xi\,dt\end{aligned}$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}} \left(\int_0^\infty e^{-t(1+|\xi|^\alpha)}\, dt \right) e^{ix\cdot\xi}\, d\xi$$
$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}} \frac{e^{ix\cdot\xi}}{1+|\xi|^\alpha}\, d\xi = \mathcal{F}^{-1}\left(\frac{1}{1+|\cdot|^\alpha} \right)(x).$$

(2) The function g_α is radially symmetric and takes positive values since so does P_α. Moreover,

$$\|g_\alpha\|_{L^1} = \int_0^\infty e^{-t} \|G_\alpha(t,\cdot)\|_{L^1}\, dt = \int_0^\infty e^{-t}\, dt = 1.$$

(3) To prove (A.4) we remark first that

$$|x|^{d+\alpha} g_\alpha(x) = \int_0^\infty e^{-t} t \left| \frac{x}{t^{1/\alpha}} \right|^{d+\alpha} P_\alpha\left(\frac{x}{t^{1/\alpha}} \right) dt, \tag{A.5}$$

then the equality in (A.1) and the dominated convergence theorem allow us to conclude.

□

Lemma A.3. *For any given $\phi \in C_c^\infty(\mathbb{R}^d)$ the fractional Rosenau operator appearing in the regularized problem* (1.6)

$$R_\varepsilon^\alpha(\phi)(x) = -\mathcal{F}^{-1}\left(\frac{|\cdot|^\alpha}{1+\varepsilon^\alpha|\cdot|^\alpha} \mathcal{F}(\phi) \right)(x),$$

admits the following integral representation

$$R_\varepsilon^\alpha(\phi)(x) = \int_{\mathbb{R}^d} (\phi(x+z) - \phi(x)) \frac{g_\alpha^\varepsilon(z)}{\varepsilon^\alpha}\, dz.$$

Moreover, R_ε^α is a Lévy diffusion operator associated to the measure $\nu(z) = \varepsilon^{-\alpha} g_\alpha^\varepsilon(z)\, dz$, where

$$g_\alpha^\varepsilon(x) = \frac{g_\alpha(x/\varepsilon)}{\varepsilon^d} = \mathcal{F}^{-1}\left(\frac{1}{1+\varepsilon^\alpha|\cdot|^\alpha} \right)(x).$$

Finally, we have that

$$\lim_{\varepsilon\to 0} R_\varepsilon^\alpha(\phi) = \Delta^{\alpha/2}\phi, \qquad \text{strongly in } L^1(\mathbb{R}^d) \text{ and a.e. in } \mathbb{R}^d. \tag{A.6}$$

Proof.

$$\begin{aligned} R_\varepsilon^\alpha(\phi)(x) &= -\mathcal{F}^{-1}\left(\frac{|\cdot|^\alpha}{1+\varepsilon^\alpha|\cdot|^\alpha} \mathcal{F}(\phi) \right)(x) \\ &= -\frac{\phi(x)}{\varepsilon^\alpha} + \frac{1}{\varepsilon^\alpha}\left(\mathcal{F}^{-1}\left(\frac{1}{1+\varepsilon^\alpha|\cdot|^\alpha} \right) * \phi \right)(x) \\ &= -\frac{\phi(x)}{\varepsilon^\alpha} + \frac{1}{\varepsilon^{\alpha+d}} \left(g_\alpha^\varepsilon * \phi \right)(x) \\ &= -\frac{\phi(x)}{\varepsilon^\alpha} + \frac{1}{\varepsilon^{\alpha+d}} \int_{\mathbb{R}}^d \phi(x+z) g_\alpha^\varepsilon(z)\, dz \\ &= \int_{\mathbb{R}^d} (\phi(x+z) - \phi(x)) \frac{g_\alpha^\varepsilon(z)}{\varepsilon^\alpha}\, dz, \end{aligned}$$

since $\|g_\alpha^\varepsilon\|_{L^1} = 1$, see Lemma A.2. Due to the radial symmetry of g_α^ε we have that for any given $\phi \in C_c^\infty(\mathbb{R}^d)$ and $r > 0$

$$\begin{aligned} R_\varepsilon^\alpha(\phi)(x) &= \int_{\mathbb{R}^d} (\phi(x+z) - \phi(x)) \frac{g_\alpha^\varepsilon(z)}{\varepsilon^\alpha}\, dz \\ &= \int_{|z|>r} (\phi(x+z) - \phi(x)) \left(\frac{|z|}{\varepsilon}\right)^{d+\alpha} g_\alpha\left(\frac{z}{\varepsilon}\right) \frac{dz}{|z|^{d+\alpha}} \\ &\quad + \int_{|z|<r} (\phi(x+z) - \phi(x) - \nabla\phi(x)\cdot z) \left(\frac{|z|}{\varepsilon}\right)^{d+\alpha} g_\alpha\left(\frac{z}{\varepsilon}\right) \frac{dz}{|z|^{d+\alpha}}. \end{aligned} \tag{A.7}$$

Using the estimates in (A.4) we observe that the integrands in (A.7) are dominated by integrable functions, so that

$$\begin{aligned} \lim_{\varepsilon\to 0} R_\varepsilon^\alpha(\phi)(x) =& c_d(\alpha) \int_{|z|>r} (\phi(x+z) - \phi(x)) \frac{dz}{|z|^{d+\alpha}} \\ &+ c_d(\alpha) \int_{|z|<r} (\phi(x+z) - \phi(x) - \nabla\phi(x)\cdot z) \frac{dz}{|z|^{d+\alpha}}, \end{aligned}$$

which gives us the a.e. convergence in (A.6). The same argument shows that the convergence holds with respect to the strong topology of $L^1(\mathbb{R}^d)$. □

A.2. **Bessel functions.** We recall that the Bessel function $\mathcal{J}_\gamma$, is the function of $C^\infty(\mathbb{C},\mathbb{C})$ defined as

$$\mathcal{J}_\gamma(z) = \sum_{n=0}^\infty \frac{(-1)^n}{n!\,\Gamma(n+\gamma+1)} \left(\frac{z}{2}\right)^{2n+\gamma}.$$

We are only interested in the case in which $\gamma = d/2$ for $d \in \mathbb{N}$, and in this[2] case $z \in \mathbb{R}$ implies $\mathcal{J}_{d/2}(z) \in \mathbb{R}$. In the following we will make use of some classical properties of $\mathcal{J}_\gamma$, in particular

$$(z^\gamma \mathcal{J}_\gamma)' = z^\gamma \mathcal{J}_{\gamma-1}, \tag{A.8}$$

and the following Theorem on the Fourier transform of radially symmetric functions

Theorem A.1. *Let $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ be a radially symmetric function then*

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{\frac{d}{2}} |x|^{\frac{d}{2}-1}} \int_{\mathbb{R}^+} f(r) r^{\frac{d}{2}} \mathcal{J}_{\frac{d}{2}-1}(r|x|)\, dr. \tag{A.9}$$

Proof of Lemma A.1. Since $P_\alpha(x) = \mathcal{F}^{-1}(e^{-|\cdot|^\alpha})$, from Theorem A.1, we have

$$\begin{aligned} |x|^{d+\alpha} P_\alpha(x) &= \frac{|x|^{d+\alpha}}{(2\pi)^{\frac{d}{2}} |x|^{\frac{d}{2}-1}} \int_{\mathbb{R}^+} e^{-r^\alpha} r^{\frac{d}{2}} \mathcal{J}_{\frac{d}{2}-1}(r|x|)\, dr \\ &= \frac{|x|^\alpha + 1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-r^\alpha} \underbrace{(r|x|)^{\frac{d}{2}} \mathcal{J}_{\frac{d}{2}-1}(r|x|)}_{= \frac{1}{|x|}\left((r|x|)^{\frac{d}{2}} \mathcal{J}_{\frac{d}{2}}(r|x|)\right)'} dr \\ &\overset{\text{integrating by parts}}{=} \frac{\alpha |x|^\alpha}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} r^{\alpha-1} e^{-r^\alpha} (r|x|)^{\frac{d}{2}} \mathcal{J}_{\frac{d}{2}}(r|x|)\, dr \\ &\overset{\text{change variable } s = r|x|}{=} \frac{\alpha}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-\frac{s^\alpha}{|x|^\alpha}} s^{\frac{d}{2}+\alpha-1} \mathcal{J}_{\frac{d}{2}}(s)\, ds \end{aligned} \tag{A.10}$$

[2]But we also have that $\mathcal{J}_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos(z) \in \mathbb{R}$ is $z \in \mathbb{R}$.

From a result on Bessel functions of the third kind in [4, Sect. 7.7.3, (27)] we have

$$\begin{aligned}\lim_{|x|\to\infty} |x|^{d+\alpha} P_\alpha(x) &= \lim_{p\to 0^+} \frac{\alpha}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^+} e^{-ps^\alpha} s^{\frac{d}{2}+\alpha-1} \mathcal{J}_{\frac{d}{2}}(s)\, ds \\ &= \frac{\alpha 2^{\alpha-1}}{\pi^{\frac{d}{2}+1}} \sin\left(\frac{\alpha\pi}{2}\right) \Gamma\left(\frac{d+\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2}\right).\end{aligned} \tag{A.11}$$

We use Euler Reflection Formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \qquad \text{for } z \notin \mathbb{Z},$$

and we get

$$\lim_{|x|\to\infty} |x|^{d+\alpha} P_\alpha(x) = \frac{\alpha 2^{\alpha-1}}{\pi^{\frac{d}{2}}} \frac{\Gamma\left(\frac{d+\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)} = c_d(\alpha). \tag{A.12}$$

□

A.3. **Fractional gradient.** In this last part of the Appendix we collect some results on the fractional gradient.

Lemma A.4. *For $\alpha \in (0,1)$, we define the function $\xi \mapsto h(\xi) = \xi|\xi|^{\alpha-1}$, for $\xi \in \mathbb{R}^d$, and the fractional gradient of $\phi \in L^\infty(\mathbb{R}^d)$ by*

$$\nabla^\alpha \phi = \mathcal{F}^{-1}\left(ih(\cdot)\mathcal{F}(\phi)\right). \tag{A.13}$$

This is a Lévy operator and, for any x in $\mathbb{R}^d$, the following integral representation holds

$$\nabla^\alpha \phi(x) = c_d(\alpha) PV \int_{\mathbb{R}^d} \left(\phi(x+z) - \phi(x)\right) \frac{z}{|z|^{d+\alpha+1}}\, dz. \tag{A.14}$$

Lemma A.5. *Fix $\alpha \in (0,1]$. Then for any $0 < \beta < \alpha < 1$, $\nabla^\beta g_\alpha \in L^1(\mathbb{R}^d)$.*

Proof. From the explicit formula for ∇^β in Lemma A.4, equation (A.14), is not difficult to check that $\nabla^\beta P_\alpha \in L^1(\mathbb{R}^d)$. Then we have

$$\nabla^\beta g_\alpha(x) = \int_0^\infty e^{-t} t^{-\frac{1+\beta}{\alpha}} \nabla^\beta P_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dt,$$

so that

$$\|\nabla^\beta g_\alpha\|_{L^1} = \int_0^\infty e^{-t} t^{-\frac{\beta}{\alpha}}\, dt \|\nabla^\beta P_\alpha\|_{L^1} = \Gamma\left(1-\frac{\beta}{\alpha}\right) \|\nabla^\beta P_\alpha\|_{L^1} < +\infty.$$

□

Appendix B. Alternative proofs of Lemmas 3.1 and 3.2 in 1-D

This appendix contains alternative proofs of Lemmas 3.1 and 3.2 which rely on a purely PDE approach which does not involve Bessel functions but only works in one space dimension.

Proof of Lemma 3.1. For any $x \in \mathbb{R}^+$ we have, by definition of P_α and basic symmetry properties

$$x^{\alpha+1} P_\alpha(x) = x^{\alpha+1} \mathcal{F}^{-1}\left(e^{-|\cdot|^\alpha}\right)(x) = x^{\alpha+1} \frac{1}{\pi} \int_0^{+\infty} e^{-\xi^\alpha} \cos(x\xi)\, d\xi.$$

Integrating by parts we get

$$\begin{aligned} x^{\alpha+1} P_\alpha(x) &= \frac{1}{\pi} x^\alpha \int_0^{+\infty} \alpha \xi^{\alpha-1} e^{-\xi^\alpha} \sin(x\xi)\, d\xi \\ &= (x\xi)^\alpha) = \frac{1}{\pi} \int_0^{+\infty} e^{-\tau/x^\alpha} \sin(\tau^{1/\alpha})\, d\tau = \mathfrak{Im}\left(\frac{1}{\pi} \int_0^{+\infty} e^{i\tau^{1/\alpha} - \tau/x^\alpha}\, d\tau\right), \end{aligned} \tag{B.1}$$

where we use the change of variable $\tau = (x\xi)^\alpha$. We fix a representation for the logarithm, so to precise the meaning of the function $\tau \mapsto \tau^{1/\alpha} = \exp\left(\dfrac{\ln(\tau)}{\alpha}\right)$, then we observe that the function $\tau \mapsto e^{i\tau^{1/\alpha} - \tau/x^\alpha}$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}^-$. Therefore, we can compute the value of the integral on the right hand side of (B.1) by a change of path. Consider $\theta > 0$ small enough, $R > 0$ fixed and the closed path $\mathbb{R} \ni s \mapsto \gamma(s) \in \mathbb{C}$ given by

$$\gamma(s) = \begin{cases} s, & s \in [0, R], \\ Re^{i(s-R)}, & s \in (R, R+\theta], \\ (2R + \theta - s)e^{i\theta}, & s \in (R+\theta, 2R+\theta]. \end{cases}$$

We clearly have that

$$\begin{aligned} 0 &= \int_\gamma e^{i\tau^{1/\alpha} - \tau/x^\alpha}\, d\tau \\ &= \int_0^R e^{is^{1/\alpha} - s/x^\alpha}\, ds + \int_R^{R+\theta} \exp\left(i(Re^{i(s-R)})^{1/\alpha} - \frac{Re^{i(s-R)}}{x^\alpha}\right) iRe^{i(s-R)}\, ds \\ &\quad - \int_{R+\theta}^{2R+\theta} \exp\left(i((2R+\theta-s)e^{i\theta})^{1/\alpha} - \frac{(2R+\theta-s)e^{i\theta}}{x^\alpha}\right) e^{i\theta}\, ds \\ &= \int_0^R e^{is^{1/\alpha} - s/x^\alpha}\, ds + \int_0^\theta \exp\left(i(Re^{is})^{1/\alpha} - \frac{Re^{is}}{x^\alpha}\right) iRe^{is}\, ds \\ &\quad + \int_R^0 \exp\left(i(se^{i\theta})^{1/\alpha} - \frac{se^{i\theta}}{x^\alpha}\right) e^{i\theta}\, ds. \end{aligned} \tag{B.2}$$

If θ is sufficiently small we have that

$$\begin{aligned} &\int_0^\theta \left| \exp\left(i(Re^{is})^{1/\alpha} - \frac{Re^{is}}{x^\alpha}\right) iRe^{is} \right| ds \\ &= \int_0^\theta R \exp\left(-R^{1/\alpha}\sin(s/\alpha) - \frac{R}{x^\alpha}\cos(s)\right) ds \le \theta R \exp\left(-\frac{R}{x^\alpha}\cos(\theta)\right), \end{aligned}$$

which clearly converges to zero as $R \to \infty$. A similar argument shows that the other two integrals in (B.2) are convergent, so that

$$\int_0^{+\infty} e^{i\tau^{1/\alpha} - \tau/x^\alpha}\, d\tau = \int_0^{+\infty} \underbrace{\exp\left(i(\tau e^{i\theta})^{1/\alpha} - \frac{\tau e^{i\theta}}{x^\alpha}\right) e^{i\theta}}_{=f(x,\tau)}\, d\tau = F(x), \tag{B.3}$$

and we can rewrite (B.1) as

$$x^{\alpha+1} P_\alpha(x) = \mathfrak{Im}\left(\frac{F(x)}{\pi}\right).$$

From now on we fix θ so small that $\theta \in (0, \pi/2)$ and $\theta/\alpha \in (0, \pi)$.

For any fixed $x \in \mathbb{R}^+$ the function $\tau \mapsto f(x,\tau)$ belongs to $L^1(\mathbb{R})$, because

$$|f(x,\tau)| = \exp\left(-\tau^{1/\alpha}\sin(\theta/\alpha) - \frac{\tau}{x^\alpha}\cos(\theta)\right) \le \exp\left(-\tau^{1/\alpha}\sin(\theta/\alpha)\right) \in L^1(\mathbb{R}),$$

therefore we can pass to the limit as $x \to +\infty$ in the right hand side of (B.3) and call F_∞ the value we obtain.

We have that

$$\left|x^{\alpha+1}P_\alpha(x) - c_1(\alpha)\right| = \left|\mathfrak{Im}\left(\frac{F(x)}{\pi}\right) - \mathfrak{Im}\left(\frac{F_\infty}{\pi}\right)\right| \le \frac{1}{\pi}\int_0^{+\infty} \left|\mathfrak{Im}\left(f(x,\tau) - \lim_{x\to+\infty} f(x,\tau)\right)\right| d\tau,$$

and

$$\mathfrak{Im}\left(f(x,\tau)\right) = \underbrace{\exp\left(-\tau^{1/\alpha}\sin\left(\frac{\theta}{\alpha}\right) - \frac{\tau}{x^\alpha}\cos(\theta)\right)}_{=A(x,\tau)} \sin\underbrace{\left(-\tau^{1/\alpha}\cos\left(\frac{\theta}{\alpha}\right) - \frac{\tau}{x^\alpha}\sin(\theta)+\theta\right)}_{=B(x,\tau)}.$$

We denote $A_\infty(\tau) = \lim_{x\to+\infty} A(x,\tau)$ and $B_\infty(\tau) = \lim_{x\to+\infty} B(x,\tau)$, and we compute

$$\begin{aligned}
\left|\mathfrak{Im}\left(f(x,\tau) - \lim_{x\to+\infty} f(x,\tau)\right)\right| &= \left|A(x,\tau)\sin(B(x,\tau)) - A_\infty(\tau)\sin(B_\infty(\tau))\right| \\
&\le |A(x,\tau) - A_\infty(\tau)|\underbrace{|\sin(B(x,\tau))|}_{\le 1} + |A_\infty(\tau)|\,|B(x,\tau) - B_\infty(\tau)| \\
&\le \exp\left(-\tau^{1/\alpha}\sin\left(\frac{\theta}{\alpha}\right)\right)\left|1-\exp\left(-\frac{\tau}{x^\alpha}\cos(\theta)\right)\right| + \exp\left(-\tau^{1/\alpha}\sin\left(\frac{\theta}{\alpha}\right)\right)\frac{\tau}{x^\alpha} \\
&\le 2\exp\left(-\tau^{1/\alpha}\sin\left(\frac{\theta}{\alpha}\right)\right)\frac{\tau}{x^\alpha}.
\end{aligned}$$

Finally

$$\left|x^{\alpha+1}P_\alpha(x) - c_1(\alpha)\right| = \frac{2}{\pi x^\alpha}\int_0^\infty \tau e^{-\tau^{1/\alpha}\sin(\theta/\alpha)}\,d\tau \le \frac{C}{x^\alpha}.$$

□

Proof of Lemma 3.2. Let H_α be the unique solution of the Cauchy problem

$$\begin{cases} \partial_t H_\alpha = \partial_{xx}^{\alpha/2} H_\alpha, & (t,x)\in\mathbb{R}^+\times\mathbb{R}, \\ H_\alpha(0,x) = \mathrm{Heav}(x) = \begin{cases} 1, & \text{for } x>0, \\ 0, & \text{otherwise,} \end{cases} & x\in\mathbb{R}. \end{cases}$$

We know that

$$H_\alpha(t,x) = \int_{-\infty}^x G_\alpha(t,y)\,dy,$$

so that straightforward computations give us

$$\partial_t H_\alpha(t,x) = -\frac{1}{\alpha}\frac{x}{t^{(1+\alpha)/\alpha}}P_\alpha\left(\frac{x}{t^{1/\alpha}}\right).$$

Then from the fractional heat equation we have that

$$\left(\frac{x}{t^{1/\alpha}}\right)^{1+\alpha} P_\alpha\left(\frac{x}{t^{1/\alpha}}\right) = -\alpha x^\alpha \partial_{xx}^{\alpha/2} H_\alpha(t,x).$$

This equality allows us to show that for any fixed $x\in\mathbb{R}$, the function

$$\mathbb{R}^+ \ni t \mapsto f(t) = \left(\frac{x}{t^{1/\alpha}}\right)^{1+\alpha} P_\alpha\left(\frac{x}{t^{1/\alpha}}\right),$$

is monotone decreasing and satisfies $f(0) = c_1(\alpha)$. We have that

$$\begin{aligned}
f'_\alpha(t) &= -\alpha x^\alpha \partial_{xx}^{\alpha/2}\partial_t H_\alpha(t,x) = -\alpha x^\alpha \partial_{xx}^{\alpha/2}\left(\partial_{xx}^{\alpha/2} H_\alpha\right)(t,x) \\
&= -\alpha x^\alpha \mathcal{F}^{-1}\left(-|\cdot|^\alpha \mathcal{F}\left(\mathcal{F}^{-1}\left(-|\cdot|^\alpha \mathcal{F}(H_\alpha)\right)\right)\right)(t,x) \\
&= -\alpha x^\alpha \mathcal{F}^{-1}\left(|\cdot|^{2\alpha}\mathcal{F}(H_\alpha)\right)(t,x) = \alpha x^\alpha \partial_{xx}^\alpha H_\alpha(t,x).
\end{aligned}$$

For every $t>0$, the function $x\mapsto H_\alpha(t,x) - 1/2$ is odd, non decreasing and concave on $\mathbb{R}^+$, see [3]. Then for any $\alpha\in(0,1]$

$$\partial_{xx}^\alpha H_\alpha(t,x) = c_1(\alpha)\underbrace{\lim_{r\to+\infty}\int_{|z|<r}\left(H_\alpha(t,x+z) - H_\alpha(t,x) - \partial_x H_\alpha(t,x)z\right)\frac{dz}{|z|^{1+\alpha}}}_{\le 0} \le 0.$$

On the other hand we can compute explicitely

$$f(0) = -\alpha x^{\alpha}\partial_{xx}^{\alpha/2}\mathrm{Heav}(x) = -\alpha x^{\alpha}c_1(\alpha)\underbrace{\mathrm{PV}\int_{\mathbb{R}}(\mathrm{Heav}(x+z)-\mathrm{Heav}(x))\,\frac{dz}{|z|^{1+\alpha}}}_{=-\frac{1}{\alpha x^{\alpha}}} = c_1(\alpha).$$

The properties of f directly lead to the estimate we are looking for, by a simple application of the equality in (A.5). □

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