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Classifying bulk-edge anomalies in the Dirac Hamiltonian

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Abstract

We study the Dirac Hamiltonian in dimension two with a mass term and a large momentum regularization, and show that bulk-edge correspondence fails. Despite a well defined bulk topological index –the Chern number–, the number of edge modes depends on the boundary condition. The origin of this anomaly is rooted in the unbounded nature of the spectrum. It is detected with Levinson’s theorem from scattering theory and quantified via an anomalous winding number at infinite energy, dubbed ghost charge. First we classify, up to equivalence, all self-adjoint boundary conditions, using Schubert cell decomposition of a Grassmanian. Then, we investigate which ones are anomalous. We expand the scattering amplitude near infinite energy, for which a dominant scale captures the asymptotic winding number. Remarkably, this can be achieved for every self-adjoint boundary condition, leading to an exhaustive anomaly classification. It shows that anomalies are ubiquitous and stable. Boundary condition with a ghost charge of 2 is also revealed within the process.

1 Introduction

Bulk-edge correspondence is a profound result of topological insulators. It has been used as a paradigm for more than 40 years to understand the exotic properties of topological materials. The bulk index, defined for an infinite sample without boundary, predicts the number of states appearing at the edge of a half-infinite sample with a sharp boundary. Such edge states are confined near the boundary and have remarkable properties, like robust unidirectional propagation in two dimensions. Their number is counted by the edge index, which equals the bulk index via the bulk-edge correspondence.

Such a result was established with various techniques from spectral analysis to K-theory and for a very broad class of models, originally in the context of the quantum Hall effect [16, 24] and then extended to various condensed matter systems [2, 5, 33, 19, 8, 21, 1, 15, 29, 26, 6, 14, 17, 10, 9, 4] and actually to almost any energy-preserving classical wave phenomenon [31, 11, 28, 13]. Yet, it has its limitation: Bulk-edge correspondence has been shown to fail in the context of shallow-water waves [35, 20]. There, the system has a well-defined bulk index and yet the number of edge modes depends on the choice of boundary condition. The origin of the anomaly was identified in the unbounded nature of the operator spectrum. Using scattering theory and a relative version of Levinson’s theorem, a winding number captures the anomalous asymptotic properties of the

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spectrum and compensates for the lack or excess of edge modes in the finite spectrum. Such a winding number was dubbed “ghost” charge in [35], with possible physical consequences.

The purpose of this paper is to show that this anomalous situation is not isolated. First, it is not restricted to water waves. As noticed in [35], we show that it also appears in the celebrated Dirac Hamiltonian, a canonical model to describe graphene near Fermi energy [7]. Its massive version is used as an effective theory for various topological materials, like the Haldane or Kane-Mele model [23, 25]. However, because its bulk index is ill-defined, we consider instead its regularized version which compactifies the problem at large momenta. The regularized and massive Dirac Hamiltonian appears for example in [38] in the context of superfluid Helium. Second, and more importantly, we do not study a specific anomalous boundary condition as in [20] but classify them all.

The first result of this paper is to classify all self-adjoint boundary conditions for the massive and regularized Dirac Hamiltonian. We identify them with stable subspaces of 2×4 matrices of maximal rank. Up to equivalence, we get seven classes based on the Schubert cell decomposition of the Grassmannian $\text{Gr}(2, 4)$. The main result of this paper is then to compute the asymptotic winding number, denoted $w_\infty \in \mathbb{Z}$ below, for any boundary condition among these classes. Remarkably, the classification can be done exhaustively. One consequence is that anomalies are actually ubiquitous and very stable: this is not a fine-tuning effect. Nevertheless, many boundary conditions (e.g. Dirichlet) are also anomaly-free and stable as well.

To do so, we develop a formalism to extract precisely the dominant scale of the scattering amplitude in the asymptotic part of the spectrum, in order to compute the winding number w_∞ . Then we manage to apply this strategy to any boundary condition, class by class. In some cases the expressions contain up to 60 terms, but using a symbolic computation software –Mathematica– our approach reduces them to small and tractable complex curves. Surprisingly, among the many parameters in each class, only one or two drive the anomaly, so that w_∞ can be always computed exactly and moreover the whole classification can be summarized in a rather compact way. Finally, the classification also reveals a family of anomalous boundary conditions where the ghost charge is 2 instead of 1, which was unexpected and never noticed before.

The general message of this paper is that one should always consider with extra care apparent topological edge modes of unbounded operators, as they might be anomalous. However, there are several ways to circumvent anomalous boundary conditions. Considering a smooth interface or domain wall instead of a sharp boundary usually removes the anomaly and restores the bulk-edge correspondence [34, 32]. Moreover, there are situations where an interface index can be properly defined even though there is no bulk index [13, 12, 3, 30]. Furthermore it is also possible to avoid the asymptotic part of the spectrum even with a sharp boundary condition, for example via a spectral flow of boundary conditions [36] or using an external scalar field [27]. Finally, we also mention a related anomalous problem for the magnetic Dirac Hamiltonian in graphene [37], in a slightly different context.

The rest of the paper is organized as follows: In Section 2 we describe the model, explain the scattering theory to define w_∞ and present the main results: Theorem 2 and 11, associated with Table 1 and 2, respectively. Section 3 proves the boundary condition classification. Section 4 develops the main tools to expand the scattering amplitude near infinity, and Section 5 applies these tools to classify all the anomalies.

2 Setting and main results

2.1 Dirac Hamiltonian and its topological index

The massive and regularized Dirac Hamiltonian is a self-adjoint operator on $L^2(\mathbb{R}^2)$ with domain $H^1(\mathbb{R}^2) \oplus H^1(\mathbb{R}^2)$ given by

$$\mathcal{H} = \begin{pmatrix} m + \epsilon(\partial_x^2 + \partial_y^2) & i\partial_x + \partial_y \\ i\partial_x - \partial_y & -m - \epsilon(\partial_x^2 + \partial_y^2) \end{pmatrix}. \quad (2.1)$$

In the following we assume $m > 0$, $\epsilon > 0$ and $\epsilon < 1/2m$. The mass parameter m opens a gap in the spectrum, see (2.4) below, and the term $\epsilon(\partial_x^2 + \partial_y^2)$ regularizes the problem at high energy. It also has a physical interpretation in the context of superfluid Helium [38]. By translation invariance in space, the stationary solutions to the Schrödinger equation $i\partial_t\psi = \mathcal{H}\psi$ are given by normal modes

$$\psi := \widehat{\psi}(k_x, k_y, \omega)e^{i(k_x x + k_y y - \omega t)}. \quad (2.2)$$

They have momentum $\mathbf{k} = (k_x, k_y) \in \mathbb{R}^2$, energy $\omega \in \mathbb{R}$ and correspond to the eigenvalue problem:

$$H\widehat{\psi} = \omega\widehat{\psi}, \quad \widehat{\psi} = \begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \end{pmatrix}, \quad H(\mathbf{k}) = \begin{pmatrix} m - \epsilon\mathbf{k}^2 & -k_x + ik_y \\ -k_x - ik_y & -m + \epsilon\mathbf{k}^2 \end{pmatrix}, \quad (2.3)$$

with $\mathbf{k}^2 = k_x^2 + k_y^2$ and $H(\mathbf{k})$ a Hermitian matrix. The system admits two energy bands

$$\omega_{\pm} = \pm\sqrt{\mathbf{k}^2 + (m - \epsilon\mathbf{k}^2)^2} \quad (2.4)$$

separated by a gap of size m . We denote by $P_{\pm}(\mathbf{k})$ the corresponding eigenprojections. Since $\epsilon \neq 0$, these projections are single-valued at $\mathbf{k} \rightarrow \infty$, which allows to consider the associated fibre bundle over the compactified plane $\mathbb{R}^2 \cup \{\infty\} \cong S^2$. See [20] or Appendix A for more details. Consequently, each projection P has an associated fibre bundle index –the Chern number– given by

$$C(P) = \frac{1}{2\pi i} \int_{S^2} dk_x dk_y \operatorname{tr} (P [\partial_{k_x} P, \partial_{k_y} P]). \quad (2.5)$$

A short computation shows that

$$C_{\pm} := C(P_{\pm}) = \pm \frac{\operatorname{sign} m + \operatorname{sign} \epsilon}{2} = \pm 1.$$

Thus, each energy band has an associated non-trivial bulk topological index.

2.2 Self-adjoint boundary conditions

We restrict the spatial domain to $\{(x, y) \mid y \geq 0\} \subset \mathbb{R}^2$ and study the restriction of \mathcal{H} to it, named \mathcal{H}^{\sharp} . This domain preserves translation invariance in x -direction, so we focus on normal modes of the form $\psi := \widetilde{\psi}(y; k_x, \omega)e^{i(k_x x - \omega t)}$. The Schrödinger equation $i\partial_t\psi = \mathcal{H}^{\sharp}\psi$ is reduced to the study of a one-parameter family of operators on $L^2(\mathbb{R}^+)$:

$$H^{\sharp}\widetilde{\psi} = \omega\widetilde{\psi}, \quad H^{\sharp}(k_x) = \begin{pmatrix} m - \epsilon k_x^2 + \epsilon\partial_y^2 & -k_x + \partial_y \\ -k_x - \partial_y & -m + \epsilon k_x^2 - \epsilon\partial_y \end{pmatrix}, \quad (2.6)$$

for $k_x, \omega \in \mathbb{R}$. We consider any local boundary conditions for this problem:

$$(B_0 + ik_x B_1)\widetilde{\psi} + B_2\widetilde{\psi}' \Big|_{y=0} = 0 \quad (2.7)$$

where $B_0, B_1, B_2 \in M_2(\mathbb{C})$ are k_x and y -independent and

$$\tilde{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \tilde{\psi}' = \begin{pmatrix} \partial_y \psi_1 \\ \partial_y \psi_2 \end{pmatrix}. \quad (2.8)$$

For example, Dirichlet's boundary condition corresponds to $B_0 = I_2$ and $B_1 = B_2 = 0$. However, not all matrices in (2.7) define a self-adjoint operator associated to (2.6). The first result of this paper is to classify all local self-adjoint boundary conditions. It appears convenient to regroup the B_i matrices in 2×4 -matrices $A, A_0, A_1 \in M_{2,4}(\mathbb{C})$:

$$A := A_0 + ik_x A_1, \quad A_0 := [B_0|B_2], \quad A_1 := [B_1|0], \quad (2.9)$$

so that (2.7) is equivalent to

$$A\Psi|_{y=0} = 0, \quad \Psi = \begin{pmatrix} \tilde{\psi} \\ \tilde{\psi}' \end{pmatrix} \in \mathbb{C}^4. \quad (2.10)$$

Definition 1. Two boundary conditions given by A and \tilde{A} are equivalent if $\tilde{A} = BA$ for some $B \in GL_2(\mathbb{C})$.

Equivalent boundary conditions preserve self-adjointness as well as anomalies, see below.

Theorem 2. For almost every $k_x \in \mathbb{R}$ the operator $H^\sharp(k_x)$ together with boundary condition (2.10) is self-adjoint on $L^2(\mathbb{R}_+)$ if and only if $A \in M_{2,4}(\mathbb{C})$ is a rank-2 matrix and is equivalent to one of the seven classes $\mathfrak{A}_{i,j}$, \mathfrak{B} and \mathfrak{C} given in Table 1.

The proof of this theorem can be found in Section 3. We relate self-adjoint extensions of H^\sharp with stable subspaces generated by $\ker A$, then use the GL_2 -invariance to identify A with elements of the Grassmanian $Gr(4, 2)$. The subclasses $\mathfrak{A}_{i,j}$ correspond to a Schubert cell decomposition which is compatible with the self-adjoint constraint.

Remark 3. A_0 and A_1 are k_x -independent, so each couple (A_0, A_1) provides a self-adjoint operator \mathcal{H}^\sharp on $L^2(\mathbb{R} \times \mathbb{R}^+)$. It may happen exceptionally that $A = A_0 + ik_{x,0}A_1$ is not of rank 2 for some $k_{x,0} \in \mathbb{R}$ (e.g. $k_{x,0} = 0$ in class \mathfrak{B}) so that that $H^\sharp(k_{x,0})$ is not self-adjoint. However this would happen only for a finite number of k_x so that \mathcal{H}^\sharp remains globally self-adjoint.

2.3 Edge modes

For each boundary condition, equation (2.6) is a system of two coupled ODEs in y which is exactly solvable for every k_x, ω . We consider the spectrum of H^\sharp , namely solutions to (2.6) which are bounded in y , and plot it in the (k_x, ω) -plane. We distinguish two kinds of solutions. For $|\omega| \geq \omega_+(k_x, 0)$, see (2.4), the solutions are oscillatory modes delocalized in the upper half-plane, they correspond to the essential spectrum of \mathcal{H} . We call them bulk modes. On the other hand, for $|\omega| \leq \omega_+(k_x, 0)$, there may exist solutions which decay exponentially when $y \rightarrow \infty$. We call them edge modes. As k_x varies, edge modes draw continuous curves that we count as follows.

Definition 4. The number n_b of edge modes below a bulk band is the signed number of edge mode branches emerging (+) or disappearing (−) at the lower band limit, as $k_x \in \mathbb{R}$ increases. The number n_a of edge modes above a band is counted likewise up to a global sign change.

Class	A_0	A_1
$\mathfrak{A}_{1,2}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \end{pmatrix}$
$\mathfrak{A}_{1,4}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 1 \end{pmatrix}$ $\alpha \in \mathbb{R}$	$\begin{pmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & i\beta & 0 & 0 \end{pmatrix}$ $\beta \in \mathbb{R}$
$\mathfrak{A}_{2,3}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & 0 \end{pmatrix}$ $\alpha \in \mathbb{R}$	$\begin{pmatrix} 0 & b_{12} & 0 & 0 \\ i\beta & b_{22} & 0 & 0 \end{pmatrix}$ $\beta \in \mathbb{R}$
$\mathfrak{A}_{2,4}$	$\begin{pmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & 0 & (a_{11}^*)^{-1} & 1 \end{pmatrix}$ $a_{11} \in \mathbb{C} \setminus \{0\}, a_{21} = \alpha a_{11} + \epsilon^{-1}, \alpha \in \mathbb{R}$	$\begin{pmatrix} b_{11} & b_{11}(a_{11})^{-1} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \end{pmatrix}$ $b_{22} - b_{21}a_{11}^{-1} = i\beta, \beta \in \mathbb{R}$
$\mathfrak{A}_{3,4}$	$\begin{pmatrix} \alpha_1 & a_{12} & 1 & 0 \\ \epsilon^{-1} - a_{12}^* & \alpha_2 & 0 & 1 \end{pmatrix}$ $\alpha_1, \alpha_2 \in \mathbb{R}$	$\begin{pmatrix} i\beta_1 & b_{12} & 0 & 0 \\ b_{12}^* & i\beta_2 & 0 & 0 \end{pmatrix}$ $\beta_1, \beta_2 \in \mathbb{R}$
\mathfrak{B}	$\begin{pmatrix} a_1 & a_2 & i\alpha & -i\mu^*\alpha \\ \mu a_1 & \mu a_2 & i\mu\alpha & -i \mu ^2\alpha \end{pmatrix}$ $\alpha \in \mathbb{R}, \alpha(\alpha \operatorname{Im}(\mu) - \epsilon \operatorname{Re}(a_1 - a_2\mu)) = 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
\mathfrak{C}	$\begin{pmatrix} a_1 & a_2 & 0 & a_4 \\ \mu a_1 & \mu a_2 & 0 & \mu a_4 \end{pmatrix}$ $(a_2, a_4) \neq 0, \mu \in \mathbb{C} \setminus \{0\}, \operatorname{Im}(a_2 a_4^*) = 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Table 1: Classification of local self-adjoint boundary conditions from Theorem 2, up to GL_2 -invariance. All unconstrained parameters are arbitrary complex numbers: $a_{ij}, b_{ij}, a_i \in \mathbb{C}$.

Example 5. To illustrate the diversity of situations we plot three examples of the spectrum of $H^\#$ in Figure 1 for the following boundary condition:

- Dirichlet boundary condition: class \mathfrak{A}_{12} with $A_0 = [I_2|0]$ and $A_1 = [0|0]$. This corresponds to $\psi_1 = \psi_2 = 0$ at $y = 0$.
- Condition a: class $\mathfrak{A}_{1,4}$ with $A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$ which corresponds to $\psi_1 = 0$ and $\psi_2' + k_x \psi_2 = 0$ at $y = 0$.
- Condition b: class $\mathfrak{A}_{3,4}$ with $A_0 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 9 & 1 & 0 & 1 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 4i & -i & 0 & 0 \\ i & -4i & 0 & 0 \end{pmatrix}$.

We focus on the edge modes below the upper band. For Dirichlet boundary condition, we get $n_b = 1 = C_+$, in agreement with the bulk-edge correspondence. However, such a correspondence is violated for boundary conditions (a) and (b), for which we get $n_b = 2$ and 3, respectively. Changing

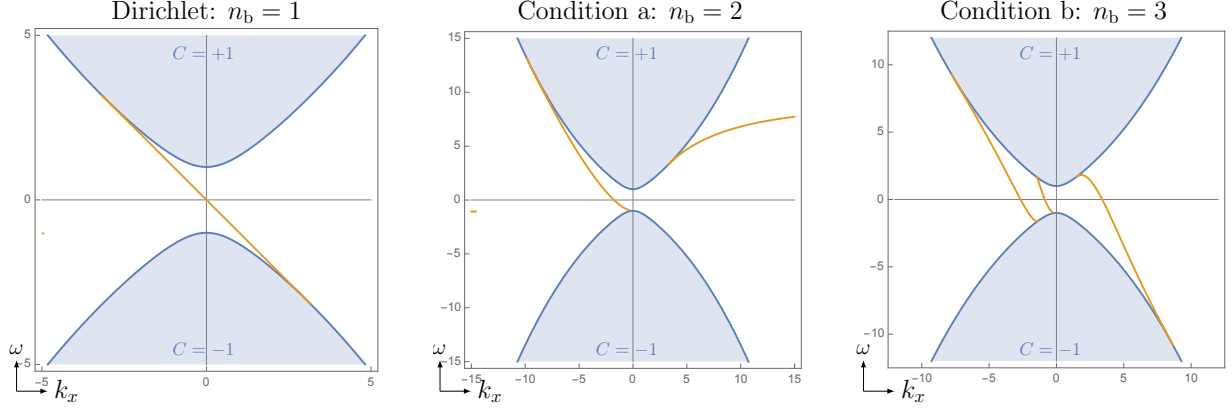


Figure 1: Edge spectrum of H^\sharp for $m = 1$, $\epsilon = 0.1$ and various boundary conditions from Example 5. Blue regions correspond to bulk modes and yellow curves to edge modes. Their number below the upper band is respectively $n_b = 1, 2$ and 3 .

$-i$ to i in A_1 of Condition (a) leads to a spectrum with no edge modes, so that $n_b = 0$. As we shall see, for this model all the possible values are $n_b = -1, 0, 1, 2, 3$, see Remark 12 below.

Similarly to [20], a boundary condition with $C_+ \neq n_b$ is said to be anomalous. Notice that the mismatch cannot be cured by looking at higher –but finite– values of k_x and ω , possibly revealing extra edge modes that are not visible in Figure 1. The anomaly is actually rooted in a singularity at $k_x \rightarrow \infty$, and can be precisely detected for any boundary condition via the scattering amplitude formalism.

2.4 Scattering amplitude

For the rest of the paper we focus on the upper part of the spectrum $\omega > 0$. A similar analysis could be performed on the lower part¹. Let $k_x \in \mathbb{R}$ and fix $\omega > \omega_+(k_x, 0) = \sqrt{k_x^2 + (m - \epsilon k_x^2)^2}$. The bulk band equation

$$k_x^2 + k_y^2 + (m - \epsilon(k_x^2 + k_y^2))^2 = \omega^2 \quad (2.11)$$

has four solutions in k_y , two are opposite and real and two are opposite and purely imaginary. Let $\kappa = \kappa(k_x, \omega)$ be the real positive one. Then the other solutions are: $-\kappa < 0$ and

$$\kappa_{\text{ev/div}} = \pm i \sqrt{\kappa^2 + 2k_x^2 + \frac{1 - 2\epsilon m}{\epsilon^2}} \in i\mathbb{R}_+. \quad (2.12)$$

Reciprocally, for fixed $k_x \in \mathbb{R}$ and $\kappa > 0$, $k_y = \kappa, -\kappa, \kappa_{\text{ev}}$ and κ_{div} are all solution to (2.11) for the same $\omega = \omega_+(k_x, \kappa)$. Let $\hat{\psi}_{\text{in}}, \hat{\psi}_{\text{out}}$ and $\hat{\psi}_{\text{ev}}$ be bulk eigenstates, namely solutions of $H\hat{\psi} = \omega\hat{\psi}$, see (2.3), with momentum k_x and $k_y = -\kappa, \kappa, \kappa_{\text{ev}}$, respectively. The associated normal modes are not solutions of the half-space problem, but a linear combination can be. We define the scattering state as

$$\psi_s := \tilde{\psi}_s e^{i(k_x x - \omega t)}, \quad \tilde{\psi}_s := \hat{\psi}_{\text{in}} e^{-i\kappa y} + S\hat{\psi}_{\text{out}} e^{i\kappa y} + T\hat{\psi}_{\text{ev}} e^{i\kappa_{\text{ev}} y}. \quad (2.13)$$

Such a state has well-defined momentum k_x and energy $\omega = \omega_+(k_x, \kappa)$. Each term is respectively interpreted as an incoming, outgoing and evanescent state with respect to the boundary $y = 0$. The

¹In general, a boundary condition does not preserve the symmetry of the bulk, so the nature –anomalous or not– of the upper band may be different than the lower one. However the analysis of the lower band would be very similar.

coefficients S and T are tuned so that $\widehat{\psi}_s$ satisfies the boundary condition (2.10), and S is called the scattering amplitude².

We are interested in the properties of S as (k_x, κ) varies. Let $U_{\text{out}} \subset \mathbb{R}^2$ be an open subset and let U_{in} and U_{ev} be the images of it by the maps $(k_x, \kappa) \mapsto (k_x, -\kappa)$ and $(k_x, \kappa) \mapsto (k_x, \kappa_{\text{ev}})$. We assume $\widehat{\psi}_{\text{in}}$, $\widehat{\psi}_{\text{out}}$ and $\widehat{\psi}_{\text{ev}}$ to be regular and non-vanishing on their respective domains, and also of amplitude one. Namely $\|\widehat{\psi}_{\text{out}}(k_x, \kappa)\| = 1$ for $k_x, \kappa \in U_{\text{out}}$, and similarly for the others. In that case, the scattering state (2.13) is uniquely defined and moreover $S(k_x, \kappa) \in U(1)$.

The relevance of the scattering amplitude comes from the following result:

Theorem 6. [20, Thm. 9] *Let Γ be any smooth, counter-clockwise and not self-intersecting loop in the half-plane $\{(k_x, \kappa) \in \mathbb{R}^2, \kappa > 0\}$. Then*

$$\frac{1}{2\pi i} \int_{\Gamma} S^{-1} dS = C_+ \quad (2.14)$$

Moreover,

$$\lim_{\kappa \rightarrow 0} \int_{\mathbb{R}} dk_x (S^{-1} \partial_{k_x} S)(k_x, \kappa) = n_b \quad (2.15)$$

The scattering amplitude appears as a pivotal point to understand the bulk-edge correspondence. On the one hand, S can be interpreted as a transition function between two bulk sections, and its winding is naturally related to the Chern number. Eq. (2.14) is sometimes called the bulk-scattering correspondence. On the other hand, $\omega(k_x, \kappa) \rightarrow \omega(k_x, 0)$ as $\kappa \rightarrow 0$ so that Eq. (2.15) is the change of argument of S near the bottom of the bulk spectrum. This quantity detects merging events of edge mode branches with the bulk band, thank to a relative version of Levinson's theorem, originally developed in [21].

2.5 Detecting anomalies: winding number

One sees from Theorem 6 above that in order for the bulk-edge correspondence to hold, the scattering amplitude needs to be regular and not winding at infinity. In that case, we can deform the loop Γ to the real line $\kappa = 0$ and recover $C_+ = n_b$. In order to investigate the infinite part of the spectrum, we need more on the scattering matrix. For a bulk section $\widehat{\psi}(k_x, k_y)$, solution of $H\widehat{\psi} = \omega\widehat{\psi}$, see (2.3), we denote by

$$\widehat{\Psi}(k_x, k_y) = \begin{pmatrix} \widehat{\psi}(k_x, k_y) \\ ik_y \widehat{\psi}(k_x, k_y) \end{pmatrix} \in \mathbb{C}^4 \quad (2.16)$$

and notice that $A\widehat{\Psi}$ is a column vector of size two for $A \in M_{2,4}(\mathbb{C})$.

Proposition 7. *Let $\widehat{\psi}^0$ a bulk section that is regular and non-vanishing in $\mathbb{R}^2 \setminus \{0\} \cup \{\infty\}$ and let $\widehat{\psi}^\infty$ be a bulk section that is regular and non-vanishing in \mathbb{R}^2 . Let $A \in M_{2,4}(\mathbb{C})$ of rank 2 be a boundary condition as in (2.10). The scattering amplitude reads:*

$$S(k_x, \kappa) = -\frac{g(k_x, -\kappa)}{g(k_x, \kappa)}, \quad g(k_x, \kappa) = \det([A\widehat{\Psi}^0(k_x, \kappa), A\widehat{\Psi}^\infty(k_x, \kappa_{\text{ev}})]), \quad (2.17)$$

and is regular in the limit $k_x^2 + \kappa^2 \rightarrow \infty$. In particular, equivalent boundary conditions have the same scattering amplitude S .

²In this framework, S is actually a pure reflection amplitude, since the incoming wave cannot be transmitted across the sharp boundary.

This proposition is proved in Section 4.1. The last sentence is obvious from the expression of g above and ensures that the boundary condition classification from Theorem 2 is relevant for the investigation of the anomalous bulk-edge correspondence. To study the winding phase of S near infinity, we consider the following path for $\lambda > 0$ and $\delta > 0$:

$$\Gamma_\delta(\lambda) = \left\{ \left(k_x = \frac{-\lambda_x}{\lambda_x^2 + \delta^2}, \kappa = \frac{\delta}{\lambda_x^2 + \delta^2} \right), \lambda_x \in [-\lambda, \lambda] \right\} \quad (2.18)$$

where λ_x, δ are called the dual variables to k_x, κ .

Definition 8. We say that S winds at ∞ if $w_\infty \neq 0$ with

$$w_\infty := \lim_{\lambda \rightarrow 0} \lim_{\delta \rightarrow 0} \int_{\Gamma_\delta(\lambda)} S^{-1} dS$$

For small δ , $\Gamma_\delta(\lambda)$ explores the region near $k_x = 0$, $\kappa = \infty$. The limit $\lambda \rightarrow 0$ ensures that we are not counting other edge modes at finite but large k_x .

Lemma 9. $C_+ = n_b + w_\infty$.

Proof. According to Theorem 6 one has

$$C_+ = \frac{1}{2\pi i} \int_{\Gamma} S^{-1} dS$$

for any simple and counterclockwise loop Γ . Fix $\delta > 0$ and consider

$$\mathcal{C}_\delta = \left\{ \left(k_x = \frac{-\lambda_x}{\lambda_x^2 + \delta^2}, \kappa = \frac{\delta}{\lambda_x^2 + \delta^2} \right), \lambda_x \in \mathbb{R} \right\}$$

This loop naturally splits into $\mathcal{C}_\delta = \Gamma_\delta(\lambda) \cup (\Gamma_\delta(\lambda))^c$ for any $\lambda > 0$, with

$$(\Gamma_\delta(\lambda))^c = \left\{ \left(k_x = \frac{-\lambda_x}{\lambda_x^2 + \delta^2}, \kappa = \frac{\delta}{\lambda_x^2 + \delta^2} \right), \lambda_x \in]\lambda, +\infty[\cup]-\infty, \lambda[\right\}$$

In particular, $(\Gamma_\delta(\lambda))^c$ becomes the real line $\kappa = 0$, $k_x \in \mathbb{R}$ as $\delta \rightarrow 0$ and $\lambda \rightarrow 0$, and S is regular in the upper half-plane $\kappa > 0$ so that

$$\lim_{\lambda \rightarrow 0} \lim_{\delta \rightarrow 0} \int_{(\Gamma_\delta(\lambda))^c} S^{-1} dS = \lim_{\kappa \rightarrow 0} \int_{\mathbb{R}} dk_x (S^{-1} \partial_{k_x} S)(k_x, \kappa) = n_b$$

The remaining part of the integral along Γ_δ leads to w_∞ . □

Consequently, w_∞ detects when a boundary condition is anomalous at infinity, but also predicts the number of edge modes below the upper bulk band by $n_b = C_+ - w_\infty$ since $C_+ = 1$ is fixed. The main idea to compute w_∞ is to expand S and g from Proposition 7 near $k_x^2 + \kappa^2 \rightarrow \infty$, then extract the leading term \mathcal{S} along the path $\Gamma_\delta(\lambda)$. Ultimately, the winding number is reduced to

$$w_\infty = \int_{-\infty}^{+\infty} \mathcal{S}^{-1}(u) \mathcal{S}'(u) du, \quad \mathcal{S}(u) := \frac{G_-(u)}{G_+(u)}.$$

with $G_\pm : \mathbb{R} \rightarrow \mathbb{C}^*$ two complex curves whose winding phases can be computed explicitly.

Example 10. We continue Example 5.

- Dirichlet boundary condition: one has $G_+ = G_- = 1$ so that $w_\infty = 0$.
- Condition a: one has $G_-(u) = -u - i$ and $G_+(u) = G_-^*(u)$. As u goes from $-\infty$ to $+\infty$, G_- goes from $+\infty - i$ to $-\infty - i$ so that its change of argument is $-\pi$. Consequently G_+ argument change is π , so that S winds by -2π and $w_\infty = -1$. We recover $n_b = 1 - (-1) = 2$.
- Condition b: one has

$$G_-(u) = -15u^2 + 4u\sqrt{2u^2 + 1} + i\left(\sqrt{2u^2 + 1} - 4u\right), \quad G_+(u) = G_-^*(u).$$

A detailed analysis of this complex curve (see Section 5.5) shows that it is dominated by its real part as $u \rightarrow \pm\infty$. $\text{Re}(G_-)$ goes from $-\infty$ (with $\text{Im}(G_-) > 0$) to $-\infty$ (with $\text{Im}(G_-) < 0$). Its imaginary part is strictly decreasing and vanishes at u_0 with $\text{Re}(G_-(u_0)) > 0$. Thus G_- winds once clockwise around zero as $u \in \mathbb{R}$, and $G_+ = G_-^*$ winds once anti-clockwise. Thus $w_\infty = -2$, and we recover $n_b = 3$.

What is remarkable for this problem is that G_\pm , S and w_∞ can be computed not only for specific examples, but actually for any boundary condition from Table 1. Moreover, among the many free parameters in each class, only a few of them survive near infinity. The anomalous nature (and the value of w_∞) is driven by one or two parameters only, so that an exhaustive classification can be formulated in a rather compact way. This is the central result of this paper.

Theorem 11. *w_∞ can be systematically computed for almost every self-adjoint boundary condition. Its value is given in Table 2. Each expression for S can be found in Section 5.*

Class	w_∞	Condition
$\mathfrak{A}_{1,2}$	0	None
$\mathfrak{A}_{1,4}$	$\text{sign}_0(\beta)$	None
$\mathfrak{A}_{2,3}$	$-\text{sign}(\beta)$	$\beta \neq 0$
	1	$\beta = 0$ and $ \alpha - \epsilon^{-1} < 1/\sqrt{2}$
$\mathfrak{A}_{2,4}$	0	$\beta = 0$ and $ \alpha - \epsilon^{-1} > 1/\sqrt{2}$
	$\text{sign}(\beta)$	$\beta a_{11} ^2 > \sqrt{2}$ or $\beta a_{11} ^2 < -\sqrt{2}$
$\mathfrak{A}_{3,4}$	0	$-\sqrt{2} < \beta a_{11} ^2 < \sqrt{2}$
	$\text{sign}(B_+)$	$b_{12} \neq 0$ and $B_+B_- < 0$
	0	$b_{12} \neq 0$ and $B_\pm > 0$
	$2 \times \text{sign}(\sqrt{2} - \beta_1)$	$b_{12} \neq 0$ and $B_\pm < 0$
\mathfrak{B}	$\text{sign}_0(\beta)$	$b_{12} = 0$ and $\beta_1^2 \neq 2$
	0	None
\mathfrak{C}	1	$a_2 = 0$
	0	$a_2 \neq 0$

Table 2: Anomaly classification for any self-adjoint boundary condition. The class and parameters refer to Table 1. For class $\mathfrak{A}_{3,4}$ we also define $B_\pm := \beta_2(\beta_1 \pm \sqrt{2}) + |b_{12}|^2$. We distinguish the sign function $\text{sign} : \mathbb{R} \setminus \{0\} \rightarrow \{\pm 1\}$ from its extended version $\text{sign}_0 : \mathbb{R} \rightarrow \{0, \pm 1\}$ which sends 0 to 0.

Remark 12. Beyond the classification in itself, several important consequences can be inferred:

- Class $\mathfrak{A}_{1,2}$ and \mathfrak{B} are never anomalous: $w_\infty = 0$, whereas all other classes are of both nature with $w_\infty \in \{0, \pm 1, \pm 2\}$. By Lemma 9, we deduce that $n_b \in \{-1, 0, 1, 2, 3\}$.
- Among the many parameters in each class, only a few (up to three) actually drive the value of w_∞ . Thus each phase is rather stable: in both cases (anomalous or not), several free parameters can vary without changing w_∞ . In particular, anomalies are not fine-tuned effects among the boundary condition, but are as stable as the non-anomalous regime.
- The possibility that $w_\infty = \pm 2$ was unexpected from previous works and could be revealed only through the full classification.
- There is another possibility for the upper band to be anomalous, with $w_\infty = 0$ and an edge mode branch merging exactly at infinity, see [20] for an example. The detection of asymptotic edge modes is another distinct classification that we postpone to future work.
- Class $\mathfrak{A}_{1,2}, \mathfrak{A}_{1,4}, \mathfrak{B}$ and \mathfrak{C} are completely classified. Some threshold cases remain: in $\mathfrak{A}_{2,3}$ ($\alpha - \epsilon^{-1} = \pm 1/\sqrt{2}$), $\mathfrak{A}_{2,4}$ ($\beta|a_{11}|^2 = \pm\sqrt{2}$) and $\mathfrak{A}_{3,4}$ ($B_+ = 0$ or $B_- = 0$), where we suspect w_∞ to be ill-defined. In that sense, the classification is almost exhaustive.

The rest of the paper is organized as follows. In Section 3 we prove Theorem 2 and establish Table 1 of self-adjoint boundary conditions. In Section 4 we prove Proposition 7 and expand the scattering amplitude S near infinity. Then we develop a general formalism to extract the dominant scale \mathcal{S} of S . There, the main technical results are Proposition 17 and 19. Finally, we apply this strategy to each class in Section 5. Due to the large number of parameters in some classes, some expression for S are very large and tedious to manipulate (up to 60 terms, see e.g. (5.22)). Thus, some computation were assisted with a symbolic computation software (Mathematica), but without any numerical approximation. We also give a detailed example in Section 4.4 where all computations can be done by hand to facilitate the reading.

3 Self-adjoint boundary conditions

In this section we prove Theorem 2 and establish Table 1. We adapt the method from Appendix B of [20] for shallow-water wave operator to the Dirac Hamiltonian. We study the edge operator H^\sharp from (2.6) associated to states $\tilde{\psi}(y; k_x)$ from (2.8). In this section, we drop the k_x dependence as well as \sharp and $\tilde{}$ notations, so that we work instead with H and $\psi(y)$.

Lemma 13. *Self-adjoint realizations of the Hamiltonian H correspond to subspaces $M \subset \mathbb{C}^4$ with*

$$\Omega M = M^\perp, \quad (3.1)$$

where

$$\Omega = \begin{pmatrix} 0 & 1 & \epsilon & 0 \\ -1 & 0 & 0 & -\epsilon \\ -\epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \end{pmatrix}. \quad (3.2)$$

The domain $\mathcal{D}(H)$ is a subspace of the Sobolev space $H^2 \oplus H^2$ which is characterized by

$$\mathcal{D}(H) := \left\{ \psi \in H^2 \oplus H^2 \mid \Psi = \begin{pmatrix} \psi(0) \\ \psi'(0) \end{pmatrix} \in M \right\}. \quad (3.3)$$

Proof. Let $\psi, \phi \in H^2 \oplus H^2$. A partial integration of $\langle \phi, H\psi \rangle$ yields a boundary term at $y = 0$:

$$\langle \phi, H\psi \rangle - \langle H\phi, \psi \rangle = \phi_1^* \psi_2 - \phi_2^* \psi_1 + \epsilon (\phi_1^* \psi_1' - \phi_2^* \psi_2' - \psi_1 (\phi_1^*)' + \psi_2 (\phi_2^*)') \quad (3.4)$$

$$= \Phi^* \Omega \Psi, \quad (3.5)$$

with Ψ and Φ as in (3.3), Ω as in (3.2). In order for H to be self-adjoint, we require $\Psi \in M \subset \mathbb{C}^4$ with the following properties

- i) $\Phi^* \Omega \Psi = 0, \Psi \in M \Rightarrow \Phi \in M,$
- ii) $\Psi \in M \Rightarrow \Phi^* \Omega \Psi = 0$ for all $\Phi \in M.$

By (3.5), the properties (i) and (ii) imply $H^* \subset H$ and $H \subset H^*$ respectively, and thus $H = H^*$.

Moreover these two properties imply $(\Omega M)^\perp = M$. Indeed, let $\Phi \in (\Omega M)^\perp$ and $\Psi \in M$. One has $\Phi^* \Omega \Psi = 0$ so that $\Phi \in M$ by (i). Thus $(\Omega M)^\perp \subset M$. Conversely, let $\Psi \in M$. For any $\Phi \in M$ one has $\Phi^* \Omega \Psi = 0$ by (ii), which implies $\Psi^* \Omega \Phi = 0$ since $\Omega^* = -\Omega$. Thus $\Psi \in (\Omega M)^\perp$ and $M \subset (\Omega M)^\perp$.

Reciprocally, it is not hard to check that $(\Omega M)^\perp = M$ implies (i) and (ii), so that H is self-adjoint if and only if $(\Omega M)^\perp = M$, namely $\Omega M = M^\perp$. \square

Remark 14. The matrix Ω defines a symplectic form on \mathbb{C}^4 and the subspace M is sometimes called a Lagrangian space, a general structure that naturally appears for self-adjoint extensions of elliptic operators, see e.g. [18].

Proposition 15. *i) Subspaces $M \subset \mathbb{C}^4$ with $\dim M = 2$ are determined by 2×4 matrices A of maximal rank, i.e. $\text{rk } A = 2$ by means of*

$$M = \{\Psi \in \mathbb{C}^4 \mid A\Psi = 0\} = \ker A, \quad (3.6)$$

and conversely. Two such matrices A, \tilde{A} determine the same subspace if and only if $A = B\tilde{A}$ with $B \in \text{GL}_2(\mathbb{C})$.

ii) Self-adjoint boundary conditions as in Lemma 13 are characterized by matrices A as in i) with

$$A\Omega^{-1}A^* = 0. \quad (3.7)$$

Proof. i) The first part is clear as $A\Psi = 0$ gives two conditions if A is of $\text{rk } 2$. In other words Ω is invertible and thus $\ker \Omega = \{0\}$. For the second part (the last sentence) suppose that A and \tilde{A} determine the same subspace, then there is a map $B : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, which is well-defined by $Av \mapsto \tilde{A}v, v \in \mathbb{C}^4$ as $\ker A = \ker \tilde{A}$. If the kernel is to be preserved then $B \in \text{GL}(2)$.

ii) Self-adjoint boundary conditions correspond to subspaces $M \subset \mathbb{C}^4$ with $\Omega M = M^\perp$. Moreover, Ω is invertible so that $\Omega M = M^\perp$ is equivalent to $\Omega^{-1}M^\perp \subset M$ and $\dim M = 2$. Thus, by (i), $M = \ker A$ for some $A \in \text{M}_{2,4}(\mathbb{C})$ of rank 2. Moreover one has $M^\perp = \text{ran } A^* = \{A^*v \mid v \in \mathbb{C}^2\}$. Let $v_1, v_2 \in \mathbb{C}^2$. One has

$$\langle v_1, A\Omega^{-1}A^*v_2 \rangle = \langle A^*v_1, \Omega^{-1}A^*v_2 \rangle = 0$$

since $A^*v_1 \in M^\perp$ and $\Omega^{-1}A^*v_2 \in \Omega^{-1}M^\perp \subset M$.

Reciprocally, if $M = \ker A$ with $A \in \text{M}_{2,4}(\mathbb{C})$ of rank 2 then $\dim M = 2$ and one can check that $\Omega^{-1}M^\perp \subset (M^\perp)^\perp = M$. \square

3.1 Grassmannian

Self-adjoint boundary conditions are given by 2×4 matrices of rank 2 up to GL_2 -invariance. This set of matrices corresponds to the Grassmannian $\text{Gr}(4, 2)$, which parametrizes 2-dimensional subspaces in the 4-dimensional vector space \mathbb{C}^4 :

$$\text{Gr}(4, 2) \cong \frac{\{A \in \text{M}_{2,4}(\mathbb{C}) \mid \text{rk}(A) = 2\}}{\text{GL}_2(\mathbb{C})}. \quad (3.8)$$

Grassmannians are complex manifolds for which a lot is known, see e.g. [22]. We shall use the following facts: since $\text{rk}(A) = 2$ there are at least 2 columns among the 4 of $A \in \text{M}_{2,4}(\mathbb{C})$ which are linearly independent vectors in \mathbb{C}^2 . Then one can use the $\text{GL}_2(\mathbb{C})$ invariance to set the first one to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the second to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and reduce the matrix A to its row echelon form (from the right). This provides the following partition

$$\text{Gr}(4, 2) = \bigsqcup_j \mathcal{C}_j. \quad (3.9)$$

where $\mathbf{j} = \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}$ and $\{3, 4\}$, with

$$\begin{aligned} \mathcal{C}_{\{1,2\}} &: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \mathcal{C}_{\{1,3\}} &: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \star & 1 & 0 \end{pmatrix}, & \mathcal{C}_{\{1,4\}} &: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \star & \star & 1 \end{pmatrix}, \\ \mathcal{C}_{\{2,3\}} &: \begin{pmatrix} \star & 1 & 0 & 0 \\ \star & 0 & 1 & 0 \end{pmatrix}, & \mathcal{C}_{\{2,4\}} &: \begin{pmatrix} \star & 1 & 0 & 0 \\ \star & 0 & \star & 1 \end{pmatrix}, & \mathcal{C}_{\{3,4\}} &: \begin{pmatrix} \star & \star & 1 & 0 \\ \star & \star & 0 & 1 \end{pmatrix}. \end{aligned} \quad (3.10)$$

The \mathcal{C}_j are called the Schubert cells of $\text{Gr}(4, 2)$. Other Schubert cells of $\text{Gr}(4, 2)$ exist but the six above are enough to describe each element uniquely. Notice that $\dim(\mathcal{C}_j) = \sum_{i=1}^2 j_i - i$ and $\mathcal{C}_j \cong \mathbb{C}^{\dim(\mathcal{C})}$ so that the Schubert cells above have complex dimensions 0, 1, 2, 2, 3 and 4, respectively.

3.2 Local boundary conditions

According to (2.7) we are interested in local boundary condition of the form $A\Psi = 0$ with $A = A_0 + ik_x A_1 \in \text{M}_{2,4}(\mathbb{C})$. We shall start with

$$A_0 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix} = [A_1 \quad A_2], \quad A_1 = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \end{pmatrix} = [B \quad 0] \quad (3.11)$$

with A_1, A_2 and $B \in \text{M}_2(\mathbb{C})$. According to Proposition 15, A has to be a rank-2 matrix. Thus we consider the various cases with respect to the rank of A_0 and A_1 . Depending on their rank, such matrices can be simplified further using the $\text{GL}_2(\mathbb{C})$ -invariance.

Moreover, A satisfies

$$A\Omega^{-1}A^* = 0$$

with

$$\Omega^{-1} = \epsilon^{-2} \begin{pmatrix} 0 & 0 & -\epsilon & 0 \\ 0 & 0 & 0 & \epsilon \\ \epsilon & 0 & 0 & -1 \\ 0 & -\epsilon & 1 & 0 \end{pmatrix} := \epsilon^{-2} \begin{pmatrix} 0 & -\Omega_1 \\ \Omega_1 & \Omega_2 \end{pmatrix}. \quad (3.12)$$

with $\Omega_1, \Omega_2 \in \text{M}_2(\mathbb{C})$ and $\Omega_1^* = \Omega_1$ and $\Omega_2^* = \Omega_2$. The condition $A\Omega^{-1}A^* = 0$ becomes

$$-A_1\Omega_1A_2^* + A_2\Omega_1A_1^* + A_2\Omega_2A_2^* - ik_x(B\Omega_1A_2^* + A_2\Omega_1B^*) = 0.$$

This relation has to be valid for every $k_x \in \mathbb{R}$ so we infer

$$-A_1\Omega_1A_2^* + A_2\Omega_1A_1^* + A_2\Omega_2A_2^* = 0, \quad B\Omega_1A_2^* + A_2\Omega_1B^* = 0. \quad (3.13)$$

3.3 Class \mathfrak{A} : $\text{rk}(A_0) = 2$

In this section we assume $\text{rk}(A_0) = 2$ and A_1 arbitrary:

$$A_1 = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \end{pmatrix} = [B \quad 0]$$

The $\text{GL}_2(\mathbb{C})$ -invariance allows to reduce A_0 to one of the six Schubert cells from (3.10). For each of them we investigate (3.13) and possibly restrict some parameters.

Class $\mathfrak{A}_{1,2}$. In that case $A_0 \in \mathcal{C}_{\{1,2\}}$, namely

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

One can check that (3.13) is satisfied for any $B \in M_2(\mathbb{C})$.

Class $\mathfrak{A}_{1,3}$. In that case $A_0 \in \mathcal{C}_{\{1,3\}}$, namely

$$A_0 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & a_{22} & 0 & 0 \end{pmatrix}.$$

In that case we have

$$-A_1\Omega_1A_2^* + A_2\Omega_1A_1^* + A_2\Omega_2A_2^* = \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix}$$

which never vanishes for $\epsilon > 0$ so that (3.13) is never satisfied. Thus there is no self-adjoint boundary condition in this class, this is why it is not appearing in Table 1.

Class $\mathfrak{A}_{1,4}$. In that case $A_0 \in \mathcal{C}_{\{1,4\}}$, namely

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & 1 \end{pmatrix}.$$

In that case we have

$$-A_1\Omega_1A_2^* + A_2\Omega_1A_1^* + A_2\Omega_2A_2^* = \begin{pmatrix} 0 & -\epsilon(a_{23})^* \\ a_{23}\epsilon & -\epsilon(a_{22})^* + (a_{23})^* + a_{22}\epsilon - a_{23} \end{pmatrix},$$

from which we infer $a_{23} = 0$ and then $a_{22} \in \mathbb{R}$. Moreover, with that knowledge, we compute

$$B\Omega_1A_2^* + A_2\Omega_1B^* = \begin{pmatrix} 0 & -\epsilon b_{12} \\ -\epsilon(b_{12})^* & -\epsilon(b_{22} + (b_{22})^*) \end{pmatrix},$$

from which we infer $b_{12} = 0$ and $b_{22} \in i\mathbb{R}$. Denoting $a_{22} = \alpha$ and $b_{22} = i\beta$ we deduce the self-adjoint matrices in that case

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & i\beta & 0 & 0 \end{pmatrix},$$

with $\alpha, \beta \in \mathbb{R}$ and $b_{11}, b_{21} \in \mathbb{C}$.

Class $\mathfrak{A}_{2,3}$. In that case $A_0 \in \mathcal{C}_{\{2,3\}}$, namely

$$A_0 = \begin{pmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & 0 & 1 & 0 \end{pmatrix}.$$

In that case we have

$$-A_1\Omega_1A_2^* + A_2\Omega_1A_1^* + A_2\Omega_2A_2^* = \begin{pmatrix} 0 & -\epsilon a_{11} \\ \epsilon (a_{11})^* & \epsilon (a_{21})^* - a_{21}\epsilon \end{pmatrix},$$

from which we infer $a_{11} = 0$ and $a_{21} = \alpha \in \mathbb{R}$. Moreover one has

$$B\Omega_1A_2^* + A_2\Omega_1B^* = \begin{pmatrix} 0 & b_{11}\epsilon \\ \epsilon (b_{11})^* & \epsilon (b_{21})^* + b_{21}\epsilon \end{pmatrix}$$

from which we infer $b_{11} = 0$ and $b_{21} = i\beta \in i\mathbb{R}$. The self-adjoint matrices in that case are

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & b_{12} & 0 & 0 \\ i\beta & b_{22} & 0 & 0 \end{pmatrix},$$

with $\alpha, \beta \in \mathbb{R}$ and $b_{12}, b_{22} \in \mathbb{C}$.

Class $\mathfrak{A}_{2,4}$. In that case $A_0 \in \mathcal{C}_{\{2,4\}}$, namely

$$A_0 = \begin{pmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & 0 & a_{23} & 1 \end{pmatrix}.$$

In that case we have

$$-A_1\Omega_1A_2^* + A_2\Omega_1A_1^* + A_2\Omega_2A_2^* = \begin{pmatrix} 0 & \epsilon(1 - a_{11}(a_{23})^*) \\ -\epsilon(1 - a_{23}(a_{11})^*) & \epsilon(a_{23}(a_{21})^* - a_{21}(a_{23})^*) + (a_{23})^* - a_{23} \end{pmatrix}$$

from which we infer $1 - a_{11}(a_{23})^* = 0$, that can be rewritten $a_{11} \neq 0$ and $a_{23} = (a_{11}^{-1})^*$. The lower right coefficient of the matrix above then implies

$$\epsilon \left(\left(\frac{a_{21}}{a_{11}} \right)^* - \frac{a_{21}}{a_{11}} \right) + \frac{1}{a_{11}} - \frac{1}{(a_{11})^*} = 0$$

which can be rephrased as

$$\operatorname{Im} \left(\frac{a_{21} - \epsilon^{-1}}{a_{11}} \right) = 0.$$

Thus we have $a_{21} = \alpha a_{11} + \epsilon^{-1}$ with $\alpha \in \mathbb{R}$.

The second part of (3.13) reads

$$B\Omega_1A_2^* + A_2\Omega_1B^* = \begin{pmatrix} 0 & \epsilon(b_{11}a_{11}^{-1} - b_{12}) \\ \epsilon(b_{11}a_{11}^{-1} - b_{12})^* & \epsilon \left((b_{21}a_{11}^{-1})^* + b_{21}a_{11}^{-1} - (b_{22})^* - b_{22} \right) \end{pmatrix}$$

which implies $b_{12} = b_{11}a_{11}^{-1}$ and

$$\operatorname{Re} (b_{21}a_{11}^{-1} - b_{22}) = 0.$$

Thus we have $b_{21}a_{11}^{-1} - b_{22} = i\beta$ with $\beta \in \mathbb{R}$.

The self-adjoint matrices in that case are

$$A_0 = \begin{pmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & 0 & (a_{11}^{-1})^* & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} b_{11} & b_{11}a_{11}^{-1} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \end{pmatrix},$$

with $a_{11} \in \mathbb{C} \setminus \{0\}$, $a_{21}, b_{11}, b_{21}, b_{22} \in \mathbb{C}$ and $a_{21} = \alpha a_{11} + \epsilon^{-1}$, $\alpha \in \mathbb{R}$ as well as $b_{21}a_{11}^{-1} - b_{22} = i\beta$, $\beta \in \mathbb{R}$.

Class $\mathfrak{A}_{3,4}$. In that case $A_0 \in \mathcal{C}_{\{3,4\}}$, namely

$$A_0 = \begin{pmatrix} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \end{pmatrix}.$$

In that case we have

$$0 = -A_1\Omega_1A_2^* + A_2\Omega_1A_1^* + A_2\Omega_2A_2^* = \begin{pmatrix} \epsilon(a_{11})^* - a_{11}\epsilon & \epsilon(a_{21})^* + a_{12}\epsilon - 1 \\ -\epsilon(a_{12})^* - a_{21}\epsilon + 1 & a_{22}\epsilon - \epsilon(a_{22})^* \end{pmatrix},$$

from which we infer $a_{11} = \alpha_1 \in \mathbb{R}$, $a_{22} = \alpha_2 \in \mathbb{R}$ and $a_{21} = \epsilon^{-1} - (a_{12})^*$. Moreover, we have

$$0 = B\Omega_1A_2^* + A_2\Omega_1B^* = \begin{pmatrix} \epsilon(b_{11})^* + b_{11}\epsilon & \epsilon(b_{21})^* - b_{12}\epsilon \\ b_{21}\epsilon - \epsilon(b_{12})^* & -\epsilon(b_{22})^* - b_{22}\epsilon \end{pmatrix},$$

from which we infer $b_{11} = i\beta_1 \in i\mathbb{R}$, $b_{22} = i\beta_2 \in i\mathbb{R}$ and $b_{21} = (b_{12})^*$.

The self-adjoint matrices in that case are

$$A_0 = \begin{pmatrix} \alpha_1 & a_{12} & 1 & 0 \\ \epsilon^{-1} - (a_{12})^* & \alpha_2 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} i\beta_1 & b_{12} & 0 & 0 \\ (b_{12})^* & i\beta_2 & 0 & 0 \end{pmatrix},$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and $a_{12}, b_{12} \in \mathbb{C}$.

3.4 Class \mathfrak{B} : $\text{rk}(A_1) = 2$

In this section we assume $\text{rk}(A_1) = 2$. Since $A_1 = [B | 0]$ this means that $\text{rk}(B) = 2$ and thus by the $\text{GL}_2(\mathbb{C})$ -invariance we can reduce the study to

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

To avoid any overlap with class \mathfrak{A} it is sufficient to consider the case where $\text{rk}(A_0) \leq 1$. Thus we write

$$A_0 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ \mu a_1 & \mu a_2 & \mu a_3 & \mu a_4 \end{pmatrix}$$

with $a_1, a_2, a_3, a_4, \mu \in \mathbb{C}$.

In that case we have

$$0 = B\Omega_1A_2^* + A_2\Omega_1B^* = \begin{pmatrix} \epsilon(a_3)^* + a_3\epsilon & \epsilon(\mu a_3)^* - a_4\epsilon \\ a_3\mu\epsilon - \epsilon(a_4)^* & -\epsilon(\mu a_4)^* - a_4\mu\epsilon \end{pmatrix},$$

from which we infer $a_3 = i\alpha \in i\mathbb{R}$ and $a_4 = i\alpha\mu^*$ (in particular $\mu a_4 \in i\mathbb{R}$). Moreover, we have

$$\begin{aligned} 0 &= -A_1\Omega_1A_2^* + A_2\Omega_1A_1^* + A_2\Omega_2A_2^* \\ &= \alpha(\alpha(\mu^* - \mu) + i\epsilon\alpha(a_1^* + a_1 - a_2\mu - a_2^*\mu^*)) \begin{pmatrix} 1 & \mu^* \\ \mu & |\mu|^2 \end{pmatrix}, \end{aligned}$$

from which we infer

$$\alpha(\alpha \text{Im}(\mu) - \epsilon \text{Re}(a_1 - a_2\mu)) = 0.$$

The self-adjoint matrices in that case are

$$A_0 = \begin{pmatrix} a_1 & a_2 & \alpha & \alpha\mu^* \\ \mu a_1 & \mu a_2 & \alpha\mu & \alpha|\mu|^2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

with $a_1, a_2, \mu \in \mathbb{C}$, $\alpha \in \mathbb{R}$ and $\alpha(\alpha \text{Im}(\mu) - \epsilon \text{Re}(a_1 - a_2\mu)) = 0$. This last condition is equivalent to one of the three cases:

1. $\alpha = 0$
2. $\alpha \neq 0, \mu \in \mathbb{R}, a_1 = a_2\mu + i\beta, \beta \in \mathbb{R}$
3. $\mu \in \mathbb{C} \setminus \mathbb{R}$ and $\alpha = \frac{\epsilon}{\text{Im}(\mu)} \text{Re}(a_1 - a_2\mu) \in \mathbb{R}$

but we shall not use them explicitly below so we keep the general constraint instead.

3.5 Class \mathfrak{C} : $\text{rk}(A_0) = \text{rk}(A_1) = 1$

In this section we consider the case where A_0 and A_1 are exactly of rank 1, with $\text{rk}(A_0 + ik_x A_1) = 2$. Since $A_1 = [B | 0]$ this means that $\text{rk}(B) = 1$ and thus by the $\text{GL}_2(\mathbb{C})$ -invariance we can reduce the study to

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then, since $\text{rk}(A_0) = 1$ we write

$$A_0 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ \mu a_1 & \mu a_2 & \mu a_3 & \mu a_4 \end{pmatrix}$$

with $a_1, a_2, a_3, a_4, \mu \in \mathbb{C}$. Moreover one has $\mu \neq 0$ and $(a_2, a_3, a_4) \neq 0$, otherwise $\text{rk}(A_0 + ik_x A_1) = 1 < 2$.

In that case we have

$$0 = B\Omega_1 A_2^* + A_2 \Omega_1 B^* = \begin{pmatrix} \epsilon(a_3)^* + a_3\epsilon & \epsilon(\mu a_3)^* \\ a_3\mu\epsilon & 0 \end{pmatrix}$$

from which we infer $a_3 = 0$. Moreover, we have

$$0 = -A_1 \Omega_1 A_2^* + A_2 \Omega_1 A_1^* + A_2 \Omega_2 A_2^* = \epsilon(a_2(a_4)^* - a_4(a_2)^*) \begin{pmatrix} 1 & \mu^* \\ \mu & |\mu|^2 \end{pmatrix}$$

from which we infer $\text{Im}(a_2 a_4^*) = 0$.

The self-adjoint matrices in that case are

$$A_0 = \begin{pmatrix} a_1 & a_2 & 0 & a_4 \\ \mu a_1 & \mu a_2 & 0 & \mu a_4 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with $a_1, a_2, a_4, \mu \in \mathbb{C}$ such that $\mu \neq 0, (a_2, a_4) \neq 0$ and $\text{Im}(a_2 a_4^*) = 0$.

4 Asymptotic expansion of the scattering amplitude

4.1 Bulk eigensections

Here we prove Proposition 7 and also provide explicit expression for bulk sections which are used further below. Boundary condition (2.10) reads

$$A\Psi|_{y=0} = 0, \quad \Psi = \begin{pmatrix} \tilde{\psi} \\ \tilde{\psi}' \end{pmatrix}, \quad (4.1)$$

for $A \in M_{2,4}(\mathbb{C})$. We apply it to the scattering state from (2.13) at $y = 0$,

$$\begin{aligned}\tilde{\psi}_s &= \widehat{\psi}_{\text{in}} + S\widehat{\psi}_{\text{out}} + T\widehat{\psi}_{\text{ev}} \\ \tilde{\psi}'_s &= -i\kappa\widehat{\psi}_{\text{in}} + i\kappa S\widehat{\psi}_{\text{out}} + i\kappa_{\text{ev}}T\widehat{\psi}_{\text{ev}}\end{aligned}\quad (4.2)$$

where $\widehat{\psi}_{\text{in/out/ev}}$ are bulk solution at momentum k_x and $k_y = -\kappa, \kappa, \kappa_{\text{ev}}$ respectively. Thus at $y = 0$ we have

$$\Psi_s = \widehat{\Psi}_{\text{in}} + S\widehat{\Psi}_{\text{out}} + T\widehat{\Psi}_{\text{ev}}$$

as in (2.16). This amounts to

$$A\Psi_s = \begin{pmatrix} -a_1 & - \\ -a_2 & - \end{pmatrix} \Psi_s = \begin{pmatrix} a_1 \cdot \Psi_s \\ a_2 \cdot \Psi_s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (4.3)$$

where $a_i \in M_{1,4}(\mathbb{C})$ is the i -th row of A for $i = 1, 2$. This means that we get a 2×2 linear system of equations to be solved for S and T

$$a_1 \cdot (\widehat{\Psi}_{\text{in}} + S\widehat{\Psi}_{\text{out}} + T\widehat{\Psi}_{\text{ev}}) = 0, \quad (4.4)$$

$$a_2 \cdot (\widehat{\Psi}_{\text{in}} + S\widehat{\Psi}_{\text{out}} + T\widehat{\Psi}_{\text{ev}}) = 0. \quad (4.5)$$

Its solution for S is

$$S = -\frac{\begin{vmatrix} a_1 \cdot \widehat{\Psi}_{\text{in}} & a_1 \cdot \widehat{\Psi}_{\text{ev}} \\ a_2 \cdot \widehat{\Psi}_{\text{in}} & a_2 \cdot \widehat{\Psi}_{\text{ev}} \end{vmatrix}}{\begin{vmatrix} a_1 \cdot \widehat{\Psi}_{\text{out}} & a_1 \cdot \widehat{\Psi}_{\text{ev}} \\ a_2 \cdot \widehat{\Psi}_{\text{out}} & a_2 \cdot \widehat{\Psi}_{\text{ev}} \end{vmatrix}} = -\frac{\begin{vmatrix} A\widehat{\Psi}_{\text{in}} & A\widehat{\Psi}_{\text{ev}} \end{vmatrix}}{\begin{vmatrix} A\widehat{\Psi}_{\text{out}} & A\widehat{\Psi}_{\text{ev}} \end{vmatrix}} \quad (4.6)$$

Let $\widehat{\psi}^0$ a bulk section that is regular and non-vanishing in $\mathbb{R}^2 \setminus \{0\} \cup \{\infty\}$ and let $\widehat{\psi}^\infty$ be a bulk section that is regular and non-vanishing in \mathbb{R}^2 . Then for $k_x \in \mathbb{R}$ and $\kappa > 0$ we set $\widehat{\psi}_{\text{in}}(k_x, -\kappa) = \widehat{\psi}_0(k_x, -\kappa)$, $\widehat{\psi}_{\text{out}}(k_x, \kappa) = \widehat{\psi}_0(k_x, \kappa)$ and $\widehat{\psi}_{\text{ev}}(k_x, \kappa_{\text{ev}}) = \widehat{\psi}_\infty(k_x, \kappa_{\text{ev}})$, which leads to the expression (2.17) for S and g from Proposition 7.

To expand S near ∞ we provide explicit expressions for the sections $\widehat{\psi}_0$ and $\widehat{\psi}_\infty$. We concentrate on the upper part of the spectrum and only discuss the eigensections corresponding to $\omega_+(k) = +\sqrt{k^2 + (m - \epsilon k^2)^2}$. Assuming $m, \epsilon > 0$ the corresponding Chern number is $C_+ = 1$, and therefore it is impossible to find a global bulk eigensection $\widehat{\psi}$ regular for all points on the compactified k -plane, $\mathbb{C} \cup \{\infty\} \cong S^2$. To cover the sphere we need at least two distinct ones overlapping in a region and regular where they are defined. Explicitly, one section satisfying $H\widehat{\psi} = \omega_+\widehat{\psi}$ is given by

$$\widehat{\psi}_0(k_x, k_y) = \frac{1}{\sqrt{2}\sqrt{1-q(\mathbf{k})}} \begin{pmatrix} \frac{k_x - ik_y}{\omega_+(k)} \\ q(\mathbf{k}) - 1 \end{pmatrix}, \quad q(\mathbf{k}) := \frac{m - \epsilon k^2}{\omega_+}. \quad (4.7)$$

One can check that $\left\| \widehat{\psi}_0(k_x, k_y) \right\| = 1$. Notice that $\omega_+(k) \sim \epsilon k^2$ and $q \rightarrow -\epsilon/|\epsilon| = -1$ as $k \rightarrow \infty$. On the other hand, $\omega_+(k) \rightarrow m$ and

$$q(k) = 1 - \frac{k^2}{2m^2} + o(k^2)$$

as $k \rightarrow 0$. Thus, writing $k_x + ik_y = ke^{i\phi}$,

$$\lim_{k \rightarrow 0} \widehat{\psi}_0(k_x, k_y) = \begin{pmatrix} e^{-i\phi} \\ 0 \end{pmatrix}, \quad \lim_{k \rightarrow \infty} \widehat{\psi}_0(k_x, k_y) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \quad (4.8)$$

Therefore, ψ_0 is regular at $k = \infty$ but not at $k = 0$. There, it has a removable phase singularity up to a gauge transformation. We define $\psi_\infty = \lambda\psi_0$ with $\lambda = e^{i\phi} = k^{-1}(k_x + ik_y)$, or explicitly

$$\widehat{\psi}_\infty(k_x, k_y) = \frac{1}{k\sqrt{2}\sqrt{1-q(\mathbf{k})}} \begin{pmatrix} \frac{k^2}{\omega_+(k)} \\ (k_x + ik_y)(q(\mathbf{k}) - 1) \end{pmatrix}. \quad (4.9)$$

In particular, ψ_∞ is regular at $k = 0$ but not at $k = \infty$:

$$\lim_{k \rightarrow 0} \widehat{\psi}_\infty(k_x, k_y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lim_{k \rightarrow \infty} \widehat{\psi}_\infty(k_x, k_y) = \begin{pmatrix} 0 \\ -e^{i\phi} \end{pmatrix}.$$

Now for $k_x \in \mathbb{R}$ and $\kappa > 0$ consider $k_y = \kappa_{\text{ev}}(k_x, \kappa) \in i\mathbb{R}$, see (2.12). Define

$$\tilde{k}^2 = k_x^2 + \kappa_{\text{ev}}^2 = -(k_x^2 + \kappa^2) - \frac{1 - 2\epsilon m}{\epsilon^2} < 0,$$

as well as $\tilde{\omega}_+ = \sqrt{\tilde{k}^2 + (m - \epsilon\tilde{k}^2)^2}$ and $\tilde{q} = (m - \epsilon\tilde{k}^2)\tilde{\omega}_+^{-1}$. One can check that $\tilde{q} > 1$ so that $\tilde{k}^2(1 - \tilde{q}) > 0$ and thus $\tilde{k}\sqrt{1 - \tilde{q}}$ is well defined, and so is $\widehat{\psi}_\infty(k_x, \kappa_{\text{ev}}(k_x, \kappa))$. In particular

$$\lim_{(k_x, \kappa) \rightarrow \infty} \tilde{q} = 1, \quad \lim_{(k_x, \kappa) \rightarrow \infty} \tilde{k}\sqrt{1 - \tilde{q}} = \frac{1}{\sqrt{2}\epsilon},$$

so that

$$\lim_{(k_x, \kappa) \rightarrow \infty} \widehat{\psi}_\infty(k_x, \kappa_{\text{ev}}(k_x, \kappa)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Notice that $\kappa_{\text{ev}} \rightarrow i\epsilon^{-1}\sqrt{1 - 2\epsilon m} \neq 0$ as $(k_x, \kappa) \rightarrow 0$ so that $\widehat{\psi}_\infty(k_x, \kappa_{\text{ev}}(k_x, \kappa))$ is defined for all $\kappa > 0$ and $k_x \in \mathbb{R}$, including at ∞ . We then use $\widehat{\psi}_{\text{in}}(k_x, -\kappa) = \widehat{\psi}_0(k_x, -\kappa)$, $\widehat{\psi}_{\text{out}}(k_x, \kappa) = \widehat{\psi}_0(k_x, \kappa)$ and $\widehat{\psi}_{\text{ev}}(k_x, \kappa_{\text{ev}}) = \widehat{\psi}_\infty(k_x, \kappa_{\text{ev}}(k_x, \kappa))$ in the definition (2.13) of the scattering state. Then recall that

$$\widehat{\Psi}_0(k_x, \pm\kappa) = \begin{pmatrix} \widehat{\psi}_0(k_x, \pm\kappa) \\ \pm i\kappa \widehat{\psi}_0(k_x, \pm\kappa) \end{pmatrix}, \quad \widehat{\Psi}_\infty(k_x, \kappa_{\text{ev}}) = \begin{pmatrix} \widehat{\psi}_\infty(k_x, \kappa_{\text{ev}}) \\ i\kappa_{\text{ev}} \widehat{\psi}_\infty(k_x, \kappa_{\text{ev}}) \end{pmatrix}, \quad (4.10)$$

which provides a fully explicit expression for the scattering amplitude S . In particular, it is regular near infinity.

4.2 Expansion near ∞

In order to expand the scattering amplitude near ∞ , we focus on the expansion of g . The latter requires to expand $\widehat{\psi}_0(k_x, \kappa)$ and $\widehat{\psi}_\infty(k_x, \kappa_{\text{ev}})$ near ∞ . Such an expansion is a bit tricky. Even though the limits are single-valued, the next terms are not, as they are k_x, k_y dependent. Some of these terms are relevant in the expansion of g and have to be properly included. We shall then distinguish in each section the radial part, which depends only on the variable $k = \sqrt{k_x^2 + \kappa^2}$ and can be simply expanded, from the directional part which depends on k_x, κ and κ_{ev} . Thus we get

$$\widehat{\psi}_0(k_x, \kappa) = \begin{pmatrix} (k_x - i\kappa) \left(\frac{1}{2\epsilon k^2} + o\left(\frac{1}{k^3}\right) \right) \\ -1 + \frac{1}{8\epsilon k^2} + o\left(\frac{1}{k^3}\right) \end{pmatrix}$$

For the expansion of $\widehat{\psi}_\infty$, we first notice that \tilde{k} , \tilde{q} and $\tilde{\omega}_\pm$ are all functions of $k^2 = k_x^2 + \kappa^2$ only, so that they can be expanded as well as a radial part. The remaining terms depends on k_x and κ_{ev} . Explicitly we get

$$\widehat{\psi}_\infty(k_x, \kappa_{\text{ev}}) = \begin{pmatrix} 1 + \frac{1}{8\epsilon k^2} + o\left(\frac{1}{k^3}\right) \\ (k_x + i\kappa_{\text{ev}}) \left(-\frac{1}{2\epsilon k^2} + o\left(\frac{1}{k^3}\right) \right) \end{pmatrix}$$

Notice that $\kappa_{\text{ev}} \sim \sqrt{2k_x^2 + \kappa^2}$ as $k_x, \kappa \rightarrow \infty$. This way one has for example

$$\frac{k_x - i\kappa}{k^2} \rightarrow 0, \quad \frac{k_x + i\kappa_{\text{ev}}}{k^2} \rightarrow 0$$

as $k_x, \kappa \rightarrow \infty$, independent from the direction towards infinity, whereas expressions like $k_x(k_x - i\kappa)k^{-2}$ have a limit which depends on the (k_x, κ) direction. Such expressions naturally appear in the scattering amplitude through $\widehat{\Psi}_0$ and $\widehat{\Psi}_\infty$, see (4.10), which involve $\kappa\widehat{\psi}_0$ and $\kappa_{\text{ev}}\widehat{\psi}_\infty$, and through $A\widehat{\Psi}_0$ and $A\widehat{\Psi}_\infty$, which involve $k_x\widehat{\psi}_0$ and $k_x\widehat{\psi}_\infty$. Consequently, an expansion of the radial part of $\widehat{\psi}_0$ and $\widehat{\psi}_\infty$ up to order k^{-3} is required to get and expansion of g up to order 1 in the limit $k_x, \kappa \rightarrow \infty$. The non radial part of the expansion will be first kept as it is and specified later when passing to dual variables. Thus we will use

$$\widehat{\psi}_0(k_x, \kappa) = \begin{pmatrix} \frac{k_x - i\kappa}{2\epsilon k^2} \\ -1 + \frac{1}{8\epsilon k^2} \end{pmatrix} + o\left(\frac{1}{k^2}\right), \quad \widehat{\psi}_\infty(k_x, \kappa_{\text{ev}}) = \begin{pmatrix} 1 + \frac{1}{8\epsilon k^2} \\ -\frac{k_x + i\kappa_{\text{ev}}}{2\epsilon k^2} \end{pmatrix} + o\left(\frac{1}{k^2}\right) \quad (4.11)$$

4.3 Detecting anomalies

From the explicit expression of g from Proposition 7 we compute its expansion near ∞ using the expansion of $\widehat{\psi}_0$ and $\widehat{\psi}_\infty$ from the previous section and keeping all the terms that do not vanish in the limit:

$$g(k_x, \kappa) = g_\infty(k_x, k) + o(1), \quad (4.12)$$

with $o(1)$ in the limit $k_x^2 + \kappa^2 \rightarrow \infty$, independent from the direction. The exact expression for g_∞ depends on the boundary condition. The winding number of S near infinity is thus reduced to the study the winding phase of g_∞ computed along the curve $\Gamma_\delta(\lambda)$ with dual variables $\lambda_x \in [-\lambda, \lambda]$ and $\delta > 0$, see (2.18). Then consider

$$g_0(\lambda_x, \delta) = g_\infty\left(-\frac{\lambda_x}{\lambda_x^2 + \delta^2}, \frac{\delta}{\lambda_x^2 + \delta^2}\right). \quad (4.13)$$

Lemma 16. *Assume the existence of $M > 0$ as well as $\lambda_0 > 0$ and $\delta_0 > 0$ sufficiently small such that $|g_0| > M$ and g_0 is continuously differentiable on $(0, \delta_0] \times [-\lambda_0, \lambda_0]$. Then*

$$\lim_{\lambda \rightarrow 0} \lim_{\delta \rightarrow 0} \int_{\Gamma_\delta(\lambda)} g^{-1} dg = \lim_{\lambda \rightarrow 0} \lim_{\delta \rightarrow 0} \int_{-\lambda}^{\lambda} g_0^{-1} \partial_{\lambda_x} g_0 d\lambda_x$$

Proof. The winding of g reads

$$\int_{\Gamma_\delta(\lambda)} g^{-1} dg = \int_{-\lambda}^{\lambda} \tilde{g}^{-1} \partial_{\lambda_x} \tilde{g} d\lambda_x, \quad \tilde{g}(\lambda_x, \delta) = g\left(-\frac{\lambda_x}{\lambda_x^2 + \delta^2}, \frac{\delta}{\lambda_x^2 + \delta^2}\right).$$

Since $k_x^2 + \kappa^2 = (\lambda_x^2 + \delta^2)^{-1}$, one has $\tilde{g}(\lambda_x, \delta) = g_0(\lambda_x, \delta) + o(1)$ with $o(1)$ in the limit $\lambda_x^2 + \delta^2 \rightarrow 0$, independent from the direction. Thus we decompose

$$\tilde{g} = g_0 + R = g_0 \left(1 + \frac{R}{g_0}\right).$$

with $R(\lambda_x, \delta) \rightarrow 0$ as $\lambda_x^2 + \delta^2 \rightarrow 0$. Since $|g_0| > M$ on $(0, \delta_0] \times [-\lambda, \lambda]$ then $\frac{R}{g_0} \rightarrow 0$ as $\lambda_x^2 + \delta^2 \rightarrow 0$.

Moreover, the multiplicative property of winding phase gives

$$\int_{-\lambda}^{\lambda} \tilde{g}^{-1} \partial_{\lambda_x} \tilde{g} d\lambda_x = \int_{-\lambda}^{\lambda} g_0^{-1} \partial_{\lambda_x} g_0 d\lambda_x + \int_{-\lambda}^{\lambda} \left(1 + \frac{R}{g_0}\right)^{-1} \partial_{\lambda_x} \left(1 + \frac{R}{g_0}\right) d\lambda_x.$$

Let $\varepsilon > 0$ and δ_1, λ_1 such that $\left|\frac{R}{g_0}\right| < \varepsilon$ on $(0, \delta_1] \times [-\lambda_1, \lambda_1]$.

Consequently, for $|\lambda| < \min(\lambda_0, \lambda_1)$ and $\delta < \min(\delta_0, \delta_1)$ one has

$$\left|\arg\left(1 + \frac{R}{g_0}\right)\right| \leq \frac{\varepsilon}{1 - \varepsilon}$$

so that

$$\left|\int_{-\lambda}^{\lambda} \left(1 + \frac{R}{g_0}\right)^{-1} \partial_{\lambda_x} \left(1 + \frac{R}{g_0}\right) d\lambda_x\right| \leq \frac{2\varepsilon}{1 - \varepsilon}$$

Thus the winding phase of $\left(1 + \frac{R}{g_0}\right)$ goes to 0 as $\delta \rightarrow 0$ and $\lambda \rightarrow 0$. \square

One can check that the assumption of the lemma above is satisfied for all explicit g_0 below by checking that the limits of $|g_0|$ as $\delta \rightarrow 0$ or $\lambda \rightarrow 0$ are either finite and non vanishing or infinite. Thus, to compute w_∞ we can simply replace S by S_0 with

$$S_0(\lambda_x, \delta) = \frac{g_0(\lambda_x, -\delta)}{g_0(\lambda_x, \delta)}.$$

The next step is to extract the dominant scale from g_0 in the limit $\delta \rightarrow 0$ and $\lambda \rightarrow 0$. First, due to the expression of S_0 we can replace g_0 by any function that is even in δ , typically $g_0 \mapsto g_0 f$ with $f(\delta, \lambda) = (\lambda_x^2 + \delta^2)^2$ or $(\lambda_x^2 + \delta^2)$, see other examples below. We shall keep the same symbol for the simplified expression of g_0 .

Finally, we use the following proposition to extract the leading contribution from g_0 .

Proposition 17. *Let $g_0 : \mathbb{R} \times \mathbb{R}_+^* \rightarrow \mathbb{C}^*$ continuously differentiable. Let $s > 0$, $u = \frac{\lambda_x}{\delta^s}$ and $r > 0$. Decompose*

$$\delta^{-r} g_0(u\delta^s, \delta) = G(u) + R(u, \delta)$$

with $G : \mathbb{R} \rightarrow \mathbb{C}^*$ and $R : \mathbb{R} \times \mathbb{R}_+^* \rightarrow \mathbb{C}$ continuously differentiable and such that

1. For any $u \in \mathbb{R}$, $\lim_{\delta \rightarrow 0} R(u, \delta) = 0$,

2. For $\delta > 0$,

$$\frac{R(u, \delta)}{G(u)} \sim C u^\gamma \delta^{s\gamma}$$

as $u \rightarrow +\infty$, with $C \in \mathbb{C}^*$ independent from δ , and $\gamma > 0$.

3. Assumption 2 also holds as $u \rightarrow -\infty$ with possibly distinct constants \tilde{C} and $\tilde{\gamma}$.

Then,

$$\lim_{\lambda \rightarrow 0} \lim_{\delta \rightarrow 0} \int_{-\lambda}^{\lambda} g_0^{-1} \partial_{\lambda_x} g_0 d\lambda_x = \int_{-\infty}^{\infty} G^{-1}(u) G'(u) du. \quad (4.14)$$

Proof. First notice that for $\delta > 0$ the winding phase of g_0 and $\delta^{-r}g_0$ are the same. We change the integration variable λ_x to $u = \frac{\lambda_x}{\delta^s}$ and replace g_0 by $g_0\delta^{-r} = G+R$. Then we rewrite $G+R = G(1+\frac{R}{G})$ and use the additive property of winding phase:

$$\begin{aligned} \int_{-\lambda}^{\lambda} g_0^{-1} \partial_{\lambda_x} g_0 d\lambda_x &= \int_{-\frac{\lambda}{\delta^s}}^{\frac{\lambda}{\delta^s}} g_0^{-1} \partial_u g_0 du \\ &= \int_{-\frac{\lambda}{\delta^s}}^{\frac{\lambda}{\delta^s}} G^{-1}(u) G'(u) du + \int_{-\frac{\lambda}{\delta^s}}^{\frac{\lambda}{\delta^s}} \left(1 + \frac{R}{G}\right)^{-1}(u, \delta) \partial_u \left(1 + \frac{R}{G}\right)(u, \delta) du. \end{aligned} \quad (4.15)$$

The integrand of the first term does not depend on δ , so in the limit $\delta \rightarrow 0$ we get the right hand side of (4.14).

Then we claim that for all $\varepsilon > 0$ there exist $\delta_0 > 0$ and $\lambda_0 > 0$ such that

$$\forall u \in \left[-\frac{\lambda}{\delta^s}, \frac{\lambda}{\delta^s}\right], \quad \forall \delta < \delta_0, \quad \forall \lambda < \lambda_0, \quad \left| \frac{R(u, \delta)}{G(u)} \right| \leq \varepsilon,$$

which implies, similarly to the proof of the Lemma above, that the winding phase of $u \mapsto 1 + \frac{R}{G}$ is arbitrarily small for $u \in [-\frac{\lambda}{\delta^s}, \frac{\lambda}{\delta^s}]$ if δ, λ are sufficiently small. Consequently, the second term in the right hand side of (4.15) vanishes in the limit $\delta \rightarrow 0, \lambda \rightarrow 0$, and we get (4.14).

Let $\varepsilon > 0$. In order to prove the claim above, we split the study of the ratio by into the asymptotic and finite regions. First, by Assumption 2, there exists a $u_1 > 0$ such that,

$$\forall u > u_1, \quad \left| \frac{R(u, \delta)}{G(u)} \right| \leq (1 + \varepsilon) C u^\gamma \delta^{s\gamma}$$

Let $\delta_1 > 0$ such that $\frac{\lambda}{\delta_1^s} > u_1$. For $\delta < \delta_1$ and u such that $u_1 < u \leq \frac{\lambda}{\delta^s}$ one has

$$\left| \frac{R(u, \delta)}{G(u)} \right| \leq (1 + \varepsilon) C \lambda^\gamma,$$

so that for $\lambda < \lambda_1$ sufficiently small the right hand side is smaller than ε .

Similarly, at $-\infty$ there exists $u_2 > 0$ and $\delta_2 > 0$ such that for $\delta < \delta_2$ and u such that $-\frac{\lambda}{\delta^s} \leq u < -u_2$ one has

$$\left| \frac{R(u, \delta)}{G(u)} \right| \leq (1 + \varepsilon) \tilde{C} \lambda^{\tilde{\gamma}}.$$

So that for $\lambda < \lambda_2$ sufficiently small the right hand side is smaller than ε .

Finally, G and R are continuous on $[-u_2, u_1]$ and $G(u) \neq 0$. Let $M > 0$ such that $|G(u)| > M$ for $u \in [-u_2, u_1]$. Moreover, by Assumption 1 one has $R(u, \delta) \rightarrow 0$ as $\delta \rightarrow 0$. For $u \in [-u_2, u_1]$, this limit is uniform in δ . So there exists a $\delta_3 > 0$ such that

$$\forall 0 < \delta < \delta_3, \forall u \in [-u_2, u_1], \quad |R(u, \delta)| < \frac{\varepsilon}{M},$$

which implies

$$\forall 0 < \delta < \delta_3, \forall u \in [-u_2, u_1], \quad \left| \frac{R(u, \delta)}{G(u)} \right| < \varepsilon.$$

Putting all together, we get the claim if $\delta < \min(\delta_1, \delta_2, \delta_3)$ and $\lambda < \min(\lambda_1, \lambda_2)$. \square

This lemma drastically simplifies the computation of the winding number since the leading contribution G usually consists of only a few terms of the numerous ones of g_0 .

Corollary 18. *Assume*

$$\delta^{-r} g_0(u\delta^s, \delta) = G_+(u) + R_+(u, \delta), \quad \delta^{-r} g_0(u\delta^s, -\delta) = G_-(u) + R_-(u, \delta)$$

with G_{\pm}, R_{\pm} satisfying the assumption of Proposition 17. Then

$$w_{\infty} = \int_{-\infty}^{\infty} \mathcal{S}^{-1}(u) \mathcal{S}'(u) du, \quad \mathcal{S}(u) = \frac{G_-(u)}{G_+(u)}.$$

A detailed application of this result with a specific boundary condition is given in Section 4.4 below. Before to move on with the full classification, we need to anticipate one possible issue: It may occasionally happen for specific boundary conditions below that $G_+(u_0) = 0 = G_-(u_0)$ for some $u_0 \in \mathbb{R}$. In that case Proposition 17 does not apply for two reasons:

1. The winding phases of G_+ and G_- are ill-defined
2. $\frac{R_+}{G_+}$ and $\frac{R_-}{G_-}$ may be very large near u_0 .

However, if G_+ and G_- are such that the ratio $\mathcal{S} = \frac{G_-}{G_+}$ has a well defined the winding phase then point 1 is actually not an issue to compute w_{∞} . Thus we only need to check that R_- and R_+ do not contribute to the winding phase of \mathcal{S} in that case. The proposition below provides another criterion and generalizes Proposition 17.

Proposition 19. *Let $g_0 : \mathbb{R} \times \mathbb{R}_+^* \rightarrow \mathbb{C}^*$ continuously differentiable. Let $s > 0$, $u = \frac{\lambda x}{\delta^s}$ and $r > 0$. Decompose*

$$\delta^{-r} g_0(u\delta^s, \delta) = G_+(u) + R_+(u, \delta), \quad \delta^{-r} g_0(u\delta^s, -\delta) = G_-(u) + R_-(u, \delta)$$

with $G_+, G_- : \mathbb{R} \rightarrow \mathbb{C}$ and $R_+, R_- : \mathbb{R} \times \mathbb{R}_+^* \rightarrow \mathbb{C}$ continuously differentiables and such that Assumptions 1, 2 and 3 of Proposition 17 hold. Assume moreover that $G_+(u_0) = 0 = G_-(u_0)$ for some $u_0 \in \mathbb{R}$, but such that the ratios

$$\mathcal{S}(u) = \frac{G_-(u)}{G_+(u)}, \quad F(u, \delta) = \frac{1 + \frac{R_-(u, \delta)}{G_-(u)}}{1 + \frac{R_+(u, \delta)}{G_+(u)}}$$

remain continuously differentiables near u_0 and that

$$\lim_{\delta \rightarrow 0} \lim_{u \rightarrow u_0} F(u, \delta) = 1.$$

Then,

$$\lim_{\lambda \rightarrow 0} \lim_{\delta \rightarrow 0} \int_{-\lambda}^{\lambda} S_0^{-1} \partial_{\lambda_x} S_0 d\lambda_x = \int_{-\infty}^{\infty} \mathcal{S}^{-1}(u) \mathcal{S}'(u) du. \quad (4.16)$$

Proof. We proceed as in the proof of Proposition 17: multiply numerator and denominator by δ^{-r} , change the variable and split the integral:

$$\begin{aligned} \int_{-\lambda}^{\lambda} S_0^{-1} \partial_{\lambda_x} S_0 d\lambda_x &= \int_{-\frac{\lambda}{\delta^s}}^{\frac{\lambda}{\delta^s}} S_0^{-1} \partial_u S_0 du \\ &= \int_{-\frac{\lambda}{\delta^s}}^{\frac{\lambda}{\delta^s}} \mathcal{S}^{-1}(u) \mathcal{S}'(u) du + \int_{-\frac{\lambda}{\delta^s}}^{\frac{\lambda}{\delta^s}} F^{-1}(u, \delta) \partial_u F(u, \delta) du. \end{aligned} \quad (4.17)$$

The first term gives the expected result and we show that the second has no winding phase in the limit. We split as before between small and large u

$$\begin{aligned}
& \int_{-\frac{\lambda}{\delta^s}}^{\frac{\lambda}{\delta^s}} F^{-1}(u, \delta) \partial_u F(u, \delta) du \\
&= \int_M^{\frac{\lambda}{\delta^s}} \left(1 + \frac{R_+}{G_+}\right)^{-1} \partial_u \left(1 + \frac{R_+}{G_+}\right) du + \int_M^{\frac{\lambda}{\delta^s}} \left(1 + \frac{R_-}{G_-}\right)^{-1} \partial_u \left(1 + \frac{R_-}{G_-}\right) du \\
&+ \int_{-\frac{\lambda}{\delta^s}}^{-M} \left(1 + \frac{R_+}{G_+}\right)^{-1} \partial_u \left(1 + \frac{R_+}{G_+}\right) du + \int_{-\frac{\lambda}{\delta^s}}^{-M} \left(1 + \frac{R_-}{G_-}\right)^{-1} \partial_u \left(1 + \frac{R_-}{G_-}\right) du \\
&+ \int_{-M}^M F^{-1}(u, \delta) \partial_u F(u, \delta). \tag{4.18}
\end{aligned}$$

We treat the four first terms as before: the asymptotic properties are the same, so similarly to the proof of Proposition 17, we show that for any $\varepsilon > 0$ there exist δ and λ sufficiently small and some $M > |u_0|$ such that

$$|u| > M \quad \Rightarrow \quad \left| \frac{R_{\pm}(u, \delta)}{G_{\pm}(u)} \right| \leq \varepsilon$$

so that in these regions the associated winding phase of $1 + \frac{R_{\pm}}{G_{\pm}}$ vanish as $\delta \rightarrow 0$ and $\lambda \rightarrow 0$.

Finally, for the fifth term, we use the fact that F is continuous for all $u \in [-M, M]$ and that $F(u, \delta) \rightarrow 1$ and $\delta \rightarrow 0$. Moreover, the limit can be taken uniformly on $u \in [-M, M]$ so that $|F(u, \delta) - 1| \leq \varepsilon$ for all $u \in [-M, M]$ and δ, λ sufficiently small. Again, F has no winding phase in that region in the limit. \square

4.4 Case study

Example 20. We illustrate Lemma 16, Proposition 17 and Corollary 18 above in the following simple case of the classification where the boundary condition is

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_{21} & i\beta & 0 & 0 \end{pmatrix} := (B \ 0), \tag{4.19}$$

with $b_{12} \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$, which is part of class $\mathfrak{A}_{1,4}$. We further assume $\beta \neq 0$ here. Notice that for $b_{12} = \alpha = 0$ and $\beta = -1$ we recover condition (b) from Examples 5 and 10. Inserting (4.11) and (4.19) into (2.17) we get

$$\begin{aligned}
g(k_x, \kappa) &= \frac{1}{64k^4 \epsilon^4} \left(-\alpha - i\kappa + \beta k_x + 64k^4 \epsilon^4 (\alpha + i\kappa - \beta k_x) \right. \\
&\quad \left. + 16\epsilon^2 (\kappa + ik_x) (\kappa_{\text{ev}} - ik_x) (\alpha + i\kappa_{\text{ev}} - \beta k_x) \right) + o(1) \tag{4.20}
\end{aligned}$$

Many terms vanish in the limit $k \rightarrow \infty$ so this further simplifies to $g(k_x, \kappa) = g_{\infty}(k_x, \kappa) + o(1)$ with

$$g_{\infty}(k_x, \kappa) = \alpha + i\kappa - ik_x \beta.$$

Passing to dual variables we get

$$g_0(\lambda_x, \delta) = g_{\infty} \left(-\frac{\lambda_x}{\lambda_x^2 + \delta^2}, \frac{\delta}{\lambda_x^2 + \delta^2} \right) = \frac{\alpha \lambda_x^2 + \beta \lambda_x + \alpha \delta^2 + i\delta}{\delta^2 + \lambda_x^2},$$

and one can check that Lemma 16 applies. Then as discussed above, the denominator does not contribute to the winding phase so we replace g_0 by $(\delta^2 + \lambda_x^2)g_0$ (and keep the same symbol for g_0), leading to

$$g_0(\lambda_x, \delta) = \alpha\lambda_x^2 + \beta\lambda_x + \alpha\delta^2 + i\delta \quad (4.21)$$

Now, the dominant scale is given by $s = r = 1$, namely

$$\delta^{-1}g(u\delta, \pm\delta) = G_{\pm}(u) + R_{\pm}(u, \delta)$$

with

$$G_+(u) = i + \beta u, \quad G_-(u) = -i + \beta u, \quad R_+(u) = R_-(u) = \alpha\delta(u^2 + 1) \quad (4.22)$$

One can check that Proposition 17 applies. In particular, since we assume $\beta \neq 0$

$$\frac{R_{\pm}}{G_{\pm}} \underset{u \rightarrow +\infty}{\sim} \frac{\alpha}{\beta} u\delta,$$

namely $\gamma = 1$, and similarly as $u \rightarrow -\infty$. Thus we are left with the winding number of

$$\mathcal{S} = \frac{G_-(u)}{G_+(u)} = \frac{-i + \beta u}{i + \beta u}$$

which can be computed as twice the winding phase of $G_-(u)$ since $G_+(u) = G_-^*(u)$. The real part of G_- is constant negative so that the winding phase is simply computed by the limits of $\arg(G_-(u))$ as $u \rightarrow \pm\infty$. For $\beta > 0$, the winding phase of G_- is π and for $\beta < 0$ it is $-\pi$. Thus in that example we infer $w_{\infty} = \text{sign}(\beta)$, regardless of the values of α and b_{21} . The relevant quantities of this example are illustrated in Figure 2 below.

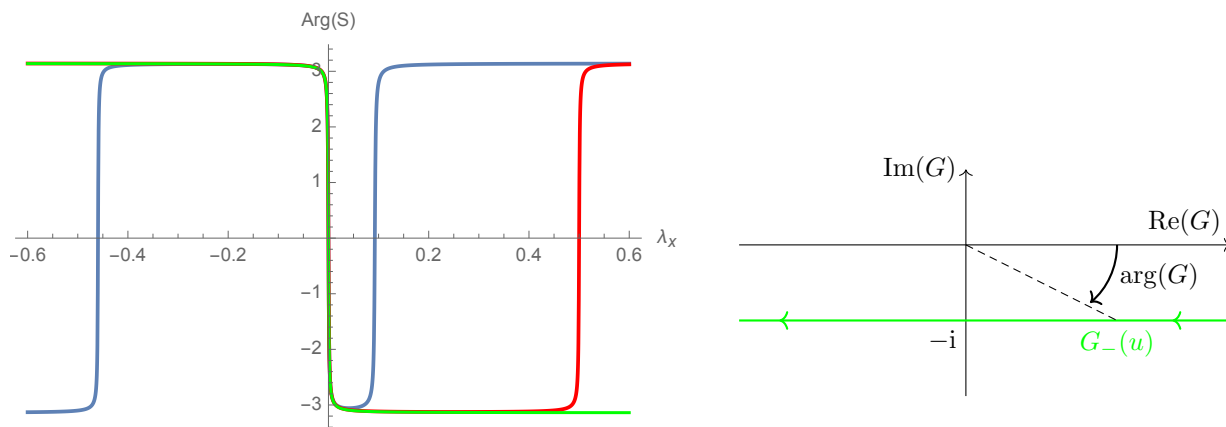


Figure 2: Argument of the scattering amplitude near $\lambda_x = 0$ for $\alpha = 2$, $\beta = -1$ and $b_{12} = 0$. Moreover, $m = 1$, $\epsilon = 0.1$ and $\delta = 0.001$. (Left) Argument of S with respect to λ_x . The blue curve corresponds to the exact S , which winds at $\lambda_x = 0$ but also at near -0.5 and 0.1 , detecting edge modes at finite $k_x \approx 2$ and -10 (compare with Figure 1, middle). The red curve corresponds to S_0 , computed with g_0 , which approximates g well near 0 but also keeps track (poorly) of unwanted finite edge modes. The green curve corresponds to \mathcal{S} computed with G_{\pm} (actually $\delta G_{\pm}(\lambda_x/\delta)$ here, to compare with other curves), which exactly captures only the winding of S near infinity. (Right) Complex curve $G_-(u) = -i + \beta u$ for $\beta < 0$. Its argument changes by $-\pi$ as u goes from $-\infty$ to $+\infty$.

5 Anomaly classification

General strategy. We apply the statements from Section 4.3 to each class of self-adjoint boundary conditions from Table 1, going from g to its leading contribution G , and inferring w_∞ . More precisely:

1. Pick a matrix $A = A_0 + ik_x A_1$ from Table 1 and insert it in expression (2.17) for g , together with the asymptotic expansion of the bulk eigensections (4.11). This leads to $g(k_x, \kappa) = g_\infty(k_x, k) + o(1)$. We shall not detail this step below and give only the final expression.
2. Pass from (k_x, κ) to dual variables via (4.13), compute $g_0(\lambda_x, \delta)$ and simplify it.
3. Find the dominant scale $\delta^{-r} g_0(u\delta^s, \pm\delta) = G_\pm(u) + R_\pm(u)$ and check that G_\pm and R_\pm satisfy either the assumptions of Proposition 17 or that of Proposition 19. We shall not always give the explicit expressions of R_\pm but they can be inferred from g_0 and G_\pm .
4. Study the complex curves G_\pm and compute their winding phases, from which we deduce w_∞ .
5. For specific values of the parameters in A , Assumption 2 of Proposition 17 may fail to hold so that another scale has to be found instead. This leads to various subcases for which we repeat steps 4 and 5.

5.1 Class $\mathfrak{A}_{1,2}$

In that case we have

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \end{pmatrix} := (B \ 0),$$

where $B \in M_2(\mathbb{C})$ is arbitrary. The expansion of g near ∞ leads to $g(k_x, \kappa) = g_\infty(k_x, k) + o(1)$ where $o(1)$ in the limit $k_x^2 + \kappa^2 \rightarrow \infty$, independent from the direction, and with

$$g_\infty(k_x, \kappa) = -\det(B)k_x^2 + i \operatorname{tr}(B)k_x + \det(B) \frac{k_x^2(k_x^2 + \kappa\kappa_{\text{ev}} + ik_x(\kappa_{\text{ev}} - \kappa))}{4\epsilon^2 k^4} + 1. \quad (5.1)$$

We then move to the dual variables λ_x, δ via (4.13), leading to

$$\begin{aligned} g_0(\lambda_x, \delta) &= \frac{1}{4\epsilon^2(\delta^2 + \lambda_x^2)^2} \left(4\delta^4\epsilon^2 + i\delta \det(B)\lambda_x^2\sqrt{\delta^2 + 2\lambda_x^2} + \det(B)\lambda_x^3\sqrt{\delta^2 + 2\lambda_x^2} \right. \\ &\quad \left. + i\delta \det(B)\lambda_x^3 + \det(B)\lambda_x^4 - 4\det(B)\epsilon^2\lambda_x^2 - 4i\delta^2 \operatorname{tr}(B)\epsilon^2\lambda_x \right. \\ &\quad \left. - 4i \operatorname{tr}(B)\epsilon^2\lambda_x^3 + 8\delta^2\epsilon^2\lambda_x^2 + 4\epsilon^2\lambda_x^4 \right) \end{aligned} \quad (5.2)$$

Case $\det B = 0$. In that case one has $g(\lambda_x, \delta) = g_0(\lambda_x, -\delta)$ so that $S_0 = 1$ and $w_\infty = 0$.

Case $\det B \neq 0$. The denominator does not contribute to w_∞ so we replace g_0 by $(\delta^2 + \lambda_x^2)^2 g_0$, and we extract the dominant scale by considering

$$\delta^{-4} g_0(u\delta^2, \delta) = G_+(u) + R_+(u, \delta)$$

with

$$G_+(u) = 1 - i \operatorname{tr}(B)u - \det(B)u^2, \quad (5.3)$$

$$\begin{aligned} R_+(u, \delta) &= \frac{\delta^4 \det(B)u^4}{4\epsilon^2} + \frac{i\delta^3 \det(B)u^3}{4\epsilon^2} + \frac{\delta^2 \det(B)u^3\sqrt{\delta^2 + 2\delta^4 u^2}}{4\epsilon^2} \\ &\quad + \frac{i\delta \det(B)u^2\sqrt{\delta^2 + 2\delta^4 u^2}}{4\epsilon^2} - i\delta^2 \operatorname{tr}(B)u^3 + \delta^4 u^4 + 2\delta^2 u^2. \end{aligned} \quad (5.4)$$

One has $R_+(u, 0) = 0$ and

$$\frac{R(u, \delta)}{G(u)} \sim -\frac{(1 \pm \sqrt{2}) \det(B) + 4\epsilon^2}{4 \det(B) \epsilon^2} u^2 \delta^4$$

as $u \rightarrow \pm\infty$, so that Proposition 17 applies³ when G_+ has no zero.

Similarly, replacing δ by $-\delta$ in (5.2) one gets $\delta^{-4} g_0(u\delta^2, -\delta) = G_-(u) + R_-(u, \delta)$ with

$$G_-(u) = G_+(u), \tag{5.5}$$

$$R_-(u, \delta) = \frac{\delta^4 \det(B) u^4}{4\epsilon^2} - \frac{i\delta^3 \det(B) u^3}{4\epsilon^2} + \frac{\delta^2 \det(B) u^3 \sqrt{\delta^2 + 2\delta^4 u^2}}{4\epsilon^2} - \frac{i\delta \det(B) u^2 \sqrt{\delta^2 + 2\delta^4 u^2}}{4\epsilon^2} - i\delta^2 \operatorname{tr}(B) u^3 + \delta^4 u^4 + 2\delta^2 u^2. \tag{5.6}$$

and one can check that Proposition 17 also applies. In particular

$$\mathcal{S}(u) = \frac{G_-(u)}{G_+(u)} = 1$$

so that $w_\infty = 0$.

Remark 21 (About vanishing cases). G_+ and G_- may occasionally vanish at some $u_0 \in \mathbb{R}$ for exceptional values of B , but w_∞ remains the same. Let $\det(B) = d_r + id_i$ and $\operatorname{tr}(B) = t_r + it_i$ with d_r, d_i, t_r and $t_i \in \mathbb{R}$. One has

$$\operatorname{Re}(G_+(u)) = 1 + t_i u - d_r u^2, \quad \operatorname{Im}(G_+(u)) = -u(t_r + u d_i).$$

The function G_+ vanishes in the following three cases :

1. If $d_r = 0, t_i \neq 0, d_i \neq 0$ and $d_i = -t_r t_i$ at $u_0 = -\frac{1}{t_i}$
2. If $d_r \neq 0, \Delta = t_i^2 + 4d_r > 0$ and

$$(a) \quad t_r = d_i = 0 \text{ at } u_{0\pm} = \frac{t_i \pm \sqrt{\Delta}}{2d_r},$$

$$(b) \quad d_i \neq 0 \text{ with } 2t_r d_r = -d_i(t_i + \sqrt{\Delta}) \text{ or } 2t_r d_r = -d_i(t_i - \sqrt{\Delta}), \text{ at } u_0 = -\frac{t_r}{d_i}.$$

In all cases above one could check that Proposition 19 applies. However, one still has $G_+ = G_-$, so we can instead replace g_0 by $g_0 + z_0$ for some $z_0 \in \mathbb{C}^*$ with $|z_0|$ sufficiently small so that it does not change the winding phase of g_0 . The computation above leads to

$$g_0(\lambda_x, \delta) + z_0 = G_+(u) + z_0 + R_+(u, \delta), \quad g_0(\lambda_x, -\delta) + z_0 = G_-(u) + z_0 + R_-(u, \delta)$$

with again $G_-(u) = G_+(u)$. Moreover, z_0 can be chosen such that $G_+(u) + z_0 \in \mathbb{C}^*$ for all $u \in \mathbb{R}$ and, consequently, $G_-(u) + z_0 \in \mathbb{C}^*$. The asymptotic properties are preserved as $u \rightarrow \pm\infty$ so that Proposition 17 now applies to $(G_+ + z_0, R_+)$ and $(G_- + z_0, R_-)$. Thus $\mathcal{S} = \frac{G_- + z_0}{G_+ + z_0} = 1$ and we get $w_\infty = 0$ also in that case.

Summarizing. Thus S does not wind and $w_\infty = 0$ for any boundary condition in this class.

³One can check that Proposition 17 also applies in the fine-tuned case where $\det(B) = -\frac{1 \pm \sqrt{2}}{4\epsilon^2}$, then a lower order computation leads to $\frac{R(u, \delta)}{G(u)} \sim -\frac{i(\sqrt{2}+1)\delta^2 u(\delta + \operatorname{tr}(B))}{4\epsilon^2}$

5.2 Class $\mathfrak{A}_{1,4}$

In this case we have

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & i\beta & 0 & 0 \end{pmatrix} := (B \ 0),$$

with $b_{11}, b_{21} \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$. We get $g(k_x, \kappa) = g_\infty(k_x, \kappa) + o(1)$ with

$$g_\infty(k_x, \kappa) = -\det(B)k_x^2 - b_{11}k_x\kappa + i\kappa + ik_x(\alpha b_{11} + i\beta) + \alpha \\ + \det(B) \frac{k_x^2(k_x^2 + \kappa\kappa_{\text{ev}} + ik_x(\kappa_{\text{ev}} - \kappa))}{4\epsilon^2 k^4} + b_{11} \frac{k_x\kappa_{\text{ev}}(k_x^2 + \kappa\kappa_{\text{ev}} + ik_x(\kappa_{\text{ev}} - \kappa))}{4k^4\epsilon^2}. \quad (5.7)$$

Passing to dual variables via (4.13) we get

$$g_0(\lambda_x, \delta) = \frac{1}{4\epsilon^2(\lambda_x^2 + \delta^2)^2} \left(4\alpha\delta^4\epsilon^2 - 4i\alpha b_{11}\delta^2\epsilon^2\lambda_x - 4i\alpha b_{11}\epsilon^2\lambda_x^3 + b_{11}\delta^3\lambda_x - ib_{11}\delta^2\lambda_x^2 \right. \\ \left. + b_{11}\delta\lambda_x^2\sqrt{\delta^2 + 2\lambda_x^2} - ib_{11}\lambda_x^3\sqrt{\delta^2 + 2\lambda_x^2} + 2b_{11}\delta\lambda_x^3 + 4b_{11}\delta\epsilon^2\lambda_x - 2ib_{11}\lambda_x^4 \right. \\ \left. + 4i\delta^3\epsilon^2 + i\delta\det(B)\lambda_x^2\sqrt{\delta^2 + 2\lambda_x^2} + \det(B)\lambda_x^3\sqrt{\delta^2 + 2\lambda_x^2} + i\delta\det(B)\lambda_x^3 + \det(B)\lambda_x^4 \right. \\ \left. - 4\det(B)\epsilon^2\lambda_x^2 + 8\alpha\delta^2\epsilon^2\lambda_x^2 + 4\alpha\epsilon^2\lambda_x^4 + 4\beta\delta^2\epsilon^2\lambda_x + 4\beta\epsilon^2\lambda_x^3 + 4i\delta\epsilon^2\lambda_x^2 \right). \quad (5.8)$$

The denominator does not contribute to w_∞ so we replace g_0 by $(\delta^2 + \lambda_x^2)^2 g_0$. Then the dominant scale of g_0 depends on some parameter vanishing or not.

Case $\det(B) \neq 0$. Since $\det(B) = i\beta b_{11}$ one has $b_{11} \neq 0$ and $\beta \neq 0$. The dominant scale is

$$\delta^{-2}g_0(u\delta, \pm\delta) = G_\pm(u) + R_\pm(u)$$

with

$$G_+(u) = ub_{11}(1 - i\beta u), \quad G_-(u) = -ub_{11}(1 + i\beta u). \quad (5.9)$$

G_+ and G_- vanish both at $u = 0$ but one can check that Proposition 19 applies, in particular that the ratio $F(u, \delta) \rightarrow 1$ as $u \rightarrow 0$ and $\delta \rightarrow 0$. Thus we are left with the winding number of

$$\mathcal{S}(u) = \frac{G_-(u)}{G_+(u)} = -\frac{1 + i\beta u}{1 - i\beta u} = \frac{i - \beta u}{i + \beta u} \quad (5.10)$$

Numerator and denominator are conjugated so w_∞ is twice the winding phase of the numerator, which has constant positive imaginary part. Its winding phase is the difference of limit of argument between $+\infty$ and $-\infty$. If $\beta > 0$ this winding phase is π . If $\beta < 0$ it is $-\pi$. Thus we infer $w_\infty = \text{sign}(\beta)$.

Case $b_{11} = 0$ and $\beta \neq 0$. This case has been completely treated already in Example 20, and also leads to $w_\infty = \text{sign}(\beta)$.

Case $b_{11} \neq 0$ and $\beta = 0$. In that case $\det(B) = 0$ and we go back to expression (5.8) for g_0 and replace it again by $(\delta^2 + \lambda_x^2)^2 g_0$. The dominant scale is

$$\delta^{-3/2}g_0(u\delta^{1/2}, \pm\delta) = G_\pm(u) + R_\pm(u)$$

with

$$G_+(u) = b_{11}u(1 - i\alpha u^2), \quad G_-(u) = b_{11}u(-1 - i\alpha u^2) = -G_+^*(u) \quad (5.11)$$

G_+ and G_- vanish both at $u = 0$ but one can check that Proposition 19 applies, in particular that the ratio $F(u, \delta) \rightarrow 1$ as $u \rightarrow 0$ and $\delta \rightarrow 0$. Thus we are left with the winding number of

$$\mathcal{S}(u) = \frac{G_-(u)}{G_+(u)} = \frac{1 - i\alpha u^2}{1 + i\alpha u^2}.$$

Notice that $\text{Re}(1 - i\alpha u^2) = 1 > 0$ so that its argument goes continuously from $\pi/2$ to $\pi/2$ as u goes from $-\infty$ to ∞ . Consequently, its winding phase vanishes, and similarly for $1 + i\alpha u^2$. Thus $w_\infty = 0$.

Case $b_{11} = 0$ and $\beta = 0$ In that case we go back to (5.7) which simplifies to $g_\infty(k_x, \kappa) = \alpha + i\kappa$ leading to

$$g_0(\lambda_x, \delta) = \frac{\alpha\lambda_x^2 + \alpha\delta^2 + i\delta}{\delta^2 + \lambda_x^2},$$

Replacing g_0 by $g_0(\delta^2 + \lambda_x^2)$, the dominant scale is

$$\delta^{-1}g(u\delta^{1/2}, \pm\delta) = G_\pm(u) + R_\pm(u, \delta)$$

with $G_\pm(u) = i \pm u^2\alpha$ and $R_\pm(u, \delta) = \pm\alpha\delta$. Notice that Proposition 17 does not apply here but the situation is simpler since R_\pm do not depend on u and G_\pm do not vanish hence one has $\left|\frac{R_\pm}{G_\pm}\right| \rightarrow 0$ as $\delta \rightarrow 0$, for any $\lambda > 0$. Thus we are left with the winding number of

$$\mathcal{S}(u) = \frac{G_-(u)}{G_+(u)} = \frac{i - \alpha u^2}{i + \alpha u^2}.$$

which is 0 as discussed in the case above. Consequently $w_\infty = 0$.

Summarizing. In class $\mathfrak{A}_{1,4}$ one has $w_\infty = \text{sign}(\beta)$ if $\beta \neq 0$ and $w_\infty = 0$ if $\beta = 0$.

5.3 Class $\mathfrak{A}_{2,3}$

In this case we have

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & b_{12} & 0 & 0 \\ i\beta & b_{22} & 0 & 0 \end{pmatrix} := (B \ 0),$$

with $b_{12}, b_{22} \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$. We get $g(k_x, \kappa) = g_\infty(k_x, \kappa) + o(1)$ with

$$\begin{aligned} g_\infty(k_x, \kappa) &= -\alpha + k_x(\beta - i\alpha b_{12}) - \det(B)k_x^2 \\ &\quad + i\det(B)\frac{k_x^2(k_x - i\kappa)(\kappa_{\text{ev}} - ik_x)}{4k^4\epsilon^2} - \frac{i(\kappa_{\text{ev}} - \kappa)(b_{12}k_x - i)(\kappa_{\text{ev}} - ik_x)}{2k^2\epsilon} \end{aligned} \quad (5.12)$$

Passing to dual variables via (4.13) we get

$$\begin{aligned} g_0(\lambda_x, \delta) &= \frac{1}{4\epsilon^2(\lambda_x^2 + \delta^2)^2} \left(-4\alpha\delta^4\epsilon^2 + 4i\alpha b_{12}\delta^2\epsilon^2\lambda_x + 4i\alpha b_{12}\epsilon^2\lambda_x^3 - 2ib_{12}\delta^2\epsilon\lambda_x \right. \\ &\quad + 2b_{12}\delta\epsilon\lambda_x\sqrt{\delta^2 + 2\lambda_x^2} - 2ib_{12}\epsilon\lambda_x^2\sqrt{\delta^2 + 2\lambda_x^2} + 2b_{12}\delta\epsilon\lambda_x^2 - 4ib_{12}\epsilon\lambda_x^3 + 2\delta^4\epsilon \\ &\quad + i\delta\det(B)\lambda_x^2\sqrt{\delta^2 + 2\lambda_x^2} + \det(B)\lambda_x^3\sqrt{\delta^2 + 2\lambda_x^2} + i\delta\det(B)\lambda_x^3 + \det(B)\lambda_x^4 - 8\alpha\delta^2\epsilon^2\lambda_x^2 \\ &\quad - 4\det(B)\epsilon^2\lambda_x^2 - 4\alpha\epsilon^2\lambda_x^4 - 4\beta\delta^2\epsilon^2\lambda_x - 4\beta\epsilon^2\lambda_x^3 + 2i\delta^3\epsilon\sqrt{\delta^2 + 2\lambda_x^2} + 2i\delta^3\epsilon\lambda_x + 6\delta^2\epsilon\lambda_x^2 \\ &\quad \left. + 2\delta^2\epsilon\lambda_x\sqrt{\delta^2 + 2\lambda_x^2} + 2i\delta\epsilon\lambda_x^2\sqrt{\delta^2 + 2\lambda_x^2} + 2\epsilon\lambda_x^3\sqrt{\delta^2 + 2\lambda_x^2} + 2i\delta\epsilon\lambda_x^3 + 4\epsilon\lambda_x^4 \right) \end{aligned} \quad (5.13)$$

The denominator does not contribute to w_∞ so we replace g_0 by $2(\delta^2 + \lambda_x^2)^2 g_0$. Then the dominant scale of g_0 depends on some parameter vanishing or not.

Case $\det(B) \neq 0$. Since $\det(B) = -i\beta b_{12}$ then $\beta \neq 0$ and $b_{12} \neq 0$. The dominant scale is

$$\delta^{-4} g_0(u\delta^2, \pm\delta) = G_\pm(u) + R_\pm(u)$$

with

$$\begin{aligned} G_+(u) &= (1+i) - 2\alpha\epsilon + (2iab_{12}\epsilon - 2\beta\epsilon + (1-i)b_{12})u - 2\epsilon \det(B)u^2, \\ G_-(u) &= (1-i) - 2\alpha\epsilon + (2iab_{12}\epsilon - 2\beta\epsilon - (1+i)b_{12})u - 2\epsilon \det(B)u^2. \end{aligned} \quad (5.14)$$

The two expressions above factorize to

$$\begin{aligned} G_+(u) &= ((1+i)b_{12}u + (-1+i))((1+i)\alpha\epsilon + (1+i)\beta u\epsilon - 1), \\ G_-(u) &= ((1+i)b_{12}u + (-1+i))((1+i)\alpha\epsilon + (1+i)\beta u\epsilon - i). \end{aligned} \quad (5.15)$$

The first factor is common to G_+ and G_- and vanishes at $u_0 = -i/b_{12}$. The second factors do not vanish for $u \in \mathbb{R}$. Thus if $b_{12} \notin i\mathbb{R}$ then Proposition 17 applies and otherwise Proposition 19 applies, as long as $\det(B) \neq 0$. In both cases the ratio simplifies to

$$\frac{G_-(u)}{G_+(u)} = \frac{2\epsilon\beta u + 2\alpha\epsilon - 1 + i}{2\epsilon\beta u + 2\alpha\epsilon - 1 - i}. \quad (5.16)$$

Finally, the map $u \rightarrow 2\epsilon\beta u + 2\alpha\epsilon - 1 + i$ winds by $-\text{sign}(\beta)\pi$ as y goes from $-\infty$ to $+\infty$ whereas $u \rightarrow 2\epsilon\beta u + 2\alpha\epsilon - 1 - i$ winds by $\text{sign}(\beta)\pi$, so that

$$w_\infty = -\text{sign}(\beta).$$

Case $\beta = 0$ and $b_{12} \neq 0$. The dominant scale is

$$\delta^{-3} g_0(u\delta, \pm\delta) = G_\pm(u) + R_\pm(u)$$

with

$$\begin{aligned} G_+(u) &= b_{12}u(u+i) \left(2\alpha\epsilon - 1 - i(2u(1-\alpha\epsilon) - \sqrt{1+2u^2}) \right) \\ G_-(u) &= b_{12}u(u-i) \left(2\alpha\epsilon - 1 + i(2u(1-\alpha\epsilon) - \sqrt{1+2u^2}) \right) = G_+^*(u). \end{aligned} \quad (5.17)$$

These two functions vanish at $u = 0$, and we can check that Proposition 19 applies when $b_{12} \neq 0$. Thus we are left with studying the winding phase of

$$G(u) = (u-i) \left(2\alpha\epsilon - 1 + i(2u(1-\alpha\epsilon) - \sqrt{1+2u^2}) \right) := (u-i)\tilde{G}(u).$$

The first factor always winds by π . As for the second factor $\tilde{G}(u)$, its real part is constant. Moreover, one has

$$\tilde{G}(u) \underset{+\infty}{\sim} i(2(1-\alpha\epsilon) + \sqrt{2})u, \quad \tilde{G}(u) \underset{-\infty}{\sim} i(2(1-\alpha\epsilon) - \sqrt{2})u.$$

Thus if $\alpha < \frac{1}{\epsilon} \left(1 - \frac{1}{\sqrt{2}}\right)$ or $\alpha > \frac{1}{\epsilon} \left(1 + \frac{1}{\sqrt{2}}\right)$ one can check that \tilde{G} winds by $-\pi$, so that G does not wind. Otherwise, if

$$\frac{1}{\epsilon} \left(1 - \frac{1}{\sqrt{2}}\right) < \alpha < \frac{1}{\epsilon} \left(1 + \frac{1}{\sqrt{2}}\right)$$

then \tilde{G} does not wind, so that G winds by π . To summarize one has

$$w_\infty = \begin{cases} 1 & \text{if } \frac{1}{\epsilon} \left(1 - \frac{1}{\sqrt{2}}\right) < \alpha < \frac{1}{\epsilon} \left(1 + \frac{1}{\sqrt{2}}\right), \\ 0 & \text{if } \alpha < \frac{1}{\epsilon} \left(1 - \frac{1}{\sqrt{2}}\right) \quad \text{or} \quad \alpha > \frac{1}{\epsilon} \left(1 + \frac{1}{\sqrt{2}}\right). \end{cases}$$

Case $\beta = 0$ and $b_{12} = 0$. The dominant scale is

$$\delta^{-4}g_0(u\delta, \pm\delta) = G_{\pm}(u) + R_{\pm}(u)$$

with

$$\begin{aligned} G_{-}(u) = & 1 - 2\alpha\epsilon - 2\alpha u^4\epsilon - 4\alpha u^2\epsilon + 2u^4 + u^3\sqrt{2u^2 + 1} + 3u^2 + u\sqrt{2u^2 + 1} \\ & + i\left(-u^3 - u^2\sqrt{2u^2 + 1} - \sqrt{2u^2 + 1} - u\right) \end{aligned} \quad (5.18)$$

and $G_{+}(u) = G_{-}^{*}(u)$. It is worth noticing here that

$$R_{+} = R_{-} = 0$$

so that Proposition 17 is not even required. $G_{-}(u)$ never vanishes and its imaginary part is always strictly negative. Moreover,

$$G(u) \underset{+\infty}{\sim} (2(1 - \alpha\epsilon) + \sqrt{2})u^4, \quad G(u) \underset{-\infty}{\sim} (2(1 - \alpha\epsilon) - \sqrt{2})u^4$$

A similar analysis may be performed as in the previous case, and we end up again with

$$w_{\infty} = \begin{cases} 1 & \text{if } \frac{1}{\epsilon}\left(1 - \frac{1}{\sqrt{2}}\right) < \alpha < \frac{1}{\epsilon}\left(1 + \frac{1}{\sqrt{2}}\right), \\ 0 & \text{if } \alpha < \frac{1}{\epsilon}\left(1 - \frac{1}{\sqrt{2}}\right) \quad \text{or} \quad \alpha > \frac{1}{\epsilon}\left(1 + \frac{1}{\sqrt{2}}\right). \end{cases}$$

Case $b_{12} = 0$ and $\beta \neq 0$. In that case we go back to

$$g_{\infty}(k_x, \kappa) = -\alpha + k_x\beta - \frac{(\kappa_{\text{ev}} - \kappa)(\kappa_{\text{ev}} - ik_x)}{2k^2\epsilon} \quad (5.19)$$

Passing to dual variables via (4.13) we get

$$g_0(\lambda_x, \delta) = \frac{-2\alpha\delta^2\epsilon + \delta^2 - 2\alpha\epsilon\lambda_x^2 - 2\beta\epsilon\lambda_x + i\delta\sqrt{\delta^2 + 2\lambda_x^2} + \lambda_x\sqrt{\delta^2 + 2\lambda_x^2} + i\delta\lambda_x + 2\lambda_x^2}{2\epsilon(\lambda_x^2 + \delta^2)}. \quad (5.20)$$

The denominator does not contribute to w_{∞} so we replace g_0 by $2(\delta^2 + \lambda_x^2)\epsilon g_0$. The dominant scale is

$$\delta^{-2}g_0(u\delta^2, \pm\delta) = G_{\pm}(u) + R_{\pm}(u)$$

with

$$G_{+}(u) = -2\alpha\epsilon - 2\beta u\epsilon + 1 + i, \quad G_{-}(u) = G_{+}^{*}(u).$$

Proposition 17 applies and we infer that G_{-} has a winding phase of $-\pi \text{sign}(\beta)$ so that

$$w_{\infty} = -\text{sign}(\beta).$$

Summarizing. If $\beta \neq 0$ then $w_{\infty} = -\text{sign}(\beta)$. If $\beta = 0$ then

$$w_{\infty} = \begin{cases} 1 & \text{if } \frac{1}{\epsilon}\left(1 - \frac{1}{\sqrt{2}}\right) < \alpha < \frac{1}{\epsilon}\left(1 + \frac{1}{\sqrt{2}}\right), \\ 0 & \text{if } \alpha < \frac{1}{\epsilon}\left(1 - \frac{1}{\sqrt{2}}\right) \quad \text{or} \quad \alpha > \frac{1}{\epsilon}\left(1 + \frac{1}{\sqrt{2}}\right). \end{cases}$$

Remark 22. At the threshold cases $\beta = 0$ and $\alpha = \frac{1}{\epsilon}(1 \pm \frac{1}{\sqrt{2}})$, one can check that the winding phase of G_{+} is not a multiple of π so that $w_{\infty} \notin \mathbb{Z}$. We suspect some collapse of edge mode branch occurring at infinity, similarly to the case $a = \pm\sqrt{2}$ in [20]. We do not investigate further this fine-tuned case and consider it out of the w_{∞} -classification.

5.4 Class $\mathfrak{A}_{2,4}$

In this case we have

$$A_0 = \begin{pmatrix} a_{11} & 1 & 0 & 0 \\ a_{21} & 0 & (a_{11}^*)^{-1} & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} b_{11} & b_{11}(a_{11})^{-1} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \end{pmatrix} := (B \ 0),$$

with $a_{11} \in \mathbb{C}^*$, $b_{11}, b_{21}, b_{22} \in \mathbb{C}$, such that $a_{21} = \alpha a_{11} + \epsilon^{-1}$ and $b_{22} - b_{21}a_{11}^{-1} = i\beta$ with $\alpha, \beta \in \mathbb{R}$. We get $g(k_x, \kappa) = g_\infty(k_x, \kappa) + o(1)$ with

$$\begin{aligned} g_\infty(k_x, \kappa) = & \\ & - \det(B)k_x^2 - \frac{i\kappa_{\text{ev}}}{a_{11}^*} + k_x \left(\frac{b_{11}\kappa_{\text{ev}}}{|a_{11}|^2} - \frac{ia_{21}b_{11}}{a_{11}} + ia_{11}b_{22} + b_{11}(-\kappa) - ib_{21} \right) + ia_{11}\kappa - a_{21} \\ & - \frac{(\kappa_{\text{ev}} - \kappa)(\kappa_{\text{ev}} - ik_x)}{2k^2\epsilon} + \frac{(\kappa_{\text{ev}} - \kappa)(\kappa + ik_x)(a_{11} + ib_{11}k_x)}{2a_{11}^*k^2\epsilon} - \frac{ib_{11}(\kappa_{\text{ev}} - \kappa)k_x(\kappa_{\text{ev}} - ik_x)}{2a_{11}k^2\epsilon} \\ & + \frac{b_{11}k_x(k_x - i\kappa)(k_x + i\kappa_{\text{ev}})(b_{22}k_x + \kappa_{\text{ev}})}{4k^4\epsilon^2} - \frac{b_{11}k_x(k_x - i\kappa)(k_x + i\kappa_{\text{ev}})(b_{21}a_{11}^*k_x + \kappa)}{4|a_{11}|^2k^4\epsilon^2}. \end{aligned} \quad (5.21)$$

Passing to dual variables via (4.13) we get

$$\begin{aligned} g_0(\lambda_x, \delta) = & \\ & \frac{1}{4|a_{11}|^2\epsilon^2(\delta^2 + \lambda_x^2)^2} \left(2\epsilon|a_{11}|^2\delta^4 - 2\epsilon a_{11}^2\delta^4 - 4\epsilon^2|a_{11}|^2a_{21}\delta^4 + 2i\epsilon\sqrt{\delta^2 + 2\lambda_x^2}a_{11}^2\delta^3 \right. \\ & + 2i\epsilon|a_{11}|^2\sqrt{\delta^2 + 2\lambda_x^2}\delta^3 + 4i\epsilon^2|a_{11}|^2a_{11}\delta^3 + 2i\epsilon|a_{11}|^2\lambda_x\delta^3 + 2i\epsilon a_{11}^2\lambda_x\delta^3 + |a_{11}|^2b_{11}\lambda_x\delta^3 \\ & + 6\epsilon|a_{11}|^2\lambda_x^2\delta^2 - 2\epsilon a_{11}^2\lambda_x^2\delta^2 - 8\epsilon^2|a_{11}|^2a_{21}\lambda_x^2\delta^2 - i|a_{11}|^2b_{11}\lambda_x^2\delta^2 + ib_{11}\lambda_x^2\delta^2 \\ & + 4\epsilon^2\sqrt{\delta^2 + 2\lambda_x^2}a_{11}\delta^2 + 2\epsilon\sqrt{\delta^2 + 2\lambda_x^2}a_{11}^2\lambda_x\delta^2 + 2\epsilon|a_{11}|^2\sqrt{\delta^2 + 2\lambda_x^2}\lambda_x\delta^2 \\ & + i\sqrt{\delta^2 + 2\lambda_x^2}b_{11}\lambda_x\delta^2 + 2i\epsilon a_{11}b_{11}\lambda_x\delta^2 + 4i\epsilon^2|a_{11}|^2b_{21}\lambda_x\delta^2 - 2i\epsilon b_{11}a_{11}^*\lambda_x\delta^2 \\ & + 4i\epsilon^2a_{21}b_{11}a_{11}^*\lambda_x\delta^2 - 4i\epsilon^2a_{11}^2b_{22}a_{11}^*\lambda_x\delta^2 + 2i\epsilon|a_{11}|^2\lambda_x^3\delta + 2i\epsilon a_{11}^2\lambda_x^3\delta + 2|a_{11}|^2b_{11}\lambda_x^3\delta \\ & + b_{11}\lambda_x^3\delta + i|a_{11}|^2b_{11}b_{22}\lambda_x^3\delta - ib_{11}b_{21}a_{11}^*\lambda_x^3\delta + 2i\epsilon\sqrt{\delta^2 + 2\lambda_x^2}a_{11}^2\lambda_x^2\delta \\ & + 2i\epsilon|a_{11}|^2\sqrt{\delta^2 + 2\lambda_x^2}\lambda_x^2\delta + |a_{11}|^2\sqrt{\delta^2 + 2\lambda_x^2}b_{11}\lambda_x^2\delta + \sqrt{\delta^2 + 2\lambda_x^2}b_{11}\lambda_x^2\delta + 2\epsilon a_{11}b_{11}\lambda_x^2\delta \\ & + i|a_{11}|^2\sqrt{\delta^2 + 2\lambda_x^2}b_{11}b_{22}\lambda_x^2\delta + 4i\epsilon^2|a_{11}|^2a_{11}\lambda_x^2\delta + 2\epsilon b_{11}a_{11}^*\lambda_x^2\delta - ib_{11}b_{21}a_{11}^*\lambda_x^2\sqrt{\delta^2 + 2\lambda_x^2}\delta \\ & + 4\epsilon^2|a_{11}|^2b_{11}\lambda_x\delta + 2\epsilon\sqrt{\delta^2 + 2\lambda_x^2}a_{11}b_{11}\lambda_x\delta + 2\epsilon\sqrt{\delta^2 + 2\lambda_x^2}b_{11}a_{11}^*\lambda_x\delta + 4\epsilon|a_{11}|^2\lambda_x^4 \\ & - 4\epsilon^2|a_{11}|^2a_{21}\lambda_x^4 - 2i|a_{11}|^2b_{11}\lambda_x^4 + |a_{11}|^2b_{11}b_{22}\lambda_x^4 - b_{11}b_{21}a_{11}^*\lambda_x^4 + 2\epsilon\sqrt{\delta^2 + 2\lambda_x^2}a_{11}^2\lambda_x^3 \\ & + 2\epsilon|a_{11}|^2\sqrt{\delta^2 + 2\lambda_x^2}\lambda_x^3 + 4i\epsilon^2|a_{11}|^2b_{21}\lambda_x^3 + |a_{11}|^2\sqrt{\delta^2 + 2\lambda_x^2}b_{11}b_{22}\lambda_x^3 - 4i\epsilon b_{11}a_{11}^*\lambda_x^3 \\ & + 4i\epsilon^2a_{21}b_{11}a_{11}^*\lambda_x^3 - 4i\epsilon^2|a_{11}|b_{22}a_{11}\lambda_x^3 - 4\det(B)\epsilon^2|a_{11}|^2\lambda_x^2 + 4\epsilon^2\sqrt{\delta^2 + 2\lambda_x^2}a_{11}\lambda_x^2 \\ & - i|a_{11}|^2b_{11}\lambda_x^3\sqrt{\delta^2 + 2\lambda_x^2} - b_{11}b_{21}a_{11}^*\lambda_x^3\sqrt{\delta^2 + 2\lambda_x^2} - 2i\epsilon a_{11}b_{11}\lambda_x^2\sqrt{\delta^2 + 2\lambda_x^2} \\ & \left. - 2i\epsilon b_{11}a_{11}^*\lambda_x^2\sqrt{\delta^2 + 2\lambda_x^2} - 4i\epsilon^2b_{11}\lambda_x\sqrt{\delta^2 + 2\lambda_x^2} \right). \end{aligned} \quad (5.22)$$

The denominator does not contribute to w_∞ so we replace g_0 by $2\epsilon|a_{11}|^2(\delta^2 + \lambda_x^2)^2g_0$. Then the dominant scale of g_0 depends on some parameter vanishing or not.

Case $b_{11} \neq 0$. If $b_{11} \neq 0$, the dominant scale is

$$\delta^{-2}g_0(u\delta, \pm\delta) = G_\pm(u) + R_\pm(u)$$

with

$$\begin{aligned} G_+(u) &= -2 \det(B) \epsilon |a_{11}|^2 u^2 - 2i \epsilon b_{11} u \sqrt{1+2u^2} + 2\epsilon b_{11} |a_{11}|^2 u \\ G_-(u) &= -2 \det(B) \epsilon |a_{11}|^2 u^2 - 2i \epsilon b_{11} u \sqrt{1+2u^2} - 2\epsilon b_{11} |a_{11}|^2 u. \end{aligned} \quad (5.23)$$

Recalling that $\det(B) = b_{11}(b_{22} - b_{21}/a_{11})$ and that $b_{22} - b_{21}/a_{11} = i\beta$ with $\beta \in \mathbb{R}$. G_+ and G_- vanish at $u = 0$ and we can check that Proposition 19 applies when $b_{11} \neq 0$. We are left with the ratio

$$\mathcal{S}(u) = \frac{G_-(u)}{G_+(u)} = \frac{\beta |a_{11}|^2 u + \sqrt{1+2u^2} - i|a_{11}|^2}{\beta |a_{11}|^2 u + \sqrt{1+2u^2} + i|a_{11}|^2} := \frac{G(u)}{G^*(u)} \quad (5.24)$$

and we are left with twice the winding of the numerator $G(u) = \beta |a_{11}|^2 u + \sqrt{1+2u^2} - i|a_{11}|^2$ which has constant negative imaginary part so its winding phase is the difference of limits of argument between $+\infty$ and $-\infty$. If $\beta |a_{11}|^2 > \sqrt{2}$ then

$$\lim_{u \rightarrow +\infty} \frac{G(u)}{G(-u)} = -1$$

and G winds by π so that $w_\infty = 1$. Similarly, if $\beta |a_{11}|^2 < -\sqrt{2}$ then G winds by $-\pi$ so that $w_\infty = -1$. Otherwise, if $|\beta| |a_{11}|^2 < \sqrt{2}$ then

$$\lim_{u \rightarrow +\infty} \frac{G(u)}{G(-u)} = 1$$

and G does not wind, so that $w_\infty = 0$.

Case $b_{11} = 0$. If $b_{11} = 0$, the dominant scale is

$$\delta^{-3} g_0(u\delta, \pm\delta) = G_\pm(u) + R_\pm(u)$$

with

$$\begin{aligned} G_+(u) &= 2a_{11} (u^2 + 1) \epsilon \left(\sqrt{2u^2 + 1} + i(|a_{11}|^2 + u(a_{11}^* b_{21} - |a_{11}|^2 b_{22})) \right) \\ G_-(u) &= 2a_{11} (u^2 + 1) \epsilon \left(-\sqrt{2u^2 + 1} + i(|a_{11}|^2 - u(a_{11}^* b_{21} - |a_{11}|^2 b_{22})) \right) \end{aligned} \quad (5.25)$$

The self-adjoint condition implies $b_{22} - b_{21}/a_{11} = i\beta$ with $\beta \in \mathbb{R}$. G_+ and G_- do not vanish and Proposition 17 applies. The ratio simplifies to:

$$\mathcal{S}(u) = \frac{G_-(u)}{G_+(u)} = \frac{-\sqrt{2u^2 + 1} + i|a_{11}|^2 - \beta |a_{11}|^2 u}{\sqrt{2u^2 + 1} + i|a_{11}|^2 + \beta |a_{11}|^2 u} = -\frac{G(u)}{G^*(u)}.$$

We are back to the same numerator and denominator from the case $b_{11} = 0$, from which we immediately infer

$$w_\infty = \begin{cases} 1, & \beta |a_{11}|^2 > \sqrt{2}, \\ 0, & -\sqrt{2} < \beta |a_{11}|^2 < \sqrt{2}, \\ -1, & \beta |a_{11}|^2 < -\sqrt{2}. \end{cases}$$

Remark 23. At the threshold cases $\beta |a_{11}|^2 = \pm\sqrt{2}$, one can check that the winding phase of G_+ is not a multiple of π so that $w_\infty \notin \mathbb{Z}$. We suspect some collapse of edge mode branch occurring at infinity, similarly to the case $a = \pm\sqrt{2}$ in [20]. We do not investigate further this fine-tuned case and consider it out of the w_∞ -classification.

5.5 Class $\mathfrak{A}_{3,4}$

In this case we have

$$A_0 = \begin{pmatrix} \alpha_1 & a_{12} & 1 & 0 \\ \epsilon^{-1} - a_{12}^* & \alpha_2 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} i\beta_1 & b_{12} & 0 & 0 \\ b_{12}^* & i\beta_2 & 0 & 0 \end{pmatrix} := (B \ 0),$$

with $\alpha_{12}, b_{12} \in \mathbb{C}$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. We get $g(k_x, \kappa) = g_\infty(k_x, \kappa) + o(1)$ with

$$\begin{aligned} g_\infty(k_x, \kappa) = & -\det(B)k_x^2 + i\alpha_2\kappa_{\text{ev}} - \kappa\kappa_{\text{ev}} + i\alpha_1\kappa + \det(A) \\ & + ik_x \left(i\alpha_2\beta_1 + i\alpha_1\beta_2 - a_{12}b_{12}^* + b_{12}a_{12}^* - \beta_1\kappa - \frac{b_{12}}{\epsilon} - \beta_2\kappa_{\text{ev}} \right) \\ & \frac{(\kappa_{\text{ev}} - \kappa)(a_{12}\epsilon(-\kappa_{\text{ev}} + ik_x) + b_{12}\epsilon k_x(-k_x - i\kappa_{\text{ev}}) + (k_x - i\kappa)(ia_{12}^*\epsilon + b_{12}^*\epsilon k_x - i))}{2k^2\epsilon^2} \\ & + \frac{(k_x - i\kappa)(-\kappa_{\text{ev}} + ik_x)(-i\det(B)k_x^2 - i\kappa\kappa_{\text{ev}} + \beta_1\kappa_{\text{ev}}k_x + \beta_2\kappa k_x)}{4k^4\epsilon^2}. \end{aligned} \quad (5.26)$$

Passing to dual variables via (4.13) we get

$$\begin{aligned} g_0(\lambda_x, \delta) = & \frac{1}{4\epsilon^2(\delta^2 + \lambda_x^2)^2} \left(4\det(A)\epsilon^2\delta^4 + 2\epsilon a_{12}\delta^4 - 2\epsilon a_{12}^*\delta^4 + \delta^4 - 2i\sqrt{\delta^2 + 2\lambda_x^2}\delta^3 + 2i\epsilon\sqrt{\delta^2 + 2\lambda_x^2}a_{12}\delta^3 \right. \\ & + 2i\epsilon\sqrt{\delta^2 + 2\lambda_x^2}a_{12}^*\delta^3 + 4i\epsilon^2\alpha_1\delta^3 - i\lambda_x\delta^3 + 2i\epsilon a_{12}\lambda_x\delta^3 + 2i\epsilon a_{12}^*\lambda_x\delta^3 + i\beta_1\lambda_x\delta^3 + 8\det(A)\epsilon^2\lambda_x^2\delta^2 \\ & + 6\epsilon a_{12}\lambda_x^2\delta^2 - 2\epsilon a_{12}^*\lambda_x^2\delta^2 + \beta_1\lambda_x^2\delta^2 + \beta_2\lambda_x^2\delta^2 - 4\epsilon^2\alpha_2\sqrt{\delta^2 + 2\lambda_x^2}\delta^2 - 3\lambda_x\sqrt{\delta^2 + 2\lambda_x^2}\delta^2 \\ & + 2\epsilon\sqrt{\delta^2 + 2\lambda_x^2}a_{12}\lambda_x\delta^2 + 2i\epsilon b_{12}\lambda_x\delta^2 + 2\epsilon\sqrt{\delta^2 + 2\lambda_x^2}a_{12}^*\lambda_x\delta^2 - 4i\epsilon^2 b_{12}a_{12}^*\lambda_x\delta^2 - 2i\epsilon b_{12}^*\lambda_x\delta^2 \\ & + 4i\epsilon^2 a_{12}b_{12}^*\lambda_x\delta^2 + 4\epsilon^2\alpha_2\beta_1\lambda_x\delta^2 + \sqrt{\delta^2 + 2\lambda_x^2}\beta_2\lambda_x\delta^2 + 4\epsilon^2\alpha_1\beta_2\lambda_x\delta^2 + \det(B)i\lambda_x^3\delta + 2i\epsilon a_{12}\lambda_x^3\delta \\ & + 2i\epsilon a_{12}^*\lambda_x^3\delta + 2i\beta_1\lambda_x^3\delta - i\beta_2\lambda_x^3\delta + \det(B)i\sqrt{\delta^2 + 2\lambda_x^2}\lambda_x^2\delta + 2i\epsilon\sqrt{\delta^2 + 2\lambda_x^2}a_{12}\lambda_x^2\delta + 2\epsilon b_{12}\lambda_x^2\delta \\ & + 2i\epsilon\sqrt{\delta^2 + 2\lambda_x^2}a_{12}^*\lambda_x^2\delta - 2\epsilon b_{12}^*\lambda_x^2\delta + 4i\epsilon^2\alpha_1\lambda_x^2\delta + i\sqrt{\delta^2 + 2\lambda_x^2}\beta_1\lambda_x^2\delta - 4i\epsilon^2\sqrt{\delta^2 + 2\lambda_x^2}\delta \\ & - i\lambda_x^2\sqrt{\delta^2 + 2\lambda_x^2}\delta - i\beta_2\lambda_x^2\sqrt{\delta^2 + 2\lambda_x^2}\delta - 2\epsilon b_{12}^*\lambda_x\sqrt{\delta^2 + 2\lambda_x^2}\delta + 2\epsilon\sqrt{\delta^2 + 2\lambda_x^2}b_{12}\lambda_x\delta + 4i\epsilon^2\beta_1\lambda_x\delta \\ & + 4\det(A)\epsilon^2\lambda_x^4 + \det(B)\lambda_x^4 + 4\epsilon a_{12}\lambda_x^4 + 2\beta_1\lambda_x^4 + \det(B)\sqrt{\delta^2 + 2\lambda_x^2}\lambda_x^3 + 2\epsilon\sqrt{\delta^2 + 2\lambda_x^2}a_{12}\lambda_x^3 \\ & + 2\epsilon\sqrt{\delta^2 + 2\lambda_x^2}a_{12}^*\lambda_x^3 - 4i\epsilon^2 b_{12}a_{12}^*\lambda_x^3 + 4i\epsilon^2 a_{12}b_{12}^*\lambda_x^3 + \sqrt{\delta^2 + 2\lambda_x^2}\beta_1\lambda_x^3 + 4\epsilon^2\alpha_2\beta_1\lambda_x^3 + 4\epsilon^2\alpha_1\beta_2\lambda_x^3 \\ & - 4\det(B)\epsilon^2\lambda_x^2 + 2i\epsilon\sqrt{\delta^2 + 2\lambda_x^2}b_{12}^*\lambda_x^2 - 2\lambda_x^3\sqrt{\delta^2 + 2\lambda_x^2} - 2i\epsilon b_{12}\lambda_x^2\sqrt{\delta^2 + 2\lambda_x^2} - 4\epsilon^2\alpha_2\lambda_x^2\sqrt{\delta^2 + 2\lambda_x^2} \\ & \left. - 4\epsilon^2\beta_2\lambda_x\sqrt{\delta^2 + 2\lambda_x^2} \right). \end{aligned} \quad (5.27)$$

The denominator does not contribute to w_∞ so we replace g_0 by $(\delta^2 + \lambda_x^2)^2 g_0$. Then the dominant scale of g_0 depends on some parameter vanishing or not.

5.5.1 Case $b_{12} \neq 0$

If $b_{12} \neq 0$ the dominant scale is

$$\delta^{-2}g_0(u\delta, \pm\delta) = G_\pm(u) + R_\pm(u)$$

with

$$\begin{aligned} G_+(u) &= -\det(B)u^2 - \beta_2u\sqrt{2u^2 + 1} - i\sqrt{2u^2 + 1} + i\beta_1u \\ G_-(u) &= -\det(B)u^2 - \beta_2u\sqrt{2u^2 + 1} + i\sqrt{2u^2 + 1} - i\beta_1u \end{aligned} \quad (5.28)$$

Notice that

$$\det(B) = -\beta_1\beta_2 - |b_{12}|^2 \in \mathbb{R}.$$

Since $\beta_1 \in \mathbb{R}$ then $G_+(u) = G_-^*(u)$. Moreover G_+ and G_- do not vanish and Proposition 17 applies as long as $\Delta \neq 0$ with

$$\Delta := (\det B - \sqrt{2}\beta_2)(\det B + \sqrt{2}\beta_2) = B_+B_-$$

with

$$B_{\pm} := \beta_2(\beta_1 \pm \sqrt{2}) + |b_{12}|^2.$$

We are left with studying the argument of

$$G(u) = -G_-(u) = \det(B)u^2 + \beta_2u\sqrt{2u^2 + 1} + i(\beta_1u - \sqrt{2u^2 + 1}) := G_r(u) + iG_i(u). \quad (5.29)$$

Case 1: $\Delta < 0$. In that case one can check that $G'_r(u)$ is nowhere vanishing so that G_r is strictly monotonic. Moreover, the real part dominates the behavior of G asymptotically:

$$G(u) \underset{+\infty}{\sim} (\det(B) + \sqrt{2}\beta_2)u^2, \quad G(u) \underset{-\infty}{\sim} (\det(B) - \sqrt{2}\beta_2)u^2.$$

In particular

$$\lim_{u \rightarrow \infty} \frac{G(u)}{G(-u)} = \frac{\Delta}{(\det(B) - \sqrt{2}\beta_2)^2} < 0$$

so that G crosses the whole complex plane near the real line. Consequently, its argument changes by $\pm\pi$.

The exact value of the winding sign of G requires a detailed study of $G_i(u)$, which behaves asymptotically as

$$G_i(u) \underset{+\infty}{\sim} (\beta_1 - \sqrt{2})u, \quad G_i(u) \underset{-\infty}{\sim} (\beta_1 + \sqrt{2})u.$$

Moreover $G_i(u)$ vanishes only if $\beta_1^2 > 2$, in which case $G_i(u_0) = 0$ with

$$u_0 = \frac{\text{sign}(\beta_1)}{\sqrt{\beta_1^2 - 2}}. \quad (5.30)$$

and

$$G_r(u_0) = -\frac{|b_{12}|^2}{\beta_1^2 - 2} < 0$$

since we assume $b_{12} \neq 0$.

Case 1a: $B_+ > 0$ and $B_- < 0$. This implies $B_+ - B_- = 2\sqrt{2}\beta_2 > 0$ so that $\beta_2 > 0$. Moreover, $B_- < 0$ implies $\beta_2(\beta_1 - \sqrt{2}) < -|b_{12}|^2 < 0$ and therefore $\beta_1 - \sqrt{2} < 0$, so that $G_i(u) < 0$ near $+\infty$. Moreover, in this case, one has $G_r(u) > 0$ near $+\infty$ and $G_r(u) < 0$ near $-\infty$. Finally, if $\beta_1 < -\sqrt{2}$ then $G_i(u) > 0$ near $-\infty$ and $G_i(u)$ vanishes at u_0 given above, with $G_r(u_0) < 0$. Putting all together, G winds by $+\pi$. Otherwise, if $\beta_1 > -\sqrt{2}$ then $G_i(u) < 0$ near $-\infty$ and $G_i(u)$ never vanishes. Again, G winds by $+\pi$.

Case 1b: $B_+ < 0$ and $B_- > 0$. Similarly $B_+ - B_- < 0$ so that $\beta_2 < 0$. Moreover $B_+ < 0$ implies $\beta_2(\beta_1 + \sqrt{2}) < -|b_{12}|^2 < 0$ and therefore $\beta_1 + \sqrt{2} > 0$ so that $G_i(u) < 0$ near $-\infty$. Moreover, in this case, one has $G_r(u) < 0$ near $+\infty$ and $G_r(u) > 0$ near $-\infty$. Finally, if $\beta_1 > \sqrt{2}$ then $G_i(u) > 0$ near $+\infty$ and $G_i(u)$ vanishes at u_0 given above, with $G_r(u_0) < 0$. Putting all together, G winds by $-\pi$. Otherwise, if $\beta_1 < \sqrt{2}$ then $G_i(u) < 0$ near $+\infty$ and $G_i(u)$ never vanishes. Again, G winds by $-\pi$.

Summarizing cases 1a and 1b we get

$$\int_{-\infty}^{\infty} G^{-1}(u) \partial_u G(u) du = \text{sign}(B_+) \pi$$

so that

$$w_\infty = \text{sign}(B_+) = -\text{sign}(B_-)$$

Case 2: $\Delta > 0$. In that case one can check that $G'_r(u)$ vanishes exactly once. Moreover, similarly to case 1, we compute

$$\lim_{u \rightarrow \infty} \frac{G(u)}{G(-u)} = \frac{\Delta}{(\det(B) - \sqrt{2}\beta)^2} > 0$$

This time G starts from $+\infty$ or $-\infty$ and comes back to the same direction.

Case 2a: $B_+ < 0$ and $B_- < 0$. These inequalities imply $\beta_2(\beta_1 \pm \sqrt{2}) < -|b_{12}|^2 < 0$ which means that $\beta_2 \neq 0$ and $\beta_1 + \sqrt{2}$ and $\beta_1 - \sqrt{2}$ have the same sign. In particular $\beta_1^2 - 2 > 0$ and $G_i(u)$ vanishes at u_0 given in (5.30), with $G_r(u_0) < 0$. Moreover, in this case one has $G_r(u) > 0$ near $+\infty$ and $-\infty$, so that G fully winds around 0 as u goes from $-\infty$ to $+\infty$.

If $\beta_1 + \sqrt{2} > 0$ and $\beta_1 - \sqrt{2} > 0$ then $G_i(u) > 0$ near $+\infty$ and $G_i(u) < 0$ near $-\infty$. Thus G winds by -2π around zero, so that $w_\infty = -2$. Otherwise, if $\beta_1 + \sqrt{2} < 0$ and $\beta_1 - \sqrt{2} < 0$ then $G_i(u) < 0$ near $+\infty$ and $G_i(u) > 0$ near $-\infty$. Thus G winds by 2π , so that $w_\infty = +2$. Summarizing,

$$w_\infty = -2 \text{sign}(\beta_1 + \sqrt{2}) = -2 \text{sign}(\beta_1 - \sqrt{2}).$$

Case 2b: $B_+ > 0$ and $B_- > 0$. In this case one has $G_r(u) < 0$ near $+\infty$ and $-\infty$. Either $G_i(u)$ vanishes once at u_0 , with $G_r(u_0) < 0$, or $G_i(u)$ never vanishes. In both cases, G never crosses the real positive axis, so that its change of argument is zero. Consequently

$$w_\infty = 0.$$

Notice that this case is also valid when $\beta_2 = 0$, or $\beta_1 = \pm\sqrt{2}$.

Summarizing If $b_{12} \neq 0$ and $\Delta \neq 0$ then we have the following table:

B_+	B_-	w_∞
> 0	< 0	1
< 0	> 0	-1
> 0	> 0	0
< 0	< 0	± 2

(5.31)

with $\pm 2 = 2 \text{sign}(\sqrt{2} - \beta_1)$.

5.5.2 Case $b_{12} = 0$, $\beta_2 \neq 0$ and $\beta_1^2 \neq 2$.

In that case the dominant scale, G_{\pm} and R_{\pm} are the same as before, except that G further simplifies to

$$G(u) = (\beta_1 u - \sqrt{2u^2 + 1})(-\beta_2 u + i).$$

The first factor vanishes at $u_0 = \text{sign } \beta_1 (\beta_1^2 - 2)^{-1/2}$, only if $\beta_1^2 > 2$, in which case Proposition 19 applies. Otherwise Proposition 17 applies as long as $\beta_1^2 < 2$. In both cases we are left with

$$\mathcal{S}(u) = \frac{G(u)}{G^*(u)} = \frac{-\beta_2 u + i}{-\beta_2 u - i}$$

so that, as in previous classes, we infer

$$w_{\infty} = \text{sign}(\beta_2).$$

5.5.3 Case $b_{12} = 0$, $\beta_2 = 0$ and $\beta_1^2 \neq 2$.

Notice that in that case one has $\Delta = 0$. The dominant scale is

$$\delta^{-3/2} g_0(u\delta^{1/2}, \pm\delta) = G_{\pm}(u) + R_{\pm}(u)$$

with

$$G_+(u) = (\alpha_2 u^2 + i) \left(\beta_1 u - \sqrt{2}\sqrt{u^2} \right), \quad G_-(u) = G^*(u).$$

The function G_+ and G_- vanish at $u_0 = 0$. One can check that Proposition 19 applies as long as $\beta_1^2 \neq 2$, in which case we are left with the winding of

$$\mathcal{S}(u) = \frac{G_-(u)}{G_+(u)} = \frac{\alpha_2 u^2 - i}{\alpha_2 u^2 + i},$$

from which we infer $w_{\infty} = 0$.

Remaining cases. We did not deal with the cases where $\Delta = 0$ with

$$\Delta = ((\beta_1 + \sqrt{2})\beta_2 + |b_{12}|^2)((\beta_1 - \sqrt{2})\beta_2 + |b_{12}|^2)$$

except if $\beta_2 = b_{12} = 0$, which is treated in the last case above. The four remaining cases would be

1. $\beta_2 \neq 0$, $b_{12} = 0$ and $\beta_1 = \pm\sqrt{2}$,
2. $\beta_1^2 \neq 2$, $b_{12} \neq 0$ and

$$\beta_2 = -\frac{|b_{12}|^2}{\beta_1 \pm \sqrt{2}}.$$

The same issue with all this cases is that the dominant scale is correct at $u \rightarrow +\infty$ and fails at $u \rightarrow -\infty$ (or conversely), so that neither Proposition 17 or 19 apply. One could still compute w_{∞} by splitting the integral into two parts (positive and negative λ_x), study them separately with distinct dominant scales, then compute “half-winding” phases and finally glue them together properly, but we do not expect $w_{\infty} \in \mathbb{Z}$. We rather suspect some collapse of edge mode branch occurring at infinity, similarly to the case $a = \pm\sqrt{2}$ in [20]. We do not investigate further this fine-tuned case and consider it out of the w_{∞} -classification.

5.6 Class \mathfrak{B}

In this case we have

$$A_0 = \begin{pmatrix} a_1 & a_2 & i\alpha & -i\mu^*\alpha \\ \mu a_1 & \mu a_2 & i\mu\alpha & -i|\mu|^2\alpha \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

with $a_1, a_2, \mu \in \mathbb{C}$ and $\alpha \in \mathbb{R}$. The self-adjoint condition further requires $\alpha = 0$ or $\alpha \operatorname{Im}(\mu) - \epsilon \operatorname{Re}(a_1 - a_2\mu) = 0$ but we shall keep it as an implicit constraint and keep general a_2, α for the computations. We get $g(k_x, \kappa) = g_\infty(k_x, \kappa) + o(1)$ with

$$g_\infty(k_x, \kappa) = -k_x^2 + ia_2\mu k_x + ia_1 k_x + \frac{i\kappa_{\text{ev}}k_x^3 + \kappa\kappa_{\text{ev}}k_x^2 - i\kappa k_x^3 + k_x^4}{4k^4\epsilon^2}. \quad (5.32)$$

Passing to dual variables via (4.13) we get

$$g_0(\lambda_x, \delta) = \frac{1}{4\epsilon^2(\delta^2 + \lambda_x^2)^2} \left(-4ia_2\delta^2\mu\epsilon^2\lambda_x - 4ia_1\delta^2\epsilon^2\lambda_x - 4ia_2\mu\epsilon^2\lambda_x^3 - 4ia_1\epsilon^2\lambda_x^3 + \lambda_x^3\sqrt{\delta^2 + 2\lambda_x^2} \right. \\ \left. + i\delta\lambda_x^2\sqrt{\delta^2 + 2\lambda_x^2} + i\delta\lambda_x^3 + \lambda_x^4 - 4\epsilon^2\lambda_x^2 \right). \quad (5.33)$$

The denominator, as well as a common λ_x factor, do not contribute to w_∞ so we replace g_0 by $(\delta^2 + \lambda_x^2)^2\lambda_x^{-1}g_0$. The dominant scale is

$$\delta^{-2}g_0(u\delta^2, \pm\delta) = G_\pm(u) + R_\pm(u)$$

with

$$G_+(u) = -ia_2\mu - ia_1 - u, \quad G_-(u) = G_+(u). \quad (5.34)$$

G_+ and G_- do not vanish if $a_2\mu + a_1 \notin i\mathbb{R}$, and we can check that Proposition 17 applies, from which we immediately infer $\mathcal{S} = 1$ and $w_\infty = 0$. If $a_2\mu + a_1 \in i\mathbb{R}$, since $G_+ = G_-$ then we can replace g_0 by $g_0 + i\eta$ with $\eta \in \mathbb{R}^*$ sufficiently small so that the winding phases of $g_0(\lambda_x, \delta)$ and $g_0(\lambda_x, -\delta)$ are unchanged. Then Proposition 17 applies to $G_+ + i\eta = G_- + i\eta$. Again, we get $w_\infty = 0$.

5.7 Class \mathfrak{C}

In this case we have

$$A_0 = \begin{pmatrix} a_1 & a_2 & 0 & a_4 \\ \mu a_1 & \mu a_2 & 0 & \mu a_4 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with $a_1 \in \mathbb{C}$, $a_2, a_4 \in \mathbb{C}^2 \setminus \{0\}$ and $\mu \in \mathbb{C}^* \setminus \{0\}$. The self-adjoint condition further requires $\operatorname{Im}(a_2a_4^*) = 0$, which implies $a_2 = 0$, or $a_4 = 0$ or $a_4 = ra_2$ with $r \in \mathbb{R}^*$ and $a_2 \neq 0$. We get $g(k_x, \kappa) = g_\infty(k_x, \kappa) + o(1)$ with

$$g_\infty(k_x, \kappa) = -a_4\mu k_x\kappa + ia_2\mu k_x + \frac{a_4\mu k_x(\kappa\kappa_{\text{ev}}^2 + \kappa_{\text{ev}}k_x^2 + i\kappa_{\text{ev}}^2k_x - i\kappa\kappa_{\text{ev}}k_x)}{4k^4\epsilon^2}. \quad (5.35)$$

Passing to dual variables via (4.13) we get

$$g_0(\lambda_x, \delta) = \frac{1}{4\epsilon^2(\lambda_x^2 + \delta^2)^2} \left(-ia_4\mu\lambda_x^3\sqrt{\delta^2 + 2\lambda_x^2} - ia_4\delta^2\mu\lambda_x^2 + a_4\delta\mu\lambda_x^2\sqrt{\delta^2 + 2\lambda_x^2} + a_4\delta^3\mu\lambda_x \right. \\ \left. - 4ia_2\delta^2\mu\epsilon^2\lambda_x + 2a_4\delta\mu\lambda_x^3 + 4a_4\delta\mu\epsilon^2\lambda_x - 2ia_4\mu\lambda_x^4 - 4ia_2\mu\epsilon^2\lambda_x^3 \right). \quad (5.36)$$

The denominator, as well as some common $\lambda_x\mu$ factor, do not contribute to w_∞ so we replace g_0 by $\lambda_x^{-1}\mu^{-1}(\delta^2 + \lambda_x^2)^2g_0$. Then the dominant scale of g_0 depends on some parameter vanishing or not.

Case $a_2 \neq 0$ and $a_4 = ra_2$ with $r \in \mathbb{R}^*$. The dominant scale is

$$\delta^{-1}g_0(u\delta^{1/2}, \pm\delta) = G_{\pm}(u) + R_{\pm}(u)$$

with

$$\begin{aligned} G_+(u) &= a_2(1 - iru^2), \\ G_-(u) &= a_2(1 + iru^2) = G_+^*(u). \end{aligned} \quad (5.37)$$

Proposition 17 applies, and $G_-(u)$ has no winding phase as u goes from $-\infty$ to $+\infty$, so that $w_{\infty} = 0$.

Case $a_2 = 0$ and $a_4 \neq 0$. In that case we also divide g_0 by a_4 . The dominant scale is

$$\delta^{-1}g_0(u\delta^{1/3}, \pm\delta) = G_{\pm}(u) + R_{\pm}(u)$$

with $G_+(u) = 1 - i\frac{2u + \sqrt{2}|u|}{4\epsilon^2}u^2$, $G_-(u) = 1 + i\frac{2u + \sqrt{2}|u|}{4\epsilon^2}u^2 = G_+^*(u)$. Proposition 17 does not apply but G_+ and G_- do not vanish and

$$\frac{R_+(u, \delta)}{G_+(u)} \underset{u \rightarrow \pm\infty}{\sim} \frac{i\delta^{2/3}}{u}, \quad \frac{R_-(u, \delta)}{G_-(u)} \underset{u \rightarrow \pm\infty}{\sim} \frac{-i\delta^{2/3}}{u}$$

so that $\left|\frac{R_{\pm}}{G_{\pm}}\right| \rightarrow 0$ as $\delta \rightarrow 0$ and all $u \in [-\frac{\lambda}{\delta^{1/3}}, \frac{\lambda}{\delta^{1/3}}]$, and we get the same conclusion than Proposition 17. The real part of G_- is constant and positive, and

$$G_-(u) \underset{u \rightarrow \pm\infty}{\sim} \frac{2 \pm \sqrt{2}}{4\epsilon^2}u^3$$

so that the winding phase of G_- is π , and since $G_+ = G_-^*$ we infer $w_{\infty} = 1$ for any $a_4 \neq 0$.

Case $a_2 \neq 0$ and $a_4 = 0$. The dominant scale is

$$\delta^{-2}g_0(u\delta, \pm\delta) = G_{\pm}(u) + R_{\pm}(u)$$

with

$$G_+(u) = -ia_2(u^2 + 1), \quad G_-(u) = G_+(u)$$

and $R_{\pm} = 0$ here. Thus we immediately infer $w_{\infty} = 0$.

Summarizing. In that class, $w_{\infty} = 1$ if $a_2 = 0$ and $a_4 \neq 0$, and $w_{\infty} = 0$ otherwise.

A Chern number

We rewrite the Hamiltonian in terms of Pauli matrices as

$$H = \vec{d} \cdot \vec{\sigma}, \quad \vec{d} = \begin{pmatrix} -k_x \\ -k_y \\ m - \epsilon \mathbf{k}^2 \end{pmatrix}, \quad (A.1)$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (A.2)$$

The eigenprojections of H are shared with those of the flat Hamiltonian $H' = \vec{e} \cdot \vec{\sigma}$, where $\vec{e} = \vec{d}/|\vec{d}|$. They are

$$P_{\pm} = \frac{1}{2} (1 \pm \vec{e} \cdot \vec{\sigma}) , \quad (\text{A.3})$$

$((\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = (\vec{a} \cdot \vec{b}) + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma})$. Note that $\vec{e} = \vec{e}(\mathbf{k})$ is convergent for $k \rightarrow \infty$

$$\vec{e} \rightarrow \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix} \quad (k \rightarrow \infty), \quad \text{for } \epsilon \gtrsim 0. \quad (\text{A.4})$$

Therefore, also the eigenprojections converge and the Chern number is a well-defined topological invariant

$$C(P) = \frac{1}{2\pi i} \int_{\mathbb{R}^2} dk_x dk_y \text{tr} (P [\partial_{k_x} P, \partial_{k_y} P]) . \quad (\text{A.5})$$

If the regulator $\epsilon \neq 0$ we can compactify the momentum plane to the 2-sphere S^2 and we can compute the r.h.s. on a closed manifold with the map $\vec{e}: S^2 \rightarrow S^2$. According to Prop. 1 of [20], we get

$$C_{\pm} = \pm \frac{1}{4\pi} \int_{S^2} \vec{e} \cdot (\partial_1 \vec{e} \wedge \partial_2 \vec{e}) dx_1 dx_2 ,$$

which, in our case, leads to

$$C_{\pm} = \pm \frac{\text{sign } m + \text{sign } \epsilon}{2} . \quad (\text{A.6})$$

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