PROPAGATION OF MOMENTS AND REGULARITY FOR THE VLASOV-MAXWELL-BOPP-PODOLSKY MODEL

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PROPAGATION OF MOMENTS AND REGULARITY FOR THE VLASOV-MAXWELL-BOPP-PODOLSKY MODEL

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Abstract. We consider a class of hyperbolic systems which can be interpreted as approximations of the relativistic Vlasov-Maxwell system. These equations are derived by taking into account radiation-reaction effects occurring at a microscopic level. They involve a small length $\ell \in \mathbb{R}_+^*$, called the Bopp-Podolsky parameter. In this context, we address common issues in kinetic equations such as the propagation of moments, regularity properties and well-posedness. We also investigate semiclassical limits appearing when ℓ goes to zero.

Keywords. Hyperbolic systems of conservation laws ; Relativistic Vlasov-Maxwell system ; Bopp-Podolsky electrodynamics ; Well-posedness of Cauchy problems ; Propagation of moments ; Asymptotic analysis.

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1. Introduction

Subsection 1.1 introduces the model ; Subsection 1.2 presents our main results.

1.1. The Vlasov-Maxwell-Bopp-Podolsky model.

In the next Paragraph 1.1.1, we explain the physics underlying our study. In Paragraph 1.1.2, we set up a Cauchy problem. Then, in Paragraph 1.1.3, we present our motivations.

1.1.1. *Physical background.* The plasmas contain charged particles generating an electric field E, a magnetic field B, a displacement D and a magnetic intensity H. The pairs of vector fields $(\mathrm{E},\mathrm{B})(t,x) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $(\mathrm{D},\mathrm{H})(t,x) \in \mathbb{R}^3 \times \mathbb{R}^3$ depend on the time $t \in \mathbb{R}$ and the spatial position $x \in \mathbb{R}^3$. They must satisfy the following (premetric) Maxwell field equations

$$\left\{ \begin{array}{ll} \partial_t \mathrm{B} + c\, \nabla_x \times \mathrm{E} = 0\,, & \nabla_x \cdot \mathrm{B} = 0\,, \\ \partial_t \mathrm{D} - c\, \nabla_x \times \mathrm{H} = -\jmath\,, & \nabla_x \cdot \mathrm{D} = \rho\,, \end{array} \right. \tag{1.1}$$

where $c \simeq 3 \times 10^8\, m\, s^{-1}$ is the speed of light, $\rho(t,x) \in \mathbb{R}$ is a charge density and $\jmath(t,x) \in \mathbb{R}^3$ is a current vector-density.

The equations (1.1) are familiar from theories in material media. They can be considered at all scales, including short distances. In particular, they can be studied for $|x| \lesssim \ell$ where $\ell \in \mathbb{R}_+^*$ is the Bopp-Podolsky parameter [12, 20], which in practice [3, 17] is a length in the order of $\ell \leq 10^{-15}\, m$, and which can be interpreted as a cut-off distance or as an effective radius for the electron [14]. At such microscopic scales, the electron is viewed as a dressed particle. It is the source of an electromagnetic field which it creates and with which it interacts. A way to describe this interconnection is to keep the notion of point particle, and to impose constitutive relations between B, E, D and H, called electromagnetic vacuum laws [15]. Given an integer $n \in \mathbb{N}_+$ and a nonnegative number $\beta \in \mathbb{R}_+$, we consider that

$$\mathrm{D} = (1+\ell^2\,\Box)^n\,(1-\ell^2\,\Delta)^{\beta/2}\,\mathrm{E}\,, \qquad \mathrm{H} = (1+\ell^2\,\Box)^n\,(1-\ell^2\,\Delta)^{\beta/2}\,\mathrm{B}\,, \tag{1.2}$$

where Δ and $\Box$ are respectively the Laplacian and the d'Alembertian ($\Box := c^{-2}\,\partial_{tt}^2 - \Delta$). Following are a few historical and general comments concerning the choice of n and β.

The law of electromagnetic vacuum (called also « law of the pure ether »), which was originally introduced by Maxwell, corresponds to the case of $(n,\beta) = (0,0)$, that is $(\mathrm{D},\mathrm{H}) = (\mathrm{E},\mathrm{B})$. It raises a well-known *radiation-reaction* problem [3, 15], originated in the preliminary works of Lorentz, Abraham, Dirac, ... However, nowadays there is a renewed interest to solve this difficulty in the context of generalized continuum theories [13, 20] involving situations where $n \in \mathbb{N}^*$. The case $n = 1$ was first suggested by Bopp ([1]-1940), just after studied by Landé and Thomas ([18, 19]-1941), and taken up again by Podolsky ([24]-1942). The system (1.1)-(1.2) with $n = 1$ (and $\beta = 0$) is the cornerstone of Bopp-Podolsky electrodynamics which has many interesting features : It provides a smoothing of underlying singularities [26], and it solves the infinite self-energy difficulty [15] ; It is issued from a Lorentz and gauge invariant Lagrangian [7, 13] (containing higher order derivatives), and it is a continuation of gradient theories [20] ; It is amenable to quantization, and it is a natural way of providing the Pauli-Villars procedure in quantum electrodynamics [14]. For $n \in \mathbb{N}$, we can introduce

$$\mathrm{G}_0 := -(1-\ell^2\,\Delta)^{-\beta/2}\,\jmath \quad \text{and} \quad \mathrm{G}_j := (1+\ell^2\,\Box)^{n-j}\,(\partial_t \mathrm{E} - c\,\nabla_x \times \mathrm{B})\,, \quad j \in \{1,\cdots,n\}\,.$$

We can reformulate (1.1)-(1.2) according to

$$\left\{\begin{array}{l} (1+\ell^2\,\Box)\mathrm{G}_1 = \mathrm{G}_0 = -(1-\ell^2\,\Delta)^{-\beta/2}\,\jmath\,, \\ \quad \cdots\cdots\cdots\cdots \\ (1+\ell^2\,\Box)\mathrm{G}_j = \mathrm{G}_{j-1}\,, \\ \quad \cdots\cdots\cdots\cdots \\ (1+\ell^2\,\Box)\mathrm{G}_n = \mathrm{G}_{n-1}\,, \\ \partial_t \mathrm{E} - c\,\nabla_x \times \mathrm{B} = \mathrm{G}_n\,, \\ \partial_t \mathrm{B} + c\,\nabla_x \times \mathrm{E} = 0\,. \end{array}\right. \tag{1.3}$$

In a quantized interpretation, this means that the n first equations describe hypothetical massive photons, whereas the last two are for the usual (massless) photons. The n vector fields G_j, with $\mathrm{G}_j(t,x) \in \mathbb{R}^3$ and $j \in \{1,\cdots,n\}$ may be grouped together to form $\mathrm{G} := (\mathrm{G}_1,\cdots,\mathrm{G}_n)$. The system (1.3) appears clearly as an extension of Maxwell's equations. As will be seen, the introduction of the n supplementary equations has smoothing properties, which are all the more pronounced when n is large. The same applies to β. The limit case $\beta = 0$ is perhaps more relevant from a physical standpoint. Now, the transition to $\beta > 0$ is mathematically interesting since it serves to bring some additional (fractional) regularizing effects.

1.1.2. *The Cauchy problem.* There is another important consequence of the above microscopic considerations, which is related to the kinetic description of plasmas at large scales. By looking at a system of N charged point particles interacting through (1.2) with $n = 1$, Y. Elskens and M.K-H. Kiessling [9] have recently laid down (in a scaling limit $N \to +\infty$) a microscopic foundation of the Vlasov-Maxwell-Bopp-Podolsky (VMBP) model (called in [9] the Vlasov-Maxwell-BLTP model with LT in memory of Landé and Thomas).

In this description of the interactions of radiation with matter, they keep (1.3) with $n = 1$ and they add a coupling with a continuum density (of charged particles) denoted by $\mathrm{f}(t, x, \xi)$, defined on the phase space $\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$. Let e be the electron charge and m be the electron mass. The function f is positive, and it must satisfy the transport equation

$$\partial_t \mathrm{f} + c\, \nu(\xi) \cdot \nabla_x \mathrm{f} + (e/m\, c)\, \mathrm{F}(t, x, \xi) \cdot \nabla_{(\xi/m\, c)} \mathrm{f} = 0\,, \tag{1.4}$$

where we have introduced the relativistic speed $\nu(\xi)$ of momentum ξ and the Lorentz force F which are given by

$$\nu(\xi) := \frac{\xi/m\, c}{\sqrt{1 + |\xi|^2/m^2\, c^2}}\,, \qquad \mathrm{F}(t, x, \xi) := \mathrm{E}(t, x) + \nu(\xi) \times \mathrm{B}(t, x)\,. \tag{1.5}$$

Retain that the sup norm of f remains unchanged

$$\| \mathrm{f}(t, \cdot) \|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} = \| \mathrm{f}^0 \|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}\,, \qquad \forall\, t \in \mathbb{R}_+\,. \tag{1.6}$$

Given $p \in \mathbb{R}$ with $1 \leq p < +\infty$, the same holds true for all L^p-norms

$$\| \mathrm{f}(t, \cdot) \|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)} = \| \mathrm{f}^0 \|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}\,, \qquad \forall\, t \in \mathbb{R}_+\,. \tag{1.7}$$

Starting from (1.4), we can check that the charge density ρ and the current vector-density $\jmath$ which are defined by

$$\rho(t, x) := -\, e \int_{\mathbb{R}^3} \mathrm{f}(t, x, \xi)\, d\xi\,, \qquad \jmath(t, x) := -\, e \int_{\mathbb{R}^3} \nu(\xi)\ \mathrm{f}(t, x, \xi)\ d\xi\,, \tag{1.8}$$

must satisfy the continuity equation

$$\partial_t \rho + c\, \nabla_x \cdot \jmath = 0\,. \tag{1.9}$$

The VMBP model (of length ℓ, order n and index β) is denoted by $BP^n_{\beta,\ell}$. It is built with (1.3)-(1.4) together with (1.8). The unknown is therefore

$$\mathrm{U}(t, x, \xi) := \big(\mathrm{G}_1(t, x), \cdots, \mathrm{G}_n(t, x), \mathrm{E}(t, x), \mathrm{B}(t, x), \mathrm{f}(t, x, \xi)\big)\,. \tag{1.10}$$

Inside (1.3), each Klein-Gordon equation can be written as a hyperbolic system involving G_j, $\partial_t \mathrm{G}_j$ and $\nabla_x \mathrm{G}_j$. Thus, we can alternatively consider that the unknown is

$$\tilde{\mathrm{U}} := (\mathrm{G}_1, \partial_t \mathrm{G}_1, \nabla_x \mathrm{G}_1, \cdots, \mathrm{G}_n, \partial_t \mathrm{G}_n, \nabla_x \mathrm{G}_n, \mathrm{E}, \mathrm{B}, \mathrm{f})\,. \tag{1.11}$$

With this in mind, the initial data should be given by $\tilde{\mathrm{U}}_{|t=0} = \tilde{\mathrm{U}}^0$. But $\nabla_x \mathrm{G}_{j\,|t=0}$ can be deduced from $\mathrm{G}_{j\,|t=0}$. Thus, it suffices to start with

$$\left\{ \begin{array}{ll} \mathrm{U}_{|t=0} \equiv \mathrm{U}^0 = (\mathrm{G}^0, \mathrm{E}^0, \mathrm{B}^0, \mathrm{f}^0)\,, & \mathrm{G}^0 = (\mathrm{G}^0_1, \cdots, \mathrm{G}^0_n)\,, \\ \partial_t \mathrm{G}_{|t=0} = \mathrm{H}^0\,, & \mathrm{H}^0 = (\mathrm{H}^0_1, \cdots, \mathrm{H}^0_n)\,. \end{array} \right. \tag{1.12}$$

The system $BP^n_{\beta,\ell}$ must be supplemented with the compatibility conditions inherited from (1.1), which read

$$C^n_{\beta,\ell} \qquad \nabla_x \cdot \mathrm{B} = 0\,, \qquad (1 + \ell^2\, \Box)^n\, (1 - \ell^2\, \Delta)^{\beta/2}\, \nabla_x \cdot \mathrm{E} = \rho\,.$$

We can exploit the evolution equations to express $(\partial^2_{tt})^j\, \nabla_x \cdot \mathrm{E}$ with $j \leq n$ in terms of spatial derivatives of U and $\partial_t \mathrm{G}$. At time $t = 0$, this furnishes a constraint on U^0 and H^0. In what follows, we assume that it is satisfied. Then, due to (1.9), it is propagated. This is why, it can be forgotten. From now on, the focus is on the Cauchy problem built with the multidimensional (one order) hyperbolic system of conservation laws (inherited from) $BP^n_{\beta,\ell}$ associated with the initial data (1.12). To our knowledge, the mathematical study of $BP^n_{\beta,\ell}$, even when $(\beta, n) = (0, 1)$, has not yet been investigated. Observe that the changes

$$t \to \tilde{t} := \frac{c\, t}{\ell}\,, \qquad x \to \tilde{x} := \frac{x}{\ell}\,, \qquad \xi \to \tilde{\xi} := \frac{\xi}{m\, c}\,, \tag{1.13}$$

together with

$$\mathrm{G}_j \to \tilde{\mathrm{G}}_j := \frac{\ell^2\, e}{m\, c^3}\ \mathrm{G}_j\,, \qquad (\mathrm{E},\mathrm{B}) \to (\tilde{\mathrm{E}},\tilde{\mathrm{B}}) := \frac{\ell\, e}{m\, c^2}\ (\mathrm{E},\mathrm{B})\,, \qquad \mathrm{f} \to \tilde{\mathrm{f}} := \ell^2\, e^2\, m^2\ \mathrm{f} \tag{1.14}$$

provide with a dimensionless version of $BP^n_{\beta,\ell}$, which is denoted by BP^n_β. In other words, the system BP^n_β is the same as $BP^n_{\beta,1}$ with all coefficients c, e and m normalized to 1. For simplicity, we can drop the tilde $\tilde{\cdot}$ everywhere when formulating BP^n_β. There remains

$$BP^n_\beta \qquad \begin{cases} (1+\Box)\mathrm{G}_1 = -(1-\Delta)^{-\beta/2}\,\jmath\,, & \Delta := \sum \partial^2_{x_i}\,, \\ \quad \cdots\cdots\cdots\cdots & \Box := 1+\partial^2_{tt}-\Delta\,, \\ (1+\Box)\mathrm{G}_n = \mathrm{G}_{n-1}\,, & \\ \partial_t \mathrm{E} - \nabla_x \times \mathrm{B} = \mathrm{G}_n\,, & (1+\Box)^n\,(1-\Delta)^{\beta/2}\,\nabla_x\cdot \mathrm{E} = \rho\,, \\ \partial_t \mathrm{B} + \nabla_x \times \mathrm{E} = 0\,, & \nabla_x \cdot \mathrm{B} = 0\,, \\ \partial_t \mathrm{f} + \nu(\xi)\cdot\nabla_x \mathrm{f} + \mathrm{F}\cdot\nabla_\xi \mathrm{f} = 0\,. \end{cases}$$

By construction, with $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$, we deal here with

$$\nu(\xi) := \frac{\xi}{\langle\xi\rangle} < 1\,, \qquad \mathrm{F}(t,x,\xi) := \mathrm{E}(t,x) + \nu(\xi)\times \mathrm{B}(t,x)\,, \tag{1.15}$$

and the coupling is now achieved through

$$\rho(t,x) := -\int_{\mathbb{R}^3} \mathrm{f}(t,x,\xi)\, d\xi\,, \qquad \jmath(t,x) := -\int_{\mathbb{R}^3} \nu(\xi)\ \mathrm{f}(t,x,\xi)\ d\xi\,. \tag{1.16}$$

In other words, the family of systems $BP^n_{\beta,\ell}$ with $\ell \in \mathbb{R}^*_+$ admits the rescaled version BP^n_β. When discussing about global (strong or weak) well-posedness associated with any (small or large) data, it makes no sense to differentiate between $BP^n_{\beta,\ell}$ and BP^n_β because we can pass from one system to the other. For this reason, our related results will be expressed only in terms of BP^n_β. By contrast, when dealing with local existence or with the limit $\ell \to 0+$ (for fixed scales or data), it is crucial to think in terms of $BP^n_{\beta,\ell}$. This allows to keep track of the lifespan or to study the asymptotic behavior of the solutions (when $\ell \to 0+$).

For $\ell = 0$, we find that $\mathrm{G}_j = -\jmath$ for all j, and we recover the Relativistic Vlasov-Maxwell (RVM) system. As well, for $n = 0$ and $\beta = 0$, the system $BP^0_{0,\ell}$ coincides with RVM. Clearly, there are links to explore between $BP^n_{\beta,\ell}$ and RVM. It turns out that VMBP (with $0 < \ell \ll 1$) and RVM are (almost) identical in terms of (relatively) large wavelengths (Subsection 4.2). But they can strongly differ out of such regimes, where VMBP (especially for $n = 1$ and $\beta = 0$) is built on stronger theoretical foundations. The VMBP system can be viewed as a kind of large frequency hyperbolic deformation of RVM, operating as some hyperbolic regularization of RVM, which is clearly driven by physics.

1.1.3. *Motivations.* Our aim is to explore the consequences of the modification of (1.2) from $\ell = 0$ to $\ell > 0$, $\beta = 0$ to $\beta > 0$ and from $n = 0$ to $n \geq 1$. Three topics are particularly relevant :

(1) The first is about the *propagation of moments*

$$\mathcal{M}_a(t) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \langle\xi\rangle^a\ \mathrm{f}(t,x,\xi)\ dx\, d\xi\,, \qquad a \in \mathbb{R}_+\,. \tag{1.17}$$

(a) For Vlasov-Poisson, this question has received much attention. Historically, progress has been made step by step : starting from [21], by going through [4, 10] up to more recent refinements [23].

(b) For RVM, we have the conservation of mass (that is of $\mathcal{M}_0$)

$$\mathcal{M}_0(t) = \mathcal{M}_0(0) = \|\ \mathrm{f}^0\ \|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} = \int_{\mathbb{R}^3} \rho(0,x)\, dx\,, \qquad \forall\, t \in \mathbb{R}_+\,. \tag{1.18}$$

Moreover, we have $\mathrm{f} \geq 0$ and the conservation of the total energy

$$\mathfrak{E}(t) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \langle\xi\rangle\ \mathrm{f}(t,x,\xi)\ dx\,d\xi + \frac{1}{2}\int_{\mathbb{R}^3} (|\mathrm{E}|^2+|\mathrm{B}|^2)(t,x)\,dx = \mathfrak{E}(0)\,. \tag{1.19}$$

This ensures that $\mathcal{M}_1(t)$ remains bounded if $\mathfrak{E}(0) < +\infty$. And therefore the same applies to all expressions $\mathcal{M}_a(t)$ as soon as $a \leq 1$. However, this issue remains still completely open for $a > 1$. Recall that controls on $\mathcal{M}_a$ (for $a > 1$ sufficiently large) are crucial for a number of purposes. For instance, they can furnish continuation criteria for RVM, see for instance [16, 22].

(c) For $BP^n_{\beta,\ell}$, one of the challenges is precisely to gain bounds on $\mathcal{M}_a$, with $a > 1$. Note that both (1.6) and (1.18) are still verified. By contrast, the expression (1.19) is no more a conserved (or decreasing) quantity. This is due to microscopic exchanges of energy between the fields G_j and the electromagnetic field (E,B). On the one hand, the regularization provides advantages. On the other hand, the loss of (1.19), without a substitute to control $\mathcal{M}_1(t)$, leads to new difficulties.

(2) The second theme is concerned with the local (or global) and strong (or weak) existence of solutions. This includes the problem of global existence of smooth solutions (propagation of the regularity) as outlined below.

(a) For Vlasov-Poisson, it is a well known fact. For example, we refer the reader to [21].

(b) For RVM, this is a longstanding open problem [22] going back to [11].

(c) For $BP^n_{\beta,\ell}$, we will complete this program when $n \geq 2$ (Theorem 2) or when $n = 1$ and $\beta > 1$ (Theorem 3), but only partially in other situations. In particular, when $n = 1$, $\beta = 0$ and $a = 1$, due to the absence of (1.19), even local weak solutions may be difficult to access.

(3) The third direction of research relates to the passage to the limit $\ell \to 0+$. Connections between $BP^n_{\beta,\ell}$ and RVM are established in Section 4. Incidentally, the passage through $BP^n_{\beta,\ell}$ furnishes a new way to construct global weak solutions to the RVM system.

1.2. **Main results.** As explained before, the VMBP model is physically relevant. It looks like the RVM system when dealing with long wavelength regimes, while including microscopic radiation corrections (which could be essential to get accurate dissipation effects). The aim here is to deliver outcomes concerning VMBP related to the three preceding key aspects (1), (2) and (3). In our presentation, a distinction is drawn between the two situations $n \geq 2$ (Theorem 2) and $n = 1$ (Theorem 3). In each case, the questions (1) and (2) are addressed simultaneously. The third issue (3) is discussed separately in Theorem 4.

Let H^s_ℓ with $s \in \mathbb{R}$ be the weighted Sobolev space based on L^2 and $\ell\,\partial_x$ derivatives, that is

$$H^s_\ell \equiv H^s_\ell(\mathbb{R}^3) := \big\{\mathrm{G} \in \mathcal{D}'(\mathbb{R}^3)\,;\, (1+\ell^2\,|\xi|^2)^{s/2}\ \hat{\mathrm{G}}(\xi) \in L^2(\mathbb{R}^3)\big\}\,.$$

As usual, we simply denote by H^s the space H^s_1, and we say that $\mathrm{G} \in H^{s-}_\ell$ when $\mathrm{G} \in H^{s-\delta}_\ell$ for all $\delta > 0$. The field system (1.3) can be endowed with the following energy

$$\mathcal{E}^n_\ell(t) := \sum_{j=1}^{n} \Big(\|\mathrm{G}_j(t,\cdot)\|^2_{H^j_\ell} + \|\partial_t \mathrm{G}_j(t,\cdot)\|^2_{H^{j-1}_\ell}\Big) + \|\mathrm{E}(t,\cdot)\|^2_{H^n_\ell} + \|\mathrm{B}(t,\cdot)\|^2_{H^n_\ell}\,. \tag{1.20}$$

For $n = 0$, we just recover the L^2-energy of Maxwell's equations. For $n = 1$, we find the sum of the energy associated with the Klein-Gordon equation (imposed on G_1) plus the L^2-energies corresponding to Maxwell's equations on E, B, $\partial_{x_i}\mathrm{E}$ and $\partial_{x_i}\mathrm{B}$ with $i \in \{1,2,3\}$. For $n > 1$, the definition follows the same logic, which means implicitly that the equations on G_j are (spatially) derived $j-1$ times, while Maxwell's equations are derived n times. We simply write $\mathcal{E}^n$ when dealing with $\mathcal{E}^n_1$.

Definition 1 (Strong solution). *Fix $n \in \mathbb{N}^*$ and select $T \in \mathbb{R}_+^*$. We say that $\mathrm{U} \in L^\infty([0,T]; L^2)$ is a strong solution to BP^n_β on $[0,T]$ if it satisfies BP^n_β in a weak sense, and if moreover*

$$(\mathrm{E}, \mathrm{B}) \in L^\infty\big([0,T], H^1(\mathbb{R}^3;\mathbb{R}^3)\big)^2 . \tag{1.21}$$

From [6], we know that (1.21) appears as a minimal condition to ensure the existence of a Lagrangian flow with corresponding stability properties. It is a guarantee of uniqueness, and that is why it is selected. Retain that we do not impose that $\mathrm{f}(t,\cdot)$ is in H^s for some $s > 0$ but only for $s = 0$. Thus, care must be taken with the above notion of strong solution. The adjective *strong* does not mean that the whole solution U is « smooth » (but only, up to some extent, its components E and B). It is rather to emphasize a class of solutions for which the *uniqueness* becomes available.

Theorem 2 (Global strong well-posedness of BP^n_β when $n \geq 2$). *Select any $(a,\beta) \in \mathbb{R}_+ \times \mathbb{R}_+$. Assume that the initial condition $(\mathrm{U}^0, \mathrm{H}^0)$ with $\mathrm{U}^0 = (\mathrm{E}^0, \mathrm{B}^0, \mathrm{f}^0)$ is adjusted in such a way that*

$$\mathrm{f}^0 \in (L^\infty \cap L^1)(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}_+)\,, \qquad \mathcal{M}_a(0) < +\infty\,, \qquad \mathcal{E}^n(0) < \infty\,. \tag{1.22}$$

As soon as $(n,a,\beta) \neq (2,0,0)$, there exists a unique global strong solution to BP^n_β satisfying

$$\begin{cases} \mathrm{G}_j \in L^\infty(\mathbb{R}_+, H^{j-1})\,, & \forall\, j \in \{1,\cdots,n\}\,, \\ (\mathrm{E},\mathrm{B}) \in L^\infty(\mathbb{R}_+, H^{n-1})^2\,, \\ \mathrm{f} \in L^\infty\big(\mathbb{R}_+, (L^\infty \cap L^1)(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}_+)\big)\,. \end{cases} \tag{1.23}$$

In addition, this solution enjoys the propagation of its moment, meaning that

$$\mathcal{M}_a(t) < \infty, \qquad \forall\, t \in \mathbb{R}_+\,. \tag{1.24}$$

When $(n,a,\beta) = (2,0,0)$, there is a global weak solution to BP^n_β satisfying $(\mathrm{E},\mathrm{B}) \in \mathcal{C}(\mathbb{R}_+, H^{1-})^2$.

Thus, for $n \geq 2$, resorting to BP^n_β in place of RVM implies a gain of moments : we can reach any value of $a \in \mathbb{R}_+$. Recall that the same type of result does apply for $BP^n_{\beta,\ell}$ for all $\ell \in \mathbb{R}_+^*$, just by a change of scaling. Observe also that, at the level of (1.23) with $n = 2$ and $\beta = 0$, we recover a functional configuration which is very similar to [5], with $\mathrm{E}(t,\cdot)$ and $\mathrm{B}(t,\cdot)$ in H^1. But there are significant improvements. First, we can get rid of the compactness condition on the support of $\mathrm{f}(t,x,\cdot)$. Secondly, the « strong » solutions exist for all times.

Now, for $n = 1$, the situation seems to be less favorable. Define

$$a(\beta) := \frac{9 - 6\,\beta}{3 + 2\,\beta} \leq 3\,, \qquad \tilde{a}(\beta) := \frac{9 - 6\,\beta}{2\,\beta - 1}\,. \tag{1.25}$$

Theorem 3 (Local vs. global or strong vs. weak well-posedness of BP^1_β). *Assume that the initial data $\mathrm{U}^0 := (\mathrm{E}^0, \mathrm{B}^0, \mathrm{f}^0)$ is chosen in such a way that*

$$\mathrm{f}^0 \in (L^\infty \cap L^1)(\mathbb{R}^3 \times \mathbb{R}^3)\,, \qquad \mathcal{M}_a(0) < +\infty\,, \qquad \mathcal{E}^1(0) < \infty\,. \tag{1.26}$$

As soon as $\beta > 0$, there exists a global weak solution to BP^1_β satisfying

$$\begin{cases} \mathrm{G}_1 \in L^\infty(\mathbb{R}_+, H^{\beta-})\,, \\ (\mathrm{E},\mathrm{B}) \in L^\infty(\mathbb{R}_+, H^{\beta-})^2 \\ \mathrm{f} \in L^\infty\big(\mathbb{R}_+, (L^\infty \cap L^1)(\mathbb{R}^3 \times \mathbb{R}^3)\big)\,. \end{cases} \tag{1.27}$$

This weak solution is strong on $\mathbb{R}_+$ as soon as $1 < \beta$. When $0 \leq \beta \leq 1$, it remains locally strong on $[0,T]$ for some $T > 0$ as long as $a \geq 3$. For all $\beta \in [0, 3/2[$ and a with $0 < a(\beta) \leq a \leq 3$, we can find some $T > 0$ (which is the same as above) such that

$$\mathcal{M}_a(t) < \infty, \quad \forall\, t \in [0,T]\,. \tag{1.28}$$

For $3/2 \leq \beta$ and $0 \leq a \leq 3$ or for $1 \leq \beta < 3/2$ and $\tilde{a}(\beta) \leq a \leq 3$, we have (1.28) with $T = +\infty$.

In particular, for $\beta = 0$ and $a = 3$, this furnishes the local strong well-posedness, while global weak well-posedness is not covered. On the contrary, for $\beta = 0$ and $a < 3$, local strong solutions in the sense of Definition 1 may not exist at all. There are still many interesting questions to solve concerning BP^1_β, for completion of the missing information. One of the difficulties is that f^0 is assumed to be just bounded and integrable. Of course, for more regular initial data (say C^s or H^s with s sufficiently large), such information is achievable as a consequence of general results related to nonlinear hyperbolic systems.
We conclude this introduction by giving some informal result (called a Fact) about the third axis (3). More precise statements (see especially Proposition 12) will be made explicit in Section 4.

Fact 4 (Semiclassical limit)**.** *Given $a \in \mathbb{R}_+$, consider a family $\{\mathrm{U}^0_\ell\}_\ell$ of initial data satisfying (1.22) for all $\ell \in \mathbb{R}^*_+$. We work either in a weak-field regime or below semiclassical frequencies, in contexts that are specified respectively in Subsections 4.1 and 4.2. Then, for all $\ell \in \mathbb{R}^*_+$, under the assumptions of Theorems 2 or 3, the Cauchy problem built with $BP^n_{\beta,\ell}$ and U^0_ℓ has a global weak solution U_ℓ. Moreover, when $\ell \to 0+$, we can extract a subsequence converging to a global weak solution to the Vlasov equation (in the first case) and to the relativistic Vlasov-Maxwell system (in the second case).*

The plan of our text is as follows. In Section 2, we start with a list of a priori estimates. The proofs of Theorems 2 and 3 are achieved in Section 3 : In Subsection 3.1, assuming that smooth solutions exist, we concentrate on (1.24) and (1.28) ; In Subsection 3.2, we solve the Cauchy problem in the framework of (1.23) or (1.27). Section 4 is aimed at clarifying passages to the limit $l \to 0+$ as outlined in Fact 4 : In Subsection 4.1, we consider a small amplitude framework ; In Subsection 4.2, we work in the broader context $\mathrm{U}^0_\ell = O(1)$, but the analysis is limited to what happens below semiclassical frequencies.

2. Preliminary work

The goal of this section is to derive several preliminary inequalities which will be used later. In Subsection 2.1, the emphasis is on f. In Subsection 2.2, the focus is on the remaining components G_j, E and B.

2.1. Computations involving the density.

Here, we seek to estimate quantities related to f. To this end, we adapt techniques that can be found in [2, 4, 10, 21]. We start by exploiting the Vlasov equation in order to study the time evolution of $\mathcal{M}_a$.

Lemma 5 (Control of $\mathcal{M}_a$ through E)**.** *For all $a \geq 0$, we have*

$$\partial_t \mathcal{M}_a(t) \lesssim \|\mathrm{E}(t,\cdot)\|_{L^{a+3}} \, \mathcal{M}_a(t)^{\frac{a+2}{a+3}} . \tag{2.1}$$

Proof. From the rescaled version of (1.4), namely BP^n_β, we get

$$\begin{aligned} \partial_t \mathcal{M}_a(t) &= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \langle\xi\rangle^a \, \nabla_{x,\xi} \cdot \big(\mathrm{f}\nu(\xi), \mathrm{f}\,\mathrm{F}\big) \, dxd\xi = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \nabla_\xi \langle\xi\rangle^a \cdot (\mathrm{f}\,\mathrm{F}) \, dxd\xi \\ &= a\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \langle\xi\rangle^{a-2} \, \mathrm{f}\, \xi \cdot \mathrm{F} \, dxd\xi = a\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \langle\xi\rangle^{a-2} \, \mathrm{f}\, \xi \cdot \mathrm{E} \, dxd\xi \\ &\leq a\int_{\mathbb{R}^3} |\mathrm{E}(t,x)| \, \Big(\int_{\mathbb{R}^3} \langle\xi\rangle^{a-1} \, \mathrm{f} \, d\xi\Big) \, dx . \end{aligned} \tag{2.2}$$

On the other hand, for any $R > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} \langle\xi\rangle^{a-1} \, \mathrm{f} \, d\xi &\leq \int_{|\xi|\leq R} \langle\xi\rangle^{a-1} \, \mathrm{f} \, d\xi + \int_{|\xi|\geq R} \langle\xi\rangle^{a-1} \, \mathrm{f} \, d\xi \\ &\lesssim R^{a+2} \, \|\mathrm{f}(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} + \frac{1}{R}\int_{|\xi|\geq R} \langle\xi\rangle^a \, \mathrm{f} \, d\xi . \end{aligned} \tag{2.3}$$

Optimizing in R by choosing $R^{a+3} = \int_{|\xi|\geq R} \langle\xi\rangle^a f \, d\xi / \|\mathrm{f}^0\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}$ gives

$$\int_{\mathbb{R}^3} \langle\xi\rangle^{a-1} \, \mathrm{f} \, d\xi \leq \|\mathrm{f}_0\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}^{\frac{1}{a+3}} \left(\int_{\mathbb{R}^3} \langle\xi\rangle^a \, \mathrm{f} \, d\xi \right)^{\frac{a+2}{a+3}} .$$

Plugg this into (2.2), and use Hölder's inequality to obtain (2.1).

□

It is clear that $\mathcal{M}_b(t) \leq \mathcal{M}_c(t)$ when $b \leq c$. But, we can do better.

Lemma 6 (Comparison of moments)**.** *Let $(b,c) \in \mathbb{R}_+^2$ with $b \leq c$. Knowing (1.18), we find that*

$$\mathcal{M}_b(t) \lesssim \mathcal{M}_c(t)^{\frac{b}{c}} . \tag{2.4}$$

Proof. Observe that

$$\begin{aligned} \mathcal{M}_b(t) &\leq \int_{\mathbb{R}^3}\int_{|\xi|\leq R} \langle\xi\rangle^b \, \mathrm{f}(t,x,\xi) \, d\xi + \frac{1}{R^{c-b}} \int_{\mathbb{R}^3}\int_{|\xi|\geq R} \langle\xi\rangle^c \, \mathrm{f}(t,x,\xi) \, d\xi \\ &\lesssim R^b \, \|\mathrm{f}_0\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} + R^{b-c} \, \mathcal{M}_c(t) . \end{aligned}$$

It suffices to optimize the choice of R through $R^c = \mathcal{M}_c(t)/\|\mathrm{f}_0\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}$.

□

From (1.18), we can easily deduce that

$$\|\jmath(t,\cdot)\|_{L^1(\mathbb{R}^3)} \leq \|\mathrm{f}(t,\cdot)\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} = \|\mathrm{f}^0\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} . \tag{2.5}$$

But L^p-norms of $\jmath$ with $p > 1$ are more problematic to estimate. Unlike (2.5), we do not have bounds that depend only on f^0.

Lemma 7 (Control of the electric current by moments)**.** *For any $a \geq 0$, we have*

$$\|\jmath(t,\cdot)\|_{L^{\frac{3+a}{3}}(\mathbb{R}^3)} \lesssim \mathcal{M}_a(t)^{\frac{3}{a+3}} . \tag{2.6}$$

Moreover, for any $a \geq 3$, knowing (2.5), we can assert that

$$\|\jmath(t,\cdot)\|_{L^2(\mathbb{R}^3)} \lesssim \mathcal{M}_a(t)^{\frac{3}{2a}} . \tag{2.7}$$

Proof. The proof is organized along the same lines as in Lemma 5. For any $R > 0$,

$$\begin{aligned} |\jmath(t,x)| &\leq \int_{|\xi|\leq R} \mathrm{f}(t,x,\xi) \, d\xi + \int_{|\xi|\geq R} \mathrm{f}(t,x,\xi) \, d\xi \\ &\leq R^3 \, \|\mathrm{f}(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} + \frac{1}{R^a} \int_{|\xi|\geq R} \langle\xi\rangle^a \, \mathrm{f}(t,x,\xi) \, d\xi. \end{aligned} \tag{2.8}$$

Choosing $R^{a+3} = \int_{|\xi|\geq R} \langle\xi\rangle^a \mathrm{f} \, d\xi / \|\mathrm{f}^0\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}$ gives

$$|\jmath(t,x)| \leq \|\mathrm{f}^0\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}^{\frac{a}{a+3}} \left(\int_{\mathbb{R}^3} \langle\xi\rangle^a \, \mathrm{f}(t,x,\xi) \, d\xi \right)^{\frac{3}{a+3}} .$$

After integration with respect to x, this pointwise estimate on $\jmath$ raised to the power $(a+3)/3$ furnishes (2.6). In particular, for $a = 3$, this yields

$$\|\jmath(t,\cdot)\|_{L^2(\mathbb{R}^3)} \lesssim \mathcal{M}_3(t)^{\frac{1}{2}} .$$

For $a \geq 3$, applying Lemma 6 with $(b,c) = (3,a)$, we recover (2.7) as desired.

□

2.2. **Computations involving the fields.** We refer to works of Brenner, Ginibre, Velo, Keel and Tao for Lemma 8 below, which is directly extracted from [27].

Lemma 8 (Strichartz estimates for Klein-Gordon's equation)**.** *Suppose that the function* $\mathrm{G}_1(t,x)$ *solves* $(1+\Box)\mathrm{G}_1 = \jmath$ *on* $[0,T]$ *for a given source term* $\jmath$ *and with smooth initial data. Then, the solution* G_1 *must satisfy the following space time integrability properties (modulo constants that do not depend on the time* T*)*

$$\begin{aligned}(2.9)\qquad &\|\mathrm{G}_1(t,x)\|_{L^q_t([0,T],L^r_x(\mathbb{R}^3))} + \|\mathrm{G}_1(t,x)\|_{C([0,T];H^s(\mathbb{R}^3))}\\ &\quad\lesssim \|\mathrm{G}_1(0,\cdot)\|_{H^s(\mathbb{R}^3)} + \|\partial_t\mathrm{G}_1(0,\cdot)\|_{H^{s-1}(\mathbb{R}^3)} + \|\jmath\|_{L^{\tilde q'}_t([0,T],L^{\tilde r'}_x(\mathbb{R}^3))}\,,\end{aligned}$$

where for some $s\geq 0$ *and parameter* θ *with* $0\leq\theta\leq 1$*, the two pairs* (q,r) *and* $(\tilde q,\tilde r)$ *in* $[2,\infty[^2$ *must be admissible*

$$(2.10)\qquad \frac{2}{q}+\frac{2+\theta}{r}\leq\frac{2+\theta}{2}\,,\qquad \Big(q,r,\frac{2+\theta}{2}\Big)\neq(2,\infty,1)$$

and must satisfy the gap condition

$$(2.11)\qquad \frac{1}{q}+\frac{3+\theta}{r}=\frac{3+\theta}{2}-s=\frac{1}{\tilde q'}+\frac{3+\theta}{\tilde r'}-2\,.$$

The control (2.5) on $\jmath$ indicates that the choice $\tilde r'=1$ would be the most convenient. But this cannot be done. As a matter of fact, the right part of (2.11) together with the preliminary restriction $\tilde q'\leq 2$ would imply $(s,\theta,\tilde q')=(0,0,2)$ giving rise to $\big(\tilde q,\tilde r,(\theta+2)/2\big)=(2,\infty,1)$. The last condition inside (2.10) would not be satisfied. Now, the limiting Strichartz estimate for $(\tilde q,\tilde r,\theta)=(2,\infty,0)$ is known to be false. This particularity plays an important role in what follows. It induces some technicalities (in order to circumvent this difficulty) which strongly motivate the introduction of the parameter $\beta>0$ and the passage below through negative Sobolev estimates.

Lemma 9 (Negative Sobolev estimates on G_1)**.** *Select any* $T\in\mathbb{R}_+$ *and any* $\beta\in\mathbb{R}_+$*. Given a source term* $\jmath$ *satisfying (2.5), the solution* G_1 *to*

$$(2.12)\qquad (1+\Box)\mathrm{G}_1=-(1-\Delta)^{-\beta/2}\jmath\,,\qquad \mathrm{G}_1(0,\cdot)\in H^{\beta-}\,,\qquad \partial_t\mathrm{G}_1(0,\cdot)\in H^{-1+\beta-}\,,$$

is subject to $\mathrm{G}_1\in L^\infty([0,T],H^{\beta-})$*. More precisely, for any* δ *with* $0<\delta\leq 1/2$*, we have*

$$(2.13)\qquad \|\mathrm{G}_1\|_{L^\infty([0,T],H^{\beta-\delta})}\lesssim\|\mathrm{G}_1(0,\cdot)\|_{H^{\beta-\delta}}+\|\partial_t\mathrm{G}_1(0,\cdot)\|_{H^{-1+\beta-\delta}}+\|\jmath\|_{L^{\tilde q'}_t([0,T],L^1_x(\mathbb{R}^3))}\,.$$

Proof. Fix δ as indicated. Using Adam's definition of the Sobolev space as well as the Sobolev embedding, we find that

$$(2.14)\qquad (1-\Delta)^{-\delta/2}:L^1(\mathbb{R}^3)\hookrightarrow W^{\delta,1}\hookrightarrow L^{\tilde r'}(\mathbb{R}^3)\,,\qquad \tilde r':=\frac{3}{3-\delta}\in]1,6/5]\,.$$

In other words, for any $t\in[0,T]$

$$\exists\, C_1(\delta)\in\mathbb{R}_+\,;\qquad \|\,(1-\Delta)^{-\delta/2}\jmath(t,\cdot)\,\|_{L^{\tilde r'}}\leq C_1(\delta)\;\|\,\jmath(t,\cdot)\,\|_{L^1}\;.$$

We consider (2.10) and (2.11) with $(q,r)=(\infty,2)$ and $(s,\theta)=(0,0)$. The left part of (2.11) is satisfied ; the right part of (2.11) furnishes $\tilde q'=2\,\tilde r'/(7\,\tilde r'-6)\in[1,2]$. It follows that the conjugate indices $\tilde q$ and $\tilde r$ of $\tilde q'$ and $\tilde r'$ are defined according to

$$\tilde q:=\frac{2}{1-2\delta}\in[2,+\infty[\,,\qquad \tilde r:=\frac{3}{\delta}\in[2,+\infty[\,.$$

As required in (2.10), we have

$$\frac{2}{q}+\frac{2}{r}=1\leq 1\,,\qquad \frac{2}{\tilde q}+\frac{2}{\tilde r}=1-\frac{4\,\delta}{3}\leq 1\,.$$

We apply the Fourier multiplier $\langle D\rangle^{(\beta-\delta)/2}$ to both sides of (2.12) to obtain that

$$(1+\Box)\tilde{\mathrm{G}}_1=-(1-\Delta)^{-\delta/2}\jmath\,,\qquad \tilde{\mathrm{G}}_1:=(1-\Delta)^{(\beta-\delta)/2}\,\mathrm{G}_1\,.$$

From Lemma 8, we can assert that there exists a constant C_2 such that

$$\begin{aligned}\|\tilde{\mathrm{G}}_1\|_{L^\infty([0,T];L^2)} &\leq C_2(\delta)\,\big(\|\tilde{\mathrm{G}}_1(0,\cdot)\|_{L^2}+\|\partial_t\tilde{\mathrm{G}}_1(0,\cdot)\|_{H^{-1}}+\|(1-\Delta)^{-\delta/2}\jmath\|_{L^{\tilde q'}([0,T],L^{\tilde r'}(\mathbb{R}^3))}\big)\\ &\leq C_2(\delta)\,\big(\|\tilde{\mathrm{G}}_1(0,\cdot)\|_{L^2}+\|\partial_t\tilde{\mathrm{G}}_1(0,\cdot)\|_{H^{-1}}+T^{1/\tilde q'}\;C_1(\delta)\;\|\mathrm{f}_0\|_{L^1(\mathbb{R}^3)}\big)\,.\end{aligned}$$

Exploiting (2.5) and coming back to G_1, we easily recover (2.13). It should also be noted that, even if $\mathrm{G}_1(0,\cdot)\in H^\beta$ and $\partial_t\mathrm{G}_1(0,\cdot)\in H^{-1-\beta}$, the bound (2.13) is not uniform with respect to $\delta\in]0,1]$. Indeed, as a consequence of the decay of $(\tilde q,\tilde r)$ towards the forbidden couple $(2,\infty)$, the constant $C_2(\delta)$ should tend to $+\infty$ when δ goes to $0+$.

□

The fields G, E and B are depending on $\jmath$ through (1.3). As is well known, linear hyperbolic systems such as (1.3) are amenable to L^2-estimates. Two strategies are possible to take advantage of such L^2-information :

*S*1**:** Knowing directly that $(1-\ell^2\Delta)^{-\beta/2}\jmath\in L^2$, for instance through a control on $\mathcal{M}_a$, we can take derivatives of (1.3), and then perform L^2-energy estimates giving access to the expression $\mathcal{E}^n_{\beta,\ell}$ which has been introduced at the level of (1.20) ;

*S*2**:** Knowing only that $\jmath\in L^1$, as a corollary of Lemma 9, we can assert that the solution G_1 to the first equation of (1.3) falls in $H^{\beta-}$. Then, by successive energy estimates on the Klein-Gordon equations, we get that $\mathrm{G}_j\in H^{j-1+\beta-}$ (this is the regularizing effect), which implies in particular that $\mathrm{G}_j\in L^2$ for all j such that $j>1-\beta$. When $\beta=0$, we do not have $\mathrm{G}_1\in L^2$, and this component must be excluded. Thus, given $m\in\mathbb{N}$ with $m<n$, we can select the $m+2$ last equations of (1.3). This furnishes a truncated system on G_{n-m+1}, $\cdots$, G_n, E and B with all source terms in L^2, and thereby some L^2-energy estimates become available. With this in mind, we introduce the auxiliary energy

$$\begin{aligned}\mathscr{E}^m(t) := \sum_{j=n-m+1}^{n}\;\sum_{|\gamma|\leq j-n+m-1}\Big(\|\partial_x^\gamma\mathrm{G}_j(t,\cdot)\|^2_{L^2}+\|\nabla_x\partial_x^\gamma\mathrm{G}_j(t,\cdot)\|^2_{L^2}+\|\partial_t\partial_x^\gamma\mathrm{G}_j(t,\cdot)\|^2_{L^2}\Big)\\ +\|\mathrm{E}(t,\cdot)\|^2_{H^m}+\|\mathrm{B}(t,\cdot)\|^2_{H^m}\,.\end{aligned}$$

Observe that $\mathscr{E}^m(t)$ provides a control on G_j in H^{j-n+m} and on $\partial_t\mathrm{G}_j$ in $H^{j-n+m-1}$. Recall also that we can pass from $\ell>0$ to $\ell=1$ through (1.13), and therefore it suffices to investigate the case $\ell=1$.

Lemma 10 (Energy estimate on the fields)**.** *From (1.3) with $\ell=1$, we can deduce that for all $m\in\mathbb{N}$ with $m\leq n$, we have*

$$\mathscr{E}^m(t)\leq e^t\,\Big[\mathscr{E}^m(0)+\sup\,\big\{\,\|\mathrm{G}_{n-m}(s,\cdot)\|^2_{L^2}\,;\,s\in[0,t]\,\big\}\Big]\,. \tag{2.15}$$

Proof. Given a multi-index $\gamma=(\gamma_1,\gamma_2,\gamma_3)\in\mathbb{N}^3$, we differentiate the component G_j with respect to ∂_x^γ with $|\gamma|\leq j-n+m-1$. This furnishes

$$\left\{\begin{array}{ll}(1+\Box)\mathrm{G}_{n-m+1}=\mathrm{G}_{n-m}\,,&\\ \quad\cdots\cdots\cdots\cdots&\\ (1+\Box)\partial_x^\gamma\mathrm{G}_j=\partial_x^\gamma\mathrm{G}_{j-1}\,,&|\gamma|\leq j-n+m-1\,,\\ \quad\cdots\cdots\cdots\cdots&\\ (1+\Box)\partial_x^\gamma\mathrm{G}_n=\partial_x^\gamma\mathrm{G}_{n-1}\,,&|\gamma|\leq m-1\,,\\ \partial_t(\partial_x^\gamma\mathrm{E})-\nabla_x\times\partial_x^\gamma\mathrm{B}=\partial_x^\gamma\mathrm{G}_n\,,&|\gamma|\leq m\,,\\ \partial_t(\partial_x^\gamma\mathrm{B})+\nabla_x\times\partial_x^\gamma\mathrm{E}=0\,,&|\gamma|\leq m\,.\end{array}\right. \tag{2.16}$$

We proceed by standard energy estimates which means to multiply the equations on $\partial_x^\gamma\mathrm{G}_j$, $\partial_x^\gamma\mathrm{E}$ and $\partial_x^\gamma\mathrm{B}$ respectively by $\partial_t(\partial_x^\gamma\mathrm{G}_j)$, $\partial_x^\gamma\mathrm{E}$ and $\partial_x^\gamma\mathrm{B}$. Then, we integrate with respect to $x\in\mathbb{R}^3$ and we perform integration by parts.

This yields

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\mathscr{E}^m(t) \le & \sum_{j=n-m+1}^{n}\ \sum_{|\gamma|\le j-n+m-1} \|\partial_x^\gamma \mathrm{G}_{j-1}(t,\cdot)\|_{L^2}\ \|\partial_t(\partial_x^\gamma \mathrm{G}_j)(t,\cdot)\|_{L^2}\\ & + \sum_{|\gamma|\le m} \|\partial_x^\gamma \mathrm{G}_n(t,\cdot)\|_{L^2}\ \|\partial_x^\gamma \mathrm{E}(t,\cdot)\|_{L^2}\\ \le & \ \frac{1}{2}\,\|\mathrm{G}_{n-m}(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\,\|\partial_t \mathrm{G}_{n-m+1}(t,\cdot)\|_{L^2}^2\\ & + \frac{1}{2}\sum_{j=n-m+2}^{n}\ \sum_{|\gamma|\le j-n+m-1} \big(\|\partial_x^\gamma \mathrm{G}_{j-1}(t,\cdot)\|_{L^2}^2 + \|\partial_t(\partial_x^\gamma \mathrm{G}_j)(t,\cdot)\|_{L^2}^2\big)\\ & + \frac{1}{2}\sum_{|\gamma|\le m} \big(\|\partial_x^\gamma \mathrm{G}_n(t,\cdot)\|_{L^2}^2 + \|\partial_x^\gamma \mathrm{E}(t,\cdot)\|_{L^2}^2\big)\\ \le & \ \frac{1}{2}\,\|\mathrm{G}_{n-m}(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\ \mathscr{E}^m(t)\,.\end{aligned}$$

This implies after integration

$$\mathscr{E}^m(t) \le e^t\ \mathscr{E}^m(0) + \int_0^t e^{t-s}\ \|\mathrm{G}_{n-m}(s,\cdot)\|_{L^2}^2\ ds\,.$$

From there, it is easy to derive (2.15).

□

For $\beta = 0$, by convention, we have $\mathrm{G}_0 = -\jmath$. Recall that $\mathcal{E}^n \equiv \mathcal{E}_1^n$ has been defined in (1.20). In comparison with $\mathcal{E}^n$, some derivatives may be counted twice inside $\mathscr{E}^n$. But $\mathscr{E}^n$ and $\mathcal{E}^n$ are equivalent, with $\mathcal{E}^n \le \mathscr{E}^n \le 2\,\mathcal{E}^n$. Applying (2.15) with $m = n$, we find in particular that

$$\mathcal{E}^n(t) \lesssim e^t\ \Big[\mathcal{E}^n(0) + \sup\ \big\{\ \|(1-\Delta)^{-\beta/2}\jmath(s,\cdot)\|_{L^2}^2\ ;\ s\in[0,t]\ \big\}\Big]\,. \tag{2.17}$$

As explained in [15], the Bopp-Podolsky system BP_0^1 does inherit some field energy density, but this is without evident sign conditions, see more precisely [15]-(47), which seems to make it useless. On the other hand, keeping in mind that we cannot exploit (1.19) when dealing with BP_β^n, we must stick to (2.15).

3. Proof of Theorems

3.1. **Propagation of moments.** In this section, we work with a smooth solution $\mathbf{U}$ of BP_β^n defined on $[0,T]$ for some $T \in \mathbb{R}_+^*$. We assume (1.22) with $n = 2$ and (1.26) when $n = 1$. The function $\mathcal{M}_a(\cdot)$ may not be increasing on $[0,T]$. For this reason, we sometimes replace it by

$$\mathcal{SM}_a(t) := \sup\ \big\{\mathcal{M}_a(s)\,;\ s\in[0,t]\ \big\}\,,\qquad t\in[0,T]\,.$$

In Paragraphs 3.1.1 and 3.1.2, we present two different methods which are based respectively on the strategies $S1$ and $S2$.

3.1.1. *Sobolev estimates on the fields through a control on the moments.* The idea here is to exploit (2.6) in the context of (2.17). From (2.6), we get that $(1-\Delta)^{-\beta/2}\jmath$ is in $W^{\beta,(3+a)/3}$. Then, by Sobolev embedding theorem, we can assert that $a \ge a(\beta)$ with $a(\beta)$ as in (1.25) implies that $(1-\Delta)^{-\beta/2}\jmath \in L^2$. Define $\check{a}(\beta) := \max\big(a(\beta);0\big)$. We first apply (2.6) with $a = \check{a}(\beta)$. Then, we follow $S1$, and we use (2.17) to get

$$\|\ \mathrm{E}(t,\cdot)\ \|_{H^n} \lesssim e^{t/2}\ \Big[\mathcal{E}^n(0)^{1/2} + \sup\ \big\{\ \mathcal{M}_{\check{a}(\beta)}(s)^{\frac{3}{\check{a}(\beta)+3}}\ ;\ s\in[0,t]\ \big\}\Big]\,.$$

The next step is to catch the L^{a+3}-norm of E in order to implement (2.1).

By Sobolev embedding theorem again, we have $H^n \subset L^{a+3}$ as soon as $n \geq 3\,(a+1)/2\,(a+3)$. This is always the case if $n \geq 2$. However, when $n = 1$, we have to work with $\check{a}(\beta) \leq a \leq 3$. Then, from Lemma 6 applied with $b = \check{a}(\beta)$ and $c = a \geq \check{a}(\beta)$, we get

$$\| \mathrm{E}(t,\cdot) \|_{L^{a+3}} \lesssim e^{t/2} \left[\mathcal{E}^n(0)^{1/2} + \sup \left\{ \mathcal{M}_a(s)^{\frac{3}{\check{a}(\beta)+3}\frac{\check{a}(\beta)}{a}} \,;\, s \in [0,t] \right\} \right]. \tag{3.1}$$

We can plug this inside (2.1) and integrate in time to get

$$\mathcal{SM}_a(t) \leq \mathcal{M}_a(0) + \int_0^t e^{s/2} \left[\mathcal{E}^n(0)^{1/2} + \mathcal{SM}_a(s)^{\frac{3}{\check{a}(\beta)+3}\frac{\check{a}(\beta)}{a}} \right] \mathcal{SM}_a(s)^{\frac{a+2}{a+3}}\, ds\,.$$

From there, Bihari-Lasalle inequality furnishes already a local (in time) control on $\mathcal{SM}_a$ (and thereby on $\mathcal{M}_a$) for all $a \geq 0$ if $n \geq 2$, and for $\check{a}(\beta) \leq a \leq 3$ if $n = 1$. Moreover, this control becomes global if the exponent is less than 1, that is

$$\frac{3}{\check{a}(\beta)+3}\,\frac{\check{a}(\beta)}{a} + \frac{a+2}{a+3} \leq 1 \quad \Longleftrightarrow \quad \frac{3\,\check{a}(\beta)}{\check{a}(\beta)+3} \leq \frac{a}{a+3}\,. \tag{3.2}$$

If $\beta \geq 3/2$, we find $\check{a}(\beta) = 0$, and (3.2) is verified. There remains $0 \leq a \leq 3$.

Otherwise, when $0 \leq \beta < 3/2$, we have $0 < \check{a}(\beta) = a(\beta)$, and (3.2) is achievable only on condition that $3\,a(\beta)/\big(\check{a}(\beta)+3\big) < 1$, which implies that $1/2 < \beta$. Then, (3.2) is the same as $\tilde{a}(\beta) \leq a$ with $\tilde{a}(\beta)$ as in (1.25). Combined with the preceding constraint $a \leq 3$, this yields in fact $\tilde{a}(\beta) \leq 3$ or equivalently $\beta \geq 1$ together with $\tilde{a}(\beta) \leq a \leq 3$. When $n = 1$, we recover the conditions which are listed in Theorem 3.

For $n \geq 2$, this argumentation is not optimal. This is because the condition $\mathrm{E} \in H^n$ may not be indispensable. Instead, in line with $S2$, we can try to estimate E just in H^1. In the next paragraph, we seek for such less demanding estimate. When doing this, the challenge is to exploit the regularizing properties of Klein-Gordon equations to convert the a priori L^1- bound on $\jmath$ into H^s-estimates on E.

3.1.2. *Sobolev estimates on the fields through a control on the electric current.* Throughout this paragraph, we assume that $n \geq 2$. From the system BP^n_β, we can extract (2.12). The context is as in Lemma 9. But from (1.22), we only know that $\mathrm{G}^0_1 \in H^1$ and $\mathrm{H}^0_1 \in L^2$. Thus, with $\tilde{\beta} := \min(\beta;1) \in \mathbb{R}_+$, we have to work with

$$(1+\Box)\mathrm{G}_1 = -(1-\Delta)^{-\tilde{\beta}/2}\,\tilde{\jmath}\,, \qquad \tilde{\jmath} := (1-\Delta)^{-(\beta-\tilde{\beta})/2}\,\jmath\,, \qquad \|\tilde{\jmath}(t,\cdot)\|_{L^1(\mathbb{R}^3)} \leq \|\mathrm{f}^0\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}\,,$$

together with $\mathrm{G}^0_1 \in H^{\tilde{\beta}}$ and $\mathrm{H}^0_1 \in H^{-1+\tilde{\beta}}$. From Lemma 9 applied with $\tilde{\beta}$, we can infer that G_1 is in $L^\infty\big([0,T];H^{\tilde{\beta}-}(\mathbb{R}^3)\big)$. More precisely, from (2.13), for all $\delta > 0$, we have

$$\|\mathrm{G}_1\|_{L^\infty([0,T],H^{\tilde{\beta}-\delta})} \lesssim \|\mathrm{G}^0_1\|_{H^1} + \|\mathrm{H}^0_1\|_{L^2} + \|\mathrm{f}^0\|_{L^1(\mathbb{R}^3)} \lesssim 1\,. \tag{3.3}$$

From Lemma 10 with $m = n-1$, we can deduce that (the same holds for B)

$$\|\mathrm{E}(t,\cdot)\|_{H^{n-1+\tilde{\beta}-\delta}(\mathbb{R}^3)} \lesssim k(t) := e^{t/2}\,\big(\mathscr{E}^{n-1}(0)+1\big)^{1/2}\,. \tag{3.4}$$

Thus, resorting to BP^l_n is a way of performing a sort of (hyperbolic) regularization procedure on the electromagnetic field. The increase of n from 1 to 2 (and more generally from n to $n+1$) is associated with a gain of one degree of regularity on (E,B).

For $a \in [0,3[$, we can select δ (small enough) so that $a+3 = 6/(1+2\,\delta)$. Then, by the Sobolev embedding theorem, we find that

$$\|\mathrm{E}(t,\cdot)\|_{L^{a+3}(\mathbb{R}^3)} \leq k(t)\,, \qquad \forall t \in [0,T]\,. \tag{3.5}$$

Coming back to (2.1), this implies that

$$\mathcal{M}_a(t) \leq \Big(\mathcal{M}_a(0)^{\frac{1}{a+3}} + \frac{1}{a+3}\int_0^t k(s)\,ds\Big)^{a+3}\,, \qquad \forall t \in [0,T]\,.$$

which is sufficient to prevent the explosion of $\mathcal{M}_a$. We have (1.24).

For $a \geq 3$ and $\mathcal{M}_a$ bounded, as already seen, Lemma 7 guarantees that $\jmath \in L^2$. Thus, we can directly apply Lemma 10 (with $m = n$) to deduce with (2.7) that

$$\|\mathrm{E}(t,\cdot)\|_{H^2} \leq \|\mathrm{E}(t,\cdot)\|_{H^n} \lesssim 1 + \mathcal{M}_a(t)^{\frac{3}{2\,a}}\,, \qquad \forall\, t \in [0,T]\,.$$

On the other hand, we still have (3.4). Now, the idea is to combine Sobolev embedding theorem

$$\|\mathrm{E}(t,\cdot)\|_{L^{a+3}(\mathbb{R}^3)} \leq \|\mathrm{E}(t,\cdot)\|_{H^{s(a)}(\mathbb{R}^3)}\,, \qquad s(a) := \frac{3}{2}\,\frac{a+1}{a+3}\,,$$

with the interpolation inequality

$$\|\mathrm{E}(t,\cdot)\|_{H^{s(a)}(\mathbb{R}^3)} \leq \|\mathrm{E}(t,\cdot)\|_{H^{1-\delta}}^{1-\sigma}\,\|\mathrm{E}(t,\cdot)\|_{H^2}^{\sigma}\,, \qquad s(a) = (1-\sigma)\,(1-\delta) + 2\,\sigma\,,$$

in order to reduce the exponent of $\mathcal{M}_a$ according to

$$\|\mathrm{E}(t,\cdot)\|_{L^{a+3}(\mathbb{R}^3)} \leq k(t)^{1-\sigma}\,\big(1 + \mathcal{M}_a(t)^{\frac{3}{2\,a}}\big)^{\sigma(a,\delta)}\,, \qquad \sigma(a,\delta) := \frac{1}{1+\delta}\,\Big(\frac{a-3}{2\,(a+3)} + \delta\Big)\,.$$

Contrary to (3.5), this bound implies a loss of a positive power of $\mathcal{M}_a(t)$. But, if we plug this control inside (2.1), we find

$$\partial_t \mathcal{M}_a(t) \lesssim 1 + \mathcal{M}_a(t)^{\frac{3\,\sigma(a,\delta)}{2\,a} + \frac{a+2}{a+3}}\,,$$

while

$$\lim_{\delta \to 0+}\ \frac{3\,\sigma(a,\delta)}{2\,a} + \frac{a+2}{a+3} = \frac{3}{4\,a}\,\frac{a-3}{a+3} + \frac{a+2}{a+3} \leq 1 - \frac{1}{4}\,\frac{1}{a+3} < 1\,.$$

Thus, for δ small enough, we recover a sublinear growth which leads to (1.24).

3.2. **Well-posedness properties.** In this section, we investigate the construction of solutions. System $BP^n_{\beta,\ell}$ is nonlinear due to the presence of quadratic terms in the Vlasov part. As usual in such nonlinear situation, an iterative scheme is needed. Given $\varepsilon \in \mathbb{R}_+^*$, in Paragraph 3.2.1, we seek approximate solutions U_ε. Then, in Paragraph 3.2.2, we pass to the limit ($\varepsilon \to 0+$).

3.2.1. *A regularization procedure.* Given smooth initial conditions, the standard hyperbolic theories furnish local smooth solutions (say H^s with s large enough). The discussion becomes more problematic when dealing in the mild context of (1.22) or concerning the issue of global existence. Then, a preliminary stage is to construct approximate solutions by regularizing the data (initial conditions and source term). Without loss of generality, we can work with $\ell = 1$.

Let $\varphi \in C_c^\infty(\mathbb{R}^3;\mathbb{R})$ be a (spatial) cut-off function : it is such that $\varphi \equiv 1$ on $B(0,1] \subset \mathbb{R}^3$ and $\varphi \equiv 0$ on $B(0,2]^c$. Let $\psi \in C_c^\infty(\mathbb{R};\mathbb{R})$ be a (time) cut-off function : it is such that $\psi \equiv 1$ on the interval $[-1,1]$ and $\psi \equiv 0$ on $[-2,2]^c$. Given $\varepsilon \in \mathbb{R}_+^*$ and $\alpha \in \mathbb{R}_+$, define

$$\begin{aligned} &\varphi_\varepsilon^\alpha(\eta) := \varphi(\varepsilon^\alpha\,\eta)\,, \qquad \check{\varphi}_\varepsilon^\alpha(x) := \mathcal{F}_\eta^{-1}(\varphi_\varepsilon^\alpha)(x) = \varepsilon^{-3\alpha}\,(\mathcal{F}_\eta^{-1}\varphi)(\varepsilon^{-\alpha}\,x)\,,\\ &\psi_\varepsilon^\alpha(\tau) := \psi(\varepsilon^\alpha\,\tau)\,, \qquad \check{\psi}_\varepsilon^\alpha(t) := \mathcal{F}_\tau^{-1}(\psi_\varepsilon^\alpha)(t) = \varepsilon^{-\alpha}\,(\mathcal{F}_\eta^{-1}\psi)(\varepsilon^{-\alpha}\,t)\,. \end{aligned} \tag{3.6}$$

From now on, we denote by $\mathcal{F} \equiv \mathcal{F}_{\tau,\eta}$ the Fourier transform with respect to both variables t and x. We look at φ and ψ as Fourier multipliers. With this in mind, we replace $BP^n_\beta \equiv BP^n_{\beta,1}$ by

$$BP^{n,\alpha}_{\beta,\varepsilon} \qquad \left\{\begin{array}{l} (1+\Box)\mathrm{G}_1 = -\tilde{\jmath} *_x \check{\varphi}_\varepsilon^\alpha *_t \check{\psi}_\varepsilon^\alpha\,, \qquad \tilde{\jmath} := (1-\Delta)^{-\beta/2}\,\jmath \\ \quad \cdots\cdots\cdots\cdots \\ (1+\Box)\mathrm{G}_n = \mathrm{G}_{n-1}\,, \\ \partial_t \mathrm{E} - \nabla_x \times \mathrm{B} = \mathrm{G}_n\,, \\ \partial_t \mathrm{B} + \nabla_x \times \mathrm{E} = 0\,, \\ \partial_t \mathrm{f} + \nu(\xi)\cdot\nabla_x \mathrm{f} + \big(\mathrm{E}(t,x) + \nu(\xi)\times \mathrm{B}(t,x)\big)\cdot\nabla_\xi \mathrm{f} = 0\,, \end{array}\right.$$

and we substitute $C^n_\beta \equiv C^n_{\beta,1}$ for

$$C^{n,\alpha}_{\beta,\varepsilon} \qquad \mathrm{div}\,\mathrm{B} = 0\,, \qquad (1+\Box)^n\,(1-\Delta)^{\beta/2}\,\mathrm{div}\,\mathrm{E} = \rho *_x \check{\varphi}_\varepsilon^\alpha *_t \check{\psi}_\varepsilon^\alpha\,.$$

We complete $BP^{n,\alpha}_{\beta,\varepsilon}$ with regularized versions of $\tilde{\mathrm{U}}(0,\cdot)$, say built with

$$\mathrm{H}^0_{j\varepsilon} := \mathrm{H}^0_j *_x \check{\varphi}^\alpha_\varepsilon\,, \qquad \mathrm{G}^0_{j\varepsilon} := \mathrm{G}^0_j *_x \check{\varphi}^\alpha_\varepsilon\,, \qquad \mathrm{E}^0_\varepsilon := \mathrm{E}^0 *_x \check{\varphi}^\alpha_\varepsilon\,, \qquad \mathrm{B}^0_\varepsilon := \mathrm{B}^0 *_x \check{\varphi}^\alpha_\varepsilon\,.$$

as well as $\mathrm{f}^0_\varepsilon := \mathrm{f}^0 *_{x,\xi} \big(\check{\varphi}^\alpha_\varepsilon(x) \otimes \check{\varphi}^\alpha_\varepsilon(\xi)\big)$. Again, the conditions inside $C^{n,\alpha}_{\beta,\varepsilon}$ are propagated in time, and therefore they can be forgotten.

Given $\tilde{\jmath}$ issued from the Vlasov part, the solution to the field equations is (by superposition principle) the sum of the solution generated by the initial data (which is global and smooth) plus the solution with zero initial data and nonzero source term (which we study now).

Denote by $\mathcal{N}$ the 3D-Green function associated with the Klein-Gordon equation, given for instance in (31)-Lazar. At any time t, the expression $\mathcal{N}(t,\cdot)$ is a compactly supported (with support in the ball of radius $|t|$) measure on $\mathbb{R}^3$ of finite total mass. By construction, we have

$$\mathrm{G}_j = -\mathcal{N}^{*j} *_{t,x} \tilde{\jmath} *_x \check{\varphi}^\alpha_\varepsilon *_t \check{\psi}^\alpha_\varepsilon\,, \qquad \mathcal{N}^{*j} = \underbrace{\mathcal{N} * \cdots * \mathcal{N}}_{j \text{ times}}\,. \tag{3.7}$$

Let $m \in \mathbb{N}$. By Young's convolution inequality, we have

$$\| \mathcal{N}^{*j} *_{t,x} \tilde{\jmath} *_x \check{\varphi}^\alpha_\varepsilon *_t \check{\psi}^\alpha_\varepsilon(t,\cdot) \|_{H^m} \leq \| \check{\varphi}^\alpha_\varepsilon \|_{H^m} \| \check{\psi}^\alpha_\varepsilon \|_{L^1} \| \tilde{\jmath} \|_{L^\infty_t(L^1_x)} \int_0^t \| \mathcal{N}(s,\cdot) \|^j_{\mathcal{M}}\, ds\,.$$

The right hand side depends on $t \in \mathbb{R}_+$ (just as on ε and α). But, as a consequence of (2.5), it is locally bounded on $\mathbb{R}_+$. Through energy estimates, we get

$$\| \partial^k_t \mathrm{G}_j(s) \|_{H^{m-k}} \lesssim_t 1\,, \qquad \| \partial^k_t (\mathrm{E},\mathrm{B})(s) \|_{H^{m-k}} \lesssim_t 1\,, \qquad k = 0, 1 \cdot\cdot n\,, \qquad \forall\, s \in [0,t]\,,$$

and thereby, for m large enough, we can assert that

$$\| \mathrm{G}_j \|_{C^1([0,t]\times\mathbb{R}^3)} \lesssim_t 1\,, \qquad \| (\mathrm{E},\mathrm{B}) \|_{C^1([0,t]\times\mathbb{R}^3)} \lesssim_t 1\,, \qquad \forall\, t \in \mathbb{R}^*_+\,.$$

Let (X,Ξ) be the Vlasov flow. Remark also that

$$\| \Xi(t) \| \leq \| \Xi(0) \| + \int_0^t \| \mathrm{E}(s,\cdot) \|_{L^\infty}\ ds\,.$$

Thus, all the characteristics (which are unique due to the regularity) remain locally (in time) bounded. No explosion can occur, and the criterion for a finite time blow up is not satisfied. Thus, the classical solutions obtained by standard arguments can be extended for all times. This implies that the system $BP^{n,\alpha}_{\beta,\varepsilon}$ has global smooth solutions $\mathrm{U}_\varepsilon \in C^1(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$. Note that the arguments of Section 2 are unaffected by the regularization procedure. We can exploit the preceding estimates on $\mathcal{M}_a$ when dealing with $\mathrm{U}_\varepsilon = (\mathrm{f}_\varepsilon, \mathrm{G}_\varepsilon, \mathrm{E}_\varepsilon, \mathrm{B}_\varepsilon)$.

3.2.2. *Passage to the limit ($\varepsilon \to 0+$).* We fix $\alpha \in \mathbb{R}^*_+$, and we would like to pass to the weak limit $\varepsilon \to 0+$. Since the family $\{\jmath_\varepsilon\}_\varepsilon$ is bounded in $L^\infty_t(L^1_x)$, it is therefore bounded in the space $L^\infty_t(H^s_x)$ for some negative $s < 0$ small enough. By energy estimates, it follows that $\{\mathrm{G}_{1\varepsilon}\}_\varepsilon$ is bounded in $L^\infty_{loc}(H^s_x)$. The same applies to $\{\mathrm{G}_{j\varepsilon}\}_\varepsilon$ for all $1 \leq j \leq n$, and consequently to $\{(\mathrm{E}_\varepsilon, \mathrm{B}_\varepsilon)\}_\varepsilon$. Thus, we can extract from $\{\mathrm{U}_\varepsilon\}_\varepsilon$ a subsequence converging weakly in $L^\infty_{loc}(H^s_x)$ to some $\bar{U}$. Since the field equations are linear, and because $\{\varphi^\alpha_\varepsilon\}_\varepsilon$ and $\{\psi^\alpha_\varepsilon\}_\varepsilon$ are mollifiers, the limit $\bar{U}$ is a solution to the Maxwell's part of BP^n_β. In fact, the difficulty is to pass to the limit in the quadratic terms of the Vlasov equation. To this end, we need a uniform $L^\infty_t(H^s_x)$-bound on $\{(\mathrm{E}_\varepsilon, \mathrm{B}_\varepsilon)\}_\varepsilon$, this time with $s \geq 0$.

i) *Case $n \geq 2$.* We can rely on (3.4) with $n - 1 \geq 1$. For $\beta > 0$, we find that $\tilde{\beta}/2 > 0$. So we can take $\delta = \tilde{\beta}/2$ because this is positive as required. For $n > 2$, just select $\delta = 1/2$ so that $n - 1 - \delta > 1$. In these situations, observe that

$$\|\mathrm{E}_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R}^3)} \lesssim 1\,, \qquad \|\mathrm{B}_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R}^3)} \lesssim 1\,, \qquad \forall\, (\varepsilon,t) \in]0,1] \times \mathbb{R}_+\,. \tag{3.8}$$

For $(\beta,n)=(0,2)$ and $a>0$, we can exploit (1.24) to ensure with (2.6) and Lemma 8 (with $\tilde{r}'>1$) that $\mathrm{G}_{1\varepsilon}(t,\cdot)$ is bounded in L^2, and therefore we still have (3.8). Now, in the limiting case $(a,\beta,n)=(0,0,2)$, given any $\delta>0$, we have to be satisfied with

$$\|\mathrm{E}_\varepsilon(t,\cdot)\|_{H^{1-\delta}(\mathbb{R}^3)}\lesssim 1\,,\qquad \|\mathrm{B}_\varepsilon(t,\cdot)\|_{H^{1-\delta}(\mathbb{R}^3)}\lesssim 1\,,\qquad \forall\,(\varepsilon,t)\in]0,1]\times\mathbb{R}_+\,. \tag{3.9}$$

ii) *Case* $n=1$. For $\beta>0$, we can exploit (2.13). As soon as $0<\breve{\beta}\leq\beta$, this yields

$$\|\mathrm{E}_\varepsilon(t,\cdot)\|_{H^{\breve{\beta}}(\mathbb{R}^3)}\lesssim 1\,,\qquad \|\mathrm{B}_\varepsilon(t,\cdot)\|_{H^{\breve{\beta}}(\mathbb{R}^3)}\lesssim 1\,,\qquad \forall\,(\varepsilon,t)\in]0,1]\times\mathbb{R}_+\,. \tag{3.10}$$

For $\beta>1$, we can even take $\breve{\beta}=1$ to recover (3.8). For $0\leq\beta\leq 1$, assuming (1.26) with $a=3$, with T as in (1.28), from (1.28) together with (2.6) and (2.17), we can still obtain

$$\|\mathrm{E}_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R}^3)}\lesssim 1\,,\qquad \|\mathrm{B}_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R}^3)}\lesssim 1\,,\qquad \forall\,(\varepsilon,t)\in]0,1]\times[0,T]\,. \tag{3.11}$$

The common point of all the cases examined above is to furnish uniform H^s-estimates on $(\mathrm{E}_\varepsilon,\mathrm{B}_\varepsilon)$ for some $s>0$. This is enough to guarantee the L^2-compactness of subsequences extracted from $(\mathrm{E}_\varepsilon,\mathrm{B}_\varepsilon)_\varepsilon$, and then to pass to the limit in the quadratic terms. Note that the usual (more elaborated) argument also prevails. Since uniform L^2-estimates on $(\mathrm{E}_\varepsilon,\mathrm{B}_\varepsilon)$ are available, we have access to averaging lemmas. For all test function $\phi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^3;\mathbb{R})$, the family

$$\rho_\phi^\varepsilon(t,x):=\int_{\mathbb{R}^3}\mathrm{f}_\varepsilon(t,x,\xi)\;\phi(t,\xi)\;d\xi\,,\qquad \varepsilon\in]0,1]$$

is bounded in $H^{1/4}(\mathbb{R}\times\mathbb{R}^3)$.

The notion of weak solutions requires to multiply the equations by test functions which are compactly supported in both time, space and momentum. By Rellich-Kondrachov theorem, which furnishes the compact embedding result $H^{1/4}(\Omega)\hookrightarrow L^2(\Omega)$ on bounded domains Ω, we can extract from $(\rho_\phi^\varepsilon)_\varepsilon$ a subsequence that converges (locally) strongly in L^2. Then, following the main lines of [8, 25], we can pass to the limit in the equations. In short, except when $(\beta,n)=(0,1)$, there exists global weak solutions to BP^n_β.

After passing to the limit, a uniform H^1-estimate on $(\mathrm{E}_\varepsilon,\mathrm{B}_\varepsilon)$ provides with (1.21) on the weak limit $(\bar{\mathrm{E}},\bar{\mathrm{B}})$. The corresponding weak solution $\bar{\mathrm{U}}$ is therefore a strong solution in the sense of Definition 1. In the analysis above of cases i) and ii), we can identify the situations allowing to get such uniform H^1-bound. In doing this, we recognize the criteria which in Theorems 2 and 3 guarantee the existence of strong solutions.

4. Semiclassical limits

The length ℓ is a small positive parameter controlling microscopic effects which, for phenomena observed at a fixed scale, should become invisible when ℓ goes to 0 (hence the qualifying term *semiclassical*). To keep track of the influence of ℓ, we go through the following process :

- We work at the level of $BP^n_{\beta,\ell}$ with fixed values of c, e and m (say with $c=e=m=1$).
- We select some initial condition U^0_ℓ which is subject to (1.22) for some $a\in\mathbb{R}_+$.

Throughout this section, we assume that $n\geq 2$ or that $n=1$ and $\beta>0$. By applying (after rescaling) Theorems 2 or 3, for all $\ell\in\mathbb{R}_+^*$, we recover a global weak solution U_ℓ to $BP^n_{\beta,\ell}$. The question is about what happens when ℓ goes to 0. When doing this, a crucial role is played by the structure (amplitude and oscillations) involved by the family $\{\mathrm{U}^0_\ell\}_\ell$. Two regimes will be investigated, from the easier (implying weak interactions) to the more complicated (involving stronger interactions) :

WFP : *Weak Field Phenomena.* As the name suggests, this is when $\tilde{\mathrm{U}}_\ell(0,\cdot)=\tilde{\mathrm{U}}^0_\ell$ with $\tilde{\mathrm{U}}_\ell$ as in (1.11) is of small size $o(1)$. This corresponds to a small amplitude context. In practice, given some exponent $\gamma>0$, this means that the family $\{\ell^{-\gamma}\,\tilde{\mathrm{U}}^0_\ell\}_\ell$ is bounded in some adequate space. Consequently, we can seek the solution in the form $\ell^\gamma\,\mathrm{U}_\ell$.

LFR : *Low Frequency Regime.* This is when $\mathrm{U}_\ell^0 \equiv \mathrm{U}^0$ is fixed. However, the parameter ℓ appears in the formulation of $BP^n_{\beta,\ell}$. Thus, the solution U_ℓ does depend on ℓ, and it should oscillate strongly with respect to ℓ. To remedy this, we will have to perform a cutoff below frequencies of size ℓ^{-1}.

The question is whether it is possible to extract from $\{\mathrm{U}_\ell\}_\ell$ a subsequence having a weak limit. And, if any, it is to identify the equation satisfied by it. The WFP and LFR situations are studied respectively in Subsections 4.1 and 4.2. We will limit ourselves here to preliminary results. Indeed, strong nonlinear effects are avoided in Subsection 4.1 ; and the frequency cutoff is far below ℓ^{-1} in Subsection 4.2.

4.1. **Weak-field phenomena.** Select some γ with $\gamma > 2$. Given an initial data satisfying

$$\mathrm{G}_j^0 \in H^j\,, \quad \mathrm{H}_j^0 \in H^{j-1}\,, \quad (\mathrm{E}^0, \mathrm{B}^0) \in L^2\,, \quad \mathrm{f}^0 \in L^1 \cap L^\infty\,, \quad \forall j \in \{1, \cdots, n\}\,,$$

we adjust $\tilde{\mathrm{U}}_\ell^0$ in such a way that

$$\tilde{\mathrm{U}}_\ell^0 = (\ell^\gamma\,\mathrm{G}^0, \ell^{\gamma-1}\,\mathrm{H}^0, \ell^\gamma\,\nabla_x \mathrm{G}^0, \ell^\gamma\,\mathrm{E}^0, \ell^\gamma\,\mathrm{B}^0, \ell^\gamma\,\mathrm{f}^0)\,.$$

By this way, for all $\ell \in \mathbb{R}_+^*$, we recover (1.22) with $a = 0$. We assume here that $n \geq 2$ or that $n = 1$ and $\beta > 0$. Then, by applying Theorem 2 or 3, we get a global strong solution to $BP^n_{\beta,\ell}$. We seek this solution in the form $\ell^\gamma\,\mathrm{U}_\ell$ with $\mathrm{U}_\ell = (\mathrm{G}_\ell, \mathrm{E}_\ell, \mathrm{B}_\ell, \mathrm{f}_\ell)$. When there is no ambiguity, we will omit the subscript $\cdot_\ell$. After substitution of $\ell^\gamma\,\mathrm{U}_\ell$ inside the system $BP^n_{\beta,\ell}$, we find that the components of U_ℓ must satisfy

$$WFP^n_{\beta,\ell} \qquad \left\{\begin{array}{ll}
(1+\ell^2\,\Box)\mathrm{G}_1 = -(1-\ell^2\,\Delta)^{-\beta/2}\,\jmath\,, & \mathrm{G}_1(0,\cdot) = \mathrm{G}_1^0\,, \quad \partial_t \mathrm{G}_1(0,\cdot) = \ell^{-1}\,\mathrm{H}_1^0\,, \\
\quad \cdots\cdots\cdots\cdots & \\
(1+\ell^2\,\Box)\mathrm{G}_n = \mathrm{G}_{n-1}\,, & \mathrm{G}_n(0,\cdot) = \mathrm{G}_n^0\,, \quad \partial_t \mathrm{G}_n(0,\cdot) = \ell^{-1}\,\mathrm{H}_n^0\,, \\
\partial_t \mathrm{E} - \nabla_x \times \mathrm{B} = \mathrm{G}_n\,, & \mathrm{E}(0,\cdot) = \mathrm{E}^0\,, \\
\partial_t \mathrm{B} + \nabla_x \times \mathrm{E} = 0\,, & \mathrm{B}(0,\cdot) = \mathrm{B}^0\,, \\
\partial_t \mathrm{f} + \nu(\xi)\cdot\nabla_x \mathrm{f} + \ell^\gamma\ \mathrm{F}\cdot\nabla_\xi \mathrm{f} = 0 & \mathrm{f}(0,\cdot) = \mathrm{f}^0\,,
\end{array}\right.$$

together with (1.16). We impose (1.22) on the above initial data made of $(\mathrm{f}^0, \mathrm{G}^0, \ell^{-1}\,\mathrm{H}^0, \mathrm{E}^0, \mathrm{B}^0)$. The main (important) point of difference with the system $BP^n_{\beta,\ell}$ is the small weight ℓ^γ which is put in factor of the Lorentz force F, and which marks the presence of weak nonlinearities. Introduce

$$\check{\mathrm{G}}(\tau, y) := \mathrm{G}(\ell\,\tau, \ell\,y)\,, \quad \check{\mathrm{G}}^0(y) := \mathrm{G}^0(\ell\,y)\,, \quad \check{\mathrm{H}}^0(y) := \mathrm{H}^0(\ell\,y)\,, \quad \check{\jmath}(\tau, y) := \jmath(\ell\,\tau, \ell\,y)\,.$$

Observe that $\check{\mathrm{G}}$ satisfies a system whose coefficients do not depend on ℓ, namely

$$\left\{\begin{array}{ll}
(1+\Box_{\tau,y})\check{\mathrm{G}}_1 = -(1-\Delta_y)^{-\beta/2}\,\check{\jmath}\,, & \check{\mathrm{G}}_1(0,\cdot) = \check{\mathrm{G}}_1^0\,, \quad \partial_\tau \check{\mathrm{G}}_1(0,\cdot) = \check{\mathrm{H}}_1^0\,, \\
\quad \cdots\cdots\cdots\cdots & \\
(1+\Box_{\tau,y})\check{\mathrm{G}}_n = \check{\mathrm{G}}_{n-1}\,, & \check{\mathrm{G}}_n(0,\cdot) = \check{\mathrm{G}}_n^0\,, \quad \partial_\tau \check{\mathrm{G}}_n(0,\cdot) = \check{\mathrm{H}}_n^0\,.
\end{array}\right.$$

From the proof of Lemma 9, we already know that

$$\| \check{\mathrm{G}}_1 \|_{L^\infty([0,\mathcal{T}];H^{\beta-\delta}_y)} \lesssim \| \check{\mathrm{G}}_1^0 \|_{L^2_y(\mathbb{R}^3)} + \| \check{\mathrm{H}}_1^0 \|_{L^2_y(\mathbb{R}^3)} + \|(1-\Delta_y)^{-\delta/2}\check{\jmath}\|_{L^{\tilde{q}'}([0,\mathcal{T}],L^{\tilde{r}'}(\mathbb{R}^3))}\,. \tag{4.1}$$

That is true for all δ with $0 < \delta \leq 1/2$, for all $\mathcal{T} > 0$, and for all $\tilde{r}' > 1$ with $\tilde{q}' = 2\,\tilde{r}'/(7\,\tilde{r}' - 6)$. When $n = 1$ and $\beta > 0$, by choosing $\beta = \delta$, we recover some L^2-estimate on $\check{\mathrm{G}}_1$ which can be converted in terms of G_1 according to

$$\begin{aligned}
\| \mathrm{G}_1 \|_{L^\infty([0,T];L^2_x)} &= \| \check{\mathrm{G}}_1(\cdot/\ell, \cdot/\ell) \|_{L^\infty([0,T];L^2_x)} = \ell^{3/2}\ \| \check{\mathrm{G}}_1 \|_{L^\infty([0,T/\ell];L^2_y)} \\
&\lesssim \ell^{3/2}\ \| \check{\mathrm{G}}_1^0(0,\cdot) \|_{L^2_y(\mathbb{R}^3)} + \ell^{3/2}\ \| \check{\mathrm{H}}_1^0(0,\cdot) \|_{L^2_y(\mathbb{R}^3)} \\
&\quad + \ell^{3/2}\ \|(1-\Delta_y)^{-\delta/2}\check{\jmath}\|_{L^{\tilde{q}'}([0,T/\ell],L^{\tilde{r}'}(\mathbb{R}^3))}\,.
\end{aligned}$$

In view of (2.14), it follows that

$$\| \mathrm{G}_1 \|_{L^\infty([0,T];L^2_x)} \lesssim \| \mathrm{G}_1^0 \|_{L^2_x(\mathbb{R}^3)} + \| \mathrm{H}_1^0 \|_{L^2_x(\mathbb{R}^3)} + \ell^{3/2}\ \|\check{\jmath}\|_{L^{\tilde{q}'}([0,T/\ell],L^1(\mathbb{R}^3))}\,.$$

Compute

$$\|\check\jmath\|_{L^{\tilde q'}([0,T/\ell],L^1(\mathbb{R}^3))} = \left[\int_0^{T/\ell}\Big(\int_{\mathbb{R}^3}|\check\jmath(\tau,y)|\,\ell^{-3}\,d(\ell y)\Big)^{\tilde q'}\ell^{-1}\,d(\ell\tau)\right]^{1/\tilde q'}$$
$$= \ell^{-3-1/\tilde q'}\left[\int_0^{T}\Big(\int_{\mathbb{R}^3}|\jmath(t,x)|\,dx\Big)^{\tilde q'}\,dt\right]^{1/\tilde q'} = \ell^{-3-1/\tilde q'}\ \|\jmath\|_{L^{\tilde q'}([0,T],L^1(\mathbb{R}^3))}.$$

We can adjust $\tilde r'\in]1,+\infty[$ small enough to ensure that $2<3/2+1/\tilde q'<\gamma$. This implies that

$$\|\,\ell^\gamma\,\mathrm{G}_1(t,\cdot)\,\|_{L^\infty([0,T];L^2_x)}\lesssim l^\gamma + l^{\gamma-3/2-1/\tilde q'}\ \|\jmath\|_{L^{\tilde q'}([0,T],L^1(\mathbb{R}^3))}\le 1+T^{1/\tilde q'}\ \|\mathrm{f}_0\|_{L^1(\mathbb{R}^3)}\lesssim 1\,,$$

with a bound on the right hand side which is uniform with respect to ℓ.

When $n\ge 2$, we start from (4.1) with $\beta=0$ and $\delta=1/2$. Then, by energy estimates on the Klein-Gordon equations, we deduce that

$$\|\,\check{\mathrm{G}}_n\,\|_{L^\infty([0,\mathcal{T}];H^{n-3/2})}\lesssim\|\,\check{\mathrm{G}}^0\,\|_{L^2_y(\mathbb{R}^3)}+\|\,\check{\mathrm{H}}^0\,\|_{L^2_y(\mathbb{R}^3)}+\|(1-\Delta)^{-\delta/2}\check\jmath\|_{L^{\tilde q'}([0,\tau],L^{\tilde r'}(\mathbb{R}^3))}\,.$$

Following the same methods as above to come back to the initial variables, we find that

$$\|\,\ell^\gamma\,\mathrm{G}_n(t,\cdot)\,\|_{L^\infty([0,T];L^2)}\le\|\,\ell^\gamma\,\mathrm{G}_n(t,\cdot)\,\|_{L^\infty([0,T];H^{n-3/2}_\ell)}\lesssim 1\,.$$

By energy estimates on the Maxwell's part, this gives rise to some L^2-bound on $\{\ell^\gamma\,\mathrm{F}_\ell\}_\ell$. Thus, we can extract a subsequence converging weakly in L^2. Moreover, passing to the limit in the $n+2$ first equations multiplied by ℓ^2, we find that the weak limit must be 0. In parallel, we can apply averaging lemmas to pass to the limit (as in [8, 25]) in the Vlasov equation. Let $\bar{\mathrm{f}}$ be a weak L^2-limit of $\{\mathrm{f}_\ell\}_\ell$. By this way, we just recover the transport equation

$$\partial_t\bar{\mathrm{f}}+\nu(\xi)\cdot\nabla_x\bar{\mathrm{f}}=0\,,\qquad\bar{\mathrm{f}}(0,\cdot)=\mathrm{f}^0\,.$$

It is not expected that $\{\mathrm{G}_\ell\}_\ell$ has a weak limit. Presumably, the fiels G_ℓ is of amplitude ℓ^{-1}. An additional loss of ℓ^{-1} seems to be required above (due to scaling considerations) to pass from the L^1-control on $\jmath$ to some uniform L^2-estimate on G_n. The question of whether there is still a weak limit when $\gamma\le 2$, for instance when $\gamma=1$ (or less), is open.

4.2. **Low-frequency limit.** The initial condition U^0 is here $O(1)$. To progress in this large amplitude framework, we need to perform a cutoff of the system $BP^n_{\beta,\ell}$ below semi-classical frequencies of size l^{-1}. Without loss of generality (just to simplify the presentation), we can work with $n=1$ and $\beta=0$. The purpose is thus to investigate the passage from $BP^1_{0,\ell}$ to RVM when ℓ goes to zero. Denoting simply $G_1\equiv G$, we deal with

$$BP^1_{0,\ell}\qquad\begin{cases}(1+\ell^2\,\Box)\mathrm{G}=-\jmath\,,\\ \partial_t E-\nabla_x\times\mathrm{B}=\mathrm{G}\,,\\ \partial_t\mathrm{B}+\nabla_x\times\mathrm{E}=0\,,\\ \partial_t\mathrm{f}+\nu(\xi)\cdot\nabla_x\mathrm{f}+\mathrm{F}(t,x)\cdot\nabla_\xi\mathrm{f}=0\,.\end{cases}$$

Remark 11 (Energy estimates for $BP^1_{0,\ell}$ copied from RVM)**.** *The RVM system is endowed with the conserved quantity $\mathfrak{E}(t)$ which has been defined at the level of (1.19). To mimic what is done on RVM, we can multiply the Mawell's part of $BP^1_{0,\ell}$ by $2\,{}^t(\mathrm{E},\mathrm{B})$ and the Vlasov equation by $2\,\langle\xi\rangle$. Then, we integrate with respect to (x,ξ) to obtain*

$$\partial_t\mathfrak{E}(t)=2\int_{\mathbb{R}^3}\mathrm{E}(t,x)\cdot(\mathrm{G}+\jmath)(t,x)\,dx\,.$$

What makes the difference with RVM is that $\mathrm{G}+\jmath=-\ell^2\,\Box\mathrm{G}\not\equiv 0$. And it is not clear if the right hand side is bounded (or vanishing when $\ell\to 0+$).

Still, we can try to compare G with $-\jmath$. With this in mind, we perform a Fourier transform (with respect to t and x) in the first equation to see that

$$(1-\ell^2\,\tau^2+\ell^2\,|\eta|^2)\,\mathcal{F}\mathrm{G}(\tau,\eta)=-\mathcal{F}\jmath(\tau,\eta)\,,\qquad \mathcal{F}\equiv\mathcal{F}_{t,x}\,. \tag{4.2}$$

As long as $|\tau|+|\eta|=o(l^{-1})$, say $|\tau|+|\eta|\leq l^{-\alpha}$ with $\alpha\in[0,1[$, we can assert that $\mathcal{F}\mathrm{G}\sim-\mathcal{F}\jmath$, and therefore $\mathrm{G}\sim-\jmath$. Thus, it can be expected that a cutoff of $BP^1_{0,\ell}$ below semi-classical frequencies of size $|\tau|+|\eta|\sim l^{-1}$ could allow to recover the RVM system. Our aim in this subsection is to give concrete form to this idea.

To this end, we implement the mollifiers φ^α_ℓ and ψ^α_ℓ which have been introduced in (3.6). Let us also introduce a function $\chi\in C^\infty_c(\mathbb{R};[0,1])$ which is such that $\chi\equiv1$ on $[0,T]$ and $\chi\equiv0$ out of the interval $[-T,2T]$. A way (among others) to perform such low-frequency cutoff is to look at the solution $\mathrm{U}_\ell=(\mathrm{G}_\ell,\mathrm{E}_\ell,\mathrm{B}_\ell,\mathrm{f}_\ell)$ to the system

$$BP^{1,\alpha}_{0,\ell}\qquad\left\{\begin{array}{l}(1+\ell^2\,\Box)\mathrm{G}=-\chi(t)\;\jmath\,,\\ \partial_t E-\nabla_x\times\mathrm{B}=\mathrm{G}*_t\check{\psi}^\alpha_\ell*_x\check{\varphi}^\alpha_\ell\,,\\ \partial_t\mathrm{B}+\nabla_x\times\mathrm{E}=0\\ \partial_t\mathrm{f}+\nu(\xi)\cdot\nabla_x\mathrm{f}+\chi(t)\;\mathrm{F}*_x\check{\varphi}^\alpha_\ell\cdot\nabla_\xi(\mathrm{f}*_t\check{\psi}^\alpha_\ell)=0\,.\end{array}\right.$$

As long as $t\in[0,T]$, the time cutoff of $\jmath$ through the multiplication by χ has no impact (because the Klein-Gordon and Vlasov equations are hyperbolic evolution equations). On the other hand, the convolution of G and F by $\check{\varphi}^\alpha_\ell$ removes the influence of large (above $\ell^{-\alpha}$) spatial frequencies. The same applies (through the application of ψ^α_ℓ) to time frequencies (with resulting nonlocal time effects).

Proposition 12. *For all parameter $\alpha\in[0,4/9]$, the solutions to $BP^{1,\alpha}_{0,\ell}$ converge (modulo the extraction of a subsequence) weakly (when the parameter $l>0$ goes to 0) to a weak solution of the RVM system.*

Proof. Again, we multiply the Vlasov equation by $2\,\langle\xi\rangle$ and the Mawell's part by $2\,{}^t(\mathrm{E},\mathrm{B})$. After integration with respect to (x,ξ), there remains

$$\partial_t\mathfrak{E}(t)=2\int_{\mathbb{R}^3}\mathrm{E}*_x\check{\varphi}^\alpha_\ell\cdot\jmath*_t\check{\psi}^\alpha_\ell\;dx+2\int_{\mathbb{R}^3}\mathrm{G}*_t\check{\psi}^\alpha_\ell*_x\check{\varphi}^\alpha_\ell\cdot\mathrm{E}\;dx\,.$$

By Plancherel theorem, denoting as before by $\mathcal{F}_\eta$ the Fourier transform with respect to x, this is the same as

$$\begin{aligned}\partial_t\mathfrak{E}(t)\;&=2\int_{\mathbb{R}^3}\overline{\mathcal{F}_\eta\mathrm{E}}(t,\eta)\cdot\big[\mathcal{F}_\eta(\chi\,\jmath+\mathrm{G})(\cdot,\eta)*_t\check{\psi}^\alpha_\ell(\cdot)\big](t,\eta)\;\varphi^\alpha_\ell(\eta)\;d\eta\\&=2\int_{\mathbb{R}^3}\int_{\mathbb{R}}e^{-i\tau t}\;\overline{\mathcal{F}_\eta\mathrm{E}}(t,\eta)\cdot\mathcal{F}(\chi\,\jmath+G)(\tau,\eta)\;\psi^\alpha_\ell(\tau)\;\varphi^\alpha_\ell(\eta)\;d\tau\,d\eta\,.\end{aligned}$$

The advantage of working with $\chi\,\jmath$ instead directly with $\jmath$ is to make possible a Fourier transform with respect to t. We find that

$$\mathcal{F}(\chi\,\jmath+G)(\tau,\eta)=\frac{-\ell^2\,\tau^2+\ell^2\,|\eta|^2}{1-\ell^2\,\tau^2+\ell^2\,|\eta|^2}\;\mathcal{F}(\chi\,\jmath)(\tau,\eta)\,,\qquad\mathcal{F}=\mathcal{F}_{\tau,\eta}\,.$$

After substitution, changing (τ,η) into $(\tilde\tau,\tilde\eta):=(\ell^\alpha\tau,\ell^\alpha\eta)$, this means that

$$\partial_t\mathfrak{E}(t)=2\int_{\mathbb{R}^3}\int_{\mathbb{R}}e^{-i\ell^{-\alpha}\tilde\tau\,t}\;\frac{\ell^{2-6\alpha}\,(-\tilde\tau^2+|\tilde\eta|^2)}{1+\ell^{2-2\alpha}\,(-\tilde\tau^2+|\tilde\eta|^2)}\;\overline{\mathcal{F}_\eta\mathrm{E}}(t,\ell^{-\alpha}\tilde\eta)\cdot\mathcal{F}(\chi\,\jmath)(\ell^{-\alpha}\tilde\tau,\ell^{-\alpha}\tilde\eta)\;\psi(\tilde\tau)\;\varphi(\tilde\eta)\;d\tilde\tau\,d\tilde\eta\,.$$

Observe that

$$|\mathcal{F}(\chi\,\jmath)(\tau,\eta)|=|\int_{\mathbb{R}}e^{-i\tau t}\;\chi(t)\,\mathcal{F}_\eta\jmath(t,\eta)\;dt|\leq\int_{-T}^{2T}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}f(t,x,\xi)\;dt\,dx\,d\xi\leq3\,T\;\parallel f_0\parallel_{L^1}\;.$$

For $0 < \ell \le \ell_0$ with ℓ_0 small enough (say $4\,\ell_0^{1-\alpha} \le 1$), we have

$$\begin{aligned}
|\partial_t \mathfrak{E}(t)| &\le 48\ \ell^{2-6\alpha}\ T\ \|\ f_0\ \|_{L^1}\ \|\ \psi\ \|_{L^1}\ \int_{\mathbb{R}^3} \overline{\mathcal{F}_\eta \mathrm{E}}(t, \ell^{-\alpha}\tilde{\eta})|\ \varphi(\tilde{\eta})\ d\tilde{\eta} \\
&\lesssim \ell^{2-3\alpha}\ \int_{\mathbb{R}^3} |\overline{\mathcal{F}_\eta \mathrm{E}}(t,\eta)|\ \varphi(\ell^\alpha\,\eta)\ d\eta \\
&\lesssim \ell^{2-3\alpha}\ \|\ E(t,\cdot)\ \|_{L^2}\ \Big(\int_{\mathbb{R}^3} \varphi(\ell^\alpha\,\eta)^2\ d\eta\Big)^{1/2} \\
&\lesssim \ell^{2-(9\alpha/2)}\ \|\ E(t,\cdot)\ \|_{L^2} \lesssim 1 + \mathfrak{E}(t)\,.
\end{aligned}$$

It follows that

$$\mathfrak{E}(t) \lesssim (1+\mathfrak{E}_0)\ e^t\,, \qquad \forall\, t \in [0,T]\,, \qquad \mathfrak{E}_0 := \mathfrak{E}(0)\,. \tag{4.3}$$

As before, working in H^s with $s < 0$ small enough, it is easy to show that the family $\{\mathrm{U}_\ell\}_\ell$ has a subsequence converging weakly to some $\bar{\mathrm{U}} = (\bar{\mathrm{G}}, \bar{\mathrm{E}}, \bar{\mathrm{B}}, \bar{\mathrm{f}})$, which must be such that

$$\begin{cases} \partial_t \bar{\mathrm{E}} - \nabla_x \times \bar{\mathrm{B}} = \bar{\mathrm{G}} = -\bar{\jmath} = -\displaystyle\int_{\mathbb{R}^3} \bar{\mathrm{f}}(t,x,\xi)\ d\xi\,, \\ \partial_t \bar{\mathrm{B}} + \nabla_x \times \bar{\mathrm{E}} = 0\,. \end{cases}$$

On the other hand, by following [8, 25], the uniform estimate on $\{(\mathrm{E}_\ell, \mathrm{B}_\ell, \mathrm{f}_\ell)\}_\ell$ deduced from (4.3) is enough (by averaging lemma) to pass to the weak limit (when ℓ goes to zero) in the quadratic terms of the Vlasov equation. Thus, at the limit, we do recover that $\bar{\mathrm{U}}$ is a weak solution to the RVM system.

□

A few remarks are in order.

Remark 13 (Other types of cutoffs)**.** *The main difficulties in the preceding proof come from the zeros of the function* $(\tau,\eta) \to 1 - \ell^2\,\tau^2 + \ell^2\,|\eta|^2$*. The choice of* $BP^{1,\alpha}_{0,\ell}$ *is only illustrative. It is certainly not optimal. There are other ways to perform the cutoff. For instance, we could refine the discussion by cutting in a neighborhood of the cone* $|\tau| = |\eta|$ *for* $|\tau| \sim |\eta| \sim l^{-1}$*.*

Remark 14 (Large-frequency discrepancies)**.** *At microscopic scales, the Bopp-Podolsky model takes over with a (presumably) more realistic description than RVM. Thus, it would be interesting to compare the behaviors of the solutions to* $BP^1_{0,\ell}$ *and RVM at frequencies of size* ℓ^{-1}*, knowing that they could strongly differ.*

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