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Abstract

In this paper we prove that the time dependent solutions of a large class of Smoluchowski coagulation equations for multicomponent systems concentrate along a particular direction of the space of cluster compositions for long times. The direction of concentration is determined by the initial distribution of clusters. These results allow to prove the uniqueness and global stability of the self-similar profile with finite mass in the case of coagulation kernels which are not identically constant, but are constant along any direction of the space of cluster compositions.

Keywords: multicomponent Smoluchowski's equation; localization; time-dependent solutions; self-similarity; stability.

Mathematics Subject Classification: 45K05, 35Q82

Contents

1 Introduction

1.1 Motivation

In this work, we are concerned with the multicomponent Smoluchowski coagulation equation

$$\partial_t f(x,t) = \mathbb{K}_d[f](x,t), \quad x \in \mathbb{R}_*^d, \quad t \geq 0 \tag{1.1}$$

where the coagulation operator reads

$$\mathbb{K}_d[f](x,t) := \frac{1}{2}\int_{\{0<\xi<x\}} d\xi K(x-\xi,\xi) f(x-\xi,t) f(\xi,t) - \int_{\mathbb{R}_*^d} d\xi K(x,\xi) f(x,t) f(\xi,t) \tag{1.2}$$

with $x = (x_1, x_2, \ldots, x_d)$, $y = (y_1, y_2, \ldots, y_d)$. We use the the notation $\mathbb{R}_* = (0,\infty)$ and $\mathbb{R}_*^d = [0,\infty)^d \backslash \{O\}$, with $O = (0,0,...,0)$, as well as, the comparison notation: $x < y$ whenever $x \leq y$ componentwise and $x \neq y$. In particular, we thus have

$$\int_{\{0<\xi<x\}} d\xi = \int_0^{x_1} d\xi_1 \int_0^{x_2} d\xi_2 \cdots \int_0^{x_d} d\xi_d .$$

Equation (1.1)-(1.2) describes the evolution of the number density (or concentration), $f(x,t)$, of particles characterized by the composition vector, x, interacting by binary coalescence. The rate of coagulation between two particles is given by the coagulation kernel K. Equation (1.1)-(1.2) generalizes the well-known (one-component) Smoluchowski coagulation equation [23] where each particle is described by a scalar, instead of a vector. Contrarily to the one-component equation, multicomponent equations have been much less studied in the mathematics literature.

The multicomponent coagulation equation arises in applications as a model for systems in which the size is not the only characteristic of the particles affecting the rate of coagulation. It has been extensively used to analyse the evolution of chemical properties of aerosol particles in atmospheric science (cf. [21, 25]). See, for instance, [21] where the rate is also affected by the particle charge in the context of aerosol growth. Additional details about the physics of these systems can be found in [8], [9], [13].

Coagulation equations with two-components, i.e., $d = 2$, have been introduced in [15]. In that paper, the solutions to the coagulation equations with kernels of the form $K(x_1, x_2; y_1, y_2) = K_1(x_1 + x_2; y_1 + y_2)$ have been written in terms of the solutions of the one-component coagulation equation with kernel K_1. In particular, the solutions with the constant kernel $K(x,y) = 1$ are computed explicitly.

This paper is centred around a phenomenon that is specific to multicomponent coagulation equations and that we termed in [9] as *localization.* We say that the solutions to equation (1.1) localize if for large particle sizes they tend to concentrate along a line, more precisely, along a ray starting from the origin O, in the space of compositions. In the physics literature, it has been noticed that localization can be observed in the solutions of the discrete version of equation (1.1) for some particular kernels for which the solutions of the time dependent problem can be explicitly computed (cf. [14] and also [7] for further details). We also remark that the solutions of the continuous equation (1.1) can be computed in the case of the constant and the additive kernels using multicomponent Laplace transform [5, 15]. In all these cases the solutions concentrate for long times along a ray of the space of cluster concentrations.

In this paper we prove that the localization of the solutions of the time dependent problem (1.1), as well as of the solutions of the corresponding discrete problem where $x \in \mathbb{N}_*^d$, hold under very general conditions on the coagulation kernels $K(x,y)$ and initial conditions. Our results imply in particular that localization is a universal property of multicomponent coagulation systems. The direction of localization is determined by the initial distribution of clusters. This could be expected, due to the following reasoning. First we note that, formally, the following mass conservation property holds (1.1)

$$\partial_t \left(\int_{\mathbb{R}_*^d} x f(x,t)\, dx \right) = 0. \tag{1.3}$$

This vector identity represents the conservation of the different types of monomers for the solutions of (1.1). We will restrict our attention to coagulation kernels for which the so-called gelation phenomenon does not take place. Therefore, the mass-conservation identity (1.3) will be satisfied for the solutions considered in this paper. Assuming this conservation law (1.3) and localization, we can then deduce the direction of localization. For this, let us consider the standard 1-norm $|\cdot|$ on $\mathbb{R}^d$, using $|x| = \sum_{j=1}^d x_j$ for $x \in \mathbb{R}_*^d$. Now, if f at time t is concentrated along a ray in some fixed direction θ_0 with $|\theta_0| = 1$, then to each x in this region of concentration $x \approx |x|\theta_0$. But then the conserved total mass vector satisfies $m(t) := \int_{\mathbb{R}_*^d} x f(x,t)\, dx \approx \theta_0 \int_{\mathbb{R}_*^d} |x| f(x,t)\, dx$ which implies that $(\theta_0)_i \approx m_i(0)/|m(0)|$. Hence, for completely localized solutions the direction θ_0 is already determined by the initial mass vector. In other words, the conservation law (1.3) implies that the monomer composition of clusters, relative to the total number of monomers, along the ray follows the relative composition of monomers in the initial data. Interestingly, the localization direction is not encoded in any property of the coagulation kernel. Indeed, we show in this paper that localization takes place for a large class of coagulation kernels for which there is not any strongly preferred direction in the space of cluster compositions. Due to this we can think of localization as an emergent property of the coagulation dynamics.

The phenomenon of localization takes place also for multicomponent coagulation equations for which an additional source term is included. In [9] we provide a detailed study of the localization properties of stationary solutions of the continuous equation

$$\mathbb{K}_d[f](x) + \eta(x) = 0\,, \tag{1.4}$$

and its discrete counterpart, where the source $\eta(x) \geq 0$ satisfies suitable integrability conditions. Solutions to equation (1.4) have been proven to exist in [10] under general conditions on the kernels $K(x,y)$ satisfying the condition (1.6). It has been proved in [9] that those solutions are concentrated along rays in the space $\mathbb{R}_*^d$ for large values of $|x|$. We emphasize that, differently from the case studied in this paper, in the stationary regime the direction of the localization line depends on the first moments of the source term $\eta(x)$.

An important consequence of localization results is that they allow to reduce multicomponent coagulation equations to one-component equations. An example of how this idea can be applied to specific cases is given later in Theorem 1.5 where existing results for one-component explicitly solvable kernels are used to characterize the long-time behaviour of multicomponent systems with kernels for which it does not seem possible to obtain explicit solutions. Another important application of localization results pertains to numerical methods for multicomponent equations. The large computational effort and storage needed for the simulations in the multidimensional composition space make a straightforward implementation very challenging

[16]. Localization results allow reducing the dimensionality of the problem by means of approximating the multidimensional composition space by a region around a one-dimensional space, thus, greatly improving the numerical efficiency of the algorithms.

In the rest of this paper, we assume that the coagulation kernel K satisfies

$$K \in C((\mathbb{R}^d_*)^2),\ K(x,y) = K(y,x),\ K(x,y) \geq 0. \tag{1.5}$$

We require continuity of the kernels K in order to obtain meaningful formulas for measure-valued solutions f. In addition, we will assume that

$$K(x,y) \geq c_1(|x|+|y|)^\gamma \Phi_p\left(\frac{|x|}{|x|+|y|}\right) \quad , \quad x,y \in \mathbb{R}^d_* \tag{1.6}$$

$$K(x,y) \leq c_2(|x|+|y|)^\gamma \Phi_p\left(\frac{|x|}{|x|+|y|}\right) \quad , \quad x,y \in \mathbb{R}^d_* \tag{1.7}$$

with $\gamma \in \mathbb{R}$, and for some $p \in \mathbb{R}$ and $\Phi_p \in C(0,1)$ such that

$$\Phi_p(s) = \frac{1}{s^p(1-s)^p}, \quad 0 < s < 1, \text{ with } \gamma + 2p \geq 0\,, \tag{1.8}$$

and some constants $0 < c_1 \leq c_2 < \infty$. Note that then $\Phi_p(s) = \Phi_p(1-s)$ and thus the bounds are symmetric functions, due to $\frac{|y|}{|x|+|y|} = 1 - \frac{|x|}{|x|+|y|}$.

We remark that the estimates (1.6)–(1.8) are invariant under the permutation of the components $x_1, x_2, ..., x_d$. In particular, the kernels satisfying (1.6)–(1.8) cannot have different power law behaviour along any two different variables, say $x_j,\ x_k$ with $j \neq k$.

In order to avoid gelation, we will assume also the following conditions for the parameters γ and p in (1.6), (1.7), (1.8), (cf. [1])

$$\gamma + p < 1, \quad \gamma < 1. \tag{1.9}$$

We would like to point out that the class of kernels considered here is equivalent to another class commonly found in the literature, namely, of those kernels satisfying the power law bounds

$$\tilde{c}_1(|x|^{\gamma+\lambda}|y|^{-\lambda} + |y|^{\gamma+\lambda}|x|^{-\lambda}) \leq K(x,y) \leq \tilde{c}_2(|x|^{\gamma+\lambda}|y|^{-\lambda} + |y|^{\gamma+\lambda}|x|^{-\lambda})\,, \tag{1.10}$$

$$-\lambda < 1, \quad \gamma + \lambda < 1 \quad , \quad \gamma < 1\,,$$

for some constants $\tilde{c}_1$ and $\tilde{c}_2$. Given (1.10), we may choose $p = \max\{\lambda, -\gamma - \lambda\}$, for which $\gamma + p \geq -p$ due to $\gamma + 2p = |\gamma + 2\lambda|$. Conversely, given (1.6)-(1.7), we may choose $\lambda = p$ to obtain (1.10). We recall that the coagulation kernels depend on the specific mechanism which is responsible for the aggregation of the clusters at the microscopic level. In particular, the class of kernels (1.10) contains the physically relevant kernels that are often used in aerosol science, such as the free molecular (ballistic) kernel and the Brownian kernel (cf. [13, 25] as well as [9] for a more detailed discussion).

Notice that the kernels K satisfying (1.6), (1.7), (1.8) are bounded from above and below by homogeneous functions, but they are not necessarily themselves homogeneous. In some of the results presented later we will need to assume homogeneity, i.e., then we additionally require that

$$K(rx, ry) = r^\gamma K(x,y) \quad , \quad r > 0 \quad , \quad x,y \in \mathbb{R}^d_*\,. \tag{1.11}$$

In the one component case ($d = 1$), there are several results about solutions to equation (1.1). For example, assuming that (1.6), (1.7), (1.8), (1.9) are satisfied and that the initial mass ($\int_{\mathbb{R}^d_*} xf\,(x,t)\,dx$, respectively) is finite, then the mass becomes concentrated in the region of cluster sizes of order $\alpha \approx t^{\frac{1}{1-\gamma}}$ or $x \approx t^{\frac{1}{1-\gamma}}$, respectively, as $t \to \infty$. Moreover, if $d = 1$, $0 \le \gamma < 1$, $0 \le \gamma + p < 1$, and (1.11) holds, it is well known that self-similar solutions of (1.1) with the form $f\,(x,t) = \frac{1}{t^{\frac{2}{1-\gamma}}} F\left(\frac{x}{t^{\frac{1}{1-\gamma}}}\right)$ exist (cf. [3, 4, 11]). These self-similar solutions are expected to represent the long time asymptotics of the solutions of (1.1) in great generality, although this has been rigorously proven only for particular kernels from the class defined by (1.6), (1.7), (1.8), (1.9), specifically, only if the kernel K is constant [17] or a perturbation of a constant [2, 24]. We note though that it is also possible to obtain rigorously the long time asymptotics for other special kernels that do not satisfy (1.9), namely for the additive kernel $K(x,y) = x + y$ and the multiplicative kernel $K(x,y) = xy$ (cf. [17], [18]). In the multicomponent case ($d \ge 1$), representation formulas for the solutions of the initial value problem associated to (1.1) can also be obtained for the constant kernel $K(x,y) = 1$, the additive kernel $K(x,y) = |x| + |y|$ and the product kernel $K(x,y) = |x||y|$ (cf. [5, 6, 15]) using multicomponent Laplace transform methods.

Analogous estimates, which show that the mass of the clusters is concentrated in the self-similar region (i.e. $|x| \approx t^{\frac{1}{1-\gamma}}$) for large times t, can be derived in the case of multicomponent coagulation systems, by adapting in a suitable manner the methods used to prove these results in the case $d = 1$. In addition to such estimates, we further prove that the self-similar profiles are completely concentrated along a ray of the cluster space. As a final remark, we note that, except for particular kernels such as the constant kernel [18] or kernels that are 'close' to constant [19, 24], in general there are no uniqueness results for self-similar profiles available in the literature. Nevertheless, our result implies that all self-similar profiles are localized. Notably, for a special class of kernels that are constant only along rays, it is possible to obtain uniqueness and stability results by employing earlier results for the one-component equation with constant kernel (cf. Theorem 1.5).

1.2 Notations

We collect here for the reader's convenience the main notations and definitions which will be repeatedly used throughout the paper, some of these having been already introduced above.

We recall the already defined sets $\mathbb{R}_* = (0,\infty)$ and $\mathbb{R}^d_* = [0,\infty)^d \backslash \{O\}$. We use $|\cdot|$ and $\|\cdot\|$ to denote the following norms on $\mathbb{R}^d$,

$$|x| = \sum_{j=1}^{d} |x_j|\,, \quad \|x\| = \sqrt{\sum_{j=1}^{d} (x_j)^2}\,, \qquad \text{for } x \in \mathbb{R}^d_*,\ x = (x_1, x_2, ..., x_d)\,.$$

We denote by $C_c\left(\mathbb{R}^d_*\right)$ the set of compactly supported continuous functions in $\mathbb{R}^d_*$, and by $C^k_c\left(\mathbb{R}^d_*\right)$, for $k = 1, 2, \ldots$, the set of such compactly supported functions with k continuous derivatives. We use the notation $\mathscr{M}_1\left(\mathbb{R}^d_*\right)$ to denote the space of non-negative Radon measures on $\mathbb{R}^d_*$ with finite first moment, i.e., satisfying $\int_{\mathbb{R}^d_*} |x| f(dx) < \infty$. We will use indistinctly the notation $f\,(dx)$, $f\,(x)\,dx$, or f to denote a measure $f \in \mathscr{M}_1\left(\mathbb{R}^d_*\right)$. The former notation will be preferred when the measure is integrated against a test function. We stress that we will use the notation $f\,(x)\,dx$ to denote a measure on $\mathbb{R}^d_*$ even if this measure is not absolutely

continuous with respect to the Lebesgue measure on $\mathbb{R}^d_*$. We denote by $C\left([0,\infty);\mathscr{M}_1\left(\mathbb{R}^d_*\right)\right)$ the space of continuous functions on $[0,\infty)$ with values in $\mathscr{M}_1\left(\mathbb{R}^d_*\right)$ which is endowed with the weak-$*$ topology.

We denote by Δ^{d-1} the simplex

$$\Delta^{d-1} = \left\{\theta \in \mathbb{R}^d_* : |\theta| = 1\right\}, \tag{1.12}$$

and by $\mathscr{M}_+\left(\Delta^{d-1}\right)$ the space of non-negative Radon measures on Δ^{d-1}. Let $d\nu(\theta)$ denote the $(d-1)$ dimensional Hausdorff measure restricted to Δ^{d-1}. Then we denote by $\mathscr{M}_1\left(\mathbb{R}_* \times \Delta^{d-1}\right)$ the space of non-negative Radon measures, G, on $\mathbb{R}_* \times \Delta^{d-1}$ satisfying the moment bound $\int_{\mathbb{R}_*\times\Delta^{d-1}} \rho G(\rho,\theta) d\rho d\nu(\theta) < \infty$. We will denote by δ_{θ_0} or $\delta(\cdot - \theta_0)$ the Dirac measure supported at $\theta_0 \in \Delta^{d-1}$.

We will denote by C a generic constant which can depend on d and on the properties of the kernels (specifically, γ, p, as well as c_1 and c_2 in (1.6), (1.7)) but which is independent of the solution under consideration. The value of C may also change from line to line.

1.3 Main results

We now state the main results proved in this paper. The precise definitions will be given later in Section 2. We begin with our main localization result.

Theorem 1.1 *Let $f_0 \in \mathscr{M}_1(\mathbb{R}^d_*)$ satisfy $\int_{\mathbb{R}^d_*}(|x| + |x|^{1+r}) f_0(dx) < \infty$ for some $r > 0$. Define $m(0) := \int_{\mathbb{R}^d_*} x f_0(dx) \in \mathbb{R}^d$, denote $m_0 = |m(0)|$, and suppose that $m_0 > 0$. Let the coagulation kernel K satisfy the assumptions (1.5), (1.6), (1.7), (1.8) with $0 \le \gamma < 1$, and $0 \le \gamma + p < 1$. Then there exists a weak solution $f \in C\left([0,\infty);\mathscr{M}_1\left(\mathbb{R}^d_*\right)\right)$ to (1.1), (1.2) such that $f(\cdot,0) = f_0$ with the following properties. This solution is mass-conserving: $\int_{\mathbb{R}^d_*} x f(x,t)\, dx = m(0)$ for all $t \ge 0$, and in addition it satisfies*

$$\int_{\mathbb{R}^d_*} |x|^k f(x,t)\, dx \le C_0 t^{\frac{k-1}{1-\gamma}}, \quad t \ge 1, \tag{1.13}$$

for some $k > 1$ and $C_0 > 0$. Moreover, there exists a function $\delta(\cdot) \in C([0,\infty))$ such that $\delta(t) > 0$ for $t \in [1,\infty)$ and $\lim_{t\to\infty} \delta(t) = 0$ and for which

$$\lim_{t\to\infty} \left| \int_{\left\{\delta(t) t^{\frac{1}{1-\gamma}} \le |x| \le (\delta(t))^{-1} t^{\frac{1}{1-\gamma}}\right\} \cap \left\{\left|\frac{x}{|x|} - \theta_0\right| \le \delta(t)\right\}} |x| f(x,t)\, dx - m_0 \right| = 0 \tag{1.14}$$

where

$$\theta_0 = \frac{\int_{\mathbb{R}^d_*} x f_0(x)\, dx}{m_0} \in \Delta^{d-1} \quad , \quad |\theta_0| = 1. \tag{1.15}$$

Remark 1.2 *The crucial information about the function $\delta(t)$ is that it converges to zero. Therefore, (1.14) implies that the mass is localized along a particular direction in distances $|x|$ of order $t^{\frac{1}{1-\gamma}}$ for all the large times t as $t \to \infty$, i.e., localization of the measure $|x| f(x,t)\, dx$ takes place as $t \to \infty$. Notice that the vector θ_0 defined in (1.15) only depends on the conserved quantities, just as was discussed in the Introduction.*

We remark that a discrete version of Theorem 1.1 will be presented in Section 7.

Notice that Theorem 1.1 yields localization for a particular weak solution of the initial value problem (1.1), (1.2) with initial value $f(0,.) = f_0$. The reason why the localization result is not stated for every weak solution is due to the lack of a uniqueness theory. Indeed, the arguments used in the proof of Theorem 1.1 rely on the results of [4] that only ensure existence of a weak solution to (1.1), (1.2) satisfying (1.13) with initial value f_0, but no uniqueness is proved in [4]. A theory of uniqueness of weak solutions combined with Theorem 1.1 would then imply localization for all weak solutions of (1.1), (1.2). The derivation of such results for weak solutions is not the goal of this paper. Uniqueness results in the one-component case $d = 1$, for some kernels satisfying the upper bound (1.7) with $\gamma \leq 1$ and $p = 0$ as well as additional regularity conditions have been obtained in [12]. The condition (1.13) ensures that most of the mass of the solution remains in the self-similar region. We expect the estimate (1.13) to hold for all weak solutions to (1.1), (1.2) which decay sufficiently fast for large $|x|$ and for the physically relevant kernels with homogeneity smaller than 1. It turns out that it is possible to obtain a slightly weaker localization result for all solutions to (1.1), (1.2) satisfying the moment estimate (1.13) for a more general class of kernels than the one considered in Theorem 1.1. More precisely we have the following result.

Theorem 1.3 *Suppose that $f \in C\left([0,\infty); \mathscr{M}_1\left(\mathbb{R}^d_*\right)\right)$ is a weak solution of (1.1) such that $0 < \int_{\mathbb{R}^d_*} |x| f(x,t)\, dx = m_0 < \infty$ and such that the assumptions* (1.5)*, (1.6), (1.7), (1.8) hold with γ, p satisfying (1.9). Assume that there are $a > 1$ and $C_0 > 0$ such that f satisfies (1.13) for all $k \in [1/a, a]$. Then, there exists a function $\delta(\cdot) \in C([0,\infty))$ such that $\delta(t) > 0$ for $t \in [1,\infty)$ and $\lim_{t\to\infty} \delta(t) = 0$ as well as a Borel set $I \subset [0,\infty)$ with the property that $\lim_{T\to\infty} \frac{|I\cap[T,2T]|}{T} = 0$ such that*

$$\lim_{T\to\infty} \left(\sup_{t\in[T,2T]\setminus I} \left| \int_{\left\{\delta(t) t^{\frac{1}{1-\gamma}} \leq |x| \leq (\delta(t))^{-1} t^{\frac{1}{1-\gamma}}\right\} \cap \left\{\left|\frac{x}{|x|} - \theta_0\right| \leq \delta(t)\right\}} |x| f(x,t)\, dx - m_0 \right| \right) = 0 \quad (1.16)$$

where θ_0 is as in (1.15).

Notice that the main difference between Theorems 1.1 and 1.3 is that in the first case we assume a more restricted set of parameters γ and p. On the other hand, we obtain stronger localization results in the case of Theorem 1.1. In (1.16) we allow for the existence of a set of times $I \subset (1,\infty)$ whose density converges to zero for large values of t and for which the localization property could fail. We do not know if it is possible to have solutions of (1.1) for which localization does not take place for a small set of large times. Most likely such a type of behaviour does not take place for any solution of (1.1). However, only the estimate (1.16) can be obtained from the assumptions on the solutions to (1.1) considered in Theorem 1.3. In fact, estimates ensuring that the mapping $t \mapsto (1+t)^{\frac{d+1}{1-\gamma}} f((t+1)^{\frac{1}{1-\gamma}}\cdot, t)$ is uniformly continuous in the weak-$*$ topology would yield the stronger localization result (1.14), but this would require analysis going much beyond the currently available well posedness results.

We will discuss in Sections 4 and 5 sufficient conditions for the moment estimates (1.13) to be satisfied for the range of parameters $0 \leq \gamma < 1$, $0 \leq \gamma + p < 1$, which in particular is contained in the range defined by (1.9). In the case of moments $k > 1$ we need to assume suitable conditions on the initial data f_0.

We will also study localization properties for the self-similar solutions of (1.1) with $d > 1$. The mass conserving self-similar solutions are solutions of (1.1) with the form:

$$f(x,t) = (\varepsilon_t)^{1+d} F(x\varepsilon_t) \quad , \quad \xi = x\varepsilon_t \quad , \quad \varepsilon_t = (t+1)^{-\frac{1}{1-\gamma}}. \tag{1.17}$$

The existence of solutions of (1.1) with the form (1.17) under the assumptions (1.6), (1.7), (1.8), (1.9), (1.11) and for $0 \leq \gamma < 1$, $0 \leq \gamma + p < 1$ has been proved in the case $d = 1$ in [3, 4, 11]. Using these results it is possible to prove the existence of self-similar solutions in the multicomponent case $d > 1$ under analogous assumptions on the collision kernels and having the particular form

$$F(\xi) = \frac{\sqrt{d}}{|\xi|^{d-1}} F_0(|\xi|)\, \delta\left(\frac{\xi}{|\xi|} - \theta_0\right) \tag{1.18}$$

where $\theta_0 \in \Delta^{d-1}$, $\delta \in \mathscr{M}_+(\Delta^{d-1})$ is supported at θ_0 and F_0 is a self-similar profile for a suitable one-dimensional coagulation equation. The existence of self-similar profiles with the form (1.18) will be seen in Section 4.

It turns out that all the solutions of (1.1) with the form (1.17) and satisfying suitable integrability conditions for both small and large $|\xi|$, have the form (1.18). This result can be interpreted as a localization result analogous to the Theorems 1.1, 1.3 for solutions of (1.1). The precise localization result for self-similar solutions that we will prove in this paper is the following.

Theorem 1.4 *Suppose that the kernel K satisfies the assumptions* (1.5)*, (1.6), (1.7), (1.8), (1.9), as well as the homogeneity condition* (1.11)*. Suppose that $F \in \mathscr{M}_1\left(\mathbb{R}^d_*\right)$ is a self-similar profile with finite mass for (1.1) in the sense of Definition 2.3. Then, there exists $\theta_0 \in \Delta^{d-1}$ such that F has the form (1.18) where F_0 is a self-similar profile associated to the one component coagulation equation (i.e. $d = 1$) and coagulation kernel $K_{\theta_0}(s,r) = K(s\theta_0, r\theta_0)$, $s, r \in \mathbb{R}_*$.*

An interesting consequence of the localization results contained in Theorem 1.1 is that they allow to characterize the long time asymptotics for a class of coagulation kernels for which it does not seem feasible to obtain an explicit representation formula for the solutions. We recall that in the case of one component systems a complete characterization of the long time asymptotics for arbitrary initial data has been obtained only for coagulation kernels with homogeneity smaller than one for which it is possible to obtain representation formulas of the solutions using Laplace transform methods (cf. [17, 18]), or for kernels K which are close to the constant kernel (cf. [24]).

We will combine the localization results obtained in this paper (cf. Theorem 1.1) with the characterization of the long time asymptotics obtained in [17, 18] to characterize the long time behaviour of the solutions of coagulation equations with kernels that are constant along each ray that passes through the origin. This is due to the fact that the localization of the solutions along a ray allows to approximate the behaviour of the solutions by a one-component coagulation equation with a constant kernel. It does not seem feasible to derive an explicit formula using Laplace transform methods for the class of kernels with the form (1.19) below, except for some very particular choices of the function Q. We have the following result.

Theorem 1.5 *Suppose that the kernel K satisfies* (1.5) *and has the form*

$$K(r\theta, s\theta) = Q(\theta) \tag{1.19}$$

for any $r, s > 0$ and for any $\theta = (\theta_1, \theta_2, ..., \theta_d) \in \Delta^{d-1}$. Here Q is a continuous function defined on Δ^{d-1} and $0 < c_1 \leq Q(\theta) \leq c_2$. Let $f_0 \in L^1(\mathbb{R}^d_*)$ be a nonnegative function satisfying

$$m_0 := \int_{\mathbb{R}^d_*} |x| f_0(x)\, dx > 0 \tag{1.20}$$

and also

$$\int_{\mathbb{R}^d_*} |x|^a f_0(x)\, dx < \infty \tag{1.21}$$

for some $a > 1$. Then there exists a function $f \in C([0,\infty); L^1(\mathbb{R}^d_*)) \cap C^1((0,\infty); L^1(\mathbb{R}^d_*))$ that solves (1.1) in the classical sense, satisfying $f(x,0) = f_0(x)$ and

$$\int_{\mathbb{R}^d_*} x f(x,t) dx = m := \int_{\mathbb{R}^d_*} x f_0(x) dx, \ t > 0.$$

Moreover, using the variable $\xi = \frac{x}{t}$, we have

$$\lim_{t\to\infty} t^2 f(t\xi, t) = F_0(|\xi|; \theta_0)\, \delta\left(\frac{\xi}{|\xi|} - \theta_0\right)$$

where the convergence takes place in the weak$-*$ topology of $\mathscr{M}_1(\mathbb{R}^d_*)$ and where

$$\theta_0 := \frac{m}{m_0} \in \Delta^{d-1} \quad \text{and}$$

$$F_0(|\xi|; \theta_0) := \frac{4\sqrt{d}}{(Q(\theta_0))^2 m_0} \frac{1}{|\xi|^{d-1}} \exp\left(-\frac{2|\xi|}{Q(\theta_0) m_0}\right).$$

Notice that the class of kernels considered here satisfies the growth bounds (1.6)-(1.7) with $\gamma = p = 0$. Notice also that the mass vector $\int_{\mathbb{R}^d_*} f(x,t)\, x dx = m$ remains constant for arbitrary values of $t \geq 0$. Theorem 1.5 states that for each value of $m \in \mathbb{R}^d_*$ there exists a unique self-similar solution of the form (1.17), (1.18) with $\theta_0 = \frac{m}{m_0}$ which is a global attractor for the solutions of (1.1) satisfying $m = \int_{\mathbb{R}^d_*} f_0(x)\, x dx$ as well as the moment estimate (1.21).

It seems possible to extend Theorem 1.5 to initial values f_0 in some measure spaces. A technical problem which arises if we try to replace the space $L^1(\mathbb{R}^d_*)$ by the space $\mathscr{M}_1(\mathbb{R}^d_*)$ is that the kernels K with the form (1.19) are not necessarily continuous at $x = y = 0$ and therefore it is not possible to define the products $K(x,y) f(dx) f(dy)$. In order to avoid these technicalities we prefer to use the space $L^1(\mathbb{R}^d_*)$.

1.4 Plan of the paper

The plan of this paper is the following. In Section 2, we introduce several definitions and notation that will be used in the rest of the paper. In Section 3, we prove the localization results for the time dependent solutions and for the self-similar solutions. Specifically, we prove Theorems 1.1, 1.3, and 1.4. The proof of these results is based on the use of particular test functions that allow to measure the dispersion of the solutions around the localization

line. Similar test functions have also been used to obtain localization results for stationary solutions in [9]. A difference with the stationary case is that time-dependent solutions exhibit an additional localization property. Besides concentrating around a ray in the composition space, time-dependent solutions also concentrate around the self-similar region $|x| \approx t^{\frac{1}{1-\gamma}}$. This requires the derivation of new estimates to ensure that the contribution of the regions $|x| \ll t^{\frac{1}{1-\gamma}}$ and $|x| \gg t^{\frac{1}{1-\gamma}}$ is negligible. On the contrary, in the stationary solutions treated in [9] there is no characteristic cluster size in which most of the mass of the solutions is concentrated.

Section 4 collects several well posedness results and moment estimates for the solution of the coagulation equation (1.1). These results are well known in the case of one component coagulation systems and their proof can be readily adapted to the multicomponent case. The results in Section 4 ensure the existence of solutions with the properties required in Theorems 1.1 and 1.3 for a suitable set of parameters γ and p and a large class of initial data. We also prove in this section that the measures with the form (1.18) yield self-similar solutions of the multicomponent coagulation equation, obtained in terms of a self-similar solution F_0 of a suitable one-component coagulation equation. Section 5 contains the proof of certain moment estimates which provide some of the key properties of the time-dependent solutions necessary to prove the localization results. In particular, these estimates ensure that the mass of the solutions remains within the self-similar region, $|x| \approx t^{\frac{1}{1-\gamma}}$ as $t \to \infty$, for a large class of initial data. Although they are well known in the case of one-component coagulation systems, these estimates play a crucial role in the proof of our localization results, and we have written in detail a generalization of their proof to the multicomponent coagulation case. In Section 6, we study the long time asymptotics to the solutions of the multicomponent coagulation equation with kernels satisfying (1.19). In particular, the proof of Theorem 1.5 is given in this section. In Section 7 we present an application of the results obtained in this paper to the case of discrete systems, which are more commonly found in the applied literature. More precisely, we formulate a discrete version of Theorem 1.3. The results for the discrete equation (7.1) are of interest to several applications, for instance, in the study of aerosol growth [13].

2 Definitions and auxiliary results

In this section we provide the definition of weak solutions and self-similar profiles that will be used in the following. We also collect, without proof, several results for the multicomponent coagulation equation (1.1) that are well known for one-component coagulation systems and can be proved for multicomponent coagulation systems by means of simple adaptations of the methods used to derive them in the one-component case.

We now introduce the definitions of solutions to (7.1), (1.1). We formulate the definition of solution in the continuous case (1.1) since the discrete case (7.1) can be considered as a particular case of solutions f having the form (7.2).

Definition 2.1 *Let K be as in (1.5) and satisfy the upper bound* (1.7), (1.8). *Let $f_0 \in \mathscr{M}_1(\mathbb{R}^d_*)$. A function $f \in C([0,\infty); \mathscr{M}_1(\mathbb{R}^d_*))$ is called a weak solution to* (1.1) *with initial value f_0 if $f(0,\cdot) = f_0(\cdot)$ and for each $1 < T < \infty$*

$$\sup_{t\in[1/T,T]} \Big[\int_{\{|x|\geq 1\}} |x|^{\gamma+p} f(dx,t) + \int_{\{|x|\leq 1\}} |x|^{1-p} f(dx,t)\Big] < \infty, \tag{2.1}$$

$$\int_{\mathbb{R}^d_*} x f(dx,t) = \int_{\mathbb{R}^d_*} x f_0(dx), \quad t>0\,, \tag{2.2}$$

and, for all test functions $\varphi \in C^1_c(\mathbb{R}^d_* \times (0,\infty))$ *the following identity holds*

$$\begin{aligned} 0 &= \int_0^\infty \int_{\mathbb{R}^d_*} f(dx,t)\partial_t\varphi(x,t)dt \\ &+ \frac{1}{2}\int_0^\infty \int_{\mathbb{R}^d_*}\int_{\mathbb{R}^d_*} K(x,y) f(dx,t) f(dy,t)[\varphi(x+y,t)-\varphi(x,t)-\varphi(y,t)]dt. \end{aligned} \tag{2.3}$$

Remark 2.2 *Notice that the condition* (2.1) *is equivalent to*

$$\sup_{t\in[1/T,T]}\Big[\int_{\{|x|\geq a\}} |x|^{\gamma+p} f(dx,t) + \int_{\{|x|\leq a\}} |x|^{1-p} f(dx,t)\Big] < \infty, \tag{2.4}$$

for any $a>0$. *In order to define the solutions it would be enough to impose an integrability condition in* t *and* x. *However we decided to stick to the stronger condition* (2.1) *as it allows us to use estimates derived in [4].*

Note that in Definition 2.1 we allow only solutions with finite mass that is conserved over time for each component due to condition (2.2). The assumption (2.1) ensures that all the integrals appearing in (2.3) are well-defined for kernels satisfying the upper bound (1.7). Indeed, the last term in (2.3) can be estimated by splitting the domain of integration into two regions defined by $\{|y|\leq|x|\}$ and $\{|x|>|y|\}$. Using a symmetrization argument, the integral over the second region can be estimated by the integral over the first region and therefore, (2.3) can be estimated as

$$\int_0^\infty \int_{\{|y|\leq|x|\}} \int K(x,y) f(dx,t) f(dy,t)\,|\varphi(x+y,t)-\varphi(x,t)-\varphi(y,t)|\,dt. \tag{2.5}$$

Since $\varphi \in C^1_c(\mathbb{R}^d_*\times(0,\infty))$, there exists a function $\psi \in C_c(\mathbb{R}^d_*)$ such that $|\varphi(x+y,t)-\varphi(x,t)|\leq\psi(x)|y|$. Given a positive constant $L>1$, let $\operatorname{supp}\varphi(\cdot,t)\subset\{x\in\mathbb{R}^d_* \mid \frac{1}{L}<|x|<L\}$, for any $t\in(0,\infty)$, and $\operatorname{supp}\psi\subset\{x\in\mathbb{R}^d_* \mid \frac{1}{L}<|x|<L\}$. We can then rewrite the integrals using a smaller integration region, yielding,

$$\begin{aligned} &\iint_{\{|y|\leq|x|\}} K(x,y) f(dx,t) f(dy,t)|\varphi(x+y,t)-\varphi(x,t)| \\ &\qquad + \iint_{\{|y|\leq|x|\}} K(x,y) f(dx,t) f(dy,t)\,|\varphi(y,t)| = \\ &= \int_{\{\frac{1}{L}\leq|x|\leq L\}}\int_{\{|y|\leq|x|\}} (\cdots) + \int_{\{\frac{1}{L}\leq|y|\leq L\}}\int_{\{|y|\leq|x|\}} (\cdots) \\ &\leq \int_{\{\frac{1}{L}\leq|x|\leq L\}}\int_{\{|y|\leq|x|\}} K(x,y)\psi(x)|y| f(dx,t) f(dy,t) + \\ &+ \int_{\{\frac{1}{L}\leq|y|\leq L\}}\int_{\left\{\frac{1}{L}\leq|x|\right\}} K(x,y) f(dx,t) f(dy,t)|\varphi(y,t)|. \end{aligned}$$

Using now the upper bound (1.10) with $\lambda=p$ and using the property that Radon measures are locally finite, we obtain that there is a positive constant C, depending on L, $\tilde{c_2}$, γ, p, φ

and ψ, such that

$$\begin{aligned}
&\int_{\{\frac{1}{L}\le|x|\le L\}}\int_{\{|y|\le|x|\}} K(x,y)\psi(x)|y|f(dx,t)f(dy,t)+\\
&+\int_{\{\frac{1}{L}\le|y|\le L\}}\int_{\left\{\frac{1}{L}\le|x|\right\}} K(x,y)f(dx,t)f(dy,t)|\varphi(y,t)|\\
&\le C\int_{\{|y|\le L\}}\left(|y|^{-p}|y|+|y|^{\gamma+p}|y|\right)f(dy,t)+C\int_{\left\{\frac{1}{L}\le|x|\right\}}\left(|x|^{\gamma+p}+|x|^{-p}\right)f(dx,t)\\
&\le C\int_{\{|y|\le L\}}|y|^{1-p}f(dy,t)+C\int_{\left\{\frac{1}{L}\le|x|\right\}}|x|^{\gamma+p}f(dx,t)<\infty\,.
\end{aligned}$$

The finiteness of the integrals follows from the assumption (2.1) (more precisely it follows from (2.4) with $a=1/L, a=L$). This concludes the proof that the coagulation term in (2.3) is well-defined.

We remark that using a standard limit argument, we obtain that (2.3) implies that the following identity holds for test functions $\varphi\in C^1_c(\mathbb{R}^d_*\times[0,\infty))$,

$$\begin{aligned}
0=&\int_{\mathbb{R}^d_*}f_0(dx)\varphi(x,0)+\int_0^\infty\int_{\mathbb{R}^d_*}f(dx,t)\partial_t\varphi(x,t)dt\\
&+\frac{1}{2}\int_0^\infty\int_{\mathbb{R}^d_*}\int_{\mathbb{R}^d_*}K(x,y)f(dx,t)f(dy,t)[\varphi(x+y,t)-\varphi(x,t)-\varphi(y,t)]dt.
\end{aligned}\tag{2.6}$$

The localization result in Theorem 1.4 concerns self-similar solutions. In the next definition we collect the properties required for the self similar solutions, and more specifically for the self-similar profiles F (cf. (1.17)) which we need in order to derive the localization result.

Definition 2.3 *Let the kernel K satisfy* (1.5)*, (1.6), (1.7), (1.8), (1.9) as well as the homogeneity condition (1.11). We say that a measure $F\in\mathscr{M}_1(\mathbb{R}^d_*)$ satisfying*

$$\int_{\{|\xi|\ge1\}}|\xi|^{\gamma+p}F(d\xi)+\int_{\{|\xi|<1\}}|\xi|^{1-p}F(d\xi)+\int_{\mathbb{R}^d_*}|\xi|F(d\xi)<\infty\tag{2.7}$$

is a self-similar profile to (1.1) *if the following identity holds*

$$\begin{aligned}
0=&\frac{1}{2}\int_{\mathbb{R}^d_*}\int_{\mathbb{R}^d_*}F(d\xi)F(d\eta)K(\xi,\eta)[\psi(\xi+\eta)-\psi(\xi)-\psi(\eta)]\\
&+\frac{1}{1-\gamma}\int_{\mathbb{R}^d_*}F(d\xi)\left[\psi-\xi\cdot\partial_\xi\psi\right]
\end{aligned}\tag{2.8}$$

for all $\psi\in C^1_c(\mathbb{R}^d_*)$.

The finiteness of the integrals in (2.7) provide the integrability required to ensure that the integrals in (2.8) are well defined. On the other hand, the finiteness of the last integral in (2.7) implies that the total number of monomers associated to the function f defined by means of (1.17) is finite. Notice that a self-similar profile can be interpreted as a weak solution of the equation

$$\mathbb{K}[F]+\frac{1}{1-\gamma}\xi\cdot\partial_\xi F+\frac{d+1}{1-\gamma}F=0\quad,\quad\xi\in\mathbb{R}^d_*\tag{2.9}$$

with $\mathbb{K}[F]$ as in (1.2). Equation (2.9) may be obtained formally from the coagulation equation (1.1) using the change of variables (1.17). This change of variables can be made precise by noticing that given a self-similar profile F in the sense of Definition 2.3 we can obtain a weak solution f of (1.1) in the sense of Definition 2.1 by requiring

$$\iint_{\mathbb{R}_*^d\times[0,\infty)} f(x,t)\varphi(x,t)dxdt = \iint_{\mathbb{R}_*^d\times[0,\infty)} \varepsilon_t F\left(\xi\right)\varphi(\xi(\varepsilon_t)^{-1},t)d\xi dt, \tag{2.10}$$

for any $\varphi \in C_c(\mathbb{R}_*^d \times [0,\infty))$ and where $\varepsilon_t = (1+t)^{-\frac{1}{1-\gamma}}$. We have the following result.

Proposition 2.4 *Suppose that K satisfies (1.5), (1.6), (1.7), (1.8), (1.9) as well as the homogeneity condition (1.11). Let us assume also that $F \in \mathscr{M}_1(\mathbb{R}_*^d)$ is a self-similar profile in the sense of Definition 2.3. We define f for $t \geq 0$ as in (1.17) (cf. also (2.10)). Then, f is a weak solution of (1.1) in the sense of Definition 2.1 with initial value $f_0 = F$, satisfying the moment bounds for all $T > 0$*

$$\sup_{t\in[0,T]}\left[\int_{\{|x|\geq 1\}} |x|^{\gamma+p} f(dx,t) + \int_{\{|x|\leq 1\}} |x|^{1-p} f(dx,t)\right] < \infty.$$

Moreover, f is invariant under the following group of transformations:

$$f_\lambda\left(x,t\right) = \lambda^{d+1} f\left(\lambda x, \lambda^{1-\gamma}\left(t+1\right)-1\right) \quad , \quad \lambda > 0. \tag{2.11}$$

Remark 2.5 *We notice that f_λ and f are measures and (2.11) should be interpreted as follows, for any $\varphi \in C_c(\mathbb{R}_*^d \times [0,\infty))$,*

$$\iint_{\mathbb{R}_*^d\times[0,\infty)} f_\lambda(x,t)\varphi(x,t)dxdt = \lambda^\gamma \iint_{\mathbb{R}_*^d\times[0,\infty)} f(x,t)\varphi\left(\frac{x}{\lambda}, \frac{t+1}{\lambda^{1-\gamma}} - 1\right)dxdt.$$

Proof: Given that f is defined in (1.17) it follows from (2.7) that (2.1) holds. We now compute the right-hand side of (2.6). Notice that (1.17) implies that $f_0\left(x\right) = F\left(x\right)$. Then

$$\begin{aligned}
&\int_{\mathbb{R}_*^d} f_0(dx)\varphi(x,0) + \int_0^\infty \int_{\mathbb{R}_*^d} f(dx,t)\partial_t\varphi(x,t) \\
&+ \frac{1}{2}\int_0^\infty \int_{\mathbb{R}_*^d}\int_{\mathbb{R}_*^d} K(x,y)f(dx,t)f(dy,t)[\varphi(x+y,t)-\varphi(x,t)-\varphi(y,t)]dt \\
&= \int_{\mathbb{R}_*^d} F(dx)\varphi(x,0) + \int_0^\infty \int_{\mathbb{R}_*^d} f(dx,t)\partial_t\varphi(x,t) \\
&+ \frac{1}{2}\int_0^\infty \int_{\mathbb{R}_*^d}\int_{\mathbb{R}_*^d} K(x,y)f(dx,t)f(dy,t)[\varphi(x+y,t)-\varphi(x,t)-\varphi(y,t)]dt \\
&:= J.
\end{aligned}$$

Given a test function $\varphi \in C_c^1(\mathbb{R}_*^d \times [0,\infty))$ we define $\psi\left(\xi,\tau\right)$ by means of

$$\varphi\left(x,t\right) = (\varepsilon_t)^{-1}\psi\left(\xi,\tau\right) \tag{2.12}$$

where

$$\xi = x\varepsilon_t \quad , \quad \tau = \log\left(t+1\right) \quad , \quad \varepsilon_t = (1+t)^{-\frac{1}{1-\gamma}}.$$

Then, using also (1.17) as well as the homogeneity condition (1.11) and $d\tau = \frac{dt}{t+1}$ we obtain

$$J = \int_{\mathbb{R}^d_*} F(d\xi)\psi(\xi,0) + \int_0^\infty d\tau \int_{\mathbb{R}^d_*} F(d\xi)\left[\partial_\tau\psi + \frac{\psi}{1-\gamma} - \frac{1}{1-\gamma}\xi\cdot\partial_\xi\psi\right](\xi,\tau)$$
$$+ \frac{1}{2}\int_0^\infty d\tau \int_{\mathbb{R}^d_*}\int_{\mathbb{R}^d_*} K(\xi,\eta)\,F(d\xi)F(d\eta)[\psi(\xi+\eta,\tau) - \psi(\xi,\tau) - \psi(\eta,\tau)].$$

Employing (2.8), we then find

$$J = \int_{\mathbb{R}^d_*} F(d\xi)\psi(\xi,0) + \int_0^\infty d\tau \int_{\mathbb{R}^d_*} F(d\xi)\partial_\tau\psi = \int_{\mathbb{R}^d_*} F(d\xi)\psi(\xi,0) - \int_{\mathbb{R}^d_*} F(d\xi)\psi(\xi,0) = 0$$

and the result follows. □

3 Proof of the localization results (Theorems 1.1, 1.3, 1.4)

3.1 Mass localization along a ray in time-dependent solutions

In this subsection we prove Theorems 1.1 and 1.3. To this end, it is convenient to rewrite the function f in the theorems using the set of self-similar variables

$$f(x,t) = (\varepsilon_t)^{1+d} F(\xi,\tau), \quad \xi = x\varepsilon_t, \quad \tau = \log(t+1), \quad \varepsilon_t = (t+1)^{-\frac{1}{1-\gamma}} \tag{3.1}$$

where f is a weak solution of (1.1) in the sense of definition 2.1. In order to prove the localization results it will be convenient to further define a set of simplicial coordinates

$$(\rho,\theta) = \left(|\xi|, \frac{\xi}{|\xi|}\right) \quad \text{with } \rho\in\mathbb{R}_* \quad , \quad \theta\in\Delta^{d-1} \tag{3.2}$$

where Δ^{d-1} is as in (1.12). A similar system of coordinates has been used also in [9]. We denote as $d\nu(\theta)$ the $(d-1)$ dimensional Hausdorff measure restricted to Δ^{d-1}. We thus have that $d\xi = \frac{\rho^{d-1}}{\sqrt{d}} d\rho d\nu(\theta)$ (cf. [9]). We can then define measures $G(\tau) \in \mathscr{M}_1\left(\mathbb{R}_*\times\Delta^{d-1}\right)$ by requiring that

$$\int_{\mathbb{R}_*\times\Delta^{d-1}} \psi(\rho,\theta)G(\rho,\theta,\tau)\rho^{d-1}d\rho d\nu(\theta) = \int_{\mathbb{R}^d_*} \psi(|\xi|,\xi/|\xi|)F(\xi,\tau)d\xi\,, \tag{3.3}$$

for all test functions $\psi\in C_c(\mathbb{R}_*\times\Delta^{d-1})$ and with F defined by (3.1). Notice that in the case in which $F(\cdot,\tau)$ is absolutely continuous this implies that $G(\cdot,\cdot,\tau)$ is also absolutely continuous and we have the following relation between the corresponding densities

$$F(\xi,\tau) = \sqrt{d}\,G(|\xi|,\xi/|\xi|,\tau)\,.$$

We then use (3.3) and the property $t+1 = e^\tau$ and follow an argument similar to that in the proof of Proposition 2.4. This allows us to rewrite (2.3) as

$$0 = \int_0^\infty d\tau \int_{\mathbb{R}_*\times\Delta^{d-1}} G(\rho,\theta,\tau)\left[\partial_\tau\tilde\psi + \frac{\tilde\psi}{1-\gamma} - \frac{1}{1-\gamma}\rho\partial_\rho\tilde\psi\right](\rho,\theta,\tau)d\Omega$$
$$+ \frac{1}{2}\int_0^\infty d\tau \int_{\mathbb{R}_*\times\Delta^{d-1}}\int_{\mathbb{R}_*\times\Delta^{d-1}} \tilde K(\rho,\theta,r,\sigma,\tau)G(\rho,\theta,\tau)\,G(r,\sigma,\tau)\,\Psi(\rho,r,\theta,\sigma,\tau)d\Omega d\tilde\Omega \tag{3.4}$$

with

$$\tilde{K}(\rho,\theta,r,\sigma,\tau) = e^{-\frac{\gamma}{1-\gamma}\tau} K(e^{\frac{1}{1-\gamma}\tau}\rho\theta, e^{\frac{1}{1-\gamma}\tau} r\sigma) \tag{3.5}$$

$$d\Omega = \rho^{d-1} d\rho d\nu(\theta), \quad d\tilde{\Omega} = r^{d-1} dr d\nu(\sigma),$$

$$\Psi(\rho,r,\theta,\sigma,\tau) = \tilde{\psi}\left(\rho + r, \frac{\rho}{\rho+r}\theta + \frac{r}{\rho+r}\sigma, \tau\right) - \tilde{\psi}(\rho,\theta,\tau) - \tilde{\psi}(r,\sigma,\tau) \tag{3.6}$$

where we write $\tilde{\psi}(\rho,\theta,\tau) = \psi(\xi,t)$ with ψ defined in (2.12), $\xi = \rho\theta$, $\eta = r\sigma$ and $G_0(\rho,\theta)\,\rho^{d-1} d\rho d\nu(\theta) = f_0(dx)$ for each θ, $\sigma \in \Delta^{d-1}$.

Notice that (1.6), (1.7) imply the estimate

$$c_1(\rho+r)^\gamma \Phi_p\left(\frac{\rho}{\rho+r}\right) \le \tilde{K}(\rho,\theta,r,\sigma,\tau) \le c_2(\rho+r)^\gamma \Phi_p\left(\frac{\rho}{\rho+r}\right) \tag{3.7}$$

where Φ_p is as in (1.8).

The following lemma, which will be used to obtain Theorem 1.3, has been proved in [9]. For this reason, we will just state the result and refer to [9] for the proof.

Lemma 3.1 *There is a constant $C_d > 0$ which depends only on the dimension $d \ge 1$ such that, for any probability measure $\lambda \in \mathscr{M}_+\left(\Delta^{d-1}\right)$ and any pair of parameters $\varepsilon, \delta \in (0,1)$ at least one of the following alternatives holds true:*

(i) There exists a measurable set $A \subset \Delta^{d-1}$ with $\operatorname{diam}(A) \le \varepsilon$ such that $\int_A \lambda(d\theta) > 1-\delta$.

(ii) $\int_{\Delta^{d-1}} \lambda(d\theta) \int_{\Delta^{d-1}} \lambda(d\sigma)\,\|\theta-\sigma\|^2 \ge C_d \delta \varepsilon^{d+1}$ where $\|\cdot\|$ is the Euclidean distance.

A corollary of Lemma 3.1 that will be used in the proof of Theorem 1.4 is the following result.

Lemma 3.2 *Suppose that $\lambda \in \mathscr{M}_+\left(\Delta^{d-1}\right)$ is a probability measure such that*

$$\int_{\Delta^{d-1}} \lambda(d\theta) \int_{\Delta^{d-1}} \lambda(d\sigma)\,\|\theta-\sigma\|^2 = 0. \tag{3.8}$$

Then, there exists $\bar{\theta} \in \Delta^{d-1}$ such that

$$\lambda = \delta_{\bar{\theta}}\,. \tag{3.9}$$

Proof: We apply Lemma 3.1 for a sequence of values $\varepsilon_n = \delta_n = 2^{-n}$ with $n \in \mathbb{N}$. Due to (3.8) we have that the alternative (ii) in Lemma 3.1 does not take place. Therefore (i) holds, and for each n, we can pick a point θ_n from the corresponding set A. By compactness of Δ^{d-1}, we can find a convergent subsequence with a limit point $\bar{\theta} \in \Delta^{d-1}$, and it can be checked that then also (3.9) holds. □

We can now prove Theorem 1.3.

Proof of Theorem 1.3. The conservation of the total number of monomers (2.2), combined with the definition of F in (3.1) and the definition of G in (3.3) implies that

$$\int_{\mathbb{R}_*\times\Delta^{d-1}} \rho G(\rho,\theta,\tau) d\Omega = m_0 := \int_{\mathbb{R}_*\times\Delta^{d-1}} \rho G_0(\rho,\theta)\, d\Omega > 0\,. \tag{3.10}$$

We recall that the moment estimate (1.13) holds for $t \geq 1$. Since $\tau = \log(t+1)$ we will assume $\tau \geq \ln 2$ throughout the proof.

On the other hand, the assumption (1.13) with (3.1), (3.3) yields

$$\int_{\mathbb{R}_* \times \Delta^{d-1}} \rho^k G(\rho, \theta, \tau) d\Omega \leq C, \quad k \in [1/a, a] \ \text{ for some } a > 1, \tag{3.11}$$

where $C > 0$. In addition, using again (3.1), (3.3) as well as the estimate (2.1) we obtain

$$\int_{\left[\mathbb{R}_* \times \Delta^{d-1}\right] \cap \{\rho \geq 1\}} \rho^{\gamma+p} G(\rho, \theta, \tau) d\Omega + \int_{\left[\mathbb{R}_* \times \Delta^{d-1}\right] \cap \{\rho \leq 1\}} \rho^{1-p} G(\rho, \theta, \tau) d\Omega \leq C(T), \ \ln 2 \leq \tau \leq T \tag{3.12}$$

for any given $T > 1$, with $C(T)$ a constant that depends on T.

In the definition of weak solution (cf. Definition 2.1) we have assumed that the mass vector is conserved. Using the measure G we then obtain the following form of the conservation of mass

$$\frac{1}{m_0} \int_{\mathbb{R}_* \times \Delta^{d-1}} G(\rho, \theta, \tau) \rho \theta d\Omega = \theta_0, \tag{3.13}$$

with

$$\theta_0 := \frac{1}{m_0} \int_{\mathbb{R}_* \times \Delta^{d-1}} G_0(\rho, \theta) \rho \theta d\Omega. \tag{3.14}$$

Using (3.11) and (3.12) we can readily see that all the terms appearing in (3.4) are well defined for any $\tilde{\psi} \in C_c^1\left(\mathbb{R}_* \times \Delta^{d-1} \times (0, \infty)\right)$. Moreover, using an approximation argument as well as (3.11) and (3.12) it follows that (3.4) holds true for any test function $\tilde{\psi} \in C_c^1\left(\mathbb{R}_* \times \Delta^{d-1} \times [\log(2), \infty)\right)$ whose support is contained in $\{\tau : \tau \in [\log(2), \tau^*]\}$, for some $\tau^* \in (\log(2), \infty)$, and satisfying

$$\left|\tilde{\psi}(\rho, \theta, \tau)\right| + \rho \left|\partial_\rho \tilde{\psi}(\rho, \theta, \tau)\right| + \left|\partial_\tau \tilde{\psi}(\rho, \theta, \tau)\right| + \left|\nabla_\theta \tilde{\psi}(\rho, \theta, \tau)\right| \leq C\rho \ \ , \tag{3.15}$$

for any $\rho \in \mathbb{R}_*$, $\theta \in \Delta^{d-1}$ and $\tau \in [\log(2), \tau^*]$, and for some $C > 0$. Indeed, we can consider a sequence of compactly supported test functions $\tilde{\psi}_n(\rho, \theta, \tau) = \zeta_n(\rho)\, \tilde{\psi}(\rho, \theta, \tau)$ with $\tilde{\psi}$ satisfying (3.15), $\zeta_n \in C^\infty(0, \infty)$ such that

$$\zeta_n(\rho) = \begin{cases} 0 & \text{for } \ 0 < \rho \leq \frac{1}{2n} \ \text{ and } \ \rho \geq 2n \\ 1 & \text{for } \ \rho \in \left[\frac{1}{n}, n\right] \end{cases}$$

and $0 \leq \zeta_n{}'(\rho) \leq 4n$ for $\rho \in \left[\frac{1}{2n}, \frac{1}{n}\right]$ and $0 \leq -\zeta_n{}'(\rho) \leq \frac{2}{n}$ for $\rho \in [n, 2n]$. By assumption the identity (3.4) holds with the test functions $\tilde{\psi}_n$. We consider the limit as $n \to \infty$ of this sequence of identities. Notice that (3.15) combined with the properties of ζ_n imply the estimate

$$\left|\partial_\tau \tilde{\psi}_n + \frac{\tilde{\psi}_n}{1-\gamma} - \frac{1}{1-\gamma} \rho \partial_\rho \tilde{\psi}_n\right| \leq C\rho$$

with C independent of n. We can then take the limit as $n \to \infty$ of the first term on the right-hand side of (3.4) (with $\tilde{\psi}$ replaced by $\tilde{\psi}_n$) due to Lebesgue's dominated convergence Theorem as well as (3.13). On the other hand, using (3.15) and the properties of ζ_n, we can estimate the functions $\Psi_n(\rho, r, \theta, \sigma, \tau)$ that are defined using $\tilde{\psi} = \tilde{\psi}_n$ as

$$|\Psi_n(\rho, r, \theta, \sigma, \tau)| \leq C \min\{\rho, r\} \tag{3.16}$$

where C is independent of n. To see this we can restrict ourselves to the case in which $r \leq \rho$. We can then estimate the term $\tilde{\psi}_n(r,\sigma,\tau)$ in the formula of Ψ_n (cf. (3.6)) by Cr and the difference $\tilde{\psi}\left(\rho+r, \frac{\rho}{\rho+r}\theta + \frac{r}{\rho+r}\sigma, \tau\right) - \tilde{\psi}(\rho,\theta,\tau)$ can be estimated also as Cr using Taylor's formula and (3.15). Using (3.7) and (1.8) it then follows that the integrands in the second term of (3.4) (with $\tilde{\psi} = \tilde{\psi}_n$) can be estimated by an integrable function independent of n, due to (3.12) and (3.13). Indeed, the previous estimate (3.16) yields the bound

$$\begin{aligned}
&\tilde{K}(\rho,\theta,r,\sigma,\tau)G(\rho,\theta,\tau)\,G(r,\sigma,\tau)\,\Psi_n(\rho,r,\theta,\sigma,\tau)\\
&\leq C\left(\rho^{\gamma+p}r^{-p} + r^{\gamma+p}\rho^{-p}\right) rG(\rho,\theta,\tau)\,G(r,\sigma,\tau)\\
&\leq C\left(\rho^{\gamma+p}r^{1-p} + \rho^{-p}r^{\gamma+p+1}\right) G(\rho,\theta,\tau)\,G(r,\sigma,\tau) \qquad (3.17)
\end{aligned}$$

where in the last inequality we use that $r \leq \rho$. This inequality gives an estimate for the integrand of the last term in (3.4) by means of an integrable function for $0 < r \leq \rho \leq 1$, due to the assumption (2.1). In order to estimate the regions where $0 < r \leq 1 \leq \rho$ we use the fact that $\gamma + p < 1$ and $p > -1$ to obtain the following estimate for this range of values of r and ρ

$$\begin{aligned}
&\tilde{K}(\rho,\theta,r,\sigma,\tau)G(\rho,\theta,\tau)\,G(r,\sigma,\tau)\,\Psi_n(\rho,r,\theta,\sigma,\tau)\\
&\leq C\left(\rho r^{1-p} + \rho r^{\gamma+p+1}\right) G(\rho,\theta,\tau)\,G(r,\sigma,\tau)\,.
\end{aligned}$$

We can then combine the conservation of mass estimate and (2.1) to obtain that the integrand is bounded by an uniformly integrable function independent of n for this range of parameters. It remains to estimate the range of values $1 \leq r \leq \rho$. We distinguish the two cases $p \geq 0$ and $-1 < p < 0$. In the first case, we use (3.17) to obtain, using also that $\gamma + p < 1$ and $\gamma < 1$ (cf. (1.9))

$$\begin{aligned}
&\tilde{K}(\rho,\theta,r,\sigma,\tau)G(\rho,\theta,\tau)\,G(r,\sigma,\tau)\,\Psi_n(\rho,r,\theta,\sigma,\tau)\\
&\leq C\left(\rho r + \rho^{-p}r^{\gamma+p+1}\right) G(\rho,\theta,\tau)\,G(r,\sigma,\tau)\\
&\leq C\left(\rho r + \rho r^{\gamma}\right) G(\rho,\theta,\tau)\,G(r,\sigma,\tau)
\end{aligned}$$

where in the last inequality we used that $r^{p+1} \leq \rho^{p+1}$. In the case $-1 < p < 0$ we obtain

$$\begin{aligned}
&\tilde{K}(\rho,\theta,r,\sigma,\tau)G(\rho,\theta,\tau)\,G(r,\sigma,\tau)\,\Psi_n(\rho,r,\theta,\sigma,\tau)\\
&\leq C\left(\rho^{\gamma}\rho^{-|p|}r^{|p|}r + r^{\gamma-|p|}\rho^{|p|}r\right) G(\rho,\theta,\tau)\,G(r,\sigma,\tau)\\
&\leq C\left(\rho^{\gamma}r + r^{\gamma-|p|}\rho^{|p|}\rho^{1-|p|}r^{|p|}\right) G(\rho,\theta,\tau)\,G(r,\sigma,\tau)\\
&\leq C\left(\rho^{\gamma}r + r^{\gamma}\rho\right) G(\rho,\theta,\tau)\,G(r,\sigma,\tau)\,.
\end{aligned}$$

The right-hand side of this inequality is uniformly integrable for $1 \leq r \leq \rho$ since $\gamma < 1$.

We can then take the limit in both terms of (3.4) (with $\tilde{\psi} = \tilde{\psi}_n$). It then follows that (3.4) holds for any function $\tilde{\psi}$ satisfying (3.15).

We now approximate a test function with the form $\tilde{\psi}(\rho,\theta,\tau)\chi_{[\bar{\tau}_1,\bar{\tau}_2]}$ with $\bar{\tau}_2 > \bar{\tau}_1 \geq \log(2)$ and $\tilde{\psi} \in C_c^1\left(\mathbb{R}_* \times \Delta^{d-1} \times [\log(2),\infty)\right)$ by means of a sequence of functions $\tilde{\psi}_n(\rho,\theta,\tau) = \tilde{\psi}(\rho,\theta,\tau)\zeta_n(\tau)$ with $\zeta_n \geq 0$, and $\zeta_n \in C_c^1((0,\infty))$, such that $\zeta_n(\tau) \to \chi_{[\bar{\tau}_1,\bar{\tau}_2]}(\tau)$ pointwise in τ as $n \to \infty$ and $\zeta_n' \geq 0$ in a neighborhood of $\bar{\tau}_1$ and $\zeta_n' \leq 0$ in a neighborhood of

$\bar\tau_2$. From (3.4), it then follows by means of a standard computation that, for any function $\tilde\psi \in C^1_c\left(\mathbb{R}_* \times \Delta^{d-1} \times [\log(2),\infty)\right)$ satisfying (3.15),

$$\begin{aligned}
&\int_{\mathbb{R}_*\times\Delta^{d-1}} G\left(\rho,\theta,\bar\tau_2\right)\tilde\psi(\rho,\theta,\bar\tau_2)d\Omega = \int_{\mathbb{R}_*\times\Delta^{d-1}} G\left(\rho,\theta,\bar\tau_1\right)\tilde\psi(\rho,\theta,\bar\tau_1)d\Omega \qquad (3.18)\\
&+\int_{\bar\tau_1}^{\bar\tau_2} d\tau \int_{\mathbb{R}_*\times\Delta^{d-1}} G\left(\rho,\theta,\tau\right)\left[\partial_\tau\tilde\psi + \frac{\tilde\psi}{1-\gamma} - \frac{1}{1-\gamma}\rho\partial_\rho\tilde\psi\right](\rho,\theta,\tau)d\Omega\\
&+\frac{1}{2}\int_{\bar\tau_1}^{\bar\tau_2} d\tau \int_{\mathbb{R}_*\times\Delta^{d-1}}\int_{\mathbb{R}_*\times\Delta^{d-1}} \tilde K(\rho,\theta,r,\sigma,\tau)G\left(\rho,\theta,\tau\right)G\left(r,\sigma,\tau\right)\Psi(\rho,r,\theta,\sigma,\tau)d\Omega d\tilde\Omega
\end{aligned}$$

where $\tilde K$ and Ψ are as in (3.5) and (3.6) respectively.

We now use the test function $\varphi(\rho,\theta) = \rho\|\theta\|^2$ in (3.18). Notice that we have

$$\Psi(\rho,r,\theta,\sigma,\tau) = \varphi(\rho+r, \frac{\rho}{\rho+r}\theta + \frac{r}{\rho+r}\sigma) - \varphi(\rho,\theta) - \varphi(r,\sigma) = -\frac{\rho r}{\rho+r}\|\theta-\sigma||^2.$$

Using this identity in (3.18) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_*\times\Delta^{d-1}} G\left(\rho,\theta,\bar\tau_2\right)\rho\|\theta\|^2 d\Omega = \int_{\mathbb{R}_*\times\Delta^{d-1}} G(\rho,\theta,\bar\tau_1)\rho\|\theta\|^2 d\Omega\\
&-\frac{1}{2}\int_{\bar\tau_1}^{\bar\tau_2} d\tau \int_{\mathbb{R}_*\times\Delta^{d-1}}\int_{\mathbb{R}_*\times\Delta^{d-1}} \tilde K(\rho,\theta,r,\sigma,\tau)G\left(\rho,\theta,\tau\right)G\left(r,\sigma,\tau\right)\frac{\rho r}{\rho+r}\|\theta-\sigma||^2 d\Omega d\tilde\Omega
\end{aligned} \qquad (3.19)$$

which combined with (3.10) implies, taking $\bar\tau_1 = \log(2)$, $\bar\tau_2 = \bar\tau$, that

$$\int_{\log(2)}^{\bar\tau} d\tau \int_{\mathbb{R}_*\times\Delta^{d-1}}\int_{\mathbb{R}_*\times\Delta^{d-1}} \tilde K(\rho,\theta,r,\sigma,\tau)G\left(\rho,\theta,\tau\right)G\left(r,\sigma,\tau\right)\frac{\rho r}{\rho+r}\|\theta-\sigma||^2 d\Omega d\tilde\Omega \le 2m_0 \qquad (3.20)$$

for all $\bar\tau \ge \log(2)$.

Using (3.10), (3.11) it follows that for any $\delta_0 > 0$ small there exists $M > 0$ (depending only on δ_0, m_0 and a in (3.11)) such that

$$\int_{(0,\frac{1}{M})\times\Delta^{d-1}} \rho G(\rho,\theta,\tau)d\Omega + \int_{(M,\infty)\times\Delta^{d-1}} \rho G(\rho,\theta,\tau)d\Omega \le \delta_0 m_0\ . \qquad (3.21)$$

Then

$$\int_{\left[\frac{1}{M},M\right]\times\Delta^{d-1}} \rho G(\rho,\theta,\tau)d\Omega \ge m_0\left(1-\delta_0\right). \qquad (3.22)$$

The lower estimate in (3.7) implies that there exists a constant $\eta_M > 0$ such that $\frac{\tilde K(\rho,\theta,r,\sigma,\tau)}{\rho+r} \ge \eta_M$ for $(\rho,r) \in \left[\frac{1}{M},M\right]^2$, $(\theta,\sigma) \in \left(\Delta^{d-1}\right)^2$. Using (3.20) we then obtain the estimate

$$\int_1^{\bar\tau} d\tau \int_{\left[\frac{1}{M},M\right]\times\Delta^{d-1}}\int_{\left[\frac{1}{M},M\right]\times\Delta^{d-1}} \rho G\left(\rho,\theta,\tau\right) rG\left(r,\sigma,\tau\right)\|\theta-\sigma||^2 d\Omega d\tilde\Omega \le \frac{2m_0}{\eta_M} \qquad (3.23)$$

for all $\bar\tau \ge 1 > \ln 2$. For each $\tau \ge 1$ and M as above, we define a probability measure on $\mathscr{M}_+\left(\Delta^{d-1}\right)$ by means of

$$\lambda_M(A,\tau) = \frac{\int_{\left[\frac{1}{M},M\right]\times A} \rho G\left(\rho,\theta,\tau\right)d\Omega}{\int_{\left[\frac{1}{M},M\right]\times\Delta^{d-1}} \rho G\left(\rho,\theta,\tau\right)d\Omega}, \qquad (3.24)$$

for each Borel set $A \subset \Delta^{d-1}$. Then for each $\tau \geq 1$ we have that $\lambda_M(d\theta, \tau)$ is a probability measure and (3.22) and (3.23) imply, after taking the limit $\bar{\tau} \to \infty$

$$\int_1^\infty d\tau \int_{\Delta^{d-1}} \int_{\Delta^{d-1}} \|\theta - \sigma\|^2 \lambda_M(d\theta, \tau) \lambda_M(d\sigma, \tau) \leq \frac{2m_0}{\eta_M (m_0)^2 (1-\delta_0)^2} < \infty. \tag{3.25}$$

It then follows that there exists a Borel set $\tilde{I}_M \subset (1, \infty)$ such that $\lim_{R\to\infty} \int_{(R,\infty)\cap\tilde{I}_M} d\tau = 0$ and such that

$$\lim_{\tau\to\infty} \left[\chi_{(1,\infty)\setminus\tilde{I}_M}(\tau) \int_{\Delta^{d-1}} \int_{\Delta^{d-1}} \|\theta - \sigma\|^2 \lambda_M(d\theta, \tau) \lambda_M(d\sigma, \tau) \right] = 0 \tag{3.26}$$

where $\chi_{(1,\infty)\setminus\tilde{I}_M}$ is the characteristic function of the set $(1, \infty) \setminus \tilde{I}_M$. Notice that $\tilde{I}_M$ is the set of times for which localization does not take place. More precisely, suppose that $M > 0$ is given. Our goal is to construct a set $\tilde{I}_M$ such that

(i) $|\tilde{I}_M| < \infty$,

(ii) $\lim_{R\to\infty} |\tilde{I}_M \cap (R, \infty)| = 0$,

(iii) $\lim_{\tau\to\infty} g(\tau) \chi_{(1,\infty)\setminus\tilde{I}_M}(\tau) = 0$

where the function g in item (iii) is such that $\int_1^\infty g(\tau)d\tau < \infty$ and $g(\tau) \geq 0$. Note that the function g here corresponds to the function defined by the double integral in (3.26).

To make the notation lighter we now drop the M dependence in what follows. Let $\delta > 0$ and $n \geq 0$, we define

$$J_n(\delta) = \left\{ \tau \in [2^n, 2^{n+1}] \ : \ g(\tau) \geq \delta \right\}. \tag{3.27}$$

Notice that

$$J_n(\delta_1) \supset J_n(\delta_2) \quad \text{if} \quad \delta_1 < \delta_2. \tag{3.28}$$

We have

$$\sum_{n=0}^\infty |J_n(\delta)| \leq \sum_{n=0}^\infty \frac{1}{\delta} \int_{2^n}^{2^{n+1}} g(\tau)d\tau = \frac{1}{\delta} \int_1^\infty g(\tau) < \frac{C}{\delta}.$$

Thus $\sum_{n=0}^\infty |J_n(\delta)| \leq \frac{C}{\delta}$ for any $\delta > 0$.

We now define a sequence $\delta_n \to 0$ as $n \to \infty$ such that $\sum_{n=0}^\infty |J_n(\delta_n)| < \infty$. In order to do this, for each $k \in \mathbb{N}$, we choose $N_k \in \mathbb{N}$ sufficiently large such that $\sum_{n=N_k}^\infty |J_n(2^{-k})| \leq 2^{-k}$. Note that $\{N_k\}$ is an increasing sequence. Note also that $\sum_{n=N}^\infty |J_n(\delta)|$ increases if δ decreases due to (3.28).

We set

$$\begin{aligned}
&\delta_n = 1 \quad \text{for} \quad n = 0, \ldots, N_1 \\
&\delta_n = \frac{1}{2} \quad \text{for} \quad n = N_1 + 1, \ldots, N_2 \\
&\cdots \\
&\delta_n = \frac{1}{2^k} \quad \text{for} \quad n = N_k + 1, \ldots, N_{k+1}.
\end{aligned}$$

With this choice of δ_n we obtain

$$\sum_{n=0}^{\infty} |J_n(\delta_n)| = \sum_{n=0}^{N_1} |J_n(1)| + \sum_{n=N_1+1}^{N_2} |J_n\left(\frac{1}{2}\right)| + \cdots + \sum_{n=N_\ell+1}^{N_{\ell+1}} |J_n\left(\frac{1}{2^\ell}\right)| + \ldots$$
$$\leq C_0 + \frac{1}{2} + \cdots + \frac{1}{2^\ell} + \cdots \leq C_0 + \sum_{k=1}^{\infty} \frac{1}{2^k} \leq C_1$$

Therefore $\sum_{n=0}^{\infty} |J_n(\delta_n)| \leq C_1 < \infty$. We then set $\tilde{I} = \bigcup_{n=0}^{\infty} J_n(\delta_n)$. We finally notice that $\tilde{I} = \tilde{I}_M$ satisfies the properties $(i), (ii), (iii)$ above and this concludes the argument.

We can then apply Lemma 3.1 to prove that there exists $\tau_{0,M}$ sufficiently large such that, for any $\tau \in (\tau_{0,M}, \infty) \setminus \tilde{I}_M$ there exists a Borel set $A_M(\tau) \subset \Delta^{d-1}$ such that the function defined by means of

$$f_M(\tau) = \begin{cases} \operatorname{diam}(A_M(\tau)) & \text{for } \tau \in (\tau_{0,M}, \infty) \setminus \tilde{I}_M \\ 0 & \text{for } \tau \in \tilde{I}_M \end{cases}$$

satisfies $\lim_{\tau\to\infty} f_M(\tau) = 0$ and, in addition, the function

$$g_M(\tau) = \begin{cases} \int_{A_M(\tau)} \lambda_M(d\theta, \tau) & \text{for } \tau \in (\tau_{0,M}, \infty) \setminus \tilde{I}_M \\ 1 & \text{for } \tau \in \tilde{I}_M \end{cases}$$

satisfies

$$\lim_{\tau\to\infty} g_M(\tau) = 1. \tag{3.29}$$

On the other hand, the conservation law (3.13), (3.14) combined with the fact that $\operatorname{diam}(A_M(\tau))$ becomes arbitrarily small for any $\tau \in (\tau_{0,M}, \infty) \setminus \tilde{I}_M$ and combined also with the estimate (3.22), implies that for $L > \tau_{0,M}$ large enough it holds

$$\bigcup_{\tau\in(L,\infty)\setminus\tilde{I}_M} A_M(\tau) \subset B_{6\delta_0}(\theta_0). \tag{3.30}$$

Indeed, we have for each $\tau \in (\tau_{0,M}, \infty) \setminus \tilde{I}_M$

$$\begin{aligned}
\theta_0 &= \frac{1}{m_0} \int_{\mathbb{R}_* \times \Delta^{d-1}} \rho\theta G(\rho, \theta, \tau) d\Omega \\
&= \frac{1}{m_0} \int_{\left[\frac{1}{M}, M\right] \times \Delta^{d-1}} \rho\theta G(\rho, \theta, \tau) d\Omega + \frac{1}{m_0} \int_{\left(\mathbb{R}_* \setminus \left[\frac{1}{M}, M\right]\right) \times \Delta^{d-1}} \rho\theta G(\rho, \theta, \tau) d\Omega \\
&= \frac{1}{m_0} \left[\int_{\left[\frac{1}{M}, M\right] \times \Delta^{d-1}} \rho G(\rho, \theta, \tau) \, d\Omega \right] \int_{\Delta^{d-1}} \theta \lambda_M(d\theta, \tau) \\
&\quad + \frac{1}{m_0} \int_{\left(\mathbb{R}_* \setminus \left[\frac{1}{M}, M\right]\right) \times \Delta^{d-1}} \rho\theta G(\rho, \theta, \tau) d\Omega.
\end{aligned}$$

Defining $m_{0,M}(\tau) := \int_{\left[\frac{1}{M}, M\right] \times \Delta^{d-1}} \rho G(\rho, \theta, \tau) \, d\Omega$ we obtain

$$\begin{aligned}
\theta_0 &= \frac{m_{0,M}(\tau)}{m_0} \int_{A_M(\tau)} \theta \lambda_M(d\theta, \tau) + \frac{m_{0,M}(\tau)}{m_0} \int_{\Delta^{d-1} \setminus A_M(\tau)} \theta \lambda_M(d\theta, \tau) \\
&\quad + \frac{1}{m_0} \int_{\left(\mathbb{R}_* \setminus \left[\frac{1}{M}, M\right]\right) \times \Delta^{d-1}} \rho\theta G(\rho, \theta, \tau) d\Omega.
\end{aligned} \tag{3.31}$$

We can write the third term on the right-hand side as $(1-\frac{m_{0,M}(\tau)}{m_0})$, which can be estimated by δ_0, using (3.21). Therefore, using (3.22), (3.31), we obtain for $\tau \in (\tau_{0,M},\infty)\setminus\tilde{I}_M$ sufficiently large,

$$\begin{aligned}&\left|\int_{A_M(\tau)}\theta\lambda_M(d\theta,\tau)-\theta_0\right|\leq\\&\leq\left(1-\frac{m_{0,M}(\tau)}{m_0}\right)\int_{A_M(\tau)}|\theta|\lambda_M(d\theta,\tau)+(1-g_M(\tau))+\left(1-\frac{m_{0,M}(\tau)}{m_0}\right)\\&\leq 2\delta_0+(1-g_M(\tau)).\end{aligned}$$

Thus using (3.29) we obtain, for τ sufficiently large, the estimate

$$\left|\int_{A_M(\tau)}\theta\lambda_M(d\theta,\tau)-\theta_0\right|\leq 3\delta_0.$$

Then, since $\operatorname{diam}(A_M(\tau))\to 0$ as $\tau\to\infty$, and $\lambda_M(d\theta,\tau)$ is a probability measure for each $\tau>\tau_{0,M}$, satisfying (3.29), it follows that for $\tau\geq L$ and L sufficiently large and $\tau\in(\tau_{0,M},\infty)\setminus\tilde{I}_M$ we have

$$\left|\int_{A_M(\tau)}\theta\lambda_M(d\theta,\tau)-\bar{\theta}(\tau)\right|\leq\delta_0$$

for some $\bar{\theta}(\tau)\in\Delta^{d-1}$. Then, combining the last two previous inequalities, we obtain $\left|\bar{\theta}(\tau)-\theta_0\right|\leq 4\delta_0$, for $\tau\in(\tau_{0,M},\infty)\setminus\tilde{I}_M$ sufficiently large. We have also $A_M(\tau)\subset B_{\operatorname{diam}(A_M(\tau))+\delta_0}\left(\bar{\theta}(\tau)\right)\subset B_{2\delta_0}\left(\bar{\theta}(\tau)\right)$, if $\tau\geq L$ and L sufficiently large. In addition we obtain $A_M(\tau)\subset B_{6\delta_0}(\theta_0)$ for any $\tau\in(L,\infty)\setminus\tilde{I}_M$, hence the claim (3.30) follows.

In order to conclude the proof we consider a sequence of values $\delta_0=\frac{1}{n}$, $n\in\mathbb{N}$, the corresponding sequence $M_n\to\infty$, the sets $\tilde{I}_{M_n}$ and a sequence of increasing values τ_{0,M_n}, starting at $\tau=\tau_{0,M_1}$, such that $\left|(\tau_{0,M_n},\infty)\cap\tilde{I}_{M_n}\right|\leq 2^{-n}$ for $n\in\mathbb{N}$ with $\lim_{n\to\infty}\tau_{0,M_n}=\infty$ and also $\operatorname{diam}(A_{M_n}(\tau))\leq 2^{-n}$ for $\tau\in\left(\tau_{0,M_n},\tau_{0,M_{n+1}}\right]\setminus\tilde{I}_{M_n}$ and

$$\int_{A_{M_n}(\tau)}\lambda_{M_n}(d\theta,\tau)\geq 1-2^{-n}\text{ for }\tau\in\left(\tau_{0,M_n},\tau_{0,M_{n+1}}\right]\setminus\tilde{I}_{M_n}.\tag{3.32}$$

Then, if we define $\tilde{I}=[\tau_{0,M_1},\infty)\setminus\bigcup_{n\in\mathbb{N}}\left(\left(\tau_{0,M_n},\tau_{0,M_{n+1}}\right]\setminus\tilde{I}_{M_n}\right)$ or, equivalently, $\tilde{I}=\bigcup_{n\in\mathbb{N}}\left[\tilde{I}_{M_n}\cap\left(\tau_{0,M_n},\tau_{0,M_{n+1}}\right]\right]$ it follows that $\lim_{L\to\infty}\int_{[L,\infty)\cap\tilde{I}}d\tau=0$, i.e. $|\tilde{I}\cap[L,\infty)|\to 0$ as $L\to\infty$.

We now define the sets, with n fixed

$$A(\tau)=\begin{cases}\Delta^{d-1} & \text{for }\ \tau\in[1,\tau_{0,M_1}]\\ A_{M_n}(\tau) & \text{for }\ \tau\in\left(\tau_{0,M_n},\tau_{0,M_{n+1}}\right]\setminus\tilde{I}_{M_n}\end{cases}.$$

We have also that the function defined as

$$f(\tau)=\begin{cases}\operatorname{diam}(A(\tau)) & \text{for }\ \tau\in(1,\infty)\setminus\tilde{I}\\ 0 & \text{for }\ \tau\in\tilde{I}\end{cases}$$

satisfies $\lim_{\tau\to\infty} f(\tau) = 0$. Moreover, the function g defined as

$$g(\tau) = \begin{cases} \int_{A(\tau)} \lambda_{M_n}(d\theta, \tau) & \text{for } \tau \in \left(\tau_{0,M_n}, \tau_{0,M_{n+1}}\right] \setminus \tilde{I}_{M_n} \\ 1 & \text{for } \tau \in \tilde{I} \end{cases}$$

satisfies $\lim_{\tau\to\infty} g(\tau) = 1$.

It then follows from $\lim_{\tau\to\infty} f(\tau) = 0$ that $\bigcap_{\{L\geq 1\}} \bigcup_{\tau\in(L,\infty)\setminus\tilde{I}} A(\tau) = \{\theta_0\}$. On the other hand, the definition of the function $g(\cdot)$, as well as the definition of the measures λ_{M_n} (c.f. (3.24)), imply (1.16) for some function $\delta(t)$ such that $\lim_{t\to\infty} \delta(t) = 0$. Using the fact that $d\tau = \frac{dt}{t+1}$ it follows that if we define the set $I \subset (0,\infty)$ as the image of $\tilde{I}$ by means of the mapping $\tau \to (e^\tau - 1)$, it then follows from $\int_{(L,\infty)\cap I} d\tau \to 0$ as $L \to \infty$ that $\lim_{L\to\infty} \frac{|I\cap[L,2L]|}{L} = 0$. This concludes the proof of Theorem 1.3. □

We now prove Theorem 1.1.

Proof of Theorem 1.1. The proof is similar to the one of Theorem 1.3. The main difference is that under the assumptions in Theorem 1.1 we can prove that the measure $\lambda_M(d\theta, \tau)$ changes continuously in the weak$-*$ topology as τ varies. More precisely, it turns out that if $0 \leq \gamma < 1$ and $0 \leq \gamma + p < 1$ the following estimate holds

$$\int_{\mathbb{R}_*\times\Delta^{d-1}} G(\rho,\theta,\tau)\rho^\sigma d\Omega \leq C_0 \quad , \quad \tau \geq 1 \tag{3.33}$$

where $\sigma = \gamma$ if $p > 0$ and $\sigma = \gamma + \delta$ with $\delta > 0$ arbitrarily small if $p \leq 0$. The constant C_0 on the right-hand side of (3.33) is independent of τ.

The estimate (3.33) has been proved in [4] in the case of one-component coagulation equations (i.e. $d = 1$) under the assumption $f_0 \in L^1_{loc}\left(\mathbb{R}^d_*\right)$. The proofs can be adapted to the case in which $f_0 \in \mathscr{M}_1(\mathbb{R}^d_*)$. A sketch of the ideas required to prove this estimate in the multicomponent coagulation case $d > 1$ are collected in Section 5 (cf. Proposition 5.1).

On the other hand we also have the conservation of mass identity (cf. (3.10))

$$\int_{\mathbb{R}_*\times\Delta^{d-1}} G(\rho,\theta,\tau)\rho d\Omega = m_0 \quad , \quad \tau \geq 1. \tag{3.34}$$

Using (3.33) and (3.34), as well as the fact that for the range of parameters under consideration we have $\gamma < 1 - p$ it follows that

$$\int_{\{\rho\geq 1\}\times\Delta^{d-1}} G(\rho,\theta,\tau)\rho^{\gamma+p} d\Omega + \int_{\{\rho\leq 1\}\times\Delta^{d-1}} G(\rho,\theta,\tau)\rho^{1-p} d\Omega \leq C_1 \quad , \quad \tau \geq 1 \tag{3.35}$$

where C_1 depends on C_0 and m_0 but it is independent of τ. Using (3.18) we obtain that, for any smooth test function $\tilde{\psi}$ satisfying (3.15) and any $\bar{\tau}_2 \geq \bar{\tau}_1 > 0$ we have

$$\begin{aligned} &\int_{\mathbb{R}_*\times\Delta^{d-1}} G(\rho,\theta,\bar{\tau}_2)\,\tilde{\psi}(\rho,\theta,\bar{\tau}_2) d\Omega - \int_{\mathbb{R}_*\times\Delta^{d-1}} G(\rho,\theta,\bar{\tau}_1)\,\tilde{\psi}(\rho,\theta,\bar{\tau}_1) d\Omega \\ &= \int_{\bar{\tau}_1}^{\bar{\tau}_2} d\tau \int_{\mathbb{R}_*\times\Delta^{d-1}} G(\rho,\theta,\tau) \left[\partial_\tau\tilde{\psi} + \frac{\tilde{\psi}}{1-\gamma} - \frac{1}{1-\gamma}\rho\partial_\rho\tilde{\psi}\right](\rho,\theta,\tau) d\Omega \\ &+ \frac{1}{2}\int_{\bar{\tau}_1}^{\bar{\tau}_2} d\tau \int_{\mathbb{R}_*\times\Delta^{d-1}} \int_{\mathbb{R}_*\times\Delta^{d-1}} \tilde{K}(\rho,\theta,r,\sigma,\tau) G(\rho,\theta,\tau)\, G(r,\sigma,\tau)\, \Psi(\rho,r,\theta,\sigma,\tau) d\Omega d\tilde{\Omega}. \end{aligned} \tag{3.36}$$

We now argue as in the proof of Theorem 1.3. Given $\delta_0 > 0$ arbitrarily small we select $M > 0$ sufficiently large such that (3.21), (3.22) hold. We then define $\lambda_M(d\theta, \tau)$ by means of (3.24). Then (3.25) holds.

Suppose that $\tilde{\psi}(\cdot, \cdot, \tau) \in W^{1,\infty}\left(\mathbb{R}_* \times \Delta^{d-1}\right)$ such that $\operatorname{supp}\left(\tilde{\psi}\right)$ is compact and $\tilde{\psi}(\rho, \theta, \cdot) \in C_c^1(0, \infty)$. Then, using (3.7), (3.34) and (3.35), we obtain from (3.36) the following estimate

$$\left|\int_{\mathbb{R}_* \times \Delta^{d-1}} G(\rho, \theta, \bar{\tau}_2)\, \tilde{\psi}(\rho, \theta, \bar{\tau}_2) d\Omega - \int_{\mathbb{R}_* \times \Delta^{d-1}} G(\rho, \theta, \bar{\tau}_1)\, \tilde{\psi}(\rho, \theta, \bar{\tau}_1) d\Omega\right| \leq$$
$$C_2 \sup_{\tau \geq 1}\left\{\left\|\tilde{\psi}(\cdot, \cdot, \tau)\right\|_{W^{1,\infty}\left(\mathbb{R}_* \times \Delta^{d-1}\right)}\right\} |\bar{\tau}_2 - \bar{\tau}_1|$$

for $\bar{\tau}_1, \bar{\tau}_2 \geq 1$, where C_2 depends on m_0 and C_0. It then follows that the mapping $\tau \in [1, \infty) \to G(\rho, \theta, \tau) \in \mathscr{M}_1\left(\mathbb{R}_* \times \Delta^{d-1}\right)$ is continuous in the weak$-*$ topology of $\mathscr{M}_1\left(\mathbb{R}_* \times \Delta^{d-1}\right)$. In particular, this implies that the mapping $\tau \to \int_{\left[\frac{1}{M}, M\right]} \int_{\Delta^{d-1}} \rho G(\rho, \theta, \tau)\, d\Omega$ is continuous in $\tau \geq 1$ and that the mapping $\tau \in [1, \infty) \to \lambda_M(d\theta, \tau) \in \mathscr{M}_+\left(\Delta^{d-1}\right)$ with λ_M as in (3.24) is also continuous in the weak$-*$ topology of $\mathscr{M}_+\left(\Delta^{d-1}\right)$. Moreover, the mapping from $\mathscr{M}_+\left(\Delta^{d-1}\right)$ to $[0, \infty)$ defined by means of $\lambda_M \to \int_{\Delta^{d-1}} \int_{\Delta^{d-1}} \|\theta - \sigma\|^2 \lambda_M(d\theta)\, \lambda_M(d\sigma)$ is continuous if the topology of $\mathscr{M}_+\left(\Delta^{d-1}\right)$ is given by the weak$-*$ topology. This follows from the fact that the tensor product is a continuous mapping in the weak$-*$ topology. We now claim that (3.25) implies that

$$\lim_{\tau \to \infty} \int_{\Delta^{d-1}} \int_{\Delta^{d-1}} \|\theta - \sigma\|^2 \lambda_M(d\theta, \tau)\, \lambda_M(d\sigma, \tau) = 0. \tag{3.37}$$

Indeed, suppose that $\limsup_{\tau \to \infty} \int_{\Delta^{d-1}} \int_{\Delta^{d-1}} \|\theta - \sigma\|^2 \lambda_M(d\theta, \tau)\, \lambda_M(d\sigma, \tau) > 0$. Then, there exist an increasing sequence $\{\tau_n\}_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} \tau_n = \infty$ and $\eta > 0$ such that

$$\int_{\Delta^{d-1}} \int_{\Delta^{d-1}} \|\theta - \sigma\|^2 \lambda_M(d\theta, \tau_n)\, \lambda_M(d\sigma, \tau_n) > \eta$$

for n large enough. We can assume without loss of generality that $\tau_{n+1} - \tau_n \geq 1$. Then, the uniform continuity of the mapping $\tau \to \int_{\Delta^{d-1}} \int_{\Delta^{d-1}} \|\theta - \sigma\|^2 \lambda_M(d\theta, \tau)\, \lambda_M(d\sigma, \tau)$ implies that there exists $\varepsilon_0 > 0$ small such that

$$\int_{\Delta^{d-1}} \int_{\Delta^{d-1}} \|\theta - \sigma\|^2 \lambda_M(d\theta, \tau)\, \lambda_M(d\sigma, \tau) > \frac{\eta}{2} \text{ for } \tau \in (\tau_n - \varepsilon_0, \tau_n + \varepsilon_0).$$

However, this contradicts (3.25) and implies (3.37). We can then apply Lemma 3.1 to show that there exists a family of Borel sets $\{A_M(\tau)\}_{\tau \geq 1}$ with $A_M(\tau) \subset \Delta^{d-1}$ and $\lim_{\tau \to \infty} \operatorname{diam}(A_M(\tau)) = 0$ such that

$$\lim_{\tau \to \infty} \int_{A_M(\tau)} \lambda_M(d\theta, \tau) = 1. \tag{3.38}$$

Notice that this result is similar to the claim (3.29) in the proof of Theorem 1.3, with the only difference that in the case of (3.29) there is a set of "exceptional" times with small measure for which $\int_{A_M(\tau)} \lambda_M(d\theta, \tau)$ might not be close to 1. We can now take $\delta_0 = \frac{1}{n}$, $n \in \mathbb{N}$, select the corresponding values of the sequence $M_n \to \infty$ as it was made in the proof of Theorem 1.3 and argue exactly as it was made there in order to prove (1.14). Hence the result follows. □

3.2 Complete localization along a ray in self-similar solutions

We now prove Theorem 1.4.

Proof of Theorem 1.4. Using the change of variables (3.2), (3.3) with F and G independent of time, we can rewrite (2.8) as

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}_*\times\Delta^{d-1}}\int_{\mathbb{R}_*\times\Delta^{d-1}}\hat{K}(\rho,\theta,r,\sigma)G(\rho,\theta)\,G(r,\sigma)\,\Psi(\rho,r,\theta,\sigma)d\Omega d\tilde{\Omega} \\ &+\frac{1}{1-\gamma}\int_{\mathbb{R}_*\times\Delta^{d-1}}G(\rho,\theta)\left[\tilde{\psi}-\rho\partial_\rho\tilde{\psi}\right](\rho,\theta)d\Omega=0\end{aligned} \tag{3.39}$$

where due to the homogeneity of the kernel K we have

$$\hat{K}(\rho,\theta,r,\sigma)=K(\rho\theta,r\sigma)\ ,\quad \rho,r\in[0,\infty)\ ,\quad \theta,\sigma\in\Delta^{d-1}$$

and

$$\Psi(\rho,r,\theta,\sigma)=\tilde{\psi}\left(\rho+r,\frac{\rho}{\rho+r}\theta+\frac{r}{\rho+r}\sigma\right)-\tilde{\psi}(\rho,\theta)-\tilde{\psi}(r,\sigma). \tag{3.40}$$

Notice that $\tilde{\psi}(\rho,\theta)=\psi(\xi)$ with $\xi=\rho\theta$.

Arguing as in the proof of Theorem 1.3 we can prove that (3.39) holds for any test function $\tilde{\psi}(\rho,\theta)$ satisfying (cf. (3.15))

$$\left|\tilde{\psi}(\rho,\theta)\right|+\rho\left|\frac{\partial\tilde{\psi}}{\partial\rho}(\rho,\theta)\right|+\left|\nabla_\theta\tilde{\psi}(\rho,\theta)\right|\le C\rho\ \ ,\quad \rho\in[0,\infty),\ \theta\in\Delta^{d-1}.$$

We can then choose the test function $\tilde{\psi}(\rho,\theta)=\rho\,\|\theta\|^2$ in (3.39). Then

$$\int_{\mathbb{R}_*\times\Delta^{d-1}}\int_{\mathbb{R}_*\times\Delta^{d-1}}\hat{K}(\rho,\theta,r,\sigma)G(\rho,\theta)\,G(r,\sigma)\,\frac{\rho r}{\rho+r}\|\theta-\sigma\|^2d\Omega d\tilde{\Omega}=0. \tag{3.41}$$

This identity implies that G has the form

$$G(\rho,\theta)=G_0(\rho)\,\delta(\theta-\theta_0) \tag{3.42}$$

where $G_0\in\mathscr{M}_1(\mathbb{R}_*)$.

This can be seen defining for each M sufficiently large the probability measures on Δ^{d-1} by

$$\lambda_M(A)=\frac{\int_{\left[\frac{1}{M},M\right]\times A}\rho G(\rho,\theta)\,d\Omega}{\int_{\left[\frac{1}{M},M\right]\times\Delta^{d-1}}\rho G(\rho,\theta)\,d\Omega} \tag{3.43}$$

for each Borel set $A\subset\mathscr{M}_+(\Delta^{d-1})$.

These probability measures are well defined for M large enough since G is not identically zero. Then (3.41) implies

$$\int_{\Delta^{d-1}}\int_{\Delta^{d-1}}\lambda_M(d\theta)\,\lambda_M(d\sigma)\,\|\theta-\sigma\|^2=0.$$

We can then apply Lemma 3.2 with ε and δ arbitrarily small to prove that

$$\lambda_M=\delta_{\theta_M} \tag{3.44}$$

with $\theta_M \in \Delta^{d-1}$. We now use that, since $\bigcup_{M>1} \left[\frac{1}{M}, M\right] \times \Delta^{d-1} = \mathbb{R}_*^d$

$$\int_{\left[\frac{1}{M},M\right]\times\Delta^{d-1}} G(\rho,\theta)\,\rho d\Omega \to m_0 \quad \text{as } M \to \infty.$$

Hence, combining the previous limit with (3.44) and using the change of variables (3.2), (3.3), the definition of θ_0 in (1.15) and Lebesgue dominated convergence Theorem, we obtain

$$\lim_{M\to\infty} \int_{\Delta^{d-1}} \theta \lambda_M(d\theta) = \frac{1}{m_0} \int_{\mathbb{R}_*\times\Delta^{d-1}} G(\rho,\theta)\,\theta\rho d\Omega = \theta_0.$$

Therefore, from (3.44) we obtain that $\lim_{M\to\infty} \theta_M = \theta_0$. Notice that (3.43) and (3.44) imply that there exists a measure $G_{0,M} \in \mathscr{M}_1(\mathbb{R}_*)$ such that

$$G(\rho,\theta)\,\chi_{\left[\frac{1}{M},M\right]}(\rho) = G_{0,M}(\rho)\,\delta(\theta-\theta_M).$$

This implies that the measure $G(\rho,\theta)$ is supported along the line $\{\theta = \theta_M\}$. Therefore θ_M is independent of M and we have $\theta_M = \theta_0$. This gives (3.42).

Plugging (3.42) into (3.39) we obtain that G_0 satisfies

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \hat{K}(\rho,\theta_0,r,\theta_0) G_0(\rho)\,G_0(r)\,\Psi(\rho,r,\theta_0,\theta_0) r^{d-1}\rho^{d-1} d\rho dr \\
&+ \frac{1}{1-\gamma}\int_{\mathbb{R}_*} G_0(\rho)\left[\tilde{\psi} - \rho\partial_\rho\tilde{\psi}\right](\rho,\theta_0)\rho^{d-1}d\rho = 0
\end{aligned}$$

for each $\tilde{\psi} \in C_c^1\left((0,\infty)\times\Delta^{d-1}\right)$. In particular, defining $F_0 := \rho^{d-1}G_0$ and $K_{\theta_0}(\rho,r) = \hat{K}(\rho,\theta_0,r,\theta_0)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K_{\theta_0}(\rho,r) F_0(\rho)\,F_0(r)\left[\tilde{\psi}(\rho+r) - \tilde{\psi}(\rho) - \tilde{\psi}(r)\right] d\rho dr \\
&+ \frac{1}{1-\gamma}\int_{\mathbb{R}_*} F_0(\rho)\left[\tilde{\psi} - \rho\partial_\rho\tilde{\psi}\right](\rho) d\rho = 0
\end{aligned}$$

for any $\tilde{\psi} \in C_c^1(0,\infty)$, i.e. F_0 is a self-similar profile for the one-dimensional coagulation equation with coagulation kernel $K_{\theta_0}(\rho,r)$. This concludes the proof of Theorem 1.4.

□

4 Global existence and self-similar solutions for the multicomponent problem

In this section we show that the assumptions of Theorems 1.1 and 7.1 are satisfied for some ranges of exponents γ, p as well as for some choices of initial values $f_0 \in \mathscr{M}_1(\mathbb{R}_*^d)$. The global existence of weak solutions in the sense of Definition 2.1 which in addition satisfy the conservation of mass condition (1.3) has been proved (cf. [1, 4, 12, 20]) in the case of one-component coagulation equations (i.e. $d = 1$) for product kernels of the form $x^{\gamma+\lambda}y^{-\lambda} + x^{\gamma+\lambda}y^{-\lambda}$ with $-\lambda \le \gamma+\lambda < 1$ and $\gamma < 1$. These results can be easily extended to the class of kernels considered here satisfying (1.5), (1.7), (1.8), (1.9), with $d \ge 1$.

On the other hand, the existence of self-similar profiles has been proved for one-component coagulation kernels which in addition to the previously stated conditions (cf. (1.7), (1.8), (1.9), (1.5)) satisfy also (1.6) and the homogeneity condition (1.11). These results can be used to prove the existence of self-similar solutions with the form (1.18) in the multicomponent case (i.e. $d > 1$). We will explain in this section how this can be achieved.

We first notice that the following global existence result holds.

Theorem 4.1 *Suppose that K is as in (1.5) and satisfies the homogeneity property (1.11) and the upper and lower bounds (1.6),* (1.7) *with $\gamma,\ p \in \mathbb{R}$ such that $\gamma, \gamma + p \in [0, 1)$. Suppose that $f_0 \in \mathscr{M}_1(\mathbb{R}^d_*)$ satisfies*

$$\int_{\mathbb{R}^d_*} (|x| + |x|^{1+r}) f_0(dx) < \infty \tag{4.1}$$

for some $r > 0$. Then, there exists a weak solution $f \in C([0,\infty), \mathscr{M}_1(\mathbb{R}^d_*))$ to (1.1) *in the sense of Definition 2.1 with $f(0,\cdot) = f_0$. Moreover, this solution f has the following property. For any k satisfying $k \in (\gamma, 1 + r]$ if $p \le 0$ or $k \in (-\infty, 1 + r]$ if $p > 0$, there is a constant $c > 0$ that may depend on the initial data such that*

$$\int_{\mathbb{R}^d_*} |x|^k f(dx,t) \le c t^{\frac{k-1}{1-\gamma}}, \qquad t \ge 1. \tag{4.2}$$

The existence of a solution under the conditions of Theorem 4.1 can be obtained by adapting to the multicomponent setting the results proved in [4] (see also [1]). Those results have been obtained for locally integrable initial data $f_0 \in L^1_{\rm loc}(\mathbb{R}^d_*)$, but they can be adapted to the case of more general initial data $f_0 \in \mathscr{M}_1(\mathbb{R}^d_*)$. In [22] an existence result for measure-valued solutions was obtained for $d = 1$ for a larger class of non-homogeneous kernels. The moment estimate (4.2) has been derived in [4] for $d = 1$ (cf. Theorem 2.4 and Lemma 3.1 in [4]). We will present in Section 5 the ideas that allow to generalize the proof to multicomponent coagulation equations.

On the other hand, the existence of self-similar profiles is well known for a large class of homogeneous kernels K in one component (i.e. $d = 1$) coagulation systems. Using the results obtained for the one component system in [3, 4, 11] we can immediately prove the existence of self-similar profiles for the multicomponent system in terms of the solution to the one-component equation, having the form (1.18). Moreover, Theorem 1.4 guarantees that all self-similar profiles of (1.1) in the sense of Definition 2.3 have the form (1.18). We have the following result.

Theorem 4.2 *Suppose that $K : (\mathbb{R}^d_*)^2 \to \mathbb{R}_+$ is as in (1.5) and satisfies the homogeneity property (1.11) and the upper and lower bounds (1.6),* (1.7) *with $\gamma, \gamma + p \in [0, 1)$. Let $m \in \mathbb{R}^d_*$ with the form $m = (m_k)_{k=1}^d$ with $m_k \ge 0$ for any $k = 1, 2, ..., d$ and satisfying $|m| > 0$. Let $\theta_0 = \frac{m}{|m|} \in \Delta^{d-1}$. There exists at least one measure $F_0 \in \mathscr{M}_1(\mathbb{R}_*)$ such that the measure $F \in \mathscr{M}_1(\mathbb{R}^d_*)$ defined in (1.18) is a self-similar profile to* (1.1) *in the sense of Definition 2.3. Moreover, we have*

$$\int_{\mathbb{R}^d_*} \xi_k F(d\xi) = m_k\ , \quad k = 1, 2, ..., d. \tag{4.3}$$

Proof: Suppose that F has the form (1.18) with $F_0 \in \mathscr{M}_1(\mathbb{R}_*)$ a solution to the one-dimensional problem. We then have, using the variables (ρ, θ), that

$$F(d\xi) = F_0(\rho)\, \delta(\theta - \theta_0)\, d\rho d\nu(\theta).$$

Hence, (2.7) holds if and only if

$$\int_{(1,\infty)} \rho^{\gamma+p} F_0(\rho)\, d\rho + \int_{(0,1]} \rho^{-p+1} F_0(\rho)\, d\rho + \int_{\mathbb{R}_*} \rho F_0(\rho)\, d\rho < \infty \tag{4.4}$$

and (2.8) holds if and only if the following identity is satisfied

$$\begin{aligned} &\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(\rho\theta_0, r\theta_0)\, \rho\, [\varphi(\rho+r,\theta_0) - \varphi(\rho,\theta_0)]\, F_0(\rho)\, F_0(r)\, d\rho dr \\ &- \frac{1}{1-\gamma}\int_{\mathbb{R}_*} F_0(\rho)\, \frac{\partial\varphi}{\partial\rho}(\rho,\theta_0)\rho^2 d\rho \\ &= 0 \end{aligned} \tag{4.5}$$

with $\psi(\xi) = \rho\varphi(\rho,\theta)$, $\xi = \rho\theta$ and $\varphi \in C_c^1(\mathbb{R}_*^d)$ is an arbitrary test function. Notice that since (4.4) holds, then using the definition of θ_0, (4.3) is automatically satisfied.

We define $K_{\theta_0}(\rho, r) = K(\rho\theta_0, r\theta_0)$. Notice that the kernel K_{θ_0} is homogeneous with homogeneity γ and continuous. Due to (1.6), (1.7) we have that K_{θ_0} satisfies

$$c_1(\rho+r)^\gamma \Phi_p\left(\frac{\rho}{\rho+r}\right) \le K_{\theta_0}(\rho,r) \le c_2(\rho+r)^\gamma \Phi_p\left(\frac{\rho}{\rho+r}\right) \quad , \quad \rho, r \in \mathbb{R}_*. \tag{4.6}$$

The existence of measures $F_0 \in \mathscr{M}_1(\mathbb{R}_*)$ satisfying (4.5), (4.4) for kernels satisfying (4.6) with γ, p satisfying $\gamma, \gamma+p \in [0,1)$ is ensured by the results in [3, 4, 11]. Then the result follows. □

5 Moment estimates

A crucial step in the proof of the existence of self-similar solutions for one-dimensional coagulation equations is the derivation of some estimates for the moments of f which guarantee that the mass of the monomers of the solutions of the coagulation equations remain in the self-similar region $x \approx t^{\frac{1}{1-\gamma}}$ for arbitrary long times. Since these estimates are a crucial ingredient in the proof of Theorem 1.3 we will describe in this section how these estimates are derived for the solutions of the multicomponent coagulation equation (1.1).

Proposition 5.1 *Suppose that $K : (\mathbb{R}_*^d)^2 \to \mathbb{R}_+$ is a coagulation kernel satisfying (1.5), the homogeneity property (1.11) as well as the the upper and lower bounds (1.6),* (1.7) *with $\gamma, \gamma+p \in [0,1)$. Let f be a solution satisfying $f(0,\cdot) = f_0$ and* (4.1) *whose existence was stated in Theorem 4.1. Then, for any $k \in \mathbb{R}$ satisfying $k \in (\gamma, 1+r]$ if $p \le 0$ or $k \in (-\infty, 1+r]$ if $p > 0$, there is a constant $c > 0$ that may depend on the initial data such that, for all $t \ge 1$,*

$$\int_{\mathbb{R}_*^d} |x|^k f(dx,t) \le c t^{\frac{k-1}{1-\gamma}}. \tag{5.1}$$

The proof of Proposition 5.1 follows directly from the next two lemmas, each of which provides bounds for the moments $k > 1$ and $k < 1$ respectively, of a solution F to the coagulation equation in self-similar variables. More precisely, let f be a weak solution

to (1.1) constructed in Theorem 4.1. Using the self-similar variables (3.1), we obtain that $f(x,t) = (\varepsilon_t)^{1+d} F(\xi,\tau)$ where F satisfies

$$\frac{d}{d\tau}\int_{\mathbb{R}^d_*} F(d\xi,\tau)\psi(\xi,\tau) = \int_{\mathbb{R}^d_*} F(d\xi,\tau)\left[\partial_\tau\psi - \frac{1}{1-\gamma}\xi\cdot\partial_\xi\psi + \frac{1}{1-\gamma}\psi\right](\xi,\tau) + \frac{1}{2}\int_{\mathbb{R}^d_*}\int_{\mathbb{R}^d_*} \tilde{K}(\xi,\eta,\tau)\, F(d\xi,\tau)F(d\eta,\tau)[\psi(\xi+\eta,\tau)-\psi(\xi,\tau)-\psi(\eta,\tau)] \tag{5.2}$$

for all $\psi \in C^1_c(\mathbb{R}^d_* \times (0,\infty))$, and with $\tilde{K}(\xi,\eta,\tau) = e^{-\frac{\gamma}{1-\gamma}\tau} K\left(e^{\frac{1}{1-\gamma}\tau}\xi, e^{\frac{1}{1-\gamma}\tau}\eta\right)$. Notice that this identity is satisfied for *a.e.* $\tau \in (0,\infty)$ as it might be seen using (2.3) and the self-similar variables defined in (3.1).

Lemma 5.2 *Let γ, p and the kernel K satisfy the conditions of Proposition 5.1. Let $F_0 \in \mathscr{M}_1(\mathbb{R}^d_*)$ satisfy*

$$\int_{\mathbb{R}^d_*} F_0(y)|y|dy = m_0.$$

There is a weak solution F to the coagulation equation in self-similar variables (5.2) *with initial condition $F(\cdot,0) = F_0$. Then, for all $k \in (\gamma,1)$ there is a positive constant w_k depending on k such that*

$$\int_{\mathbb{R}^d_*} F(d\xi,\tau)\min\{|\xi|,1\}^k \le w_k, \quad \textit{for all } \tau \ge 1.$$

Proof: We generalize the proof of Lemma 3.1 in [4] to the multicomponent setting.

We can replace the initial value F_0 by $F_{\varepsilon,0}$ that is supported in the region $\{|x| \ge \varepsilon\}$ such that in this region $F_{0,\varepsilon}(dx) = F_0(dx)$. Similarly, we can use an $\varepsilon-$truncated coagulation operator such that each solution F_ε to (5.2) remains supported away from the origin in $\{|x| \ge \varepsilon\}$ for all times $t \ge 0$. All computations that we do in this proof are then fully justified for the regularized problem. As we will see, the moment estimates derived next will be uniform in the parameter ε which allows to conclude their validity for the original problem taking the limit $\varepsilon \to 0$ at the end of the argument. Since this argument is standard in the study of coagulation equations we will not reproduce it here. For simplicity we write in the following F instead of F_ε.

Define the time-independent test functions $\psi(\xi,\tau) = \varphi_A(\xi) = \min(A,|\xi|)^\ell$, with $A > 0$ and $\ell \in (\gamma,1]$, and $\tilde{\varphi}_A(\xi,\eta) = \varphi_A(\xi+\eta) - \varphi_A(\xi) - \varphi_A(\eta)$. Computing $\tilde{\varphi}_A(\xi,\eta)$ yields

$$\tilde{\varphi}_A(\xi,\eta) = \begin{cases} |\xi+\eta|^\ell - |\xi|^\ell - |\eta|^\ell, & \text{for} \quad |\xi|+|\eta| \le A \\ A^\ell - |\xi|^\ell - |\eta|^\ell, & \text{for} \quad |\xi|+|\eta| > A,\ |\xi|,|\eta| \le A \\ -|\xi|^\ell, & \text{for} \quad |\xi| \le A,\ |\eta| > A \\ -|\eta|^\ell, & \text{for} \quad |y| \le A,\ |\xi| > A \\ -A^\ell, & \text{for} \quad |\xi|,|\eta| > A. \end{cases}$$

Note that $\tilde{\varphi}_A(\xi,\eta) \le 0$. Moreover, the following estimate holds

$$\tilde{\varphi}_A(\xi,\eta) \le -A^\ell \mathbb{1}_{\{|\xi|,|\eta| \ge A\}}. \tag{5.3}$$

We also have the following bound for the first terms on the right-hand side of equation (5.2)

$$\left[-\frac{1}{1-\gamma}\xi\cdot\partial_\xi\varphi_A+\frac{1}{1-\gamma}\varphi_A\right](\xi)\leq\frac{1}{1-\gamma}\varphi_A(\xi). \tag{5.4}$$

On the other hand, any kernel in the class considered satisfies the lower bound

$$K(\xi,\eta)\geq c_1(|\xi||\eta|)^{\gamma/2}. \tag{5.5}$$

This follows from the lower bound

$$\begin{aligned}K(\xi,\eta)&\geq c_1(|\xi|+|\eta|)^\gamma\left(\frac{|\xi|}{|\xi|+|\eta|}\right)^{-p}\left(\frac{|\eta|}{|\xi|+|\eta|}\right)^{-p}\\&=c_1(|\xi|+|\eta|)^{\gamma+2p}(|\xi||\eta|)^{-p}=c_1(|\xi||\eta|)^{\gamma/2}(|\xi|+|\eta|)^{\gamma+2p}(|\xi||\eta|)^{-p-\gamma/2}\end{aligned}$$

and from the fact that $(|\xi|+|\eta|)^{\gamma+2p}(|\xi||\eta|)^{-p-\gamma/2}\geq 2^{2p+\gamma}\geq 1$. To obtain the latter inequality we use that, due to the Young inequality, $|\xi|^{1/2}|\eta|^{1/2}\leq\frac{1}{2}(|\xi|+|\eta|)$, as well as the fact that $\gamma+2p\geq 0$ (see (1.8)).

Using (5.3), (5.4) and (5.5) it follows from (5.2) that, for all $A>0$,

$$\frac{d}{d\tau}\int_{\mathbb{R}^d_*}F(d\xi,\tau)\varphi_A(\xi)\leq\frac{1}{1-\gamma}\int_{\mathbb{R}^d_*}F(d\xi,\tau)\varphi_A(\xi)-\frac{A^\ell}{2}\left(\int_{\{|\xi|\geq A\}}|\xi|^{\gamma/2}F(d\xi,\tau)\right)^2. \tag{5.6}$$

The strategy now is to obtain a differential inequality for the moment $\min(1,|\xi|)^{\gamma+\delta}$ with $0<\delta<1-\gamma$.

Define $\phi(y)=\min(1,y)^{\gamma/2+\delta}$. Using integration by parts we may write

$$\int_{\mathbb{R}^d_*}\phi(|\xi|)|\xi|^{\gamma/2}F(d\xi,\tau)=\int_0^\infty\phi'(A)\left(\int_{|\xi|>A}|\xi|^{\gamma/2}F(d\xi,\tau)\right)dA$$

and from Cauchy-Schwarz inequality, we obtain the estimate

$$\left(\int_0^\infty\phi'(A)\left(\int_{|\xi|>A}|\xi|^{\gamma/2}F(d\xi,\tau)\right)dA\right)^2\leq$$
$$c\int_0^\infty\phi'(A)A^{\gamma/2}\left(\int_{|\xi|>A}|\xi|^{\gamma/2}F(d\xi,\tau)\right)^2dA,$$

with

$$c:=\int_0^\infty\phi'(A)A^{-\gamma/2}dA=\int_0^1A^{-1+\delta}dA<\infty.$$

Using now (5.6) it follows

$$\begin{aligned}&\left(\int_{\mathbb{R}^d_*}\phi(|\xi|)|\xi|^{\gamma/2}F(d\xi,\tau)\right)^2\leq c\int_0^\infty\phi'(A)A^{\gamma/2}\left(\int_{|\xi|>A}|\xi|^{\gamma/2}F(d\xi,\tau)\right)^2dA\\&\leq 2c\int_0^\infty\phi'(A)A^{\gamma/2}A^{-\ell}\left(\frac{1}{1-\gamma}\int_{\mathbb{R}^d_*}F(d\xi,\tau)\varphi_A(\xi)-\frac{d}{d\tau}\int_{\mathbb{R}^d_*}F(d\xi,\tau)\varphi_A(\xi)\right)\\&=2c\left(\frac{1}{1-\gamma}\int_{\mathbb{R}^d_*}F(d\xi,\tau)\psi(\xi)-\frac{d}{d\tau}\int_{\mathbb{R}^d_*}F(d\xi,\tau)\psi(\xi)\right)\end{aligned} \tag{5.7}$$

with $\psi(\xi)$ defined by

$$\psi(\xi) = \int_0^\infty \varphi_A(\xi)\phi'(A)A^{\gamma/2-\ell}dA.$$

Since $\varphi_A(\xi) = \min(A, |\xi|)^\ell$, then for the choice $\ell = \gamma + 2\delta \leq 1$, one easily concludes that ψ satisfies the bounds

$$\frac{1}{C}\min(1, |\xi|)^{\gamma+\delta} \leq \psi(\xi) \leq C\min(1, |\xi|)^{\gamma+\delta} \tag{5.8}$$

for some positive constant C. Then (5.7) together with (5.8) imply an inequality for the moment $\min(1, |\xi|)^{\gamma+\delta}$,

$$\frac{d}{d\tau}\int_{\mathbb{R}^d_*} F(d\xi, \tau)\min(1, |\xi|)^{\gamma+\delta} + \kappa_1\left(\int_{\mathbb{R}^d_*} F(d\xi, \tau)\min(1, |\xi|)^{\gamma+\delta}\right)^2 \leq$$
$$\kappa_2\int_{\mathbb{R}^d_*} F(d\xi, \tau)\min(1, |\xi|)^{\gamma+\delta}$$

for some positive constants κ_1, κ_2. Integrating this inequality in time yields the desired uniform in ε estimate for F_ε and $\tau \geq 1$. □

Lemma 5.3 *Let γ, p and the kernel K satisfy the conditions of Proposition 5.1. Let $F_0 \in \mathscr{M}_1(\mathbb{R}^d_*)$ satisfy*

$$M_k := \int_{\mathbb{R}^d_*} F_0(y)|y|^k dy < C$$

and F be the weak solution to the coagulation equation in self-similar variables (5.2) *with initial condition $F(\cdot, 0) = F_0$ obtained in Lemma 5.2. Then, for all $k \in (1, 1+\delta)$ there is a positive constant w_k depending on k such that*

$$\sup_{t\geq 0}\int_{\mathbb{R}^d_*} F(y, t)|y|^k dy \leq \max\{w_k, M_k\}.$$

The idea of the proof is to use the test function $\varphi(x) = |x|^k$ and to obtain an estimate in terms of the lower order moments. This idea has been widely used in the analysis of one component coagulation equation (see for instance [4] Lemma 3.4 and the book [1]), and it can be immediately adapted to the multicomponent case. The use of this test function allows to reduce the estimate for the moment $\int_{\mathbb{R}^d_*} F(t, y)|y|^k dy$ to the estimate of moments with an exponent smaller than k. We can then use the estimate obtained in Lemma 5.2. Since the argument is by now standard, we will not give more details here.

6 Long time asymptotics for kernels which are constant along any direction

In this section we prove Theorem 1.5. We need a preliminary result yielding well-posedness for (1.1) with kernels satisfying (1.19).

Lemma 6.1 *Suppose that the kernel K is as in (1.19). Then, for any $f_0 \in L^1\left(\mathbb{R}^d_*\right)$ satisfying (1.20) and (1.21) there exists a unique solution $f \in C^1\left((0,\infty); L^1\left(\mathbb{R}^d_*\right)\right) \cap C\left([0,\infty); L^1\left(\mathbb{R}^d_*\right)\right)$ to (1.1) in the classical sense with initial value $f(\cdot, 0) = f_0(\cdot)$. The function f is also a weak solution to (1.1) in the sense of Definition 2.1.*

Proof: Due to the boundedness of the kernel K we can prove the existence and uniqueness of a solution f just reformulating (1.1) as an integral equation and using a fixed point argument in the space $C\left([0,\infty);L^1\left(\mathbb{R}^d_*\right)\right)$. The fact that f is also a weak solution in the sense of Definition 2.1 follows by multiplying (1.1) by a test function $\varphi(x,t)$ and using integration by parts in the variable t as well as Fubini's Theorem. These computations are standard, we refer to the book [1] for further details. □

We now prove Theorem 1.5.

Proof of Theorem 1.5. For kernels with the form (1.19) and for initial data $f_0 \in L^1\left(\mathbb{R}^d_*\right)$ with the properties stated in Theorem 1.5, the conditions in Theorem 1.1 are satisfied. Indeed, we can apply Lemma 6.1 and Theorem 4.1 with $\gamma = p = 0$ with initial data $f_0 \in L^1(\mathbb{R}^d_*)$ satisfying the moment bound (1.21) to obtain a solution $f \in C\left((0,\infty), \mathscr{M}_1\left((0,\infty)\times\Delta^{d-1}\right)\right)$ to (1.1). Moreover, since the kernel is bounded, one can show that the solution f is unique using a standard fixed-point argument. We then define $G \in C\left((0,\infty), \mathscr{M}_1\left((0,\infty)\times\Delta^{d-1}\right)\right)$ by means of (3.1), (3.2), (3.3) with an initial condition $\bar{G} \in L^1\left((0,\infty)\times\Delta^{d-1}\right)$. We will later assume that the initial datum $\bar{G}$ belongs to a more restrictive set. Due to the uniqueness of f (and G), we can define the semigroup, $S(\tau)$, for the evolution of G. We will then write $G(\cdot,\cdot,\tau) = S(\tau)\bar{G}(\cdot,\cdot)$. Notice that (5.1) (or Proposition 5.1) implies the estimate

$$\int_{\mathbb{R}_*\times\Delta^{d-1}} \rho^k G(\rho,\theta,\tau)d\Omega \le C_1, \quad k \in \left[\frac{1}{a}, a\right] \quad \text{for some } a > 1\,. \tag{6.1}$$

We recall that $\tau = \log(t+1)$ with $t \ge 1$, and again we assume $\tau \ge \ln 2$ throughout.

Let $m_0 = |m|$. We denote as $\mathscr{N}(\theta_0; m_0; C_1)$ the subset of $\mathscr{M}_1\left((0,\infty)\times\Delta^{d-1}\right)$ that consists in the measures $\bar{G}$ supported along the line $\{\theta = \theta_0\}$ and satisfying the estimate

$$\int_{\mathbb{R}_*\times\Delta^{d-1}} \rho^k \bar{G}(\rho,\theta)d\Omega \le C_1 \tag{6.2}$$

(cf. (6.1)) and having the mass $\int_{\mathbb{R}_*\times\Delta^{d-1}} \rho\bar{G}(\rho,\theta)d\Omega = m_0$. Notice that $\mathscr{N}(\theta_0; m_0; C_1)$ is a compact subset of $\mathscr{M}_1\left((0,\infty)\times\Delta^{d-1}\right)$ in the weak$-*$ topology of $\mathscr{M}_1\left((0,\infty)\times\Delta^{d-1}\right)$. We will denote as $\mathrm{dist}(\cdot,\cdot)$ a metric which characterizes the weak$-*$ topology of bounded measures in $\mathscr{M}_1\left((0,\infty)\times\Delta^{d-1}\right)$. We can then apply Theorem 1.1 that implies that

$$\mathrm{dist}\left(G(\rho,\theta,\tau), \mathscr{N}(\theta_0; m_0; C_1)\right) \to 0 \quad \text{as} \quad \tau \to \infty. \tag{6.3}$$

We denote now as $G_0(\rho,\theta;\theta_0,m_0)$ the measure in $\mathscr{M}_1\left((0,\infty)\times\Delta^{d-1}\right)$ defined by

$$G_0(\rho,\theta;\theta_0,m_0) = \frac{4}{\left(Q(\theta_0)\right)^2 m_0}\frac{1}{\rho^{d-1}}\exp\left(-\frac{2\rho}{Q(\theta_0)m_0}\right)\delta(\theta-\theta_0)\,.$$

Then $G_0 \in \mathscr{N}(\theta_0; m_0; C_1)$. Given $\bar{G} \in \mathscr{N}(\theta_0; m_0; C_1)$ we can characterize the evolution semigroup $S(\tau)$ in terms of the corresponding evolution semigroup for the one-dimensional coagulation evolution. More precisely, we can obtain $S(\tau)\bar{G}$ as the element of $C\left((0,\infty), \mathscr{M}_1\left((0,\infty)\times\Delta^{d-1}\right)\right)$ given by $\tau \mapsto \hat{G}(\rho,\tau)\,\delta(\theta-\theta_0)$ where $\hat{G}(\rho,\tau)$ is the solution of the one-component coagulation equation with constant kernel $K = Q(\theta_0)$ and initial value $\hat{G}_0(\rho) = \bar{G}(\rho,\theta_0)$. The existence and uniqueness of $\hat{G}(\rho,\tau)$ is given in [17]. The results in [17] also provide the long-time behaviour of the solutions to the one-component equation. In particular, they imply that for any measure $\bar{G} \in \mathscr{N}(\theta_0; m_0; C_1)$ we have that $S(\tau)\bar{G} \to$

$G_0(\cdot,\cdot;\theta_0,m_0)$ as $\tau\to\infty$ in the weak$-*$ topology of $\mathscr{M}_1\left((0,\infty)\times\Delta^{d-1}\right)$. Moreover, the compactness of $\mathscr{N}(\theta_0;m_0;C_1)$ implies that the convergence is uniform. More precisely, for any $\varepsilon>0$ there exists $T=T(\varepsilon)>0$ such that

$$\operatorname{dist}\left(S(\tau)\bar{G}(\rho,\theta),G_0(\rho,\theta;\theta_0,m_0)\right)<\frac{\varepsilon}{2},\quad \text{for any } \bar{G}\in\mathscr{N}(\theta_0;m_0;C_1)\quad\text{and } \tau\geq T \tag{6.4}$$

On the other hand, the evolution equation yields an evolution semigroup that is continuous in the weak$-*$ topology of measures with respect to the initial value. We can then argue as follows in order to prove that the solution $G(\rho,\theta,\tau)$ is at a distance smaller than ε from $G_0(\rho;\theta_0,m_0)$ for sufficiently large times.

Let $\varepsilon>0$ be an arbitrarily small number. Note that, from (6.4), there exists $T=T(\varepsilon)$ such that

$$\operatorname{dist}\left(S(T)\bar{G},G_0(\rho,\theta;\theta_0,m_0)\right)<\frac{\varepsilon}{2},\quad \text{for any } \bar{G}\in\mathscr{N}(\theta_0;m_0;C_1). \tag{6.5}$$

On the other hand, the continuity of the semigroup $S(\tau)$ implies that there exists $\delta=\delta(\varepsilon,T)>0$, that we can assume to satisfy $\delta<\frac{\varepsilon}{2}$ such that, for any $G_1\in\mathscr{M}_1\left((0,\infty)\times\Delta^{d-1}\right)$ satisfying $\operatorname{dist}(G_1(\rho,\theta),\mathscr{N}(\theta_0;m_0;C_1))<\delta$, then the following inequality holds

$$\operatorname{dist}\left(S(T)G_1(\rho,\theta),S(T)\bar{G}(\rho,\theta)\right)<\frac{\varepsilon}{2},\quad \text{for any } \bar{G}\in\mathscr{N}(\theta_0;m_0;C_1). \tag{6.6}$$

In particular, notice that this continuity estimate on the evolution semigroup is uniform in the class of measures G_1 satisfying the moment bound (6.2). From (6.3) it follows that there exists $T_1=T_1(\varepsilon,T)=T_1(\varepsilon)$ such that for any $\bar{\tau}\geq T_1$ we have $\operatorname{dist}(G(\rho,\theta,\bar{\tau}),\mathscr{N}(\theta_0;m_0;C_1))<\delta$. Then, given any $\tau\geq T_1+T$ we can write $\tau=\bar{\tau}+T$, which ensures that $\tau\geq T_1$. From (6.5) and (6.6), taking $G_1(\rho,\theta)=G(\rho,\theta,\bar{\tau})$, we finally obtain

$$\begin{aligned}
&\operatorname{dist}(G(\rho,\theta,\tau),G_0(\rho,\theta;\theta_0,m_0))\\
&=\operatorname{dist}(S(T)G(\rho,\theta,\bar{\tau}),G_0(\rho,\theta;\theta_0,m_0))\\
&\leq\operatorname{dist}\left(S(T)G(\rho,\theta,\bar{\tau}),S(T)\bar{G}(\rho,\theta)\right)+\operatorname{dist}\left(S(T)\bar{G}(\rho,\theta),G_0(\rho,\theta;\theta_0,m_0)\right)\\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

Since ε is arbitrary, the result follows. □

Remark 6.2 *Combining the methods used in the previous proof with the ones used in [2, 24] it would be possible to prove convergence to a self-similar solution supported along a particular direction for coagulation kernels that are near constant along each particular direction of the space of cluster compositions.*

7 Application to discrete systems

In this section we are concerned with the discrete version of the multicomponent Smoluchowski coagulation equation (1.1). In the discrete case, clusters are characterized by the composition vector $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_d)\in\mathbb{N}_0^d\backslash\{O\}$ consisting of d different monomer types, where $\mathbb{N}_0=\{0,1,2,3,\ldots\}$ and $O=(0,0,\ldots,0)$. The concentration $n_\alpha(t)$ of particles of composition α at time $t\geq 0$ is governed by the following equation

$$\partial_t n_\alpha=\kappa_d[n_\alpha],\qquad \kappa_d[n_\alpha]:=\frac{1}{2}\sum_{\beta<\alpha}K_{\alpha-\beta,\beta}n_{\alpha-\beta}n_\beta-n_\alpha\sum_{\beta>O}K_{\alpha,\beta}n_\beta. \tag{7.1}$$

Given $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_d)$ we write $\beta < \alpha$ to indicate that $\beta_k \leq \alpha_k$ for all $k = 1, 2, \ldots, d$, and in addition $\alpha \neq \beta$. The collision kernel $K_{\alpha,\beta}$, which we assume to satisfy the symmetry condition $K_{\alpha,\beta} = K_{\beta,\alpha}$, describes the coagulation rate between clusters with compositions α and β. Equation (7.1) was first proposed in [23] in the case of particles described by a single component, corresponding here to the case $d = 1$.

Notice that (7.1) can be considered a particular case of (1.1) for measure solutions of (1.1) with the form

$$f(x,t) = \sum_{\alpha \in \mathbb{N}_0^d \setminus \{O\}} n_\alpha \delta(x - \alpha). \tag{7.2}$$

In this case, f in (7.2) solves (1.1) if $(n_\alpha)_\alpha$ solves (7.1) with the kernel $K_{\alpha,\beta} = K(\alpha, \beta)$.

We will assume that the kernel $K_{\alpha,\beta}$ can be written as

$$K_{\alpha,\beta} = K(\alpha, \beta) \ \text{ for } \alpha, \beta \in \mathbb{N}_0^d \setminus \{O\} \tag{7.3}$$

for some $K(x, y)$ satisfying the conditions (1.5)-(1.8). Notice that if $K_{\alpha,\beta}$ is a kernel for which there are c_1, c_2 such that $c_1(|\alpha| + |\beta|)^\gamma \Phi_p(\frac{|\alpha|}{|\alpha|+|\beta|}) \leq K_{\alpha,\beta} \leq c_2(|\alpha| + |\beta|)^\gamma \Phi_p(\frac{|\alpha|}{|\alpha|+|\beta|})$ for all $\alpha, \beta \in \mathbb{N}_0^d \setminus \{O\}$, we may find a function $K(x, y)$ of the required type so that (7.3) holds (changing, if needed, the values of the constants c_1, c_2).

We notice that, since the solutions of (7.1) can be interpreted as particular solutions of (1.1) with the form (7.2), the localization results for (7.1) will follow from the corresponding results for (1.1).

More precisely we can formulate the following discrete version of Theorem 1.1.

Theorem 7.1 *Suppose that $K_{(\cdot,\cdot)} : \left(\mathbb{N}_0^d \setminus \{O\}\right)^2 \to [0, \infty)$ is a mapping that can be written in the form (7.3) for some function $K : (\mathbb{R}_*^d)^2 \to [0, \infty)$ satisfying (1.5) as well as the bounds (1.6)–(1.8) for some $\gamma \in [0, 1)$ and $p \in \mathbb{R}$ such that $0 \leq \gamma + p < 1$. Let $\{n_{\alpha,0}\}_{\alpha \in \mathbb{N}_0^d \setminus \{O\}}$ satisfy $0 < \sum_{\alpha \in \mathbb{N}_0^d \setminus \{O\}} |\alpha|^{1+r} n_{\alpha,0} < \infty$ for some $r > 0$. Then there is a solution $\{n_\alpha(\cdot)\}_{\alpha \in \mathbb{N}_0^d \setminus \{O\}}$ of (1.1) such that $n_\alpha(0) = n_{\alpha,0}$ and, for each $t > 0$,*

$$\sum_{\alpha \in \mathbb{N}_0^d \setminus \{O\}} |\alpha| \, n_\alpha(t) = \sum_{\alpha \in \mathbb{N}_0^d \setminus \{O\}} |\alpha| \, n_{\alpha,0} := m_0 \in (0, \infty).$$

Moreover, it satisfies the following localization property. There exists a positive function $\delta \in C(0, \infty)$ such that $\lim_{t \to \infty} \delta(t) = 0$ and with the property that

$$\lim_{t \to \infty} \left| \sum_{\left\{\delta(t) t^{\frac{1}{1-\gamma}} \leq |\alpha| \leq (\delta(t))^{-1} t^{\frac{1}{1-\gamma}}\right\} \cap \left\{\left|\frac{\alpha}{|\alpha|} - \theta_0\right| \leq \delta(t)\right\}} |\alpha| \, n_\alpha(t) - m_0 \right| = 0 \tag{7.4}$$

where $\theta_0 \in \Delta^{d-1}$ is defined by means of

$$\theta_0 := \frac{\sum_{\alpha \in \mathbb{N}_0^d \setminus \{O\}} \alpha n_{\alpha,0}}{m_0}. \tag{7.5}$$

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Compliance with ethical standards

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M. A. Ferreira Institut de Mathématiques de Toulouse, UMR 5219, CNRS,
F-31062, Toulouse Cedex 9, France,
CMUC, Department of Mathematics, University of Coimbra,
3000-413 Coimbra, Portugal,
and
Department of Mathematics and Statistics, University of Helsinki,
FI-00014 Helsingin yliopisto, Finland.
E-mail: marina.ferreira@math.univ-toulouse.fr

J. Lukkarinen Department of Mathematics and Statistics, University of Helsinki,
P.O. Box 68, FI-00014 Helsingin yliopisto, Finland.
E-mail: jani.lukkarinen@helsinki.fi

A. Nota: Department of Information Engineering, Computer Science and Mathematics,
University of L'Aquila, 67100 L'Aquila, Italy.
E-mail: alessia.nota@univaq.it

J. J. L. Velázquez Institute for Applied Mathematics, University of Bonn,
Endenicher Allee 60, D-53115 Bonn, Germany.
E-mail: velazquez@iam.uni-bonn.de