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► **To cite this version:**

Manish Kumar, Subrata Majumdar. INSENSITIZING CONTROL PROBLEM FOR THE KAWAHARA EQUATION. 2024. hal-04568222

**HAL Id: hal-04568222**

**<https://hal.science/hal-04568222>**

Preprint submitted on 4 May 2024

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# INSENSITIZING CONTROL PROBLEM FOR THE KAWAHARA EQUATION

MANISH KUMAR<sup>†</sup> AND SUBRATA MAJUMDAR<sup>‡</sup>

ABSTRACT. This paper deals with the existence of insensitizing control for a nonlinear dispersive equation, namely Kawahara equation. Roughly speaking, the underlying problem is to find a distributed control such that  $L^2$ -norm of the state in some subregion is insensitive with respect to small perturbations in the initial data.

The problem of finding an insensitizing control is first reduced to a null controllability problem for an extended cascade system using some standard arguments. Next, to solve this null controllability problem we first establish null controllability of the associated linearized system using suitable Carleman estimates for the corresponding adjoint system, and then use the well known inverse mapping theorem to conclude the desired controllability result for the main nonlinear extended cascade system.

## 1. INTRODUCTION AND MAIN RESULTS

1.1. **Setting of the problem.** This article studies an insensitizing control problem for the following non-linear Kawahara equation

$$\partial_t u + \alpha \partial_x u + \beta \partial_x^3 u + \gamma \partial_x^5 u + u^p \partial_x u = 0$$

on a bounded domain  $(0, T) \times (0, L)$  for  $\alpha = \beta = \gamma = 1$  and  $p \in [1, 2)$ . This equation for  $\alpha = 0$  and  $p = 1$  was first introduced in [43] by T. Kawahara. The presence of fifth order derivative term  $\partial_x^5$  in the equation strengthens the dispersive effect due to the  $\partial_x^3$  term, and thus plays an important role when the coefficient  $\beta$  is small. This is the reason why this equation is also sometimes referred as fifth order KdV equation (see [9]) or singularly perturbed KdV equation (see [37]). This equation models many physical systems in fluid dynamics (such as shallow water waves [41]), plasma physics, and nonlinear optics. The Cauchy problem for this equation over  $\mathbb{R}^2$  was first studied in the work [21], followed by few other works, for instance, see [51], [23]. In the work [29], Cauchy problem over a bounded domain with homogeneous boundary conditions was studied.

Let us discuss about the recent developments on the Kawahara equation in theory of Control of PDEs. Regarding the controllability problem, it was first addressed in 2009 by O. Glass and S. Guerrero in their work [36]. They showed that the Kawahara equation is locally controllable to trajectory with two boundary controls. In the paper [20], Mo Chen has shown the local null controllability and null controllability with some constraints of the Kawahara equation with distributed control. Later in [15], R de A. Capistrano-Filho et al. have studied the well-posedness and controllability of the same equation in weighted Sobolev spaces. They also introduced the *regional controllability* of the concerned equation in  $L^2$ . Next, let us mention the work [47] of A. F. Pazoto et al., where the authors established the local exact controllability of Kawahara equation in periodic domain using the moment method. There are many other control results in the literature with different boundary conditions and/or different type of controls, for instance, [32], and [53]. Concerning the stabilization problem, a significant number of articles are present in the literature. We refer to the works [49], [50] by C. F. Vasconcellos et al., and also see [2], [28], [13], [34] for more information. The stability using time delayed feedback control has been recently addressed by R. de A. Capistrano-Filho et al. in the works [14], [24]. In [46], the authors investigated stabilizability of the Kawahara equation by a saturated internal control and boundary feedback control.

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Date: May 3, 2024.

2020 *Mathematics Subject Classification.* 35K52 - 35Q53 - 93B05 - 93B07 - 93B35.

*Key words and phrases.* Kawahara equation, insensitizing control, Carleman estimate, observability, inverse mapping theorem.

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To describe the main problem of this article, let us first introduce the relevant control system. For  $T, L > 0$ , denote  $Q := (0, T) \times (0, L)$ , and we consider the system

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u + \partial_x^5 u + u^p \partial_x u = \xi + \chi_\omega h, & \text{in } Q, \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = 0, & t \in (0, T), \\ \partial_x^2 u(t, 0) = 0, & t \in (0, T), \\ u(0, x) = u_0(x) + \tau \hat{u}_0(x), & x \in (0, L), \end{cases} \quad (1.1)$$

where  $\omega, \mathcal{O} \subset (0, L)$  are any non-empty open subsets satisfying  $\mathcal{O} \cap \omega \neq \emptyset$  and  $\tau \in \mathbb{R}$ . The function  $h = h(t, x)$  is the interior control localized on the subset  $(0, T) \times \omega$ , and  $\xi = \xi(t, x)$  is some known external force term. The initial state  $u(0)$  is partially unknown for the system (1.1), more precisely,  $u_0 \in L^2(0, L)$  is given, but  $\tau \in \mathbb{R}$  and  $\hat{u}_0 \in \{f \in L^2(0, L) : \|f\|_{L^2(0, L)} = 1\}$  are unknowns. The unknowns  $\hat{u}_0$  and  $\tau \in \mathbb{R}$  represent *uncertainty* in the initial data.

Next, let us introduce a functional  $J_\tau$  (known as *sentinel*), defined on the set of solutions to (1.1) as

$$J_\tau(u) := \frac{1}{2} \iint_{(0, T) \times \mathcal{O}} |u|^2, \quad (1.2)$$

where  $\mathcal{O}$  is the *observation domain*.

Thus, the insensitizing control problem is to guarantee the existence of a control function  $h$  such that the uncertainty in the initial data  $u(0)$  does not alter the value of functional  $J_\tau$  drastically. More precisely, we aim to find some control  $h \in L^2((0, T) \times \omega)$  so that

$$\left. \frac{\partial J_\tau(u)}{\partial \tau} \right|_{\tau=0} = 0, \quad \forall \hat{u}_0 \in L^2(0, L) \text{ with } \|\hat{u}_0\|_{L^2(0, L)} = 1. \quad (1.3)$$

The condition (1.3) indicates that the sentinel  $J_\tau$  does not detect the small perturbations  $\tau \hat{u}_0$  in the given initial data  $u_0$ . When such  $h$  exists, the sentinel  $J_\tau$  is said to be *locally insensitive* to small perturbations in initial data, and we say that the control  $h$  insensitizes  $J_\tau$ . Such insensitizing control problem was originally introduced by J.-L. Lions in his work [44]. Since then many works have been devoted in the literature to study the insensitizing problem from different perspectives.

To begin with, let us mention the pioneer works [26] and [7], concerning the existence of insensitizing control for the linear and semilinear heat equations. Similar problems for linear and semilinear heat equations with different types of nonlinearities and/or boundary conditions was further addressed in the works [5, 6, 8]. Moreover, a numerical study for the insensitizing property of semilinear heat equations has been pursued in [10]. It is worth mentioning the works [30, 45] which also study the insensitizing control problem wherein the insensitivity of some functional is established with respect to domain variation.

In [39], the author addressed an insensitizing control problem for a linear parabolic equation, where the *sentinel* depends on the gradient of the solution. The same author later studied this control problem for the Stokes equation in [38], where the sentinel is based on the curl of the solution. The paper [31] investigated the insensitizing control problem for quasi-geostrophic ocean models. For the semilinear parabolic equation with dynamic boundary conditions, one may look into the paper [54]. Moreover, in the work [52], the authors discussed insensitizing controls of a Stefan problem for a semilinear heat equation in one-dimension. Next, let us refer to the work [42] which studied insensitizing problem for the fourth-order parabolic equation. In [33], the author considered Cahn-Hilliard type equation to discuss the insensitizing control property. Concerning the study of similar insensitizing problem for scalar wave equation, one can refer to the work [1, 22]. It is worth mentioning that the paper [48] treated this issue with a gradient type sentinel associated with the solutions of a nonlinear Ginzburg-Landau equation. In a recent work [25], the insensitizing property for the fourth-order dispersive nonlinear Schrödinger equation with cubic nonlinearity has been studied.

In the context of insensitizing problems for coupled PDEs, let us cite the works [18, 19, 40] for Navier-Stokes equation; [16, 17] for Boussinesq system, and [11] for a phase field system. Finally, we mention the most recent works [3] and [4], where the insensitizing control problem for the Hirota-Satsuma system of KdV-KdV type and for the stabilized Kuramoto-Sivashinsky system have been analyzed, respectively.

**1.2. Main results.** With the goal of proving the insensitivity of the sentinel  $J_\tau$  given by (1.2) with respect to small perturbation in the initial data, we could obtain the following result.

**Theorem 1.1.** *Assume that  $\mathcal{O} \cap \omega \neq \emptyset$  and set  $u_0 \equiv 0$ . Then, there exist constants  $\delta, C > 0$  such that for any  $\xi \in L^2(Q)$  satisfying*

$$\|e^{C/t} \xi\|_{L^2(Q)} \leq \delta, \quad (1.4)$$

*there exist control  $h \in L^2((0, T) \times \omega)$  which insensitizes the functional  $J_\tau$  in the sense of (1.3).*

**Remark 1.2.** *The above theorem proves the insensitizing problem partially in the sense that we have chosen a particular initial data, i.e.,  $u_0 = 0$ . However, a study of the possible initial conditions for which the sentinel  $J_\tau$  is insensitive can also be done. Such problem has been studied for the heat equation in [27], which suggests that the answer is not immediate.*

Mimicking the arguments used in the works [7, Proposition 1] and [27, Appendix], one can easily show that proving Theorem 1.1 is equivalent to establishing null controllability for the state  $w$ , associated to the following extended cascade coupled system:

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u + \partial_x^5 u + u^p \partial_x u = \xi + \chi_\omega h, & \text{in } Q, \\ -\partial_t w - \partial_x w - \partial_x^3 w - \partial_x^5 w - u^p \partial_x w = \chi_{\mathcal{O}} u, & \text{in } Q, \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = \partial_x^2 u(t, 0) = 0, & t \in (0, T), \\ w(t, 0) = w(t, L) = \partial_x w(t, 0) = \partial_x w(t, L) = \partial_x^2 w(t, L) = 0, & t \in (0, T), \\ u(0, x) = u_0(x), w(T, x) = 0, & x \in (0, L). \end{cases} \quad (1.5)$$

More precisely, we have the following equivalence result.

**Proposition 1.3.** *A control functions  $h \in L^2((0, T) \times \omega)$  insensitizes the sentinel  $J_\tau$  given by (1.2) if and only if the associated solution to (1.5) satisfies*

$$w(0, \cdot) = 0 \text{ in } (0, L). \quad (1.6)$$

Thus, from now onwards our main goal is to prove the following null controllability theorem which subsequently establishes the insensitizing control result mentioned in Theorem 1.1.

**Theorem 1.4.** *Assume that  $\mathcal{O} \cap \omega \neq \emptyset$  and set  $u_0 \equiv 0$ . Then, there exist constants  $\delta, C > 0$  such that for any  $\xi \in L^2(Q)$  satisfying*

$$\|e^{C/t} \xi\|_{L^2(Q)} \leq \delta, \quad (1.7)$$

*there exist a control function  $h \in L^2((0, T) \times \omega)$  such that the solution  $(u, w)$  to (1.5) satisfies*

$$w(0, \cdot) = 0 \text{ in } (0, L).$$

The proof of Theorem 1.4 is done using *inverse mapping theorem*, which forces us to study global null controllability problem for the following linear system, obtained by linearizing system (1.5) about zero

$$\begin{cases} \partial_t u + \partial_x u + \partial_x^3 u + \partial_x^5 u = f_1 + \chi_\omega h, & \text{in } Q, \\ -\partial_t w - \partial_x w - \partial_x^3 w - \partial_x^5 w = f_2 + \chi_{\mathcal{O}} u, & \text{in } Q, \\ u(t, 0) = u(t, L) = \partial_x u(t, 0) = \partial_x u(t, L) = \partial_x^2 u(t, 0) = 0, & t \in (0, T), \\ w(t, 0) = w(t, L) = \partial_x w(t, 0) = \partial_x w(t, L) = \partial_x^2 w(t, L) = 0, & t \in (0, T), \\ u(0, x) = u_0(x), w(T, x) = 0, & x \in (0, L), \end{cases} \quad (1.8)$$

where the non-homogeneous terms  $f_1, f_2$  belong to some relevant space (to be specified later).

**Remark 1.5.** *Although we are concerned about the null controllability of component  $w$ , the control  $h$ , localized on the subset  $\omega$ , does not directly affect the equation of  $w$ . However, the state  $u$  can be solved independently and acts as a control for the state  $w$ , which is localized on the subset  $\mathcal{O}$ . Hence, the condition  $\mathcal{O} \cap \omega \neq \emptyset$  is necessary in the above theorem.*

The controllability of this linear system will be proved via duality approach which requires establishing some *observability property* for its associated adjoint system, given by

$$\begin{cases} -\partial_t v - \partial_x v - \partial_x^3 v - \partial_x^5 v = g_1 + \chi_{\mathcal{O}} \psi, & \text{in } Q, \\ \partial_t \psi + \partial_x \psi + \partial_x^3 \psi + \partial_x^5 \psi = g_2, & \text{in } Q, \\ \psi(t, 0) = \psi(t, L) = \partial_x \psi(t, 0) = \partial_x \psi(t, L) = \partial_x^2 \psi(t, 0) = 0, & t \in (0, T), \\ v(t, 0) = v(t, L) = \partial_x v(t, 0) = \partial_x v(t, L) = \partial_x^2 v(t, L) = 0, & t \in (0, T), \\ v(T, x) = 0, \psi(0, x) = \psi_0(x), & x \in (0, L), \end{cases} \quad (1.9)$$

where  $\psi_0 \in L^2(0, L)$  and the non-homogeneous term  $g_1, g_2$  belong to some relevant space (to be specified later). To study this observability problem, we derive a suitable Carleman estimate for the adjoint system (1.9) using the known Carleman estimates for single Kawahara equation, see [20, Proposition 3.1].

**1.3. Paper organization.** The organization of this paper has been done as follows. In Section 2, the well-posedness of aforementioned systems have been proved. Section 3 is dedicated for deriving suitable Carleman estimates for the adjoint system (1.9). Using this Carleman, the relevant observability inequality for the adjoint system (1.9) has been obtained in Section 4. Finally, the null controllability result for linear control system (1.8) and main control system (1.5) has been proved in Section 5 and Section 6, respectively.

## 2. WELL-POSEDNESS RESULTS

In this section we study the well-posedness issues for the system (1.5). Let us first consider the following linearized Kawahara equation with given source term  $f$  and initial data  $y_0$

$$\begin{cases} \partial_t y + \partial_x y + \partial_x^3 y + \partial_x^5 y = f, & \text{in } Q, \\ y(t, 0) = y(t, L) = \partial_x y(t, L) = \partial_x y(t, 0) = \partial_x^2 y(t, 0) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, L). \end{cases} \quad (2.1)$$

We introduce the following spaces related to the state and source term

$$\begin{aligned} X_{s,T} &= \mathcal{C}([0, T]; H^s(0, L)) \cap L^2(0, T; H^{s+2}(0, L)), \quad s \in [0, 5], \\ \mathcal{S} &= L^2(0, T; H^{-2}(0, L)) \cup L^1(0, T; L^2(0, L)). \end{aligned}$$

Let us recall the following known well-posedness results for (2.1).

**Lemma 2.1** ([20, Proposition 2.1]). *For given  $y_0 \in L^2(0, L)$  and  $f \in \mathcal{S}$ , the system (2.1) admits a unique solution  $y \in X_{0,T} = \mathcal{C}([0, T]; L^2(0, L)) \cap L^2(0, T; H^2(0, L))$ . Moreover, there exists a constant  $C > 0$  such that*

$$\|y\|_{\mathcal{C}([0,T];L^2(0,L))} + \|y\|_{L^2(0,T;H^2(0,L))} \leq C (\|y_0\|_{L^2(0,L)} + \|f\|_{\mathcal{S}}). \quad (2.2)$$

We also have the following results for the Kawahara equation (2.1) with  $y_0 = 0$ .

**Lemma 2.2** ([20, Proposition 2.2]). *Set  $y_0 = 0$ , and assume  $f \in L^2(0, T; H_0^{s-2})$ , for  $s \in [0, 5]$ . Then the system (2.1) admits a unique solution  $y \in X_{s,T}$ . In addition, there exists a constant  $C > 0$  such that*

$$\|y\|_{X_{s,T}} \leq C \|f\|_{L^2(0,T;H_0^{s-2})}. \quad (2.3)$$

**Remark 2.3.** *Note that the above results are also applicable for the adjoint equation to (2.1) which is backward in time.*

**2.1. Well-posedness of the linearized system and its adjoint.** In this subsection, we mention the well-posedness result for the linearized coupled systems, mentioned above. The proof is immediate due to the cascade nature of the system and Lemma 2.1.

**Proposition 2.4.** *Let  $u_0 \in L^2(0, L)$ ,  $h \in L^2((0, T) \times \omega)$  and  $(f_1, f_2) \in \mathcal{S} \times \mathcal{S}$  be given. Then, the system (1.8) possesses a unique solution  $(u, w) \in [X_{0,T}]^2$ . In addition, there exists a constant  $C > 0$  such that*

$$\|(u, w)\|_{[X_{0,T}]^2} \leq C \left( \|u_0\|_{L^2(0,L)} + \|h\|_{L^2((0,T)\times\omega)} + \|(f_1, f_2)\|_{\mathcal{S}\times\mathcal{S}} \right). \quad (2.4)$$

*Proof.* We demonstrate the proof of this proposition in the following way. First, we use Lemma 2.1 to the equation (1.8)<sub>1</sub> to show that  $u \in X_{0,T}$  along with the estimate

$$\|u\|_{X_{0,T}} \leq C \left( \|u_0\|_{L^2(0,L)} + \|h\|_{L^2((0,T)\times\omega)} + \|f_1\|_{\mathcal{S}} \right). \quad (2.5)$$

Then, using  $u\chi_{\mathcal{O}}$  as source term in the equations of  $w$  given by (1.8)<sub>2</sub>, and combining with (2.5) we get the required estimate (2.4).  $\square$

Similar result holds for the adjoint system (1.9).

**Proposition 2.5.** *Let  $\psi_0 \in L^2(0, L)$  and  $(g_1, g_2) \in \mathcal{S} \times \mathcal{S}$  be given. Then, the system (1.9) admits a unique solution  $(v, \psi) \in [X_{0,T}]^2$  and moreover, there exists a constant  $C > 0$  such that*

$$\|(v, \psi)\|_{[X_{0,T}]^2} \leq C \left( \|\psi_0\|_{L^2(0,L)} + \|(g_1, g_2)\|_{\mathcal{S}\times\mathcal{S}} \right). \quad (2.6)$$

**2.2. Well-posedness of the nonlinear system.** We now prove the well-posedness of our nonlinear system (1.5), using the result of the corresponding linearized system (1.8) and a suitable fixed point theorem.

**Proposition 2.6.** *Let  $T > 0$  and  $L > 0$ . Then, there exists some positive real number  $\delta_0$  such that for every  $u_0 \in L^2(0, L)$ ,  $h \in L^2((0, T) \times \omega)$  and  $\xi \in L^2(Q)$ , satisfying*

$$\|u_0\|_{L^2(0,L)} + \|h\|_{L^2((0,T)\times\omega)} + \|\xi\|_{L^2(Q)} \leq \delta_0, \quad (2.7)$$

the system (1.5) possesses a unique solution

$$(u, w) \in [X_{0,T}]^2.$$

Before going to the proof of above proposition, we prove the following lemma.

**Lemma 2.7.** *The map  $\mathcal{M} : [X_{0,T}]^2 \rightarrow L^1(0, T; L^2(0, L))$  given by*

$$\mathcal{M}(y_1, y_2) = y_1^p \partial_x y_2 \quad (2.8)$$

is well-defined and continuous.

*Proof.* For  $(y_1, y_2) \in [X_{0,T}]^2$ , we have

$$\|y_1^p y_{2,x}\|_{L^1(0,T;L^2(0,L))} = \int_0^T \|y_1^p \partial_x y_2\|_{L^2(0,L)} \leq \int_0^T \|y_1\|_{L^\infty(0,L)}^p \|\partial_x y_2\|_{L^2(0,L)}.$$

Thanks to the estimate

$$\|y_1(t, \cdot)\|_{L^\infty(0,L)} \leq C \left( \|y_1(t, \cdot)\|_{L^2(0,L)} + \|y_1(t, \cdot)\|_{L^2(0,L)}^{1/2} \|\partial_x y_1(t, \cdot)\|_{L^2(0,L)}^{1/2} \right),$$

and the fact that  $|a + b|^p \leq C(|a|^p + |b|^p)$ , from the last inequality, we obtain

$$\|y_1^p \partial_x y_2\|_{L^1(0,T;L^2(0,L))} \leq C \left( \int_0^T \|y_1\|_{L^2}^p \|y_2\|_{H^1(0,L)} + \int_0^T \|y_1\|_{L^2}^{p/2} \|y_1\|_{H^1(0,L)}^{p/2} \|y_2\|_{H^1(0,L)} \right).$$

Lastly, we use the Hölder's inequality to get

$$\begin{aligned} \|y_1^p \partial_x y_2\|_{L^1(0,T;L^2(0,L))} &\leq C \left( \sqrt{T} \|y_1\|_{C([0,T];L^2(0,L))}^p \|y_2\|_{L^2(0,T;H^1(0,L))} + \|y_1\|_{C([0,T];L^2(0,L))}^{p/2} \right. \\ &\quad \left. \left( \int_0^T \|y_1\|_{H^1(0,L)}^2 \right)^{p/4} \left( \int_0^T \|y_2\|_{H^1(0,L)}^{4/(4-p)} \right)^{(4-p)/4} \right) \\ &\leq C(T^{(2-p)/4} + \sqrt{T}) \|y_1\|_{X_{0,T}}^p \|y_2\|_{X_{0,T}}, \end{aligned} \quad (2.9)$$

for some constant  $C > 0$ , which proves that the map  $\mathcal{M}$  is well-defined.

To establish continuity of the map  $\mathcal{M}$ , let us consider  $(y_1, y_2), (\tilde{y}_1, \tilde{y}_2) \in [X_{0,T}]^2$ . Then using the above estimate (2.9) and triangle inequality, we have

$$\begin{aligned} &\|\mathcal{M}(y_1, y_2) - \mathcal{M}(\tilde{y}_1, \tilde{y}_2)\|_{L^1(0,T;L^2(0,L))} \\ &= \int_0^T \|y_1^p \partial_x y_2 - \tilde{y}_1^p \partial_x \tilde{y}_2\|_{L^2(0,L)} \\ &= \int_0^T \|y_1^p \partial_x (y_2 - \tilde{y}_2) + (y_1^p - \tilde{y}_1^p) \partial_x \tilde{y}_2\|_{L^2(0,L)} \\ &\leq C \left( T^{(2-p)/4} + \sqrt{T} \right) \|y_1\|_{X_{0,T}}^p \|y_2 - \tilde{y}_2\|_{X_{0,T}} + \int_0^T \|(y_1^p - \tilde{y}_1^p) \partial_x \tilde{y}_2\|_{L^2(0,L)}. \end{aligned} \quad (2.10)$$

The second term can be estimated as

$$\begin{aligned} \int_0^T \|(y_1^p - \tilde{y}_1^p) \partial_x \tilde{y}_2\|_{L^2(0,L)} &\leq C \int_0^T \|(y_1 - \tilde{y}_1) (|y_1|^{p-1} + |\tilde{y}_1|^{p-1}) \partial_x \tilde{y}_2\|_{L^2(0,L)} \\ &\leq C \int_0^T \|(y_1 - \tilde{y}_1) |y_1|^{p-1} \partial_x \tilde{y}_2\|_{L^2} + C \int_0^T \|(y_1 - \tilde{y}_1) |\tilde{y}_1|^{p-1} \partial_x \tilde{y}_2\|_{L^2}. \end{aligned} \quad (2.11)$$

Note that both terms of right hand side can be estimated similarly. More precisely, we have

$$\begin{aligned}
\int_0^T \|(y_1 - \tilde{y}_1)|y_1|^{p-1} \partial_x \tilde{y}_2\|_{L^2(0,L)} &\leq C \int_0^T \|\partial_x \tilde{y}_2\|_{L^\infty(0,L)} \|y_1^{p-1}(y_1 - \tilde{y}_1)\|_{L^2(0,L)} \\
&\leq C \left( \int_0^T \|\tilde{y}_2\|_{H^2(0,L)}^2 \right)^{1/2} \left( \int_0^T \|y_1^{p-1}(y_1 - \tilde{y}_1)\|_{L^2(0,L)}^2 \right)^{1/2} \\
&\leq C \|\tilde{y}_2\|_{L^2(0,T;H^2(0,L))} \left( \int_0^T \|y_1\|_{L^\infty(0,L)}^{2(p-1)} \|y_1 - \tilde{y}_1\|_{L^2(0,L)}^2 \right)^{1/2} \\
&\leq CT^{\frac{(2-p)}{2}} \|y_1 - \tilde{y}_1\|_{C([0,T];L^2(0,L))} \|\tilde{y}_2\|_{L^2(0,T;H^2(0,L))} \|y_1\|_{L^2(0,T;H^1(0,L))}^{p-1}.
\end{aligned}$$

Thus, substituting this estimate in (2.11), we obtain

$$\int_0^T \|(y_1^p - \tilde{y}_1^p) \partial_x \tilde{y}_2\|_{L^2(0,L)} \leq CT^{\frac{(2-p)}{2}} \|y_1 - \tilde{y}_1\|_{X_{0,T}} \|\tilde{y}_2\|_{X_{0,T}} \left( \|y_1\|_{X_{0,T}}^{p-1} + \|\tilde{y}_1\|_{X_{0,T}}^{p-1} \right), \quad (2.12)$$

and hence the estimate (2.10) becomes

$$\begin{aligned}
\|\mathcal{M}(y_1, y_2) - \mathcal{M}(\tilde{y}_1, \tilde{y}_2)\|_{L^1(0,T;L^2(0,L))} &\leq C \left( T^{(2-p)/4} + \sqrt{T} \right) \|y_1\|_{X_{0,T}}^p \|y_2 - \tilde{y}_2\|_{X_{0,T}} \\
&\quad + CT^{\frac{(2-p)}{2}} \|y_1 - \tilde{y}_1\|_{X_{0,T}} \|\tilde{y}_2\|_{X_{0,T}} \left( \|y_1\|_{X_{0,T}}^{p-1} + \|\tilde{y}_1\|_{X_{0,T}}^{p-1} \right) \\
&\leq C(T) \left( \|y_1\|_{X_{0,T}}^p + \|\tilde{y}_2\|_{X_{0,T}} \left( \|y_1\|_{X_{0,T}}^{p-1} + \|\tilde{y}_1\|_{X_{0,T}}^{p-1} \right) \right) \|(y_1, y_2) - (\tilde{y}_1, \tilde{y}_2)\|_{[X_{0,T}]^2}. \quad (2.13)
\end{aligned}$$

This establishes the continuity of the map  $\mathcal{M}$ .  $\square$

Next, we prove the well-posedness of the coupled nonlinear system (1.5).

**Proof of Proposition 2.6.** Let us define the map

$$\mathcal{N} : [X_{0,T}]^2 \rightarrow [X_{0,T}]^2, \quad \mathcal{N}(\tilde{u}, \tilde{w}) = (u, w), \quad (2.14)$$

where  $(u, w)$  is the unique solution to (1.8) with  $u_0 \in L^2(0, L)$ ,  $h \in L^2((0, T) \times \omega)$  and

$$\begin{aligned}
f_1 &= \xi + \tilde{u}^p \partial_x \tilde{u} \in L^1(0, T; L^2(0, L)), \\
f_2 &= -\tilde{u}^p \partial_x \tilde{w} \in L^1(0, T; L^2(0, L)),
\end{aligned}$$

Existence of such unique  $(u, w)$  is guaranteed by Proposition 2.4. Moreover, using the bound (2.9) in the continuity estimate (2.4), we get

$$\|(u, w)\|_{[X_{0,T}]^2} \leq C_0 \left( \|u_0\|_{L^2(0,L)} + \|h\|_{L^2((0,T) \times \omega)} + \|\xi\|_{L^2(Q)} + \|\tilde{u}\|_{X_{0,T}}^{p+1} + \|\tilde{u}\|_{X_{0,T}}^p \|\tilde{w}\|_{X_{0,T}} \right), \quad (2.15)$$

for some constant  $C_0 > 0$ .

Next, for some  $R > 0$ , let us consider the ball

$$\mathcal{B}_R := \left\{ (u, w) \in [X_{0,T}]^2 : \|u\|_{X_{0,T}} + \|w\|_{X_{0,T}} \leq R \right\}. \quad (2.16)$$

Thanks to the estimate (2.15), for  $(\tilde{u}, \tilde{w}) \in \mathcal{B}_R$ , we get

$$\|(u, w)\|_{[X_{0,T}]^2} \leq C_0 \left( \|u_0\|_{L^2(0,L)} + \|h\|_{L^2((0,T) \times \omega)} + \|\xi\|_{L^2(Q)} + R^{p+1} \right). \quad (2.17)$$

Let us choose  $R$  in such a way that  $R < \frac{1}{C_0^{1/p}}$  and so for  $(u_0, h, \xi)$  satisfying

$$\|u_0\|_{L^2(0,L)} + \|h\|_{L^2((0,T) \times \omega)} + \|\xi\|_{L^2(Q)} < \frac{R - C_0 R^{p+1}}{C_0},$$

we have  $\mathcal{N}(\mathcal{B}_R) \subset \mathcal{B}_R$ .

Fix

$$\delta_0 := \frac{R - C_0 R^{p+1}}{C_0}. \quad (2.18)$$



Next, assume  $(u_0, h, \xi)$  satisfies the condition (2.7) with  $\delta_0$  given by (2.18), and let  $(\tilde{u}_1, \tilde{w}_2), (\tilde{u}_2, \tilde{w}_2) \in \mathcal{B}_R$ . Then  $(u_1, w_1) := \mathcal{N}(\tilde{u}_1, \tilde{w}_1), (u_2, w_2) := \mathcal{N}(\tilde{u}_2, \tilde{w}_2) \in \mathcal{B}_R$ . Using the linearity of system (1.8) and the continuity estimate (2.4), we have

$$\begin{aligned} \|\mathcal{N}(\tilde{u}_1, \tilde{w}_1) - \mathcal{N}(\tilde{u}_2, \tilde{w}_2)\|_{[X_0, T]^2} &\leq C \|(\tilde{u}_1^p \partial_x \tilde{u}_1 - \tilde{u}_2^p \partial_x \tilde{u}_2, -\tilde{u}_1^p \partial_x \tilde{w}_1 + \tilde{u}_2^p \partial_x \tilde{w}_2)\|_{[L^1(0, T; L^2(0, L))]^2} \\ &\leq C \left( \|\tilde{u}_1^p \partial_x \tilde{u}_1 - \tilde{u}_2^p \partial_x \tilde{u}_2\|_{L^1(L^2)} + \|\tilde{u}_1^p \partial_x \tilde{w}_1 - \tilde{u}_2^p \partial_x \tilde{w}_2\|_{L^1(L^2)} \right). \end{aligned}$$

Thanks to the continuity estimate (2.13) of the map  $\mathcal{M}$  established in Lemma 2.7, the above estimate can be further simplified as

$$\begin{aligned} \|\mathcal{N}(\tilde{u}_1, \tilde{w}_1) - \mathcal{N}(\tilde{u}_2, \tilde{w}_2)\|_{[X_0, T]^2} &\leq C_1 R^p \left( \|\tilde{u}_1 - \tilde{u}_2\|_{X_0, T} + \|(\tilde{u}_1, \tilde{w}_1) - (\tilde{u}_2, \tilde{w}_2)\|_{[X_0, T]^2} \right) \\ &\leq C_1 R^p \|(\tilde{u}_1 - \tilde{u}_2, \tilde{w}_1 - \tilde{w}_2)\|_{[X_0, T]^2}. \end{aligned}$$

Finally, we choose  $R > 0$  such that  $R < \min \left\{ \frac{1}{C_0^{1/p}}, \frac{1}{C_1^{1/p}} \right\}$  and thus the map

$$\mathcal{N} : \mathcal{B}_R \rightarrow \mathcal{B}_R$$

becomes a contraction, which guarantees the existence of a unique fixed point for the map  $\mathcal{N}$ . This completes the proof of Proposition 2.6.  $\square$

### 3. CARLEMAN ESTIMATES

This section is devoted to obtain a suitable Carleman estimate satisfied by the solution of adjoint system (1.9). To establish this Carleman inequality for the coupled system (1.9), we will use the Carleman estimate proved in [20] for the linear Kawahara equation.

**3.1. Carleman estimate for Kawahara equation.** Let us consider the following backward linear Kawahara equation

$$\begin{cases} \partial_t z + \partial_x z + \partial_x^3 z + \partial_x^5 z = g, & (t, x) \in (0, T) \times (0, L), \\ z(t, 0) = z(t, L) = \partial_x z(t, 0) = \partial_x z(t, L) = 0, & t \in (0, T), \\ \partial_x^2 z(t, L) = 0, & t \in (0, T), \\ z(T, x) = z_T(x), & x \in (0, L). \end{cases} \quad (3.1)$$

This subsection is dedicated to establish a modified version of the Carleman estimate obtained in the work [20] for the system (3.1). The proof of this modified Carleman estimate is primarily based on the original Carleman inequality obtained in [20]. For the modified Carleman estimate, we need the non-homogeneous term  $g$  of (3.1) to be in  $L^2(0, T; H_0^1(0, L))$ , unlike the one obtained in [20], where  $g \in L^2(0, T; L^2(0, L))$  is enough.

To start the study of Carleman estimate, let us first mention the relevant weight function as described in [20]. Recall that  $\mathcal{O} \cap \omega \neq \emptyset$ , so let  $\omega_0$  be any nonempty open subset of  $\mathcal{O} \cap \omega$ . Further, consider any open interval  $\omega_1 = (a, b)$  such that  $\omega_1 \Subset \omega_0 \subset \mathcal{O} \cap \omega$  with  $0 < a < b < L$ . Let  $\phi \in \mathcal{C}^8([0, L])$  be a positive function satisfying

$$\begin{cases} \phi'(0) < 0, \phi'(L) > 0; \\ |\phi'(x)| > 0, \phi''(x) < 0 \text{ for } x \in [0, L] \setminus \omega_1; \\ \min_{x \in \overline{\omega_1}} \phi(x) = \phi(c) < \max_{x \in \overline{\omega_1}} \phi(x) = \phi(a) = \phi(b), \text{ for } c \in \omega_1; \\ \max_{x \in [0, L]} \phi(x) = \phi(L) = \phi(0) < \frac{10}{9} \phi(c). \end{cases} \quad (3.2)$$

Next, let us consider a function  $\xi(t)$  given by

$$\xi(t) = \frac{1}{t(T-t)}, \quad \forall t \in (0, T). \quad (3.3)$$

Finally, we define the weight function  $\varphi$  by

$$\varphi(t, x) = \phi(x)\xi(t), \quad \forall (t, x) \in (0, T) \times [0, L]. \quad (3.4)$$

We use the symbol  $\check{\varphi}(t)$  and  $\hat{\varphi}(t)$  to denote

$$\begin{aligned} \check{\varphi}(t) &= \min_{x \in [0, L]} \varphi(t, x), \quad \forall t \in (0, T), \\ \hat{\varphi}(t) &= \max_{x \in [0, L]} \varphi(t, x), \quad \forall t \in (0, T). \end{aligned} \quad (3.5)$$



Then, using the above condition on  $\phi$ , we have

$$10\check{\varphi}(t) - 9\hat{\varphi}(t) > 0. \quad (3.6)$$

Let us now introduce the shorthand notation  $\mathcal{I}_{kw}(z)$  and  $\tilde{\mathcal{I}}_{kw}(z)$  to denote

$$\mathcal{I}_{kw}(z) = \iint_Q e^{-2s\varphi} (s^9 \xi^9 |z|^2 + s^7 \xi^7 |\partial_x z|^2 + s^5 \xi^5 |\partial_x^2 z|^2 + s^3 \xi^3 |\partial_x^3 z|^2 + s \xi |\partial_x^4 z|^2), \quad (3.7)$$

$$\tilde{\mathcal{I}}_{kw}(z) = \mathcal{I}_{kw}(z) + s \int_0^T \xi e^{-2s\hat{\varphi}} \|z\|_{H^5(0,L)}^2. \quad (3.8)$$

**Proposition 3.1.** *Assume  $T > 0$  be given, and let  $\omega_0 \subset \mathcal{O} \cap \omega$  be any nonempty open subset. Then, for any  $z_T \in L^2(0, L)$ , and  $g \in L^2(0, T; H_0^1(0, L))$ , there exists a constants  $C, s_0 > 0$  such that for all  $s \geq s_0$ , the solution of (3.1) satisfies*

$$\tilde{\mathcal{I}}_{kw}(z) \leq C \left( s^3 \iint_Q \xi e^{-2s\varphi} |g|^2 + s \int_0^T \xi^5 e^{-2s\hat{\varphi}} \|g\|_{H^1(0,L)}^2 + s^9 \int_0^T \xi^9 e^{-2s(5\check{\varphi}-4\hat{\varphi})} \|z\|_{L^2(\omega_0)}^2 \right). \quad (3.9)$$

*Proof.* Let  $\omega_1 \Subset \omega_0$  be any nonempty open interval. Then, the solution  $z$  of (3.1) satisfies the following Carleman estimate (see Corollary 3.1, [20]):

$$\mathcal{I}_{kw}(z) \leq C \iint_Q e^{-2s\varphi} |g|^2 + C \int_0^T \int_{\omega_1} e^{-2s\varphi} (s^9 \xi^9 |z|^2 + s^7 \xi^7 |\partial_x z|^2 + s^5 \xi^5 |\partial_x^2 z|^2 + s^3 \xi^3 |\partial_x^3 z|^2 + s \xi |\partial_x^4 z|^2), \quad (3.10)$$

for  $s \geq s_0$ , for sufficiently large  $s_0$  and for some  $C > 0$ .

Let  $\zeta \in C_c^\infty(\omega_0)$  be a positive function such that  $\zeta \equiv 1$  on  $\omega_1$ , and let  $\epsilon > 0$  be arbitrary. Then, using smoothness of  $\zeta, \varphi$  and Hölder's inequality, we obtain the following estimate

$$\begin{aligned} \int_0^T \int_{\omega_1} e^{-2s\varphi} s^7 \xi^7 |\partial_x z|^2 &\leq \int_0^T \int_{\omega_0} \zeta e^{-2s\varphi} s^7 \xi^7 |\partial_x z|^2 \\ &= - \int_0^T \int_{\omega_0} s^7 \xi^7 e^{-2s\varphi} (\partial_x \zeta \partial_x z - 2s \partial_x \varphi \zeta \partial_x z + \zeta \partial_x^2 z) z \\ &\leq \epsilon \int_0^T \int_{\omega_0} s^7 \xi^7 e^{-2s\varphi} |\partial_x z|^2 + C_\epsilon \int_0^T \int_{\omega_0} s^7 \xi^7 e^{-2s\varphi} |z|^2 \\ &\quad + C \int_0^T \int_{\omega_0} s^8 \xi^8 e^{-2s\varphi} (\partial_x z) z \\ &\quad + \epsilon \int_0^T \int_{\omega_0} s^5 \xi^5 e^{-2s\varphi} |\partial_x^2 z|^2 + C_\epsilon \int_0^T \int_{\omega_0} s^9 \xi^9 e^{-2s\varphi} |z|^2 \\ &\leq \epsilon \mathcal{I}(s, z) + C_\epsilon \int_0^T \int_{\omega_0} s^9 \xi^9 e^{-2s\varphi} |z|^2, \end{aligned}$$

where  $C_\epsilon > 0$  is a constant which depends on the choice of  $\epsilon$ . Similarly, the other terms of (3.10) can be estimated as

$$\begin{aligned} \bullet \int_0^T \int_{\omega_1} e^{-2s\varphi} s^5 \xi^5 |\partial_x^2 z|^2 &\leq \int_0^T \int_{\omega_0} \zeta e^{-2s\varphi} s^5 \xi^5 |\partial_x^2 z|^2 \\ &= - \int_0^T \int_{\omega_0} e^{-2s\varphi} s^5 \xi^5 (-2s \partial_x \varphi \zeta \partial_x^2 z + \partial_x \zeta \partial_x^2 z + \zeta \partial_x^3 z) (\partial_x z) \\ &\leq \epsilon \mathcal{I}(s, z) + C \int_0^T \int_{\omega_0} s^9 \xi^9 e^{-2s\varphi} |z|^2. \\ \bullet \int_0^T \int_{\omega_1} e^{-2s\varphi} s^3 \xi^3 |\partial_x^3 z|^2 &\leq \int_0^T \int_{\omega_0} \zeta e^{-2s\varphi} s^3 \xi^3 |\partial_x^3 z|^2 \\ &= - \int_0^T \int_{\omega_0} e^{-2s\varphi} s^3 \xi^3 (-2s \partial_x \varphi \zeta \partial_x^3 z + \partial_x \zeta \partial_x^3 z + \zeta \partial_x^4 z) (\partial_x^2 z) \\ &\leq \epsilon \mathcal{I}(s, z) + C_\epsilon \int_0^T \int_{\omega_0} s^9 \xi^9 e^{-2s\varphi} |z|^2 + C_\epsilon \int_0^T \int_{\omega_0} s \xi e^{-2s\varphi} |\partial_x^4 z|^2. \end{aligned}$$

Substituting these estimates on the right side of (3.10), and absorbing the terms with  $\epsilon$  with the term in the left by choosing sufficiently small  $\epsilon$ , we obtain

$$\mathcal{I}_{kw}(z) \leq C \left( \iint_Q e^{-2s\varphi} |g|^2 + \int_0^T \int_{\omega_0} s^9 \xi^9 e^{-2s\varphi} |z|^2 + \int_0^T \int_{\omega_0} s \xi e^{-2s\varphi} |\partial_x^4 z|^2 \right), \quad (3.11)$$

for some constant  $C > 0$ .

To handle the last term of right hand side, we will use the Bootstrap technique as developed in [35] (see also [12, 20]) to introduce a new term on the left side, which will effectively absorb the last term. For that, let us first define  $\bar{z}(t, x) := \bar{\rho}(t)z(t, x)$  with  $\bar{\rho}(t) = s^{1/2}\xi^{5/2}e^{-s\hat{\varphi}}$  so that  $\bar{z}$  satisfies

$$\begin{cases} \partial_t \bar{z} + \partial_x \bar{z} + \partial_x^3 \bar{z} + \partial_x^5 \bar{z} = \bar{\rho} g + \partial_t \bar{\rho} z, & \text{in } Q, \\ \bar{z}(t, 0) = \bar{z}(t, L) = \partial_x \bar{z}(t, 0) = \partial_x \bar{z}(t, L) = \partial_x^2 \bar{z}(t, 0) = \partial_x^2 \bar{z}(t, L) = 0, & \text{for } t \in (0, T), \\ \bar{z}(T, x) = 0, & \text{for } x \in (0, L). \end{cases} \quad (3.12)$$

Note that  $|\partial_t \bar{\rho}| \leq Cs^{3/2}\xi^{9/2}e^{-s\hat{\varphi}}$ , and so we deduce

$$\|\bar{\rho} g + \partial_t \bar{\rho} z\|_{L^2(Q)}^2 \leq Cs \iint_Q e^{-2s\hat{\varphi}} \xi^5 |g|^2 + Cs^3 \iint_Q e^{-2s\hat{\varphi}} \xi^9 |z|^2, \quad (3.13)$$

and therefore, by virtue of well-posedness result Lemma 2.2, the system (3.12) admits a unique solution  $\bar{z} \in \mathcal{C}([0, T]; H^2(0, L)) \cap L^2(0, T; H^4(0, L))$ , satisfying the following continuity estimate

$$\|\bar{z}\|_{L^2(0, T; H^4(0, L))}^2 \leq Cs \iint_Q e^{-2s\hat{\varphi}} \xi^5 |g|^2 + Cs^3 \iint_Q e^{-2s\hat{\varphi}} \xi^9 |z|^2. \quad (3.14)$$

Next, we define  $\tilde{z}(t, x) = \tilde{\rho}(t)z(t, x)$  with  $\tilde{\rho}(t) = s^{1/2}\xi^{1/2}e^{-s\hat{\varphi}}$ . Then  $\tilde{z}$  satisfies

$$\begin{cases} \partial_t \tilde{z} + \partial_x \tilde{z} + \partial_x^3 \tilde{z} + \partial_x^5 \tilde{z} = \tilde{\rho} g + \partial_t \tilde{\rho} z & \text{in } Q, \\ \tilde{z}(t, 0) = \tilde{z}(t, L) = \partial_x \tilde{z}(t, 0) = \partial_x \tilde{z}(t, L) = \partial_x^2 \tilde{z}(t, 0) = \partial_x^2 \tilde{z}(t, L) = 0 & \text{for } t \in (0, T), \\ \tilde{z}(T) = 0, & \text{in } (0, L). \end{cases} \quad (3.15)$$

Note that  $\partial_t \tilde{\rho} z = \partial_t \tilde{\rho} \bar{\rho}^{-1} \bar{z}$ , and  $|\partial_t \tilde{\rho} \bar{\rho}^{-1}| \leq Cs$ . Then, using the fact that  $\bar{z} \in L^2(0, T; H^4(0, L))$  and  $g \in L^2(0, T; H^1(0, L))$ , one has

$$\|\tilde{\rho} g + \partial_t \tilde{\rho} z\|_{L^2(0, T; H^1(0, L))}^2 \leq C \left( s \int_0^T \xi e^{-2s\hat{\varphi}} \|g\|_{H^1(0, L)}^2 + s^2 \int_0^T \|\bar{z}\|_{H^1(0, L)}^2 \right).$$

As a consequence, thanks to the Lemma 2.2, we get existence of the unique solution  $\tilde{z} \in X_{3, T} = \mathcal{C}([0, T]; H^3(0, L)) \cap L^2(0, T; H^5(0, L))$  of (3.15) satisfying the estimate

$$\|\tilde{z}\|_{X_{3, T}}^2 \leq C \left( s \int_0^T \xi e^{-2s\hat{\varphi}} \|g\|_{H^1(0, L)}^2 + s^2 \int_0^T \|\bar{z}\|_{H^1(0, L)}^2 \right). \quad (3.16)$$

Let us now use the estimate (3.14) in the last one to get

$$s \int_0^T \xi e^{-2s\hat{\varphi}} \|z\|_{H^5(0, L)}^2 \leq C \left( s \int_0^T \xi e^{-2s\hat{\varphi}} \|g\|_{H^1(0, L)}^2 + s^3 \iint_Q e^{-2s\hat{\varphi}} \xi^5 |g|^2 + s^5 \iint_Q e^{-2s\hat{\varphi}} \xi^9 |z|^2 \right). \quad (3.17)$$

Upon adding the estimates (3.11) and (3.17), we obtain:

$$\begin{aligned} & \mathcal{I}_{kw}(z) + s \int_0^T \xi e^{-2s\hat{\varphi}} \|z\|_{H^5(0, L)}^2 \\ & \leq C \left( \iint_Q e^{-2s\varphi} |g|^2 + s^9 \int_0^T \int_{\omega_0} e^{-2s\varphi} \xi^9 |z|^2 + s \int_0^T \int_{\omega_0} \xi e^{-2s\varphi} |\partial_x^4 z|^2 \right. \\ & \quad \left. + s \int_0^T e^{-2s\hat{\varphi}} \xi^5 \|g\|_{H^1(0, L)}^2 + s^3 \iint_Q \xi e^{-2s\hat{\varphi}} |g|^2 + s^5 \iint_Q e^{-2s\hat{\varphi}} \xi^9 |z|^2 \right). \end{aligned} \quad (3.18)$$

Note that the last term of right hand side of the above inequality can be dominated by

$$s^5 \iint_Q e^{-2s\varphi} \xi^9 |z|^2$$

as  $-\hat{\varphi} \leq -\varphi$ , which can be absorbed by  $\mathcal{I}_{kw}$  in the left hand side of (3.18) by choosing sufficiently large  $s$ . The relation  $-\hat{\varphi} \leq -\varphi$  can be used for the second last term of the right hand side of (3.18) as well to combine it with the first term of the right hand side. Thus we finally obtain

$$\begin{aligned} \mathcal{I}_{kw}(z) + s \int_0^T \xi e^{-2s\hat{\varphi}} \|z\|_{H^5(0,L)}^2 &\leq C \left( s^9 \int_0^T \int_{\omega_0} e^{-2s\varphi} \xi^9 |z|^2 + s \int_0^T \int_{\omega_0} \xi e^{-2s\varphi} |\partial_x^4 z|^2 \right. \\ &\quad \left. + s^3 \iint_Q \xi e^{-2s\varphi} |g|^2 + s \int_0^T e^{-2s\hat{\varphi}} \xi^5 \|g\|_{H^1(0,L)}^2 \right). \end{aligned} \quad (3.19)$$

Our final aim is to eliminate the local term involving  $|\partial_x^4 z|$  from right hand side of the above estimate. Noting the fact that  $\xi$  and  $\check{\varphi}$  do not depend on space variable, and using the Sobolev interpolation inequality and Young's inequality, we have

$$\begin{aligned} s \int_0^T \int_{\omega_0} \xi e^{-2s\varphi} |\partial_x^4 z|^2 &\leq s \int_0^T e^{-2s\check{\varphi}} \xi \|z\|_{H^4(\omega_0)}^2 \\ &\leq C s \int_0^T \int_{\omega_0} \xi e^{-2s\check{\varphi}} \|z(t, \cdot)\|_{H^5(\omega_0)}^{\frac{8}{5}} \|z(t, \cdot)\|_{L^2(\omega_0)}^{\frac{2}{5}} \\ &\leq \epsilon \int_0^T \xi e^{-2s\hat{\varphi}} \|z(t, \cdot)\|_{H^5(\omega_0)}^2 + C_\epsilon s^5 \int_0^T \xi e^{s(-10\check{\varphi}+8\hat{\varphi})} \|z(t, \cdot)\|_{L^2(\omega_0)}^2. \end{aligned} \quad (3.20)$$

Also, using the fact  $\check{\varphi} - \hat{\varphi} < 0$ , we have

$$\begin{aligned} s^9 \int_0^T \int_{\omega_0} \xi^9 e^{-2s\check{\varphi}} |z|^2 &= s^9 \int_0^T \xi^9 e^{s(-10\check{\varphi}+8\hat{\varphi})} e^{8s(\check{\varphi}-\hat{\varphi})} \|z(t, \cdot)\|_{L^2(\omega_0)}^2 \\ &\leq s^9 \int_0^T \xi^9 e^{s(-10\check{\varphi}+8\hat{\varphi})} \|z(t, \cdot)\|_{L^2(\omega_0)}^2. \end{aligned} \quad (3.21)$$

Thus substituting the estimates (3.20) and (3.21) in (3.19), and choosing  $\epsilon > 0$  sufficiently small, we finally get the desired estimate (3.9).  $\square$

**Remark 3.2.** *One can also consider the boundary conditions:*

$$z(t, 0) = z(t, L) = z_x(t, L) = \partial_x z(t, 0) = \partial_x^2 z(t, L) = 0 \quad \text{for } t \in (0, T), \quad (3.22)$$

as a replacement for the set of boundary conditions in (3.1), and this will not affect the Carleman estimate (3.9).

**3.2. Carleman estimate for the adjoint system.** We are now in a position to derive the main Carleman estimate for the coupled system (1.9), which is crucial to deduce the main null controllability result for the system (1.5), as stated in Theorem 1.4.

**Theorem 3.3** (Main Carleman estimate). *Let  $T > 0$  be given and  $\tilde{\omega} \subset \mathcal{O} \cap \omega$  be a nonempty open subset. Then there exist constants  $C > 0$  and  $s_0 > 0$ , depending on  $T$  such that for any given source terms  $g_1, g_2 \in L^2(0, T; H_0^1(0, L))$ , and  $\psi_0 \in L^2(0, L)$ , the solution to (1.9) satisfies*

$$\begin{aligned} \tilde{\mathcal{I}}_{kw}(v) + \tilde{\mathcal{I}}_{kw}(\psi) &\leq C \left( s^{17} \int_0^T \int_{\tilde{\omega}} \xi^{17} e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2 \right. \\ &\quad \left. + s^9 \iint_Q \xi^9 e^{-2s(10\check{\varphi}-9\hat{\varphi})} (|\partial_x g_1|^2 + |\partial_x g_2|^2) \right), \end{aligned} \quad (3.23)$$

for all  $s \geq s_0$ , where  $\tilde{\mathcal{I}}_{kw}(\cdot)$  has been defined in (3.8).

*Proof.* Let  $\hat{\omega}$  be any nonempty open subset of  $(0, L)$  such that  $\hat{\omega} \Subset \tilde{\omega} \subset \mathcal{O} \cap \omega$ . Then we apply Proposition 3.1, more specifically the estimate (3.9) to the state solution  $v$  and  $\psi$  of the adjoint system (1.9) for the local domain  $\omega_0 = \hat{\omega}$ . In what follows, for sufficiently large  $s_0$  the states  $v$  and  $\psi$  satisfies

the following estimate

$$\begin{aligned} \tilde{\mathcal{I}}_{kw}(v) + \tilde{\mathcal{I}}_{kw}(\psi) \leq & C \left( s \int_0^T \xi^5 e^{-2s\hat{\varphi}} \left( \|g_1\|_{H^1(0,L)}^2 + \|\psi\|_{H^1(0,L)}^2 + \|g_2\|_{H^1(0,L)}^2 \right) \right. \\ & + s^3 \iint_Q \xi e^{-2s\varphi} (|g_1|^2 + |\psi|^2 + |g_2|^2) \\ & \left. + s^9 \int_0^T \xi^9 e^{-2s(5\check{\varphi}-4\hat{\varphi})} \left( \|v\|_{L^2(\tilde{\omega})}^2 + \|\psi\|_{L^2(\tilde{\omega})}^2 \right) \right), \end{aligned} \quad (3.24)$$

for any  $s \geq s_0$  and for some constant  $C > 0$ .

**I. Handling the global terms of rhs.** Note that using the fact that  $\varphi \leq \hat{\varphi}$  and  $\xi(t) \geq 4/T^2$  for  $t \in (0, T)$ , we get the following estimate

$$s \int_0^T \xi^5 e^{-2s\hat{\varphi}} \|\psi\|_{H^1(0,L)}^2 + s^3 \iint_Q \xi e^{-2s\varphi} |\psi|^2 \leq C \left( s^3 \iint_Q \xi^5 e^{-2s\varphi} |\psi|^2 + s \iint_Q \xi^5 e^{-2s\varphi} |\partial_x \psi|^2 \right),$$

which can be absorbed in  $\tilde{\mathcal{I}}_{kw}$  for sufficiently large  $s$ .

For  $i \in \{1, 2\}$ , the  $g_i$  terms can also be clubbed together using same argument to get

$$s \int_0^T \xi^5 e^{-2s\hat{\varphi}} \|g_i\|_{H^1(0,L)}^2 + s^3 \iint_Q \xi e^{-2s\varphi} |g_i|^2 \leq C \left( s^3 \iint_Q \xi^5 e^{-2s\varphi} |g_i|^2 + s \iint_Q \xi^5 e^{-2s\varphi} |\partial_x g_i|^2 \right).$$

Since  $g_i \in L^2(0, T; H_0^1)$ , so we use the Poincarè inequality in the above estimate to deduce

$$s \int_0^T \xi^5 e^{-2s\hat{\varphi}} \|g_i\|_{H^1(0,L)}^2 + s^3 \iint_Q \xi e^{-2s\varphi} |g_i|^2 \leq C s^3 \iint_Q \xi^5 e^{-2s\varphi} |\partial_x g_i|^2.$$

Thus, for sufficiently large  $s_0$ , the estimate (3.24) reduces to

$$\tilde{\mathcal{I}}_{kw}(v) + \tilde{\mathcal{I}}_{kw}(\psi) \lesssim s^3 \iint_Q \xi^5 e^{-2s\varphi} (|\partial_x g_1|^2 + |\partial_x g_2|^2) + s^9 \iint_{Q_{\tilde{\omega}}} \xi^9 e^{-2s(5\check{\varphi}-4\hat{\varphi})} (|v|^2 + |\psi|^2), \quad (3.25)$$

for all  $s \geq s_0$ .

**II. Absorbing the local term of  $\psi$ .** In this part, we eliminate the local integral associated to  $\psi$ . Let us first consider a function  $\tilde{\zeta} \in C_c^\infty(\tilde{\omega})$  such that  $0 \leq \tilde{\zeta} \leq 1$  in  $\tilde{\omega}$  and  $\tilde{\zeta} = 1$  in  $\hat{\omega}$ . From the differential equation of  $v$ , that is (1.9)<sub>1</sub>, we have

$$\psi = -\partial_t v - \partial_x v - \partial_x^3 v - \partial_x^5 v - g_1 \quad \text{in } (0, T) \times \mathcal{O} \quad (\text{consequently in } (0, T) \times \tilde{\omega}),$$

which yields

$$\begin{aligned} s^9 \iint_{Q_{\tilde{\omega}}} \xi^9 e^{-2s(5\check{\varphi}-4\hat{\varphi})} |\psi|^2 & \leq s^9 \int_0^T \int_{\tilde{\omega}} \xi^9 e^{-2s(5\check{\varphi}-4\hat{\varphi})} \tilde{\zeta} \psi (-\partial_t v - \partial_x v - \partial_x^3 v - \partial_x^5 v - g_1) \\ & = \sum_{i=1}^5 \mathcal{Q}_i. \end{aligned}$$

We now estimate the terms  $\mathcal{Q}_i$ , for  $i \in \{1, 2, 3, 4, 5\}$ .

(a) **Estimate of  $\mathcal{Q}_1$ .** Integrating by parts with respect to time  $t$ , the term  $\mathcal{Q}_1$  becomes

$$\begin{aligned} \mathcal{Q}_1 & = s^9 \int_0^T \int_{\tilde{\omega}} \tilde{\zeta} \partial_t \left( \xi^9 e^{-2s(5\check{\varphi}-4\hat{\varphi})} \right) \psi v + s^9 \int_0^T \int_{\tilde{\omega}} \xi^9 e^{-2s(5\check{\varphi}-4\hat{\varphi})} \tilde{\zeta} \partial_t \psi v \\ & = \mathcal{Q}_{1,1} + \mathcal{Q}_{1,2}. \end{aligned} \quad (3.26)$$

Thanks to the fact

$$\left| \left( \xi^9 e^{-2s(5\check{\varphi}-4\hat{\varphi})} \right)_t \right| \leq C s \xi^{11} e^{-2s(5\check{\varphi}-4\hat{\varphi})}, \quad (3.27)$$

and the Young's inequality, we first deduce that

$$\begin{aligned} |\mathcal{Q}_{1,1}| & \leq C s^{10} \int_0^T \int_{\tilde{\omega}} e^{-2s(5\check{\varphi}-4\hat{\varphi})} \xi^{11} |\psi v| \\ & \leq \epsilon s^9 \iint_Q e^{-2s\varphi} \xi^9 |\psi|^2 + C_\epsilon s^{11} \int_0^T \int_{\tilde{\omega}} e^{2(-10s\check{\varphi}+9s\hat{\varphi})} \xi^{13} |v|^2, \end{aligned} \quad (3.28)$$

for any given  $\epsilon > 0$ .

Next, we use the equation (1.9)<sub>2</sub> to see

$$\begin{aligned}\mathcal{Q}_{1,2} &= s^9 \int_0^T \int_{\tilde{\omega}} \xi^9 e^{-2s(5\check{\varphi}-4\hat{\varphi})} \tilde{\zeta} v (-\partial_x^5 \psi - \partial_x^3 \psi - \partial_x \psi + g_2) \\ &= \sum_{i=1}^4 \mathcal{Q}_{1,2}^i.\end{aligned}$$

For  $\{\mathcal{Q}_{1,2}^i\}_{i=2}^4$ , we use the Young's inequality to get

$$\begin{aligned}\bullet |\mathcal{Q}_{1,2}^1| &\leq \epsilon s \int_0^T \int_{\tilde{\omega}} \xi e^{-2s\hat{\varphi}} |\partial_x^5 \psi|^2 + C_\epsilon s^{17} \int_0^T \int_{\tilde{\omega}} \xi^{17} e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2. \\ \bullet |\mathcal{Q}_{1,2}^2| &\leq \epsilon s^3 \iint_Q \xi^3 e^{-2s\varphi} |\partial_x^3 \psi|^2 + C_\epsilon s^{15} \int_0^T \int_{\tilde{\omega}} \xi^{15} e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2. \\ \bullet |\mathcal{Q}_{1,2}^3| &\leq \epsilon s^7 \iint_Q \xi^7 e^{-2s\varphi} |\psi|^2 + C_\epsilon s^{11} \int_0^T \int_{\tilde{\omega}} \xi^{11} e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2. \\ \bullet |\mathcal{Q}_{1,2}^4| &\leq s \iint_Q \xi e^{-2s\varphi} |g_2|^2 + C s^{17} \int_0^T \int_{\tilde{\omega}} \xi^{17} e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2.\end{aligned}$$

Combining all the above estimates, we have for any small  $\epsilon > 0$ ,

$$|\mathcal{Q}_1| \leq \epsilon \tilde{\mathcal{I}}_{kw}(\psi) + s \iint_Q \xi e^{-2s\varphi} |g_2|^2 + C s^{17} \int_0^T \int_{\tilde{\omega}} \xi^{17} e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2.$$

(b) **Estimate of  $\{\mathcal{Q}_i\}_{i=2}^5$ .** Integrating by parts with respect to time  $x$ , and then using Young's inequality, the terms  $\{\mathcal{Q}_i\}_{i=2}^5$  can be estimated as

$$\begin{aligned}\bullet \mathcal{Q}_2 &= s^9 \int_0^T \int_{\tilde{\omega}} \xi^9 e^{-2s(5\check{\varphi}-4\hat{\varphi})} (\partial_x \tilde{\zeta} \psi + \tilde{\zeta} \partial_x \psi) v \\ &\leq \epsilon s^9 \iint_Q e^{-2s\varphi} \xi^9 |\psi|^2 + \epsilon s^7 \iint_Q e^{-2s\varphi} \xi^7 |\partial_x \psi|^2 + C_\epsilon s^9 \int_0^T \int_{\tilde{\omega}} \xi^9 e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2 \\ &\quad + C_\epsilon s^{11} \int_0^T \int_{\tilde{\omega}} \xi^{11} e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2 \\ &\leq \epsilon \tilde{\mathcal{I}}_{kw}(\psi) + C_\epsilon s^{11} \int_0^T \int_{\tilde{\omega}} \xi^{11} e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2. \\ \bullet \mathcal{Q}_3 &= s^9 \int_0^T \int_{\tilde{\omega}} \xi^9 e^{-2s(5\check{\varphi}-4\hat{\varphi})} (\partial_x^3 \tilde{\zeta} \psi + 3\partial_x^2 \tilde{\zeta} \partial_x \psi + 3\partial_x \tilde{\zeta} \partial_x^2 \psi + \tilde{\zeta} \partial_x^3 \psi) v \\ &\leq \epsilon \tilde{\mathcal{I}}_{kw}(\psi) + C_\epsilon s^{15} \int_0^T \int_{\tilde{\omega}} \xi^{15} e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2. \\ \bullet \mathcal{Q}_4 &= s^9 \int_0^T \int_{\tilde{\omega}} \xi^9 e^{-2s(5\check{\varphi}-4\hat{\varphi})} (\partial_x^5 \tilde{\zeta} \psi + 5\partial_x^4 \tilde{\zeta} \partial_x \psi + 10\partial_x^3 \tilde{\zeta} \partial_x^2 \psi \\ &\quad + 10\partial_x^2 \tilde{\zeta} \partial_x^3 \psi + 5\partial_x \tilde{\zeta} \partial_x^4 \psi + \tilde{\zeta} \partial_x^5 \psi) v \\ &\leq \epsilon \tilde{\mathcal{I}}_{kw}(\psi) + C_\epsilon s^{17} \int_0^T \int_{\tilde{\omega}} \xi^{17} e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2. \\ \bullet \mathcal{Q}_5 &\leq \epsilon s^9 \iint_Q \xi^9 e^{-2s\varphi} |\psi|^2 + C s^9 \iint_Q \xi^9 e^{-2s(10\check{\varphi}-9\hat{\varphi})} |g_1|^2.\end{aligned}$$

Thus combining all above estimates, we have

$$\begin{aligned}s^9 \int_0^T \int_{\tilde{\omega}} \xi^9 e^{-2s(5\check{\varphi}-4\hat{\varphi})} |\psi|^2 &\leq 5\epsilon \tilde{\mathcal{I}}_{kw}(\psi) + C s^{17} \int_0^T \int_{\tilde{\omega}} \xi^{17} e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2 \\ &\quad + C s \iint_Q \xi e^{-2s\varphi} |g_2|^2 + C s^9 \iint_Q \xi^9 e^{-2s(10\check{\varphi}-9\hat{\varphi})} |g_1|^2.\end{aligned}$$

Hence, substituting this estimate in (3.25) for sufficiently small  $\epsilon$  gives

$$\begin{aligned} \tilde{\mathcal{I}}_{kw}(v) + \tilde{\mathcal{I}}_{kw}(\psi) \leq C \left( s^{17} \int_0^T \int_{\tilde{\omega}} \xi^{17} e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2 + s^3 \iint_Q \xi^5 e^{-2s\varphi} (|\partial_x g_1|^2 + |\partial_x g_2|^2) \right. \\ \left. + s^9 \iint_Q \xi^9 e^{-2s(10\check{\varphi}-9\hat{\varphi})} |g_1|^2 \right), \end{aligned} \quad (3.29)$$

for all  $s \geq s_0$ .

To this end, due to the choices of  $g_i \in L^2(0, T; H_0^1(0, L))$  for  $i \in \{1, 2\}$ , and the definitions of  $\check{\varphi}, \hat{\varphi}$ , we have

$$\begin{aligned} s^3 \iint_Q \xi^5 e^{-2s\varphi} (|\partial_x g_1|^2 + |\partial_x g_2|^2) + s^9 \iint_Q \xi^9 e^{-2s(10\check{\varphi}-9\hat{\varphi})} |g_1|^2 \\ \leq C s^9 \iint_Q \xi^9 e^{-2s(10\check{\varphi}-9\hat{\varphi})} (|\partial_x g_1|^2 + |\partial_x g_2|^2), \end{aligned}$$

and hence the estimate (3.29) finally reduces to the desired Carleman estimate (3.23).  $\square$

#### 4. THE OBSERVABILITY INEQUALITY

In this section, we derive an observability inequality to the solutions of the adjoint system (1.9) which essentially helps to establish the null controllability of the linear system (1.8). Let us first define some modified Carleman weights that do not vanish at  $t = T$ . We consider

$$\mathfrak{Z}(t) = \begin{cases} \frac{1}{t(T-t)}, & 0 < t \leq T/2, \\ \frac{4}{T^2}, & T/2 \leq t \leq T, \end{cases} \quad (4.1)$$

and the weight function

$$\mathfrak{S}(t, x) = \phi(x) \mathfrak{Z}(t), \quad \forall (t, x) \in (0, T] \times [0, L], \quad (4.2)$$

where the function  $\phi$  satisfies the properties mentioned in (3.2).

We further define

$$\begin{aligned} \check{\mathfrak{S}}(t) &= \min_{[0, L]} \mathfrak{S}(t, x), \quad \forall t \in (0, T), \\ \hat{\mathfrak{S}}(t) &= \max_{[0, L]} \mathfrak{S}(t, x), \quad \forall t \in (0, T). \end{aligned} \quad (4.3)$$

**Remark 4.1.** Recall the former weight functions  $\xi, \varphi$ , and  $\check{\varphi}, \hat{\varphi}$  from (3.3), (3.4) and (3.5). Then, by construction we observe that

$$\mathfrak{Z}(t) = \xi(t), \quad \text{for } t \in (0, T/2], \quad \mathfrak{S}(t, x) = \varphi(t, x), \quad \text{for } t \in (0, T/2] \times [0, L],$$

and further,

$$\check{\mathfrak{S}}(t) = \check{\varphi}(t), \quad \hat{\mathfrak{S}}(t) = \hat{\varphi}(t), \quad \text{for } t \in (0, T/2].$$

With all these, we derive the following observability inequality associated to the adjoint system (1.9), which is crucial for the null controllability of the system (1.8).

**Proposition 4.2.** Let  $s$  be fixed in accordance with Theorem 3.3, and let  $\tilde{\omega} \subset \mathcal{O} \cap \omega$  be any nonempty open subset. Then, there exists some constant  $C > 0$  that depends on  $s, T, \omega$  and  $\mathcal{O}$  such that for any  $g_i \in L^2(0, T; H_0^1(0, L))$  for  $i = 1, 2$  and  $\psi_0 \in L^2(0, L)$ , the solution to (1.9) satisfies the following estimate

$$\begin{aligned} \|\psi(T)\|_{L^2(0, L)}^2 + \left\| e^{-s\hat{\mathfrak{S}}} \mathfrak{Z}^{5/2}(v, \psi) \right\|_{[C([0, T]; L^2(0, L))}^2 + \iint_Q e^{-2s\hat{\mathfrak{S}}} \mathfrak{Z}^7 (|\partial_x v|^2 + |\partial_x \psi|^2) \\ \leq C \int_0^T \int_{\tilde{\omega}} \mathfrak{Z}^{17} e^{-2s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})} |v|^2 + C \iint_Q \mathfrak{Z}^9 e^{-2s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})} (|\partial_x g_1|^2 + |\partial_x g_2|^2). \end{aligned} \quad (4.4)$$

*Proof.* We break the proof in two steps, as described below:

• **Step 1.** Using the fact that  $\varphi \leq \hat{\varphi}$  in the Carleman estimate (3.23) for fixed value of  $s$  satisfying  $s \geq s_0$ , we obtain the following estimate

$$\begin{aligned} & \iint_Q e^{-2s\hat{\varphi}} \xi^9 (|v|^2 + |\psi|^2) + \iint_Q e^{-2s\hat{\varphi}} \xi^7 (|\partial_x v|^2 + |\partial_x \psi|^2) \\ & \leq C \int_0^T \int_{\tilde{\omega}} \xi^{17} e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2 + C \iint_Q \xi^9 e^{-2s(10\check{\varphi}-9\hat{\varphi})} (|\partial_x g_1|^2 + |\partial_x g_2|^2). \end{aligned} \quad (4.5)$$

Let us choose a function  $\gamma \in C^1([0, T])$  such that

$$\gamma = 0 \quad \text{in } [0, T/4], \quad \gamma = 1 \quad \text{in } [T/2, T]. \quad (4.6)$$

It is clear that  $\text{Supp}(\gamma) \subset [T/4, T]$ , and  $\text{Supp}(\gamma') \subset [T/4, T/2]$ .

Let  $(v, \psi)$  is the solution of the adjoint equations (1.9). Then the pair  $(\gamma v, \gamma \psi)$  solves the following coupled system

$$\begin{cases} -\partial_t(\gamma v) - \partial_x(\gamma v) - \partial_x^3(\gamma v) - \partial_x^5(\gamma v) = \gamma g_1 + \chi_{\mathcal{O}} \gamma \psi - \gamma' v, & (t, x) \in Q, \\ \partial_t(\gamma \psi) + \partial_x(\gamma \psi) + \partial_x^3(\gamma \psi) + \partial_x^5(\gamma \psi) = \gamma g_2 + \gamma' \psi, & (t, x) \in Q, \\ (\gamma v)(t, 0) = (\gamma v)(t, L) = \partial_x(\gamma v)(t, 0) = \partial_x(\gamma v)(t, L) = \partial_x^2(\gamma v)(t, L) = 0, & t \in (0, T), \\ (\gamma \psi)(t, 0) = (\gamma \psi)(t, L) = \partial_x(\gamma \psi)(t, 0) = \partial_x(\gamma \psi)(t, L) = \partial_x^2(\gamma \psi)(t, 0) = 0, & t \in (0, T), \\ (\gamma \psi)(0, \cdot) = 0, (\gamma v)(T, \cdot) = 0, & \text{in } (0, L). \end{cases} \quad (4.7)$$

Applying Proposition 2.5 to (4.7), we get

$$\begin{aligned} & \|(\gamma v, \gamma \psi)\|_{[L^2(0, T; H_0^2(0, L))]}^2 + \|(\gamma v, \gamma \psi)\|_{[C([0, T]; L^2(0, L))]}^2 \\ & \leq C \left( \|(\gamma g_1, \gamma g_2)\|_{[L^2(0, T; L^2(0, L))]}^2 + \|(\gamma' v, \gamma' \psi)\|_{[L^2(0, T; L^2(0, L))]}^2 \right). \end{aligned} \quad (4.8)$$

Next, we use the properties of  $\gamma$  introduced in (4.6), in the above inequality (4.8) to obtain

$$\begin{aligned} & \|(v, \psi)\|_{[L^2(T/2, T; H_0^2(0, L))]}^2 + \|\psi(T)\|_{L^2(0, L)} \\ & \leq C \left( \|(g_1, g_2)\|_{[L^2(T/4, T; L^2(0, L))]}^2 + \|(v, \psi)\|_{[L^2(T/4, T/2; L^2(0, L))]}^2 \right). \end{aligned} \quad (4.9)$$

Using the fact  $\mathfrak{Z}(t) = \frac{1}{t(T-t)}$  and  $e^{-2s\hat{\mathfrak{S}}(t)} \mathfrak{Z}(t)^9 \geq C$  for  $t \in [T/4, T/2]$  and some  $C > 0$ , one can estimate the last term of right hand sides of (4.9) as

$$\|(v, \psi)\|_{[L^2(T/4, T/2; L^2(0, L))]}^2 \leq C \int_{T/4}^{T/2} \int_0^L e^{-2s\hat{\mathfrak{S}}} \mathfrak{Z}^9 (|v|^2 + |\psi|^2) = C \int_{T/4}^{T/2} \int_0^L e^{-2s\hat{\varphi}} \xi^9 (|v|^2 + |\psi|^2).$$

Thanks to the Carleman estimate (4.5) with the fact that  $\mathfrak{Z} = \xi$  and  $\hat{\mathfrak{S}} = \hat{\varphi}$  in  $[T/4, T/2]$  (see Remark 4.1), we get

$$\begin{aligned} \|(v, \psi)\|_{[L^2(T/4, T/2; L^2(0, L))]}^2 & \leq C \int_0^T \int_{\tilde{\omega}} \xi^{17} e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2 \\ & \quad + C \iint_Q \xi^9 e^{-2s(10\check{\varphi}-9\hat{\varphi})} (|\partial_x g_1|^2 + |\partial_x g_2|^2). \end{aligned} \quad (4.10)$$

Next, note that

$$\begin{cases} \xi^{17} e^{-2s(10\check{\varphi}-9\hat{\varphi})} = \mathfrak{Z}^{17} e^{-2s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})}, \quad \xi^9 e^{-2s(10\check{\varphi}-9\hat{\varphi})} = \mathfrak{Z}^9 e^{-2s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})} & \text{in } [0, T/2], \\ \xi^{17} e^{-2s(10\check{\varphi}-9\hat{\varphi})}, \quad \xi^9 e^{-2s(10\check{\varphi}-9\hat{\varphi})} \leq C \text{ and } \mathfrak{Z}^{17} e^{-2s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})}, \quad \mathfrak{Z}^9 e^{-2s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})} \geq C & \text{in } [T/2, T]. \end{cases} \quad (4.11)$$

Utilizing the above observations in (4.10), we deduce

$$\begin{aligned} & \|(v, \psi)\|_{[L^2(T/4, T/2; L^2(0, L))]}^2 \\ & \leq C \int_0^T \int_{\tilde{\omega}} \mathfrak{S}^{17} e^{-2s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})} |v|^2 + C \iint_Q \mathfrak{S}^9 e^{-2s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})} (|\partial_x g_1|^2 + |\partial_x g_2|^2). \end{aligned} \quad (4.12)$$



Substituting this estimate in (4.9), and using the fact  $e^{-2s\hat{\mathfrak{Z}}^n}$  is bounded above in  $[T/2, T]$  for any  $n \in \mathbb{N}$  in the left hand sides of (4.9), we get

$$\begin{aligned} & \|\psi(T)\|_{L^2(0,L)}^2 + \int_{T/2}^T \int_0^L e^{-2s\hat{\mathfrak{Z}}^9} (|v|^2 + |\psi|^2) + \int_{T/2}^T \int_0^L e^{-2s\hat{\mathfrak{Z}}^7} (|\partial_x v|^2 + |\partial_x \psi|^2) \\ & \leq C \int_0^T \int_{\tilde{\omega}} \mathfrak{S}^{17} e^{-2s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})} |v|^2 + C \iint_Q \mathfrak{S}^9 e^{-2s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})} (|\partial_x g_1|^2 + |\partial_x g_2|^2). \end{aligned} \quad (4.13)$$

In the above estimate, we have clubbed the  $\|(g_1, g_2)\|_{[L^2(T/4, T; L^2(0, L))]}^2$  terms with the  $|\partial_x g_i|$  term using Poincarè inequality as  $g_i \in L^2(0, T; H_0^1(0, L))$ .

Using the fact that  $\mathfrak{Z} = \xi$  and  $\hat{\mathfrak{S}} = \hat{\varphi}$  in  $(0, T/2]$  (Remark 4.1), and using the Carleman estimate (4.5), we have

$$\begin{aligned} & \int_0^{T/2} \int_0^L e^{-2s\hat{\mathfrak{Z}}^9} (|v|^2 + |\psi|^2) + \int_0^{T/2} \int_0^L e^{-2s\hat{\mathfrak{Z}}^7} (|\partial_x v|^2 + |\partial_x \psi|^2) \\ & \leq C \int_0^T \int_{\tilde{\omega}} \xi^{17} e^{-2s(10\check{\varphi}-9\hat{\varphi})} |v|^2 + C \iint_Q \xi^9 e^{-2s(10\check{\varphi}-9\hat{\varphi})} (|\partial_x g_1|^2 + |\partial_x g_2|^2). \end{aligned}$$

Next, we use the observation (4.11) in the last estimate to conclude

$$\begin{aligned} & \int_0^{T/2} \int_0^L e^{-2s\hat{\mathfrak{Z}}^9} (|v|^2 + |\psi|^2) + \int_0^{T/2} \int_0^L e^{-2s\hat{\mathfrak{Z}}^7} (|\partial_x v|^2 + |\partial_x \psi|^2) \\ & \leq C \int_0^T \int_{\tilde{\omega}} \mathfrak{Z}^{17} e^{-2s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})} |v|^2 + C \iint_Q \mathfrak{Z}^9 e^{-2s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})} (|\partial_x g_1|^2 + |\partial_x g_2|^2). \end{aligned} \quad (4.14)$$

As a consequence of (4.13) and (4.14), we have the following estimate

$$\begin{aligned} & \|\psi(T)\|_{L^2(0,L)}^2 + \iint_Q e^{-2s\hat{\mathfrak{Z}}^9} (|v|^2 + |\psi|^2) + \iint_Q e^{-2s\hat{\mathfrak{Z}}^7} (|\partial_x v|^2 + |\partial_x \psi|^2) \\ & \leq C \int_0^T \int_{\tilde{\omega}} \mathfrak{Z}^{17} e^{-2s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})} |v|^2 + C \iint_Q \mathfrak{Z}^9 e^{-2s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})} (|\partial_x g_1|^2 + |\partial_x g_2|^2). \end{aligned} \quad (4.15)$$

• **Step 2.** Let us define  $\hat{\rho}(t) = e^{-s\hat{\mathfrak{Z}}^5/2}$  so that  $\hat{\rho}(0) = 0$ . Again, by applying Proposition 2.5 to the equations satisfied by  $(\hat{\rho}v, \hat{\rho}\psi)$ , we get

$$\|(\hat{\rho}v, \hat{\rho}\psi)\|_{[C([0, T]; L^2(0, L))]}^2 \leq C \left( \|(\hat{\rho}g_1, \hat{\rho}g_2, )\|_{[L^2(0, T; L^2(0, L))]}^2 + \|(\hat{\rho}_t v, \hat{\rho}_t \psi)\|_{[L^2(0, T; L^2(0, L))]}^2 \right). \quad (4.16)$$

Note that  $|\hat{\rho}_t| \leq Cs\mathfrak{Z}^{9/2}e^{-s\hat{\mathfrak{Z}}}$  for some constant  $C > 0$ , and therefore

$$\|(\hat{\rho}_t v, \hat{\rho}_t \psi)\|_{[L^2(0, T; L^2(0, L))]}^2 \leq C \iint_Q e^{-2s\hat{\mathfrak{Z}}} \mathfrak{Z}^9 (|v|^2 + |\psi|^2). \quad (4.17)$$

Using the above estimate in (4.16) along with (4.15), we get the desired observability inequality (4.4). This completes the proof.  $\square$

## 5. NULL-CONTROLLABILITY FOR THE LINEAR SYSTEM

This section is devoted to prove the null-controllability of the extended linearized system (1.8) with  $u_0 = 0$ , and source terms  $f_1, f_2 \in L^1(0, T; L^2(0, L))$ .

Let us denote the following operator:

$$\mathcal{L}u = \partial_t u + \partial_x^5 u + \partial_x^3 u + \partial_x u,$$

and its formal adjoint

$$\mathcal{L}^*u = -\partial_t u - \partial_x^5 u - \partial_x^3 u - \partial_x u.$$

Denote the Banach space

$$\begin{aligned} \mathcal{E} := & \left\{ (u, w, h) \mid e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-9/2}(u, w) \in [L^2(0, T; H^{-1}(0, L))]^2, \right. \\ & e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-17/2} h \in L^2(Q), \\ & e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-11}(u, w) \in [X_{0,T}]^2, \\ & e^{s\hat{\mathfrak{E}}} \mathfrak{Z}^{-5/2}(\mathcal{L}u - h\chi_\omega), e^{s\hat{\mathfrak{E}}} \mathfrak{Z}^{-5/2}(\mathcal{L}^*w - u\chi_\mathcal{O}) \in L^1(0, T; L^2(0, L)), \\ & \left. w(T, \cdot) = 0 \text{ in } (0, L) \right\} \end{aligned} \quad (5.1)$$

Then we have the following null-controllability result.

**Proposition 5.1.** *Let  $s$  be fixed parameter according to Theorem 3.3 and  $f_1, f_2$  be the functions satisfying*

$$e^{s\hat{\mathfrak{E}}} \mathfrak{Z}^{-5/2}(f_1, f_2) \in [L^1(0, T; L^2(0, L))]^2. \quad (5.2)$$

Then, there exist control  $h \in L^2(0, T; L^2(\omega))$  and the associated solution  $(u, w)$  to (1.8) such that the tuple  $(u, w, h) \in \mathcal{E}$ , in particular, we have  $w(0, \cdot) = 0$  in  $(0, L)$ .

*Proof.* We consider the space

$$\begin{aligned} \mathcal{Q}_0 := & \left\{ (v, \psi) \in [C^4(\overline{Q})]^2 \mid v(t, 0) = v(t, L) = \partial_x v(t, 0) = \partial_x v(t, L) = \partial_x^2 v(t, L) = 0, \right. \\ & \psi(t, 0) = \psi(t, L) = \partial_x \psi(t, 0) = \partial_x \psi(t, L) = \partial_x^2 \psi(t, 0) = 0, \\ & \mathcal{L}(\psi)(t, 0) = \mathcal{L}(\psi)(t, L) = 0, \\ & \left. (\mathcal{L}^*(v) - \psi\chi_\mathcal{O})(t, 0) = (\mathcal{L}^*(v) - \psi\chi_\mathcal{O})(t, L) = 0 \right\}, \end{aligned} \quad (5.3)$$

and define the bilinear form

$$\mathfrak{B} : \mathcal{Q}_0 \times \mathcal{Q}_0 \rightarrow \mathbb{R} \quad (5.4)$$

given by

$$\begin{aligned} \mathfrak{B}((v_1, \psi_1), (v_2, \psi_2)) &= \iint_Q e^{-2s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^9 \left[ (\mathcal{L}^*v_1 - \chi_\mathcal{O}\psi_1)_x (\mathcal{L}^*v_2 - \chi_\mathcal{O}\psi_2)_x + (\mathcal{L}\psi_1)_x (\mathcal{L}\psi_2)_x \right] \\ &+ \int_0^T \int_\omega e^{-2s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{17} v_1 v_2. \end{aligned} \quad (5.5)$$

Note that due to the observability inequality (4.4) and the fact that  $\tilde{\omega} \subset \omega$ , we have

$$\mathfrak{B}((v, \psi), (v, \psi)) \geq \left\| e^{-s\hat{\mathfrak{E}}} \mathfrak{Z}^{5/2}(v, \psi) \right\|_{[C([0,T]; L^2(0,L))}^2, \quad (5.6)$$

which defines a norm on  $\mathcal{Q}_0$ . We denote the closure of  $\mathcal{Q}_0$  w.r.t. the norm  $\mathfrak{B}(\cdot, \cdot)^{1/2}$  by  $\mathcal{Q}$ , which is a Hilbert space endowed with the inner product (5.5).

Next, for any given  $(f_1, f_2) \in [L^1(0, T; L^2(0, L))]^2$  we define the linear map  $\mathfrak{F}$  on  $\mathcal{Q}_0$  as

$$\mathfrak{F}(v, \psi) = \langle f^1, v \rangle_{L^1(L^2), L^\infty(L^2)} + \langle f^2, \psi \rangle_{L^1(L^2), L^\infty(L^2)}. \quad (5.7)$$

Observe that due to (5.2) and the observability inequality (4.4), the linear functional  $\mathfrak{F}$  satisfies the following estimate

$$|\mathfrak{F}(v, \psi)| \leq \left\| e^{s\hat{\mathfrak{E}}} \mathfrak{Z}^{-5/2}(f^1, f^2) \right\|_{[L^1(0,T; L^2(0,L))]} \times \left\| e^{-s\hat{\mathfrak{E}}} \mathfrak{Z}^{5/2}(v, \psi) \right\|_{[L^\infty(0,T; L^2(0,L))]} < \infty. \quad (5.8)$$

This shows that the map  $\mathfrak{F}$  is a continuous linear map on  $\mathcal{Q}$  as  $\mathcal{Q}_0$  is dense in  $\mathcal{Q}$ .

Therefore, for any given  $(f_1, f_2) \in [L^1(0, T; L^2(0, L))]^2$ , there exists unique  $(\tilde{v}, \tilde{\psi}) \in \mathcal{Q} \times \mathcal{Q}$  by Lax-Milgram's theorem which satisfies

$$\mathfrak{B}((\tilde{v}, \tilde{\psi}), (v, \psi)) = \mathfrak{F}(v, \psi), \quad \forall (v, \psi) \in \mathcal{Q}. \quad (5.9)$$

Now, we set

$$\tilde{u} = \mathfrak{Z}^9 e^{-2s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \left( \mathcal{L}^* \tilde{v} - \tilde{\psi} \chi_{\mathcal{O}} \right)_{xx}, \quad (5.10)$$

$$\tilde{w} = \mathfrak{Z}^9 e^{-2s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} (\mathcal{L} \tilde{\psi})_{xx}, \quad (5.11)$$

$$\tilde{h} = \mathfrak{Z}^{17} e^{-2s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \tilde{v} \mathbb{1}_{\omega}. \quad (5.12)$$

(a) Let us first show that  $(\tilde{u}, \tilde{w}, \tilde{h}) \in [L^2(0, T; H^{-1}(0, L))]^2 \times L^2((0, T) \times \omega)$ .

• Note that

$$\begin{aligned} \int_0^T e^{2s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-9} \|\tilde{u}\|_{H^{-1}(0, L)}^2 &= \int_0^T e^{2s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-9} \sup_{\|\vartheta\|_{H_0^1}=1} |\langle \tilde{u}, \vartheta \rangle|_{H^{-1}, H_0^1}^2 \\ &\leq \iint_Q e^{-2s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^9 \left| (L^* \tilde{v} - \tilde{\psi} \chi_{\mathcal{O}})_x \right|^2 \\ &\leq \mathfrak{B} \left( (\tilde{v}, \tilde{\psi}), (\tilde{v}, \tilde{\psi}) \right) < +\infty. \end{aligned}$$

• Similarly, one can easily show that  $\tilde{w} \in L^2(0, T; H^{-1}(0, L))$ .

• Straightforward computation gives

$$\begin{aligned} \|e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-17/2} \tilde{h}\|_{L^2((0, T) \times \omega)}^2 &= \int_0^T \int_{\omega} e^{-2s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{17} |\tilde{v}|^2 \\ &\leq \mathfrak{B} \left( (\tilde{v}, \tilde{\psi}), (\tilde{v}, \tilde{\psi}) \right) < +\infty. \end{aligned}$$

Combining the above estimates together, we get:

$$\begin{aligned} \|e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-9/2} (\tilde{u}, \tilde{w})\|_{[L^2((0, T); H^{-1}(0, L))]^2}^2 &+ \|e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-17/2} \tilde{h}\|_{L^2((0, T) \times \omega)}^2 \\ &\leq \mathfrak{B} \left( (\tilde{v}, \tilde{\psi}), (\tilde{v}, \tilde{\psi}) \right) < +\infty. \end{aligned} \quad (5.13)$$

(b) Note that this pair  $(\tilde{u}, \tilde{w})$  is the unique solution to the linearized system (1.8) in the sense of transposition with the control function  $\tilde{h}$  due to (5.9).

(c) Lastly, we show  $e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-11} (\tilde{u}, \tilde{w}) \in [X_{0, T}]^2$ . Let us define

$$(\bar{u}, \bar{w}) := e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-11} (\tilde{u}, \tilde{w}). \quad (5.14)$$

Using the expression of  $\tilde{w}$  from (5.11), we have

$$\bar{w} = e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-11} \mathfrak{Z}^9 e^{-2s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} (\mathcal{L} \tilde{\psi})_{xx} = e^{-s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-2} (\mathcal{L} \tilde{\psi})_{xx}.$$

Thus, using  $10\check{\zeta} - 9\hat{\zeta} > 0$  due to the properties of  $\phi$  as mentioned in (3.6), it is clear that  $\bar{w}(0) = 0$ . Similarly,  $\bar{u}(0) = 0$ . Furthermore, the pair  $(\bar{u}, \bar{w})$  satisfies the following system:

$$\begin{cases} \partial_t \bar{u} + \partial_x \bar{u} + \partial_x^3 \bar{u} + \partial_x^5 \bar{u} = \bar{f}_1 + \chi_{\omega} \bar{h} + (e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-11})_t \tilde{u}, & (t, x) \in Q, \\ -\partial_t \bar{w} - \partial_x \bar{w} - \partial_x^3 \bar{w} - \partial_x^5 \bar{w} = \bar{f}_2 + \chi_{\mathcal{O}} \bar{u} - (e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-11})_t \tilde{w}, & (t, x) \in Q, \\ \bar{u}(t, 0) = \bar{u}(t, L) = \partial_x \bar{u}(t, 0) = \partial_x \bar{u}(t, L) = \partial_x^2 \bar{u}(t, 0) = 0, & t \in (0, T), \\ \bar{w}(t, 0) = \bar{w}(t, L) = \partial_x \bar{w}(t, 0) = \partial_x \bar{w}(t, L) = \partial_x^2 \bar{w}(t, L) = 0, & t \in (0, T), \\ \bar{u}(0, \cdot) = 0, \bar{w}(T, \cdot) = 0, & \text{in } (0, L), \end{cases} \quad (5.15)$$

where  $\bar{h} := e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-11} \tilde{h} \in L^2((0, T) \times \omega)$  due to (5.13).

Also, we have

$$(\bar{f}_1, \bar{f}_2) := e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-11} (f_1, f_2) \in [L^1(0, T; L^2(0, L))]^2,$$

since  $e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-11} \leq C e^{s\hat{\mathfrak{E}}} \mathfrak{Z}^{-5/2}$ . Further, one can compute that

$$\left| (e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-11})_t \right| \leq C e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-9},$$

and so due to the bound (5.13), we get

$$\left( e^{s(10\check{\mathfrak{E}}-9\hat{\mathfrak{E}})} \mathfrak{Z}^{-11} \right)_t (\tilde{u}, \tilde{w}) \in [L^2(0, T; H^{-1}(0, L))]^2.$$

Altogether, we have shown that each source term in the set of equations (5.15) belongs to the space  $L^2(0, T; H^{-1}(0, L)) \subset L^2(0, T; H^{-2}(0, L))$ . As a result, by applying Proposition 2.4, we have

$$(\bar{u}, \bar{w}) \in [\mathcal{C}([0, T]; L^2(0, L))]^2 \cap [L^2(0, T; H_0^2(0, L))]^2,$$

in other words,

$$e^{s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})}\mathfrak{Z}^{-11}(\tilde{u}, \tilde{w}) \in [X_{0,T}]^2.$$

Thus, combining all the three points we conclude that the tuple  $(\tilde{u}, \tilde{w}, \tilde{h}) \in \mathcal{E}$  solves the control system (1.8), which completes the proof.  $\square$

## 6. LOCAL NULL-CONTROLLABILITY OF THE EXTENDED NONLINEAR SYSTEM

In this last section, we establish the main result of this article concerning the insensitizing property of the functional  $J_\tau$  given by (1.2), as stated in Theorem 1.1. As explained in Section 1, to establish this result we prove the equivalent local null-controllability result for the extended system (1.5), i.e., Theorem 1.4. Thus, in this section we give a proof for the local null controllability result, Theorem 1.4.

Before starting the proof, let us first recall the well-known *Inverse Mapping Theorem* on which our proof relies.

**Theorem 6.1.** *Let  $\mathcal{G}_1, \mathcal{G}_2$  be two Banach spaces and  $\mathcal{A} : \mathcal{G}_1 \rightarrow \mathcal{G}_2$  be a map satisfying  $\mathcal{A} \in \mathcal{C}^1(\mathcal{G}_1; \mathcal{G}_2)$ . Assume that  $b_1 \in \mathcal{G}_1$ ,  $\mathcal{A}(b_1) = b_2 \in \mathcal{G}_2$  and  $\mathcal{A}'(b_1) : \mathcal{G}_1 \rightarrow \mathcal{G}_2$  is surjective. Then, there exists  $\delta > 0$  such that for every  $\tilde{b} \in \mathcal{G}_2$  satisfying  $\|\tilde{b} - b_2\|_{\mathcal{G}_2} < \delta$ , there exists a solution of the equation*

$$\mathcal{A}(b) = \tilde{b}, \quad b \in \mathcal{G}_1.$$

**Proof of Theorem 1.4.** The proof is just an application of the above theorem as described below.

**Setup.** Consider the spaces

$$\mathcal{G}_1 = \mathcal{E}, \quad \mathcal{G}_2 = \mathcal{F} \times \mathcal{F},$$

where  $\mathcal{E}$  is defined by (5.1) and

$$\mathcal{F} := \left\{ f \mid e^{s\hat{\mathfrak{S}}}\mathfrak{Z}^{-5/2}f \in L^1(0, T; L^2(0, L)) \right\}. \quad (6.1)$$

Now, define the map  $\mathcal{A} : \mathcal{G}_1 \rightarrow \mathcal{G}_2$  given by

$$\begin{aligned} \mathcal{A}(u, w, h) &= \left( \partial_t u + \partial_x u + \partial_x^3 u + \partial_x^5 u + u^p \partial_x u - \chi_\omega h, -\partial_t w - \partial_x w - \partial_x^3 w - \partial_x^5 w - u^p \partial_x w - \chi_{\mathcal{O}} u \right). \end{aligned} \quad (6.2)$$

**Continuously differentiability.** Let us first check that  $\mathcal{A} \in \mathcal{C}^1(\mathcal{G}_1; \mathcal{G}_2)$ . In this regard, we denote the space

$$\mathcal{Y} := \left\{ y \mid e^{s(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})}\mathfrak{Z}^{-11}y \in L^2(0, T; H_0^1(0, L)) \right\}. \quad (6.3)$$

Observe that all the terms, except  $u^p \partial_x u$  and  $u^p \partial_x w$  in the definition of  $\mathcal{A}$  are linear. Thus, to prove  $\mathcal{A} \in \mathcal{C}^1(\mathcal{G}_1; \mathcal{G}_2)$ , it is enough to show that the map

$$(y, z) \in \mathcal{Y} \times \mathcal{Y} \mapsto y^p \partial_x z \in \mathcal{F} \quad (6.4)$$

is continuous.

Recall the construction of weight functions  $\check{\mathfrak{S}}, \check{\mathfrak{S}}$  in (4.3). Due to the last property of the function  $\phi$  mentioned in (3.2), we have

$$\left( 10s\check{\mathfrak{S}} - 9s\hat{\mathfrak{S}} \right) \geq c_0 s \mathfrak{Z}(t),$$

for all  $t \in (0, T]$  and for some  $c_0 > 0$ . Consequently,

$$e^{s\hat{\mathfrak{S}}}\mathfrak{Z}^{-5/2} \leq e^{(10s\check{\mathfrak{S}}-9s\hat{\mathfrak{S}})}\mathfrak{Z}^{-11} e^{-c_0 s \mathfrak{Z}(t)} \mathfrak{Z}^{11-\frac{5}{2}} \leq C e^{(10s\check{\mathfrak{S}}-9s\hat{\mathfrak{S}})}\mathfrak{Z}^{-11}, \quad (6.5)$$

for some constant  $C > 0$ . Thus, for any two functions  $y, z \in \mathcal{Y}$ , we have

$$\begin{aligned} \|y^p z_x\|_{\mathcal{F}} &= \int_0^T e^{s\hat{\mathfrak{S}}}\mathfrak{Z}^{-5/2} \|y^p z_x\|_{L^2(0, L)} \\ &\leq C \int_0^T e^{\frac{s}{2}(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})} \mathfrak{Z}^{-11/2} \|y\|_{L^\infty(0, L)}^p e^{\frac{s}{2}(10\check{\mathfrak{S}}-9\hat{\mathfrak{S}})} \mathfrak{Z}^{-11/2} \|z_x\|_{L^2(0, L)} \\ &\leq C(T^{(2-p)/4} + \sqrt{T}) \|y\|_{\mathcal{Y}}^p \|z\|_{\mathcal{Y}}. \end{aligned} \quad (6.6)$$

Mimicking the arguments of Lemma 2.7, one can deduce the continuity of the map (6.4) using the above estimate (6.6). This ultimately proves that the map  $\mathcal{A} \in \mathcal{C}^1(\mathcal{G}_1; \mathcal{G}_2)$ .

**Surjectivity.** Set  $b_1 = (0, 0, 0) \in \mathcal{G}_1$ , and  $b_2 = \mathcal{A}'(0, 0, 0) = (0, 0) \in \mathcal{G}_2$ . Then we aim to prove that the map  $\mathcal{A}'(0, 0, 0)$  is surjective. Note that the map  $\mathcal{A}'(0, 0, 0) : \mathcal{G}_1 \rightarrow \mathcal{G}_2$  is given by

$$\mathcal{A}'(0, 0, 0)(u, w, h) = \left( \partial_t u + \partial_x u + \partial_x^3 u + \partial_x^5 u - \chi_\omega h, -\partial_t w - \partial_x w - \partial_x^3 w - \partial_x^5 w - \chi_{\mathcal{O}} u \right),$$

which is surjective due to the controllability result given by Proposition 5.1.

**Conclusion.** Let us consider  $\tilde{b} = (\xi, 0) \in \mathcal{G}_2$ , where  $\xi$  is the given external source term in (1.5) or in (1.1). Then, according to Theorem 6.1, there is a  $\delta > 0$  such that for given  $\xi$  verifying

$$\|(\xi, 0)\|_{\mathcal{G}_2} < \delta,$$

there exists a solution-control pair  $((u, w), h) \in \mathcal{G}_1 = \mathcal{E}$  to the system (1.5). In particular,  $w(0, \cdot) = 0$  in  $(0, L)$ . This completes the proof of Theorem 1.4, and hence of Theorem 1.1.  $\square$

## 7. ACKNOWLEDGEMENT

Manish Kumar acknowledges financial support from Prime Minister Research Fellowship, Govt. of India (PMRF ID: 0501091). Subrata Majumdar received financial support from the institute post-doctoral fellowship of IIT Bombay during the initial stage of the work. Currently this work is supported by the post-doctoral scholarship of the Universidad Nacional Autónoma de México.

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