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# Hyperbolicity of smooth logarithmic and orbifold pairs in $\mathbb{P}^n$

Clara Dérand \*

## Abstract

We derive a necessary and sufficient condition on a hyperplane arrangement in  $\mathbb{P}^n$  for the associated logarithmic cotangent bundle to be ample modulo boundary. We extend this result to the orbifold setting and give some applications concerning hyperbolicity of pairs. We improve significantly the results of [DR24].

## 1 Introduction

### 1.1 Hyperbolicity of quasi-projective varieties

Let  $(X, D)$  be a smooth log pair. It has been shown by several authors that hyperbolicity properties of  $X \setminus D$  could be investigated through the positivity of the associated logarithmic cotangent bundle  $\Omega_X(\log D)$ , see e.g. [Nog86], [BD18], [BD19], [DR24], [CDR20], [CDDR21].

When studying the properties of the logarithmic cotangent bundle, one finds trivial quotients supported on the different components of  $D$ . Each such quotient gives a positive dimensional subvariety of the augmented base locus  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1))$ . In particular,  $\Omega_X(\log D)$  cannot be ample in any way. A natural question to ask then is whether these quotients are the only obstructions to ampleness. One can define two conditions on the logarithmic cotangent bundle to express this minimality property. The first notion has been introduced by the authors of [BD18].

#### Definition 1.1

Let  $(X, D)$  be a smooth log pair, with  $D = \sum_{i=1}^c D_i$ . The logarithmic cotangent bundle  $\Omega_X(\log D)$  is said almost ample if it satisfies

$$\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1)) = \bigcup_{\substack{I \subset \{1, \dots, c\} \\ |I| < n}} \mathbb{P}(\mathcal{O}_{D_I}^{\oplus |I|}).$$

The following weaker property is used instead in the paper [DR24].

#### Definition 1.2

The logarithmic cotangent bundle  $\Omega_X(\log D)$  is ample modulo  $D$  if it satisfies

$$p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1))) = D,$$

where  $p : \mathbb{P}(\Omega_X(\log D)) \rightarrow X$  is the canonical projection.

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There are now many examples for which the logarithmic cotangent bundle is big.

- $c \geq n$  and  $D_i \in |L^m|$ , for a very ample line bundle  $L$  and  $m \geq (4n)^{n+2}$  ([BD18]);
- $X = \mathbb{P}^2$ ,  $c = 2$  with small lower bounds on the degrees of the curve ([Rou09]);
- $X = \mathbb{P}^2$ ,  $c = 1$ ,  $\deg D \geq 14$  ([Rou09]);
- $X = \mathbb{P}^3$ ,  $c = 1$ ,  $\deg D \geq 593$  ([Rou07])
- $X = \mathbb{P}^n$  with  $c \geq$  ([DR24])

Nevertheless, if one asks for a stronger positivity property, there are up to now very few results. The first one in this direction is due to Noguchi. It concerns general arrangements of lines in  $\mathbb{P}^2$ . It can be formulated with our vocabulary as follows.

**Proposition 1.3** ([Nog86])

*The logarithmic tangent bundle along an arrangement  $D$  of  $c \geq 6$  lines in general position with respect to hyperplanes and quadrics is ample modulo  $D$ .*

This has been generalized to higher dimension in [DR24].

**Theorem 1.4 (Theorem A, [DR24])**

*The logarithmic cotangent bundle along an arrangement  $\mathcal{A}$  of  $c \geq \binom{n+2}{2}$  hyperplanes in  $\mathbb{P}^n$  in general position with respect to hyperplanes and to quadrics is ample modulo  $\mathcal{A}$ .*

On the other side, the logarithmic cotangent bundle associated to a smooth pair  $(X, D)$  is almost ample when the degree of the irreducible components of  $D$  is very large.

**Theorem 1.5** ([BD18])

*Let  $X$  be a smooth projective variety of dimension  $n$  and let  $c \geq n$ . Let  $L$  be a very ample line bundle on  $X$ . For any  $m \geq (4n)^{n+2}$  and for general hypersurfaces  $H_1, \dots, H_c \in |L^m|$ , writing  $D = \sum_i H_i$ , the logarithmic cotangent bundle  $\Omega_X(\log D)$  is almost ample.*

More generally, we can ask similar questions in the orbifold case, which interpolates between the compact and the logarithmic cases. Following Campana (see [Cam04]), a smooth orbifold is a pair  $(X, \Delta)$  where  $X$  is a smooth complex projective variety and  $\Delta$  is a  $\mathbb{Q}$ -divisor with normal crossings support and coefficients in  $\mathbb{Q} \cap [0, 1]$ . One can define a sheaf  $\Omega^1(X, \Delta)$  of differential forms with ‘fractional’ poles: these are actually holomorphic forms living on a suitable ramified covering of  $X$ .

## 1.2 Results

In this paper, we focus on complements of hyperplane arrangements in  $\mathbb{P}^n$ ,  $n \geq 2$ . In this very specific case, we first show that the two notions of positivity for the logarithmic cotangent bundle are in fact equivalent.

**Theorem A (Theorem 3.10)**

*Let  $D = \sum_{i=0}^{c-1} H_i$  be a hyperplane arrangement in general position in  $\mathbb{P}^n$ . Then  $\Omega_{\mathbb{P}^n}(\log D)$  is ample modulo  $D$  if, and only if, it is almost ample.*

Our motivation for this paper is the recent paper [DR24]. The authors obtained the following result.

**Theorem 1.6 (Theorem A in [DR24])**

Let  $D$  be an arrangement of  $c \geq \binom{2n+2}{2}$  hyperplanes in general position with respect to hyperplanes and quadrics. Then the associated logarithmic cotangent bundle is ample modulo  $D$ .

However, they had no evidence in favour of the optimality of the required condition.

They derived a similar result, in the orbifold setting.

**Theorem 1.7 (Theorem B in [DR24])**

The orbifold cotangent bundle along an arrangement  $D$  of  $c \geq \binom{n+2}{2}$  hyperplanes in  $\mathbb{P}^n$  in general position with respect to hyperplanes and quadrics, with multiplicities  $\geq 2n + 2$ , is ample modulo  $D$ .

In this paper, we improve the lower bound on the number of components of a general arrangement.

**Theorem B (Theorem 4.1)**

Let  $D = \sum_{i=0}^{c-1} H_i$  be a hyperplane arrangement in general position in  $\mathbb{P}^n$ . Then the corresponding logarithmic cotangent bundle is **ample modulo boundary** if, and only if the  $c$  hyperplanes impose at least  $4n - 2$  independent conditions on quadrics.

In particular, one needs at least  $4n - 2$  components. Moreover, this theorem includes the fact that the genericity condition is optimal.

We also obtain an orbifold version of Theorem B, which also improves [DR24].

**Theorem C (Theorem 5.5)**

Let  $\Delta = \sum_{i=1}^c (1 - 1/m_i) H_i$  be an orbifold divisor in  $\mathbb{P}^n$ , where the components  $H_i$  are hyperplanes in general position. Assume that  $m_i \geq 2n$  for all  $i$ . Then the orbifold cotangent bundle  $\Omega(\mathbb{P}^n, \Delta)$  is ample modulo  $D$  if, and only if, the  $c$  hyperplanes impose at least  $4n - 2$  independent conditions on quadrics.

When the logarithmic cotangent bundle is ample modulo  $D$ , it is not difficult to see that the complement  $X \setminus D$  is hyperbolic in the sens of Brody. This comes from a classical fact concerning entire curves. In our last part, we give two applications in hyperbolicity, which we are new to the best of our knowledge.

**Theorem D**

Consider an arrangement of  $c$  hyperplanes  $H_0, \dots, H_{c-1}$  in  $\mathbb{P}^n$  in general position in  $\mathbb{P}^n$ , with respective orbifold multiplicities  $m_i$ , and the associated orbifold divisor  $\Delta = \sum_{i=0}^{c-1} (1 - 1/m_i) \cdot H_i$ . If  $m_i \geq 2n$  and the hyperplanes  $H_i$  impose at least  $4n - 2$  independent conditions on quadrics, the orbifold pair  $(\mathbb{P}^n, \Delta)$  is Kobayashi-hyperbolic.

The second concerns hyperbolicity of a very specific type of Fermat complete intersections.

**Theorem E**

The Fermat cover associated to an arrangement of  $d \geq 4n - 2$  hyperplanes in  $\mathbb{P}^n$  in general position, which impose at least  $4n - 2$  independent conditions on quadrics, with ramification

$m \geq 2n + 1$ , is Kobayashi-hyperbolic.

In the last part of this paper, concerning applications of our results in hyperbolicity, the statements are exactly the same with the bounds replaced. For this reason, we refer to [DR24] for the detailed proofs.

### 1.3 Organisation of the paper

In a first section, we give some definition and basic results concerning log pairs. We prove a characterization of the augmented base locus for the augmented base locus of a globally generated line bundle. We study more carefully the structure of the logarithmic cotangent bundle associated to a hyperplane arrangement. In particular, we prove theorem A.

The second section is mainly devoted to the proof of theorem B.

In the next section, we give a brief overview of Campana's theory of orbifolds, and we prove theorem C, which is more or less an extension of theorem B in this context.

Finally, we give some applications of our results concerning hyperbolicity of orbifold pairs and Fermat-type complete intersections.

## 2 Some projective geometry

In this section, we discuss topics of classical algebraic geometry that will be used thereafter.

### 2.1 Varieties of linear spaces

Let  $\mathbb{G}(k, n) = \text{Grass}(k + 1, n + 1)$  be the Grassmannian of  $k$ -linear subspaces in  $\mathbb{P}^n$ . This is a subvariety of  $\mathbb{P}(\wedge^{k+1} \mathbb{C}^{n+1})$  of dimension  $(k + 1)(n - k)$ . We define a subvariety  $\Sigma \subset \mathbb{G}(k, n) \times \mathbb{P}^n$  by setting

$$\Sigma = \{(Z, p); p \in Z\}.$$

Let  $\pi_1, \pi_2$  be the projection maps from  $\Sigma$  to  $\mathbb{G}(k, n)$  and  $\mathbb{P}^n$  respectively. We can make use of  $\Sigma$  to construct subvarieties of  $\mathbb{P}^n$  swept out by  $k$ -planes.

#### Proposition 2.1

Let  $\Phi \subset \mathbb{G}(k, n)$  be any subvariety. Then

$$X = \bigcup_{Z \in \Phi} Z = \pi_2(\pi_1^{-1}(\Phi))$$

is a subvariety of  $\mathbb{P}^n$ .

#### 2.1.1 Rational normal scrolls

A particular case of this construction is obtained when  $\Phi \cong \mathbb{P}^1$ . Let  $k \geq 2$ . A subvariety  $X \subset \mathbb{P}^n$  given as the union

$$X = \bigcup_{\lambda \in \mathbb{P}^1} Z_\lambda$$

of a 1-parameter family of  $(k - 1)$ -planes is named a *rational normal  $k$ -fold scroll*.

One can show ([Har92], Theorem 19.9) that  $X$  has minimal degree  $1 + n - k$ .

In particular, a hypersurface scroll  $X$  is a quadric. According to [Har92], Ex 8.36,  $X$  can also be characterised equivalently as:

- a quadric of rank 3 or 4, in the sense that the symmetric matrix representing  $X$  has rank 3 or 4;
- a cone over a plane conic or a smooth quadric surface in  $\mathbb{P}^3$ ;
- an irreducible quadric containing an  $(n - 2)$ -plane.

## 2.2 Dual varieties

Consider the  $n$ -dimensional projective space  $\mathbb{P}^n$ . The *dual projective space*  $(\mathbb{P}^n)^*$  is defined as the projective space parameterising hyperplanes in  $\mathbb{P}^n$ .

Let  $X$  be a subvariety of  $\mathbb{P}^n$  of dimension  $k$ . We define the dual variety  $X^* \subset (\mathbb{P}^n)^*$  to be the closure of the set of all hyperplanes tangent to  $X$  at a smooth point.

Restricting first to the case where  $X$  is a smooth hypersurface, we may also say that  $X^*$  is the locus of hyperplanes  $H \subset \mathbb{P}^n$  such that the intersection  $H \cap X$  is singular. It is indeed a variety since it is the image of  $X$  under the *Gauss map*  $\mathcal{G}_X$ , a morphism sending a point  $p \in X$  to its embedded tangent hyperplane  $\mathbb{T}_p X$ ; in coordinates, if  $f$  is a homogeneous polynomial defining  $X$ , the Gauss morphism is given by

$$\mathcal{G}_X : p \mapsto \left[ \frac{\partial F}{\partial Z_0}(p) : \cdots : \frac{\partial F}{\partial Z_n}(p) \right].$$

If  $X \subset \mathbb{P}^n$  is a *smooth* hypersurface of degree  $d \geq 2$ , the morphism  $\mathcal{G}_X$  is birational onto its image as well as finite. The dual  $X^*$  is then again a hypersurface, though in general highly singular (never smooth if  $d \geq 3$ ). One can compute its degree  $\deg X^* = d(d - 1)^{n-1}$ . In particular, the dual of a smooth quadric hypersurface is again a smooth quadric.

In the general situation, we define the *conormal variety* as the closure  $CX$  of the incidence correspondence

$$\{(p, H) \in \mathbb{P}^n \times (\mathbb{P}^n)^* \mid p \in X_{sm} \text{ and } \mathbb{T}_p X \subset H\}.$$

The dual variety  $X^*$  is then naturally defined as the image  $\pi_2(CX) \subset \mathbb{P}^{n*}$  of the second projection.

A fundamental result concerning dual varieties is the reflexivity theorem.

### Theorem 2.2

*Let  $X \subset \mathbb{P}^n$  be an irreducible variety and  $X^* \subset \mathbb{P}^{n*}$  its dual. Then the conormal variety  $CX \subset \mathbb{P}^n \times \mathbb{P}^{n*}$  is equal to  $CX^* \subset \mathbb{P}^{n*} \times \mathbb{P}^n \cong \mathbb{P}^n \times \mathbb{P}^{n*}$ . It follows that  $(X^*)^* = X$ : the dual of a variety is the variety itself.*

Considering a (possibly singular) subvariety  $X \subset \mathbb{P}^n$ , what can the dimension of  $X^*$  be? One computes easily the dimension of  $CX$ : the fibre of the first projection map  $CX \rightarrow X$  over a smooth point  $p \in X_{sm}$  is simply the subspace  $\mathbb{P}^{n-k-1} \subset \mathbb{P}^{n*}$  of hyperplanes containing the  $k$ -plane  $\mathbb{T}_p X$ . Hence the inverse image in  $CX$  of the smooth locus of  $X$  is irreducible of dimension  $n - 1$ . We conclude that  $X^*$  is irreducible of dimension less than  $n - 1$ , and that it should be  $(n - 1)$ -dimensional in most cases. More precisely,  $X^*$  has dimension  $n - 1$  exactly when the general tangent hyperplane to  $X$  is tangent at only finitely many points.

### Example 2.3

A situation where  $X^*$  fails to be a hypersurface is when the tangent hyperplanes are constant along subvarieties of  $X$ : for example, when we have a variety ruled by linear subspaces, as cones over subvarieties of  $\mathbb{P}^n$ . Typical examples are given by rational normal scrolls of dimension  $k \geq 3$ . Recall that a  $k$ -dimensional scroll  $X = \bigcup_{\lambda \in \mathbb{P}^1} Z_\lambda$  is a variety ruled in  $\mathbb{P}^{k-1}$ 's. The tangent plane  $\mathbb{T}_p X$  at any point  $p \in X$  will naturally contain the  $(k-1)$ -plane  $Z_\lambda$  of the ruling through  $p$ . We conclude that the general fibre of the second projection  $CX \rightarrow \mathbb{P}^{n*}$  is  $(k-2)$ -dimensional, so that  $X^*$  has dimension  $n - k + 1$ .

For instance, if  $X$  is a hypersurface scroll in  $\mathbb{P}^n$ , or in other words a quadric hypersurface of rank  $r \geq 4$ , then its dual  $X^*$  is a surface inside a  $\mathbb{P}^3$  in the non-degenerate case ( $r = 4$ ) and a plane conic curve otherwise.

## 3 Positivity of the logarithmic cotangent bundle

### 3.1 Conventions

Let  $X$  be a smooth complex projective variety.

For a vector bundle  $E$  on  $X$ , we denote by  $\mathbb{P}(E)$  the bundle of rank one quotients of  $E$  and by  $\mathcal{O}_{\mathbb{P}(E)}(1)$  the tautological line bundle. We often identify  $\mathbb{P}(E)$  with the projective space of lines  $P(E^\vee)$  of the dual of  $E$ .

If  $L$  is a line bundle over  $X$ , its *stable base locus* is  $\mathbb{B}(L) = \bigcap_{m \in \mathbb{N}} \text{Bs}(L^{\otimes m})$ . For  $m$  large and divisible enough, we have  $\mathbb{B}(L) = \text{Bs}(L^{\otimes m})$ . We define its *augmented base locus* as

$$\mathbb{B}_+(L) := \bigcap_{p \in \mathbb{N}} \mathbb{B}(L^{\otimes p} \otimes A^{-1})$$

where  $A$  is any ample divisor on  $X$ . Clearly  $L$  is ample if and only if  $\mathbb{B}_+(L) = \emptyset$ , and  $L$  is big if and only if  $\mathbb{B}_+(L) \neq X$ . In this last case, its complement is the largest Zariski-open subset  $U \subset X \setminus \mathbb{B}(L)$  such that the restriction of the map

$$\Phi_{L^m}|_U : U \rightarrow \mathbb{P}(H^0(X, L^{\otimes m}))$$

is an isomorphism onto its image, for all  $m \gg 1$  divisible enough.

If  $D = \sum_i \alpha_i D_i$  is an  $\mathbb{R}$ -divisor on  $X$ , we write as  $[D] = \sum_i D_i$  the associated reduced divisor and  $|D|$  its support.

### 3.2 Positivity of logarithmic pairs

Let  $(X, D)$  be a smooth log pair, that is,  $X$  is a smooth projective variety and  $D$  is a simple normal crossings divisor. The logarithmic tangent bundle  $T_X(-\log D)$  is defined as the locally free subsheaf of  $T_X$  of vector fields tangent to  $D$ . If  $(z_1, \dots, z_n)$  are local coordinates on an open set  $U \subset X$  such that  $U \cap D = (z_1 \dots z_r = 0)$ , then  $T_X(-\log D)|_U$  is generated by

$$z_1 \frac{\partial}{\partial z_1}, \dots, z_r \frac{\partial}{\partial z_r}, \frac{\partial}{\partial z_{r+1}}, \dots, \frac{\partial}{\partial z_n}.$$

Its dual is the  $\mathcal{O}_X$ -module of meromorphic 1-forms with at most logarithmic poles, locally generated by the forms

$$\frac{dz_1}{z_1}, \dots, \frac{dz_r}{z_r}, dz_{r+1}, \dots, dz_n.$$

It is called the logarithmic cotangent bundle of the pair and denoted  $\Omega_X(\log D)$ .

Write  $D = \sum_{1 \leq i \leq c} D_i$  the decomposition of  $D$  into irreducible components, and  $D(I) = \sum_{i \in I} D_i$  for  $I \subset \{1, \dots, c\}$ . Then the logarithmic cotangent bundle fits in the residue exact sequence

$$0 \longrightarrow \Omega_X \longrightarrow \Omega_X(\log D) \xrightarrow{\text{Res}} \bigoplus_i \mathcal{O}_{D_i} \longrightarrow 0.$$

Here the residue morphism is given over any open subset  $U \subset X$  by

$$\text{Res} \left( \alpha + \sum_{i=1}^c \beta_i \frac{d\sigma_i}{\sigma_i} \right) = (\beta_i|_{D_i})_{1 \leq i \leq c},$$

where  $\sigma_i$  is a section defining  $D_i$  over  $U$ .

We have several other residue exact sequences corresponding to the various subsets of components of  $D$

$$0 \longrightarrow \Omega_X(\log D(I^c)) \longrightarrow \Omega_X(\log D) \xrightarrow{\text{Res}} \bigoplus_{i \in I} \mathcal{O}_{D_i} \longrightarrow 0.$$

Its restriction to  $D_I$  leads to a trivial quotient

$$\Omega_X(\log D)|_{D_I} \xrightarrow{\text{Res}} \mathcal{O}_{D_I}^{\oplus |I|}$$

which induces an inclusion

$$\tilde{D}_I := \mathbb{P}(\mathcal{O}_{D_I}^{\oplus r}) \cong D_I \times \mathbb{P}^{r-1} \hookrightarrow \mathbb{P}(\Omega_X(\log D)).$$

Hence we see that the logarithmic cotangent bundle is never ample if  $D$  is not empty.

The projectivisation of these quotients are all the unavoidable obstructions to the ampleness of the tautological bundle in the projectivised cotangent bundle, see [BD18]. A natural question to ask then is whether these are the only ones.

**Definition 3.1**

We say that the pair  $(X, D)$  has **almost ample logarithmic cotangent bundle** if

$$\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1)) \subset \bigcup_{|I| \leq n-1} \tilde{D}_I.$$

We will also consider the following slightly weaker notion.

**Definition 3.2**

Let  $p : \mathbb{P}(\Omega_X(\log D)) \rightarrow X$  be the natural projection. The logarithmic cotangent bundle is said **ample modulo boundary** if  $p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1))) = D$ .



### 3.3 The map $\Phi$

Let  $L \rightarrow X$  be a holomorphic line bundle on a smooth projective variety  $X$ . The evaluation map

$$H^0(X, L) \otimes \mathcal{O}_X \longrightarrow L$$

induces a rational map

$$\Phi_L : X \dashrightarrow \mathbb{P}H^0(X, L).$$

By definition,  $L$  is ample (resp. big) if and only if  $\Phi_{L^m}$  is an embedding (resp. is birational onto its image) for some  $m > 0$ . If  $L$  is globally generated,  $\Phi_L$  is a morphism, which we will denote  $\Phi$  if there is no ambiguity.

We have the following characterisation of the augmented base locus.

**Proposition 3.3**

*With the above notations, suppose that  $L$  is globally generated. Then the augmented base locus  $\mathbb{B}_+(L)$  is the union of all positive-dimensional fibres of  $\Phi_L$ , i.e.*

$$\mathbb{B}_+(L) = \{w \in X; \dim \Phi_L^{-1}(\Phi_L(w)) > 0\}.$$

**Proof:** Since  $L$  is globally generated,  $\Phi_L$  is a well-defined morphism on  $X$  and  $L = \Phi_L^* \mathcal{O}_{\mathbb{P}H^0(X, L)}(1)$ . So we can apply lemma 3.5 below with  $A = \mathcal{O}_{\mathbb{P}H^0(X, L)}(1)$ . ■

Recall a theorem of Nakamaye concerning the augmented base locus.

**Theorem 3.4 (Nakamaye)**

*Let  $L$  be a big and nef line bundle on a smooth projective variety  $X$ . Then*

$$\mathbb{B}_+(L) = \bigcup_{\substack{Z \subset X \\ Z \cdot L^{\dim Z} = 0}} Z.$$

In particular, it implies the following characterization of the augmented base locus.

**Lemma 3.5** *Let  $Y$  be a projective variety and consider a regular morphism  $f : X \rightarrow Y$ . For any ample line bundle  $A$  on  $Y$ , the augmented base locus of  $f^*A$  is the union of the positive-dimensional fibres of  $f$ , i.e.*

$$\mathbb{B}_+(f^*A) = \{x \in X; \dim_x f^{-1}(\{f(x)\}) > 0\} =: \text{NFL}(f).$$

**Proof:** By Nakamaye's theorem,

$$\mathbb{B}_+(f^*A) = \bigcup_{\substack{Z \subset X \\ Z \cdot f^*A^{\dim Z} = 0}} Z.$$

If  $Z \subset X$  is not contained in  $\text{NFL}(f)$ , then the restriction  $f|_Z$  is generically finite. But this implies that

$$Z \cdot f^*A^{\dim Z} = (\deg f|_Z) f(Z) \cdot A^{\dim Z} > 0.$$

The reverse inclusion is straightforward since by definition

$$f(\text{NFL}(f)) \cdot A^{\dim \text{NFL}(f)} = 0. \quad \blacksquare$$

### 3.4 The case of hyperplane arrangements

Let us now study the special case of hyperplane arrangements in  $\mathbb{P}^n$ . We start with two observations.

**Lemma 3.6 (Brotbek-Deng '18 [BD18], Prop. 4.1)** *Let  $D = \sum_{i=1}^c D_i$  be a simple normal crossings divisor in  $\mathbb{P}^n$  with  $c < n$ . Then for any  $m > 0$ ,*

$$H^0(\mathbb{P}^n, S^m \Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(-1)) = 0.$$

*In particular,  $\Omega_{\mathbb{P}^n}(\log D)$  is not almost ample.*

**Lemma 3.7 ([BD18], Prop. 4.2)** *If  $D = \sum_{i=1}^c D_i$  is an arrangement of  $c$  hypersurfaces in general position in  $\mathbb{P}^n$ , then the associated logarithmic cotangent bundle has irregularity  $h^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(\log D)) = c - 1$ .*

**Proof:** For  $1 \leq i \leq c$ , let  $s_i \in H^0(\mathbb{P}^n, \mathcal{O}(\lambda_i))$  be a section defining the hypersurfaces  $D_i$ . We claim that the forms  $\omega_i = \lambda_1 d \log s_i - \lambda_i d \log s_1$  form a basis of global sections of  $\Omega_{\mathbb{P}^n}(\log D)$ . First, these are indeed well-defined global sections. If we denote respectively by  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$  the standard affine coordinates over the open subsets  $U_0 = \{Z_0 \neq 0\}$  and  $U_1 = \{Z_1 \neq 0\}$ , we have over  $U_0 \cap U_1$

$$\begin{aligned} \lambda_1 d \log s_i(x) - \lambda_i d \log s_1(x) &= \lambda_1 d \log(x_1^{\lambda_1} s_i(y)) - \lambda_i d \log(x_1^{\lambda_1} s_1(y)) \\ &= \lambda_1 d \log s_i(y) - \lambda_i d \log s_1(y). \end{aligned}$$

The residue exact sequence induces a long exact sequence in cohomology

$$0 \longrightarrow H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(\log D)) \xrightarrow{\text{Res}} \bigoplus_{i=1}^c H^0(D_i, \mathcal{O}_{D_i}) \cong \mathbb{C}^c \xrightarrow{\delta} H^1(\mathbb{P}^n, \Omega_{\mathbb{P}^n}).$$

If we denote by  $e_i \in H^0(D_i, \mathcal{O}_{D_i})$  the constant section equal to 1, we compute  $\text{Res}(\omega_i) = \lambda_1 e_i - \lambda_i e_1$ . As a consequence, we get  $\lambda_1 \delta(e_i) = \lambda_i \delta(e_1)$ . In particular, the map  $\delta$  has rank 1, so that the dimension of  $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(\log D))$  is exactly  $c - 1$ . ■

**Lemma 3.8** *The logarithmic cotangent bundle associated with an arrangement of  $c \geq n + 1$  hyperplanes  $H_j$  in general position in  $\mathbb{P}^n$  is globally generated.*

**Proof:** Without loss of generality, we can assume that  $H_j = (Z_j = 0)$  for  $j = 0, \dots, n$ . The forms  $\omega_j = \frac{dZ_j}{Z_j} - \frac{dZ_0}{Z_0}$  are global sections of  $\Omega_{\mathbb{P}^n}(\log D)$ . Hence a point  $z$  in the base locus of  $\Omega_{\mathbb{P}^n}(\log D)$  would satisfy  $Z_j \xi_0 - Z_0 \xi_j = 0$  for every  $\xi \in \mathbb{P}^n$ , which is impossible. ■

From now on, we fix an arrangement  $D = \sum_{i=0}^{n+k} H_i$  of  $n + k + 1 \geq n + 2$  hyperplanes in general position in  $\mathbb{P}^n$ . Without loss of generality, we assume that the first  $n + 1$  components are the coordinate hyperplanes. For  $i = 1, \dots, k$ , let  $\ell_i = a_0^i Z_0 + \dots + a_n^i Z_n$  be a linear form defining the hyperplane  $H_{n+i}$ .

In view of proposition 3.3, we have to compute the fibres of the map  $\Phi := \Phi_{\mathcal{O}_{\mathbb{P}^n}(\log D)}(1)$  in order to study the augmented base locus of  $\Omega_{\mathbb{P}^n}(\log D)$ . To do so, we describe explicitly the morphism  $\Phi$  local in coordinates.

Let us work on the affine chart  $U_0 = \{Z_0 \neq 0\} \subset \mathbb{P}^n$  with standard coordinates  $z_1, \dots, z_n$ . By the proof of lemma 3.7, a basis of global sections of  $\Omega_{\mathbb{P}^n}(\log D)$  is given by

$$\frac{dz_1}{z_1}, \dots, \frac{dz_n}{z_n}, \frac{d\ell_1}{\ell_1}, \dots, \frac{d\ell_k}{\ell_k}.$$

On the open set  $U = U_0 \cap \bigcap_{j=1, \dots, k} (\ell_j \neq 0)$ , a frame for  $T_{\mathbb{P}^n}(-\log D)$  is given by  $z_1 \frac{\partial}{\partial z_1}, \dots, z_n \frac{\partial}{\partial z_n}$ . Therefore, we may identify  $\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D)|_U)$  with  $U \times \mathbb{P}^{n-1}$  via

$$\begin{aligned} U \times \mathbb{P}^{n-1} &\longleftrightarrow \mathbb{P}(\Omega_{\mathbb{P}^n}(\log D)|_U) \\ (z, [\xi]) &\longmapsto \left( z, \left[ \xi_1 z_1 \frac{\partial}{\partial z_1} + \dots + \xi_n z_n \frac{\partial}{\partial z_n} \right] \right) \end{aligned}$$

Since  $\Omega_{\mathbb{P}^n}(\log D)$  is globally generated with irregularity  $h^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(\log D)) = n + k$ , we have a quotient

$$H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(\log D)) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \Omega_{\mathbb{P}^n}(\log D) \rightarrow 0$$

inducing a map

$$\Phi : \mathbb{P}(\Omega_{\mathbb{P}^n}(\log D)) \rightarrow \mathbb{P}(H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(\log D)) \otimes \mathcal{O}_{\mathbb{P}^n}) \simeq \mathbb{P}^n \times \mathbb{P}^{n+k-1} \twoheadrightarrow \mathbb{P}^{n+k-1}.$$

The morphism  $\Phi$  restricted to  $U$  then has the following expression:

$$\begin{aligned} \Phi(z, [\xi_1 : \dots : \xi_n]) &= \left[ \frac{dz_1}{z_1} \cdot \xi : \dots : \frac{dz_n}{z_n} \cdot \xi : \frac{d\ell_1}{\ell_1} \cdot \xi : \dots : \frac{d\ell_k}{\ell_k} \cdot \xi \right] \\ &= \left[ \xi_1 : \dots : \xi_n : \frac{\sum_{j=1}^n a_j^1 \xi_j z_j}{a_0^1 + \sum_{j=1}^n a_j^1 z_j} : \dots : \frac{\sum_{j=1}^n a_j^k \xi_j z_j}{a_0^k + \sum_{j=1}^n a_j^k z_j} \right]. \end{aligned}$$

From this we can obtain an explicit description of the fibres of  $\Phi|_U$ . Take a vector  $V = [V_1 : \dots : V_n : W_1 : \dots : W_k] \in \mathbb{P}^{n+k-1}$ . The fibre of  $\Phi|_U$  above  $V$  is described by

$$\Phi(z, [\xi]) = V \iff \begin{cases} \xi_i = \lambda V_i, & i = 1, \dots, n \\ \sum_{j=0}^n a_j^l (V_j - W_l) z_j = 0, & l = 1, \dots, k \end{cases} \text{ for some } \lambda \in \mathbb{C} \setminus \{0\}.$$

Therefore

$$\pi(\Phi^{-1}(V)) \cap U = \left\{ (z_1, \dots, z_n); \sum_{j=0}^n a_j^l (V_j - W_l) z_j = 0 \text{ for all } l = 1, \dots, k \right\},$$

which is a linear subspace of  $\mathbb{P}^n$  (to simplify notations, we let here  $z_0 = 1$  and  $\xi_0 = V_0 = 0$ ).

Note that we can work out the same proof for any open subset  $\mathbb{P}^n \setminus \bigcup_{i \notin I} H_i$ ,  $|I| = n$ , up to a linear coordinate change. Since linear subspaces keep the same shape after a linear transformation, the result holds on the whole of  $\mathbb{P}^n$ .

Together with the results of 3.3, we have therefore proven that:

**Proposition 3.9**

The logarithmic cotangent bundle  $\Omega_{\mathbb{P}^n}(\log D)$  fails to be ample modulo  $D$  if and only if the projection of the augmented base locus  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}^n}(\log D)(1))$  contains a line not in  $\text{Supp } D$ .

In other words,  $\Omega_{\mathbb{P}^n}(\log D)$  is ample modulo  $D$  if and only if its restriction to any line  $l \not\subset \text{Supp } D$  is ample.

### 3.5 Construction of obstruction lines

In this section, we will use some vocabulary and results of [DK93]. We introduce some notations. Denote by  $(\mathbb{P}^n)^*$  the dual projective  $n$ -space, that is the projective space of hyperplanes in  $\mathbb{P}^n$ . For a linear subspace  $Z$  of  $\mathbb{P}^n$ , we denote by  $Z^*$  the dual of  $Z$ , *i.e.* the projective space of hyperplanes containing  $Z$ . This is a subspace of  $(\mathbb{P}^n)^*$  of dimension  $\text{codim}(Z) - 1$ .

As in subsection 3.4, we consider a logarithmic pair  $(\mathbb{P}^n, D)$ , where  $D = H_0 + \dots + H_{n+k}$  is an arrangement of  $n + k + 1 \geq n + 2$  hyperplanes in general position. Remark that we can already assume that we have at least  $2n + 1$  hyperplanes:  $\mathbb{P}^n \setminus D$  is indeed Kobayashi-hyperbolic if  $\Omega_{\mathbb{P}^n}(\log D)$  is ample modulo  $D$  (see e.g. [Dem97]), and it is well-known that the complement of  $2n$  hyperplanes in  $\mathbb{P}^n$  is never hyperbolic. We assume without loss of generality that the  $n + 1$  first components are the coordinates hyperplanes, *i.e.*  $H_i = (Z_i = 0)$  for  $i = 0, \dots, n$ . We aim to prove the following result.

#### Theorem 3.10

The logarithmic cotangent bundle of the pair  $(\mathbb{P}^n, D)$  fails to be ample modulo  $D$  if, and only if, there exists a line  $l \not\subset D$  such that  $l^* \cong \mathbb{P}^{n-2}$  and the points  $H_i^*$  are contained in a same quadric hypersurface in  $\mathbb{P}^{n*}$ .

Let us first explain how this condition is translated in the original space  $\mathbb{P}^n$ . Let  $X \subset \mathbb{P}^{n*}$  denote a quadric hypersurface containing an  $(n - 2)$ -plane. According to [Har92],  $X$  is a quadric of rank  $\leq 4$ , swept out by a 1-parameter family of  $(n - 2)$ -planes:  $X = \bigcup_{\lambda \in \mathbb{P}^1} Z_\lambda$ . We have seen in 2.3 that the dual variety  $X^*$  is in general a non-degenerate quadric surface contained in a  $\mathbb{P}^3 \subset \mathbb{P}^n$  (when  $X$  has rank  $\leq 3$ ,  $X$  is a cone over a plane conic, and  $X^*$  is also a cone over the dual conic curve). It is actually a ruled surface  $\mathbb{P}^1 \times \mathbb{P}^1$ , whose two rulings correspond dually to the two rulings in  $\mathbb{P}^{n-2}$ 's of  $X$ .

**Remark 3.11** *Consequently, the condition that the hyperplanes  $H_i$  belong to a same quadric rational scroll  $X$  in  $\mathbb{P}^{n*}$  means that the  $H_i$  are all tangent to the dual quadric surface  $X^* \subset \mathbb{P}^n$ .*

For simplicity, considering  $n + k + 1$  hyperplanes  $H_0, \dots, H_{n+k}$  in general position in  $\mathbb{P}^n$ , we introduce the following property. Theorem 3.10 asserts that this is equivalent to  $\Omega_{\mathbb{P}^n}(\log D)$  being ample modulo  $D$ .

**Condition ★** *The points  $H_0^*, \dots, H_{n+k}^*$  in the dual projective space  $\mathbb{P}^{n*}$  corresponding to the hyperplanes  $H_0, \dots, H_{n+k}$  do not belong to a same quadric hypersurface of rank at most 4.*

#### Corollary 3.12

The logarithmic cotangent bundle of the pair  $(\mathbb{P}^n, D)$  is ample modulo  $D$  if, and only if, the hyperplanes  $H_i$  impose at least  $4n - 2$  linearly independent conditions on quadrics. In particular, if  $D$  has no more than  $4n - 3$  components, there always exist obstructions to ampleness away from  $D$ .

Note that for  $n = 2$ , we retrieve Noguchi's result with six lines on  $\mathbb{P}^n$ . For  $n = 3$ , we obtain the same bound as in [DR24], but as soon as  $n \geq 4$ , our result is strictly better.

**Proof:** Following the previous section, the strategy of the proof relies on the search for lines contracted by the map  $\Phi$ . Since the logarithmic cotangent bundle is globally generated,  $\Phi$  is a morphism and an embedding when restricted to each fibre of the projection  $\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D)) \rightarrow \mathbb{P}^n$ . According to the proposition 3.15 below, there exists a line  $l \not\subset \text{Supp } D$  contracted by  $\Phi$  if, and only if, there exists a degree 1 rational map  $l \rightarrow H_0$  preserving each component  $H_i$ . Applying further 3.16, producing such a map amounts to the fact that the points of  $(\mathbb{P}^n)^*$  corresponding to the hyperplanes  $H_i$  lie on a same quadric  $X$  of rank  $\leq 4$ .

Moreover, we can compute the dimension of the space of quadrics of rank  $\leq 4$  on  $\mathbb{P}^n$ . Choosing such a form amounts to taking a 4-codimensional subspace of  $\mathbb{C}^{n+1}$  together with a quadratic form on  $\mathbb{C}^4$ . We find that the dimension equals  $4(n-3) + \frac{4 \times 5}{2} - 1 = 4n - 3$ .

This ends the proof of both the theorem and its corollary. ■

We shall now prove in two main steps why the existence of such obstruction lines away from  $\text{Supp } D$  is equivalent to the above condition concerning quadrics.

Let us first give some terminology.

**Definition 3.13** ([DK93])

A line  $l \subset \mathbb{P}^n$  is called *superjumping* for the pair  $(\mathbb{P}^n, D)$  if the restriction  $\Omega_{\mathbb{P}^n}(\log D)|_l$  is not ample.

Using the residue exact sequence, it is clear that all the lines belonging to  $\text{Supp } D$  are superjumping.

In other words, the logarithmic cotangent bundle  $\Omega_{\mathbb{P}^n}(\log D)$  is ample modulo boundary if and only if there is no superjumping line away from  $\text{Supp } D$ . The following characterisation of superjumping lines is straightforward.

**Lemma 3.14**

A line  $l \subset \mathbb{P}^n$  is superjumping if, and only if, the restriction  $T_{\mathbb{P}^n}(-\log D)|_l$  of the logarithmic tangent bundle has a non-vanishing global section.

**Proof:** Recall that a vector bundle on  $\mathbb{P}^1$  can be decomposed as a direct sum of line bundles. Since  $\Omega_{\mathbb{P}^n}(\log D)|_l$  is globally generated, there exist positive integers  $a_1, \dots, a_r$  such that

$$\Omega_{\mathbb{P}^n}(\log D)|_l = \mathcal{O}_l(a_1) \oplus \dots \oplus \mathcal{O}_l(a_r) \oplus \mathcal{O}_l^{\oplus n-r},$$

so that

$$T_{\mathbb{P}^n}(-\log D)|_l = \mathcal{O}_l(-a_1) \oplus \dots \oplus \mathcal{O}_l(-a_r) \oplus \mathcal{O}_l^{\oplus n-r}.$$

By definition, the line  $l$  is superjumping if  $r < n$ , *i.e.* if there is a trivial factor in the decomposition of  $T_{\mathbb{P}^n}(-\log D)|_l$ . ■

The next proposition is the key argument in the proof of the theorem. Note that even though this results appears as Prop. 7.5 in [DK93], our proof is completely different and self-contained. Ours is geometric and constructs explicitly the map whereas the authors of [DK93] use algebraic constructions called *elementary transformations*, in order to reconstruct inductively the logarithmic cotangent bundle, adding each component of  $D$  one by one.

**Proposition 3.15**

Let  $l$  be a line not lying in any hyperplane  $H_i$ . Denote by  $p_i$  the point corresponding to  $H_i$  in the dual projective space  $(\mathbb{P}^n)^*$ , and by  $Z$  the 2-codimensional linear subspace of  $(\mathbb{P}^n)^*$  corresponding to  $l$ . Then the following are equivalent:

1. there exists a non-zero (even non-vanishing) global section of the bundle  $T_{\mathbb{P}^n}(-\log D)|_l$ ;
2. there exists a regular map  $\psi : l \rightarrow H_0$  of degree 1 such that for all  $1 \leq i \leq n+k$ ,  $\psi(l \cap H_i) \subset H_0 \cap H_i$ .

**Proof:** We are going to construct in coordinates the desired map. Let  $X_0, \dots, X_n$  be homogeneous coordinates on  $\mathbb{P}^n$  such that  $H_0 = (X_0 = 0)$  and  $l = (X_2 = \dots = X_n = 0)$ . On the open subset  $U_0 = \mathbb{P}^n \setminus H_0$ , consider the inhomogeneous coordinates  $(z_1, \dots, z_n) = (X_1/X_0, \dots, X_n/X_0)$ . Then  $z_1$  defines a coordinate on  $l \setminus H_0 \simeq \mathbb{C}$ .

Assume first that there exists  $\xi \in H^0(l, T_{\mathbb{P}^n}(-\log D)|_l) \setminus \{0\}$ . We can write

$$\xi = \xi_1(z_1) \frac{\partial}{\partial z_1} + \dots + \xi_n(z_1) \frac{\partial}{\partial z_n}$$

for some holomorphic functions  $\xi_j : \mathbb{C} \rightarrow \mathbb{C}$ .

Define now inhomogeneous coordinates on  $U_1 = \mathbb{P}^n \setminus \{X_1 = 0\}$  by

$$(x_0, x_2, \dots, x_n) = (X_0/X_1, X_2/X_1, \dots, X_n/X_1) = (1/z_1, z_2/z_1, \dots, z_n/z_1).$$

The Jacobian matrix associated to this change of coordinates in  $U_0 \cap U_1$  is given by

$$\begin{pmatrix} -1/z_1^2 & 0 & \dots & 0 \\ -z_2/z_1^2 & 1/z_1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ -z_n/z_1^2 & \dots & \dots & 1/z_1 \end{pmatrix}.$$

On  $l \cap U_0 \cap U_1$ , the vector field  $\xi$  writes as

$$\xi(1/x_0) = -x_0^2 \xi_1(1/x_0) \frac{\partial}{\partial x_0} + x_0 \xi_2(1/x_0) \frac{\partial}{\partial x_2} + \dots + x_0 \xi_n(1/x_0) \frac{\partial}{\partial x_n}.$$

Since  $\xi$  needs to be holomorphic, the functions  $x_0^2 \xi_1(1/x_0), x_0 \xi_2(1/x_0), \dots, x_0 \xi_n(1/x_0)$  are holomorphic in the variable  $x_0$ . This shows that the holomorphic functions  $x_j : \mathbb{C} \rightarrow \mathbb{C}$  are in fact polynomials, of degree  $\leq 2$  for  $j = 1$  and  $\leq 1$  otherwise. Moreover,  $\xi \in H^0(l, T_{\mathbb{P}^n}(-\log H_0)|_l)$ , so we can write, for some holomorphic function  $g$ ,

$$-x_0^2 \xi_1 \left( \frac{1}{x_0} \right) = x_0 g(x_0).$$

It follows that  $\xi_1$  also has degree  $\leq 1$ .

Define naturally a map by

$$\begin{aligned} \psi : l &\rightarrow H_0 \\ z &\mapsto [0 : \xi_1(z) : \dots : \xi_n(z)] \end{aligned}$$

By lemma 3.14, a global section of  $T_{\mathbb{P}^n}(-\log D)$  come from the trivial factors of the decomposition, so vanishes nowhere. Hence  $\psi$  is well-defined. By definition,  $\xi$  is tangent to every component of  $D$ ; this directly implies that  $\psi$  sends each  $H_j$  into itself.

Conversely, assume that we are given a degree 1 map  $\psi : l \rightarrow H_0$  as above. Over  $U_0 = \mathbb{P}^n \setminus H_0$ , write  $\psi = [0 : \xi_1 : \cdots : \xi_n]$ , and choose representatives  $\xi_j$ , which are degree 1 polynomials in the variable  $z_1$ . Then we can define a non-trivial logarithmic vector field by

$$\xi = \sum_{j=1}^n \xi_j(z_1) \frac{\partial}{\partial z_j} \in H^0(l \setminus H_0, T_{\mathbb{P}^n}(-\log H_0)|_l).$$

By construction, one has  $\xi(l \cap H_j) \in T_{l \cap H_j} H_j$  for all  $j$ , hence

$$\xi \in H^0(l \setminus H_0, T_{\mathbb{P}^n}(-\log D)|_l). \quad \blacksquare$$

More concretely, given a global logarithmic vector field  $\xi \in H^0(l, T_{\mathbb{P}^n}(-\log D)|_l)$ , we construct the map  $\psi : l \rightarrow H_0$  in the following way (see figures 1 and 2 below for illustrations of the 2- and 3-dimensional cases): starting from a point  $q$  in  $l$ , we have a tangent vector to  $l$  at  $x$ , which uniquely defines a line  $L_x$  through  $x$ . Denote by  $\psi(x)$  the intersection point of  $L_x$  with  $H_0$ . By definition, the sections of  $T_{\mathbb{P}^n}(-\log D)|_l$  are those vector fields tangent to all  $H_i$ 's. Hence, if  $x \in H_i \cap l$ , the point  $\psi(x)$  is also in  $H_i$ .

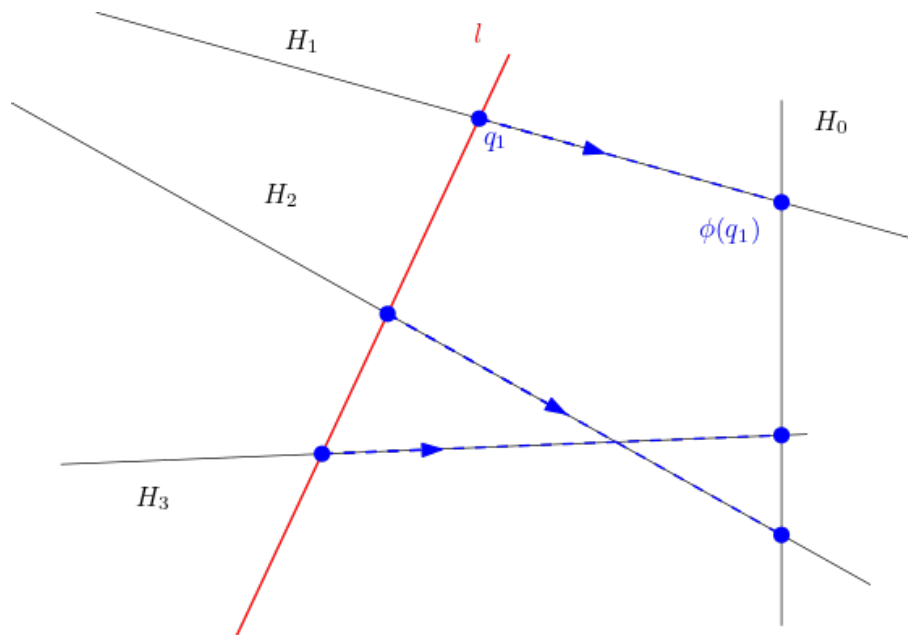


Figure 1: Lines in  $\mathbb{P}^2$

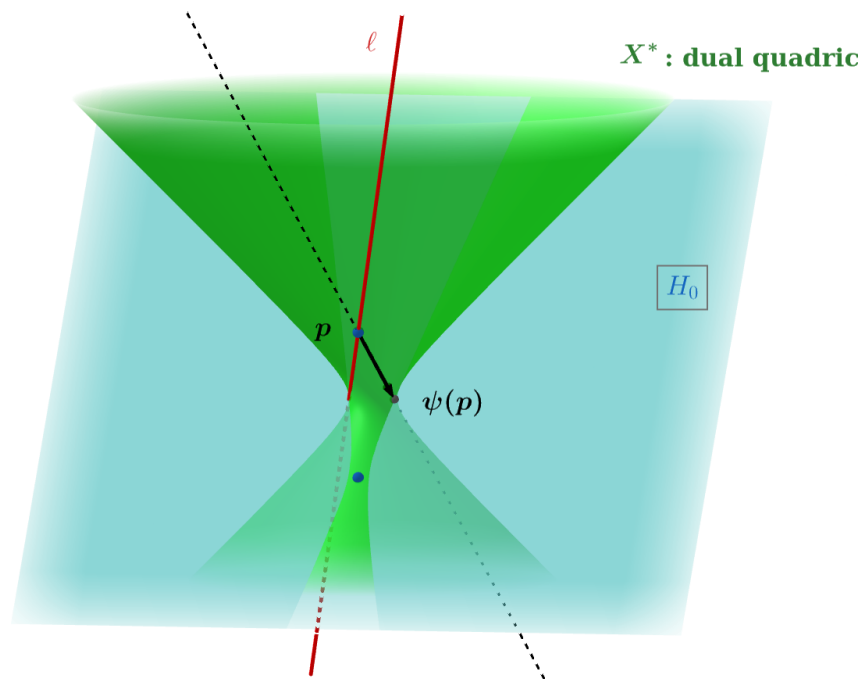


Figure 2: Construction for planes in  $\mathbb{P}^3$



Let  $p_0, \dots, p_{n+k}$  be the points of  $(\mathbb{P}^n)^*$  corresponding dually to the hyperplanes  $H_0, \dots, H_{n+k}$  and let  $W = l^* \subset (\mathbb{P}^n)^*$  be the  $(n-2)$ -plane of hyperplanes in  $\mathbb{P}^n$  containing  $l$ . Assume that there exists a quadric hypersurface  $X \subset (\mathbb{P}^n)^*$  containing  $W, p_0, \dots, p_{n+k}$ .

The inclusion  $W \cup \{p_0, \dots, p_c\} \subset X$  means that the line  $l$  and all the hyperplanes  $H_i$  are tangent to the dual surface  $X^*$  at some points.

The next proposition provides a necessary and sufficient condition for the existence of such a quadric hypersurface.

**Proposition 3.16**

Consider  $c+1$  points  $p, p_1, \dots, p_c$  in general position in  $\mathbb{P}^n$ , and a  $(n-2)$ -plane  $W$  disjoint from  $\{p_0, p_1, \dots, p_c\}$ . For  $i = 1, \dots, c$ , denote by  $L_i = \langle W, p_i \rangle$  the hyperplane generated by  $W$  and  $p_i$ . The following two assertions are equivalent.

1. There exists a quadric hypersurface scroll  $X \subset \mathbb{P}^n$  containing  $W$  and all the points  $p_i$ .
2. There exists a degree 1 map  $\psi : \mathbb{P}^1 \cong W^* \rightarrow \mathbb{P}^{n-1} \cong p_0^*$  such that  $\psi(L_i)$  contains the point  $p_i$  and  $\psi(L_0) \neq L_0$ .

**Proof:** 1. Consider a degree 1 map  $\psi$  as described in the statement. Then  $\psi$  sends  $W^* \cong \mathbb{P}^1$  to a line in  $\mathbb{P}^{n-1} \cong p_0^*$ .

The intersection  $W' = \bigcap_{L \in W^*} \psi(L)$  is therefore another  $(n-2)$ -plane.

Define a subset of  $\mathbb{P}^n$  by

$$X(\psi) = \bigcup_{L \in W^*} L \cap \psi(L).$$

It contains  $W$  as well as the points  $p_i$ . Indeed, for any  $q \in W \setminus W'$ , there exists a unique hyperplane  $\psi(L)$  containing both  $W'$  and  $q$ . It follows that  $q \in L \cap \psi(L) \subset X(\psi)$ . Moreover, by definition of  $\psi$ , we know that  $p_i \in L_i \cap \psi(L_i)$  for all  $i$ , so that  $p_i \in X(\psi)$ .

By construction,  $X(\psi)$  is an  $(n-1)$ -dimensional scroll of degree 2 in  $\mathbb{P}^n$ , which contains  $W \cup \{p_0, \dots, p_c\}$ .

2. Assume the existence of the quadric  $X$ . We will show that  $X$  is necessarily of the form  $X(\psi)$  for some map  $\psi$ .

The set  $W^*$  of hyperplanes containing  $W$  is a projective line  $\mathbb{P}^1$ . Let  $L_\infty \in W^*$  be the tangent hyperplane to  $X$  at any point of  $W$ . Since  $p_0 \in X \setminus W$ ,  $p_0 \notin L_\infty$ .

For any  $L \in W^*$ , the set  $L \cap X$  is a quadric hypersurface inside  $L$ , which contains  $W$ . Hence there exists an  $(n-2)$ -plane  $W(L)$  such that  $L \cap X = W \cup W(L)$ . One has  $W(L) \neq W$  except for  $L = L_\infty$ . Moreover,  $p_0 \in W(L)$  only if  $L = L_0$ .

We define a map  $\psi : W^* \rightarrow p_0^*$  by setting  $\psi(L) = \langle W(L), p_0 \rangle$  for  $L \neq L_0$ , and  $\psi(L_0) = \mathbb{T}_{p_0} X$  (embedded tangent hyperplane to  $X$  at  $p_0$ ). Note that if since  $p_i \in X \setminus W$ ,  $p_i \in W(L_i)$ . Then  $\psi$  has the desired properties, with  $\psi(L_\infty) = L_0$ .

Moreover, for any hyperplane  $L$ , one has  $Z(L) = L \cap \psi(L)$ , and  $X$  is described as the union

$$\bigcup_{L \in W^*} L \cap \psi(L). \quad \blacksquare$$

**Remark 3.17** We can also understand this construction in the original space  $\mathbb{P}^n$ .

Consider a ruled quadric surface  $X \cong \mathbb{P}^1 \times \mathbb{P}^1$  containing the line  $l$  and tangent to all hyperplanes  $H_i$ .

A point  $p \in l$  belongs to two lines of  $X$ , one in each ruling. Let  $l(p) \subset X$  be the other line through  $p$ . The map  $\psi$  sends  $p$  to the intersection point of  $l(p)$  with  $H_0$ .

### 3.6 Almost ampleness

Let  $(X, D)$  be a smooth log pair. Whereas being ample modulo  $D$  only involves the image of  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1))$  in  $X$ , the stronger notion of almost ampleness reads on the augmented base locus itself. In general, this last property is strictly stronger. However, in the particular case of hyperplane arrangements, both are equivalent.

#### Theorem 3.18

Let  $D = \sum_{i=0}^{n+k} D_i$  be a arrangement of hyperplanes in general position in  $\mathbb{P}^n$ . Then  $\Omega_{\mathbb{P}^n}(\log D)$  is ample modulo  $D$  if, and only if, it is almost ample.

**Proof:** We only need to treat one direction. Assume that the logarithmic cotangent bundle associated to  $D$  is ample modulo  $D$ . Then  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1)) \subset \pi^{-1}(D)$ .

We will study more carefully the fibres of  $\Phi$  above the strata of  $D$ . According to lemma 3.3, their union forms the augmented base locus  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_X(\log D))}(1))$ .

We need to show that  $\Phi$  restricted to  $\pi^{-1}(D) \setminus \bigcup_{I:|I|<n} \tilde{D}_I$  is injective.

Let  $I \subset \{0, \dots, n+k\}$  be a subset of indices of cardinality  $< n$ . Up to a linear coordinate change, we can assume that  $I = \{1, \dots, r\}$  with  $r \leq n$ .

Again, let us work on the open subset  $U = (Z_0 \neq 0) \cap \bigcap_{1 \leq j \leq k} (\ell_j \neq 0)$  with standard affine coordinates  $(z_1, \dots, z_n)$ . In these coordinates,  $D_I = \{(0, \dots, 0, z_{r+1}, \dots, z_n)\} \cong \mathbb{P}^{n-r-1}$ . Moreover, the restricted residue map  $\text{Res} : \Omega_{\mathbb{P}^n}(\log D)|_{D_I \cap U} \rightarrow \mathcal{O}_{D_I}^{\oplus r}$  is given by the expression

$$\text{Res}|_{D_I \cap U} \left( \sum_{i=1}^n \eta_i \frac{dz_i}{z_i} \right) = (\eta_1, \dots, \eta_r).$$

Under the above trivialisation,  $\tilde{D}_I$  is described as

$$\tilde{D}_I \cap \pi^{-1}(U) \cong \left\{ (z, [\xi_1 : \dots : \xi_n]) \in U \times \mathbb{P}^{n-1} \left| \begin{array}{l} z_i = 0 \ \forall 1 \leq i \leq r \\ \text{and } \xi_i = 0 \\ \forall i \geq r+1 \end{array} \right. \right\}.$$

Remark first that for any  $J \subset \{0, \dots, n+k\}$ ,  $\pi^{-1}(D_I) \cap \tilde{D}_J = \emptyset$  unless  $J \subset I$ . This can be easily checked in coordinates.

For all  $(z, [\xi]) \in \pi^{-1}(D_I)$ , we have

$$\Phi(z, [\xi]) = \left[ \xi_1 : \dots : \xi_n : \frac{\sum_{j=r+1}^n a_j^1 \xi_j z_j}{a_0^1 + \sum_{j=r+1}^n a_j^1 z_j} : \dots : \frac{\sum_{j=r+1}^n a_j^k \xi_j z_j}{a_0^k + \sum_{j=r+1}^n a_j^k z_j} \right]. \quad (\diamond)$$

Note that for  $J \subset I$  and  $(z, [\xi]) \in \pi^{-1}(D_I) \cap \tilde{D}_J$ , one has

$$\Phi(z, [\xi]) = [\mathbf{1}_J(1)\xi_1 : \dots : \mathbf{1}_J(r)\xi_r : 0 : \dots : 0].$$

In particular, this image is independent of the  $z$ -coordinates and  $\Phi^{-1}(\Phi(z, [\xi])) \cong D_J$ .

The restriction  $\Phi|_{\pi^{-1}(D_I)}$  is injective out of  $\bigcup_{J \subset I} \tilde{D}_J$  if and only if the corresponding map

$$\Phi^\dagger : \mathbb{C}^{n-r} \times (\mathbb{C}^n \setminus \mathbb{C}^r \times \{0\}^{n-r}) \longrightarrow \mathbb{C}^r \times \mathbb{C}^{n-r} \times \mathbb{C}^k$$

is so.

Since the first coordinate map of  $\Phi^\dagger$  is given by the identity  $\mathbb{C}^r \rightarrow \mathbb{C}^r$ , this is equivalent to the injectivity of

$$\begin{aligned} \Phi^\dagger : \mathbb{C}^{n-r} \times (\mathbb{C}^{n-r} \setminus \{0\}) &\longrightarrow \mathbb{C}^{n-r+k} & (\spadesuit) \\ (z, \xi) &\longmapsto \left( \xi_{r+1}, \dots, \xi_n, \frac{\sum_{j=r+1}^n a_j^1 \xi_j z_j}{a_0^1 + \sum_{j=r+1}^n a_j^1 z_j}, \dots, \frac{\sum_{j=r+1}^n a_j^k \xi_j z_j}{a_0^k + \sum_{j=r+1}^n a_j^k z_j} \right). \end{aligned}$$

We see that  $\Phi|_{\pi^{-1}(D_I)}$  is injective out of  $\bigcup_{J \subset I} \tilde{D}_J$  whenever the above map  $\Phi^\dagger : \mathbb{C}^{n-r} \times \mathbb{C}^{n-r} \rightarrow \mathbb{C}^{n-r+k}$  is injective out of the corresponding subset. We recognise in equation  $\spadesuit$  the expression for the morphism associated to the pair  $(D_I, D(I^{\mathbb{C}})|_{D_I})$ . Hence we are reduced to showing that this arrangement of  $(n-r) + k + 1$  hyperplanes in  $D_I \cong \mathbb{P}^{n-r}$  has almost ample logarithmic cotangent bundle.

By lemma 3.19 below, we know that this pair already has ample modulo boundary logarithmic cotangent bundle.

The conclusion follows by induction on the dimension, provided that we prove the case  $n = 2$ .

For any  $(z = (0, z_2), [\xi_1 : \xi_2]) \in \pi^{-1}(D_1 \cap U_0)$ ,  $\spadesuit$  reads as

$$\Phi(z, [\xi]) = \left[ \xi_1 : \xi_2 : \frac{a_2^1 \xi_2 z_2}{a_0^1 + a_2^1 z_2} : \dots \right].$$

A quick computation shows that this map is injective in restriction to

$$\pi^{-1}(D_1 \cap U_0) \setminus \tilde{D}_1 = \{((0, z_2), [\xi_1 : 1])\}. \quad \blacksquare$$

### Lemma 3.19

Let  $D = \sum_{0 \leq i \leq n+k} D_i$  be an arrangement of  $n+k+1$  hyperplanes in general position in  $\mathbb{P}^n$  satisfying Condition  $\star$ . Then so does the arrangement  $D(I^{\mathbb{C}})|_{D_I}$  in  $D_I \cong \mathbb{P}^{n-|I|}$ , for any subset of indices  $I$  of cardinality  $< n$ .  
Conversely

**Proof:** We assume for simplicity that  $I = \{0, \dots, r\}$  and  $D_i = (Z_i = 0)$  for  $i = 0, \dots, r$ .

First, it is clear that the arrangement  $D(I^{\mathbb{C}})|_{D_I}$  is in general position (*i.e.* is a normal crossings divisor).

Assume that the hyperplanes  $D_j \cap D_I$ ,  $i = r+1, \dots, n+k$  do not satisfy Condition  $\star$ . Then one can find a symmetric matrix  $Q'$  of size  $n-r$  and rank less than 4 such that

$$A_I^\top Q' A_I = 0,$$

then the matrix

$$Q = \left( \begin{array}{c|c} 0_{r+1} & 0 \\ \hline 0 & Q' \end{array} \right)$$

satisfies

$$A^\top Q A = 0$$

and Condition  $\star$  does not hold for the hyperplanes  $D_i$ 's in  $\mathbb{P}^n$ . \blacksquare

### 3.7 Description of the augmented base locus in terms of the number of hyperplanes

The proof of theorem 3.10 actually provides a complete description of the image

$$p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)))$$

of the augmented base locus of the logarithmic cotangent bundle  $\Omega_{\mathbb{P}^n}(\log D)$ .

#### Theorem 3.20

The projection

$$p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)))$$

is the union of all ruled quadric surfaces which are simultaneously tangent to all hyperplanes  $H_i$ . Hence, for  $4n - 3 \geq c \geq 3n$  hyperplanes, this locus has dimension  $4n - 1 - c$ , and

$$p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))) = \mathbb{P}^n$$

if  $c \leq 3n - 1$ .

**Proof:** We know indeed that the locus  $p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)))$  is made of lines. According to the construction of the previous section, the  $(n - 2)$ -plane, dual of any such line, is contained in some quadric scroll in  $\mathbb{P}^{n*}$ , which also contains the hyperplanes  $H_i$ 's as points of the dual projective space. Conversely, for any quadric scroll  $X$  containing the  $H_i$ 's in  $\mathbb{P}^{n*}$ , the dual of any  $\mathbb{P}^{n-2}$  of its rulings is a 'bad' line. If  $n \geq 3$ , it is a line of the quadric surface dual to  $X$ .

Consequently, the image of  $\mathbb{B}_+(\mathcal{O}_{\Omega_{\mathbb{P}^n}(\log D)}(1))$  in  $\mathbb{P}^n$  is made of  $\text{Supp } D$  and all the quadric surfaces that are simultaneously tangent to all the components  $H_i$ .

Recall that the dimension of the space of quadrics of rank  $\leq 4$  in  $\mathbb{P}^{n*}$  equals  $4n - 3$ . In the limit case  $c = 4n - 3$ , for a general choice of hyperplanes (here, the term *general* is to be understood as *imposing linearly independent conditions on quadrics*), there exists a unique quadric  $X$  of rank exactly 4 containing all of them in  $\mathbb{P}^{n*}$ . Thus  $\mathbb{B}_+(\mathcal{O}_{\Omega_{\mathbb{P}^n}(\log D)}(1))$  projects onto  $D \cup S$ , where  $S = X^*$  is a smooth quadric surface.

Similarly, the variety of rank  $\leq 4$  quadrics containing  $c$  general points in  $\mathbb{P}^n$  has dimension  $4n - 3 - c$ . We derive that for a general choice of  $c$  hyperplanes,  $p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))) \setminus \text{Supp } D$  has dimension  $2 + 4n - 3 - c = 4n - 1 - c$ . For  $c < 3n$ , this augmented base locus is not anymore dominant.

We deal separately with the 2-dimensional case, since we do not get surfaces. The 'problematic' lines showing up in the proof are the points of any conic in  $\mathbb{P}^{2*}$  containing all the points  $H_i$ . In the original plane  $\mathbb{P}^n$ , if there exists a conic  $\mathcal{C}$  osculating the lines  $H_i$ , then any other tangent line is superjumping. We see that these lines shape exactly  $p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^2}(\log D))}(1)))$ . If  $D$  is made of five general lines, there exist a unique osculating conic  $\mathcal{C}$ . In this situation,

$$p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^2}(\log D))}(1))) = \text{Supp } D \cup \mathcal{C}$$

is still a proper subset of  $\mathbb{P}^2$ .

Denote by  $\Sigma$  the closure

$$\overline{p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))) \setminus \text{Supp } D}$$

of the complement of  $\text{Supp } D$  in the projection of the augmented base locus of the logarithmic cotangent bundle. We have seen that  $\Sigma$  consists of a union of lines. In addition, the superjumping

lines were identified, through the construction of the previous section, as the lines of all quadric surfaces  $S$  contained in some  $\mathbb{P}^3 \subset \mathbb{P}^n$ , which are simultaneously tangent to all components of  $D$ .

It is known that any irreducible quadric surface is ruled (it is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ ). Any surface  $S$  as above is the union of its lines, and is thus entirely contained in  $\Sigma$ . We have ultimately described the locus  $\Sigma$  as a union of quadric surfaces.

To be more precise, we will determine the dimension of the family of such surfaces  $S$ , depending on the number of components of  $D$ . Recall from the previous sections that such a surface  $S$  is the dual variety of a quadric hypersurface in  $\mathbb{P}^{n*}$  of rank  $\leq 4$  containing all the points  $p_i = H_i^*$ . First, let us compute the dimension of the variety  $\Xi$  of quadric hypersurfaces of rank  $\leq 4$  in  $\mathbb{P}^n$ . To do so, we use the characterisation of quadrics of rank at most 4 as quadric hypersurfaces containing an  $(n-2)$ -plane. The variety  $\Xi$  can then be represented as the image under the second projection of the incidence variety

$$\mathcal{I} = \{(Z, X); Z \subset X\} \subset \mathbb{G}(n-2, n) \times \mathbb{P}^{\binom{n+2}{2}-1}.$$

Fix an  $(n-2)$ -plane  $Z \subset \mathbb{P}^n$ . What can be the dimension of the fibre  $\pi_1^{-1}(Z)$  of the first projection above  $Z$ ? Observe that this fibre is actually the kernel of the linear map

$$\mathbb{C}^{\binom{n+2}{2}} \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2)) \rightarrow H^0(Z, \mathcal{O}_Z(2)) \cong \mathbb{C}^{\binom{n}{2}}.$$

Hence  $\pi_1^{-1}(Z)$  has dimension  $\binom{n+2}{2} - \binom{n}{2} - 1 = 2n$ .

In the end, we find  $\dim \mathcal{I} = 2n + \dim \mathbb{G}(n-2, n) = 2n + 2(n-1) = 4n-2$ . Since the fibre above a general point of  $\Xi$  has dimension 1, we compute  $\dim \Xi = 4n-3$ .

We see now that for a general choice of  $c \geq 4n-2$  hyperplanes in  $\mathbb{P}^n$ , the image  $p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)))$  of the augmented base locus is empty, since  $4n-2$  general points can not belong to a same quadric of rank  $\leq 4$ . If  $D$  is made of  $c \leq 4n-3$  general components, then the locus  $\Sigma$  is swept out by a  $(4n-3-c)$ -dimensional family of quadric surfaces, and thus has dimension  $4n-3-c+2 = 4n-1-c$  for  $c \geq 3n$ . When we have strictly less than  $3n$  components,  $\Sigma$  is the whole space  $\mathbb{P}^n$ .  $\blacksquare$

When  $c \leq 3n-1$ ,  $\mathbb{P}^n$  is entirely covered by superjumping lines. However, the augmented base locus  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))$  upstairs can still be a proper subset of  $\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))$ . Is it possible to describe it more precisely?

Assume that  $2n \leq c \leq 3n-1$ . The image  $p(\mathbb{B}_+(\mathcal{O}(1)))$  is swept out by a  $(4n-3-c)$ -dimensional family of  $\mathbb{P}^1 \times \mathbb{P}^1$ 's tangent to all the hyperplanes. Therefore, the set of superjumping lines is a subvariety of  $\mathbb{G}(1, n) \cong \mathbb{P}^{2n-2}$  of dimension  $4n-2-c$ . We remark that for  $c \leq 2n$ , all lines in  $\mathbb{P}^n$  are superjumping.

### Theorem 3.21

Assume that we have  $n+2 \leq c \leq 3n-1$  hyperplanes  $H_i$  in general position in  $\mathbb{P}^n$ . Then the augmented base locus

$$\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)) \subset \mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))$$

has dimension  $\leq 4n-1-c$ .

**Proof:** We first show that  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))$  equals exactly the union over all superjumping lines  $l$  of the curves  $\mathbb{P}(\mathcal{O}_l)$  induced by trivial quotients of  $\Omega_{\mathbb{P}^n}(\log D)$ .

Write  $c = n+k+1$  with  $k \leq 1$  and let the notation be as in subsection 3.4. We work in the standard affine coordinates associated to the open subset  $\mathbb{P}^n \setminus H_0 = \{Z_0 \neq 0\}$ .

Let

$$(x, \xi = \sum_{i=1}^n \xi_i \frac{\partial}{\partial z_i}) \in \mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)) \setminus p^{-1}(D)$$

and denote for all  $i = 1, \dots, n$  and  $j = 1, \dots, k$ ,

$$V_i = \xi_i, V_j = \frac{\sum_{i=1}^n a_i^j x_i \xi_i}{\sum_{i=0}^n a_i^j x_i},$$

so that  $\Phi(x, \xi) = [V_1 : \dots : V_{n+k}]$ . Then the fibre  $Z = \Phi^{-1}(\Phi(x, \xi))$  is positive-dimensional and  $x$  belongs to some superjumping line  $l \subset p(Z)$ . The inclusion  $l \subset p(Z)$  shows that any  $z = (z_1, \dots, z_n) \in l \setminus H_0$  satisfies the equations

$$\sum_{i=0}^n a_i^j z_i (V_{n+j} - V_i) = 0.$$

One can extend the vector field

$$\xi(z) = \sum_{i=1}^n \frac{\xi_i}{x_i} z_i \frac{\partial}{\partial z_i}$$

defined on  $l \setminus H_0$  as a never-vanishing holomorphic global section of  $T_{\mathbb{P}^n}(-\log D)|_l$ . In other words, there is a trivial quotient  $\mathcal{O}_l$  of  $\Omega_{\mathbb{P}^n}(\log D)|_l$  such that  $(x, \xi) \in \mathbb{P}(\mathcal{O}_l)$ .

In order to compute the dimension of  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))$ , we will ‘count’ all such curves  $\mathbb{P}(\mathcal{O}_l)$ . Above each surface  $\mathbb{P}^1 \times \mathbb{P}^1$  composing  $p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)))$  lives a 1-parameter family of  $\mathbb{P}(\mathcal{O}_l)$  inside  $\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))$ . Therefore, the augmented base locus has dimension at most  $4n - 2 - c + 1 = 4n - 1 - c$ . ■

If  $c \leq 2n$ , the morphism  $\Phi : \mathbb{P}(\Omega_{\mathbb{P}^n}(\log D)) \rightarrow \mathbb{P}^{c-2}$  cannot be generically finite, since  $c - 2 \leq 2n - 2 < 2n - 1 = \dim \mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))$ . We deduce

**Corollary 3.22** *The logarithmic cotangent bundle associated to  $c$  general hyperplanes is big if and only if  $c \geq 2n + 1$ .*

**Remark 3.23** *Note that  $c = 2n + 1$  is also the bound for Brody-hyperbolicity of the complement  $\mathbb{P}^n \setminus D$ .*

## 4 Orbifolds

### 4.1 Campana’s orbifold category

In what follows, the term *orbifold* will refer to a geometric orbifold pair as defined by Campana. We refer for the basic notions concerning orbifolds to the papers [Cam11], [CP15] or [CDR20], for instance.

**Definition 4.1**

*A smooth orbifold pair is a pair  $(X, \Delta)$  where  $X$  is a smooth projective variety and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor with coefficients in  $[0, 1]$  and normal crossings support. In analogy with ramification divisors, we will write it under the form  $\Delta = \sum_{i \in I} (1 - \frac{1}{m_i}) D_i$  where the  $D_i$  are prime irreducible divisors and  $m_i \in \mathbb{Q}_{\geq 1} \cup \{\infty\}$ . Here, we will only consider integer multiplicities  $m_i$ .*

*We denote  $|\Delta| = \text{Supp } \Delta = \sum_{i \in I, m_i > 1} D_i$  and  $[\Delta] = \sum_i D_i$ .*

When the  $m_i$  are all infinite (resp. equal to 1), we recover the classical logarithmic (resp. compact) case. One can then naturally regard orbifolds as an interpolation between the compact and the logarithmic cases.

As in the logarithmic case, we would like to define *orbifold differential forms* as meromorphic differential forms over  $X$  having singularities of order “at most  $1 - 1/m_i$  along each  $D_i$ ”; formally, in some adapted local coordinates, the bundle  $\Omega_{(X, \Delta)}$  of orbifold 1-forms would be generated over  $\mathcal{O}_X$  by the forms  $\frac{dz_i}{z_i^{1-1/m_i}}$ . Even though this is not directly possible, one can define these bundles through appropriate ramified coverings turning  $\Delta$  into an integral divisor.

**Definition 4.2**

Let  $Y$  be a smooth projective variety. A Galois covering  $\pi : Y \rightarrow X$  is adapted to the orbifold pair  $(X, \Delta)$  if it satisfies the following conditions:

1. for any irreducible component  $D_i$ , we have  $\pi^* D_i = p_i \tilde{D}_i$ , where  $p_i$  is a multiple of  $m_i$  and  $\tilde{D}_i$  has at most simple normal crossings;
2. both the supports of  $\pi^* \Delta + \text{Ram}(\pi)$  and the branch locus of  $\pi$  have at most normal crossings.

In addition,  $\pi$  is said strictly adapted to  $(X, \Delta)$  if  $p_i = m_i$  for all  $i$ .

For a smooth orbifold pair  $(X, \Delta)$ , there always exists an adapted covering (see Prop. 4.1.12 in [Laz04a]).

Let  $\pi : Y \rightarrow (X, \Delta)$  be an adapted covering. For any point  $y \in Y$ , there exists an open neighbourhood  $U \ni y$ , invariant under the action of the isotropy group of  $y$  in  $\text{Aut}(\pi)$ . Hence, there exist local coordinates  $w_i$  on  $U$  centered at  $y$  such that  $\pi(U)$  has coordinates  $z_i$  centered at  $\pi(y)$  satisfying  $|\Delta| \cap \pi(U) \subset \{z_1 \dots z_n = 0\}$  and

$$\pi(w_1, \dots, w_n) = (z_1^{p_1}, \dots, z_n^{p_n}),$$

where  $p_i$  is an integer multiple of the multiplicity  $m_i$  in  $\Delta$  of  $(z_i = 0)$ .

If all multiplicities are infinite ( $\Delta = [\Delta]$ ), for any adapted covering  $\pi : Y \rightarrow X$ , we denote

$$\Omega(\pi, \Delta) := \pi^* \Omega_X(\log \Delta).$$

The orbifold cotangent bundle associated with  $\pi$  is then defined as the locally free subsheaf  $\Omega(\pi, \Delta)$  of  $\Omega(\pi, [\Delta])$  fitting in the short exact sequence

$$0 \rightarrow \Omega(\pi, \Delta) \rightarrow \Omega(\pi, [\Delta]) \rightarrow \bigoplus_{i \in I, m_i < \infty} \mathcal{O}_{\pi^* D_i / m_i} \rightarrow 0.$$

Here the quotient is the composition of the pullback of the residue map

$$\pi^* \text{Res} : \pi^* \Omega_X(\log |\Delta|) \rightarrow \bigoplus_{i \in I, m_i < \infty} \mathcal{O}_{\pi^* D_i}$$

with the quotients  $\mathcal{O}_{\pi^* D_i} \twoheadrightarrow \mathcal{O}_{\pi^* D_i / m_i}$ .

It is locally generated in coordinates as above by the elements

$$w_i^{\frac{p_i}{m_i}} \pi^*(dz_i/z_i) = w_i^{-p_i(1-\frac{1}{m_i})} \pi^*(dz_i). \tag{*}$$

We are now able to construct orbifold differential forms on  $X$ .

**Definition 4.3**

Given an adapted covering  $\pi : Y \rightarrow (X, \Delta)$ , the sheaf of orbifold symmetric differential forms of order  $q$  is the direct image

$$S^{[q]}\Omega_{(X,\Delta)} := \pi_* \left( (S^q\Omega(\pi, \Delta))^{\text{Aut}(\pi)} \right) \subseteq S^q\Omega_X(\log[\Delta]).$$

One says that  $\Omega_{(X,\Delta)}$  is big if  $\Omega(\pi, \Delta)$  is a big vector bundle over  $Y$  for some adapted covering  $\pi : Y \rightarrow X$ . This is equivalent to saying that for some (or any) ample line bundle over  $X$ , there exists an integer  $N$  such that  $H^0(X, S^{[N]}\Omega_{(X,\Delta)} \otimes A^{-1}) \neq \{0\}$ . By definition, the *augmented base locus* of  $\Omega_{(X,\Delta)}$  is

$$\mathbb{B}_+(\Omega_{(X,\Delta)}) = \bigcap_{N \geq 1} \bigcap_{p/q \in \mathbb{Q}} \text{Bs} \left( S^{[Nq]}\Omega_{(X,\Delta)} \otimes A^{-Np} \right)$$

for some ample line bundle  $A$  over  $X$ . Away from  $[\Delta]$ , this set turns out to be independent of the covering  $\pi$ .

**Proposition 4.4** ( [DR24] )

Over  $X \setminus |\Delta|$ , the image of the augmented base locus  $\mathbb{B}_+(\mathcal{O}_{Y'}(1))$  by the natural projection coincides with the orbifold augmented base locus  $\mathbb{B}_+(\Omega_{(X,\Delta)})$ .

**Theorem 4.5 (Fundamental vanishing theorem)**

Let  $(X, \Delta)$  be a smooth orbifold with  $X$  projective. Fix an ample line bundle  $A$  on  $X$  and a global orbifold symmetric differential

$$\omega \in H^0(X, S^{[N]}\Omega_{(X,\Delta)} \otimes A^{-1}).$$

Then for any orbifold entire curve  $f : \mathbb{C} \rightarrow (X, \Delta)$ ,

$$f^*\omega \equiv 0.$$

**Definition 4.6**

The orbifold cotangent bundle  $\Omega_{(\mathbb{P}^n, \Delta)}$  is said ample modulo  $|\Delta|$  if  $\mathbb{B}_+(\Omega_{(X,\Delta)}) \subset |\Delta|$ .

As in the logarithmic case, it is important to note that the orbifold cotangent bundle cannot be ample when  $\text{Supp } \Delta \neq \emptyset$ .

**Lemma 4.7** (see [DR24], Lemma 3.7) *Let  $(\mathbb{P}^n, \Delta)$  be a smooth orbifold pair. Then for any strictly adapted covering  $\pi : Y \rightarrow \mathbb{P}^n$ , the orbifold cotangent bundle  $\Omega_{(\mathbb{P}^n, \Delta)}$  has negative quotients supported on each component of  $\Delta$  with finite multiplicity and trivial quotients supported on each component with infinite multiplicity.*

Thus ampleness modulo boundary is the strongest property we can hope for.

## 4.2 The Fermat cover for hyperplane arrangements

Assume that the hyperplanes have equal orbifold multiplicities. In this particular case, we can construct a global strictly adapted covering as follows.

For any  $i = 0, \dots, n+k$ , let  $\ell_i$  be a linear form defining  $H_i$ . Without loss of generality, we may assume that  $\ell_i(Z) = Z_i$  for  $i_0, \dots, n$ , so that  $H_0, \dots, H_n$  are the coordinate hyperplanes, and we write

$$\ell_{n+j}(Z) = \sum_{i=0}^n a_i^j Z_i, j = 1, \dots, k.$$



In the projective space  $\mathbb{P}^{n+k}$ , one can identify  $\mathbb{P}^n$  with the linear subspace

$$\bigcap_{j=1}^k \{X_{n+j} = \sum_{i=1}^n a_{i,j} X_i\}.$$

Let  $Y$  be the complete Fermat intersection

$$\bigcap_{j=1}^k \{X_{n+j}^m = \sum_{i=1}^n a_{i,j} X_i^m\}.$$

This is an  $n$ -dimensional subvariety of  $\mathbb{P}^{n+k}$ . Then the holomorphic map

$$\pi: \begin{array}{ccc} Y & \longrightarrow & \mathbb{P}^n \\ [X_0 : \cdots : X_{n+k}] & \longmapsto & [X_0^m : \cdots : X_n^m] \end{array}$$

realises  $Y$  as a strictly adapted cover of the pair  $(\mathbb{P}^n, \Delta)$ , that is, a Galois covering ramifying exactly over the hyperplanes  $H_i$  with ramification order  $m$ . This is the *Fermat cover* associated with the pair  $(\mathbb{P}^n, \Delta)$ .

### 4.3 Ampleness of the orbifold cotangent bundle

In this context, we will prove the following analogue of 3.12.

#### Theorem 4.8

If the arrangement satisfies  $\star$  and for all  $i$ ,  $m_i \geq 2n$ , then the orbifold cotangent bundle  $\Omega_{(\mathbb{P}^n, \Delta)}$  is ample modulo  $|\Delta|$ .

**Proof:** To begin with, increasing the multiplicities  $m_i$  will always increase the ampleness of  $\Omega(\mathbb{P}^n, \Delta)$  (see [DR24]). Indeed, consider two orbifold divisors with the same support

$$\Delta = \sum_{i=0}^{n+k} (1 - 1/m_i) H_i, \Delta' = \sum_{i=0}^{n+k} (1 - 1/m'_i) H_i$$

such that for each  $i$ , one has  $m_i \leq m'_i$ . We can find a ramified covering  $\pi : X \rightarrow \mathbb{P}^n$  which is adapted for both  $\Delta$  and  $\Delta'$ . Namely, we have for each  $i$ ,

$$\pi^* D_i = p_i m_i E_i = p'_i m'_i E_i.$$

Thanks to the local expression of the sections given in  $(\star)$ , we see that  $\Omega(\pi, \Delta)$  is a subsheaf of  $\Omega(\pi, \Delta')$ , and similarly  $S^{[q]} \Omega_{(\mathbb{P}^n, \Delta)}$  is a subsheaf of  $S^{[q]} \Omega_{(\mathbb{P}^n, \Delta')}$ . This implies

$$\mathbb{B}_+(\Omega_{(X, \Delta')}) \subset \mathbb{B}_+(\Omega_{(X, \Delta)}).$$

Therefore, we can assume without loss of generality that all the  $m_i$  are equal.

Recall that in subsection 3.4, we characterised a point  $[V_1 : \cdots : V_{n+k}] \in \mathbb{P}^{n+k-1}$  in the image  $W$  of the map  $\Phi$  of  $\Omega_{\mathbb{P}^n}(\log |\Delta|)$  by the following equations, for some  $(z, [\xi]) \in \Omega_{\mathbb{P}^n}(\log |\Delta|)$ . We

work here in the the open subset  $\{Z_0 \neq 0\}$  with standard affine coordinates.

$$\left\{ \begin{array}{l} V_1 = \xi_1 \\ \vdots \\ V_n = \xi_n \\ (a_0^j + \sum_{i=1}^n a_i^j z_i) V_{n+j} = \sum_{i=1}^n a_i^j z_i \xi_i \end{array} \right\} \iff \left\{ \begin{array}{l} V_1 = \xi_1 \\ \vdots \\ V_n = \xi_n \\ a_{0,j} V_{n+j} + \sum_{i=1}^n a_i^j z_i (V_{n+j} - V_i) = 0 \end{array} \right.$$

$$\iff \ker M \neq \{0\}$$

$$\iff \operatorname{rk}(M) < n + 1.$$

Here  $M$  is the  $(n+1) \times k$ -matrix

$$\begin{pmatrix} a_0^{n+1}(V_{n+1} - V_0) & \cdots & a_n^{n+1}(V_{n+1} - V_n) \\ \vdots & \ddots & \vdots \\ a_0^{n+k}(V_{n+k} - V_0) & \cdots & a_{n,n+k}(V_{n+k} - V_n) \end{pmatrix},$$

setting  $z_0 = 1, \xi_0 = V_0 = 0$ . Hence  $W$  is described by the vanishing of the maximal minors of  $M$ . Moreover, these computations characterise the positive dimensional fibres of  $\Phi$  as the points  $V$  where

$$\operatorname{rk}(M) < n.$$

Denote by  $p$  the projection  $\mathbb{P}(\Omega_{\mathbb{P}^n}(\log|\Delta|)) \rightarrow \mathbb{P}^n$ . Our goal is to exhibit global orbifold forms vanishing on an ample divisor, whose base locus is contained in

$$p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1)))).$$

Consider a component of  $D$ , for instance  $H_1 = (z_1 = 0)$ .

The image  $\Phi(p^{-1}(H_1))$  is described by the equations

$$\left\{ \begin{array}{l} V_1 = \xi_1 \\ \vdots \\ V_n = \xi_n \\ (a_0^j + \sum_{i=2}^n a_i^j z_i) V_{n+j} = \sum_{i=2}^n a_i^j z_i \xi_i \end{array} \right\} \iff \left\{ \begin{array}{l} V_1 = \xi_1 \\ \vdots \\ V_n = \xi_n \\ a_0^j V_{n+j} + \sum_{i=2}^n a_i^j z_i (V_{n+j} - V_i) = 0 \end{array} \right.$$

$$\iff \ker M' \neq \{0\}$$

$$\iff \operatorname{rk}(M') < n$$

where  $M'$  is the  $n \times k$ -matrix

$$\begin{pmatrix} a_0^1(V_{n+1} - V_0) & a_2^1(V_{n+1} - V_2) & \cdots & a_n^1(V_{n+1} - V_n) \\ \vdots & \ddots & \vdots & \vdots \\ a_0^k(V_{n+k} - V_0) & a_2^k(V_{n+k} - V_2) & \cdots & a_n^k(V_{n+k} - V_n) \end{pmatrix}.$$

Again, a point  $V \in \mathbb{P}^{n+k-1}$  belongs to  $\Phi(\pi^{-1}(H_1))$  if all the  $n \times n$ -minors vanish at  $V$ . Hence, any  $n \times n$ -minor  $\Pi$  satisfies  $\Phi^* \Pi|_{\pi^{-1}(H_1)} \cong 0$ , so that

$$(p^* \ell_1)^{-1} \Phi^* \Pi \in H^0(\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D)), \mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(n) \otimes p^* \mathcal{O}_{\mathbb{P}^n}(-H_1)).$$

We deduce that for each such  $\Pi$

$$\text{Bs}(\Phi^*\Pi) \subset p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}^n(\log D)}(1))).$$

For instance, let  $\Pi$  be the minor of  $M'$  made of the first  $n$  rows.

Then  $(p^*\ell_1)^{-1}\Phi^*\Pi$  corresponds to a global symmetric form

$$\omega \in H^0(\mathbb{P}^n, S^n \Omega_{\mathbb{P}^n}(\log D)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1).$$

The form  $\omega$  has poles only along the hyperplanes  $H_0, H_2, \dots, H_{2n}$ . Moreover, each monomial in  $\omega$  has at most a simple pole along each component  $H_i$ .

Denote

$$\eta = \ell_0 \ell_2 \cdots \ell_{2n} \omega^{2n} \in H^0(\mathbb{P}^n, S^{2n^2} \Omega_{\mathbb{P}^n}(\log D)).$$

We will show that  $\eta$  is an orbifold form as soon as  $m \geq 2n$ .

Indeed, recall the Fermat cover  $\pi : Y \subset \mathbb{P}^{n+k} \rightarrow (\mathbb{P}^n, \Delta)$  given by

$$\pi([X_0 : \cdots : X_{n+k}]) = [X_0^m : \cdots : X_n^m].$$

In these coordinates, for all  $i$ , we have  $\pi^*\ell_i = X_i^m$  and  $\pi^*\frac{d\ell_i}{\ell_i} = m\frac{dX_i}{X_i}$ . Hence

$$\pi^*\eta = X_0^m X_2^m \cdots X_{2n}^m (\pi^*\omega)^{2n}.$$

But we know that each monomial in  $\omega^{2n}$  has pole order at most  $2n$  along each component  $H_i$ , so that we can write

$$\pi^*\eta = (X_0 X_2 \cdots X_{2n})^{m-2n} \tilde{\omega} \in H^0(Y, S^{2n^2} \Omega_Y)$$

for some holomorphic  $\tilde{\omega}$ . Hence, we have constructed an orbifold form vanishing along an ample divisor as

$$\omega^{2n} = (\ell_0 \ell_2 \cdots \ell_{2n})^{-1} \eta \in H^0(\mathbb{P}^n, S^{[2n^2]} \Omega_{(\mathbb{P}^n, \Delta)} \otimes \mathcal{O}_{\mathbb{P}^n}(-2n)).$$

As a consequence of this computation, we get the inclusion

$$\mathbb{B}_+(\Omega_{(\mathbb{P}^n, \Delta)}) \subset \text{Bs}(\omega^{2n}) \subset p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}^n(\log D)}(1))).$$

We conclude that for multiplicity  $m \geq 2n$ , the orbifold cotangent bundle of the pair  $(\mathbb{P}^n, \Delta)$  is ample modulo boundary whenever  $\Omega_{\mathbb{P}^n}(\log |\Delta|)$  is, and the statement of the theorem follows from [3.12](#). ■

Note the difference between our orbifold forms and the ones of [\[DR24\]](#). While the latter are explicitly constructed in coordinates, ours have a geometric interpretation and directly come from some specific logarithmic forms.

## 5 Applications

In this section, we give some applications of our results to complex hyperbolicity. Note that we do not claim any originality in some parts of the proofs below, which are exactly the same that in [\[DR24\]](#).

## 5.1 An orbifold Brody theorem

We recall the following definition for the Kobayashi pseudo-distance on a smooth manifold  $X$ : it is the largest pseudo-distance  $d_X: X \times X \rightarrow \mathbb{R}_+$  such that  $d_X \leq h^*(d_{\mathbb{D}})$ , for any holomorphic map  $h: \mathbb{D} \rightarrow X$ ,  $d_{\mathbb{D}}$  being the Poincaré distance on the unit disk  $\mathbb{D}$ . We can copy this definition and adapt it directly to the orbifold setting as below.

### Definition 5.1

The orbifold Kobayashi pseudo-distance  $d_{(X,\Delta)}$  on an orbifold  $(X, \Delta)$  is the largest pseudo-distance on  $X \setminus \text{Supp}[\Delta]$  such that any orbifold morphism  $h: (\mathbb{D}, \emptyset) \rightarrow (X, \Delta)$  is distance-decreasing with respect to the Poincaré distance on the unit disk, i.e.

$$d_{(X,\Delta)} \leq h^*d_{\mathbb{D}}.$$

**Remark 5.2** If one considers only classical morphisms  $h: \mathbb{D} \rightarrow (X, \Delta)$ , one obtains the classical Kobayashi pseudo-distance  $d_{(X,\Delta)}^*$ , with clearly  $d_{(X,\Delta)} \leq d_{(X,\Delta)}^*$ . Over  $X \setminus |\Delta|$ , one has

$$d_X \leq d_{(X,\Delta)} \leq d_{(X,\Delta)}^* \leq d_{X \setminus |\Delta|}.$$

As in the usual setting, there is an equivalent definition for the Kobayashi pseudo-distance using chains of holomorphic disks.

### Proposition 5.3

For any  $z, z' \in X \setminus \text{Supp}[\Delta]$ , the Kobayashi pseudo-distance  $d_{(X,\Delta)}$  is the infimum of the sums

$$\sum_i d_{\mathbb{D}}(p_i, q_i)$$

over all finite chains  $f_1, \dots, f_r$  of orbifold morphisms from  $\mathbb{D}$  to  $(X, \Delta)$  such that  $f_1(p_1) = z$ ,  $f_i(q_i) = f_{i+1}(p_i)$ ,  $f_r(q_r) = z'$ .

An immediate consequence of the definition is the distance-decreasing property of orbifold morphisms.

### Proposition 5.4

Let  $h: (X, \Delta) \rightarrow (X', \Delta')$  is an orbifold morphism (resp. classical orbifold morphism). Then

$$h^*d_{(X',\Delta')} \leq d_{(X,\Delta)}$$

(resp.  $h^*d_{(X',\Delta')}^* \leq d_{(X,\Delta)}^*$ ).

### Definition 5.5

The orbifold  $(X, \Delta)$  is hyperbolic (resp. classically hyperbolic) if the pseudo-distance  $d_{(X,\Delta)}$  (resp.  $d_{(X,\Delta)}^*$ ) is non-degenerate.

A corollary of 5.4 is

**Corollary 5.6** Let  $(X, \Delta)$  be a hyperbolic orbifold. Then every orbifold morphism  $f: (\mathbb{C}, \emptyset) \rightarrow (X, \Delta)$  is constant.

An orbifold entire curve inside  $(X, \Delta)$  will naturally be a non-constant orbifold morphism  $f: \mathbb{C} \rightarrow (X, \Delta)$ .

In [CW09], we can find the following generalisation of Brody's reparametrisation lemma to orbifolds.

**Theorem 5.7**

Let  $(X, \Delta)$  be a compact orbifold. Assume that  $(X, \Delta)$  is not hyperbolic, i.e.  $d_{(X, \Delta)}$  is not a distance. Then there exists a non-constant holomorphic map  $f : \mathbb{C} \rightarrow X$  which either is an orbifold morphism or satisfies  $f(\mathbb{C}) \subset |\Delta|$ . Furthermore,

$$\sup \|f'(z)\| = \|f'(0)\| > 0.$$

**Proposition 5.8** ([DR24])

Let  $(X, \Delta)$  be a smooth orbifold pair, with  $\Delta = \sum_i (1 - 1/m_i) \Delta_i$ . Assume that a sequence of orbifold maps  $h_n : D \rightarrow (X, \Delta)$  from the unit disk to  $(X, \Delta)$  converges locally uniformly to a holomorphic map  $h : D \rightarrow X$ . Let

$$X_h = \bigcap_{h(D) \subset \Delta_i} \Delta_i, \Delta_h = \bigcap_{h(D) \not\subset \Delta_i} \Delta_i.$$

Then  $h$  is an orbifold map  $D \rightarrow (X_h, \Delta_h)$ .

As an immediate consequence, reasoning exactly as in [CW09], we obtain the following result.

**Theorem 5.9**

Consider a smooth orbifold pair  $(X, \Delta)$  as above. For a subset of indices  $I$ , let  $\Delta_I = \bigcap_{i \in I} \Delta_i$ , and let

$$\Delta(I^c) = \sum_{j \notin I} (1 - 1/m_j) \Delta_j.$$

If all pairs  $(\Delta_I, \Delta(I^c)|_{\Delta_I})$  are Brody-hyperbolic, then the pair  $(X, \Delta)$  is Kobayashi-hyperbolic.

## 5.2 Application to orbifold hyperbolicity

We have proved the inclusion

$$\mathbb{B}_+(\Omega_{(\mathbb{P}^n, \Delta)}) \subseteq p(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(\Omega_{\mathbb{P}^n}(\log D))}(1))).$$

From the discussion in section 3.7, we deduce a bound on the dimension of the orbifold augmented base locus  $\mathbb{B}_+(\Omega_{(\mathbb{P}^n, \Delta)})$  away from  $\text{Supp } \Delta$ , in case  $\Omega_{(\mathbb{P}^n, \Delta)}$  is not ample modulo  $\text{Supp } \Delta$ .

As in [DR24], we infer from our Theorem 4.8 a result about algebraic degeneracy of orbifold curves.

**Corollary 5.10**

Consider the orbifold pair

$$\left( \mathbb{P}^n, \Delta = \sum_{1 \leq i \leq c} \left(1 - \frac{1}{m_i}\right) H_i \right)$$

formed by an arrangement of  $c$  hyperplanes in general position with orbifold multiplicities  $m_i \geq 2n$ . If the arrangement satisfies  $\star$ , then the orbifold pair  $(\mathbb{P}^n, \Delta)$  is Brody-hyperbolic. In fact,  $(\mathbb{P}^n, \Delta)$  is even Kobayashi-hyperbolic.

This result is not new and follows from Nochka's Theorem below. Since our proof completely differs, though, we have chosen to include it.

**Theorem 5.11 (Nochka [Noc83])**

Let  $f : \mathbb{C} \rightarrow \mathbb{P}^n$  be a holomorphic map and let  $d$  be the minimal integer such that the image of  $f$  is contained in a  $d$ -dimensional subspace. Let  $H_1, \dots, H_q$  be hyperplanes in general position in  $\mathbb{P}^n$ . Assume that the curve  $f$  intersects each  $H_i$  with multiplicity  $m_i$ . Then

$$\sum_{i=1}^q \left(1 - \frac{d}{m_i}\right) < 2n - d + 1.$$

As a consequence, if  $q \geq 2n + 2$  and  $m_i \geq 2n$ , the orbifold pair  $(\mathbb{P}^n, \Delta = \sum_{i=1}^q (1 - 1/m_i)H_i)$  is Brody-hyperbolic.

**Proof:** According to Theorem 5.9, if  $(\mathbb{P}^n, \Delta)$  is not hyperbolic, there exists either a non-constant orbifold curve  $f : \mathbb{C} \rightarrow (\mathbb{P}^n, \Delta)$  or an orbifold curve inside a stratum  $(\Delta_I, \Delta(I^{\mathbb{C}})|_{\Delta_I})$ . By 4.8 and 4.5, all orbifold entire curves  $\mathbb{C} \rightarrow (\mathbb{P}^n, \Delta)$  are constant. Hence we are left with the second possibility. But according to Lemma 3.19,  $(\Delta_I, \Delta(I^{\mathbb{C}})|_{\Delta_I})$  is again an orbifold pair satisfying the conditions of 4.8, so that it cannot contain any non-constant orbifold curve. ■

We use the description of the augmented base locus provided in 3.7 to refine 5.10.

**Theorem 5.12**

With the same assumptions, if the hyperplanes impose at least  $3n + k$  independent conditions on quadrics, with  $1 \leq k \leq n - 3$ , then there exists a Zariski-closed subset  $\Sigma$  of dimension  $k + 2$  containing all orbifold curves  $f : \mathbb{C} \rightarrow (\mathbb{P}^n, \Delta)$ .

**5.3 Hyperbolicity of Fermat covers**

Fermat hypersurfaces are one class of varieties for which several hyperbolicity results have been obtained. For instance, one has the two results of Green.

**Theorem 5.13 ( [Kob98], Ex. 3.10.21)**

Let

$$F(n, d) = \{[Z_0 : \dots : Z_{n+1}]; Z_0^d + \dots + Z_{n+1}^d = 0\} \subset \mathbb{P}^{n+1}$$

be the Fermat hypersurface of degree  $d$  in  $\mathbb{P}^{n+1}$ .

1. If  $d \geq (n+1)^2$ , then every entire curve  $f : \mathbb{C} \rightarrow F(n, d)$  has its image contained in a linear subspace of dimension  $\lfloor n/2 \rfloor$ .
2. If  $d > (n+1)(n+2)$ , then every entire curve  $f : \mathbb{C} \rightarrow \mathbb{P}^{n+1} \setminus F(n, d)$  has its image contained in a linear subspace of dimension  $\lfloor (n+1)/2 \rfloor$ .

These results are consequences of *Cartan's truncated defect relation* (see [Kob98], 3.B.42) which, with the orbifold terminology, gives the linear degeneracy of orbifold curves inside an orbifold pair  $(\mathbb{P}^n, \sum_{0 \leq i \leq n+1} (1 - 1/m)H_i)$  attached to an arrangement of  $n + 2$  general hyperplanes, provided that  $\sum_i (1 - n/m)^+ > n + 1$ . The hypersurface  $F(n, m)$  is precisely the Fermat cover associated with this orbifold.

We use now 4.8 to prove the hyperbolicity of Fermat covers with different assumptions.

### Theorem 5.14

The Fermat cover associated with an arrangement of hyperplanes in  $\mathbb{P}^n$  imposing at least  $4n - 2$  linearly independent conditions on quadrics, with ramification  $m \geq 2n$ , is Kobayashi-hyperbolic.

**Proof:** Let  $\pi : Y \rightarrow (\mathbb{P}^n, \Delta)$  be the Fermat cover. It suffices to prove the Brody-hyperbolicity. Let  $f : \mathbb{C} \rightarrow Y$  be an entire curve. According to 4.8, its image  $f(\mathbb{C})$  lies in the ramification locus of the covering  $\pi$ .

Note that the ramification locus can be seen as the Fermat cover associated to an arrangement of  $d \geq 4n - 3$  hyperplanes in  $\mathbb{P}^{n-1}$ . By lemma 3.19 above, one can still assume that the genericity conditions of 4.8 are satisfied. Hence we obtain the hyperbolicity of  $Y$ . ■

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