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Combinatorial Games with Incomplete Information*

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Abstract

Games with incomplete information model multi-agent interaction in which players do not have common knowledge of the game they play. We propose a minimal generalisation of combinatorial games to incorporate incomplete information, called combinatorial games with incomplete information (CGII). The most important feature of CGIIs is that all actions are public, which allows better visualisation of each player’s knowledge and incomplete information. To further motivate the study of this new formalism, we show that computing optimal strategies for CGIIs has the same computational complexity as for general extensive-form games.

1 Introduction

Game theory is a mathematical framework for studying multi-agent interactions. We focus on *extensive-form games* (EFG), in which the interaction between agents takes place sequentially, i.e. every agent takes turns to make a move. Prominent examples of such games are Chess and Go.

Of particular interest to us is the notion of games with *incomplete information*, which are games in which agents do not have common knowledge of the game they play. For instance, an agent does not know the number of participants in an auction, or how much these participants value the object to be sold; a Poker player does not see the cards in their opponent’s hidden hands, hence cannot know for sure the exact consequence (i.e. payoff) of calling and raising bets; a Bridge or Hearts player does not know the cards that their opponent can play during a trick since this depends on their hidden hand; etc.

The notion of (in)complete information is frequently confused with the one of *(im)perfect information*. Complete information describes situations in which the whole structure of a game (the number of players, the game tree, the information sets of each player, the owner of each node, the payoff for each player at each leaf node, etc.) is common knowledge among all the players of the game. On the other hand, perfect information is a more stringent requirement than complete information. Not only the structure of the game is common knowledge, but all players have full observability and

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perfect recall of the history (which is essentially a record of every decision made by every player so far). In other words, players always know their exact position in the game tree when asked to make the next decision. To summarise, incomplete information is an example of imperfect information; see Faliszewski *et al.* [15, Sec. 2.4.2].

We propose a new and minimal formalism for EFGs with incomplete information that we call combinatorial games with incomplete information (CGIIs). In such a game, Nature picks a world from a universe according to some common prior; each player may have different observability of this world. Then, the game proceeds sequentially, during which there is no chance factor and all moves by the players are publicly observable. This formalism is designed to be a minimal generalisation of the notion of combinatorial games (which are Boolean games of no chance and with perfect information; see Siegel [39]) and to closely capture the epistemic aspect of games with incomplete information.

For such games, we are interested in knowing how much reward an agent or a team of agents can guarantee for themselves; this corresponds to the notion of maxmin value, well known in optimisation under uncertainty, in which we aim to ensure that the worst possible outcome is not too bad.

By design, our new formalism seems particularly restrictive when compared to general EFGs, where hidden actions and arbitrary chance nodes are allowed. However, we show that the computational complexity of computing optimal strategies (with respect to the maxmin value) for CGIIs is as hard as for EFGs, which allows concluding that the difficulty of playing games comes from incomplete information/knowledge alone, not from hidden actions or mid-game chance factors. This also justifies that restricting algorithmic studies to CGIIs is without loss of generality. We also give a construction to enforce coordination between players in CGIIs under the constraint of public actions, which allows modelling situations similar to concurrent actions.

2 Related Work

Game theory. The study of games with incomplete information was pioneered by Harsanyi [22, 23, 24], who proposes a formalism to model games of incomplete information as EFGs with imperfect information. This formalism, called the Harsanyi model of incomplete information, introduces *types* of players, or equivalently, a universe of worlds for which each player has a potentially different partial observability (also called the Aumann model of incomplete information). For detailed and formal definitions, see textbooks on game theory, e.g. Maschler *et al.* [31, Chapter 9].

Combinatorial games, the inspiration of our formalism of CGII, are studied in the field of combinatorial game theory, established in the 70s by two books by Conway [13] and Berlekamp *et al.* [3, 4, 5, 6]. For recent advances in this field, see Nowakowski [33, 34, 35], Albert and Nowakowski [1], and Larsson [29].

Our formalism is also inspired by Frank and Basin [16], who, in order to model the card play phase of the card game Bridge, propose a game with public actions and one-sided incomplete information in which the opponent has complete information. Frank and Basin [17] show that finding optimal pure strategies for these games is NP-complete. Ginsberg [20] proposes the first exact algorithm for these games, and implements it for Bridge robots. Parallely, Chu and Halpern [11] study a model of games with incomplete information with common payoffs, and only one round of concurrent interaction after Nature picks the world; they show that it is NP-complete to play such games optimally.

Like us, Kovarik *et al.* [27] highlight the distinction between public and private

actions. They also argue that this distinction, essential for recent search algorithms, is partially lost when we model sequential multi-agent interaction with EFGs, which do not *explicitly* tell whether an action is public or not. They propose an alternative model for stochastic games that makes this distinction prominent, and show how to transform such models to augmented EFGs and vice versa.

Complexity of games. Most work in the literature on the computational complexity of games concerns the complexity of finding Nash equilibria, especially for normal-form games [18, 14]. For more references, see Conitzer and Sandholm [12], who also show that it is NP-complete to decide whether Nash Equilibria with certain natural properties exist.

Koller and Megiddo [25], Koller *et al.* [26], and von Stengel [41] make seminal contributions to understanding the complexity of two-player zero-sum EFGs. They also give polynomial-time algorithms for computing behaviour maxmin strategies of EFGs with perfect recall, based on linear programming.

Maxmin for a team of players with common payoffs is called team maxmin equilibrium (TME) in the literature, and was first proposed by von Stengel and Koller [42]. Basilico *et al.* [2] and Celli and Gatti [8] propose another notion called TMECor (“Cor” stands for “correlation”), which allows agents in the same team to access a correlation device in order to coordinate their mixed strategies. Building on these works, Gimbert *et al.* [19] and Zhang *et al.* [43] study the complexity of TME and TMECor, thereby yielding a relatively complete picture of the complexity of behaviour and mixed maxmin for two-team EFGs.

The complexity of other models of decision making have also been extensively studied, e.g. Markov decision process [32, 7, 21], propositional planning [37], graph games [10, 9]. Similar to these works, we confirm the intuition that partial observability and multi-agent coordination increases the difficulty of optimal decision making.

3 Combinatorial Games with Incomplete Information

3.1 Definitions

Combinatorial games are EFGs of no chance and with perfect information. To generalise this formalism minimally to allow incomplete information, we propose the following definition.

Definition 3.1 (CGII). A combinatorial game with incomplete information (CGII) is a tuple of the following elements:

- An Aumann model $\langle U, A, (\mathcal{R}_i)_{i \in A}, \rho \rangle$, where U is a finite set of worlds called universe, A is a set of agents, \mathcal{R}_i is an equivalence relation over U for each agent $i \in A$, and $\rho \in \Delta(U)$ is a probability distribution over the universe called common prior;
- A tree T called public tree, the nodes of which $(N(T))$ are partitioned into $\{N_i(T)\}_{i \in A} \cup L(T)$, where $L(T)$ is the set of all leaves;
- A reward function $u_i : L(T) \times U \rightarrow \mathbb{R}$ for each $i \in A$.

Note that the children of a node (available actions at that node) do not depend on the real (and partially observable) world ω ; only the rewards depend on ω .

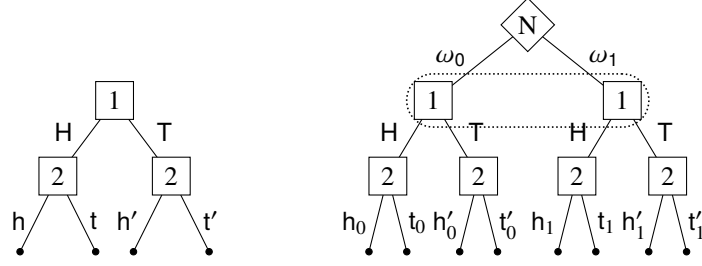


Figure 1: The public tree of a CGII and the game tree of its corresponding EFG, both with rewards omitted.

A CGII is said to be *Boolean* if all its reward functions have values in \mathbb{B} .¹ The Aumann model of a CGII defines each agent's observability over the universe, which characterises their incomplete information.

Pure strategies in a CGII. A CGII as an EFG with incomplete information proceeds as follows. First, Nature picks the *real world* $\omega \in U$ according to ρ . Then the *state game* in ω proceeds from the root of the public tree T ; agents take turns to pick a child of the current node, depending on their equivalence class of the real world. This continues until a leaf l is reached, and agent i receives a payoff $u_i(l, \omega)$.

Definition 3.2 (Pure strategy). A pure strategy of an agent $i \in A$ is a mapping $s_i : N_i(T) \times U \rightarrow N(T)$ such that for all $v \in N_i(T)$:

- For all $\omega \in U$, $s_i(v, \omega)$ is a child of v ;
- $\forall \omega, \omega' \in U, \omega \mathcal{R}_i \omega' \implies s_i(v, \omega) = s_i(v, \omega')$.²

The set of all pure strategies of agent i is denoted by Σ_i^P .

From the definition of a strategy, one can see that the actions of every agent are indeed public: when making a decision at a node, an agent knows perfectly where the node is in the public tree, which in particular means they observe and remember the actions picked by every agent in the past, starting from the root of the public tree. In addition, compared to general games with incomplete information, the state games of a CGII have the particularity that they share the same game tree T , which does not have chance nodes. These three features (public actions, unique game tree, and no chance) are the defining features of our formalism CGII.

Each CGII describes an EFG in which Nature picks the real world at the root, and information sets are determined by the players' observability of the world.

Example. Consider the following CGII: the public tree is shown in Figure 1 (on the left); the universe reads $\{\omega_0, \omega_1\}$; agent 2 can distinguish these two worlds while agent 1 cannot. This CGII models a variant of Matching Pennies with incomplete information. The game tree of its corresponding EFG is also shown in Figure 1 (on the right).

On the right, since agent 1 cannot observe the real world, they must play H in both state games, or T in both. This constraint is respected by the notion of strategies in a

¹In Boolean games, the rewards 0 and 1 are interpreted as a loss and a win, respectively.

²This means agent i must pick the same child for a node in any two worlds indistinguishable by them.

CGII: on the left, agent 1 only has two pure strategies H and T since ω_0 and ω_1 are indistinguishable by agent 1.

Similarly, on the right, agent 2 can pick between heads or tails, depending on the choices of Nature and agent 1. On the left, agent 2 can again pick between heads or tails, depending on agent 1's choice and the real world, since agent 2 can distinguish between ω_0 and ω_1 ; note that the latter point is not reflected by the public tree, but by the Aumann model.

Teams and Information in a CGII. In a CGII, agents i and j are said to be in the same *team* if $u_i = u_j$.

Definition 3.3 (Team). A team is an inclusion-wise maximal group of agents with the same reward function.

We now define a team's *degree of incomplete information*.

- *Multi-agent incomplete information (MA-II):* an arbitrary team.
- *Single-agent incomplete information (SA-II):* a team of agents with the same equivalence relation (i.e. $\mathcal{R}_i = \mathcal{R}_j$ for all agents i and j in the team).
- *Complete information (CI):* a team whose agents all have the finest equivalence relation (i.e. $\mathcal{R}_i = \{(\omega, \omega) \mid \omega \in U\}$ for all agents i in the team).

In particular, CI implies SA-II, which implies MA-II. Intuitively, a team is a group of decentralised agents with shared interests working cooperatively. In a CGII, a team with SA-II can be regarded as one single agent since every agent in this team has the same information and all actions are public.

Example. In the CGII in Figure 1, if the two agents have the same reward function, then they are in a team with MA-II; otherwise, each is a (single-agent) team with SA-II.³

Due to the public actions property, there is a close link between the degree of incomplete information of a team in a CGII and the degree of imperfect information of the corresponding team in the EFG defined by the CGII:

- a team with CI in the CGII is a player with perfect information in the EFG;
- a team with SA-II in the CGII can be seen as a single player with perfect recall in the EFG;
- a team with MA-II in the CGII is a team of players who all have perfect recall in the EFG.

This correspondence will allow us to establish upper bounds on the complexity of solving CGIIs.

Team maxmin in a CGII. Let $(s_1, \dots, s_n) \in \Sigma_1^P \times \dots \times \Sigma_n^P$, where $n = |A|$, be a pure *strategy profile*. We write $(s_1, \dots, s_n)(\omega)$ for the unique leaf reached under this profile when the real world is ω .

³The team of agent 2 even has CI.

Definition 3.4 (Expected utility). *The expected utility for an agent $i \in A$ under a pure strategy profile (s_1, \dots, s_n) is defined to be:*

$$\mathcal{U}_i(s_1, \dots, s_n) = \sum_{\omega \in U} \rho(\omega) u_i((s_1, \dots, s_n)(\omega), \omega).$$

Let $\mathcal{T} \subseteq A$ be a team. Notice that all agents in a team share the same expected utility function, which we denote by $\mathcal{U}_{\mathcal{T}}$. A pure strategy of the team is uniquely defined by the pure strategy of each of its players. In particular, the set of pure strategies of a team \mathcal{T} , denoted by $\Sigma_{\mathcal{T}}^P$, is in bijection with $\prod_{i \in A} \Sigma_i^P$. In the following, we also write $\Sigma_{-\mathcal{T}}^P = \prod_{i \notin \mathcal{T}} \Sigma_i^P$, the set of pure strategy profiles of the players not in \mathcal{T} .

Definition 3.5 (Pure maxmin for a team). *The pure maxmin value for a team $\mathcal{T} \subseteq A$ is defined to be*

$$\underline{v}_{\mathcal{T}} := \max_{s_{\mathcal{T}} \in \Sigma_{\mathcal{T}}^P} \min_{s_{-\mathcal{T}} \in \Sigma_{-\mathcal{T}}^P} \mathcal{U}_{\mathcal{T}}(s_{\mathcal{T}}, s_{-\mathcal{T}}).$$

Intuitively, this value is the largest expected reward that a team can guarantee to get by playing a pure strategy.

The notion of behaviour/mixed strategy can be defined similarly to the one for EFGs: a mixed strategy of an agent i is a probability mixture of pure strategies of i ; a behaviour strategy of i picks, at each node and for each equivalence class of \mathcal{R}_i , a probability mixture of children (instead of just a child as for pure strategies). Hence, expected utility with respect to behaviour/mixed strategy profiles and behaviour/mixed maxmin for a team can be similarly defined.⁴

In the following, we focus on zero-sum two-team CGIIs. We call the two teams *player MAX* and *player MIN*, and denote them by $+$ and $-$, respectively.

3.2 Motivation for CGIIs

Our motivations for introducing CGII as a subclass of games with incomplete information are multiple. First and foremost, the formalism of CGII aims to be a minimal generalisation of combinatorial games to allow incomplete information. Indeed, it is clear that a CGII with a singleton universe is a combinatorial game. This new formalism allows modelling a number of card games, notably Bridge.⁵

But more importantly, the formalism of CGII also aims to minimally capture the notion of knowledge and incomplete information. Due to the public actions property, the only source of the imperfect (in particular, incomplete) information of every agent comes from their partial observability of the real world, drawn at the beginning of a game.

In contrast, we argue that the distinction between perfect and imperfect information does not completely capture the essence of players' knowledge. For example, in the game Matching Pennies, MAX and MIN pick a side of a coin concurrently; this can be modelled by two different EFGs, shown in Figure 2. In the EFG on the left, MAX has perfect information while MIN has imperfect information but perfect recall; and in the EFG on the right, the situation is reversed. However, the roles of MAX and MIN are symmetric in Matching Pennies; MAX also has exactly the same information/knowledge in both EFGs when they need to choose an action. Hence, considering CGIIs allows one to focus on an unambiguous notion of knowledge of the players, as captured by the Aumann model and the initial drawing of a world.

⁴Behaviour maxmin and mixed maxmin are commonly known as TME and TMECor in the literature [8].

⁵The card play of Bridge can be described as a CGII in which MAX has SA-II and MIN has MA-II.

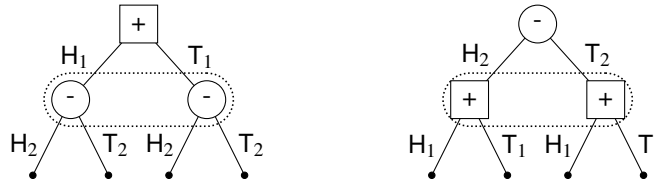


Figure 2: Two EFGs for Matching Pennies.

Expressiveness. At first sight, the requirements of public actions and no chance seem particularly restrictive: many popular tabletop games with incomplete information allow private actions (e.g. concealed Kong in Mahjong, pass in Hearts) or have randomness and chance factors besides the initial drawing (e.g. dice rolls during a game). One may worry that, due to these restrictions, CGII is not expressive enough to be conceptually or algorithmically interesting. However, we argue that this impression is not correct.

First, an initial drawing over the universe is actually quite expressive. For example, for the dice rolls we evoke above, if their number and occasions are fixed in advance, then their results can be encoded into the initial drawing of worlds.⁶ Another example is given by video games, which typically use a random seed as the sole source of randomness for all procedurally generated levels and random events during a playthrough. Similar ideas have been investigated in automated planning [36, Sec. 10].

Second, even with only public actions, we show in Subsection 4.1 that we can still design a game to force a team of players to coordinate their actions. This means that we can essentially encode concurrent actions (as in standard Matching Pennies) using only public actions (and no chance except the initial drawing).

All in all, we suggest that at least as far as computation of optimal strategies is concerned, CGII, rather than EFG, be the right model for studying sequential multi-agent interactions depending on each player’s knowledge. Moreover, as we will show, CGIIs are as hard to solve as EFGs, which confirms our intuition that the difficulty of a game actually comes from the incomplete information of a player, and not from their inability to observe the moves made by the other players.

4 Complexity of PURE MAXMIN for CGIIs

The decision problem PURE MAXMIN is defined as follows.

Definition 4.1 (PURE MAXMIN). *Let \mathcal{G} be a class of zero-sum CGIIs. Then PURE MAXMIN(\mathcal{G}) is the following decision problem.*

INPUT A CGII $G \in \mathcal{G}$ and a rational number m .

OUTPUT Decide whether the pure maxmin value for team MAX in G satisfies $v_+(\Sigma_+^P, \Sigma_-^P) \geq m$.

We study the complexity of PURE MAXMIN for CGIIs depending on the degrees of incomplete information for MAX and MIN: complete information (CI), single-agent incomplete information (SA-II), multi-agent incomplete information (MA-II). For complexity analyses, we consider the parameters $|T|$ (number of nodes in the public

⁶This will only enlarge the game tree by a polynomial factor.

	MIN	CI	SA-II	MA-II
MAX	CI	P	NP-c	Σ_2^P -c
	SA-II	NP-c	NP-c	Σ_2^P -c
	MA-II	NP-c	NP-c	Σ_2^P -c

Table 1: Complexity of PURE MAXMIN for CGIIs.

tree), $|U|$ (number of worlds), and possibly the number of bits to encode the utilities, the common prior, and the threshold m .

The complexity of PURE MAXMIN is summarised in Table 1. By definition, the complexity of each case is increasingly monotone in both MAX's and MIN's degree of incomplete information (CI/SA-II/MA-II). Hence, only a few hardness results have to be proved to establish the table. The results written in bold font are new from this work; the others can be directly deduced from the literature.

The membership results in Table 1 follow from results by Koller and Megiddo [25, Sec. 3.3]; in particular, memberships in NP and in Σ_2^P follow from the fact that given a strategy of MAX, computing MIN's best response is a problem in coNP, and even linear time when MIN has perfect recall.

Hence, we focus on hardness results. The following result is by Frank and Basin [17, Sec. 6].⁷

Proposition 4.2. *PURE MAXMIN is NP-hard for Boolean CGIIs in which MAX has SA-II and MIN has CI.*

The symmetric case does not trivially follow from this result (since the minimax theorem does not hold for pure strategies) and necessitates a proof:

Proposition 4.3. *PURE MAXMIN is NP-hard for Boolean CGIIs in which MAX has CI and MIN has SA-II.*

Proof sketch. By a reduction from VERTEX COVER. Given a graph (V, E) , consider the universe $U = \{\omega_e \mid e \in E\}$; the worlds are observable by MAX but not by MIN. During the game, MAX picks a vertex $v \in V$, then MIN picks an edge $e' \in E$. In a world $\omega_e \in U$, MAX gets a payoff of 1 if v covers e and MIN does not correctly guess this edge (i.e. $e' \neq e$); otherwise, MAX gets 0. One can verify that the pure maxmin value of this game is at least $1 - k/|E|$ if and only if the graph has a vertex cover of size at most k . \square

4.1 Multi-Agent Coordination in CGIIs

Coordination game. Now we turn our attention to CGIIs with multi-agent teams. We first show how to construct CGIIs to impose a perfect coordination between agents from the same team (à la Matching Pennies).

Consider the following Boolean CGII, which we call *coordination game*. This game has two agents of MAX, referred to as MAX 1 and MAX 2, and no agent of MIN; its universe has 4 worlds and reads $U = \{(b_1, b_2) \mid b_1, b_2 \in \mathbb{B}\}$; its Aumann model has the uniform common prior and is such that for $i = 1, 2$, MAX i only observes b_i ; its public

⁷In their setting, there is no prior over the worlds; they are interested in the strategies that win in the greatest number of worlds. This is equivalent to finding maxmin strategies with respect to the uniform prior in our setting.

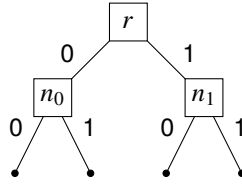


Figure 3: The public tree of the coordination game.

tree is shown in Figure 3; the reward for MAX is 1 if and only if $a_1 \oplus b_1 = a_2 \oplus b_2$, where \oplus is the exclusive or of two bits and a_i is the action chosen by MAX i .

We refer to b_i as the *hidden bit* of MAX i since it is only observable by MAX i . The coordination game is designed in such a way that MAX 1 and MAX 2 must perfectly coordinate their answer in order to win. Intuitively, MAX 1 and MAX 2 need to agree on the same answer $A \in \mathbb{B}$, then stick to it during the game by playing $a_i = A \oplus b_i$. Indeed, if they employ this strategy, then they guarantee a win since

$$a_1 \oplus b_1 = (A \oplus b_1) \oplus b_1 = A = (A \oplus b_2) \oplus b_2 = a_2 \oplus b_2.$$

Remark. Under these two winning strategies (one for each value of A), both MAX 1 and 2 pick the actions 0 and 1 with equal probability. Indeed, once the common answer A is fixed, which action to play by MAX i is dictated by their hidden bit b_i . Hence, the bits b_1 and b_2 act as the keys of a one-time pad to encrypt/mask the intended answer (i.e. A) of MAX 1 and 2. This is the key element to ensure that MAX 1 and 2 must cooperate without cheating.

Proposition 4.4. In a coordination game, the only winning pure strategies of team MAX are of the following form: for some $A \in \mathbb{B}$, MAX 1 plays $A \oplus b_1$ and MAX 2 plays $A \oplus b_2$.

Proof. Notice that the pure strategies of MAX 1 can be written in the form (a_1^0, a_1^1) , which means they choose a_1^0 if $b_1 = 0$ and a_1^1 if $b_1 = 1$. As for MAX 2, they have the right to pick a_2 as a function of a_1 and b_2 . If MAX 1 plays (A, A) (i.e. they play A regardless of b_1) for some $A \in \mathbb{B}$, then MAX 2 has no winning strategy, since the winning condition $A \oplus b_1 = a_2 \oplus b_2$ cannot be satisfied for both values of b_1 . Now if MAX 1 plays $(A, A \oplus 1)$ for some $A \in \mathbb{B}$, then to satisfy the winning condition, MAX 2 is forced to play $a_2 = A \oplus b_2$; hence $a_i = A \oplus b_i$. \square

The same reasoning also shows that these pure strategies are also the only winning behaviour strategies, and that the winning mixed strategies are exactly the mixtures of them.

Remark. From this proof, one can see that if MAX 1 cheats by using their hidden bit b_1 incorrectly (i.e. does not use b_1 to encrypt their intended answer and always picks the same action), then MAX 2 cannot cooperate perfectly since they cannot observe the value of b_1 .

In addition, when MAX 1 plays correctly (i.e. chooses a strategy of the form $(A, A \oplus 1)$), then MAX 2 must also pick A as their intended answer and mask it with their own bit b_2 in order to win. Notice that in this case, the action picked by these two agents are uniformly and independently distributed. This is an important feature since agents (of MAX or MIN) in the following part of the game tree cannot deduce any information about the intended answer of these two agents by observing only their actions.

Interrogation game. We now generalise the coordination game to the following situation: we have a finite set of questions Q , and MAX has a Boolean answer for each question $\{A_q \in \mathbb{B}\}_{q \in Q}$, or equivalently a mapping from Q to \mathbb{B} . We wish to verify whether MAX's mapping satisfies some given binary constraints $\{C_{qq'} \subseteq \mathbb{B}^2 \mid q, q' \in Q, q \neq q'\}$: MAX's mapping is said to be *valid* if it satisfies all the constraints, that is, $(A_q, A_{q'}) \in C_{qq'}$ for all $C_{qq'}$.

Example. For cliques of a given graph, the questions are the vertices of this graph; MAX's mapping induces a subgraph (MAX's answer to a vertex corresponds to whether to include this vertex in their intended subgraph); the binary constraints impose the requirement that all vertices in this subgraph be connected. Then MAX's mapping is valid if and only if it describes a clique of the graph.

To model this situation, consider the following Boolean CGII, which we call *interrogation game*: two agents of MAX (MAX 1 and MAX 2), and no agent of MIN; the universe reads $U = \{(q_1, b_1, q_2, b_2) \mid q_1, q_2 \in Q, b_1, b_2 \in \mathbb{B}\}$; the Aumann model is such that for $i = 1, 2$, MAX i only observes q_i and b_i ; the common prior is uniform; the public tree is the same one as for the coordination game (i.e. the one in Figure 3); MAX loses if and only if either (1) $q_1 = q_2$ but $a_1 \oplus b_1 \neq a_2 \oplus b_2$ or (2) $q_1 \neq q_2$ but $(a_1 \oplus b_1, a_2 \oplus b_2) \notin C_{q_1 q_2}$.

This CGII has size $O(|Q|^2)$: the universe has size $O(|Q|^2)$, while the public tree has size $O(1)$. Notice that a coordination game is just an interrogation game with only one question (hence no binary constraint). We refer to (q_i, b_i) as the *hidden information* of MAX i . Inspired by the coordination game, we propose the following definition.

Definition 4.5 (Perfect coordination). *In an interrogation game, a perfect coordination of team MAX is a pure strategy of MAX of this form: there is a set $\{A_q \in \mathbb{B}\}_{q \in Q}$ such that for all i , MAX i will play the action $a_i = A_{q_i} \oplus b_i$ in all worlds in which their hidden information is (q_i, b_i) . For such a strategy, the set $\{A_q\}_{q \in Q}$ is called the intended mapping or intended answer of the perfect coordination.*

By a similar argument to the one for the coordination game, the reward condition (1) ensures that MAX 1 and 2 have an incentive to implement a perfect coordination, which is a dominant strategy. In other words, (1) imposes non-adaptivity of MAX's answers. As for the reward condition (2), it ensures that all binary constraints are satisfied by the intended mapping of a perfect coordination, since by (1) we have $a_i \oplus b_i = A_{q_i}$ for all i . In summary, we have established the following result.

Proposition 4.6. *In an interrogation game, a pure strategy of team MAX is winning if and only if it is a perfect coordination with a valid intended mapping.*

It is straightforward to construct interrogation games involving team MIN such that if MIN does not cooperate, MAX receives a large reward. Similarly, we can also extend the construction above to allow k -ary constraints with $k \geq 2$. The interrogation game will then involve k agents of MAX, each with their hidden information (q_i, b_i) , and has size $O(2^k |Q|^k)$. Such an interrogation game can be used to encode problems such as k -SAT.⁸

⁸Contrastingly, we leave open the problem of constructing an interrogation game in which MAX's answers are not binary.

4.2 Hardness for Two-Team CGIs

With the gadgets of interrogation game, it is straightforward to show that PURE MAXMIN is Σ_2^P -hard for CGIs in which both MAX and MIN are multi-agent teams, for instance by a reduction from the canonical problem $\exists\forall 3\text{SAT}$. However, we provide a stronger result: Σ_2^P -hardness holds even when MAX has complete information.

Proposition 4.7. *PURE MAXMIN is Σ_2^P -hard for CGIs in which MAX has CI and MIN has MA-II.*

Proof sketch. By a reduction from the Σ_2^P -complete problem SUCCINCT SET COVER [40]: given a collection of 3-DNF formulae and an integer k , decide whether there is a subset S of size at most k the disjunction of which is a tautology.

We design a game in which Nature draws a DNF formula from the collection, 3 variables, and 4 hidden bits, according to the uniform common prior. The DNF is known to MAX, who plays 1 or 0 according to whether it should be in S . This answer is masked (to MIN) by the hidden bit of MAX, as in a coordination game. Then MIN chooses either to verify the size of S or to verify that the disjunction of S is a tautology. The other 3 hidden bits are used in the latter verification: MIN plays an interrogation game over the 3 variables, with the constraint to falsify the disjunction of S .

Finally, since MAX is designed to have CI, they know the variables and the hidden bits of MIN. To ensure that MAX does not play a strategy that depends on MIN's information, we introduce one additional agent of MIN whose role is to punish MAX whenever MAX plays such a strategy. \square

Remark. *The construction shows something stronger: Σ_2^P -hardness holds even when MIN has joint complete information (i.e. if the agents of MIN could pool their information, then they would have complete information).*

5 Complexity of BEHAVIOUR MAXMIN and MIXED MAXMIN

The decision problems BEHAVIOUR MAXMIN and MIXED MAXMIN can be defined similarly to PURE MAXMIN, the only difference being that MAX can use behaviour/mixed strategies instead of just pure strategies.⁹

The complexity of BEHAVIOUR MAXMIN and MIXED MAXMIN is summarised in Table 2. Again, the complexity is increasingly monotone in both dimensions, and results written in bold font are new. The only case where the complexity differs between behaviour and mixed strategies is the case in which both MAX and MIN have MA-II; in this case, BEHAVIOUR MAXMIN and MIXED MAXMIN are respectively Σ_2^P - and Δ_2^P -complete.

The membership results follow from those for EFGs, which are superclasses of CGIs: the results for P are by Koller and Megiddo [25, Sec. 3.5], and the others by Zhang *et al.* [43, Appx. C].

Therefore, we only have to establish the hardness results when MAX and/or MIN have MA-II. We first adapt a reduction from 3-SAT by Chu and Halpern [11].

⁹In our definition for all these decision problems, MIN only uses pure strategies, which is without loss of generality. Indeed, MIN is a team of agents with perfect recall, hence every MIN's behaviour strategy has an equivalent mixed strategy [31, Theorem 6.11]. In addition, the best responses in mixed strategies are no better than the best responses in pure strategies due to the linearity of expected utility with respect to mixtures of strategies.

	MIN	CI	SA-II	MA-II
MAX	CI	P	P	coNP-c
	SA-II	P	P	coNP-c
	MA-II	NP-c	NP-c	$\Sigma_2^P\text{-c}/\Delta_2^P\text{-c}$

Table 2: Complexity of BEHAVIOUR MAXMIN and of MIXED MAXMIN for CGIIs.

Proposition 5.1. *Both BEHAVIOUR MAXMIN and MIXED MAXMIN are NP-hard for Boolean CGIIs in which MAX has MA-II and MIN has CI.*

Proof. For a 3-CNF with N clauses, consider the following CGII. The universe consists of the clauses, which are observable by MAX 1 but not by MAX 2, and with the uniform prior. During the game, MAX 1 picks a variable, then MAX 2 observes this variable and picks a truth value. They win if and only if the variable picked by MAX 1 is in the clause picked by Nature, and the truth value picked by MAX 2 for this variable renders this clause true.

Since there is no agent of MIN in this game, playing behaviour or mixed strategies is no better than pure ones. It is also straightforward to verify that MAX can guarantee an expected payoff of 1 if the 3-CNF is satisfiable; otherwise, the maxmin value for MAX is at most $1 - 1/N$. \square

Now, coNP-hardness for the symmetric case (when MAX has CI and MIN has MA-II) essentially follows from this result. For mixed strategies, the minimax theorem ensures that when switching the roles of MIN and MAX, and negating the utilities in the game from the proof of Proposition 5.1, the maxmin value for MAX is at least $-(1 - 1/N)$ if the 3-CNF is unsatisfiable, and -1 otherwise. The hardness for BEHAVIOUR MAXMIN follows from the fact that mixed maxmin and behaviour maxmin have the same value due to MAX's perfect recall.

Proposition 5.2. *BEHAVIOUR MAXMIN is Σ_2^P -hard for CGIIs in which both MAX and MIN have MA-II.*

Proof sketch. By a reduction from $\exists\forall 3\text{SAT}$ (for a 3-DNF formula $\varphi(x, y)$, decides whether $\exists x\forall y \varphi(x, y)$ holds) which is known to be Σ_2^P -hard [38]. Given such a formula, we construct a CGII with 3 agents of MAX and 3 agents of MIN. The worlds consist of one existential (resp. universal) variable and one hidden bit for each agent of MAX (resp. of MIN); the common prior is uniform; each agent only observes their variable and hidden bit. During the game, the agents of MAX take turns to choose between 0 and 1, then so do the agents of MIN. The total payoff for MAX is computed as follows: (1) an inconsistency among the agents of MAX (in the sense of an interrogation game) yields $-N$ for MAX, where N is a large real number; (2) an inconsistency among the agents of MIN yields $+N$ for MAX; (3) if at least one term in $\varphi(x, y)$ is satisfied by the assignment picked by the agents of MAX and MIN, then MAX receives $+1$.

By choosing N large enough, agents of MAX have an incentive to perform a perfect coordination, and the same goes for agents of MIN. In particular, MAX has no incentive to play non-pure behaviour strategies, which would cause inconsistency to happen with a non-zero probability. It is then straightforward to verify that $\exists x\forall y \varphi(x, y)$ holds if and only if MAX can guarantee an expected utility of at least $+1/n^3$, where n is the maximum between the number of existential variables and the number of universal ones. \square

Proposition 5.3. MIXED MAXMIN is Δ_2^P -hard for CGIIs in which both MAX and MIN have MA-II.

Proof sketch. By a reduction from LAST SAT (for a 3-CNF, decide whether the lexicographically maximum satisfying assignment has value 1 for the last variable), which is Δ_2^P -hard [28]. The construction is very similar to the last proof. Given a 3-CNF, we write the variables as x_1, \dots, x_n , and we construct a CGII with 3 agents of MAX and 3 agents of MIN. The worlds consist of one variable and one hidden bit for each agent of MAX or MIN; the common prior is uniform; each agent only observes their variable and hidden bit. During the game, the agents of MAX take turns to choose between 0 and 1, then so do the agents of MIN. The total payoff for MAX is computed as follows: (1) an inconsistency among the agents of MAX or a clause violated by their assignment yields $-2N$ for MAX, where N is a large real number; (2) an inconsistency among the agents of MIN or a clause violated by their assignment yields $+N$ for MAX; (3) for the first agent of MAX (resp. of MIN), if their hidden variable and bit are x_k and b , and they pick $1 \oplus b$, then MAX receives $+2^{n-k}$ (resp. -2^{n-k}); (4) MAX receives a bonus $+1$ if the variable x_n is assigned $1 \oplus b_1^+$ by the first agent of MAX.

By choosing N large enough, both MAX and MIN have an incentive to perform a perfect coordination (which can be pure or mixed for MAX) with a satisfying assignment. Let $x = (x_1, \dots, x_n)$ be the lexicographically maximum satisfying assignment (if there is no such assignment, then MAX is bound to get a large negative expected utility). If $x_n = 1$, then MAX can guarantee an expected utility of $+1/n$ by choosing this assignment for their perfect coordination; the best MIN can do is to choose this assignment. If $x_n = 0$, MAX has an expected utility of at most 0 when MIN plays this assignment: MAX gets 0 by playing the same assignment, and possibly less when playing other satisfying (hence lexicographically smaller) assignments with a non-zero probability. \square

6 Conclusion

We have proposed a new formalism for extensive-form games with incomplete information that we name combinatorial games with incomplete information. Compared to EFGs, CGIIs only have public actions and one chance node at the beginning of the game, thereby putting better emphasis on the aspect of incomplete information/knowledge of the players.

Apart from the conceptual simplicity, the interests in this new formalism are also justified by the complexity results. Indeed, all the upper bounds for CGIIs are provided by membership results that also hold for EFGs, while all the lower bounds, proven by hardness results, coincide with the upper bounds. In particular, for every degree of observability, CGIIs have the same complexity as EFGs.

We have also shown how to model binary concurrent actions to enforce multi-agent coordination in CGIIs. We leave to future work how to model other types of hidden actions, in particular non-binary concurrent actions. Future work also includes tightening the complexity results to show that hardness holds even for Boolean CGIIs with a minimum number of agents and distributed knowledge of the real world for each team; designing a generic polynomial transformation from an arbitrary two-team EFG into a CGII; and extending the study to general-sum multi-team CGIIs with respect to solution concepts that generalise maxmin (e.g. strategies to commit to [30]). Algorithmic studies adapted to CGIIs will also be of interest, with the long-term goal to implement better AIs for games such as Bridge.

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A Auxiliary constructions

Interrogation game with k -ary constraints. For all fixed $k \geq 2$, the construction of an interrogation game with k -ary constraints is straightforward. Concretely, we can consider the following Boolean CGII with k agents of MAX: the universe reads

$$U = \{(q_1, b_1, \dots, q_k, b_k) \mid q_1, \dots, q_k \in Q, b_1, \dots, b_k \in \mathbb{B}\};$$

the Aumann model is such that for all $1 \leq i \leq k$, MAX i only observes q_i and b_i ; the common prior is uniform; the public tree is such that each agent of MAX sequentially choose between 0 and 1; the reward for MAX is 0 if either (1) $q_i = q_j$ but $a_i \oplus b_i \neq a_j \oplus b_j$ for some i and j , or (2) $(a_1 \oplus b_1, \dots, a_k \oplus b_k) \notin C_{q_1 \dots q_k}$ for some $q_1, \dots, q_k \in Q$, and 1 otherwise.

Such an interrogation game has size $O(2^k |Q|^k)$, which is still polynomial in $|Q|$ for all fixed k .

Interrogation game for MIN. For all $k \geq 2$, one can also construct interrogation games with k -ary constraints for MIN. The construction, which now involves k agents of MIN and no agent of MAX, is very similar to the one for MAX. It suffices to modify the reward function such that the reward for MAX is 1 if either (1) $q_i = q_j$ but $a_i \oplus b_i \neq a_j \oplus b_j$ for some i and j , or (2) $(a_1 \oplus b_1, \dots, a_k \oplus b_k) \notin C_{q_1 \dots q_k}$ for some $q_1, \dots, q_k \in Q$, and 0 otherwise.

In such a game, if MIN does not implement a perfect coordination, then they are caught with a probability of at least $1/(2|Q|)^2$; while if MIN implements a perfect coordination, but their intended mapping is not valid, they are caught with a probability of at least $1/|Q|^k$. Hence, if there is no valid mapping for an interrogation, MAX has a guarantee payoff of at least $\min(1/(2|Q|)^2, 1/|Q|^k) > 0$; if there is a valid mapping, then MIN can guarantee a payoff of 0 for MAX by implementing a perfect coordination with a valid mapping.

B Proofs

B.1 Pure, CI MAX, SA-II MIN

Proposition B.1. PURE MAXMIN is NP-hard for CGIIs in which MAX has CI and MIN has SA-II. The result holds even under the restriction to Boolean games.

Proof. We give a reduction from VERTEX COVER, which is defined as follows:

INPUT A non-directed graph (V, E) , a natural number k .

OUTPUT Decide whether the graph has a vertex cover of size at most k .¹⁰

Let $((V, E), k)$ be an instance of VERTEX COVER. Without loss of generality, we assume that $k \leq |E|$, otherwise, the problem is trivial. Construct the following instance of PURE MAXMIN:

PLAYERS MAX with CI; MIN with SA-II.

AUMANN MODEL The universe reads $U = \{\omega_e \mid e \in E\}$; the common prior is uniform; MAX observes the real world while MIN does not.

PUBLIC GAME TREE The game proceeds as follows: MAX chooses a vertex $v \in V$, then MIN chooses an edge $e' \in E$.

PAYOFFS FOR MAX In world ω_e , MAX gets a payoff of 1 if v is an endpoint of e and $e \neq e'$; otherwise, MAX gets 0.

MAXMIN The threshold for the pure maxmin value is $1 - \frac{k}{|E|}$.

This polynomial-time construction yields a Boolean CGII in which MAX has CI and MIN has SA-II. We now show that the graph (V, E) has a vertex cover of size at most k if and only if the maxmin value of this CGII is at least $1 - \frac{k}{|E|}$.

\implies Suppose first that (V, E) has a vertex cover of size at most k . Let V' be such a vertex cover. Notice that MAX can choose a vertex according to the edge e since MAX can observe the real world. Now consider MAX's pure strategies that consist in choosing v to be in the intersection of V' and the endpoints of e , which is always non-empty because V' is a vertex cover. Now we need to show that these strategies guarantee a payoff of at least $1 - \frac{k}{|E|}$.

Notice that under these strategies, only vertices in V' will ever be chosen by MAX; in other words, in no world will MIN observe that MAX picks a vertex not in V' .¹¹

Since MIN does not observe e , they can only base their choice of e' on v , the vertex chosen by MAX. Intuitively, MIN needs to guess the edge chosen by the initial drawing, based on the vertex picked by MAX, to yield a reward of 0 for MAX. However, regardless of what MIN's pure strategy is, they can only guess correctly in at most one world in which MAX picks v ; in other such worlds in which MIN makes a wrong guess, MAX gets 1. Since this holds for each v chosen by MAX in at least one world, which is necessarily in V' , we deduce that among all $|E|$ worlds, MAX gets 0 in at most $|V'| \leq k \leq |E|$ worlds. This means the reward for such strategies is at least $1 - \frac{k}{|E|}$, therefore the maxmin value is also at least $1 - \frac{k}{|E|}$.

\impliedby Conversely, suppose that (V, E) has no vertex cover of size at most k , i.e. all vertex covers of the graph have size at least $k + 1$. It is clear that $|E| \geq k + 1$; otherwise, the graph must have a vertex cover of smaller size.

Notice that it is a dominating strategy for MAX to always choose a vertex v that is an endpoint of the edge e . Indeed, if MAX does not choose an endpoint of e , they get 0; if they do, they get 1 or 0, depending on MIN's choice e' . Hence, we only need to show that for all such strategies, MAX cannot guarantee a reward of at least $1 - \frac{k}{|E|}$.

¹⁰A vertex cover of a graph is a subset of vertices that contains at least one endpoint of every edge of the graph.

¹¹On the other hand, it is not necessarily the case that every vertex in V' is picked by MAX in at least one world; this is only true if V' is a minimal (by inclusion) vertex cover of the graph.

Consider one such strategy, and let V' be the set of vertices that are picked by MAX in at least one world. Then by definition of this strategy, V' is a vertex cover of the graph, hence has a size of at least $k + 1$. Now, for each $v \in V'$, let MIN play an edge e_v which is such that MAX picks v in world ω_{e_v} . Then for each $v \in V'$, MAX gets 0 in world ω_{e_v} . Since $e_v \neq e_{v'}$ for $v \neq v'$ (by definition MAX picks v in world ω_{e_v} but v' in world $\omega_{e_{v'}}$), this means MAX gets 0 in at least $|V'| \geq k + 1$ worlds; hence MAX has an expected reward of at most $1 - \frac{k+1}{|E|}$ against this strategy of MIN. This being true for all dominant strategies of MAX, the maxmin value for MAX is strictly less than $1 - \frac{k}{|E|}$. \square

B.2 Pure, CI MAX, MA-II MIN

Our proof will be based on a reduction from the Σ_2^P -complete problem **SUCCINCT SET COVER** defined as follows [40]:

INPUT A collection¹² $S = \{\varphi_1, \dots, \varphi_m\}$ of 3-DNF formulae on n variables, and an integer k .

OUTPUT Decide whether there exists a subcollection $S' \subseteq S$ of size at most k and the disjunction of which is a tautology: $\bigvee_{\varphi \in S'} \varphi \equiv 1$.

Remark. We can duplicate the elements in the initial collection S to form a new collection \mathcal{S} that contains each 3-DNF three times. It is clear that for all $1 \leq k \leq |\mathcal{S}|$, \mathcal{S} has a tautological disjunction of at most k of its DNFs if and only if the new collection S has a tautological disjunction of at most k of its DNFs. As a result, every instance (S, k) of **SUCCINCT SET COVER** can be reduced to another instance (\mathcal{S}, k) such that $k/|\mathcal{S}| \leq |S|/|\mathcal{S}| = 1/3$. Therefore, without loss of generality, we may suppose that all instances (S, k) of **SUCCINCT SET COVER** satisfy $1 \leq k \leq |S|/3$.

Proposition B.2. **PURE MAXMIN** is Σ_2^P -hard for CGIs in which MAX has CI and MIN has MA-II.

Proof. Let (S, k) be an instance of **SUCCINCT SET COVER**, where S is a collection of 3-DNF formulae on a set X of n variables. Without loss of generality, we assume that $1 \leq k \leq |S|/3$. Construct the following instance of **PURE MAXMIN**:

PLAYERS MAX with CI; MIN with MA-II consisting of 5 agents, referred to as MIN 1, 2, 3, and Controller 1, 2.

AUMANN MODEL The universe is $U = S \times \mathbb{B} \times M$, where $M = (X \times \mathbb{B})^3$, i.e.

$$U = \{(\varphi, b_+, x_1, b_1, x_2, b_2, x_3, b_3) \mid \varphi \in S, x_1, x_2, x_3 \in X, b_+, b_1, b_2, b_3 \in \mathbb{B}\};$$

MAX observes everything; for $i \in \{1, 2, 3\}$, MIN i only observes x_i and b_i ; Controller 1 only observes φ and b_+ ; Controller 2 observes nothing; the common prior is uniform.

PUBLIC GAME TREE Shown in Figure 4. MAX chooses $a_+ \in \mathbb{B}$; then Controller 1 does nothing (i.e. **Pass**), or plays **CheckM** and picks a $m^* \in M$; in the former case, Controller 2 does nothing (i.e. **Pass**), or plays **Size**, or plays **CheckPhi** and picks a $\varphi^* \in S$; finally, if Controller 2 does nothing, then MIN 1, 2, and 3 participate in an interrogation game with 3 agents, and they play sequentially a_1, a_2 , and a_3 .

¹²A collection can contain two identical elements.

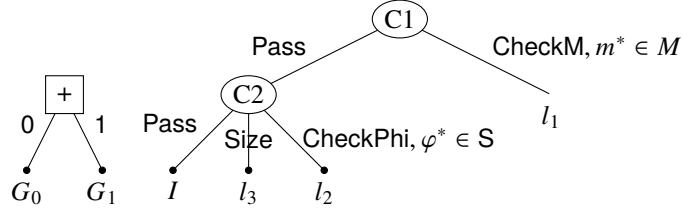


Figure 4: The public tree for playing **Succinct Set Cover**. On the right, the figure shows the subtree that repeats as G_0 and G_1 in the figure on the left. l_1 , l_2 , and l_3 are leaves; I is an interrogation game with three agents of MIN.

PAYOFFS FOR MAX We write l_i^j , where $i \in \{1, 2, 3\}$ and $j \in \mathbb{B}$, for the leaf l_i in the subtree G_j . Let $(\varphi, b_+, m) = (\varphi, b_+, x_1, b_1, x_2, b_2, x_3, b_3)$ denote the real world, and let $N = |U|$. Then for all $j \in \mathbb{B}$:

- At l_1^j , MAX receives $+N^2$ if $m^* = m$, and $-N$ otherwise.
- At l_2^j , MAX receives $+N^2$ if $\varphi^* = \varphi$, and $-N$ otherwise.
- At l_3^j , MAX receives $1 - (a_+ \oplus b_+)$.¹³
- At each leaf of the interrogation game I^j :
 - If $x_i = x_j$ for some $i \neq j$ but $a_i \oplus b_i \neq a_j \oplus b_j$,¹⁴ MAX receives $+N$;
 - Otherwise, if MAX has chosen $1 \oplus b_+$ and a term in the 3-DNF φ is satisfied by the assignments chosen by the 3 agents of MIN to x_1 , x_2 , and x_3 , MAX receives $+N$.
 - Otherwise, MAX receives 0.

MAXMIN The threshold for the pure maxmin value is $1 - k/|S|$.

The construction above is polynomial in the size of the **Succinct Set Cover** instance (S, k) . Indeed, the CGII has a size $\mathcal{O}(|S|n^3)$. In addition, MAX has CI, and MIN has MA-II. For the following, it will be helpful to notice that in a world (φ, b_+, m) :

- MAX picks a_+ as a function of φ , b_+ , and m ;
- Controller 1 picks m^* for CheckM, or plays Pass, as a function of φ , b_+ , and a_+ ;
- Controller 2 picks φ^* for CheckPhi, or plays Pass, as a function of a_+ .

Intuition behind this construction. We first motivate the construction by illustrating the intuition behind it; its correctness will be shown in the next paragraph.

Let us write an arbitrary pure strategy of MAX as a mapping of the form $a_+ : (\varphi, b_+, m) \in U \mapsto a_+(\varphi, b_+, m) \in \mathbb{B}$. We would like MAX's dominant pure strategies to be in bijection with the subcollections of S . To this end, we introduce the two controllers of MIN to punish MAX if one of the following two conditions are violated:

1. $a_+(\varphi, b_+, m)$ depends solely on φ and b_+ , not m .

¹³ $(a_+ \oplus b_+)$ represents the intended answer of MAX; an intended answer of 1 means that MAX includes this DNF in the subcollection.

¹⁴This means MIN i and j intend to choose a different assignment for this same variable.

2. For all $\varphi \in S$ and $m \in M$, $a_+(\varphi, 0, m) \neq a_+(\varphi, 1, m)$.

These conditions ensure that MAX does not pick their answer based on their knowledge of m (condition 1), and MAX always encrypts their answer using their hidden bit b_+ (condition 2). Furthermore, MAX's pure strategies satisfying both conditions are in bijection with the subcollections of S ; each such strategy corresponds to the subcollection

$$\{\varphi \mid \text{MAX plays 1 in every world } (\varphi, 0, m) \text{ with } m \in M\}.$$

Whenever one of these conditions is violated, a controller of MIN can gain information about something that they cannot observe, which they can exploit to punish MAX:

- If condition 1 is violated, Controller 1 can deduce information about m by observing φ , b_+ , and a_+ ; they can find a $m^* \in M$ that cannot be part of the real world.
- If condition 1 is satisfied but not condition 2, Controller 2 can deduce information about φ from a_+ ; they can find a $\phi^* \in S$ that cannot be part of the real world.

When both conditions are satisfied, Controller 2 can play Size to end the game with an expected utility of $1 - |S'|/|S|$ for MAX, where $S' \subseteq S$ is the subcollection corresponding to the pure strategy employed by MAX.

Correctness of the construction. We now show that there exists a subcollection $S' \subseteq S$ of size at most $k \leq |S|/3$ such that its disjunction is a tautology if and only if this game has a maxmin value of at least $1 - k/|S| \geq 2/3$.

\implies Suppose there is a subcollection $S' \subseteq S$ of size at most $k \leq |S|/3$ and the disjunction of which is a tautology. Let us consider MAX's pure strategy corresponding to S' , which means in $(\varphi, b_+, m) \in U$, MAX plays $1 \oplus b_+$ if $\varphi \in S'$, b_+ otherwise.

We first show that MIN's dominant strategy is playing Pass by Controller 1 then Size by Controller 2 in both G_0 and G_1 . This strategy of MIN yields for MAX an expected reward of $1 - |S'|/|S|$, which is at least $1 - k/|S|$ but strictly smaller than 1.

CheckM in G_0 or G_1 Let $\varphi \in S$ and $b_+ \in \mathbb{B}$, and let a_+ be MAX's action in all the worlds (φ, b_+, m) with $m \in M$, which is well-defined since a_+ depends solely on φ and b_+ . If Controller 1 plays CheckM and picks $m^* \in M$ after observing φ , b_+ , and a_+ , then MAX receives $-N$ in all worlds (φ, b_+, m) with $m \neq m^*$, but $+N^2$ in the world (φ, b_+, m^*) . Hence, the expected payoff for MAX is at least 1, which means in neither G_0 nor G_1 does Controller 1 have an incentive to play CheckM.

CheckPhi in G_0 or G_1 The argument is similar to the last one: for all $\varphi \in S$, there is at least one world in which MAX plays 0, and one in which MAX plays 1, so there is never a safe φ^* for Controller 2 to pick. If Controller 2 plays CheckPhi, the expected payoff of MAX is at least 1. Thus, in neither G_0 nor G_1 does Controller 2 have an incentive to play CheckPhi.

I_0 or I_1 In the interrogation games I_0 or I_1 , the agents of MIN has an incentive to implement a perfect coordination since otherwise the discrepancy between the response of two agents of MIN will be discovered in at least one world, yielding

+ N for MAX hence an expected reward of at least 1.¹⁵ However, since MAX picks a subcollection S' such that the disjunction is tautological, in at least one world it will be discovered that one term from the 3-DNFs of S' is satisfied by the assignment of the agents of MIN to their variables. This again means MAX gets + N in at least one world, hence an expected reward of at least 1 for the whole game. Therefore, in neither G_0 nor G_1 does Controller 2 have an incentive to choose Pass and let the game continue into an interrogation game.

In conclusion, MIN's dominant strategy is playing Pass by Controller 1 then Size by Controller 2 in both G_0 and G_1 . Therefore, MAX's strategy guarantees an expected payoff of at least $1 - |S'|/|S|$, which means the maxmin value is at least $1 - k/|S|$.

⇐ Suppose there is no subcollection of S of size at most $k \leq |S|/3$ such that its disjunction is a tautology. We will show that no strategy of MAX can guarantee a payoff of at least $1 - k/|S| \geq 2/3$.

Notice that by playing Pass then Size in both G_0 and G_1 , MIN limits MAX's reward to be at most 1 in all worlds. We first consider the case in which MAX implements a strategy that violates one of conditions 1 and 2.

Condition 1 If MAX's strategy violates condition 1, then there are $\varphi \in S$, $b_+ \in \mathbb{B}$, and $m', m'' \in M$, such that MAX plays 1 in world (φ, b_+, m') and 0 in (φ, b_+, m'') . Then in the worlds (φ, b_+, m) with $m \in M$, Controller 1 can play CheckM with $m^* = m''$ if they observe that MAX plays 1 (i.e. in G_1), and $m^* = m'$ if MAX plays 0 (i.e. in G_0). In the other worlds, Controller 1 plays Pass and Controller 2 plays Size in both G_0 and G_1 . Then in no world MAX gets a reward of + N^2 , but they get $-N$ for all worlds of the form (φ, b_+, m) and at most 1 in the other worlds. Hence, the expected world for MAX is at most $0 < 1 - k/|S|$.

Condition 2 Now suppose MAX's strategy respects condition 1 but not condition 2, then there is $\varphi' \in S$ such that MAX plays 1 (or 0) in world (φ', b_+, m) for all $b_+ \in \mathbb{B}$ and $m \in M$ (the latter being true by condition 1). Without loss of generality, suppose that MAX plays 1 in all these worlds.

- If MAX actually plays 1 in all the worlds of the universe, then Controller 1 plays Pass and Controller 2 plays Size in both G_0 and G_1 so that MAX's expected payoff is restricted to be $1/2 < 1 - k/|S|$. Indeed, with probability $1/2$, $b_+ = 0$ is drawn, yielding $a_+ \oplus b_+ = 1 \oplus b_+ = 1$, hence a payoff of 0 when Controller 2 plays Size.
- Otherwise, MAX plays 0 in at least one world in which the DNF is not φ' . Then Controller 1 plays Pass in both G_0 and G_1 , Controller 2 plays Size in G_1 , and CheckPhi with $\varphi^* = \varphi'$ in G_0 . Then in the worlds in which MAX plays 1, they get at most 1 as reward; in all worlds (there is at least one) in which MAX plays 0, they always get $-N$ as reward since the DNF in these worlds is never φ' . Hence, the expected reward for MAX is again at most $0 < 1 - k/|S|$.

Suppose now that MAX plays a strategy that respects both conditions 1 and 2. Recall that such strategies are in bijection with the subcollections of S .

¹⁵Note that there is no negative reward for MAX at any leaf of I .

- If MAX chooses a subcollection S' such that the disjunction is a tautology, then the size of this subcollection is at least $k + 1$. Then Controller 1 plays Pass and Controller 2 plays Size in both G_0 and G_1 to give MAX an expected payoff of $1 - |S'|/|S| < 1 - k/|S|$.
- Otherwise, the disjunction of S' is not a tautology, then Controller 1 and 2 play Pass in both G_0 and G_1 , and MIN 1, 2, and 3 implement a perfect coordination of an assignment that does not satisfy this disjunction in both of the interrogation games I_1 and I_2 . As a result, MAX has an expected payoff of $0 < 1 - k/|S|$.

In conclusion, no strategy of MAX can assure a payoff of at least $1 - k/|S|$, which means the maxmin value is strictly less than $1 - k/|S|$. \square

B.3 Behaviour, MA-II MAX, MA-II MIN

Proposition B.3. BEHAVIOUR MAXMIN is Σ_2^P -hard for CGIs in which both MAX and MIN have MA-II.

Proof. We give a reduction from $\exists\forall$ 3SAT (for a 3-DNF formula $\varphi(x, y)$ over a set of existential variables X and a set of universal variables Y , decide whether $\exists x\forall y \varphi(x, y)$ holds). Let $\varphi(x, y)$ be an instance of $\exists\forall$ 3SAT. Construct the following instance of BEHAVIOUR MAXMIN:

PLAYERS MAX with MA-II consisting of 3 agents, referred to as MAX 1, 2, 3; MIN with MA-II consisting of 3 agents, referred to as MIN 1, 2, 3.

AUMANN MODEL The universe is $(X \times \mathbb{B})^3 \times (Y \times B)^3$:

$$\{(x_1, b_1^+, x_2, b_2^+, x_3, b_3^+, y_1, b_1^-, y_2, b_2^-, y_3, b_3^-) \mid \\ \forall i \in \{1, 2, 3\}, \forall j \in \{+, -\}, x_i \in X, y_i \in Y, b_i^j \in \mathbb{B}\};$$

for all $i \in \{1, 2, 3\}$, MAX i only observes x_i and b_i^+ while MIN i only observes y_i and b_i^- ; the common prior is uniform.

PUBLIC GAME TREE The six agents (MAX 1, 2, 3, MIN 1, 2, 3) take turns to pick between 0 and 1; their action will be denoted by a_i^j , where $i \in \{1, 2, 3\}$ and $j \in \{+, -\}$.

PAYOFFS FOR MAX MAX's payoff is defined to be the sum of three terms of payoff, defined as follows.

- If there is an inconsistency among the agents of MAX, i.e. for some $i \neq i'$, $a_i^+ \oplus b_i^+ \neq a_{i'}^+ \oplus b_{i'}^+$ but $x_i = x_{i'}$, then MAX receives $-N$, where N is a large natural number.
- If there is an inconsistency among the agents of MIN, i.e. for some $i \neq i'$, $a_i^- \oplus b_i^- \neq a_{i'}^- \oplus b_{i'}^-$ but $y_i = y_{i'}$, then MAX receives $+N$.
- MAX receives $+1$ if at least one term in $\varphi(x, y)$ is satisfied by the assignment picked by the agents of MAX and MIN (i.e. for $x_1, x_2, x_3, y_1, y_2, y_3$).

MAXMIN The threshold for the behaviour maxmin value is $+1/n^3$, where $n = \max(|X|, |Y|)$.

This CGII is basically an interrogation game for MAX, followed by an interrogation game for MIN. The construction is polynomial in the size of the $\exists\forall$ 3SAT instance $\varphi(x, y)$. Indeed, the CGII has size $O(|\varphi|^6)$. We now show that $\exists x\forall y \varphi(x, y)$ holds if and only if the behaviour maxmin value is at least $+1/n^3$.

To simplify the discussion, we first argue that when N is large enough, perfect coordinations are dominant strategies for both MIN and MAX.

- For MIN, we only have to consider their pure strategies.¹⁶ Hence, it is clear that perfect coordinations are dominant strategies.
- For MAX, notice first that under a behaviour strategy, every probability mixture is local. Intuitively, every such mixture with probability $0 \leq p \leq 1$ at some reachable node leads to an inconsistency among the agents of MAX to be detected with a probability cp , where c is a constant independent of p , which causes an expected penalty of $-cpN$ in the first term of payoff. By linearity, such a mixture cannot gain more than $+c'$ from the last term of payoff, where c' is another constant independent of p . Hence for N large enough, reducing p to 0 is a dominant strategy. A rigorous proof can be given by showing (sequentially) that no agent has a unilateral incentive to deviate from a perfect coordination; a sufficient lower bound for N can be expressed as a polynomial function of n .

Therefore, in the following, we may assume that MAX and MIN only consider strategies that are perfect coordinations.

\implies Suppose $\exists x\forall y \varphi(x, y)$ holds. Then MAX can perform a perfect coordination with a mapping corresponding to an assignment to the variables in X that satisfies this formula. Then for every assignment to Y , there is at least one term in φ that is satisfied. In particular, for every perfect coordination of MIN, there is a term satisfied and detected with a probability of at least $1/n^3$ (whenever Nature picks the variables in this term to interrogate MAX and/or MIN). This means MAX gets 0 from the first term of payoff, at least 0 from the second term, and at least $+1/n^3$ from the last term; the expected payoff is therefore at least $+1/n^3$, which is therefore a lower bound for the behaviour maxmin value.

Also, notice that the ability to observe the actions picked by the agents of MAX does not help MIN. Indeed, since MAX implements a perfect coordination, it is straightforward to see that $\mathbb{P}(x_1, x_2, x_3) = \mathbb{P}(x_1, x_2, x_3 \mid a_1^+, a_2^+, a_3^+)$. Therefore, MIN may well perform a perfect coordination with different mappings after observing different sequences of actions by MAX, but it does not change the fact that the satisfied term is detected with a probability of at least $1/n^3$ after each sequence of actions by MAX.

\impliedby Now suppose $\exists x\forall y \varphi(x, y)$ does not hold. Consider an arbitrary perfect coordination of MAX. Then it corresponds to some assignment to X , for which there exists an assignment to Y such that no term in φ is satisfied. By performing a perfect coordination with a mapping corresponding to this assignment to Y , MIN restricts MAX's expected payoff to be 0. Hence, the behaviour maxmin value is at most 0. \square

B.4 Mixed, MA-II MAX, MA-II MIN

Proposition B.4. *MIXED MAXMIN is Δ_2^P -hard for CGIIs in which both MAX and MIN have MA-II.*

¹⁶Recall that without loss of generality, we define the maxmin value with respect to MIN's pure strategies.

Proof. We give a reduction from LAST SAT (for a 3-CNF, decide whether the lexicographically maximum satisfying assignment has value 1 for the last variable), which is Δ_2^P -hard [28]. Consider an instance of LAST SAT φ , a 3-CNF over the set of variables $X = \{x_1, \dots, x_n\}$. Construct the following instance of MIXED MAXMIN:

PLAYERS MAX with MA-II consisting of 3 agents, referred to as MAX 1, 2, 3; MIN with MA-II consisting of 3 agents, referred to as MIN 1, 2, 3.

AUMANN MODEL The universe is $(X \times \mathbb{B})^6$:

$$\{(x_1^+, b_1^+, x_2^+, b_2^+, x_3^+, b_3^+, x_1^-, b_1^-, x_2^-, b_2^-, x_3^-, b_3^-) \mid \\ \forall i \in \{1, 2, 3\}, \forall j \in \{+, -\}, x_i^j \in X, b_i^j \in \mathbb{B}\};$$

for all $i \in \{1, 2, 3\}$, MAX i only observes x_i and b_i^+ while MIN i only observes x_i^- and b_i^- ; the common prior is uniform.

PUBLIC GAME TREE The six agents (MAX 1, 2, 3, MIN 1, 2, 3) take turns to pick between 0 and 1; their action will be denoted by a_i^j , where $i \in \{1, 2, 3\}$ and $j \in \{+, -\}$.

PAYOFFS FOR MAX MAX's payoff is defined to be the sum of five terms of payoff, defined as follows.

- If there is an inconsistency among the agents of MAX, i.e. for some $i \neq i'$, $a_i^+ \oplus b_i^+ \neq a_{i'}^+ \oplus b_{i'}^+$ but $x_i^+ = x_{i'}^+$, or the assignments $a_i^+ \oplus b_i^+$ to x_i^+ for $i \in \{1, 2, 3\}$ render a clause in φ false, then MAX receives $-2N$, where N is a large natural number.
- If there is an inconsistency among the agents of MIN, i.e. for some $i \neq i'$, $a_i^- \oplus b_i^- \neq a_{i'}^- \oplus b_{i'}^-$ but $x_i^- = x_{i'}^-$, or the assignments $a_i^- \oplus b_i^-$ to x_i^- for $i \in \{1, 2, 3\}$ render a clause in φ false, then MAX receives $+N$.
- If $x_1^+ = x_k \in X$ and $a_1^+ \oplus b_1^+ = 1$, MAX receives $+2^{n-k}$.
- If $x_1^- = x_k \in X$ and $a_1^- \oplus b_1^- = 1$, MAX receives -2^{n-k} .
- If $x_1^+ = x_n$ and $a_1^+ \oplus b_1^+ = 1$, MAX receives a bonus $+1$.

MAXMIN The threshold for the mixed maxmin value is $+1/n$.

This CGII is basically an interrogation game for MAX, followed by an interrogation game for MIN. The construction is polynomial in the size of the LAST SAT instance φ . Indeed, the CGII has a size $O(|\varphi|^6)$.

Notice that MAX's payoff is the sum of two parts, the first part (the first, third, and fifth terms above) depends solely on MAX's strategy, while the second part (the second and fourth terms above) depends solely on MIN's strategy. Hence, MAX's mixed maxmin strategies can be taken to be pure strategies. In addition, the reasoning for MAX's best strategies is similar to MIN's. In the following, we will show that if N is large enough, the mixed maxmin value is $+1/n$ if and only if φ is a yes-instance of LAST SAT.

\Leftarrow If φ is a yes-instance of LAST SAT, then by implementing a perfect coordination with the last satisfying assignment, MAX receives a payoff of at least $+1/n$. Indeed, MIN's best strategy is to do the same:

- If MIN does not implement a perfect coordination or implements a perfect coordination with an invalid assignment, then the second term of payoff is non-zero for a probability of at least $1/8n^3$, yielding an expected payoff of at least $+N/8n^3$ to MAX. Such a strategy cannot gain more than 2^n from the fourth term, hence if $N > 8n^3 2^n$, it is dominant for MIN to implement a perfect coordination with a satisfying assignment.
- Among perfect coordinations with a satisfying assignment, the one with the last satisfying assignment is dominant due to the fourth term of payoff.

The conclusion then follows from the fact that when both MAX and MIN play this strategy, the first and second terms yield zero, the third and fourth terms cancel out, and the fifth term yields an expected payoff of $+1/n$.

\Rightarrow Assume now that the mixed maxmin value is $+1/n$. By a similar argument to the one above, the maxmin strategy must be a mixture of perfect coordinations with a satisfying assignment. Indeed, if not so, MIN can play the same strategy as MAX; MAX then suffers a net penalty of $-N$ (notice that the payoff is $-2N$ in the first term of payoff while $-N$ in the second) with a probability of at least $1/8n^3$, which means the total expected payoff is strictly negative. Recall that among these perfect coordinations, the one with the last satisfying assignment is dominant. It is now straightforward to see that the last bit in the satisfying assignment is 1, since otherwise when MAX and MIN both play the perfect coordination with this assignment, MAX gets an expected payoff of exactly 0. \square