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Grégoire Misguich. Quantum spin liquids and frustrated magnets. Doctoral. Les Houches School of Physics, France. 2024, pp.58. hal-04569036

HAL Id: hal-04569036

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Quantum spin liquids and frustrated magnets

Grégoire MISGUICH

Institut de Physique Théorique [IPhT], Saclay
CEA/DRF & CNRS Physique (UMR 3681)

gregoire.misguich@ipht.fr

École de Physique des Houches

Topological Order : Anyons and Fractons

topoanyons.sciencesconf.org

April 2024

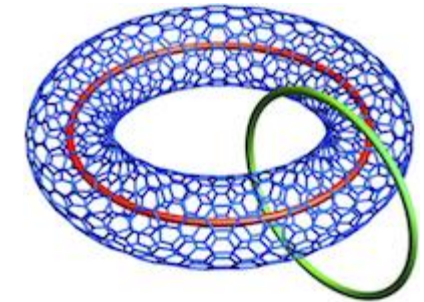




Image generated by DALL-E:
Researchers studying the physics of quantum spin liquids at a summer school located in the Alps

What is a quantum spin liquid (QSL) ?

- Historical definition:

« **a system of interacting quantum spins which remains magnetically disordered down to zero temperature** »
 [or, more generally, no spontaneously broken symmetry].

ex: spin-S Heisenberg model

$$H = J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j \quad (+ \text{ with symmetry-breaking field: } +hS_0^z)$$

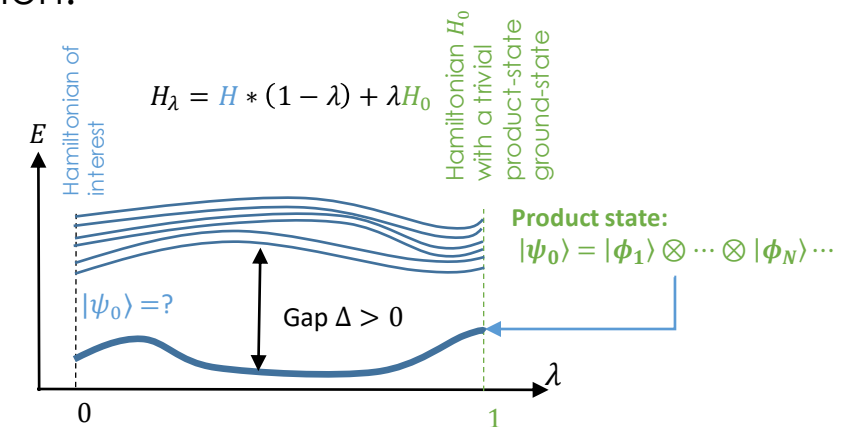
$$\vec{m}_i = \lim_{h \rightarrow \infty} \lim_{N \rightarrow \infty} \langle \text{g. s.} | \vec{S}_i | \text{g. s.} \rangle$$

$$\vec{m}_i = \vec{0} \Leftrightarrow \text{magnetically disordered}$$

The definition above includes various states which do *not* host unconventional excitations (anyons) or special topological properties. This definition is now often considered as being too broad.

- Modern point of view: a (gapped) QSL is a quantum state which **cannot be deformed into a product state** without crossing gap-closing phase transition. Such states intrinsically have some long-range quantum entanglement.

- Connection between “absence of SSB” and “no product-state limit” ?
 → Lieb-Schultz-Mattis-Hastings theorem

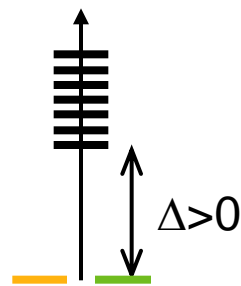
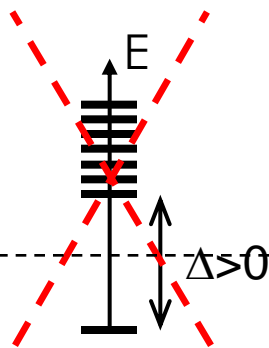


Lieb-Schultz-Mattis-Hastings theorem

- Dimension 1
 - E. Lieb, T. Schultz & D. Mattis, Annals of Phys. 16 ([1961](#))
- Dimension >1
 - M. Hastings Phys. Rev. B 69, 104431 ([2004](#))
See also M. Oshikawa, Phys. Rev. Lett. 84, 1535 ([2000](#))
 - B. Nachtergaele & R. Sims, Com. Math. Phys. 276, 437 ([2007](#))

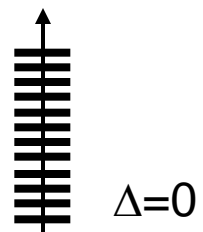
Conditions:

- **half-odd-integer spin in the unit cell**
(more precisely: $S + m^z \in \frac{1}{2} + \mathbb{N}$, where m^z is the magnetization per site)
- short-range interactions
- **Global U(1) symmetry:** $[S_{\text{tot}}^z, \mathcal{H}] = 0$
- dimensions $L_1 \times L_2 \times \dots \times L_D$ with $C=L_2 \times \dots \times L_D = \text{odd}$
- **translation invariance** in direction 1
- periodic bound. conditions in direction 1
- thermodynamic limit



Ground-state degeneracy

1. discrete spontaneous symmetry breaking (SSB)
2. **Topological degeneracy - QSL**



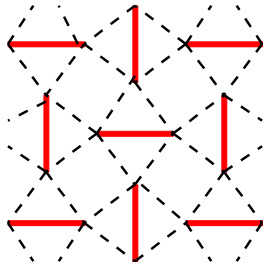
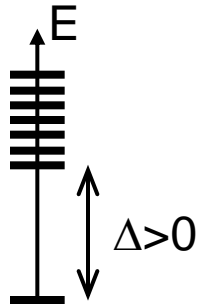
Gapless spectrum

1. Continuous SSB (Goldstone mode)
2. Critical phase or critical point

Lieb-Schultz-Mattis-Hastings theorem

Gap + unique ground-state

LSMH \rightarrow only possible for integer spin / unit cell
 ex 1: weakly coupled dimers (or plaquettes)

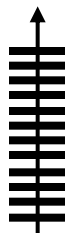


$\text{SrCu}_2(\text{BO}_3)_2$ & Shastry-Sutherland model.

Kageyama *et al.* (1999)

See also: CaV_4O_9 , TlCuCl_3 ,

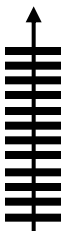
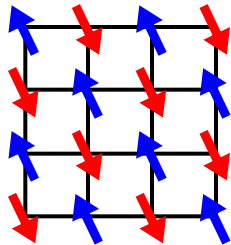
ex 2: AKTL spin-1 model, integer-S chain (Haldane)



Gapless – Goldstone

ex: Square lattice Heisenberg AF

$\Delta=0$



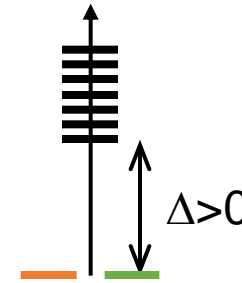
Gapless – critical

ex: spin $\frac{1}{2}$ Heisenberg chain (Bethe Ansatz)
 XY chain (Jordan-Wigner \rightarrow free fermions)

$\Delta=0$

Gap + discrete SSB

ex 1: Majumdar-Gosh chain
 J. Math. Phys. 10, 1399 (1969)



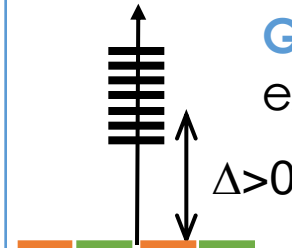
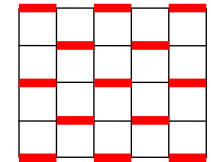
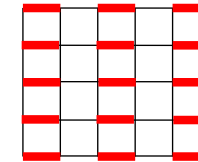
$$H_{\text{MG}} = 2 \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + \sum_i \vec{S}_i \cdot \vec{S}_{i+2}$$

$$|\psi_1\rangle = \text{---} \text{---} \text{---} \text{---} \text{---}$$

$$|\psi_2\rangle = \text{---} \text{---} \text{---} \text{---} \text{---}$$

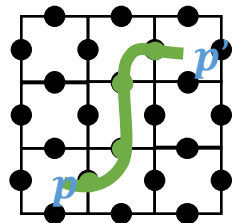
$$\text{---} = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

ex 2: 2D valence-bond crystal phases in an Heisenberg model & 4-spin ring exchange, A. Läuchli *et al.*, PRL 2005



Gap + topological deg.

ex : Kitaev's toric code(*)



(*) The toric code is mentioned here as the simplest possible example of topologically ordered state, but its Hamiltonian in fact does *not* fulfil the LSMH conditions

Outline

- Introduction
Lieb-Shultz-Mattis-Hastings theorem
 - Quantum frustrated Hamiltonians: a few examples
 - Magnetic long-range order
 - T=0 phase diagrams of a few frustrated spin-1/2 Heisenberg models (square, triangular, honeycomb & kagome)
 - Gapped QSL & short-range Resonating Valence-Bond (RVB) picture
 - Quantum dimer models (QDM)
 - From RVB to QDM
 - triangular lattice QDM: phase diagram, RK point, topo. degeneracy, indistinguishability of the ground-states, topological entanglement entropy
 - solvable kagome QDM: toric code & Z_2 gauge theory
 - Partons: Schwinger bosons, mean field and beyond
 - Examples of microscopic spin models realizing Z_2 liquids
 - Chiral spin liquids (Kalmeyer & Laughlin)
- Remark: there exists a vast zoo of quantum spin liquids but the present lectures will focus on
- simplest microscopic models
 - T=0, gapped phases (mostly Z_2 liquids)
 - 2D systems

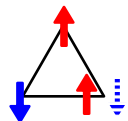
a few reviews & lecture notes

- *Exotic quantum phases and phase transitions in correlated matter*
F. Alet, A. Walczak & M. P.A. Fisher, *Physica A* 369, 122 ([2006](#))
- *Quantum spin liquids: a review*
L. Savary & L. Balents, *Rep. Prog. Phys.* 80 016502 ([2017](#))
- *Quantum spin liquid states*
Y. Zhou, K. Kanoda & T.-K. Ng, *Rev. Mod. Phys.* 89, 025003 ([2017](#))
- *Colloquium: Zoo of quantum-topological phases of matter*
X.-G. Wen, *Rev. Mod. Phys.* 89, 041004 ([2017](#))
- *Quantum Spin Liquids*
C. Broholm, R. J. Cava, S. A. Kivelson, D. G. Nocera, M. R. Norman, T. Senthil,
Science 367, eaay0668 ([2020](#))

Frustrated Ising models

- Ingredients: non-partite lattice + AF interactions

$$H = J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z$$



- Classical frustration
 - many low-energy states (sometimes degenerate)
 - new/emerging low energy degrees of freedom

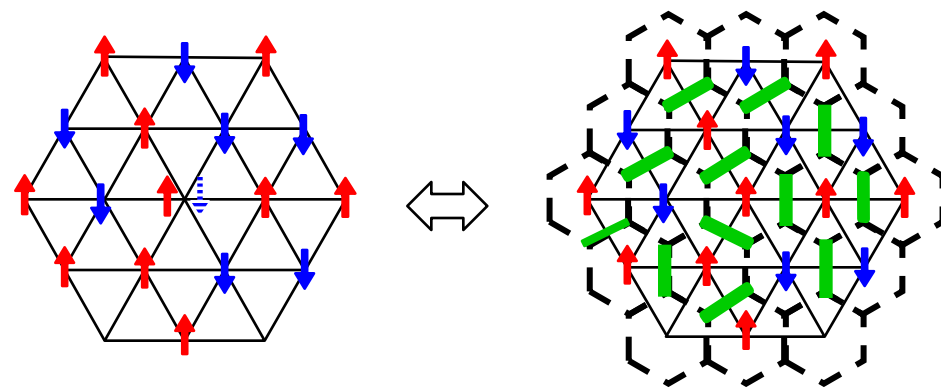
- Introduce some quantum dynamics ?

- Frustrated Ising model + **transverse field**

$$H = J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z + h \sum_i \sigma_i^x$$

Ising models of quantum frustration,
R. Moessner & S. Sondhi, Phys. Rev. B 63, 224401 ([2001](#))

ex: triangular lattice Ising AF
G. H. Wannier Phys. Rev. 79, 357 ([1950](#))



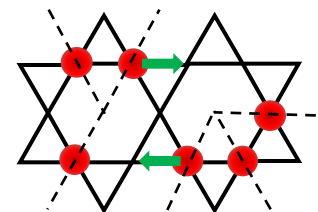
low energy configurations
↔ dimer coverings of the hex. lattice

- Frustrated Ising model + **xy exchange** (→ $U(1)$ sym.)

$$H = J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z + J^\perp \sum_{\langle ij \rangle} (S_i^+ S_j^- + S_i^- S_j^+)$$

$$H = \sum J_{ij} \sigma_i^z \sigma_j^z + J^{\text{ring}} \sum (S_i^+ S_j^- S_k^+ S_l^- + S_i^- S_j^+ S_k^- S_l^+)$$

ex.: of quantum spin liquid in an easy-axis XXZ model:
L. Balents, M. P. A. Fisher & S. Girvin, PRB [2002](#)



Frustrated models with $SU(2)$ symmetry (a few examples)

- Heisenberg models – $SU(2)$ symmetry

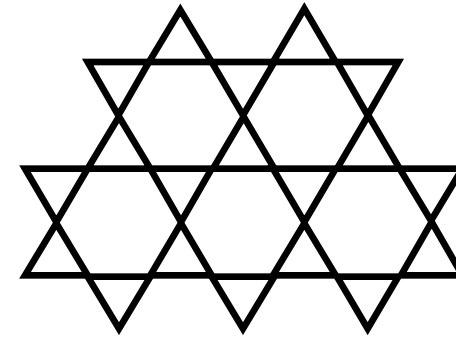
$$H = J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j$$

Spin: from $S = \frac{1}{2}$ (most quantum) to $S = \infty$ (classical)

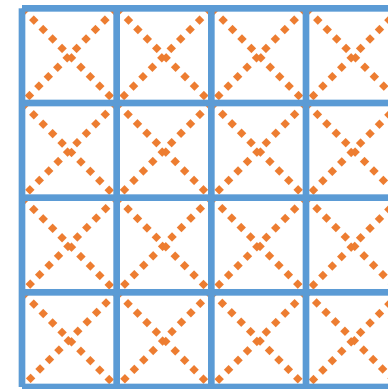
- $J_1 - J_2 - (J_3)$ Heisenberg models
competing interactions/orders
square, triangular, honeycomb lattices, ...

- Cyclic ring-exchange models

$$H = J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j + K \sum_{\langle i,j,k,l \rangle} P_{ijkl} + P_{lkji}$$



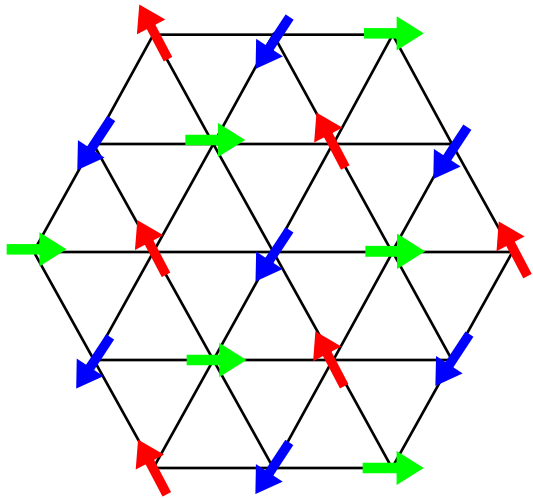
kagome



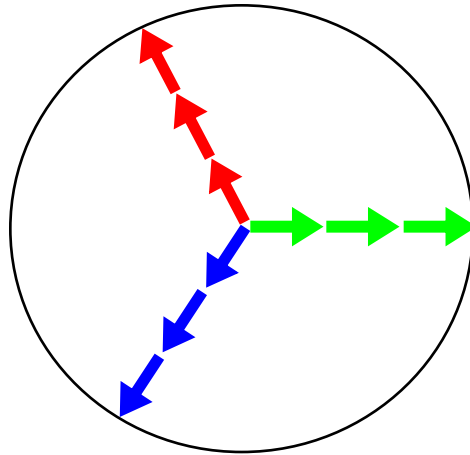
Square lattice
 $J_1 - J_2$

Antiferromagnetic/Néel long-range order

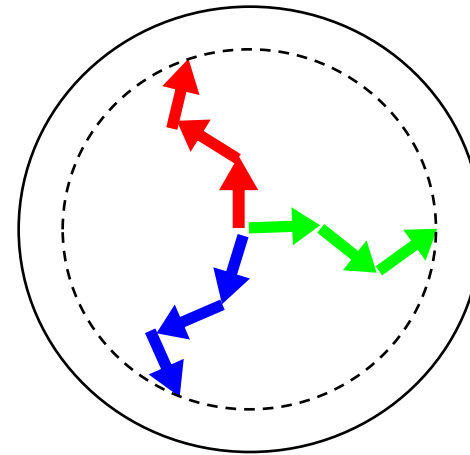
$$H = J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j$$



Classical 120° structure



Quantum zero point motion

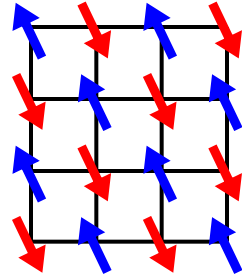


Magnetic long-range order & spontaneously broken sym.

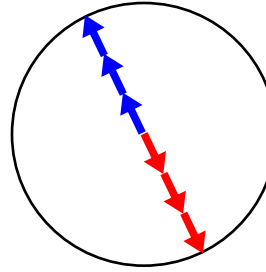
- Spin-1/2 Heisenberg model

$$H = J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j$$

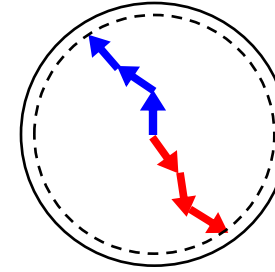
- Collinear antiferromagnetic structure on bi-partite lattices



Classical



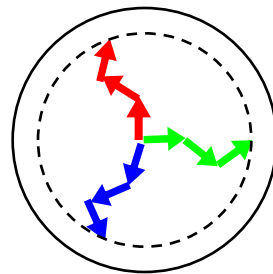
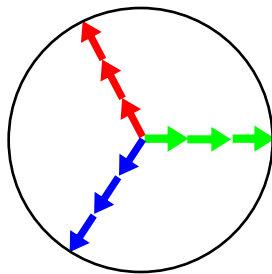
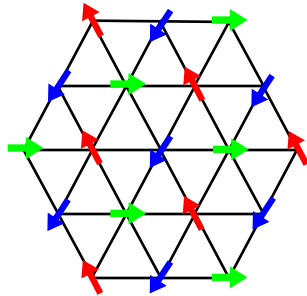
Quantum zero point motion



- Mag. long-range order on the **triangular** lattice

Classical 120° structure

Quantum zero point motion



- Gapless** spin waves, linearly dispersing, = Goldstone modes

1/S **spin-wave expansion**, based on the **Holstein-Primakoff** representation of the spin operators in terms of bosons

$$S_i^z = S - a_i^\dagger a_i$$

z axis: direction of the mean local magnetization. Assume $\langle a_i^\dagger a_i \rangle \ll 2S$:

$$S_i^+ = S_i^x + iS_i^y = \sqrt{2S - a_i^\dagger a_i} a_i$$

$$\approx \sqrt{2S} a_i$$

$H = J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j \rightarrow$ coupled harmonic oscillators

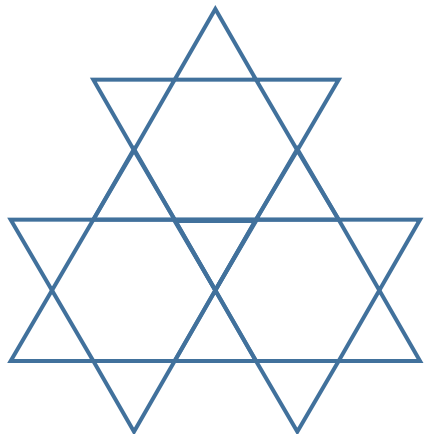
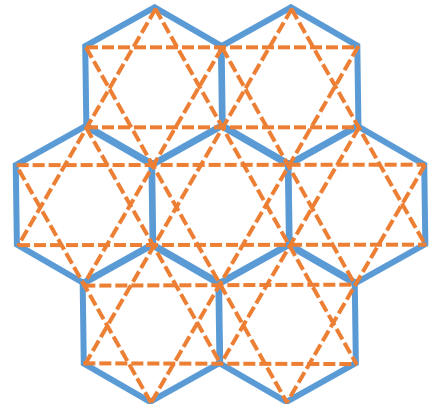
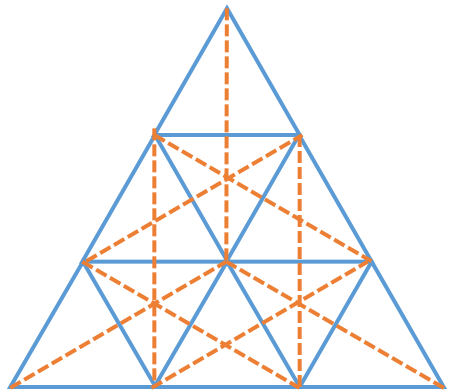
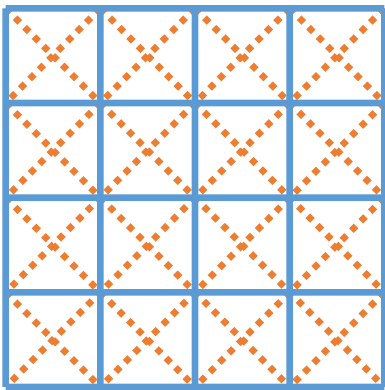
A few historical papers concerning the spin-1/2 triangular lattice AF model

- Spin-wave calculations: Jolicoeur & Le Guillou, [1989](#) [$m \approx 0.239$]
- Variational calculation: Huse & Elser [1988](#)
- Exact diagonalization: B. Bernu, C. Lhuillier & L. Pierre, [1992](#)

Sublattice magnetization (=Néel order parameter) from DMRG:

- Néel Order in Square and Triangular Lattice Heisenberg Models*
S. R. White & A. L. Chernyshev, Phys. Rev. Lett. 99 127004 ([2007](#)) [$m \approx 0.205(15)$]
- J. Huang *et al.*, J. Phys.: Condens. Matter 36 185602 ([2024](#)) [$m \approx 0.208(8)$]

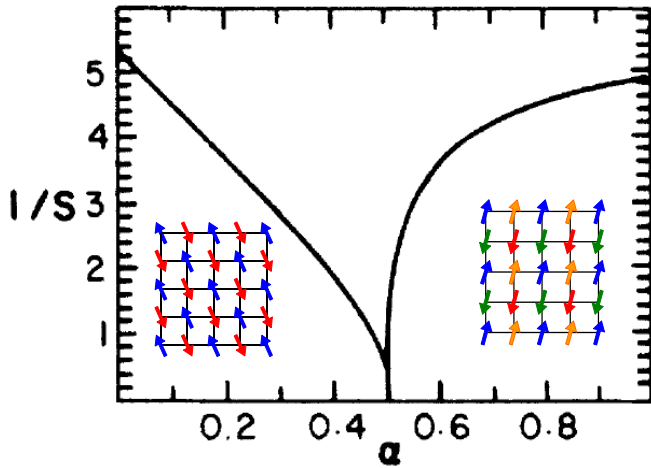
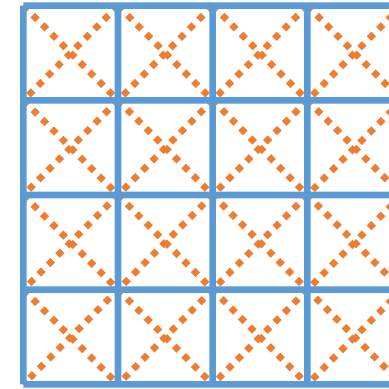
T=0 phase diagrams of a few frustrated spin-1/2 Heisenberg models



Square lattice J_1 - J_2 Heisenberg model

Spin wave theory for the $J_1 - J_2$ model

P. Chandra & B. Douçot, Phys. Rev. B 38, 9335 (1988)

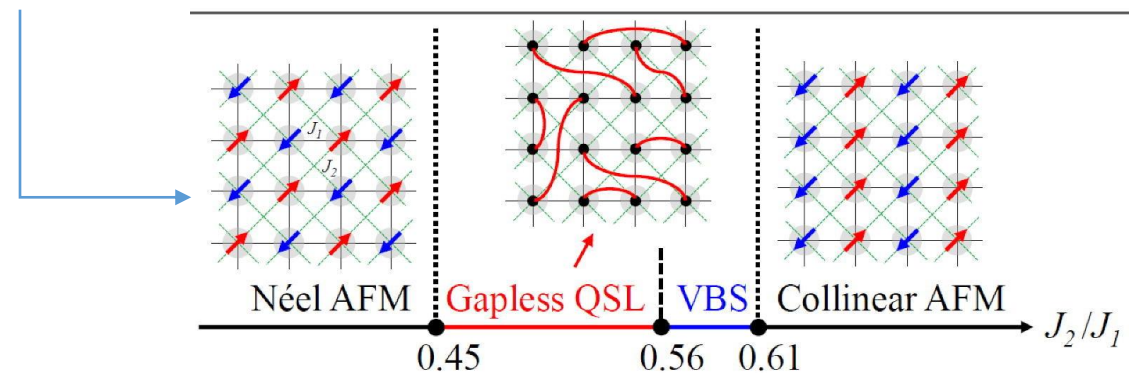


$$\alpha = \frac{J_2}{J_1}$$

What is the nature of the ground-state in highly frustrated region ?

a few selected refs. :

- DMRG → gapped Z2 liquid in the intermediate region):
Spin liquid ground state of the spin-1/2 square J_1 - J_2 Heisenberg model, H.-C. Jiang, H. Yao & L. Balents, Phys. Rev. B 86, 024424 (2012)
- DMRG → gapped plaquette VBC phase.
Plaquette ordered phase and quantum phase diagram in the spin-1/2 J_1 - J_2 square Heisenberg model, S.-S. Gong et al. Phys Rev Lett, 113 (2014)
- DMRG → 2 phases, gapless QSL & gapped VBC.
Critical level crossings and gapless spin liquid in the square-lattice spin-1/2 J_1 - J_2 Heisenberg antiferromagnet, L. Wang, A.W. Sandvik, Phys Rev Lett, 121 107202 (2018)
- 2D tensor method (PEPS) → gapless QSL.
Gapless quantum spin liquid and global phase diagram of the spin-1/2 J_1 - J_2 square antiferromagnetic Heisenberg model, W.Y. Liu et al., Science Bulletin, 67, 1034 (2022)



Triangular lattice J_1 - J_2 Heisenberg model

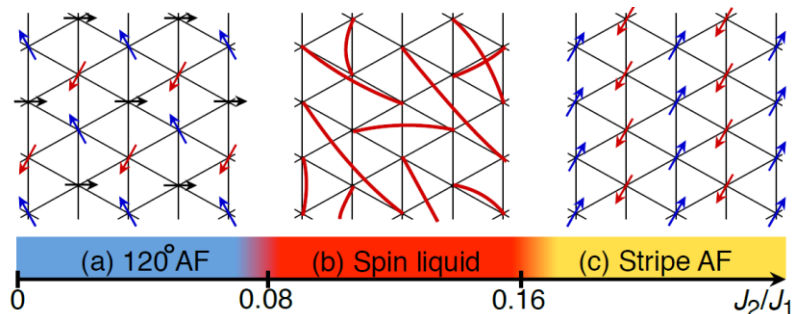
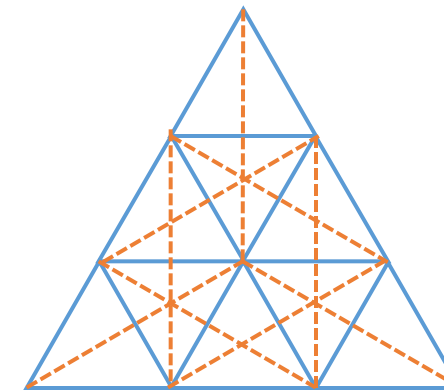


FIG. 1. (Color online) Schematic illustrations of the coplanar three-sublattice (black, blue, and red) magnetic order on the triangular lattice (a), the resonating-valence bond spin liquid (b) and the collinear two-sublattice (blue and red) stripe magnetic order (c). The phase diagram, as obtained by using variational Monte Carlo is also reported. Note that the DSL found here can be represented as a resonating-valence bond spin liquid with a power-law distribution of bond amplitudes.

QSL phase – but nature still controversial

Spin liquid nature in the Heisenberg J_1 - J_2 triangular antiferromagnet
Y. Iqbal, W.-J. Hu, R. Thomale, D. Poilblanc & F. Becca
Phys. Rev. B **93**, 144411 ([2016](#))

→ variational Monte Carlo simulations in favor of a gapless QSL

Spin liquid phase of the $S=1/2$ J_1 - J_2 Heisenberg model on the triangular lattice
Z. Zhu & S. R. White, Phys. Rev. B **92**, 041105(R) ([2015](#))

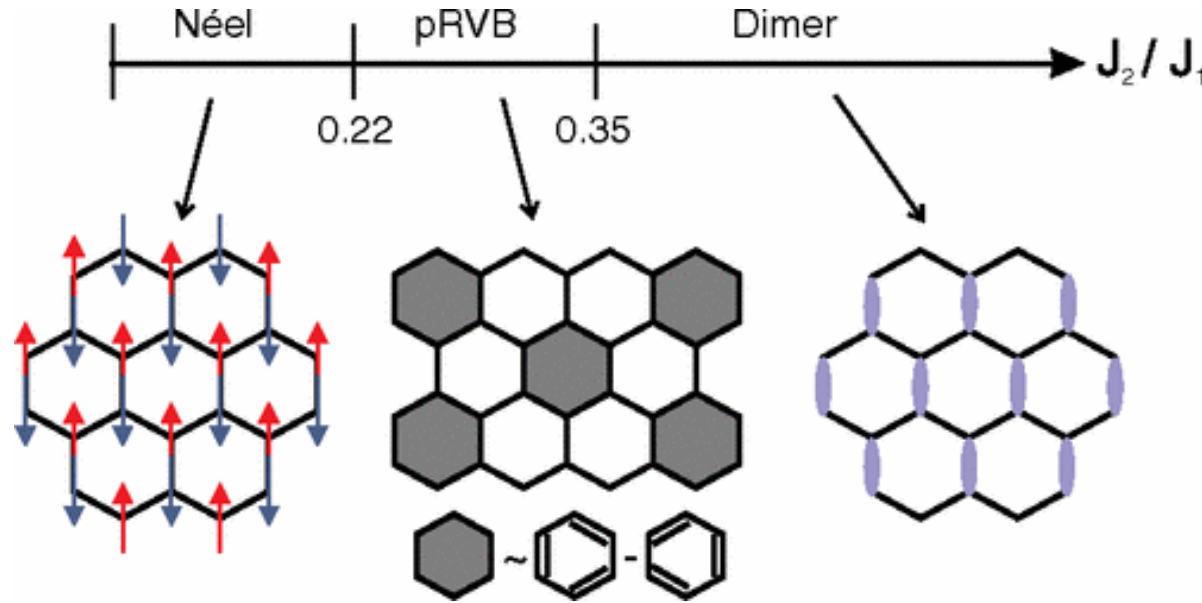
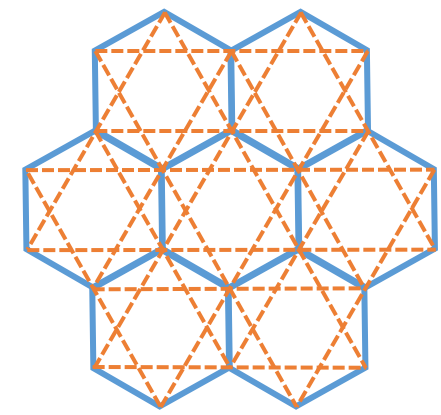
→ DMRG study in favor of a gapped QSL

Competing spin-liquid states in the spin-1/2 Heisenberg model on the triangular lattice,

W.-J. Hu, S.-S. Gong, W. Zhu & D. N. Sheng, Phys. Rev. B **92**, 140403(R) ([2015](#))

→ DMRG study in favor of a gapped QSL

Honeycomb lattice J_1 - J_2 Heisenberg model



from: Ganesh, Van den Brink & Nishimoto,
Phys. Rev. Lett. 110, 127203 (2013)

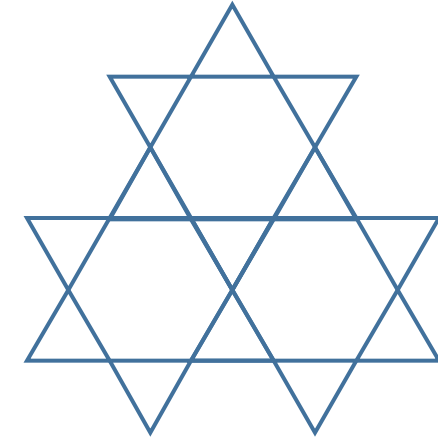
No QSL, but possibility of some deconfined critical point
separating the Néel and the plaquette phase
T. Senthil et al., Science 303 (2004)

See also

- *An investigation of the quantum $J_1 - J_2 - J_3$ model on the honeycomb lattice.*
J. Fouet, Ph. Sindzingre, & C. Lhuillier, Eur. Phys. J. B 20, 241 (2001)
- *Weak Plaquette Valence Bond Order in the $S=1/2$ Honeycomb J_1 - J_2 Heisenberg Model*
Z. Zhu, Huse, & White, Phys. Rev. Lett. 110, 127205 (2013)
- *Competition between spin liquids and valence-bond order in the frustrated spin-1/2 Heisenberg model on the honeycomb lattice.* F. Ferrari, S Bieri & F Becca, Phys. Rev. B 96 (10), 104401 (2017)
- *Dynamical properties of Néel and valence-bond phases in the J_1 - J_2 model on the honeycomb lattice*
F. Ferrari & F. Becca, J. Phys.: Cond. Matt. 32 274003 (2020)

Kagome lattice Heisenberg model

- Intensively studied since 1990 !
- Consensus on the absence of magnetic order
- But the nature of the QSL is still not established:
gapped Z_2 or gapless (Dirac) ?

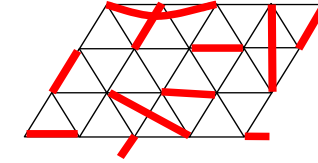


A few selected references (numerics ≥ 2011)

- *Exact diagonalizations* ($N=48$)
 $S=1/2$ kagome Heisenberg antiferromagnet revisited
A. M. Läuchli, J. Sudan & R. Moessner, Phys. Rev. B 100, 155142 ([2019](#)) → no clear indication of Z_2 character
- *DMRG*
S. Yan, D. Huse & S. White, Science ([2011](#)) → gapped QSL
S. Depenbrock, I. P. McCulloch, & U. Schollwöck PRL ([2012](#)) → gapped Z_2 QSL
H.-C. Jiang, Z. Wang & L. Balents, Nature ([2012](#)) → gapped Z_2 QSL
Y.-C. He, M. P. Zaletel, M. Oshikawa & F. Pollmann, Phys. Rev. X 7 ([2017](#)) → gapless Dirac QSL
- *2D tensor network*
J.-W. Mei, J.-Y. Chen, H. He, & X.-G. Wen, Phys. Rev. B **95**, 235107 ([2017](#)) → gapped Z_2 QSL (but $\xi \sim 10$)
- *Variational QMC*
Y. Iqbal, F. Becca, S. Sorella & D. Poilblanc, Phys. Rev. B 87, 060405(R) ([2013](#)) → gapless Dirac QSL
Y. Iqbal, D. Poilblanc & F. Becca, Phys. Rev. B 91, 020402(R) ([2015](#)) → gapless Dirac QSL
- *combination* of methods: functional RG, var. QMC, DMRG
D. Kiese *et al.*, Phys. Rev. Research 5, L012025 ([2023](#)) → gapless Dirac QSL

\mathbb{Z}_2 spin liquids

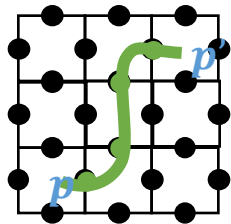
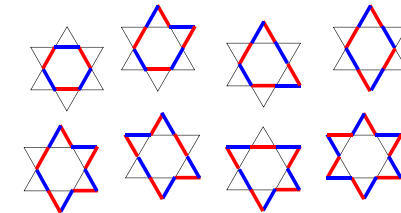
(short-range) Resonating Valence-Bond



$$H = -J \sum |\triangle\rangle\langle\triangle| + |\triangle\rangle\langle\triangle| + |\triangle\rangle\langle\triangle| + |\triangle\rangle\langle\triangle| + v \sum |\triangle\rangle\langle\triangle| + |\triangle\rangle\langle\triangle| + |\triangle\rangle\langle\triangle| + |\triangle\rangle\langle\triangle|$$

Quantum dimer models

an exactly solvable QDM
(kagome lattice)



Toric code, topological order
 \mathbb{Z}_2 gauge theory

Resonating valence bonds

P. W. Anderson, Mater. Res. Bull. 8, 153 (1973); P. Fazekas & P. W. Anderson, Phil. Mag. 30, 432 (1974)

Mat. Res. Bull. Vol. 8, pp. 153-160, 1973. Pergamon Press, Inc. Printed in the United States.

RESONATING VALENCE BONDS: A NEW KIND OF INSULATOR?*

P. W. Anderson
Bell Laboratories, Murray Hill, New Jersey 07974
and
Cavendish Laboratory, Cambridge, England

(Received December 5, 1972; Invited**)

$$H = J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j$$

Some variational energies

Triangular lattice:

- $E_{VB}/(JN) = -3/8$
- $E_{\text{Néel}}/(JN) = -3/8$ (120° degrees Néel state)

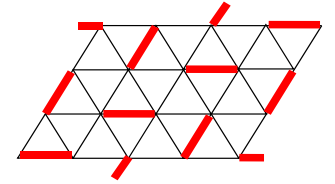
Kagome:

- $E_{VB}/(JN) = -3/8$
- $E_{\text{Néel}}/(JN) = -1/4$ (120° degrees Néel state)

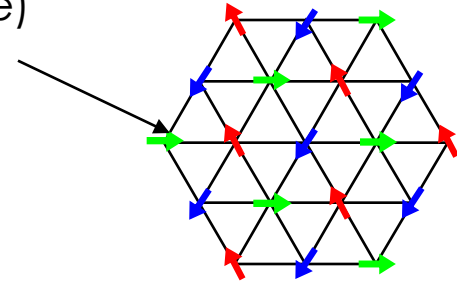
On the cubic lattice in dimension d

- $E_{VB}/(JN) = -3/8$
- $E_{\text{Néel}}/N = -d/4$ (collinear Néel)

→ **frustration, low connectivity and low dimension favor valence-bond states compared to magnetically ordered Néel states**



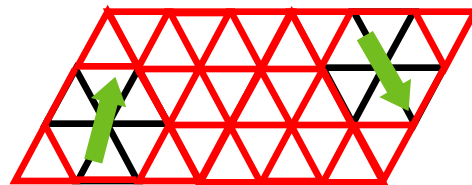
$$\text{red bond} = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$



$$|\psi_{\text{RVB}}\rangle \sim \text{triangle 1} + \text{triangle 2} + \text{triangle 3} + \dots = \text{resonating state}$$

Spatially & rotationally ($S=0$) invariant state

expect deconfined spin-1/2 excitations, called spinons



Variational RVB states with adjustable valence-bond length distribution

Some *new variational Resonating-Valence-Bond-type wave functions* for the Spin-1/2 Antiferromagnetic Heisenberg Model on a Square Lattice

S. Liang, B. Douçot, and P. W. Anderson, Phys. Rev. Lett. 61, 365 (1988)

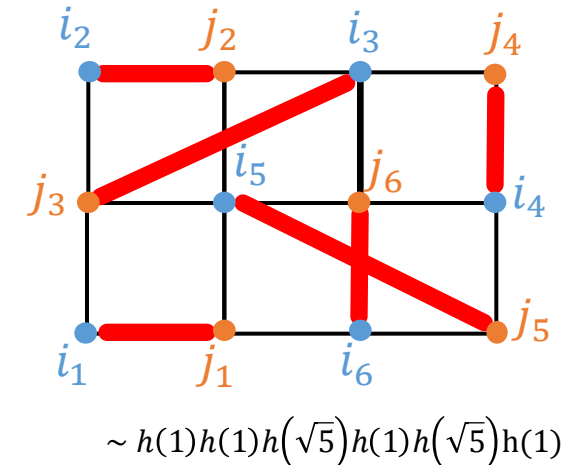
$$|\psi_{\text{RVB-LDA}}\rangle = \sum_{i_\alpha \in A, j_\alpha \in B} h(|\vec{r}_{i_1} - \vec{r}_{j_1}|) h(|\vec{r}_{i_2} - \vec{r}_{j_2}|) \cdots h(|\vec{r}_{i_n} - \vec{r}_{j_n}|) (-1)^{N_{A\downarrow}} |(i_1, j_1)\rangle |(i_2, j_2)\rangle \cdots |(i_n, j_n)\rangle$$

$|(i, j)\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ valence bond

$h(r) \geq 0$ function of the val. bond length

$(-1)^{N_{A\downarrow}}$: insures the so-called Marshal-sign

$|\psi_{\text{RVB-LDA}}\rangle$ has some AF/Néel long range order if $h(r)$ decays more slowly than r^{-5} , it otherwise has some finite spin-spin correlation length.



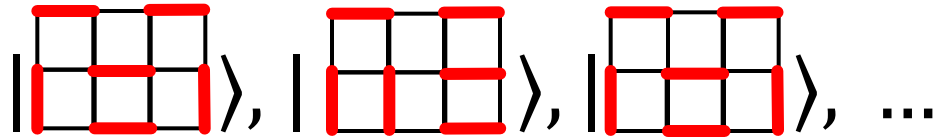
First-neighbor valence-bond only ($h(1) = 1$ and 0 otherwise) $\rightarrow \langle \vec{S}_i \cdot \vec{S}_j \rangle_{\text{short range RVB}} = -0.302$

optimized $h(r)$ (power-law decay) $\rightarrow \langle \vec{S}_i \cdot \vec{S}_j \rangle_{\text{long range RVB}} = -0.3344$ (state with Néel LRO)

compare with exact ground-state : $\langle \vec{S}_i \cdot \vec{S}_j \rangle_{\text{ground state}} \sim -0.335$

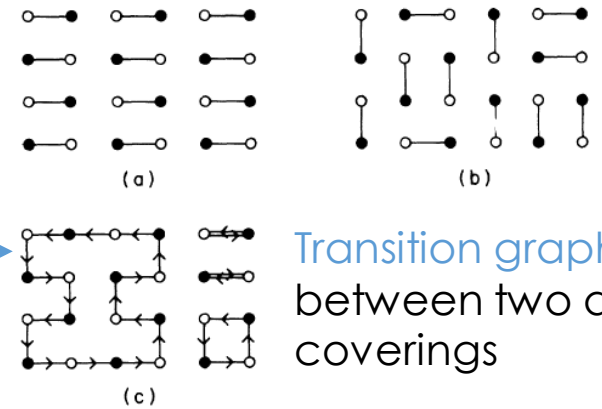
Dynamics in the short-range RVB subspace

- A valence-bond state is not an eigensate of simple Heisenberg-like models (exceptions: J_1 - J_2 chain at the Majumdar-Gosh point, 2D Shastry-Sutherland model, Klein models, ...)
- ... but following Anderson's intuition one may consider the **subspace generated by all nearest-neighbor valenced bond states as a variational subspace** to describe rotationally invariant states with short-ranged spin-spin correlations (spin liquid candidate).



$$H = \sum_{i < j} J_{ij} \vec{S}_i \cdot \vec{S}_j$$

- VB: Linearly independent on many lattices. But **non-orthogonal**. Scalar product? Look at the *transition graph*



Transition graph between two dimer coverings

- Largest non-trivial ($\neq \pm 1$) scalar product: $\langle = | \parallel \rangle = \pm 1/2$
- **Overlap matrix:** $\Omega_{ab} = \langle a | b \rangle$ $|\Omega_{ab}| = \prod_{l \in \text{loops}} \alpha^{\text{length}(l)-2}$ with $\alpha = 1/\sqrt{2}$.
- Orthogonalization: $|\tilde{a}\rangle = \sum_c [\Omega^{-\frac{1}{2}}]_{ac} |c\rangle \rightarrow \langle \tilde{a} | \tilde{b} \rangle = \delta_{ab}$
- H_{eff} : **project the spin Hamiltonian in the RVB subspace:** $(H_{\text{eff}})_{a,b} \stackrel{\text{def}}{=} \langle \tilde{a} | H | \tilde{b} \rangle$
- Overlapp expansion: formal expansion in powers of α
D. Rokhsar & S. Kivelson Phys. Rev. Lett. 61, 2376 (1988)

Dynamics in the short-range RVB subspace

Generalized hard-core dimer model approach to low-energy Heisenberg frustrated antiferromagnets: General properties and application to the kagome antiferromagnet

D. Schwandt, M. Mambrini, and D. Poilblanc,
Phys. Rev. B 81, 214413 (2010)

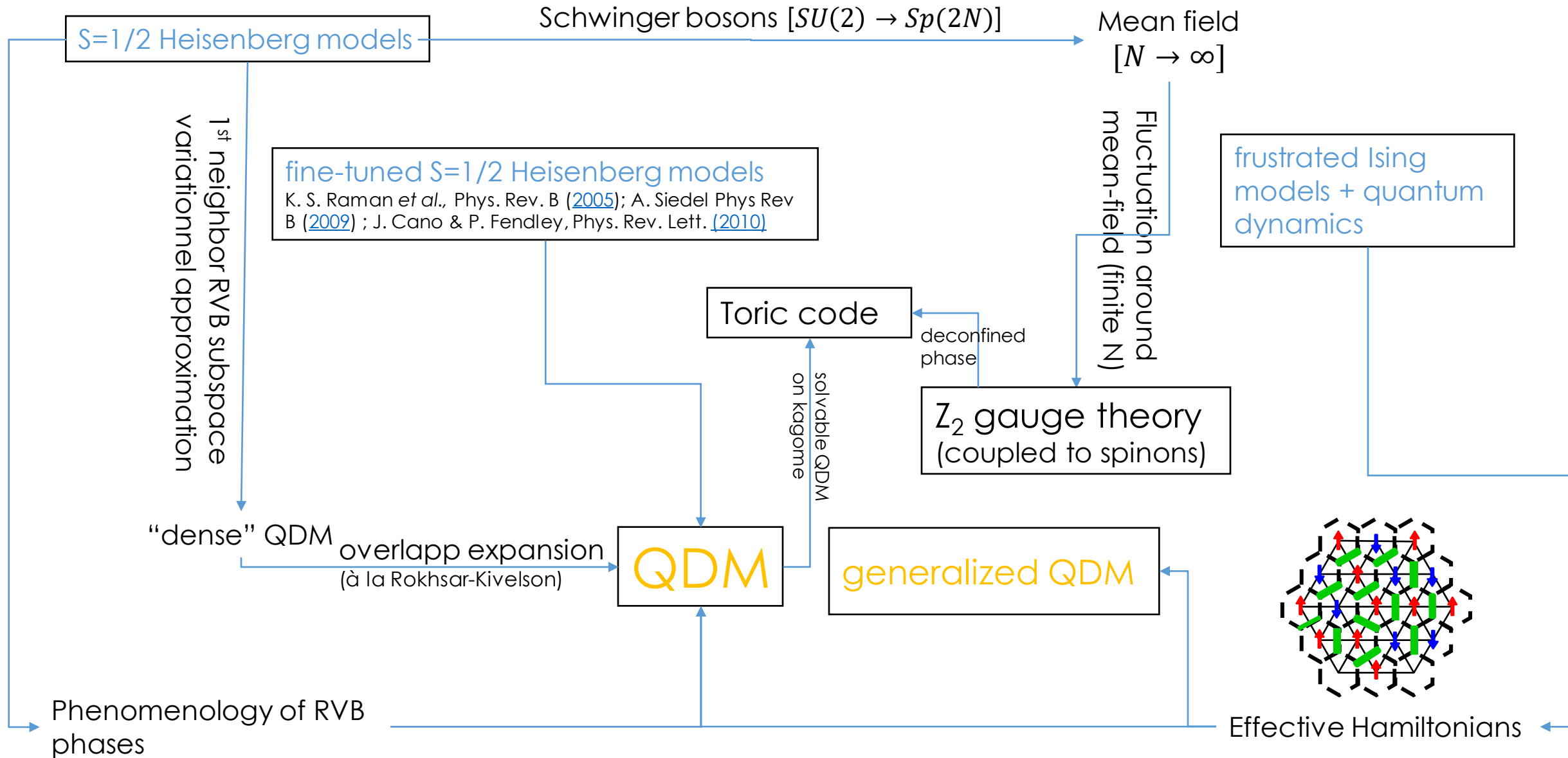
Effective Hamiltonian describing the projection of the kagome lattice Heisenberg model into the 1st neighbor RVB subspace
(in powers of $\alpha = 1/\sqrt{2}$)



Processes	$\hat{\mathcal{H}}_{\text{eff}}/J$		Processes	$\hat{\mathcal{H}}_{\text{eff}}/J$	
	LO	∞		LO	∞
	$-3\alpha^4$	$-\frac{3\alpha^4}{1-\alpha^8}$		$-\alpha^8$	$-\frac{\alpha^8}{1-\alpha^{16}}$
	$3\alpha^8$	$\frac{3\alpha^8}{1-\alpha^8}$		$-\alpha^8$	$-\frac{\alpha^8}{1-\alpha^{16}}$
	$-2\alpha^6$	$-\frac{2\alpha^6}{1-\alpha^{12}}$		$-\alpha^8$	$-\frac{\alpha^8}{1-\alpha^{16}}$
	$-2\alpha^6$	$-\frac{2\alpha^6}{1-\alpha^{12}}$		α^{16}	$\frac{\alpha^{16}}{1-\alpha^{16}}$
	$-2\alpha^6$	$-\frac{2\alpha^6}{1-\alpha^{12}}$		α^{16}	$\frac{\alpha^{16}}{1-\alpha^{16}}$
	$2\alpha^{12}$	$\frac{2\alpha^{12}}{1-\alpha^{12}}$		α^{16}	$\frac{\alpha^{16}}{1-\alpha^{16}}$
	$2\alpha^{12}$	$\frac{2\alpha^{12}}{1-\alpha^{12}}$		0	0
	$2\alpha^{12}$	$\frac{2\alpha^{12}}{1-\alpha^{12}}$		0	0

Yellow loops: diagonal (or potential) terms
white loops: off-diagonal (dimer hopping or kinetic) terms

From spin models to quantum dimer models (QDM)



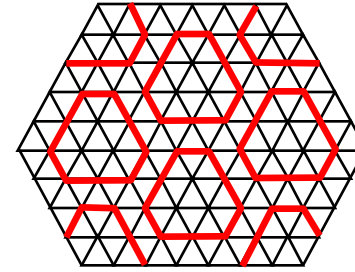
Triangular lattice QDM

Resonating Valence Bond Phase in the Triangular Lattice Quantum Dimer Model

Moessner & Sondhi, Phys. Rev. Lett. 86, 1881 (2001)

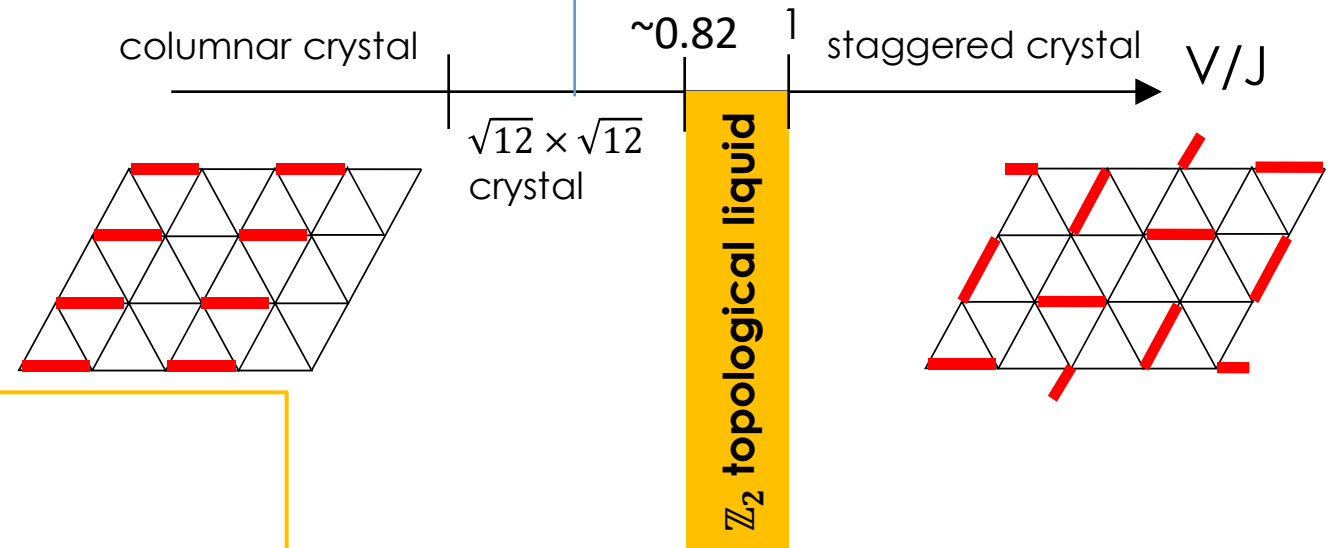
Hamiltonian :

$$H = -J \sum \left| \begin{array}{c} \triangle \\ \triangle \end{array} \right\rangle \left\langle \begin{array}{c} \triangle \\ \triangle \end{array} \right| + \left| \begin{array}{c} \triangle \\ \triangle \end{array} \right\rangle \left\langle \begin{array}{c} \triangle \\ \triangle \end{array} \right| + v \sum \left| \begin{array}{c} \triangle \\ \triangle \end{array} \right\rangle \left\langle \begin{array}{c} \triangle \\ \triangle \end{array} \right| + \left| \begin{array}{c} \triangle \\ \triangle \end{array} \right\rangle \left\langle \begin{array}{c} \triangle \\ \triangle \end{array} \right|$$



A. Ralko et al., Phys. Rev. B 74, 134301 (2006);
Phys. Rev. B 76, 140404(R) (2007)

T=0 Phase diagram :



- Short-range dimer-dimer correlations
- Gapped excitation spectrum
- The g. s. degeneracy depends on the topology
- The degenerate g. s. cannot be distinguished by any local observable
- Effective \mathbb{Z}_2 gauge theory
- Anyonic excitations: *visons*

Rokhsar-Kivelson point $J = V > 0$

D. Rokhsar & S. Kivelson, Phys. Rev. Lett. 61, 2376 (1988)

- Hamiltonian as a sum of local projectors

$$H_{\text{RK}} = \sum_{d \in \{\text{diamonds}\}} (|///\rangle\langle///| + |=\rangle\langle=| - |///\rangle\langle=| - |=\rangle\langle///|) = 2 \sum_{d \in \{\text{diamonds}\}} |\omega(d)\rangle\langle\omega(d)|$$

with local state $|\omega(d)\rangle$ of the diamond d defined as $|\omega(d)\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} (|= \rangle - |// \rangle)$.

H_{RK} = sum of local projectors \rightarrow eigenvalues ≥ 0

- Ground-state ?

$$|\text{RK state}\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{N}} \sum_{c \in \{\text{dimer coverings}\}} |c\rangle$$

Pick a diamond d and a dimer configuration c

$$\langle \omega | c \rangle = \begin{cases} +|c \text{ with } d \text{ removed}\rangle & \text{if } |c\rangle = |\cdots =_d \cdots\rangle \\ -|c \text{ with } d \text{ removed}\rangle & \text{if } |c\rangle = |\cdots //_d \cdots\rangle \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow \forall d \langle \omega | \text{RK state} \rangle = 0 \Rightarrow H_{\text{RK}} |\text{RK state}\rangle = 0 \Rightarrow |\text{RK state}\rangle$ is one ground-state of H_{RK}

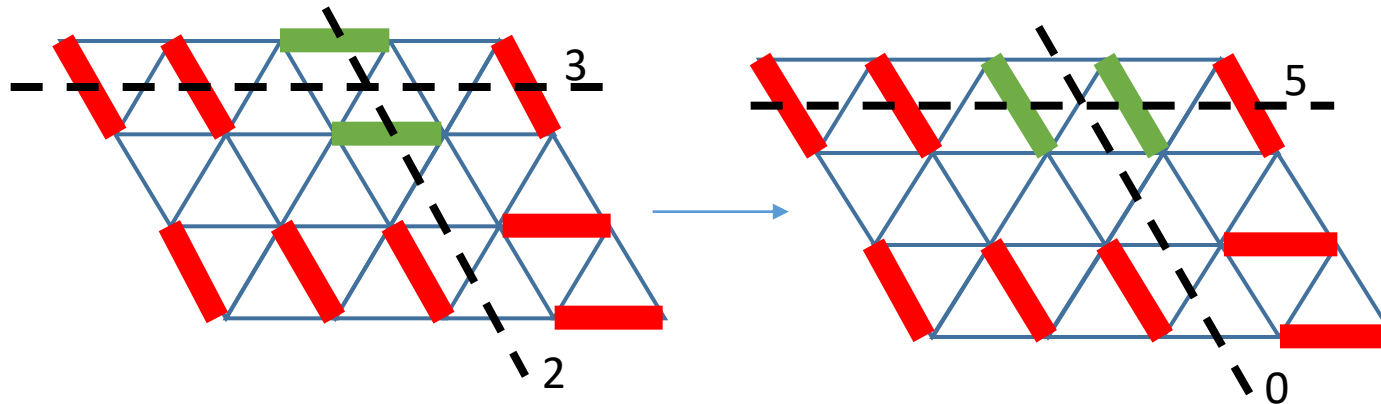
- Remarks:

- \exists one ground-state per ergodic sector
ergodic sector = set of dimer configurations which are connected by successive applications of the Hamiltonian
- Excited states are not known exactly

Rokhsar-Kivelson point $J = V > 0$

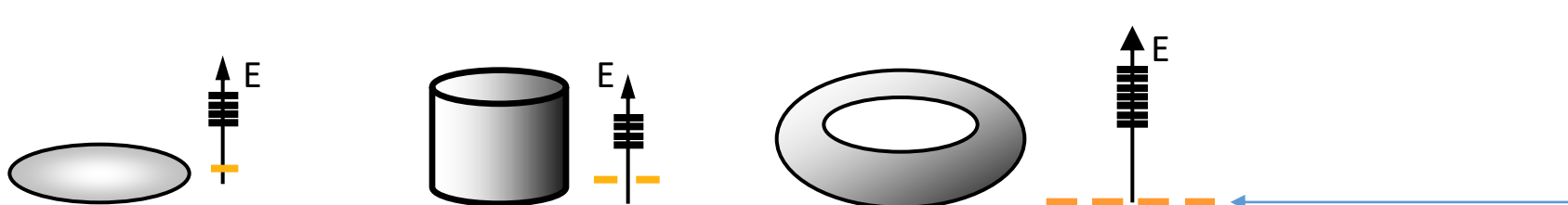
Rokhsar & Kivelson, Phys. Rev. Lett. 61, 2376 (1988)
 R. Moessner and S. L. Sondhi, Phys. Rev. Lett. 86, 1881 (2001)

- **Diagonal observables** in the dimer basis \Rightarrow classical stat. mech. problem
 \hat{O} : diagonal. Ex. : dimer-dimer correlation.
 Expectation value : $\langle \text{RK} | \hat{O} | \text{RK} \rangle = \frac{1}{N} \sum_{c \in \{\text{dimer coverings}\}} O(c)$
 Kasteleyn (1961)+Fisher: classical partition function = Pfaffian \Rightarrow exact/analytical experssion for the correlations at the RK point
- **Square lattice**: \Rightarrow algebraic dimer-dimer correlations ($\sim r^{-2}$)
- **Triangular lattice** \Rightarrow finite correlation length and exponential decay \Rightarrow **dimer liquid** [Moessner & Sondhi 2001]
- Ground-state degeneracy and topological (parity) sectors (non-bipartite lattices)



The parities of the # of dimer crossing the dashed lines is conserved under local dimer moves \Rightarrow 4 topological sector on a torus
4 degenerate ground-states at the RK points = topological degeneracy

NB: on bipartite lattices there are more sectors



Triangular lattice QDM – indistinguishability of the ground-states

(@ RK point)

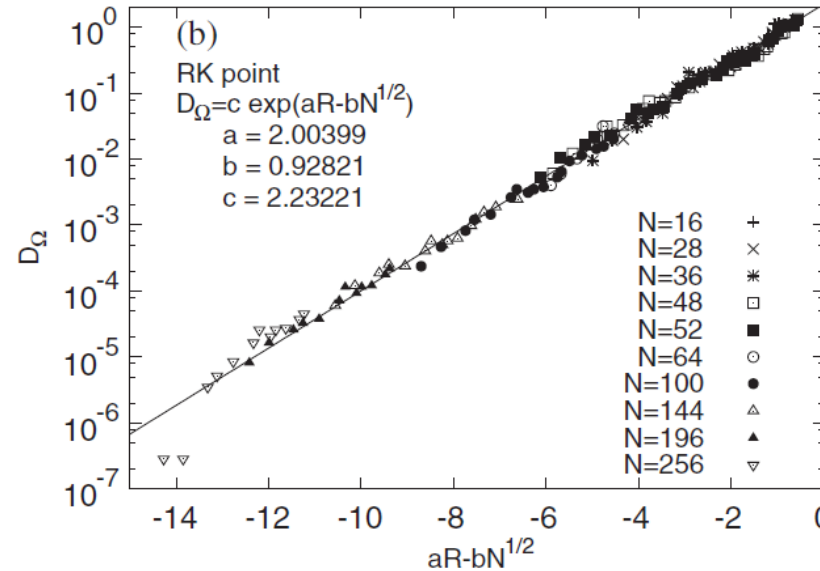
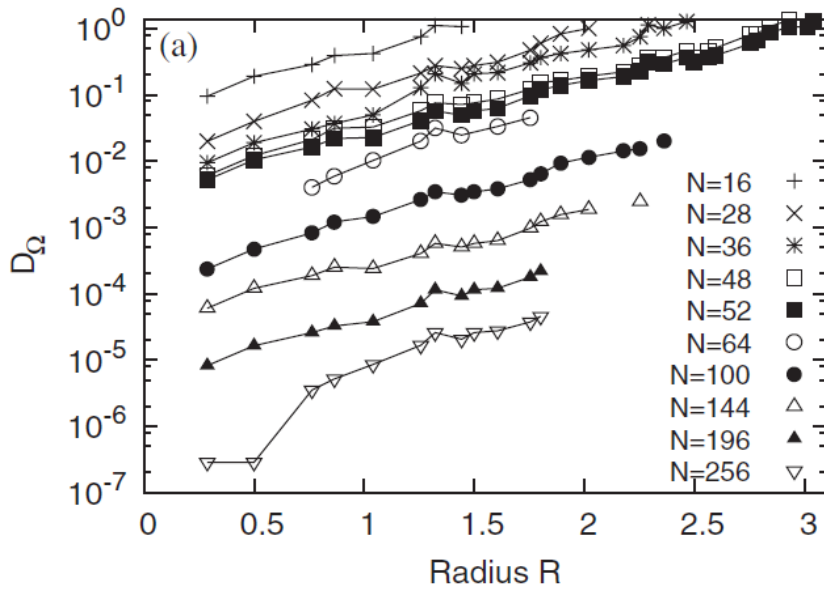
$$\text{diff}_\Omega(\rho, \rho^{\text{ref}}) \stackrel{\text{def}}{=} \max_{|\mathcal{O}| \leq 1} |\text{Tr}[\mathcal{O}\rho] - \text{Tr}[\mathcal{O}\rho^{\text{ref}}]|$$

\mathcal{O} acting on Ω

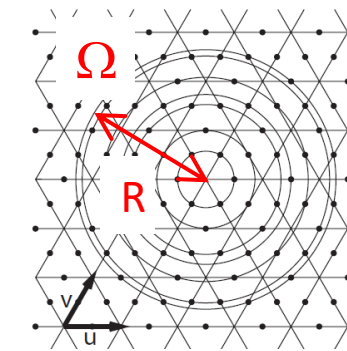
measures to which extent 2 states ρ and ρ^{ref} can be distinguished by an observable acting (only) on the region Ω

$\rho^{\text{ref}} \stackrel{\text{def}}{=} \frac{1}{q} \sum_{i=1}^q |\psi_i\rangle\langle\psi_i|$ with $|\psi_{i=1\dots 4}\rangle$ degenerate ground-states, NB: ρ^{ref} indep. of the choice of the basis

$D_\Omega \stackrel{\text{def}}{=} \max_{|\psi\rangle} \text{diff}_\Omega(|\psi\rangle\langle\psi|, \rho^{\text{ref}})$ measure of the “distinguishability” of the ground-state over the region Ω



S. Furukawa, G. M. & M. Oshikawa,
Phys. Rev. Lett. 96 047211 (2006);
J. Phys.: Cond. Mat. 19 145212 (2007)



Subsystem size

Total system size

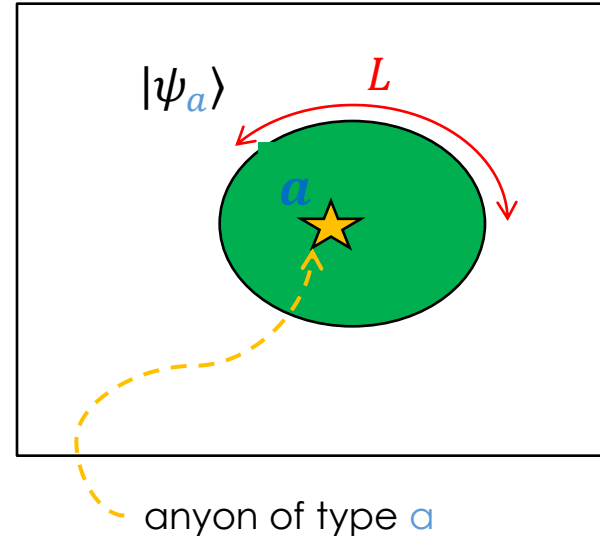
Figure 2. The result for the RK wavefunction: (a) the value of D_Ω as a function of the radius R for different system sizes N ; (b) fitting of the data using an exponential function $D_\Omega \approx ce^{aR - b\sqrt{N}}$.

$$D_\Omega \sim \exp(aR - b\sqrt{N})$$

Topological Entanglement Entropy

A. Kitaev & J. Preskill, Phys. Rev. Lett. 96, 110404 (2006)

M. Levin & X.-G. Wen, Phys. Rev. Lett. 96, 110405 (2006)



universal subleading constant

boundary law

$$S_{vN} = \alpha L - \gamma + \dots$$

$$\gamma = \log\left(\frac{\mathcal{D}}{d_a}\right)$$

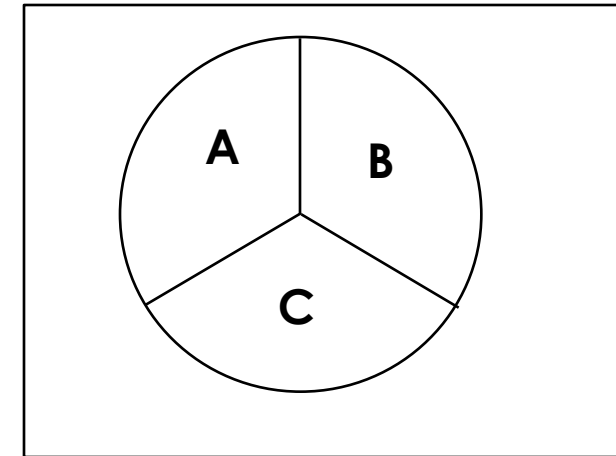
$\mathcal{D} = \sqrt{\sum_i (d_i)^2}$ total quantum dimension

d_i : quantum dimensions of the anyon of type i

ground-state: $\gamma = \log(\mathcal{D})$

Practical way to extract γ :

$$\gamma = S_{ABC} - (S_{AB} + S_{BC} + S_{AC}) + (S_A + S_B + S_C)$$

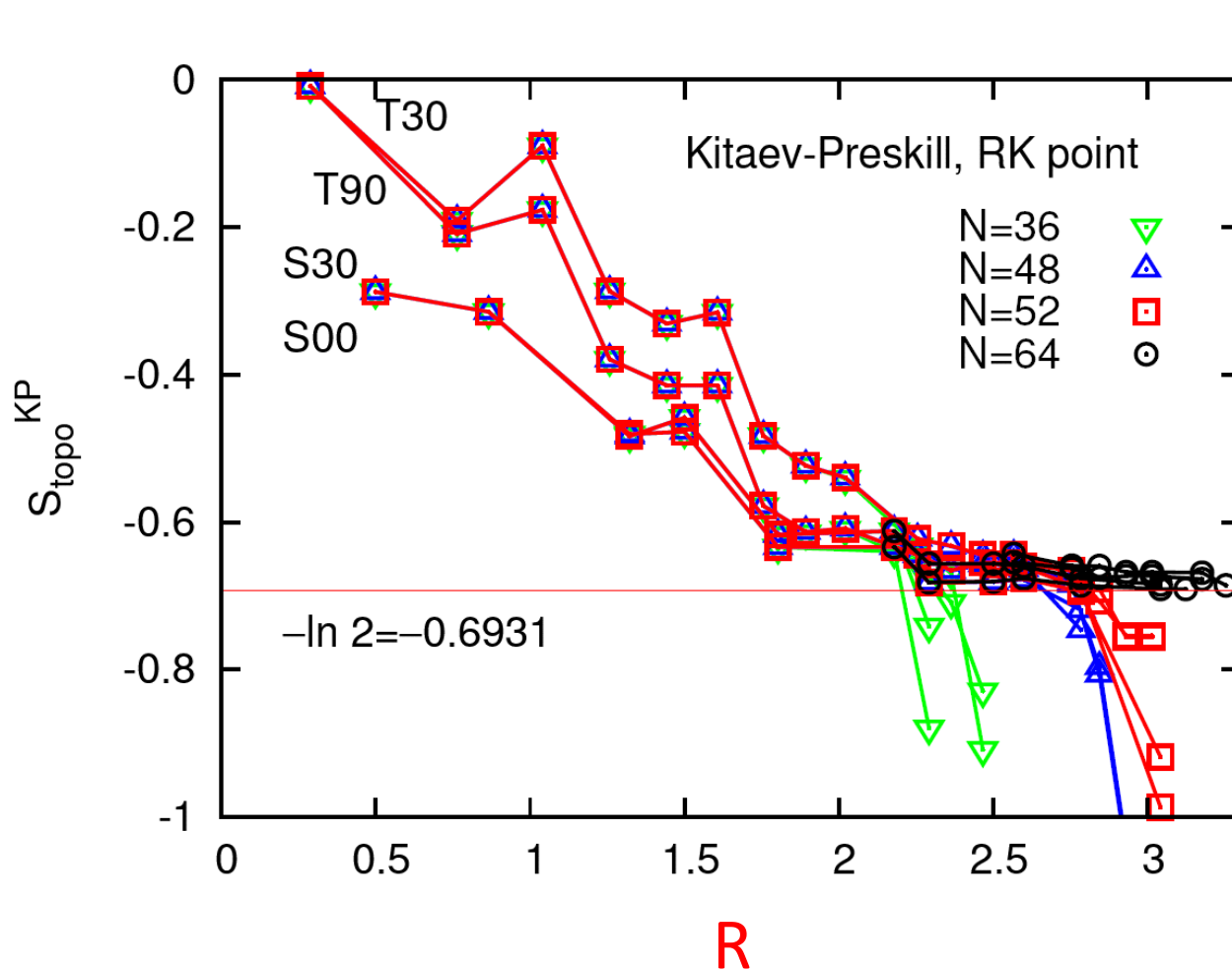


\mathbb{Z}_2 liquid / toric code example: $d_{\text{triv.}} = 1, d_{\text{vison}} = 1, d_{\text{hole}} = 1, d_{\text{vison-hole}} = 1 \rightarrow \mathcal{D} = 2$

Triangular lattice QDM – Topological Entanglement Entropy

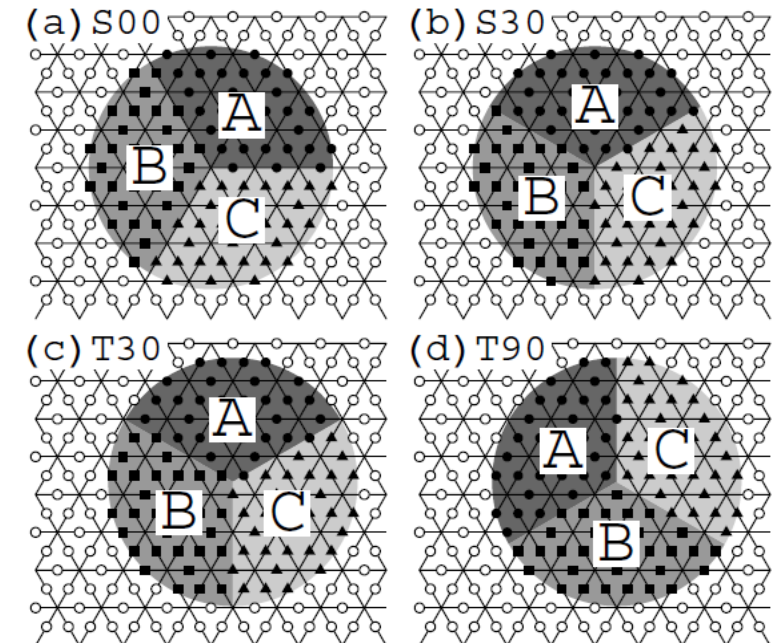
A. Kitaev & J. Preskill, Phys. Rev. Lett. 96, 110404 (2006)

M. Levin & X.-G. Wen, Phys. Rev. Lett. 96, 110405 (2006)



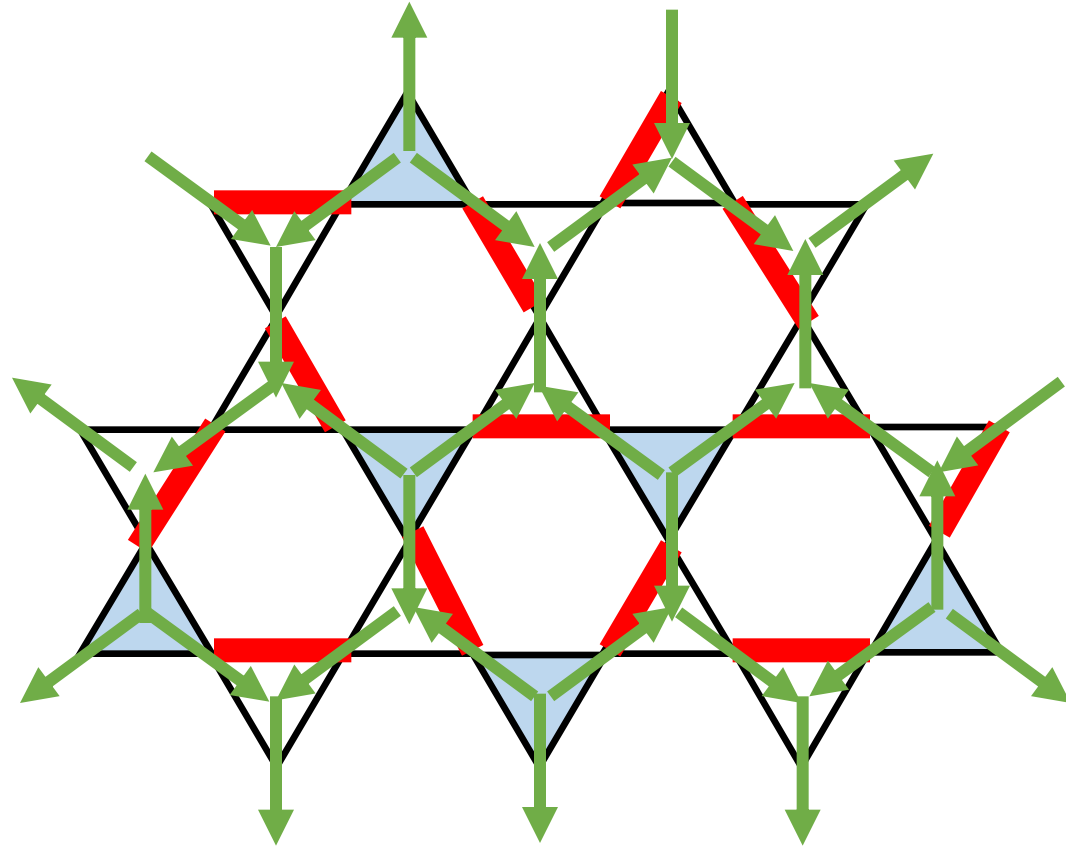
A diagram of a circle divided into three regions labeled A, B, and C. A red arrow labeled R points from the center towards the boundary between regions B and C. Below the diagram, the equation for the topological entropy is given:

$$S_{\text{topo}}^{\text{KP}} = \gamma = S_{ABC} - (S_{AB} + S_{BC} + S_{AC}) + (S_A + S_B + S_C)$$



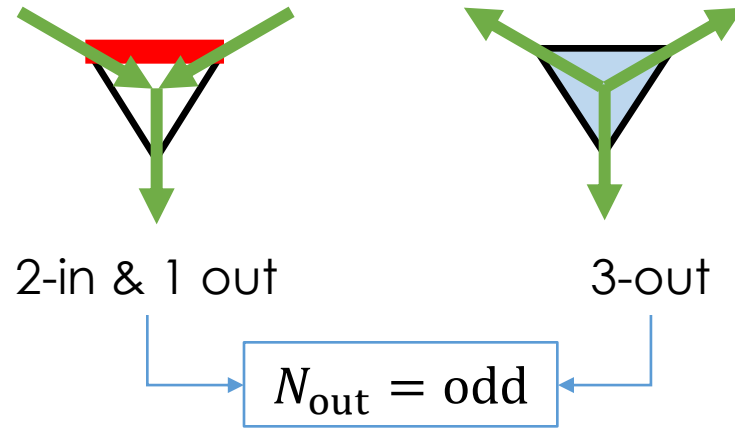
Topological entanglement entropy in the quantum dimer model on the triangular lattice, S. Furukawa & G. M., Phys. Rev. B 75, 214407 (2007)

Exactly solvable QDM with \mathbb{Z}_2 topological liquid ground state

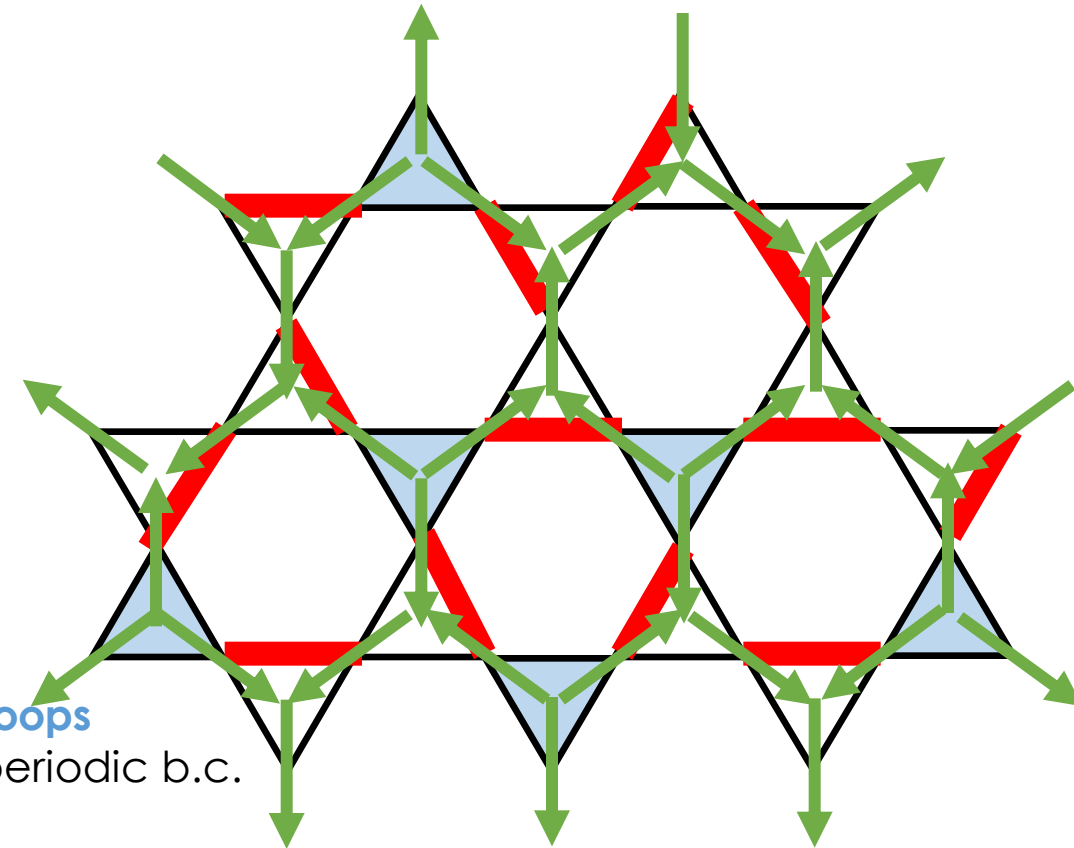


Exactly solvable QDM with \mathbb{Z}_2 topological liquid ground state

- Dimer coverings of the kagome lattice: **arrow representation**
- works for any lattice made of corner-sharing triangles



Elser & Zeng Phys. Rev. B 48, 13647 (1993)



- Arrow configuration with $N_{out} = \text{odd} \Leftrightarrow$ valid dimer covering
- How to generate all the arrow configurations?
Start from a reference config. and **flip arrows around any closed loops**
- Application: counting dimer coverings on a Kagome lattice with periodic b.c.
of dimer coverings $Z_{\text{kag.}} = 2^{N_{\text{hex}} - 1 + 2} = 2^{N_{\text{sites}} + 1}$
 - 1: flipping all hexagons=nothing (each arrow is flipped twice).
 - +2: \exists two topologically inequivalent loops on a torus
- Remark: The simplicity of the above result is a property of lattices made of corner sharing triangles. In general, counting dimer coverings is much more difficult, even on planar graphs. As a comparison, the # of coverings on the *square* lattice is:
 $Z_{\text{sq.}} \sim \exp(N_{\text{sites}} G / \pi)$ with $G \simeq 0.915$ is Catalan's constant [Kasteleyn 1961].

Exactly solvable QDM with \mathbb{Z}_2 topological liquid ground state

G.M., Serban, & Pasquier, Phys. Rev. Lett. 89, 137202 (2002)

- Arrow operators**

Pick a reference configuration (any)

$\sigma_i^Z \stackrel{\text{def}}{=} \text{flips arrow } i$ (= \mathbb{Z}_2 Gauge field)

$\sigma_i^x \stackrel{\text{def}}{=} \begin{cases} +1 & \text{if arrow } i \text{ is in the same direction as in the ref. configuration} \\ -1 & \text{if the arrow has been flipped} \end{cases}$

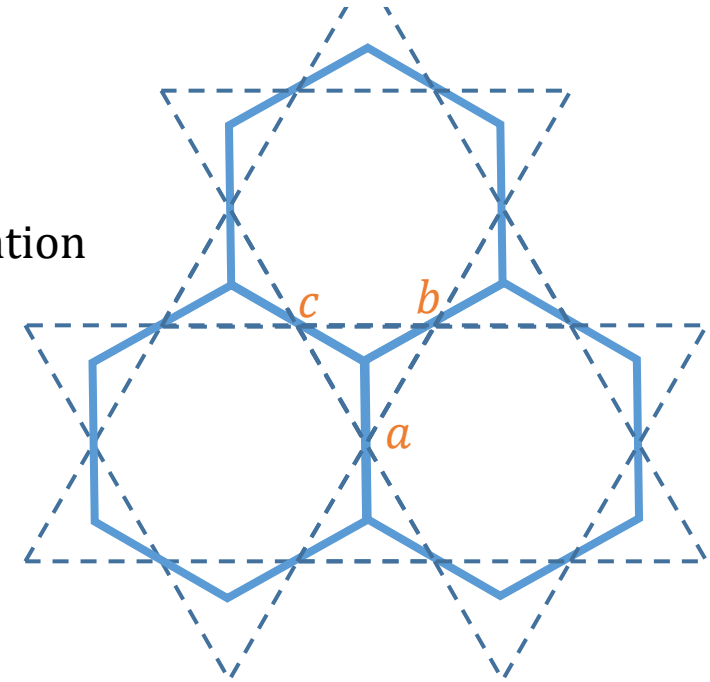
- Dimer constraint** $\sigma_a^x \sigma_b^x \sigma_c^x = 1$ for all $(abc) \in \{\text{triangles}\}$

- Elementary move (commutes with the constraints)

$$B_h \stackrel{\text{def}}{=} \sigma_{h_1}^Z \sigma_{h_2}^Z \sigma_{h_3}^Z \sigma_{h_4}^Z \sigma_{h_5}^Z \sigma_{h_6}^Z$$

$$(B_h)^2 = 1, [B_h, B_{h'}] = 0$$

+commutation with the constraint $[B_h, \sigma_a^x \sigma_b^x \sigma_c^x] = 0$



- Hamiltonian**

$$H = - \sum_{h \in \{\text{hexagons}\}} \sigma_{h_1}^Z \sigma_{h_2}^Z \sigma_{h_3}^Z \sigma_{h_4}^Z \sigma_{h_5}^Z \sigma_{h_6}^Z + \text{constraint } \tau_a^x \tau_b^x \tau_c^x = 1 \text{ on physical states}$$

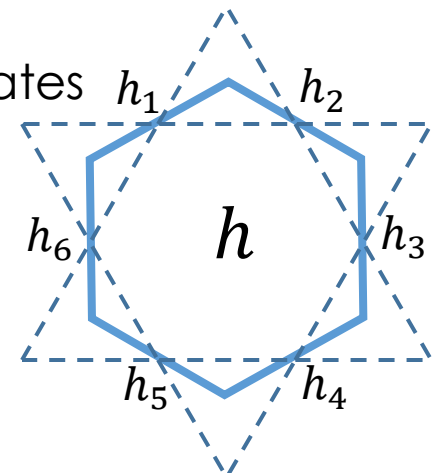
Trivial spectrum since: $H = - \sum_{h \in \{\text{hexagons}\}} B_h$ Ground-state energy $E_0 = -N_{\text{hex}}$.

- Equivalence with toric code** on the hexagonal lattice [Kitaev 1997]

- $H = H_{\text{magnetic}} + H_{\text{electric}}$

with $H_{\text{magnetic}} = - \sum_{h \in \{\text{hex.}\}} \tau_{h_1}^Z \tau_{h_2}^Z \tau_{h_3}^Z \tau_{h_4}^Z \tau_{h_5}^Z \tau_{h_6}^Z$

and $H_{\text{electric}} = - \lim_{U \rightarrow \infty} U \sum_{\text{triangles}} \tau_a^x \tau_b^x \tau_c^x$



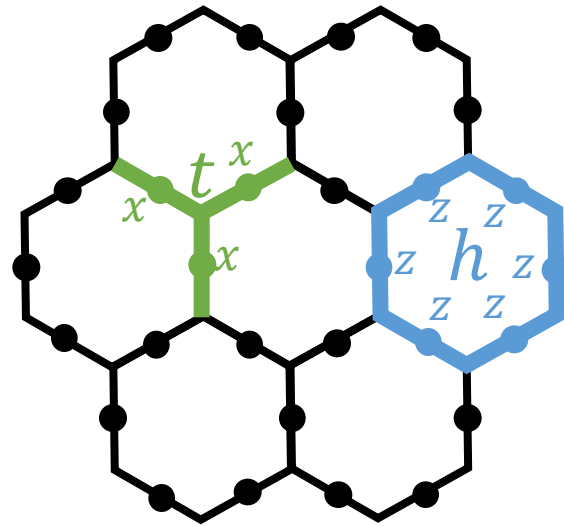
Toric code (hexagonal lattice version)

$$H = -\beta \sum_{h \in \text{plaquettes}} B_h - \alpha \sum_{t \in \text{stars}} A_t$$

$$B_h \stackrel{\text{def}}{=} \sigma_{h_1}^z \sigma_{h_2}^z \sigma_{h_3}^z \sigma_{h_4}^z \sigma_{h_5}^z \sigma_{h_6}^z$$

$$A_t \stackrel{\text{def}}{=} \sigma_t^x \sigma_{t'}^x \sigma_{t''}^x$$

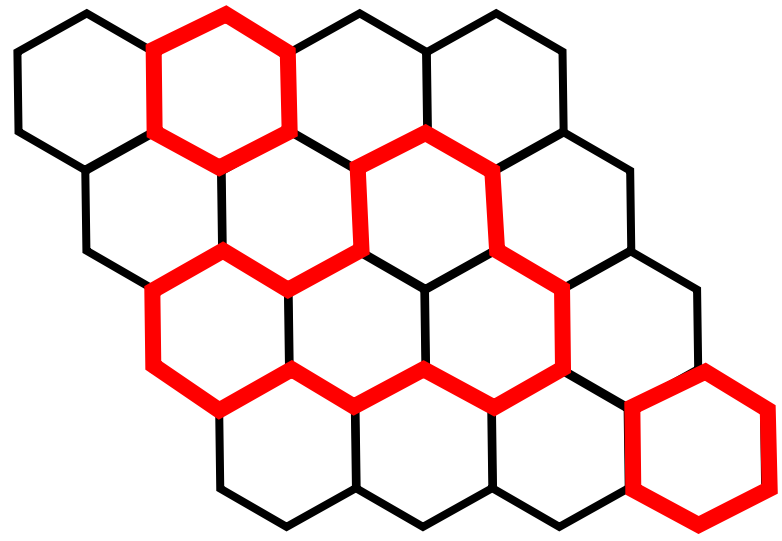
$$[A_t, A_{t'}] = [B_h, B_{h'}] = [A_t, B_h] = 0$$



Solvable dimer model on the kagome lattice \Leftrightarrow hexagonal lattice **toric code with magnetic excitations only** ($\alpha \rightarrow \infty$ and $A_t = 1$)

In the ground-state(s): $A_s = B_p = 1 \forall s$,
 |g. s⟩ = superposition of all **closed loop** config.
 with $\sigma_i^x = -1$ on **red** links

ex:



cylinder:
 2 ground-states: $\prod_{i \in \Omega} \sigma_i^z = \pm 1$
 exchanged by the application of $\prod_{j \in \Delta} \sigma_j^x$

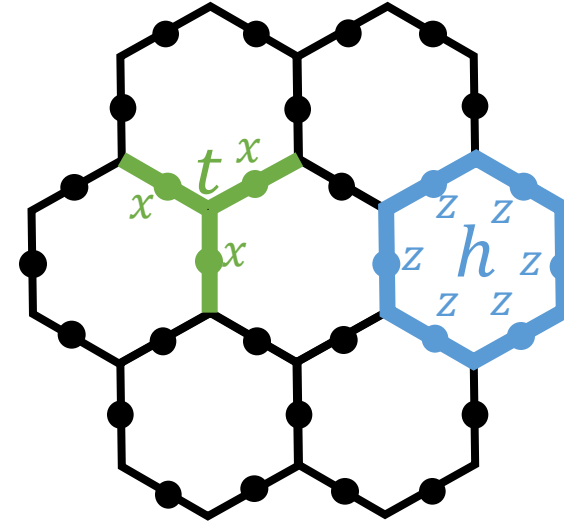
torus: 4-fold degeneracy
 $\prod_{i \in \Omega} \sigma_i^z = \pm 1$; $\prod_{j \in \Omega'} \sigma_j^z = \pm 1$

Toric code (hexagonal lattice version)

$$H = -\beta \sum_{h \in \text{plaquettes}} B_h - \alpha \sum_{t \in \text{stars}} A_t$$

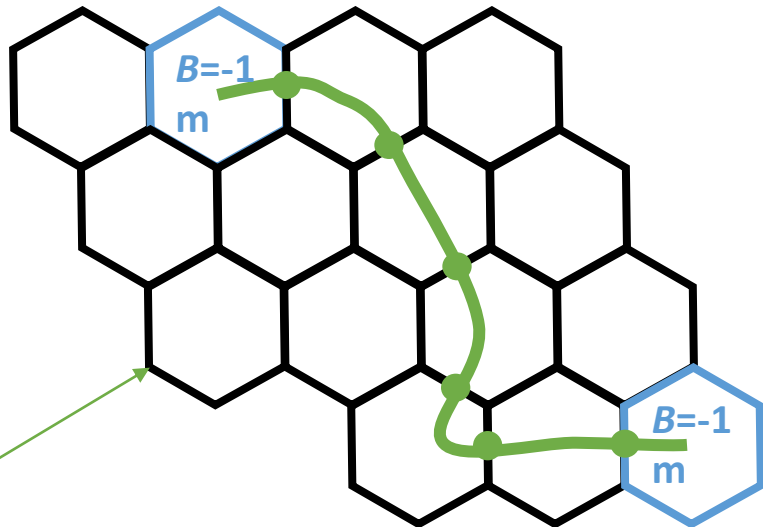
$$B_h \stackrel{\text{def}}{=} \sigma_{h_1}^z \sigma_{h_2}^z \sigma_{h_3}^z \sigma_{h_4}^z \sigma_{h_5}^z \sigma_{h_6}^z$$

$$A_t \stackrel{\text{def}}{=} \sigma_t^x \sigma_{t'}^x \sigma_{t''}^x$$

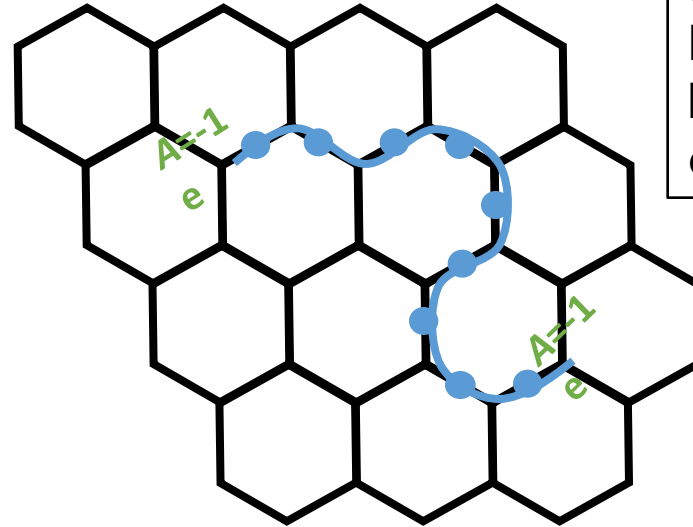


Toric code: 4 types of excitations/anyons: $1, e, m, f = e * m$
 e, m : anyons with mutual π statistics. f : fermions

Pair of magnetic excitations



Pair of electric excitations



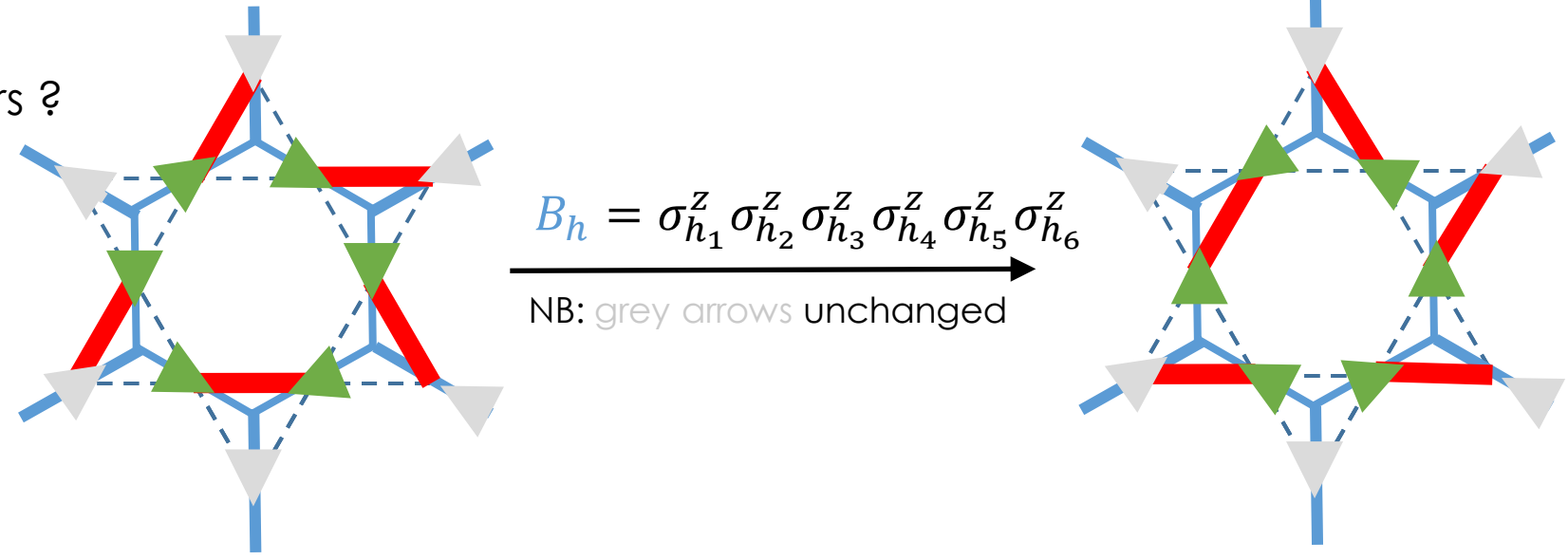
Solvable dimer model on the kagome lattice \Leftrightarrow hexagonal lattice toric code with magnetic excitations only ($\alpha \rightarrow \infty$ and $A_t = 1$)

electric excitations (e) do not exist in the 'pure' dimer model (pushed at infinite energy when $\alpha \rightarrow \infty$). They may however be introduced in a generalized QDM allowing for holes/spinons (see slide #39)

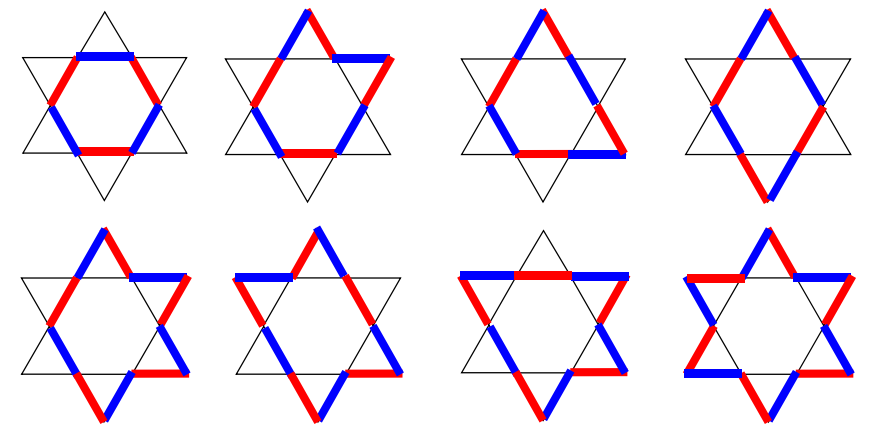
magnetic excitations (m) are called **visons** in the in the dimer model

Exactly solvable QDM with \mathbb{Z}_2 topological liquid ground state

- H in terms of dimers ?



- Such simple product of six σ^z is however complicated in terms of dimers:



32 different loops when applying the hexagon symmetries

Exactly solvable QDM with \mathbb{Z}_2 topological liquid ground state

- H in terms of dimers ?

$$\begin{aligned}
 & \sigma_1^Z \sigma_2^Z \sigma_3^Z \sigma_4^Z \sigma_5^Z \sigma_6^Z = |\text{Star}\rangle\langle\text{Star}| + \text{H. c.} \\
 & + |\text{Star}\rangle\langle\text{Star}| + \text{H. c.} \\
 & + |\text{Star}\rangle\langle\text{Star}| + \text{H. c.} \\
 & + |\text{Star}\rangle\langle\text{Star}| + \text{H. c.} \\
 & + |\text{Star}\rangle\langle\text{Star}| + \text{H. c.} \\
 & + |\text{Star}\rangle\langle\text{Star}| + \text{H. c.} \\
 & + |\text{Star}\rangle\langle\text{Star}| + \text{H. c.} \\
 & + |\text{Star}\rangle\langle\text{Star}| + \text{H. c.} \\
 & + \text{symmetric terms}
 \end{aligned}$$

Exactly solvable QDM with \mathbb{Z}_2 topological liquid ground state

- Ground-state in terms of dimers: RK state

$$= + \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]} = \sum_{c \in \{\text{dimer coverings}\}} |c\rangle$$

- Vanishing dimer-dimer correlations (easy proof using the arrow representation) \Rightarrow dimer liquid
- Topological degeneracy: the ground-states are locally indistinguishable [Furukawa *et al.* [2006](#)]

- Pair of **visons** (=mag. excitations in the toric code) in a and b
 = exact eigenstate of H . Energy is independent of the distance between the visons



$$= - \text{[Diagram 1]} + \text{[Diagram 2]} - \text{[Diagram 3]}$$

sign of each config.: $\prod_{i=a}^b \sigma_i^x = (-1)^{\Omega(a,b)}$ $\Omega(a,b) \stackrel{\text{def}}{=} \#$ of dimers crossing the path $a \rightarrow b$. Up to a global sign the resulting state does not depend on the choice of the path.

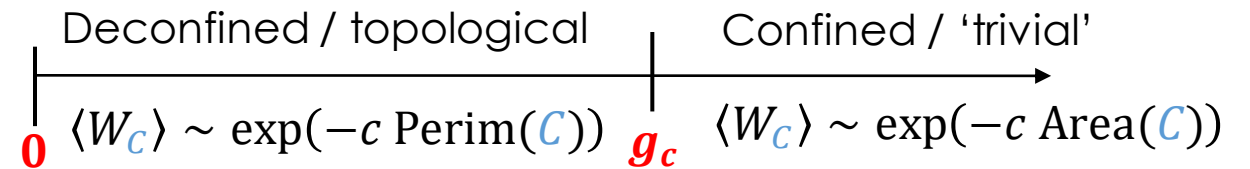
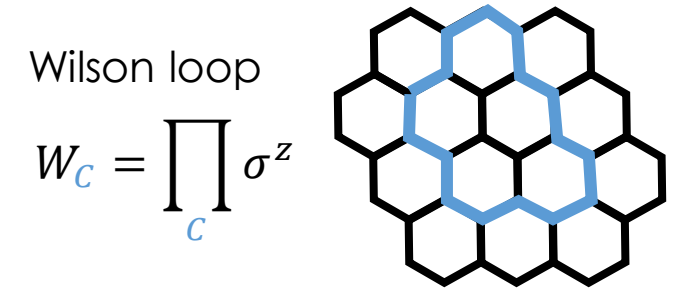
(pure) \mathbb{Z}_2 gauge theory

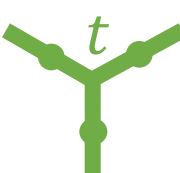
F. J. Wegner, *Duality in Generalized Ising Models and Phase Transitions without Local Order Parameters*, J. Math. Phys. 12 2259 (1971).

$$H_{\mathbb{Z}_2} \stackrel{\text{def}}{=} \underbrace{-\sum_h \prod_{i=1}^6 \sigma_{h_i}^z}_{\text{magnetic energy}} \underbrace{-g \sum_{l \in \{\text{links}\}} \sigma_l^x}_{\text{electric energy}}$$

$\sim \vec{B}^2$ in Maxwell electromagnetism $\sim \vec{E}^2$ in Maxwell electromagnetism



$$G_t = \prod_{i=1}^3 \sigma_{t_i}^x$$


Generator of Ising gauge transformations:
 $G_t \sigma_i^z G_t = -\sigma_i^z$ for i touching t . $[G_t, H_{\mathbb{Z}_2}] = 0$
 Constraint on physical states $G_t |\psi\rangle = |\psi\rangle$
Gauss law ('div $\vec{E} = 0$ ') \leftrightarrow dimer constraint

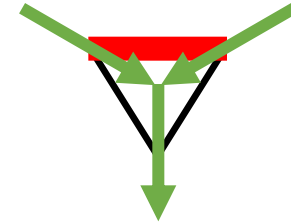
- $g < g_c$ phase: topological, magnetic excitations (visons) only appear as confined bound states. \mathbb{Z}_2 Flux expulsion. Large fluctuations of the electric field (σ^x). In dimer/RVB language: \sim liquid
- $g > g_c$ phase: Large fluctuation of the \mathbb{Z}_2 flux (and small fluctuations of the electric field σ^x) In dimer/RVB language: \sim valence-bond solid
- Transition dual/equivalent to that of an Ising model
- $g = 0$ point \leftrightarrow **Solvable QDM**

Doped QDM with holes or spinons

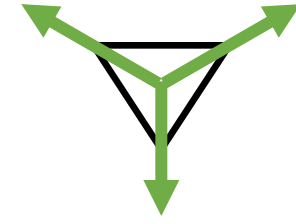
- Dimer constrain

$$N_{\text{out}} = \text{odd}, \quad G = \sigma_a^x \sigma_b^x \sigma_c^x = 1$$

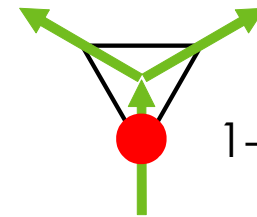
(pure gauge theory)



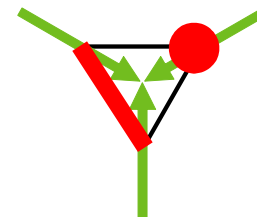
2-in & 1 out



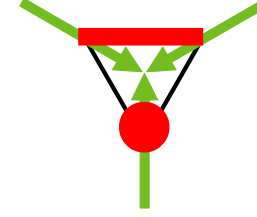
3-out



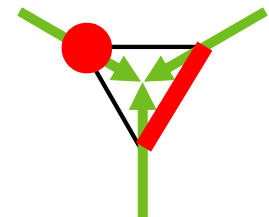
1-in & 2 out



3-in



3-in



3-in

- Monomer/spinon/hole = electric charge

$$N_{\text{out}} = \text{even}, \quad G = \sigma_a^x \sigma_b^x \sigma_c^x = -1$$

(gauge theory + matter)

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- **Exact solution of the classical dimer model on planar graphs**
 - *The statistics of dimers on a lattice*, P. W. Kasteleyn, Physica. 27 (12) 1209 ([1961](#))
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 - *Resonating Valence Bond Phase in the Triangular Lattice Quantum Dimer Model*, R. Moessner and S. L. Sondhi, Phys. Rev. Lett. 86, 1881 ([2001](#))
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- **Kagome lattice QDM, solvable point, etc.**
 - *Quantum Dimer Model on the Kagome Lattice: Solvable Dimer-Liquid and Ising Gauge Theory*, GM, D. Serban, and V. Pasquier, Phys. Rev. Lett. 89, 137202 ([2002](#))
 - *Quantum dimer model with Z_2 liquid ground state: Interpolation between cylinder and disk topologies and toy model for a topological quantum bit*, GM, V. Pasquier, F. Mila & C. Lhuillier, Phys. Rev. B 71, 184424 ([2005](#)).
 - *Phenomenological Z_2 lattice gauge theory of the spin-liquid state of the kagome Heisenberg antiferromagnet*, Y. Wan & O. Tchernyshyov, Phys. Rev. B 87, 104408 ([2013](#))
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- *Spin Hamiltonians with resonating-valence-bond ground states*, J. Cano, P. Fendley, Phys. Rev. Lett. 105, 067205 ([2010](#))
- *Effective Quantum Dimer Model for the Kagome Heisenberg Antiferromagnet: Nearby Quantum Critical Point and Hidden Degeneracy* D. Poilblanc, M. Mambrini, and D. Schwandt, Phys. Rev. B 81, 180402(R) ([2010](#))
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- *Resonating valence bond states in the PEPS formalism*, N. Schuch, D. Poilblanc, J. Ignacio Cirac, & D. Pérez-García, Phys. Rev. B 86, 115108 ([2012](#))
- *Quantum dimer model for the spin-1/2 kagome Z_2 spin liquid*, I. Rousochatzakis, Y. Wan, O. Tchernyshyov & F. Mila, Phys. Rev. B 90, 100406(R) ([2014](#))

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- **Rydberg atoms physics & QDM**

- *Quantum phases of Rydberg atoms on a kagome lattice*, Samajdar, W. W. Ho, H. Pichler, S. Sachdev PNAS (2021)
- *Prediction of Toric Code Topological Order from Rydberg Blockade*, R. Verresen, M. D. Lukin, & A. Vishwanath, Phys. Rev. X 11, 031005 (2021)
- *Probing topological spin liquids on a programmable quantum simulator*, G. Semeghini et al., Science 374 (2021)
- *Dimer states of Rydberg atoms on the Kagome lattice as resources for universal measurement-based quantum computation*, V. Crépel, AIP Adv 12, 115301 (2022)
- *Unifying Kitaev Magnets, Kagomé Dimer Models, and Ruby Rydberg Spin Liquids*, R. Verresen & A. Vishwanath, Phys. Rev. X 12, 041029 (2022)


PHYSICAL REVIEW X **11**, 031005 (2021)

Featured in Physics

Prediction of Toric Code Topological Order from Rydberg Blockade

Ruben Verresen¹,* Mikhail D. Lukin, and Ashvin Vishwanath¹

Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA

 (Received 24 November 2020; accepted 19 May 2021; published 8 July 2021)

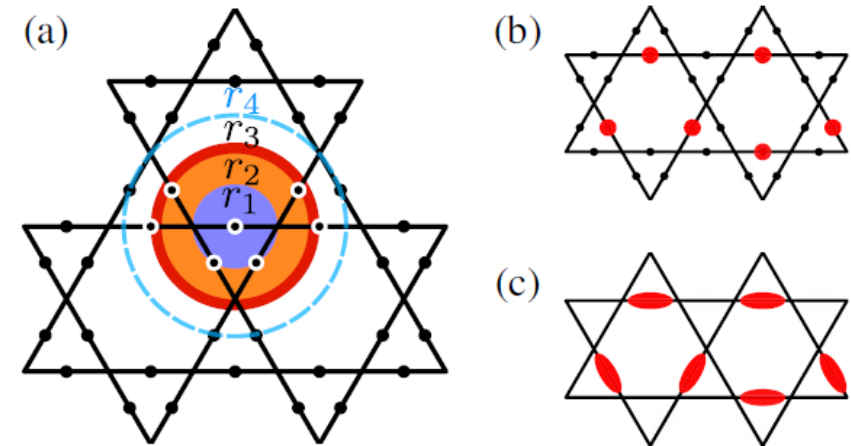


FIG. 1. Rydberg blockade model and relation to dimer model. (a) Hard-core bosons on the links of the kagome lattice (forming the ruby lattice) are strongly repelling, punishing double occupation within the disk $r \leq r_3 = 2a$. (b) An example of a state consistent with the Rydberg blockade at maximal filling. (c) Since the blockade forbids occupation of any two touching bonds, we can equivalently draw the configuration as a dimer covering on the kagome lattice.

Schwinger bosons, Z_2 gauge theory and toric code limit (more details in appendix)

- Spin operators in terms of spin-1/2 boson operators (an example of *parton* formulation of the spin model)

$$S_i^z = \frac{1}{2} \left(a_{i\uparrow}^\dagger a_{i\uparrow} - a_{i\downarrow}^\dagger a_{i\downarrow} \right), \quad S_i^+ = a_{i\uparrow}^\dagger a_{i\downarrow}, \quad S_i^- = a_{i\downarrow}^\dagger a_{i\uparrow}$$

Constraint at each site: $a_{i\uparrow}^\dagger a_{i\uparrow} + a_{i\downarrow}^\dagger a_{i\downarrow} = 2S$

- Singlet bond field $A_{ij} = a_{i\uparrow} a_{j\downarrow} - a_{i\downarrow} a_{j\uparrow}$ can be used to rewrite the Heisenberg interaction $\vec{S}_i \cdot \vec{S}_j = S^2 - \frac{1}{2} (A_{ij}^\dagger) A_{ij}$
- Mean field decoupling: $A_{ij}^\dagger A_{ij} \rightarrow A_{ij}^\dagger \langle A_{ij} \rangle + \langle A_{ij}^\dagger \rangle A_{ij} - |\langle A_{ij}^\dagger \rangle|^2$

→ Mean-field Hamiltonian is quadratic in $a_{i\sigma}$ → solvable (Bogoliubov transformation)

Mean-field state: gapped spin liquid, finite correlation length (for low enough S . ex: Sachdev [1992](#))

At the MF level: free (unconfined) bosonic spin-1/2 excitations ("spinons"). Gauge redundancy for the $\langle A_{ij} \rangle$

- Beyond mean field? Confinement of deconfinement of the spinons?

Path integral formulation. Look for **gauge degrees of freedom** describing fluctuations in the vicinity of a mean-field solution ("IGG", Wen [2002](#)). On non-bipartite lattices the associated gauge field is a Z_2 gauge field

- Simplest minimal model describing **spinons interacting with a fluctuating Z_2 gauge field** ($\sigma_{ij}^z = \pm 1 \Leftrightarrow \text{sign of } A_{ij}$)

$$H = -K \sum_{\square} \sigma_{ij}^z \sigma_{jk}^z \sigma_{kl}^z \sigma_{li}^z - \Gamma \sum_{\langle ij \rangle} \sigma_{ij}^x$$

$$-t \sum_{\langle ij \rangle, \sigma = \uparrow, \downarrow} \left(b_{i\sigma}^\dagger \sigma_{ij}^z b_{j\sigma} + \text{H.c.} \right) + \Delta \sum_{i\sigma} b_{i\sigma}^\dagger b_{i\sigma}$$

$$+V \sum_i \left[\left(b_{i\uparrow}^\dagger b_{i\uparrow} + b_{i\downarrow}^\dagger b_{i\downarrow} - \frac{1}{2} \right)^2 - \frac{1}{4} \right]$$

Toric code limit: $t = 0, \Gamma = 0, V = \infty$ ($\rightarrow n_i = b_{i\downarrow}^\dagger b_{i\downarrow} + b_{i\uparrow}^\dagger b_{i\uparrow} \leq 1$)
+ gauss law (gauge invariance):

$$G_{i_0} = \exp(i\pi n_{i_0}) \prod_{j \in +} \sigma_{i_0 j}^x = 1$$

$$H = -K \sum_{\square} \sigma^z \sigma^z \sigma^z \sigma^z - \frac{\Delta}{2} \sum_{+} \sigma^x \sigma^x \sigma^x \sigma^x$$

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- *Spin-liquid states on the triangular and Kagomé lattices: A projective-symmetry-group analysis of Schwinger boson states*, F. Wang & A. Vishwanath, Phys. Rev. B 74, 174423 ([2006](#))
- Short review: G. Misguich, [arXiv:0809.2257](#)

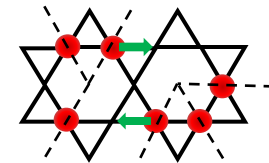
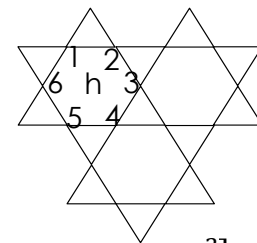
Z_2 spin liquids: examples of realizations/candidates

- **Quantum Dimer Model realization with spins – perturbed Klein models**
'SU(2)-invariant spin-1/2 Hamiltonians with resonating and other valence bond phases',
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Fractionalization in an easy-axis Kagome antiferromagnet,
 L. Balents, M. P. A. Fisher, and S. M. Girvin, Phys. Rev. B 65, 224412 (2002)
Numerical Evidences of Fractionalization in an Easy-Axis Two-Spin Heisenberg Antiferromagnet
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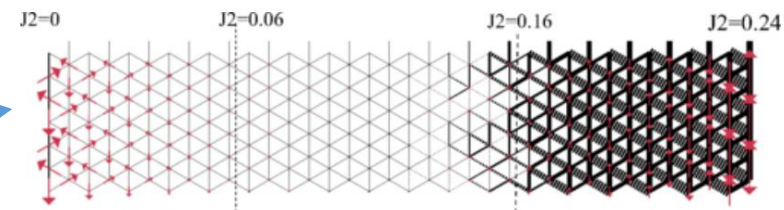
$$H = J_z \sum_{h \text{ hexagon}} (S_{h1}^z + \dots + S_{h6}^z)^2$$

$$+ J_{\perp} \sum_{h \text{ hexagon}} [(S_{h1}^x + \dots + S_{h6}^x)^2 + (S_{h1}^y + \dots + S_{h6}^y)^2]$$

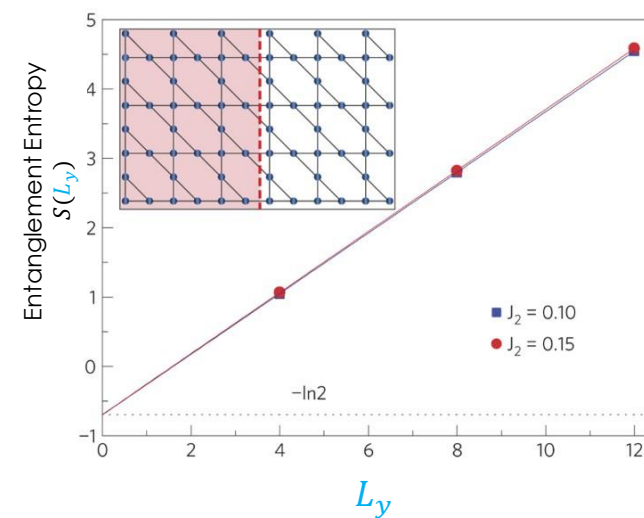
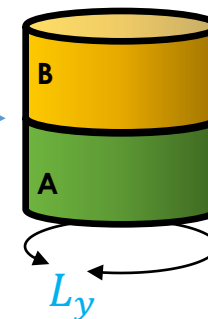
$$J_{\perp} \ll J_z$$



- [candidate] **J_1 - J_2 models on the triangular lattice ?**
 - *Spin liquid phase of the $S=1/2$ J_1 - J_2 Heisenberg model on the triangular lattice*
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 - *Competing spin-liquid states in the spin-1/2 Heisenberg model on the triangular lattice*
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 - *Spin-Liquid Ground State of the $S=1/2$ Kagome Heisenberg Antiferromagnet*
 S. Yan, D. Huse & S. White, Science (2011)
 - *Nature of the Spin-Liquid Ground State of the $S=1/2$ Heisenberg Model on the Kagome Lattice*, S. Depenbrock, I. P. McCulloch, & U. Schollwöck, Phys. Rev. Lett. 109, 067201 (2012)
 - *Identifying topological order by entanglement entropy*
 H.-C. Jiang, Z. Wang & L. Balents, Nature (2012)
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 J.-W. Mei, J.-Y. Chen, H. He, & X.-G. Wen, Phys. Rev. B 95, 235107 (2017)



Chiral spin liquids

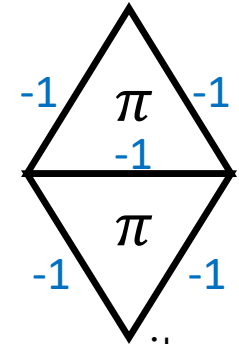
Chiral spin liquids - Kalmeyer & Laughlin's idea in a nutshell

Phys. Rev. Lett. 59, 2095 (1987)

- $S=1/2$ antiferromagnetic Heisenberg model on the triangular lattice

$$H = + \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j = \sum_{\langle ij \rangle} \left[\frac{1}{2} (S_i^+ S_j^- + S_i^- S_j^+) + S_i^z S_j^z \right] = H_{XY} + H_{ZZ}$$

- Spin-1/2 \leftrightarrow hard-core boson: $S_i^+ = a_i^\dagger$; $S_i^- = a_i$; $a_i^\dagger a_i \leq 1$; $S_i^z = a_i^\dagger a_i - \frac{1}{2}$
- H_{XY} : boson hopping with the wrong kinetic energy sign. But equivalent to a magnetic field with flux π per triangle
- $H_{XY} = - \sum_{\langle ij \rangle} [a_i^\dagger a_j \exp(iA_{ij}) + \text{H.c.}]$
- A_{ij} : lattice vector potential, equal to -1 on all nearest-neighbor bonds \Rightarrow magnetic flux π piercing each triangle \Rightarrow magnetic flux $\phi = 2\pi$ per unit cell.



unit cell of the triangular lattice

- Ground state of H has zero total magnetization $\sum_i S_i^z = 0 \Rightarrow \sum_i a_i^\dagger a_i = \frac{N_{\text{site}}}{2}$. Mean boson density is $\frac{1}{2}$ per site, or $\rho = \frac{1}{2}$ per unit cell.
- \Rightarrow Interacting (H_{ZZ} + hard-core constraint) lattice bosons in presence of a magnetic field.

- Filling fraction $\nu = \frac{\rho}{\frac{\phi}{2\pi}} = \frac{1}{2}$

- Proposal: the spins are in the same phase as the bosonic $\nu = \frac{1}{2}$ Laughlin state $\{z_i\}$: coordinate of the bosons (spin up).

$$\psi_{\text{gs}}(z_1, \dots, z_{N_b})$$

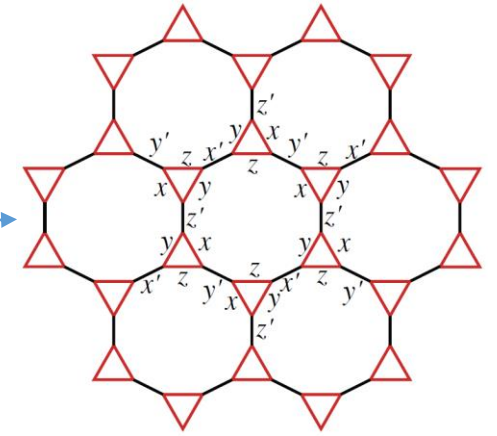
$$= \prod_{j < k} (z_j - z_k)^2 \exp \left\{ \frac{-1}{4l_0^2} \sum_{i=1}^{N_b} |z_i|^2 \right\}, \quad (6)$$

where $z_j = x_j + iy_j$ is the complex lattice coordinate of the j th particle. ψ_{gs} describes a state of density $\rho_2 = 4\pi/l_0^2$, which corresponds to $\frac{1}{2}$ boson per unit cell. In other words, the number of up spins is equal to the number of down spins. We must emphasize that the only reason for considering such a wave function is the physical precedent of its success in describing the FQH states.

- This turned out to be incorrect for the simple AF model, but this proposal launched the field of chiral spin liquids, and such liquids we later found to be realized in some other spin models (see next slide).

Chiral spin liquids – microscopic models

- Kitaev model on a [decorated honeycomb lattice](#) :
Exact Chiral Spin Liquid with Non-Abelian Anyons,
 H. Yao & S. A. Kivelson, Phys. Rev. Lett. 99, 247203 ([2007](#))



- [Kagome Heisenberg with explicit scalar chirality term](#) which breaks the time-reversal symmetry :

Chiral spin liquid and emergent anyons in a Kagome lattice Mott insulator
 B. Bauer, et al., Nat. Com. 5, ([2014](#))

$$H_{\chi} = J_1 \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j + J_{\chi} \sum_{\Delta/\nabla} \vec{S}_i \cdot (\vec{S}_j \times \vec{S}_k)$$

- [J₁-J₂-J₃ models on the kagome lattice](#) :

Chiral Spin Liquid in a Frustrated Anisotropic Kagome Heisenberg Model
 Y.-C. He, D. N. Sheng & Y. Chen, Phys. Rev. Lett. 112, 137202 ([2014](#))

$$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j + J_2 \sum_{\langle\langle ij \rangle\rangle} S_i^z S_j^z + J_3 \sum_{\langle\langle\langle ij \rangle\rangle\rangle} S_i^z S_j^z$$

Emergent Chiral Spin Liquid: Fractional Quantum Hall Effect in a Kagome Heisenberg Model,
 S. S. Gong, W. Zhu & D. N. Sheng, Sci Rep 4, 6317 ([2014](#))

$$H = J_1 \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j + J_2 \sum_{\langle\langle ij \rangle\rangle} \vec{S}_i \cdot \vec{S}_j + J_3 \sum_{\langle\langle\langle ij \rangle\rangle\rangle} \vec{S}_i \cdot \vec{S}_j$$

Variational Monte Carlo study of a chiral spin liquid in the extended Heisenberg model on the kagome lattice
 W.-J. Hu, W. Zhu, Y. Zhang, S. Gong, F. Becca, D. N. Sheng, Phys. Rev. B 91, 041124(R) ([2015](#))

Conclusions

- The theoretical understanding of QSL have made huge progress in the last ~15 years
- Growing list of microscopic models
- Important progress on the numerical front (tensor network methods, ...)
- Relatively good understanding of the classification of 2d gapped SL in 2d (SET, ...)
- Entanglement is a key to defined and to probe for these phases of matter:
Entanglement spectrum, Topological Entanglement Entropy, Minimally Entnagled states, ...
- Experimentally many QSL candidates (Herbertsmithite, triangular organics, 3d Irridate, ..)
- Many candidates seem to be gapless
- We lack clear examples of gapped (Z_2 like) QSL
- Exciting developments to expect in the field of artificial systems (cold atoms, Rydberg atoms, ...)

Thank you !

Appendix

Lieb-Schultz-Mattis-Hastings theorem

M. Oshikawa's argument: Phys. Rev. Lett. 84, 1535 ([2000](#))

$$H_\theta = \frac{1}{2} \sum_{ij} J_{ij} \left[S_i^z S_j^z + \frac{1}{2} \left(e^{i\theta(x_i - x_j)/L_x} S_i^+ S_j^- + \text{H.c.} \right) \right]$$

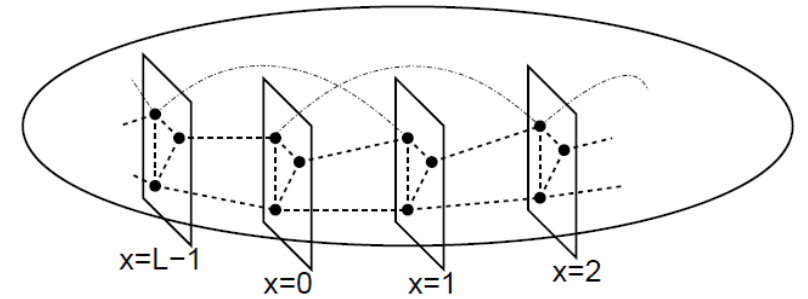
= spin- S XXZ Hamiltonian with a magnetic flux θ

$$U = \prod_i \exp \left(2i\pi \frac{x_i}{L_x} S_i^z \right) = \text{large gauge transformation}$$

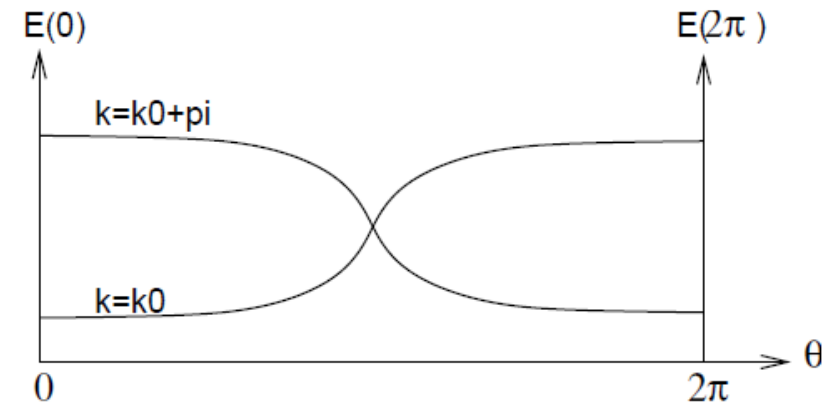
$$U H_0 U^{-1} = H_{2\pi} \quad \rightarrow \text{same spectrum for } H_0 \text{ and } H_{2\pi}$$

$$T U = U T \exp \left(2i\pi \frac{S_{\text{tot}}^z}{L_x} \right) \exp (2i\pi C S) \quad \rightarrow U \text{ does in general not commute with the translation } T \text{ (in the } x \text{ dir.)}$$

$$k_0 = k_{2\pi} + 2\pi C (S + m^z) \quad \rightarrow \text{non trivial momentum shift if } C(S + m^z) \text{ is odd}$$

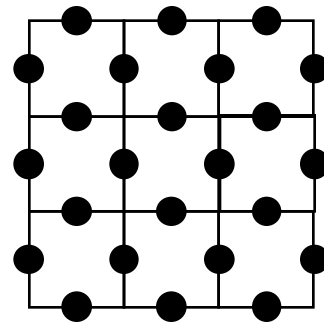


Translation invariance in the x direction
 C : Number of site in each 'section'



Toric code (square lattice version)

Spin 1/2 living on the edges of a square lattice



Fault-tolerant quantum computation by anyons,
A. Y. Kitaev, arXiv [1997](#)
(Annals of Phys. [2003](#))

$$H = - \sum_s A_s - \sum_p B_p$$

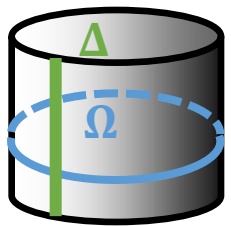
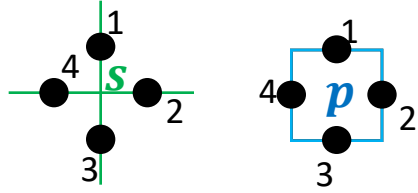
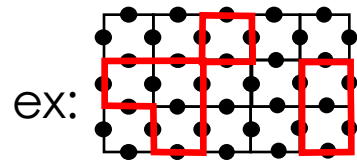
$$A_s = \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x$$

$$B_p = \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z$$

$$[A_s, A_{s'}] = [B_p, B_{p'}] = [A_s, B_p] = 0$$

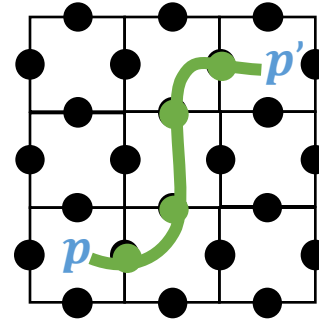
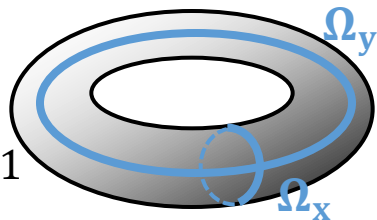
In the ground-state(s): $A_s = B_p = 1 \forall s, p$,

|g. s⟩ = superposition of all **closed loop** config.
with $\sigma_i^x = -1$ on **red** links



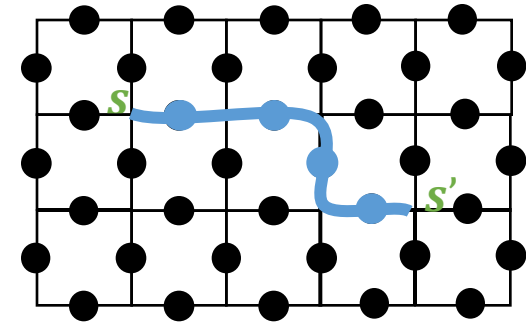
cylinder:
2 ground-states: $\prod_{i \in \Omega} \sigma_i^z = \pm 1$
exchanged by the application of $\prod_{j \in \Delta} \sigma_j^x$

torus: 4-fold degeneracy
 $\prod_{i \in \Omega} \sigma_i^z = \pm 1; \prod_{j \in \Omega'} \sigma_j^z = \pm 1$



Pair of 'magnetic' excitations in p and p'
created by acting with $\prod_{i=p}^{p'} \sigma_i^x$ along a path
connecting the plaquette p and the plaquette p' .
In this state $B_p = B_{p'} = -1$ (and +1 elsewhere)

Pair of 'electric' excitations in s and s'
created by acting with $\prod_{i=s}^{s'} \sigma_i^z$ along
a path connecting the star s and the
star s' . In this state $A_s = A_{s'} = -1$ (and
+1 elsewhere)



4 types of excitations/anyons: 1, e , m , $f=e^*m$
 e , m : anyons with mutual π statistics.
 f : fermions

Schwinger boson
mean-field theory
and gauge fluctuations

Schwinger bosons mean-field theory

- An example of 'parton' formulation (other possible choice: Abrikosov fermions)
- Spin operators in terms of boson operators

$$S_i^z = \frac{1}{2} \left(a_{i\uparrow}^\dagger a_{i\uparrow} - a_{i\downarrow}^\dagger a_{i\downarrow} \right) , \quad S_i^+ = a_{i\uparrow}^\dagger a_{i\downarrow} , \quad S_i^- = a_{i\downarrow}^\dagger a_{i\uparrow}$$

- Constraint at each site: $a_{i\uparrow}^\dagger a_{i\uparrow} + a_{i\downarrow}^\dagger a_{i\downarrow} = 2S$

- Heisenberg interaction in terms of bond operators A_{ij} : $\vec{S}_i \cdot \vec{S}_j = S^2 - \frac{1}{2}(A_{ij})^\dagger A_{ij}$
with $A_{ij} = a_{i\uparrow} a_{j\downarrow} - a_{i\downarrow} a_{j\uparrow}$.

- Mean-field decoupling: $A_{ij}^\dagger A_{ij} \longrightarrow A_{ij}^\dagger \langle A_{ij} \rangle + \langle A_{ij}^\dagger \rangle A_{ij} - |\langle A_{ij}^\dagger \rangle|^2$

→ Quadratic mean-field Hamiltonian (can be diagonalized by a Bogoliubov transformation).

- Self-consistency:

$$\langle a_{i\uparrow}^\dagger a_{i\uparrow} + a_{i\downarrow}^\dagger a_{i\downarrow} \rangle = 2S.$$

$$\langle A_{ij} \rangle = \langle a_{i\uparrow} a_{j\downarrow} - a_{i\downarrow} a_{j\uparrow} \rangle_{\text{ground-state of the quadratic H}}$$

For many frustrated models this mean-field approx. predicts **a gapped spin liquid with short-ranged spin-spin correlations** and (free) bosonic spinons. Ex: Kagome [Sachdev [1992](#)]

Schwinger bosons / $Sp(2\mathcal{N})$ – beyond mean-field

Imaginary-time path integral formulation

$SU(2) \rightarrow Sp(2\mathcal{N})$ generalization by introducing a ‘flavor’ index $m = 1.. \mathcal{N}$

Mean-field is recovered as a saddle point when $\mathcal{N} \rightarrow \infty$

$$\mathcal{Z} = \int \mathcal{D}[z_{im\sigma}(\tau), \lambda_i(\tau)] \exp\left(-\int_0^\beta L_0 d\tau\right)$$

$$L_0 = \sum_{i m \sigma} \bar{z}_{im\sigma} \partial_\tau z_{im\sigma} - \frac{1}{2\mathcal{N}} \sum_{ij} J_{ij} A_{ij}^\dagger A_{ij}$$

$$+ i \sum_{i m} \lambda_i (\bar{z}_{im\uparrow} z_{im\uparrow} + \bar{z}_{im\downarrow} z_{im\downarrow} - 2S)$$

$$A_{ij} = \sum_{m=1}^{\mathcal{N}} (z_{im\uparrow} z_{jm\downarrow} - z_{im\downarrow} z_{jm\uparrow}),$$

Implementation of the constraints with Lagrange multiplier λ

NB: this is an exact reformulation of the initial spin problem (setting $\mathcal{N} = 1$);

Schwinger bosons – beyond mean-field

- Hubbard Stratonovich → introduction of some fluctuating bond field $Q_{ij}(\tau)$

$$\mathcal{Z} = \int \mathcal{D}[z_{im\sigma}(\tau), \lambda_i(\tau), Q_{ij}(\tau)] \exp\left(-\int_0^\beta L_1 d\tau\right)$$

$$L_1 = \sum_{i m \sigma} \bar{z}_{im\sigma} \partial_\tau z_{im\sigma} + \sum_{ij} \left(\frac{2\mathcal{N}}{J_{ij}} |Q_{ij}|^2 - \bar{Q}_{ij} A_{ij} - Q_{ij} \bar{A}_{ij} \right)$$

$$+ i \sum_{i m} \lambda_i (\bar{z}_{im\uparrow} z_{im\uparrow} + \bar{z}_{im\downarrow} z_{im\downarrow} - 2S)$$

$A_{ij} = \sum_{m=1}^{\mathcal{N}} (z_{im\uparrow} z_{jm\downarrow} - z_{im\downarrow} z_{jm\uparrow})$

- U(1) Gauge invariance

$z_{im\sigma}(\tau)$	\longrightarrow	$e^{i\Lambda_i(\tau)} z_{im\sigma}(\tau)$
$Q_{ij}(\tau)$	\longrightarrow	$e^{i(\Lambda_i(\tau) + \Lambda_j(\tau))} Q_{ij}(\tau)$
$\lambda_i(\tau)$	\longrightarrow	$\lambda_i(\tau) - \partial_\tau \Lambda_i(\tau)$

Schwinger bosons – effective model beyond mean-field

Important degrees of freedom for the long distance properties (spinon confinement?) are **Gauge degrees of freedom**. How to describe the Gauge fluctuations in the vicinity of a given mean-field (saddle point) state Q_{ij}^0 ?
 → Look for the Invariant Gauge Group (IGG) of that saddle point [Wen [2002](#)]

IGG: gauge transf. Λ_i such that saddle point unchanged

$$Q_{ij}^0 = Q_{ij}^0 e^{i(\Lambda_i + \Lambda_j)}$$

Generic situation for non-bipartite lattices: IGG= Z_2 group

$$Q_{ij}(\tau) = Q_{ij}^0 e^{i\mathcal{A}_{ij}(\tau)}, \quad \mathcal{A}_{ij}(\tau) \in \{0, \pi\}$$

$$\sigma_{ij}^z = e^{i\mathcal{A}_{ij}} = \pm 1 \quad \text{fluctuating sign of the bond field}$$

Simple effective model with Z_2 gauge field coupled to bosonic spinons:

$$\begin{aligned}
 H = & -K \sum_{\square} \sigma_{ij}^z \sigma_{jk}^z \sigma_{kl}^z \sigma_{li}^z - \Gamma \sum_{\langle ij \rangle} \sigma_{ij}^x \\
 & -t \sum_{\langle ij \rangle, \sigma=\uparrow, \downarrow} \left(b_{i\sigma}^\dagger \sigma_{ij}^z b_{j\sigma} + \text{H.c.} \right) + \Delta \sum_{i\sigma} b_{i\sigma}^\dagger b_{i\sigma} \\
 & +V \sum_i \left[\left(b_{i\uparrow}^\dagger b_{i\uparrow} + b_{i\downarrow}^\dagger b_{i\downarrow} - \frac{1}{2} \right)^2 - \frac{1}{4} \right]
 \end{aligned}$$

G_{i_0} : generator of gauge transf. in i_0 :
 $b_{i_0} \rightarrow -b_{i_0}$ & $\sigma_{ij}^z \rightarrow -\sigma_{ij}^z$ if (ij) contains i_0 .

$$G_{i_0} = \exp \left[i\pi (b_{i_0\uparrow}^\dagger b_{i_0\uparrow} + b_{i_0\downarrow}^\dagger b_{i_0\downarrow}) \right] \prod_{j \in +} \sigma_{i_0 j}^x$$

Constraint on physical states $G_{i_0} |\phi\rangle = |\phi\rangle$

$$G_{i_0} = 1 \Rightarrow \exp(i\pi n_{i_0}) = 1 - 2n_{i_0} = \prod_{j \in +} \sigma_{i_0 j}^x$$

Toric code limit: $t = 0, \Gamma = 0, V = \infty$

$$H = -K \sum_{\square} \sigma^z \sigma^z \sigma^z \sigma^z - \frac{\Delta}{2} \sum_{+} \sigma^x \sigma^x \sigma^x \sigma^x$$