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Evolution of Measures in Nonsmooth Dynamical Systems: Formalisms and Computation

Saroj Prasad Chhatoi, Aneel Tanwani[†], Didier Henrion[‡]

Abstract

This article develops mathematical formalisms and provides numerical methods for studying the evolution of measures in nonsmooth dynamical systems using the continuity equation. The nonsmooth dynamical system is described by an evolution variational inequality and we derive the continuity equation associated with this system class using three different formalisms. The first formalism consists of using the superposition principle to describe the continuity equation for a measure that disintegrates into a probability measure supported on the set of vector fields and another measure representing the distribution of system trajectories at each time instant. The second formalism is based on the regularization of the nonsmooth vector field and describing the measure as the limit of a sequence of measures associated with the regularization parameter. In doing so, we obtain quantitative bounds on the Wasserstein metric between measure solutions of the regularized vector field and the limiting measure associated with the nonsmooth vector field. The third formalism uses a time-stepping algorithm to model a time-discretized evolution of the measures and show that the absolutely continuous trajectories associated with the continuity equation are recovered in the limit as the sampling time goes to zero. We also validate each formalism with numerical examples. For the first formalism, we use polynomial optimization techniques and the moment-SOS hierarchy to obtain approximate moments of the measures. For the second formalism, we illustrate the bounds on the Wasserstein metric for an academic example for which the closed-form expression of the Wasserstein metric can be calculated. For the third formalism, we illustrate the time-stepping based algorithm for measure evolution on an example that shows the effect of the concentration of measures.

Keywords: Nonsmooth dynamical systems, Optimal transport, Polynomial optimization.

1 Introduction

The study of evolution of measures in finite dimensional systems has found relevance in the design of optimal control problems, understanding the system behavior under uncertainties, and

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several other applications. The primary step in this direction is to understand how the probabilistic initial conditions evolve in time under the action of a vector field. Such questions have been fairly well studied for single-valued dynamical systems with sufficient regularity (such as Lipschitz continuity) of the vector field. However, when we relax the regularity assumptions on the vector field, the question of evolution of measures brings forth some interesting questions which are of relevance for the applications as well. We are thus motivated to study the evolution of probability measures for a class of dynamical systems described by differential inclusions and in particular where the differential inclusion models the trajectories constrained to a pre-specified set. We present different mathematical formalisms to study measure evolution for such dynamical systems and provide corresponding numerical algorithms for simulations.

For an autonomous dynamical system described by an ordinary differential equation (ODE) with Lipschitz continuous vector field, the time evolution of the measure describing the initial condition is governed by a linear partial differential equation (PDE), commonly called the continuity equation or the Liouville equation [1, Section 5.4]. The solution to this PDE, that is the probability measure describing the distribution at a given time, is the push-forward or image of the initial probability measure through the flow map at that time. Lipschitz continuity of the vector field ensures that the flow map of the ODE is invertible, which in turn ensures that the push-forward measure is the unique solution to the continuity equation. The Cauchy problem for continuity equation with Sobolev fields was studied by [2]. Continuity equations corresponding to one-sided Lipschitz vector fields have been studied in [3, 4]. In [5] and [6], the authors consider fields of bounded variation and discuss the potential nonuniqueness of solutions to the continuity equation by introducing the notion of superposition principle. For the differential inclusions with convex set-valued mappings, the reference [7] provides a generalized superposition principle. For our purposes, the solutions based on the superposition principle are useful for numerical purposes. We propose a vector field selection from a time-varying differential inclusion from which we derive a continuity equation suitable for numerical algorithms. We use the converse statement of the superposition principle to characterize all possible solutions to the proposed continuity equation.

In this article, we are particularly concerned with a class of dynamical systems where the nonsmoothness arises due to the modeling of constraints on state trajectories. Such systems are described by the inclusion

$$\dot{x}(t) \in f(x(t)) - \mathcal{N}_{S(t)}(x(t)) \quad (1)$$

where $\mathcal{N}_S(x) \in \mathbb{R}^n$ denotes the outward normal cone to the set S at the point $x \in \mathbb{R}^n$. Since the normal cone takes a zero value in the interior of S , it is clear that the right-hand side of (1) is potentially discontinuous at the boundary of the set S . One can also think of (1) as an evolution variational inequality, described as

$$\langle \dot{x}(t) - f(x(t)), y - x(t) \rangle \geq 0,$$

for all $y \in S$, $x(t) \in S$, $t \in [0, T]$, where the brackets denote the inner product between vectors. Such dynamical systems have been a matter of extensive study in past decades due to their relevance in engineering and physical systems. The survey article [8], and a research monograph [9], provide an overview of different research oriented directions in the literature pertaining to system (1) and its connections to different classes of nonsmooth mathematical

models. Analysis of such systems requires tools from variational analysis, nonsmooth analysis, set-valued analysis [10, 11, 12]. For a fixed initial condition, $x(0) \in S$, the question of existence and uniqueness of solution to system (1) has already been well-established in the literature, and the origins of such works can be found in [13], see [14] for a recent exposition.

However, if we consider the initial conditions described by a probability measure, then the evolution of this measure under the dynamics of (1) has received much less attention in the literature. One can study such problems by considering stochastic versions of (1) by adding a diffusion term on the right-hand side. Such systems first came up in the study of variational inequalities arising in stochastic control [15], and in the literature, we can find results on existence and uniqueness of solutions in appropriate function space. In [16], this is done by considering Yosida approximations of the maximal monotone operator, whereas [17] provides a proof based on time-discretization of system (1). These approaches have been generalized for prox-regular set S in [18], and the case where the drift term contains Young measures [19, 20]. In [21], the authors provide a constructive approximation of measures associated with system (1) with $f \equiv 0$, which are based on a generalization of time-stepping algorithm and involves projecting the density function onto the constraint set with respect to the Wasserstein metric.

The main contribution of this article is to provide different formalisms for describing the evolution of measures for the class of systems considered in (1). In particular, our contribution lies in studying three different techniques for describing the propagation of probabilistic initial conditions for system (1) and we provide numerical methods for each of these techniques.

The first approach is based on using the previously mentioned superposition principle. Here, we consider a continuity equation where the velocity vector field is obtained by a selection of the set-valued mapping in system dynamics, which results in a (possibly non-unique) solution to the measure evolution. We develop a converse result which actually shows that all possible solutions can be associated with a selection of the vector field. The tools used in the process are similar to the ones appearing in [7], but we develop a specific representation of the continuity equation in terms of a measure which can be computed numerically using the moment-SOS (polynomial sums of squares) hierarchy [22] and semi-definite programming based techniques.

The second approach builds on our recent work in [23] where we approximate the dynamics of system (1) by ODEs with Lipschitz continuous right-hand side. The solution of the continuity equation associated with each ODE provides a sequence of measures which allows us to approximate the solution of the measure evolution problem. We show that the limiting measure can be represented by the pushforward of the unique flow map of system (1) and we develop quantitative bounds on the Wasserstein distance between the limiting measure and its approximations obtained from the regularization method.

Finally, another approach we adopt for studying the evolution of measures subject to constrained dynamics is based on computing an approximation of the transport maps for system (1) via time discretization. Time discretization based techniques are well known for constructing solutions to evolution PDEs using the gradient flow structure of the Wasserstein space [24]. Time discretization schemes have been recently used in [25], which exploits the gradient flow structure for the system class (4) in the Euclidean space to construct solutions for constrained optimization problems. For sweeping processes without the perturbation term, this approach was adopted in [21] and it generalizes the classical time-stepping algorithm proposed in [13] to the setting of measures. We use these techniques to construct the solutions of the

continuity equation associated with system (1). In particular, one computes the distribution at discrete time instants by interpolating the distribution through the perturbation term, and then projecting it onto the constraint set with respect to the Wasserstein metric. This scheme is built on the dynamic viewpoint of optimal transport problems where the absolutely continuous curves in Wasserstein space satisfy the continuity equation. Under certain conditions, we show that the sequence of time-discretized measures converges to a solution described by the push-forward of the initial distribution under the transport map.

Finally, we address the computational aspects for each of the three formalisms with the help of academic examples. The proposed continuity equation could be seen as an infinite dimensional linear program problem in the space of measures. We use the moment-SOS hierarchy to approximate the moments of the measures and we provide an illustration of this method. For the second formalism based on functional regularization, we explicitly, for a one dimensional case, compute the Wasserstein distance between the measure solutions to the nonsmooth system and to approximation obtained by regularization method. For the last formalism based on time-stepping algorithm for measures, we consider an example of a two-dimensional system based on time-space discretization and then evolving the measures using the given algorithm.

2 Preliminaries and Overview

2.1 Measure Evolution

Consider the dynamical system described by an ordinary differential equation (ODE):

$$\dot{x} = f(t, x). \quad (2)$$

If the vector field $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is such that $f(\cdot, x)$ is Lebesgue measurable for each $x \in \mathbb{R}^n$ and $f(t, \cdot)$ is Lipschitz continuous for each $t \in \mathbb{R}_{\geq 0}$, then there exists a unique absolutely continuous function $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ that solves (2). Consequently, we consider the *flow map* $X_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ parameterized by $t \in \mathbb{R}_{\geq 0}$ having the property that $x(t) = X_t(x_0)$ for each $x_0 \in \mathbb{R}^n$. It is also of interest to study the evolution of probability measures for system (2) when the initial condition is described by a probability distribution on \mathbb{R}^n , that is, $x(0) \sim \mu_0$, where $\mu_0 \in \mathcal{P}(\mathbb{R}^n)$, the set of probability measures on \mathbb{R}^n . In words, the law of the random variable $x(0)$ is the probability measure μ_0 . The resulting measure $\mu_t \in \mathcal{P}(\mathbb{R}^n)$, for $t \in \mathbb{R}_{\geq 0}$, is defined by the *continuity equation*, also called the Liouville equation, a linear partial differential equation (PDE) which models the transport of a distribution along the flow of trajectories of the underlying system and preserves the mass of the distribution. For the cases where the vector field $f(t, x)$ is Lipschitz in x for each t , the continuity equation reads

$$\partial_t \mu_t + \nabla \cdot (f(t, \cdot) \mu_t) = 0 \quad (3)$$

where $\nabla \cdot$ is the divergence operator. The equation is to be understood in the weak sense, i.e. $\int [\partial_t \varphi(t, x) + \nabla_x \varphi(t, x) \cdot f(t, x)] d\mu(t, x) = 0$ where $\varphi \in \mathcal{C}^1([0, T] \times \mathbb{R}^n)$ and ∇_x is the gradient operator. Furthermore, a measure μ_t solving (3) can be represented as the push-forward of μ_0 under the mapping X_t , denoted $\mu_t = X_{t\#} \mu_0$. Here, and throughout this article, for a function $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and a measure μ_0 supported on a set in \mathbb{R}^n the push-forward of μ_0 under the

mapping g is denoted by $g_{\#}\mu_0$ and it is defined as $g_{\#}\mu_0(A) := \mu_0(\{x \in \mathbb{R}^n : g(x) \in A\})$ for every measurable set $A \subset \mathbb{R}^m$.

In this work, we particularly consider the class of following differential inclusions:

$$\dot{x}(t) \in f(t, x) - \mathcal{N}_{S(t)}(x), \quad x(0) \sim \mu_0 \quad (4)$$

where $f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a vector field, $S : [0, T] \rightrightarrows \mathbb{R}^n$ is a set-valued mapping, and $\mathcal{N}_{S(t)}(x)$ denotes the outward normal cone to the convex set $S(t)$ at $x \in S(t)$. We impose the following assumptions on system class (4) so that the system is well-posed.

Assumption 1. *There exists $L_f > 0$ such that*

$$\begin{aligned} |f(t, x_1)| &\leq L_f(1 + |x_1|), \\ |f(t, x_1) - f(t, x_2)| &\leq L_f|x_1 - x_2| \end{aligned}$$

for all $x_1, x_2 \in \mathbb{R}^n$.

Assumption 2. *The mapping $S : [0, T] \rightrightarrows \mathbb{R}^n$ is closed and convex-valued for each $t \in [0, T]$, and $S(\cdot)$ varies in a Lipschitz continuous manner with time, i.e., there exists a constant L_s such that*

$$d_H(S(t), S(s)) \leq L_s|t - s|$$

where $d_H(A, B) := \max\left\{\sup_{x \in B} \text{dist}(x, A), \sup_{x \in A} \text{dist}(x, B)\right\}$ is the Hausdorff distance between the sets A and B .

Under these two assumptions, several references in the literature prove the existence and uniqueness of solutions to (4) with $x(0) \in S(0) \subset \mathbb{R}^n$, see for example [8] for an overview. In this article, we are interested in studying the evolution of measures for system class (4). The PDE considered in (3) cannot be readily obtained in that case and we study three different principles to describe the evolution of measures for our system (4). In the remainder of this section, we provide an overview of these techniques from the existing literature. In the later sections, we develop each of these techniques for system class (4).

2.2 Superposition Principle

In the first instance, we look at (4) as a differential inclusion with a set-valued right-hand side in the dynamics. In this regard, we see that the evolution of measures is described using the *superposition principle* for the differential inclusions of the form

$$\dot{x}(t) \in F(t, x(t)) \quad (5)$$

where $F : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping. Let us explain briefly and informally what is the superposition principle. A *selection* of F is a mapping $(t, x) \mapsto \bar{f}(t, x) \in F(t, x)$. Associated with a selection is an absolutely continuous solution $\gamma \in AC([0, T]; \mathbb{R}^n)$ with $\gamma(0) = x(0)$ such that $\dot{\gamma}(t) = \bar{f}(t, \gamma(t))$ for Lebesgue a.e. $t \in \mathbb{R}_{\geq 0}$. Let us consider the set of all admissible curves

$$\Gamma_T := \{\gamma \in AC([0, T]; \mathbb{R}^n) : \dot{\gamma} = \bar{f}(t, \gamma), \bar{f} \text{ a selection of } F\}.$$

The evaluation map is defined as a Borel measurable map $e_t : \mathbb{R}^n \times \Gamma_T \rightarrow \mathbb{R}^n$ such that

$$e_t(x, \gamma) := \gamma(t) \quad \forall t \in [0, T] \text{ and } \gamma(0) = x, \gamma \in \Gamma_T. \quad (6)$$

Let η be a probability measure such that $\eta \in \mathcal{P}(\mathbb{R}^n \times \Gamma_T)$. Under some mild integrability condition [6, Theorem 8.2.1], the measure solutions μ_t to a continuity equation associated with (5) (under some selection of vector field from $F(t, x)$) can be represented as

$$\mu_t = e_{t\#}\eta \quad (7)$$

which for any continuous function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies $\int \phi(x) d\mu_t(x) = \int \phi(e_t(x, \gamma)) d\eta(x, \gamma)$. The solutions μ_t can be understood as a superposition over solution trajectories $\gamma \in \Gamma_T$, where the superposition is captured by the measure η . The solutions to differential inclusion (5) are possibly nonunique and hence μ_t in (7) is also not necessarily unique for a given initial measure.

In this work, we are interested in using the superposition principle for deriving a continuity equation associated with (4). For differential inclusions, such problems have been studied in [7], but in comparison, we consider a specific class of non-compact time varying differential inclusions, and we derive a different form of continuity equation which is more suitable for numerical purposes discussed later in this paper.

2.3 Functional Regularization

The basic idea of the regularization is to consider a sequence of ODEs with a parameter λ :

$$\dot{x}^\lambda(t) = g_t^\lambda(x(t))$$

so that the solutions $x^\lambda(t)$ approach the solution $x(t)$ that solves (4), under the constraint $x^\lambda(0) = x(0)$. Here, for each $\lambda > 0$ and for each $t \in [0, T]$, $g_t^\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a single-valued Lipschitz continuous function, whose construction is provided in Section 4. One can derive the classical continuity equation (3) to these ODEs and obtain a parameterized sequence of measures μ_t^λ as follows:

$$\partial_t \mu_t^\lambda + \nabla \cdot (g_t^\lambda(\cdot) \mu_t) = 0.$$

An obvious candidate for describing the measure solving (4) is to take the limit of $\{\mu_t^\lambda\}$ as $\lambda \rightarrow 0$. In Section 4, we study the limit of this sequence using the Wasserstein metric to quantify the distance between μ_t^λ and the limiting measure.

To provide some background on this performance metric used to study convergence of measures, we recall that the *Wasserstein metric*, also called the Kantorovich-Rubenstein metric, is frequently employed to measure the distance between two probability measures. The more common choice, the 2-Wasserstein distance between two probability measures $\mu, \nu \in \mathcal{P}(\Omega)$ for some $\Omega \subset \mathbb{R}^n$, is defined as

$$W_2(\mu, \nu) := \frac{1}{2} \min_{\theta \in \Theta(\mu, \nu)} \int_{\Omega \times \Omega} |x - y|^2 d\theta(x, y) \quad (8)$$

where $\Theta(\mu, \nu)$ is the set of joint probability measures on $\Omega \times \Omega$ with given marginals μ and ν , i.e. such that $\theta \in \Theta(\mu, \nu)$ satisfies $\int_A \int_{\Omega} \theta(x, y) dx dy = \mu(A)$ and $\int_{\Omega} \int_A \theta(x, y) dx dy = \nu(A)$, for every measurable $A \subset \Omega$.

Similarly, 1-Wasserstein distance or $W_1(\mu, \nu)$ is defined as,

$$W_1(\mu, \nu) = \min \left\{ \int_{\Omega \times \Omega} |x - y| d\theta(x, y) : \theta \in \Theta(\mu, \nu) \right\}. \quad (9)$$

2.4 Time Discretisation and Optimal Transport

Time discretization based techniques are well known for constructing solutions to evolution PDEs by using the gradient flow structure of the Wasserstein space. It is based on partitioning a time interval into finitely many nodes (discrete times) and describing the measure at those times as a function of the initial distribution through appropriate mappings using the system data. The interpolation between the two measures (described at two consecutive times) is based on the principles of optimal transport and provides an approximation to the measure evolution problem for system (4).

To provide some background on these interpolation schemes, we recall that the original mass transportation for measures was proposed in [26] as the problem of finding a transport map $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that given two probability measures $\mu \in \mathcal{P}(\mathbb{R}^n)$ and $\nu \in \mathcal{P}(\mathbb{R}^n)$ and a cost function $c : \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, \infty)$, it solves

$$\inf_G \left\{ \int c(x, G(x)) d\mu(x) : G_{\#}\mu = \nu \right\}. \quad (10)$$

The problem is highly nonlinear with nonconvex constraints and the existence of a minimizer is difficult to prove. The problem was later reformulated by Kantorovich [27] into a convex program that corresponds to the computation of $W_2(\mu, \nu)$ by taking $c(x, y) = \frac{1}{2}|x - y|^2$. Indeed, $W_2(\cdot, \cdot)$ provides a metric structure to the space of measures $\mathcal{P}(\mathbb{R}^n)$ and the resulting subspace of $\mathcal{P}(\mathbb{R}^n)$ is known as the Wasserstein space $\mathcal{W}_2(\mathbb{R}^n)$. One interesting property which will be of interest is that any absolutely continuous curve in Wasserstein space \mathcal{W}_2 is a solution to a continuity equation [24]. In [28] the authors proved that if there exists a pair of measures $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^n)$ with μ_0 absolutely continuous with respect to the Lebesgue measure, then there exists a constant speed geodesic between these measures and such constant speed geodesics satisfy a continuity equation. It is possible to construct an approximation of an absolutely continuous curve by defining measures at discrete time instants and using an interpolation via constant speed geodesics between successive time instants. In [21], the authors use time-discretization to approximate the measure solution of continuity equation associated with (4) without the drift term $f(\cdot)$. The method is based on recursively defining measures at different time instants using an optimal transport map which transports the measures from one time instant to the next. Considering suitable interpolation schemes, one constructs the trajectory and shows that it converges to the solution of the continuity equation. We will use a time-stepping scheme for (4) to construct measures at different time instants starting from an initial distribution. Using appropriate interpolation, we will prove that the interpolated curves converge to the absolutely continuous curves which will be the measure valued solutions to the continuity equation associated to (4).

3 Superposition Principle

In this section, we consider a general system class described by a differential inclusion. Starting from a vector field selection of this differential inclusion, we propose a continuity equation driven by this selection and we characterize all possible solutions to this equation.

3.1 Differential Inclusion

Consider a dynamical system governed by the differential inclusion:

$$\dot{x} \in F(t, x) \quad (11)$$

where $F : [0, T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping. For an initial condition $x_0 \in \mathbb{R}^n$, we denote the solution to (11) at time $t \in [0, T]$ by $X_t(x_0)$, where X_t represents the flow map for system (11). For solutions to be well-defined, F satisfies the following:

Assumption 3. *The set $F(t, x)$ is convex for every $t \in [0, T]$ and every $x \in \mathbb{R}^n$.*

Assumption 4. *If there exists a solution to (11) corresponding to a selection \bar{f} of F , then it holds that,*

$$|\bar{f}(t, x)| \leq \beta(t)(1 + |x|) \quad (12)$$

where, $\beta(\cdot) \in \mathcal{L}^1([0, T]; \mathbb{R}_+)$.

In what follows, we consider a selection $\bar{f}_\omega(t, x)$ of $F(t, x)$ defined using a probability measure $\omega(\cdot|t, x) \in \mathcal{P}(F(t, x))$ as follows:¹

$$\bar{f}_\omega(t, x) := \int_{F(t, x)} v d\omega(v|t, x), \quad \omega(\cdot|t, x) \in \mathcal{P}(F(t, x)). \quad (13)$$

Due to the convexity of $F(t, x)$, it follows that $\bar{f}_\omega(t, x) \in F(t, x)$. We do not make any further assumptions on the regularity of $\bar{f}_\omega(t, x)$ and thus an ODE system $\dot{x} = \bar{f}_\omega(t, x)$ may admit multiple solution trajectories from a given initial condition. We also let Γ_T^ω denote the set of trajectories associated with the selection \bar{f}_ω , that is,

$$\Gamma_T^\omega := \{\gamma \in AC([0, T]; \mathbb{R}^n) : \dot{\gamma} = \bar{f}_\omega(t, \gamma)\}. \quad (14)$$

Remark 1. *In the case where the set $F(t, x)$ is finitely generated (that is, for each (t, x) , it is represented by a linear combination of finitely many vector fields $f_i(x)$ for $i \in \{1, \dots, n\}$), the vector field (13) reduces to a Fillipov differential inclusion. For example, given a piecewise smooth system $\dot{x}(t) = f_i(x(t))$ for $x \in \mathcal{R}_i$, where \mathcal{R}_i are disjoint regions covering \mathbb{R}^n , the Fillipov differential inclusion for such system would result in $\dot{x}(t) = \text{conv}(f_i(x))$, where conv denotes the convex*

¹If Assumptions 3 and 4 hold, there exists a measurable selection $\bar{f}_\omega(t, x) \in F(t, x) \forall (t, x)$ [29]. Measurability of the proposed vector field can be checked by first replacing the integrand with the indicator function I_S , then $\bar{f}_\omega(t, x) = \omega(S|t, x)$ which is a measurable function for every fixed $S \subset F(t, x)$. One can use standard measure theoretic arguments to approximate the integral using simple functions.

combination of the vector fields. Any absolutely continuous solution $x(t)$ would satisfy $\dot{x}(t) = \sum_{i \in I} w_i(x(t)) f_i(x(t))$ where $I(x)$ denotes the active set at $x \in \mathbb{R}^n$, and the weights are such that $\sum w_i = 1$ and $w_i \geq 0$. In [30], the authors propose a linear program to compute the weights and thus solve the differential equation using an active-set method. In this case of piecewise smooth vector fields, the measure ω will be discretely supported on the set $\{f_i(t, x)\}_{i \in I(x)}$, and the vector field in (13) yields

$$\bar{f}_\omega(x) = \sum_{v \in \{f_i(x)\}, i \in I(x)} v w_i(x) = \sum_{i \in I(x)} f_i(x) w_i(x).$$

Thus, $w = (w_1, \dots, w_m)$ with $\sum_i w_i = 1$ can be seen as the discrete version of the measure ω .

3.2 Vector Field Selection

The vector field selection in (13) is used to define the continuity equation for the measure evolution problem. Before doing so, we make some connections with related literature to provide an interpretation of $\bar{f}_\omega(t, x)$ as the weighted average of the vector fields associated with solution trajectories of (11). In [7], the authors show that the image measure μ_t obtained by applying the evaluation map to $\eta \in \mathcal{D}(\mathbb{R}^n \times \Gamma_T)$, as described in (7), are solutions to a continuity equation driven by a *mean-vector field*. By definition of evaluation operator (6), we define the set $e_t^{-1}(x)$ of trajectories passing through x at time t , i.e.,

$$e_t^{-1}(x) := \{(y, \gamma) \text{ s.t. } \gamma \in \Gamma_T, \gamma(0) = y, \gamma(t) = x\}. \quad (15)$$

A disintegration $\eta_{t,x}(y, \gamma)$ of $\eta(y, \gamma)$ w.r.t. e_t is such that, for $\psi \in \mathcal{C}(\mathbb{R}^n \times \Gamma_T; \mathbb{R})$:

$$\int_{\mathbb{R}^n \times \Gamma_T} \psi(y, \gamma) d\eta(y, \gamma) = \int_{\mathbb{R}^n} \int_{e_t^{-1}(x)} \psi(y, \gamma) d\eta_{t,x}(y, \gamma) d\mu_t(x). \quad (16)$$

Then a mean-vector field is introduced as follows:

$$\tilde{f}(t, x) := \int_{e_t^{-1}(x)} \dot{\gamma}(t) d\eta_{t,x}(y, \gamma). \quad (17)$$

The velocity vector (17) can be understood as a weighted mean of all the velocity vectors $\dot{\gamma}(t)$ over the curves γ passing through point x at time t . Note that the convexity of $F(t, x)$ ensures that the mean-velocity (17) belongs to the set $F(t, x)$. We show that the vector field defined in (17) is equivalent to the vector field defined in (13) for some appropriate choice of $\omega \in \mathcal{D}(\mathbb{R}^n)$. To establish this, we introduce a velocity evaluation operator $\mathbf{d}_t : \mathbb{R}^n \times \Gamma_T \rightarrow \mathbb{R}^n$, which is a Borel measurable map defined by

$$\mathbf{d}_t(y, \gamma) := \dot{\gamma}(t) \text{ with } \gamma(0) = y. \quad (18)$$

Using this mapping, we define:

$$\omega(\cdot | t, x) := \mathbf{d}_{t\#} \eta_{t,x}(\cdot). \quad (19)$$

Proposition 1. *Let \mathbf{d}_t be the velocity evaluation operator in (18). Then, for each $t \in [0, T]$ and $x \in \mathbb{R}^n$, it holds that $\mathbf{d}_t^{-1}(F(t, x)) = e_t^{-1}(x)$. Moreover, for the measures ω defined in (19), the associated vector field in (13) is equal to (17).*

Proof. The proof of the first claim follows from the definition, i.e.,

$$\mathbf{d}_t^{-1}(F(t, x)) = \{(\gamma(0), \gamma); \gamma(t) = x\} \quad (20)$$

$$e_t^{-1}(x) = \{(\gamma(0), \gamma); \gamma(t) = x\}. \quad (21)$$

So the two sets are the same. Next, we prove the equivalence of the two vector fields (17) and (13). Using the definition of \bar{f}_ω in (13) and the equality in (19), we get

$$\bar{f}_\omega(t, x) = \int_{F(t, x)} \dot{\gamma}(t) \, d(\mathbf{d}_{t\#}\eta_{t, x})(y, \gamma).$$

Under the change of variables in the above equation,

$$\bar{f}_\omega(t, x) = \int_{\mathbf{d}_t^{-1}(F(t, x))} \dot{\gamma}(t) \, d\eta_{t, x}(y, \gamma).$$

Now using $\mathbf{d}_t^{-1}(F(t, x)) = e_t^{-1}(x)$, we get

$$\bar{f}_\omega(t, x) = \int_{e_t^{-1}(x)} \dot{\gamma}(t) \, d\eta_{t, x}(y, \gamma) = \tilde{f}(t, x)$$

for each $t \in [0, T]$ and $x \in \mathbb{R}^n$. □

Thus, the set of trajectories $\Gamma_T^\omega \subset \Gamma_T$ for (13) and (17) are the same under the constraint prescribed in (19).

3.3 Continuity Equation and its Measure Solution

We now state the main results of this section concerning the formulation of the continuity equation. In Proposition 2, we show that, for every $\omega(\cdot|t, x) \in \mathcal{P}(F(t, x))$ and every η concentrated on $\mathbb{R}^n \times \Gamma_T^\omega$, the image measure $\mu_t = e_{t\#}\eta$ satisfies the continuity equation driven by $\bar{f}_\omega(t, x)$ in (13). Starting from this equation, in Theorem 1 we discuss the converse statement and characterize all the measure solutions to the derived continuity equation. This characterization of the solutions is especially important as later we propose a numerical method for the simulation of measure evolution through nonsmooth dynamical systems as the solution of this continuity equation.

Proposition 2. *Consider system (11) under Assumption 3 and Assumption 4. For each $t \in [0, T]$ and $x \in \mathbb{R}^n$, let $\omega(\cdot|t, x)$ be a probability measure supported on $F(t, x)$, and let \bar{f}_ω and Γ_T^ω be defined as in (13) and (14), respectively. Then, for every $\eta \in \mathcal{P}(\mathbb{R}^n \times \Gamma_T^\omega)$, the measure $\mu_t := e_{t\#}\eta$ satisfies the following continuity equation driven by $\bar{f}_\omega(t, x)$, i.e.*

$$\int_{\mathbb{R}^n} \varphi(T, x) d\mu_T(x) - \int_{\mathbb{R}^n} \varphi(0, x) d\mu_0(x) =$$

$$\int_{[0, T] \times \mathbb{R}^n} \left[\partial_t \varphi(t, x) + \nabla_x \varphi(t, x) \cdot \bar{f}_\omega(t, x) \right] d\mu_t(x) dt \quad (22)$$

for every compactly supported $\varphi \in \mathcal{C}^1([0, T] \times \mathbb{R}^n; \mathbb{R})$.

Proof. To derive (22), we start by proving that the mapping $t \mapsto \int \phi(x) d\mu_t(x)$ is absolutely continuous,² for compactly supported $\phi \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R})$. We then use the property of almost everywhere differentiability of absolutely continuous functions to differentiate $\int \phi(x) d\mu_t(x)$ w.r.t. time.

Absolute continuity of $\int \phi d\mu_t$: Consider the pairwise disjoint intervals $(\underline{t}_i, \bar{t}_i) \subset [0, T]$, such that $\sum_{i=1}^N (\bar{t}_i - \underline{t}_i) < \delta$, for a given $\delta > 0$. Choose $\phi \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R})$, then for any $\gamma \in \Gamma_T^\omega$ we have

$$\sum_{i=1}^N \phi(\gamma(\bar{t}_i)) - \phi(\gamma(\underline{t}_i)) = \sum_{i=1}^N \int_{(\underline{t}_i, \bar{t}_i)} \left(\nabla_x \phi(\gamma(t)) \right) \cdot \bar{f}_\omega(\gamma(t)) dt. \quad (23)$$

Integrating (23) with $\eta \in \mathcal{P}(\mathbb{R}, \Gamma_T^\omega)$ leads to

$$\begin{aligned} \sum_{i=1}^N \int_{\mathbb{R}^n \times \Gamma_T^\omega} \left[\phi(\gamma(\bar{t}_i)) - \phi(\gamma(\underline{t}_i)) \right] d\eta(x, \gamma) \\ = \sum_{i=1}^N \int_{(\underline{t}_i, \bar{t}_i)} \int_{\mathbb{R}^n \times \Gamma_T^\omega} \left(\nabla_x \phi(\gamma(t)) \right) \cdot \bar{f}_\omega(\gamma(t)) d\eta(x, \gamma) dt. \end{aligned} \quad (24)$$

Now using $\mu_t = e_{t\#} \eta$ on the left side of the above equation, taking the absolute values on both sides and then using Hölder's inequality we get

$$\begin{aligned} \sum_{i=1}^N \left| \int_{\mathbb{R}^n} \phi(x) d\mu_{\bar{t}_i}(x) - \int_{\mathbb{R}^n} \phi(x) d\mu_{\underline{t}_i}(x) \right| \\ \leq \|\nabla_x \phi\|_\infty \sum_{i=1}^N \int_{(\underline{t}_i, \bar{t}_i)} \int_{\mathbb{R}^n \times \Gamma_T^\omega} |\bar{f}_\omega(\gamma(t))| d\eta(x, \gamma) dt. \end{aligned} \quad (25)$$

Using the growth bounds on the vector field (Assumption 4), we can derive the estimate $\int_{\mathbb{R}^n \times \Gamma_T^\omega} |\bar{f}_\omega(\gamma(t))| d\eta(x, \gamma) \leq K\beta(t)$ for some $K > 0$; refer to Appendix A for details. Substituting this inequality in (25), we get

$$\sum_{i=1}^N \left| \int_{\mathbb{R}^n} \phi(x) d\mu_{\bar{t}_i}(x) - \int_{\mathbb{R}^n} \phi(x) d\mu_{\underline{t}_i}(x) \right| \leq K \|\nabla_x \phi\|_\infty \sum_{i=1}^N \int_{(\underline{t}_i, \bar{t}_i)} \beta(t) dt.$$

²We consider test functions $\varphi(t, x) = \rho(t)\phi(x)$ which are dense in $\mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n; \mathbb{R})$, and then the differentiability of $\int \varphi(t, x) d\mu_t(x)$ depends on the absolute continuity of $\int \phi(x) d\mu_t(x)$ since

$$\frac{d}{dt} \int_{\mathbb{R}^n} \rho(t)\phi(x) d\mu_t(x) = \int_{\mathbb{R}^n} \partial_t(\rho(t))\phi(x) d\mu_t(x) + \int_{\mathbb{R}^n} \rho(t) \frac{d}{dt} \phi(x) d\mu_t(x).$$

So we need to prove that $t \mapsto \int \phi(x) d\mu_t(x)$ is absolutely continuous.

Since β is integrable and $\sum_{i=1}^N (\bar{t}_i - \underline{t}_i) < \delta$ for an arbitrary $\delta > 0$, the right-hand side can be made arbitrarily small. This proves the absolute continuity of $t \mapsto \int \phi(x) d\mu_t(x)$.

Next, we differentiate $\int \varphi(t, x) d\mu_t(x)$ for any $\varphi(t, x) \in \mathcal{C}^1([0, T] \times \mathbb{R}^n)$ and we obtain the following (refer to Appendix B for details),

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi(T, x) d\mu_T(x) - \int_{\mathbb{R}^n} \varphi(0, x) d\mu_0(x) \\ = \int_{[0, T] \times \mathbb{R}^n} \left(\partial_t \varphi(t, x) + \nabla_x \varphi(t, x) \cdot \bar{f}_\omega(t, x) \right) d\mu_t(x) dt \end{aligned}$$

which shows the desired relation. \square

We now rewrite equation (22) in a form which we will use in the optimization formulation proposed later in Section 6. Substituting the expression for $\bar{f}_\omega(t, x)$ from (13) in (22), we get

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi(T, x) d\mu_T(x) - \int_{\mathbb{R}^n} \varphi(0, x) d\mu_0(x) = \\ \int_{[0, T] \times \mathbb{R}^n} \left(\partial_t \varphi(t, x) + \nabla_x \varphi(t, x) \cdot \int_{F(t, x)} v d\omega(v|t, x) \right) d\mu_t(x) dt. \quad (26) \end{aligned}$$

Rearranging the terms in the above equations and defining $d\hat{\mu}(t, x, v) = d\omega(v|t, x) d\mu_t(x) dt$, we get

$$\int_{\mathbb{R}^n} \varphi(T, x) d\mu_T - \int_{\mathbb{R}^n} \varphi(0, x) d\mu_0 = \int_{[0, T] \times \mathbb{R}^n} \int_{F(t, x)} \left[\partial_t \varphi(t, x) + \nabla_x \varphi(t, x) \cdot v \right] d\hat{\mu}(t, x, v). \quad (27)$$

Equation (27) will be the starting point for the next result as we will characterize all possible solutions $\hat{\mu}$ to it. The solutions $\hat{\mu}$ will determine the vector field (13) (by defining ω) and the solutions to the continuity equation driven by this vector field will lead to a measure concentrated on the trajectories of the derived vector field.

Theorem 1. *Consider system (11) under Assumption 3 and Assumption 4. Any measure $\hat{\mu}$ that solves the continuity equation (27) is of the form*

$$d\hat{\mu}(t, x, v) = d\omega(v|t, x) d\mu_t(x) dt \quad (28)$$

where $\omega(\cdot|t, x) \in \mathcal{P}(F(t, x))$ and μ_t solves (22).

Proof. In Euclidean space \mathbb{R}^n , we can use the disintegration theorem [31, Corollary 10.4.13] to write $d\hat{\mu}(t, x, v) = d\omega(v|t, x) d\bar{\mu}(t, x)$ where $\omega(\cdot|t, x) \in \mathcal{P}(F(t, x))$. Using this we can rewrite

$$\begin{aligned} \int_{\mathbb{R}^n} \varphi(T, x) d\mu_T - \int_{\mathbb{R}^n} \varphi(0, x) d\mu_0 = \\ \int_{[0, T] \times \mathbb{R}^n} \int_{F(t, x)} \left[\partial_t \varphi(t, x) + \nabla_x \varphi(t, x) \cdot \zeta \right] d\omega(\zeta|t, x) d\bar{\mu}(t, x). \quad (29) \end{aligned}$$

Rearranging the terms results in

$$\int_{\mathbb{R}^n} \varphi(T, x) d\mu_T(x) - \int_{\mathbb{R}^n} \varphi(0, x) d\mu_0(x) = \int_{[0, T] \times \mathbb{R}^n} \left(\partial_t \varphi(t, x) + \nabla_x \varphi(t, x) \cdot \bar{f}_\omega(t, x) \right) d\bar{\mu}(t, x) \quad (30)$$

where \bar{f}_ω is defined as

$$\bar{f}_\omega(t, x) = \int_{F(t, x)} \zeta d\omega(\zeta | t, x) \text{ for } \omega(\cdot | t, x) \in \mathcal{P}(F(t, x)) \quad (31)$$

and spans the set $F(t, x)$ as we have assumed $F(t, x)$ to be convex for every $t \in [0, T]$.

Decomposition of $\bar{\mu}$: Next we show that the marginal of $\bar{\mu}(t, x)$ w.r.t. time is a Lebesgue measure. This can be shown by taking $\varphi(t, x) = t^k$ for some $k \geq 0$ in (30), then we get

$$\mu_T(\mathbb{R}^n) T^k - \int_{\mathbb{R}^n} t^k d\mu_0 = \int_{[0, T] \times \mathbb{R}^n} k t^{k-1} d\bar{\mu}(t, x) \quad (32)$$

where taking $k = 0$ gives $\mu_T(\mathbb{R}^n) = \mu_0(\mathbb{R}^n)$ and for $k \geq 1$ results in $\frac{\mu_T(\mathbb{R}^n) T^k}{k} = \int_{[0, T] \times \mathbb{R}^n} t^{k-1} d\bar{\mu}(t, x)$. So, up to scaling we can write $\bar{\mu}(dt, dx) = \mu_t(dx) dt$. Substituting this we arrive at the following continuity equation

$$\int_{\mathbb{R}^n} \phi(T, x) d\mu_T - \int_{\mathbb{R}^n} \phi(0, x) d\mu_0 = \int_{[0, T] \times \mathbb{R}^n} \left[\partial_t \phi(t, x) + \nabla_x \phi(t, x) \cdot \bar{f}_\omega(t, x) \right] d\mu_t(x) dt. \quad (33)$$

Now using the results in [5], the only solutions to the continuity equation have a representation in terms of measure $\eta \in \mathcal{P}(\mathbb{R}^n \times \Gamma)$ (as defined in Section 2.2) as $\mu_t = e_{t\#} \eta$ where η are concentrated on the solution trajectories to system $\dot{x}(t) = \bar{f}_\omega(t, x(t))$. \square

3.4 Measure Evolution for Constrained Systems

Using the equations defined in (27), we arrive at the continuity equation for the system defined in (4), i.e., the equation

$$\int_{S(T)} \phi(T, x) d\mu_T(x) - \int_{S(0)} \phi(0, x) d\mu_0(x) = \int_{[0, T] \times S(t)} \int_{f(t, x) - \mathcal{N}_{S(t)}(x)} \left[\partial_t \phi(t, x) + \nabla_x \phi(t, x) \cdot \zeta \right] d\hat{\mu}(t, x, \zeta) \quad (34)$$

holds for every compactly supported $\phi \in \mathcal{C}^1([0, T] \times \mathbb{R}^n; \mathbb{R})$. Using Theorem 1, it follows that the measure solutions to the continuity equation have a representation (28). In this decomposition, $\omega(\cdot | t, x)$ represents the selection from the set $f(t, x) - \mathcal{N}_{S(t)}(x)$ so that the resulting

trajectories evolve within the set $S(t)$. Corresponding to such selections, the solutions to ODE (4) are unique with $x(0) \in S(0)$, see for example [8, Section 5]. Consequently, we have that $\eta = \delta_{X_t(x_0)}$ and due to Proposition 2, the measure $\mu_t = e_{t\#}\eta$ corresponds to

$$\mu_t := X_{t\#}\mu_0 \quad (35)$$

where X_t denotes the flow map associated with system (4).

4 Functional Regularization

As noted earlier, the right-hand side of system (4) is possibly discontinuous, and this introduces complexity in writing the transport equation for measures. The second approach that we propose relies on working with Lipschitz continuous approximations of the right-hand side of (4) to generate a sequence of approximate solutions $\{x^\lambda\}_{\lambda>0}$ parameterized by $\lambda > 0$. In particular, we work with the so-called *Moreau-Yosida regularization*, which for system (4) takes the following form:

$$\dot{x}^\lambda(t) = g_t^\lambda(x^\lambda(t)) := f(t, x^\lambda(t)) - \frac{1}{\lambda}(x^\lambda(t) - \text{proj}(x^\lambda(t), S(t))) \quad (36)$$

where we take $x^\lambda(0) \in S(0)$, and $\text{proj}(x^\lambda(t), S(t))$ refers to the projection of the vector $x^\lambda(t)$ onto the set $S(t)$ with respect to the Euclidean distance. It is well known that the solution curve μ_t^λ of the continuity equation with the Lipschitz regular vector field (36) satisfies the following pushforward relationship

$$\mu_t^\lambda = X_{t\#}^\lambda \mu_0 \quad (37)$$

where $X_t^\lambda(x_0) := x^\lambda(t, x_0)$ is the flow map associated with (36). In [23], the authors show that the solution $x^\lambda(t)$ solving (36) converges uniformly to the solution $x(t)$ of (4) when $x^\lambda(0) = x(0) \in S(0)$, and that the measures μ_t^λ converge in weak star topology to the measure solutions $\mu_t = X_{t\#}\mu_0$, with X_t being the flow map of (4). In this section, we provide quantitative bounds on the Wasserstein distance between measures μ_t and μ_t^λ .

Remark 4.1. As a parallel to the regularization technique presented, we find an approach based on mollification in [5] to study the evolution of measures for nonsmooth dynamical systems. Such mollification is carried out by using a convolution kernel $\psi : \mathbb{R}^n \rightarrow [0, \infty)$ with the properties that $\psi(x)$ is bounded, measurable with $\psi(x) = \psi(-x)$, $\int \psi(x) dx = 1$. Let $\psi_\epsilon := \frac{1}{\epsilon^n} \psi(\frac{x}{\epsilon})$, and the corresponding convolution with a measure μ as $(\psi_\epsilon * \mu)(x) := \int \psi_\epsilon(x - y) d\mu(y)$. For $\mu^\epsilon := \mu * \psi_\epsilon$, it can be shown that $W_2(\mu^\epsilon, \mu) \leq \epsilon \int |\psi(x)|^2 dx$. In [5], a similar mollification technique was used and the narrow convergence³ of the measures was proven by working with a smooth vector field $g_\epsilon := \frac{(g\mu) * \psi_\epsilon}{\mu^\epsilon}$ with the corresponding continuity equation $\partial_t \mu_t^\epsilon + \nabla \cdot (g_t^\epsilon \mu_t^\epsilon) = 0$.

³Note: Family of measures μ_n converges narrowly to measure μ if $\lim_{n \rightarrow \infty} |\int f d\mu_n - \int f d\mu| \rightarrow 0$ for a bounded $f \in \mathcal{C}(\Omega; \mathbb{R})$, where Ω is any Polish space. Note that the definition is different from weak* convergence where the convergence is defined w.r.t. compactly supported continuous functions [6]. When the underlying space Ω is compact both the notions of convergence coincide.

Theorem 2. Let $\mu_t^\lambda \in \mathcal{P}(\mathbb{R}^n)$ be defined as in (37), and let $\mu_t = X_{t\#}\mu_0$, with $\mu_0 \in \mathcal{P}(S(0))$ and X_t being the flow map of (4). Then, the W_1 distance between μ_t and μ_t^λ satisfies the following bound:

$$W_1(\mu_t, \mu_t^\lambda) \leq C_1 \sqrt{\frac{L_f \lambda (e^{L_f t} - 1)}{2}} \int_{S(0)} |x_0| d\mu_0(x_0) \quad (38)$$

where $C_1 = L_f(1 + \kappa) + L_S$ and $\kappa := (e^{2L_f T} - 1) \frac{2L_f + L_S}{2L_f}$.

Remark 4.2. Under the assumption that $\text{supp}(\mu_0)$ is compact, $\text{supp}(\mu_t)$ and $\text{supp}(\mu_t^\lambda)$ will be compact as these are the push-forwards of Lipschitz continuous operators. This results in $W_2(\mu_t^\lambda, \mu_t) \leq C_2 W_1(\mu_t^\lambda, \mu_t)$ for some $C_2 > 0$ and hence a qualitatively similar bound holds for W_2 metric as the one indicated in (38).

We will use the following lemma in the proof of Theorem 2.

Lemma 1. Let $x^\lambda(t)$ be the solution to (36) and $x(t)$ be the solution to (4) with $x^\lambda(0) = x(0) \in S(0)$. Then, for each $t \in [0, T]$, it holds that

$$|x^\lambda(t) - x(t)| \leq (L_f(1 + \kappa + |x_0|) + L_S) \sqrt{\frac{\lambda(e^{2L_f t} - 1)}{2L_f}}$$

where $\kappa = (e^{2L_f T} - 1) \frac{2L_f + L_S}{2L_f}$.

Proof. Let $x^\lambda(\cdot)$ and $x^\nu(\cdot)$ be solutions to (36) corresponding to λ and ν as regularization parameters. Then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|x^\lambda(t) - x^\nu(t)|^2) &= \langle x^\lambda(t) - x^\nu(t), \dot{x}^\lambda(t) - \dot{x}^\nu(t) \rangle \\ &= \langle x^\lambda(t) - x^\nu(t), f(t, x^\lambda(t)) - \frac{1}{\lambda} (x^\lambda(t) - \text{proj}(x^\lambda(t), S)) - f(t, x^\nu(t)) - \frac{1}{\nu} (x^\nu(t) - \text{proj}(x^\nu(t), S)) \rangle. \end{aligned}$$

Using the Lipschitz property of $f(t, x)$ and the Cauchy-Schwarz inequality for the terms involving $\langle x^\lambda(t) - x^\nu(t), f(t, x^\lambda(t)) - f(t, x^\nu(t)) \rangle$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|x^\lambda(t) - x^\nu(t)|^2) &\leq L_f |x^\lambda(t) - x^\nu(t)|^2 - \left\langle x^\lambda(t) - x^\nu(t), \right. \\ &\quad \left. - \frac{1}{\lambda} (x^\lambda(t) - \text{proj}(x^\lambda(t), S)) + \frac{1}{\nu} (x^\nu(t) - \text{proj}(x^\nu(t), S)) \right\rangle. \quad (39) \end{aligned}$$

We rewrite $x^\lambda(t) - x^\nu(t) = x^\lambda(t) - \text{proj}(x^\lambda(t), S) + (\text{proj}(x^\lambda(t), S) - \text{proj}(x^\nu(t), S)) - (x^\nu(t) - \text{proj}(x^\nu(t), S))$ and substitute it in (39) to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|x^\lambda(t) - x^\nu(t)|^2) &\leq L_f |x^\lambda(t) - x^\nu(t)|^2 + \left\langle x^\lambda(t) \pm \text{proj}(x^\lambda(t), S) - x^\nu(t) \pm \text{proj}(x^\nu(t), S), \right. \\ &\quad \left. - \frac{1}{\lambda} (x^\lambda(t) - \text{proj}(x^\lambda(t), S)) + \frac{1}{\nu} (x^\nu(t) - \text{proj}(x^\nu(t), S)) \right\rangle. \quad (40) \end{aligned}$$

For notational convenience, we denote $Y_\lambda(x^\lambda(t)) = \frac{1}{\lambda}(x^\lambda(t) - \text{proj}(x^\lambda(t), S))$, $Y_\nu(x^\nu(t)) = \frac{1}{\nu}(x^\nu(t) - \text{proj}(x^\nu(t), S))$ and substitute these in (40) to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|x^\lambda(t) - x^\nu(t)|^2) &\leq L_f |x^\lambda(t) - x^\nu(t)|^2 + \\ &\left\langle \lambda Y_\lambda(x^\lambda(t)) + (\text{proj}(x^\lambda(t), S) - \text{proj}(x^\nu(t), S)) - \nu Y_\nu(x^\nu(t)), -Y_\lambda(x^\lambda(t)) + Y_\nu(x^\nu(t)) \right\rangle. \end{aligned} \quad (41)$$

It is known that $Y_\lambda(x^\lambda(t))$ and $Y_\nu(x^\nu(t))$ satisfy the monotonicity property, i.e.

$$\langle -Y_\lambda(x^\lambda(t)) + Y_\nu(x^\nu(t)), (\text{proj}(x^\lambda(t), S) - \text{proj}(x^\nu(t), S)) \rangle \leq 0 \quad (42)$$

since $Y_\lambda(x^\lambda(t))$ and $Y_\nu(x^\nu(t))$ are Moreau-Yosida regularizations of the $\mathcal{N}_S(x)$ operator. We substitute (42) in (41) and get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|x^\lambda(t) - x^\nu(t)|^2) &\leq L_f |x^\lambda(t) - x^\nu(t)|^2 \\ &+ \langle \lambda Y_\lambda(x^\lambda(t)) - \nu Y_\nu(x^\nu(t)), -Y_\lambda(x^\lambda(t)) + Y_\nu(x^\nu(t)) \rangle. \end{aligned} \quad (43)$$

Using the Cauchy-Schwartz inequality for the second term on the right-hand side of (43), we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|x^\lambda(t) - x^\nu(t)|^2) &\leq L_f |x^\lambda(t) - x^\nu(t)|^2 - \lambda |Y_\lambda(x^\lambda(t))|^2 - \\ &\nu |Y_\nu(x^\nu(t))|^2 + (\lambda + \nu) |Y_\lambda(x^\lambda(t))| \cdot |Y_\nu(x^\nu(t))|. \end{aligned} \quad (44)$$

Next we use Young's inequality for the term $|Y_\lambda(x^\lambda(t))| \cdot |Y_\nu(x^\nu(t))|$ to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|x^\lambda(t) - x^\nu(t)|^2) &\leq L_f |x^\lambda(t) - x^\nu(t)|^2 - \lambda |Y_\lambda(x^\lambda(t))|^2 - \nu |Y_\nu(x^\nu(t))|^2 \\ &+ \frac{(\lambda + \nu)}{2} (|Y_\lambda(x^\lambda(t))|^2 + |Y_\nu(x^\nu(t))|^2) \\ &\leq L_f |x^\lambda - x^\nu(t)|^2 + \frac{\nu}{2} |Y_\lambda(x^\lambda(t))|^2 + \frac{\lambda}{2} |Y_\nu(x^\nu(t))|^2. \end{aligned} \quad (45)$$

In [23], uniform bounds on $|x^\lambda(t)|$ were obtained, which leads to $|Y_\lambda(x^\lambda(t))| \leq L_f(1 + \kappa + e^{2L_f T} |x^\lambda(0)|) + L_s$, where $\kappa = (e^{2L_f T} - 1) \frac{2L_f + L_s}{2L_f}$ and a similar bound on $Y_\nu(x^\nu(t))$ (see Appendix C). We use this bound in (45) and obtain

$$\frac{1}{2} \frac{d}{dt} (|x^\lambda(t) - x^\nu(t)|^2) \leq L_f |x^\lambda(t) - x^\nu(t)|^2 + \frac{(\nu + \lambda)}{2} |L_f(1 + \kappa + e^{2L_f T} |x^\lambda(0)|) + L_s|^2 \quad (46)$$

where we have used the fact that $x^\lambda(0) = x^\nu(0) = x(0)$. Now applying Gronwall's lemma we get

$$|x^\lambda(t) - x^\nu(t)|^2 \leq |L_f(1 + \kappa + e^{2L_f T} |x^\lambda(0)|) + L_s|^2 (\nu + \lambda) \frac{e^{2L_f t} - 1}{2L_f}$$

where the term involving $|x^\lambda(0) - x^\nu(0)|$ in the right-hand side is zero. Next we use the fact that $\lim_{\nu \rightarrow 0} x^\nu(t) = x(t)$ [23] holds pointwise and $x^\nu(0) = x(0) = x_0$ to obtain the desired bound. \square

Proof of Theorem 2. To get a bound on the distance between μ_t and μ_t^λ , we use the dual characterization of the W_1 distance [24]:

$$W_1(\mu_t^\lambda, \mu_t) = \sup_{\substack{\phi \in \mathcal{C}(\Omega; \mathbb{R}), \\ \|\phi\|_{\text{Lip}} \leq 1}} \int \phi d(\mu_t^\lambda - \mu_t) \quad (47)$$

where $\|\phi\|_{\text{Lip}}$ denotes the Lipschitz modulus of ϕ and Ω is a measurable set containing all $S(t)$ for $t \in [0, T]$. We use the representation formula for μ_t^λ and μ_t to obtain,

$$\sup_{\substack{\phi \in \mathcal{C}(\Omega; \mathbb{R}) \\ \|\phi\|_{\text{Lip}} \leq 1}} \int \phi d(\mu_t^\lambda - \mu_t) = \sup_{\substack{\phi \in \mathcal{C}(\Omega; \mathbb{R}) \\ \|\phi\|_{\text{Lip}} \leq 1}} \int \phi(x^\lambda(t, x_0)) d\mu_0(x_0) - \int \phi(x^\lambda(t, x_0)|_{\lambda=0}) d\mu_0(x_0)$$

where $x^\lambda(t, x_0)|_{\lambda=0} := \lim_{\lambda \rightarrow 0} x^\lambda(t, x_0)$. Using the first order Taylor expansion of $\phi(x^\lambda(t, x_0))$ w.r.t. λ , for each $\phi \in \mathcal{C}(\Omega; \mathbb{R})$ s.t. $\|\phi\|_{\text{Lip}} \leq 1$, we get,

$$\begin{aligned} & \int \phi(x^\lambda(t, x_0)) d\mu_0 - \int \phi(x^\lambda(t, x_0)|_{\lambda=0}) d\mu_0 \\ & \leq \int \left[\phi(x^\lambda(t, x_0)|_{\lambda=0}) + \nabla_x \phi(x^\lambda(t, x_0))|_{\lambda=0} \cdot (x^\lambda(t, x_0) - \right. \\ & \quad \left. x(t, x_0)) \right] d\mu_0 - \int \phi(x(t, x_0)) d\mu_0 \\ & = \int \nabla_x \phi(x^\lambda(t, x_0))|_{\lambda=0} (x^\lambda(t, x_0) - x(t, x_0)) d\mu_0 \\ & \leq \int |\nabla_x \phi(x^\lambda(t, x_0))|_{\lambda=0}| |x^\lambda(t, x_0) - x(t, x_0)| d\mu_0. \end{aligned} \quad (48)$$

Using the fact that ϕ is of Lipschitz constant 1, the above equation reduces to

$$W_1(\mu_t^\lambda, \mu_t) \leq \int_{S(0)} |x^\lambda(t, x_0) - x(t, x_0)| d\mu_0(x_0). \quad (49)$$

Using Lemma 1 in (49), we get the inequality in (38). \square

5 Time Discretization and Optimal Transport

In this section, we provide a construction of solutions to the continuity equation for (4) using a time discretization scheme.

5.1 Absolutely Continuous Curves of Measures

In what follows, we consider the space of probability measures, with bounded second moment and equipped with the W_2 metric, which is denoted by $(\mathcal{P}_2(\mathbb{R}^n), W_2)$, see [6] for details. This

metric space is called the Wasserstein space and it will be denoted as $\mathcal{W}_2(\mathbb{R}^n)$. We say that a curve $[0, T] \ni t \mapsto \mu_t \in (\mathcal{P}_2(\mathbb{R}^n), W_2)$ is absolutely continuous if there exists $m(t) \in L^2([0, T])$ such that

$$\lim_{h \rightarrow 0} \frac{W_2(\mu_t, \mu_{t+h})}{h} \leq |m|(t) \quad (50)$$

holds for almost every $t \in [0, T]$. From [6, Theorem 1.1.2], the metric derivative $\mu'(t)$ is defined such that $|\mu'(t)| \leq |m|(t)$ holds for all functions $m \in L^2([0, T])$ satisfying (50).

Another important characterization of absolutely continuous curves in the \mathcal{W}_2 space is as follows. A curve $\mu_t : [0, T] \rightarrow \mathcal{W}_2(\mathbb{R}^n)$ is absolutely continuous if and only if there exists a vector field v_t with $\|v_t\|_{L^2(\mu_t)} = |\mu'(t)|$ and μ satisfies the continuity equation driven by the drift term v_t .

5.2 Construction of Curves in Wasserstein Space

Next we propose a construction of curves in $\mathcal{W}_2(\mathbb{R}^n)$ through an interpolation between measures defined at discrete time instants using a discretization of the nonsmooth dynamical system (4). Consider a partition $\{0 = t_0, t_1, \dots, t_i, \dots, t_N = T\}$ of time interval $[0, T]$ such that $t_{k+1} - t_k = \tau$. For a fixed value of τ , we now define the measures $\{\mu_k^\tau\}_{k \in \mathbb{N}}$ at time instants t_k in a recursive manner. To do so, let $S_k := S(k\tau)$ which is a closed convex set under Assumption 2. We denote by P_{S_k} the projection mapping onto the set S_k , and we consider the mapping $G^k : \mathbb{R}^n \rightarrow \mathbb{R}^n$, defined as

$$x \mapsto G^k(x) := P_{S_{k+1}} \circ (\tau f_k(x) + x)$$

with $f_k(x) := f(t_k, x)$. The successor of μ_k^τ is now defined as its push-forward under the mapping G^k as follows:

$$\mu_{k+1}^\tau := G_{\#}^k \mu_k^\tau = \left[P_{S_{k+1}} \circ (\tau f_k(\cdot) + \text{id}) \right]_{\#} \mu_k^\tau. \quad (51)$$

Similarly, for each $x \in S(k\tau)$, the velocity vector at time $(k+1)\tau$ is defined as

$$v_{k+1}^\tau(x) := \frac{G^k(x) - x}{\tau} = \frac{P_C \circ (\tau f_k(\cdot) + \text{id})(x) - x}{\tau}.$$

Next we consider the following two different interpolation curves which will serve different purposes:

(1) Geodesic interpolation between μ_k^τ and μ_{k+1}^τ over the interval $(t_k, t_{k+1}]$ by defining the transport maps

$$G_t := \left(\frac{t - k\tau}{\tau} G^* + \frac{(k+1)\tau - t}{\tau} \text{id} \right)$$

for each $t \in (k\tau, (k+1)\tau]$, for some optimal transport map G^* between μ_k^τ and μ_{k+1}^τ and letting

$$\mu_t^\tau = G_{t\#} \mu_k^\tau = \left(\frac{k\tau - t}{\tau} G^* + \frac{(k+1)\tau - t}{\tau} \text{id} \right)_{\#} \mu_k^\tau. \quad (52)$$

The map G_t is injective and the proof is based on the c-cyclical monotonicity property of optimal transport maps [24]. The interpolation of velocity vector is defined as

$$v_t^\tau(x) := v_{k+1}^\tau \circ (G_t)^{-1}(x) \text{ for } t \in (k\tau, (k+1)\tau]. \quad (53)$$

The L^2 norm of velocity v_t^τ satisfies the following relation

$$\|v_t^\tau\|_{L^2(\mu_t^\tau)} = \frac{W_2(\mu_k^\tau, \mu_{k+1}^\tau)}{\tau} = |(\mu^\tau)'|(t) \text{ for all } t \in (k\tau, (k+1)\tau]. \quad (54)$$

We further define the momentum vector as

$$E_t^\tau := v_t^\tau \mu_t^\tau \quad (55)$$

and it satisfies $\partial_t \mu_t^\tau + \nabla \cdot (E_t^\tau) = 0$.

(2) Piecewise constant interpolation curve such that

$$\hat{\mu}_t^\tau = \mu_{k+1}^\tau, \quad (56)$$

$$\hat{v}_t^\tau = v_{k+1}^\tau \text{ for } t \in (k\tau, (k+1)\tau] \quad (57)$$

We also define the corresponding momentum vector as

$$\hat{E}_t^\tau := \hat{v}_t^\tau \hat{\mu}_t^\tau. \quad (58)$$

We will use piecewise constant interpolation to show that the limit velocity belongs to (4).

5.3 Convergence Result

Next we illustrate the convergence of the constructed curves $\mu_t^\tau, \hat{\mu}_t^\tau$ to μ_t which is solution to the continuity equation associated with (4).

Theorem 3. *Consider system (4) under Assumption 1 and Assumption 2, with $\mu_0 \in \mathcal{P}(S(0))$. For $\tau > 0$, and $t \in [0, T]$, let μ_t^τ and v_t^τ be defined as in (52) and (53), respectively. Then, as $\tau \rightarrow 0$, we get the following two convergence results:*

- The measures μ_t^τ in (52) and $\hat{\mu}_t^\tau$ in (56) converge uniformly in the W_2 metric to $\mu_t = X_{t\#}\mu_0$, where X_t is the flow map associated with (4);
- Momentum vectors $E_t^\tau = v_t^\tau \mu_t^\tau$ (defined in (55)) and $\hat{E}_t^\tau = \hat{v}_t^\tau \hat{\mu}_t^\tau$ (defined in (58)) converge to $E_t := v_t \mu_t$ in the weak star sense. Moreover, the velocity v_t is such that $v_t \in f(t, x) - \mathcal{N}_{S(t)}(x)$.

Proof. We split the proof into four parts: (1) Proof of convergence of μ_t^τ to μ_t and $\hat{\mu}_t^\tau \rightarrow \mu_t$; (2) Proof of convergence of E_t^τ and \hat{E}_t^τ to E_t ; (3) Absolute continuity of E_t with respect to μ_t such that $E_t = v_t \mu_t$; and (4) Convergence of v_t^τ to v_t with the property that $v_t \in f(t, x) - \mathcal{N}_{S(t)}(x)$. The first of these four items follows from the following result:

Lemma 2. Let μ_k^τ and μ_{k+1}^τ be defined as in (51) and (52). Then, it holds that

$$W_2(\mu_{k+1}^\tau, \mu_k^\tau) \leq \tau(L_f C_{\max} + L_s) \quad (59)$$

where the constants L_f, L_s are defined in Assumption 1 and C_{\max} is a constant that captures the uniform bound on $|x_k|$ at time instant t_k , independently of $k \in \mathbb{N}$.

Proof of Lemma 2. The mapping $G^k(x)$ defines a feasible transport map between μ_k^τ and μ_{k+1}^τ . As the Wasserstein distance is defined to be the infimum over all feasible transport maps, we have

$$W_2^2(\mu_{k+1}^\tau, \mu_k^\tau) \leq \int_{S_k} |P_{S_{k+1}} \circ (\tau f_k(\cdot) + \text{id})(x) - x|^2 d\mu_k^\tau(x). \quad (60)$$

Next we use the triangle inequality to obtain

$$W_2^2(\mu_{k+1}^\tau, \mu_k^\tau) \leq \int_{S_k} |P_{S_{k+1}}(\tau f_k(x) + x) - P_{S_{k+1}}(x)|^2 d\mu_k^\tau(x) + \int_{S_k} |P_{S_{k+1}}(x) - x|^2 d\mu_k^\tau(x). \quad (61)$$

Projection operators on convex sets satisfy the nonexpansive property and we use this fact for the first term. For the second term, we use the definition of the Hausdorff distance to obtain

$$W_2(\mu_{k+1}^\tau, \mu_k^\tau) \leq \left(\int_{S_k} |\tau f_k(x)|^2 d\mu_k^\tau(x) \right)^{1/2} + \tau d_H(S_k, S_{k+1}). \quad (62)$$

Using Assumption 1 for the drift term $f_k(x)$, it holds

$$W_2(\mu_{k+1}^\tau, \mu_k^\tau) \leq \left(\int_{S_k} \tau^2 |(L_f(1 + |x|))|^2 d\mu_k^\tau(x) \right)^{1/2} + \tau d_H(S_k, S_{k+1}). \quad (63)$$

Next, we establish a bound on the first term on the right-hand side of (63). We know that $\mu_k^\tau = G_{\#}^k \circ G_{\#}^{k-1} \circ \dots \circ G_{\#}^1 \mu_0 = (G^1 \circ G^2 \circ \dots \circ G^k)_{\#} \mu_0$. Let $G^{k \dots 1} := G^k \circ G^{k-1} \circ \dots \circ G^1$, then it follows that

$$\int_{S_k} (1 + |x|)^2 d\mu_k^\tau(x) = \int_{S_0} (1 + |G^{k \dots 1}(x_0)|)^2 d\mu_0^\tau(x_0). \quad (64)$$

Letting $x_k := G^{k \dots 1} x_0$, it has been shown in [8, Section 5] that

$$|x_k| \leq \tilde{C}_1 |x_0| + \tilde{C}_2 \quad (65)$$

for some constants $\tilde{C}_1, \tilde{C}_2 > 0$ depending on the system data. Now using this uniform bound on $|x_k|$ in the first term on the right-hand side of (63), we get

$$\begin{aligned} \left(\int_{S_k} \tau^2 L_f^2 (1 + |x|)^2 d\mu_k^\tau(x) \right)^{1/2} &= \tau L_f \left(\int_{S_0} (1 + |x_k|)^2 d\mu_0^\tau(x_0) \right)^{1/2} \\ &\leq \tau L_f \left(\int_{S_0} (1 + \tilde{C}_1 |x_0| + \tilde{C}_2)^2 d\mu_0^\tau(x_0) \right)^{1/2} \end{aligned} \quad (66)$$

$$=: \tau L_f C_{\max} \quad (67)$$

where the equality between (66) and (67) holds because $\mu_0^\tau \in \mathcal{P}(S_0)$. Thus

$$W_2(\mu_{k+1}^\tau, \mu_k^\tau) \leq \tau L_f C_{\max} + \tau d_H(S_k, S_{k+1}) = \tau(L_f C_{\max} + L_s)$$

where we used Assumption 2 in the last equality. \square

(1) *Proof of convergence of μ_t^τ to μ_t :* The convergence is based on computing the bounds on $W_2(\mu_t^\tau, \mu_s^\tau)$ for $s, t \in [0, T]$, where we recall that μ_t^τ are the interpolated measures. This is done by using the characterization of absolutely continuous curves (50), i.e.

$$W_2(\mu_t^\tau, \mu_s^\tau) \leq \int_s^t |(\mu^\tau)'(r)| dr \quad (68)$$

and using Hölder's inequality leads to

$$\int_s^t |(\mu^\tau)'(r)| dr \leq (t-s)^{1/2} \left(\int_s^t |(\mu^\tau)'(r)|^2 dr \right)^{1/2} \leq (t-s)^{1/2} \left(\sum_k \tau \left(\frac{W_2(\mu_{k+1}^\tau, \mu_k^\tau)}{\tau} \right)^2 \right)^{1/2}. \quad (69)$$

Using the bounds on $W_2(\mu_{k+1}^\tau, \mu_k^\tau)$ from (59), we get

$$\sum_k \tau \left(\frac{W_2(\mu_{k+1}^\tau, \mu_k^\tau)}{\tau} \right)^2 \leq \sum_k (L_s + L_f C_{\max})^2 \tau = (L_s + L_f C_{\max})^2 T.$$

Substituting (5.3) in (69) we obtain

$$W_2(\mu_t^\tau, \mu_s^\tau) \leq (t-s)^{1/2} (L_s + L_f C_{\max}) T^{1/2}. \quad (70)$$

Thus the curves are uniformly $\frac{1}{2}$ Hölder continuous. Moreover, the curves μ_t^τ lie in the W_2 space for each $t \in [0, T]$ which is compact. Thus, we can apply the Ascoli-Arzelà theorem (for the Hölder continuous functions) i.e., there exists a subsequence τ_j for which $\mu_t^{\tau_j} \rightarrow \mu_t$ uniformly in W_2 space and the limit curve μ_t is absolutely continuous.

Similar to (70) one can derive bounds for $\hat{\mu}_t^\tau$ and conclude that $\hat{\mu}_t^{\tau_j} \rightarrow \hat{\mu}_t$. Moreover the limit curves are the same as

$$W_2(\hat{\mu}_t^\tau, \mu_t^\tau) \leq (\tau)^{1/2} (L_f C_{\max} + L_s) T^{1/2}.$$

This holds because the curves $\hat{\mu}_t^\tau$ coincide with μ_t^τ at $k\tau$ and they are constant on the interval $(k\tau, (k+1)\tau]$. Thus both curves converge to the same limit curve μ_t .

(2) *Proof of convergence of E_t^τ :* In order to study the convergence properties of the velocity vector, we need to investigate the convergence properties of a family of momentum vectors $E_t^\tau = v_t^\tau \mu_t^\tau$ which is a vector measure⁴ $E_t^\tau \in \mathcal{M}^n(\Omega)$, where Ω is a measurable set which contains all $S(t)$, for all $t \in [0, T]$. We define $m^\tau \in \mathcal{M}^n([0, T] \times \Omega)$ as $m^\tau := v_t^\tau \mu_t^\tau dt$.

Lemma 3. *The norm of m^τ satisfies the following bound:*

$$|m^\tau|([0, T] \times \Omega) \leq T^{\frac{3}{2}} (L_f C_{\max} + L_s). \quad (71)$$

⁴The space of vector measures $\mathcal{M}^n(\Omega)$ is a normed space dual to $\mathcal{C}(\Omega; \mathbb{R}^n)$. Under this duality the notion of weak star convergence is defined which further implies that bounded sets in $\mathcal{M}^n(\Omega)$ are weak star compact.

Proof. By definition

$$|m^\tau|([0, T] \times \Omega) = \int_{[0, T]} dt \int_{\Omega} |v_t^\tau| d\mu_t^\tau.$$

Using the Cauchy-Schwarz inequality and then (54) we get

$$|m^\tau|([0, T] \times \Omega) \leq T^{1/2} \int_{[0, T]} \|v_t^\tau\|_{L^2(\mu_t^\tau)} dt \leq T^{1/2} \sum_k \frac{W_2(\mu_{k+1}^\tau, \mu_k^\tau)}{\tau}.$$

Using Lemma 2, we further obtain

$$|m^\tau|([0, T] \times \Omega) \leq T^{1/2} \sum_k (L_f C_{\max} + L_s) = T^{\frac{3}{2}} (L_f C_{\max} + L_s)$$

which is the desired bound. \square

So, m^τ is uniformly bounded and thus compact under weak convergence in the space of vector valued measures on $[0, T] \times \Omega$. We conclude that up to a subsequence $m^\tau \rightharpoonup m$ and thus $E_t^\tau \rightharpoonup E_t$. For $\hat{m}^\tau := \hat{v}_t^\tau \hat{\mu}_t^\tau$, a similar bound holds, i.e. $|\hat{m}^\tau| \leq T^{\frac{3}{2}} (L_f C_{\max} + L_s)$. Using the same arguments one concludes $\hat{m}^\tau \rightharpoonup \hat{m}$ and thus $\hat{E}_t^\tau \rightharpoonup \hat{E}_t$. Moreover, using [24, Lemma 8.9] we conclude that $\hat{E}_t = E_t$.

Next we discuss about the properties of the limit object E_t and show that E_t is absolutely continuous with respect to μ_t , such that $E_t = v_t \mu_t$, for each $t \in [0, T]$.

(3) *Absolute continuity of E_t :* At this point, we recall the properties of then Benamou-Brenier functional $\mathcal{B}(\mu, E)$ defined as follows. For $\mu \in \mathcal{P}(\mathbb{R}^n)$ and $E \in \mathcal{M}^n(\mathbb{R}^n)$, let

$$\mathcal{B}(\mu, E) := \sup_{\substack{a \in \mathcal{C}(\mathbb{R}; \mathbb{R}), b \in \mathcal{C}(\mathbb{R}^n; \mathbb{R}^n) \\ a, b \text{ are bounded} \\ a + \frac{1}{2}|b|^2 \geq 0 \text{ pointwise}}} \int_{\mathbb{R}^n} a(x) d\mu(x) + \int_{\mathbb{R}^n} b(x) dE(x)$$

which has the following properties:

- $\mathcal{B}(\cdot, \cdot)$ is convex, lower semicontinuous, and non-negative;
- $\mathcal{B}(\mu, E) = \frac{1}{2} \int |v|^2 d\mu$ only if $E = v\mu$ is absolutely continuous with respect to μ and $\mathcal{B}(\mu, E)$ is ∞ otherwise.

Note that $\mathcal{B}(\mu_t^\tau dt, E_t^\tau dt) = \int_{[0, T]} \int_{\Omega} |v_t^\tau|^2 d\mu_t^\tau dt$. Now using the uniform bound on $|m^\tau|$ (thus on $\int_{[0, T]} \int_{\Omega} |v_t^\tau|^2 d\mu_t^\tau dt$) in (71) and the lower semi-continuity of $\mathcal{B}(\cdot, \cdot)$, we get

$$\mathcal{B}(\mu_t dt, dm) \leq \liminf_{\tau \rightarrow 0} \mathcal{B}(\mu_t^\tau dt, dm^\tau) < \infty$$

where dm represents the limit of dm^τ , as $\tau \rightarrow 0$. We can now invoke the second property mentioned above which implies that dm is absolutely continuous with respect to $d\mu_t dt$. Thus, there exists v_t such that $dm = v_t \mu_t dt$. Similarly, E_t is absolutely continuous with respect to μ_t and $E_t = v_t \mu_t$.

(4) *Proof that $v_t(x) \in f(t, x) - \mathcal{N}_{S(t)}(x)$:* Next we show that the velocity v_t belongs to the admissible set of velocities. By construction of the velocities v_{k+1}^τ , for every $x \in S_k$, and every $y \in S_{k+1}$, we have

$$\left\langle y - P_{S_{k+1}} \circ (\tau f_k(\cdot) + \text{id})(x), \frac{(P_{S_{k+1}} \circ (\tau f_k(\cdot) + \text{id})(x) - x)}{\tau} \right\rangle \geq 0$$

which results in

$$\langle y - P_{S_{k+1}} \circ (\tau f_k(\cdot) + \text{id})(x), v_{k+1}^\tau(x) - f_k(x) \rangle \geq 0,$$

i.e., $v_{k+1}^\tau(x) - f_k(x)$ is in the normal cone to set S_{k+1} . The above condition should hold in the integral form for any smooth positive function $h(t, x)$ i.e.

$$\int_{[0, T]} \int_{\Omega} \left\langle h(t_k, x)(y - P_{S_{k+1}} \circ (\tau f_k(\cdot) + \text{id})x), (v_{k+1}^\tau(x) - f_k(x)) d\mu_k^\tau(x) \right\rangle \geq 0, \quad \forall y \in S_{k+1}. \quad (72)$$

Next we can extend (72) to piecewise constant interpolated curves (56) and corresponding velocities (57). We make a passage from discrete time $k\tau$ to $t \in (k\tau, (k+1)\tau]$ by identifying $v_{k+1}^\tau d\mu_k^\tau = \hat{v}_t^\tau d\hat{\mu}_t^\tau = d\hat{E}_t^\tau$ and similarly, $d\mu_k^\tau = d\hat{\mu}_t^\tau$. Next, we use the convergence results established in the first two steps of the proof, so that, in the limit as $\tau \rightarrow 0$, we obtain

$$\int_{[0, T]} \int_{\Omega} \left\langle h(t, x)(y - x), dE_t(x) \right\rangle - \int_{[0, T]} \int_{\Omega} \left\langle h(t, x)(y - x), f(t, x) d\mu_t(x) \right\rangle \geq 0, \quad \forall y \in S(t).$$

As we have already established that $E_t = v_t \mu_t$ and since $h(t, x)$ is an arbitrary positive function, we get

$$\langle y - x, v(t, x) - f(t, x) \rangle \geq 0, \quad \forall y \in S(t).$$

Thus, $v(t, x) - f(t, x) \in -\mathcal{N}_{S(t)}(x)$.

Uniqueness of solutions: The uniqueness of solutions follows from the same argument as presented in Section 3.4 based on the superposition principle. Furthermore, it also states that the solution has a representation formula as $\mu_t = X_{t\#} \mu_0$. \square

6 Numerical Results

6.1 Moment-SOS hierarchy

In Section 3, we derived the continuity equation (27) associated with dynamical system (4) using the superposition principle. The problem of computing the evolution of a probability measure through a dynamical system can be interpreted as a feasibility problem where the feasible

set is an affine section (modeled by the continuity equation) of the cone of non-negative measures (supported on the trajectories). For notational convenience, we rewrite the continuity equation in (27) as

$$\partial_t \mu + \nabla \cdot (\bar{f}_\omega \mu) + \delta_T \otimes \mu_T = \delta_0 \otimes \mu_0 \quad (73)$$

where \bar{f}_ω is defined in (13) and the equation should be understood in the weak sense, i.e. when integrated against sufficiently smooth test functions as in (34). Given an initial distribution μ_0 with $\text{supp}(\mu_0) \subset X$, we formulate the problem of evolution of measures in (4) as the following infinite dimensional optimization problem:

$$\begin{aligned} & \text{Find } \mu, \mu_T \text{ such that} \\ & \partial_t \mu + \nabla \cdot (\bar{f}_\omega \mu) + \delta_T \otimes \mu_T = \delta_0 \otimes \mu_0 \\ & \mu \geq 0, \mu_0 \geq 0, \mu_T \geq 0 \\ & \text{supp}(\mu) \subset [0, T] \times B, \quad \text{supp}(\mu_T) \subset X_T, \end{aligned} \quad (74)$$

where $B = \{(x, v) | x \in \Omega \subset \mathbb{R}^n, v \in F(x)\}$. This is an infinite dimensional linear program. In this section, we present numerical results obtained using the moment-SOS hierarchy to solve this linear program in terms of approximate moments of measures μ and μ_T . The moments are generated by integration on a dense set of functions ϕ in $\mathcal{C}^1([0, T], \mathbb{R}^n)$ as in (27). For notational convenience we use the monomial basis.

Numerical Example: Consider the nonsmooth system

$$\dot{\mathbf{x}}(t) \in (1, 0) - \mathcal{N}_S(\mathbf{x}(t)) \quad (75)$$

where $S = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$. For this problem, we decompose the measure solution μ of (34) into two measures: μ_S supported in the disc and $\mu_{\partial S}$ supported on the boundary of the disc. In particular, we let $X_S := \text{supp}(\mu_S) = \{0 \leq t \leq 1\} \times \{(\mathbf{x}, \mathbf{v}) : x_1^2 + x_2^2 \leq 1, v_1 = 1, v_2 = 0\}$, and $X_{\partial S} := \text{supp}(\mu_{\partial S}) = \{0 \leq t \leq 1\} \times \{(\mathbf{x}, \mathbf{v}) : x_1^2 + x_2^2 = 1, x_1 v_2 = x_2 v_1\}$. The continuity equation (34) can be expressed as

$$\begin{aligned} & \int_{X_T} \phi(T, \mathbf{x}) d\mu_T(\mathbf{x}) - \int_{X_0} \phi(0, \mathbf{x}) d\mu_0(\mathbf{x}) \\ & = \int_{X_S} \left[\partial_t \phi(t, \mathbf{x}) + \nabla_{\mathbf{x}} \phi(t, \mathbf{x}) \cdot \mathbf{v} \right] d\mu_S(t, \mathbf{x}, \mathbf{v}) + \int_{X_{\partial S}} \left[\partial_t \phi(t, \mathbf{x}) + \nabla_{\mathbf{x}} \phi(t, \mathbf{x}) \cdot \mathbf{v} \right] d\mu_{\partial S}(t, \mathbf{x}, \mathbf{v}). \end{aligned} \quad (76)$$

To approximate the moments of the measures satisfying the above equation we use the monomial basis. Let $R[\mathbf{x}]$ be the ring of multivariate polynomials and $R_k[\mathbf{x}] \subset R[\mathbf{x}]$ be the vector space of polynomials of degree not exceeding k . Then the monomial basis of $R_k[\mathbf{x}]$ can be expressed as $\phi(t, \mathbf{x}) := t^a \mathbf{x}^b := t^a x_1^{b_1} x_2^{b_2} \dots x_n^{b_n}$ where $a + b_1 + b_2 + \dots + b_n \leq k$.

Proposition 6.1. *Let $m_{a,b}^0 := \int_0^T \int_{\mathbb{R}^n} t^a \mathbf{x}^b d\mu_0(t, \mathbf{x})$ be the moments of μ_0 . Then using the monomial basis in (76) we get*

$$m_{a,b}^T - m_{a,b}^0 = am_{a-1,b}^S + am_{a-1,b}^{\partial S} + \sum_{i=1}^n \int_{X_{\partial S}} b_i t^a \mathbf{x}^{b-e_i} v_i d\mu_{\partial S}(t, \mathbf{x}, \mathbf{v}) + \sum_{i=1}^n \int_{X_S} b_i t^a \mathbf{x}^{b-e_i} v_i d\mu_S(t, \mathbf{x}, \mathbf{v}) \quad (77)$$

where $m_{a,b}^S := \int_{X_S} t^a \mathbf{x}^b d\mu(t, \mathbf{x})$, $m_{a,b}^{\partial S} := \int_{X_{\partial S}} t^a \mathbf{x}^b d\mu(t, \mathbf{x})$, $m_{a,b}^T := T^a \int_{X_T} \mathbf{x}^b d\mu_T(\mathbf{x})$ and $e_i = (0, \dots, 1, \dots, 0)$ is the vector with one at the i th entry.

Proof. The proof is a straightforward extension of [23, Proposition 4.3] \square

The moment-SOS hierarchy allows us to evaluate approximate moments $m_{a,b} := m_{a,b}^S + m_{a,b}^{\partial S}$ and $m_{a,b}^T$ related to occupation measures μ and μ_T . For details about the moment-SOS hierarchy we refer the readers to [23, Section 4.3] or [22]. The initial distribution is $\mu_0 = \delta_{(0,0.5)}$, i.e. the Dirac measure at coordinates $(0, 0.5)$ whose moments $m_{a,b}^0$ are readily available. The simulation results are displayed in Figure 1, where we plot the first order moments of terminal measures μ_T , for different terminal times T and computed for a given relaxation order using GloptiPoly and SeDuMi. We can observe the effect of the boundary on the measure even before it hits the boundary. We attribute this effect to the numerical inaccuracy due to finite order truncation. The approximate moments obtained, sometimes called pseudo-moments, may not represent the true moments but the accuracy increases as the relaxation order increases.

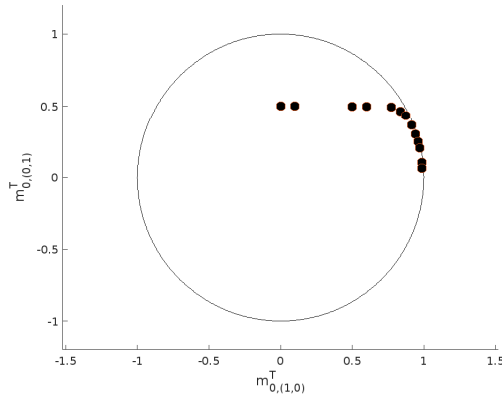


Figure 1: Moment-SOS hierarchy approximations of the first degree moments of the terminal measure μ_T solving the continuity equation (73) for the nonsmooth system (75), with different terminal times ranging from $T = 0$ (Dirac mass at $(0, 0.5)$) to $T = 3$ (Dirac mass approximately at $(1, 0)$).

6.2 Bound on Wasserstein Distance

In Section 4, we used the Moreau-Yosida regularization to approximate the solution $(\mu_t)_{t \geq 0} \in \mathcal{P}(\mathbb{R}^n)$ of the continuity equation associated with the nonsmooth dynamical system (4) with measures $(\mu_t^\lambda)_{t \geq 0} \in \mathcal{P}(\mathbb{R}^n)$ which are solutions to the continuity equation with regularized vector field (36). We showed that the measures μ_t, μ_t^λ , which have representations of the

form (35) and (37) respectively, satisfy a bound on their Wasserstein distance of the form $W_1(\mu_t^\lambda, \mu_t) < C_W \sqrt{\lambda}$ for each time $t \in [0, T]$ (the explicit expression of the constant C_W is in (38)). Now let us validate this bound for the following example:

$$\dot{x}(t) \in -1 - \mathcal{N}_{\mathbb{R}^+}(x(t)). \quad (78)$$

Its Moreau-Yosida regularization is:

$$\dot{x}^\lambda(t) = -1 - \frac{1}{\lambda}(x(t) - \max(x(t), 0)). \quad (79)$$

We consider an initial value problem for (78) and (79) with $x(0) = 0.5$. The solution to the initial value problem for (78) is

$$x(t) = \begin{cases} 0.5 - t & \text{for } 0 < t < \frac{1}{2} \\ 0 & \text{for } t \geq \frac{1}{2} \end{cases} \quad (80)$$

and the solution for (79) is

$$x^\lambda(t) = \begin{cases} 0.5 - t & \text{for } 0 < t < \frac{1}{2} \\ \lambda(\exp(-(t-0.5)/\lambda) - 1) & \text{for } t \geq \frac{1}{2}. \end{cases} \quad (81)$$

Proposition 3. Given $\mu_t^\lambda, \mu_t \in \mathcal{P}(\mathbb{R})$ and $\mu_0 = \mu_0^\lambda = \delta_{x=a}$ for some $a > 0$, then

$$W_1(\mu_t^\lambda, \mu_t) = |x^\lambda(t) - x(t)|. \quad (82)$$

Proof. It is known that in one dimension $W_1(\mu, \nu) = \int_{\mathbb{R}} |F_\mu(s) - F_\nu(s)| ds$ [24] where $F_\mu(\cdot)$ and $F_\nu(\cdot)$ are the cumulative distribution functions of measures μ and ν respectively. Using this formula, we get

$$W_1(\mu_t^\lambda, \mu_t) = \int_{\mathbb{R}} |F_{\mu_t^\lambda}(s) - F_{\mu_t}(s)| ds. \quad (83)$$

Given $\mu_0 = \mu_0^\lambda = \delta_{x=a}$, the measures μ_t^λ and μ_t can be computed as $\mu_t^\lambda = \delta_{x^\lambda(t)}$ and $\mu_t = \delta_{x(t)}$ and thus $F_{\mu_t^\lambda}(s) = \Theta_{x^\lambda(t)}(s)$ and $F_{\mu_t}(s) = \Theta_{x(t)}(s)$. Here, $\Theta : \mathbb{R} \rightarrow \mathbb{R}$ is such that $\Theta_{x(t)}(s) = 1$ for $x(t) \leq s$ and $\Theta_{x(t)}(s) = 0$ for $x(t) \geq s$. Substituting $F_{\mu_t^\lambda}$ and F_{μ_t} in (83), we get

$$W_1(\mu_t^\lambda, \mu_t) = \int_{\mathbb{R}} |\Theta_{x^\lambda(t)}(s) - \Theta_{x(t)}(s)| ds. \quad (84)$$

This quantity can be seen as the area under $|\Theta_{x^\lambda(t)}(s) - \Theta_{x(t)}(s)|$ at time t and performing the integration (84) we get the desired result. \square

Given (80), (81) and using Proposition 3 we obtain for $a = 0.5$:

$$W_1(\mu_t^\lambda, \mu_t) = \begin{cases} 0 & \text{for } 0 < t < \frac{1}{2} \\ |\lambda(e^{-(t-0.5)/\lambda} - 1)| & \text{for } t \geq \frac{1}{2}. \end{cases} \quad (85)$$

Next we obtain the bound in (38) for system (78) with $L_f = 1$, $L_s = 0$ and $\mu_0 = \delta_{x=0.5}$. Substituting these values we get

$$W_1(\mu_t^\lambda, \mu_t) \leq \frac{3}{2} e^2 \sqrt{\frac{\lambda}{2}(e^t - 1)}. \quad (86)$$

In Figure 2, we show the plots of the analytical distance (85) and its upper bound (86) for different values of λ at four different time instants.

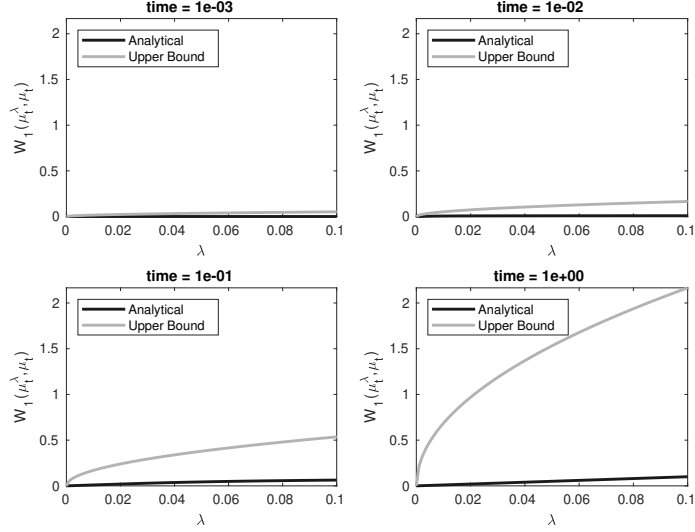


Figure 2: W_1 distance (85) (in black) and its upper bound (86) (in gray) between the image measures μ_t resp. μ_t^λ through the flow (35) resp. the regularized flow (37), at various time instants.

6.3 Time Discretized Measure Evolution

In Section 5, we used geodesic interpolation and piecewise constant interpolation for the time discretized curves and showed convergence to the solution to the continuity equation (27). Let us apply the time-stepping algorithm to implement a time discretized evolution of the measure solutions to the continuity equation associated with the nonsmooth dynamical system (4).

Let τ be the time step between two discretized measures and let μ_k^τ be the measure at time k . Then from Section 5:

$$\mu_0^\tau = \mu_0; \quad \mu_{k+1}^\tau = P_{S_{k+1}\#}(\tau f_k(\cdot) + id)_\# \mu_k. \quad (87)$$

We model the measure as mass distributed on a space discretized grid. Then the time-stepping scheme consists of the following two steps:

- Step 1: compute the pushforward measure

$$\tilde{\mu}_k := (\tau f_k(\cdot) + id)_\# \mu_k; \quad (88)$$

- Step 2: project each cell lying outside S_{k+1} back onto S_{k+1} . This operation can be formulated as

$$\mu_{k+1} = P_{S_{k+1}\#} \tilde{\mu}_k \quad (89)$$

and it can be understood as a projection of measures on the set of measures with support S_{k+1} .

In Figure 3 we illustrate the scheme for the following example

$$\dot{x}(t) \in -\mathcal{N}_{S(t)}(x(t))$$

where $S(t)$ is a 4×4 grid square, with each cell assigned with mass of $1/16$ at time $t = 0$.

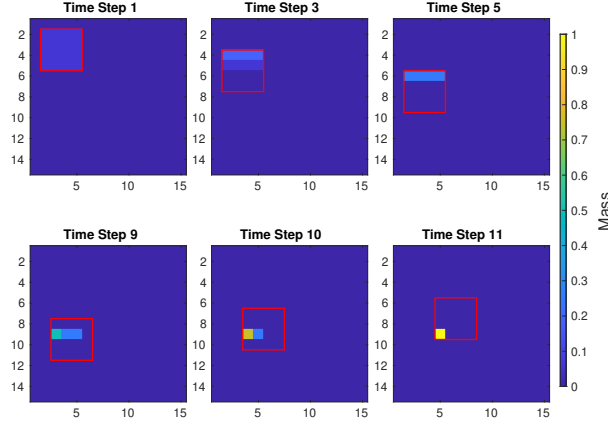


Figure 3: Snapshots of time evolution of a uniform probability distribution on the moving square. The mass is initially distributed on a 4×4 square grid with $1/16$ mass on each cell. For the first 7 time steps the square moves downward and the mass can be seen to concentrate on the top edge of the square. Afterwards the square moves diagonally and the mass concentrates on the left edge until it completely concentrates at the bottom left corner at time step 12.

7 Conclusion

We addressed the problem of evolution of measures in a nonsmooth dynamical system modeled by evolution variation inequalities using three different formalisms. In future work, we aim to study the optimal control problem for the considered system class using measure relaxation. Based on our preliminary investigation, the results of this article allow us to address the propagation of measures in the presence of control inputs. In particular, it opens up the possibility to study the convergence of discrete time optimal control problem to the continuous time optimal control problem in the space of measures.

A Bounds used in proof of Proposition 2

In this appendix, we provide estimates for the bounds $\int |\dot{\gamma}(t)| d\eta(x, \gamma)$. To do so, we first get bounds on $|\gamma(t)|$ using the linear growth in Assumption 4,

$$\begin{aligned} |\gamma(t) - \gamma(0)| &\leq \int |\dot{\gamma}(s)| ds \leq \int \beta(s)(1 + |\gamma(s)|) ds \\ &\leq \int_0^t \beta(s)(1 + |\gamma(0)|) ds + \int_0^t \beta(s)(|\gamma(s) - \gamma(0)|) ds. \end{aligned}$$

Applying Gronwall inequality by assuming first term is non-decreasing,

$$|\gamma(t) - \gamma(0)| \leq \left[\int_0^t \beta(s)(1 + |\gamma(0)|) ds \right] e^{\left(\int_0^t \beta(\sigma) d\sigma \right)}. \quad (90)$$

Let $\bar{\beta}_t := \int_0^t \beta(\sigma) d\sigma$, then

$$|\gamma(t) - \gamma(0)| \leq \bar{\beta}_t \exp(\bar{\beta}_t)(1 + |\gamma(0)|) \leq \bar{\beta}_T \exp(\bar{\beta}_T)(1 + |\gamma(0)|).$$

We use this last inequality to get the desired bound as follows:

$$\begin{aligned} \int_{\mathbb{R}^n \times \Gamma_T^\omega} |\dot{\gamma}(t)| d\eta(x, \gamma) &\leq \int_{\mathbb{R}^n \times \Gamma_T^\omega} \beta(t)(1 + |\gamma(t)|) d\eta(x, \gamma) \\ &\leq \int_{\mathbb{R}^n \times \Gamma_T^\omega} \beta(t)(1 + |\gamma(0)|)(\bar{\beta}_T \exp(\bar{\beta}_T) + 1) d\eta(x, \gamma) \leq K\beta(t), \end{aligned}$$

where $K := \int_{\mathbb{R}^n \times \Gamma_T^\omega} (1 + |\gamma(0)|)(\bar{\beta}_T \exp(\bar{\beta}_T) + 1) d\eta(x, \gamma)$.

B Differentiating first moment with time dependence

In this appendix, we derive the Liouville equation (22). For any $\varphi \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^n)$ we differentiate $\int \varphi(t, x) d\mu_t(x)$ and get,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} \varphi(t, x) d\mu_t(x) &= \frac{d}{dt} \int_{\mathbb{R}^n} \varphi(t, x) d\mathbf{e}_{t\#} \eta(x, \gamma) \\ &= \frac{d}{dt} \int_{\mathbb{R}^n \times \Gamma_T^\omega} \varphi(t, \gamma(t)) d\eta(x, \gamma) \\ &= \int_{\mathbb{R}^n \times \Gamma_T^\omega} \left(\partial_t \varphi(t, \gamma(t)) + \nabla_x \varphi(t, x) \dot{\gamma}(t) \right) d\eta(x, \gamma) \\ &= \int_{\mathbb{R}^n} \left(\partial_t \varphi(t, x) + \nabla_x \varphi(t, x) \cdot f(t, x) \right) d\mu_t(x). \end{aligned}$$

Integrating w.r.t. t on both sides gives the desired result.

C A property for Moreau-Yosida regularization for the mapping induced by normal cone

In [23], the bound on $|Y_\lambda|$ was established as follows,

$$|Y_\lambda(x^\lambda(t))| \leq \frac{1}{\lambda} \int_0^t e^{-(t-s)/\lambda} (L_f + L_f(|x^\lambda(t)|) + L_s) ds. \quad (91)$$

It was demonstrated that x^λ satisfy uniform bound $|x^\lambda(t)| \leq e^{2L_f T} |x^\lambda(0)| + \kappa$ where, $\kappa = (e^{2L_f T} - 1) \frac{2L_f + L_s}{2L_f}$ and T is the time interval for the existence of trajectory. Upon substituting this uniform bound on $|x^\lambda|$ in (91) we get,

$$|Y_\lambda(x^\lambda(t))| \leq L_f(1 + \kappa + e^{2L_f T} |x^\lambda(0)|) + L_s. \quad (92)$$

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