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Feedback stabilization of discrete-time switched systems under Büchi-constrained signals

M. Della Rossa, T. Alves Lima, and A. Girard (*IEEE Fellow*)

Abstract—This manuscript studies the feedback stabilization problem for a class of discrete-time switched systems. The goal is the design, via semidefinite optimization techniques, of feedback control rules depending only on the current state variable and on the past values of the underlying switching sequence. The resulting control policy achieves uniform exponential stabilization over a pre-constructed class of switching signals. The overall construction generalizes known approaches for stabilization over arbitrary switching sequences, but it is able to stabilize systems for which none of the defining subsystems is stabilizable. This extension is obtained employing graph-theoretic tools, introducing the Büchi automata formalism in order to specify the considered classes of admissible sequences, seen here in the general setting of ω -regular languages. The proposed construction is finally illustrated with the help of a numerical example.

I. INTRODUCTION

Switched systems form a class of hybrid dynamical systems for which the solutions are driven by a finite set of subsystems and by a switching function σ selecting which subsystem is active at any given moment. More formally, considering matrices $A_1, \dots, A_N \in \mathbb{R}^{n \times n}$ and $B_1, \dots, B_N \in \mathbb{R}^{n \times m}$, we study the *discrete-time switched control system* defined by

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \quad (1)$$

where $\sigma : \mathbb{N} \rightarrow \{1, \dots, N\}$ is a switching signal selecting among the N modes and $u : \mathbb{N} \rightarrow \mathbb{R}^m$ is a control input.

Problems related to stability and stabilizability of switched systems have been an active field of research due to their important applications in modern engineering, see [1]–[3] and references therein. Among several possible notions of stabilizability of switched linear systems, the interest of the current manuscript lies in the problem of finding stabilizing feedback maps for (1), valid uniformly over a prescribed class of switching signals.

In this setting a natural problem is the feedback stabilization under *arbitrary switching signals*, i.e. without any constraint on the external, unmodifiable, and unpredictable switching sequence. In this framework, it has been proved that a *linear* feedback gain approach, while leading to numerically-appealing conditions, is conservative, see [4]–[6]. To have more flexible conditions in studying stabilizability of (1), approaches based on piecewise linear feedback

maps (and related piecewise-defined Lyapunov functions) have been proposed. We mention the maxima and minima of quadratic functions approach in [7], [8] and polyhedral Lyapunov functions construction in [3], [6]. Another possible route to provide more general feedback stabilization schemes is provided in [9]–[12] in which the controller explicitly depends on the observation and memorization of past/future values of the switching signal. Recently, the same feedback stabilization problem was studied in [13] using graph-theoretic tools.

Considering arbitrary switching rules can, however, be conservative. First of all, requiring that *all the solutions* are driven to the equilibrium, by the *same feedback map* and for *all the possible switching sequences*, is a demanding (and often unfeasible) task. Moreover, in several real-life situations, the considered systems present some sort of constraints in the admissible switching events.

For this reason, it is rather common, both for stability and stabilizability purposes, to consider subclasses of switching signals. In this context, the framework of language and automata theory naturally arises, providing graph-theoretic tools to encode constraints on the switching signals. This idea is the core of the stability analysis provided for instance in [14]–[16] in which finite-state automata are used to define the class of admissible sequences. On the other hand, there are remarkable subclasses of switching signals that cannot be defined simply considering finite-state automata. This is the case, for instance, when one requires a persistence of activation/switching of certain modes. To model such “infinite” constraints, one has to consider the notion of ω -languages, and, as state machine counterpart, the notion of Büchi automata (see [17] for the formal definition). For the application of such ideas in the case of stability analysis, we refer to [18]–[20] and references therein.

In this work, we are interested in the *feedback stabilization* of (1) under a class of constrained switching signals σ that can be represented by ω -regular languages and formalized by Büchi automata. Many notions regarding the stability of switched systems with ω -regular switching sequences are borrowed from the recent work in [19], [20]. We introduce a class of Büchi automata, inspired by the seminal construction of De Bruijn, (see [21]), modeling the signals activating an infinite number of times some prescribed subsequences. This notion allows for relaxing the need for stabilizability of each pair (A_i, B_i) , $i \in \{1, \dots, N\}$. Based on a multiple quadratic Lyapunov functions construction, we are able to provide a composite feedback control policy, depending only on the current state, and on a finite number of past values of the

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switching sequence. Thus, the closed loop can be interpreted as a memory-based hybrid system evolving on an additional symbolic state-space, as in the formalism of [22]. By bounding from below the “frequency of occurrence” of the prescribed subsequences, the closed-loop behavior exhibits uniform exponential stability. We illustrate our proposed stabilization scheme with the help of a numerical example.

The structure of this manuscript is as follows: In Section II we recall some preliminaries from graph theory and dynamical (switched) systems framework. In Section III we present our main stabilization scheme, illustrated in Section IV by means of a numerical example. A final Section V provides some concluding remarks.

Notation: We denote by \mathbb{N} the set of natural numbers including $\{0\}$, by \mathbb{Z}^+ the set of natural numbers excluding $\{0\}$. Given $n, m \in \mathbb{Z}^+$, $\mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^m)$ is the set of continuous functions from \mathbb{R}^n to \mathbb{R}^m . We denote by $\mathbb{S}^{n \times n}$ the set of the $n \times n$ symmetric matrices, and by $\mathbb{S}_+^{n \times n}$ the set of $n \times n$ positive definite matrices.

II. PRELIMINARIES

This section introduces the studied setting and recalls the necessary notation, definitions and tools.

A. ω -Languages and Büchi automata

From now on, we consider a finite set of symbols Σ , referred to as the *alphabet*; by $n(\Sigma)$ we denote the cardinality of Σ . Given $M \in \mathbb{N}$, we denote by Σ^M the set of sequences of length M in Σ , and we denote its elements by $\hat{i} = (i_0, \dots, i_{M-1}) \in \Sigma^M$. With Σ^* we denote the *Kleene closure* of Σ , defined by

$$\Sigma^* = \bigcup_{M \in \mathbb{N}} \Sigma^M,$$

i.e., the set of all sequences of finite length of symbols of Σ . With Σ^ω we denote the ω -closure of Σ , i.e. the set of all the *infinite* sequences in Σ . More precisely, $\Sigma^\omega := \{\sigma : \mathbb{N} \rightarrow \Sigma\}$. Given any $\sigma \in \Sigma^\omega$ and any $a \leq b \in \mathbb{N}$ we denote by $[\sigma]_{[a,b]} \in \Sigma^{b-a+1}$ the *restriction of σ to the interval $[a, b]$* , i.e. $[\sigma]_{[a,b]} = (\sigma(a), \sigma(a+1), \dots, \sigma(b)) \in \Sigma^{b-a+1}$.

We then introduce some preliminaries concerning Büchi automata, i.e., a class of abstract machines used to define remarkable subsets of Σ^ω (a.k.a. ω -regular languages). Intuitively, non-deterministic Büchi automata (NBA) are a special class of automata accepting infinite sequences of inputs that repeat infinitely a set of states called accepting states. Formally, the data of a NBA is given by a tuple $\mathcal{G} = (S, \Sigma, E, S_I, S_F)$, where S is a finite set of states, Σ is a finite alphabet, $E \subseteq S \times S \times \Sigma$ is a finite set of edges defining the transitions of the form (s, s', j) , with $s \in S$ being a source state, $s' \in S$ a target state, and $j \in \Sigma$ being the symbol that activates the transition $s \xrightarrow{j} s'$. The set $S_I \subseteq S$ is the set of *initial states* and $S_F \subseteq S$ is the set of *accepting states*. A Büchi automata is a *deterministic* Büchi automata (DBA) if, for every $s \in S$ and each $j \in \Sigma$, there is at most one $s' \in S$ such that $(s, s', j) \in E$.

Definition 1: An *accepting run* in \mathcal{G} for $\sigma \in \Sigma^\omega$ is an infinite sequence of states $s_0 s_1 \dots \in S^\omega$ such that

- 1) $s_0 \in S_I$,
- 2) $(s_k, s_{k+1}, \sigma(k)) \in E$ for all $k \in \mathbb{N}$,
- 3) there exist a strictly increasing sequence $(k_n)_{n \in \mathbb{N}}$ such that $s_{k_n} = s$ for all $n \in \mathbb{N}$, for some accepting state $s \in S_F$.

If $\bar{s} \in S^\omega$ is an accepting run in \mathcal{G} for $\sigma \in \Sigma^\omega$, we write $\bar{s} \in \text{Acc}_{\mathcal{G}}(\sigma)$. Moreover, if there exists an accepting run in \mathcal{G} for $\sigma \in \Sigma^\omega$, we write $\sigma \in \mathcal{L}^\omega(\mathcal{G})$ and we say that σ is *recognized* by \mathcal{G} , or, equivalently, that σ is an element of the ω -language generated by \mathcal{G} .

Roughly speaking, a $\sigma \in \Sigma^\omega$ is recognized by \mathcal{G} only if there exists an accepting run $\bar{s} \in S^\omega$ in \mathcal{G} passing an infinite number of times through an accepting state $s \in S_F$. Given a NBA \mathcal{G} , a $\sigma \in \mathcal{L}^\omega(\mathcal{G})$ and an accepting run in \mathcal{G} for σ , denoted by $\bar{s} \in \text{Acc}_{\mathcal{G}}(\sigma)$, we denote the *sequence of accepting instants* by $\tau^{\bar{s}} : \mathbb{N} \rightarrow \mathbb{N}$, defined recursively by setting $\tau^{\bar{s}}(0) = 0$ and

$$\tau^{\bar{s}}(k+1) = \min\{h > \tau^{\bar{s}}(k) \mid \bar{s}(h) \in S_F\}, \quad \forall k \in \mathbb{N}. \quad (2)$$

Roughly speaking, $\tau^{\bar{s}}$ is the sequence of instants for which the run \bar{s} is visiting an accepting state in \mathcal{G} .

Given $\sigma \in \mathcal{L}^\omega(\mathcal{G})$, let us suppose that $\text{Acc}_{\mathcal{G}}(\sigma)$ is a finite set. The *return index of σ in \mathcal{G}* , is a function $\kappa^{\sigma, \mathcal{G}} : \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$\kappa^{\sigma, \mathcal{G}}(k) = \min_{\bar{s} \in \text{Acc}_{\mathcal{G}}(\sigma)} \max\{h \in \mathbb{N} \mid \tau^{\bar{s}}(h) \leq k\}. \quad (3)$$

Intuitively for any $k \in \mathbb{N}$, any accepting run in \mathcal{G} for σ has passed at least $\kappa^{\sigma, \mathcal{G}}(k)$ -times through an accepting node in S_F , up to time $k \in \mathbb{N}$.

B. Switched Systems: Definition and Stabilization Notions

Consider a finite alphabet Σ and a function $f : \mathbb{R}^n \times \Sigma^* \rightarrow \mathbb{R}^n$. We consider the *system*

$$x(k+1) = f(x(k), [\sigma]_{[0,k]}), \quad \forall k \in \mathbb{N}, \quad (4)$$

where $\sigma \in \Sigma^\omega$ is seen here as a *switching signal*. Given $\sigma \in \Sigma^\omega$, $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$, we denote by $\Phi_\sigma(k, x)$ the *solution to (4) with respect to σ , starting at x and evaluated at time $k \in \mathbb{N}$* .

In (4) the evolution of the state depends not only on the current value of the switching signal (i.e., $\sigma(k)$) but possibly also on the *past values* $\sigma(0), \dots, \sigma(k-1)$. It is important to note that, since it does not depend on the *future* values of σ , it is *causal*, in the classic sense of systems theory. The definition of systems depending on the whole *history* of the switching signal as in (4) is motivated by the control problem we want to tackle. In particular, we will propose the design of feedback control maps depending on the past values of the switching signal; first we recall the notion of stability for systems as in (4).

Definition 2: Given $f : \mathbb{R}^n \times \Sigma^* \rightarrow \mathbb{R}^n$ and a subset $\mathcal{L} \subseteq \Sigma^\omega$, system (4) is said to be *globally uniformly exponentially stable* on \mathcal{L} (\mathcal{L} -GUES) if there exist $C \geq 1$ and $\lambda \in [0, 1)$

such that

$$|\Phi_\sigma(k, x)| \leq C\lambda^k|x|, \quad \forall \sigma \in \mathcal{L}, \forall x \in \mathbb{R}^n, \forall k \in \mathbb{N}. \quad (5)$$

Given $\mathcal{A} = \{A_i\}_{i \in \Sigma} \subset \mathbb{R}^{n \times n}$ and $\mathcal{B} = \{B_i\}_{i \in \Sigma} \subset \mathbb{R}^{n \times m}$, we consider the associated *switched system*

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \quad (6)$$

where $\sigma \in \Sigma^\omega$ is seen here as a *switching signal*, and $u : \mathbb{N} \rightarrow \mathbb{R}^m$ is an *input signal*. Our aim is to provide stabilizing feedback laws for (6), considering signals lying in a prescribed subset of Σ^ω . We now formalize this goal.

Problem 1 (Prefix-Dependent Feedback Design):

Consider $\mathcal{A} = \{A_i\}_{i \in \Sigma} \subset \mathbb{R}^{n \times n}$, $\mathcal{B} = \{B_i\}_{i \in \Sigma} \subset \mathbb{R}^{n \times m}$ and a subset of switching signals $\mathcal{L} \subset \Sigma^\omega$. Our goal is to design a *prefix-dependent stabilizing feedback on \mathcal{L}* , i.e., find a map $\Psi : \Sigma^* \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that the system

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}\Psi([\sigma]_{[0,k]}, x(k)) \quad (7)$$

is \mathcal{L} -GUES, in the sense of Definition 2.

Remark 1: We note that in Problem 1 it is required to design a feedback law that depends, at each instant of time, only on the current state-space variable ($x(k)$), and, possibly, on the current and past values of the switching sequence. A controller $\Psi : \Sigma^* \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ as in Problem 1 is said to be of *uniform bounded memory*, if there exists a $M \in \mathbb{Z}^+$ and a $\hat{\Psi} : \cup_{k=0}^M \Sigma^k \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$\Psi(\hat{i}, x) = \hat{\Psi}([\hat{i}]_M, x), \quad \forall \hat{i} \in \Sigma^*, \quad \forall x \in \mathbb{R}^n,$$

where, for any $\hat{i} = (i_0, \dots, i_{K-1}) \in \Sigma^*$,

$$[\hat{i}]_M = \begin{cases} \hat{i}, & \text{if } \hat{i} \in \Sigma^K, K \leq M, \\ (i_{K-M}, \dots, i_{K-1}), & \text{if } \hat{i} \in \Sigma^K, K \geq M, \end{cases}$$

i.e., the map Ψ depends only on the past M values of \hat{i} . The idea of feedback controllers depending on the past values of the switching signals (a.k.a. “the memory”) is not new: it was the central contribution of, for example, [9], [11], [12]. This dependence on the past values is necessary to alleviate the limitations of classical *linear* feedback controllers (see [3], [5]) as better highlighted in what follows.

III. PREFIX-DEPENDENT FEEDBACK STABILIZATION

A. De Bruijn-based structure

In this subsection, we define a remarkable class of Büchi automata, arising by imposing the infinite recurrence of a string of modes. This construction is inspired by the seminal work of De Bruijn, see [21]. We first introduce the following notation: given $\hat{i} = (i_0, i_1, \dots, i_{M-1}) \in \Sigma^M$, we write $\hat{i}^+ = (i_1, \dots, i_{M-1}) \in \Sigma^{M-1}$.

Definition 3 (De Bruijn-based Büchi automata): Given an alphabet Σ and a $M \in \mathbb{N}$, consider a subset of *marked sequences* of length M , denoted by $\Sigma_F^M \subseteq \Sigma^M$. We define the corresponding Σ_F^M -Büchi automaton by $\mathcal{G}_{\Sigma_F^M} = (S, \Sigma, S_I, S_F, E)$ as follows:

- $S = \Sigma^M$,
- $S_I = S = \Sigma^M$,
- $S_F = \Sigma_F^M$,

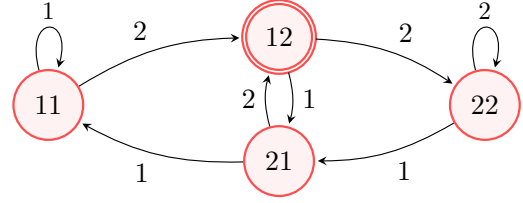


Fig. 1. De Bruijn-based Büchi-automaton on the alphabet $\Sigma = \{1, 2\}$, obtained by considering $\Sigma_F^2 = \{(1, 2)\}$.

- $(s_1, s_2, j) \in E$ if and only if $s_1 = \hat{i}$ and $s_2 = (\hat{i}^+, j)$.

Example 1: In Figure 1 we depicted the graph $\mathcal{G}_{\Sigma_F^2}$, for $\Sigma = \{1, 2\}$ and $\Sigma_F^2 = \{(1, 2)\}$. Since the only accepting state is marked by the sequence $(1, 2)$, this automaton is a finite machine representation of the class of signals with an infinite number of occurrences of the sequence $(1, 2)$.

In what follows we collect some properties regarding the De Bruijn-based Büchi automata.

Properties 1: Given an alphabet Σ , $M \in \mathbb{N}$, and a set $\Sigma_F^M \subseteq \Sigma^M$, the following statements hold:

- 1) If $\Sigma_F^M = \Sigma^M$ then $\mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M}) = \Sigma^\omega$;
- 2) For any $\sigma \in \Sigma^\omega$ there exists a (*not necessarily accepting*) run for σ in \mathcal{G} starting from any node $\hat{j} \in \Sigma^M$, i.e., there exists a sequence of nodes with all the properties of Definition 1, except *Item 3*.
- 3) For any $\sigma \in \mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M})$ there exist $|\Sigma^M|$ accepting runs in $\mathcal{G}_{\Sigma_F^M}$, each one starting at a different initial node $\hat{i} \in S$;
- 4) For any $\sigma \in \mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M})$, the accepting runs for σ coincide in $[M, +\infty)$. More precisely, given any $\bar{s} : \mathbb{N} \rightarrow S = \Sigma^M$, $\bar{s} \in \text{Acc}_{\mathcal{G}_{\Sigma_F^M}}(\sigma)$, it satisfies

$$\bar{s}(k) = [\sigma]_{[k-M, k-1]} \in \Sigma^M, \quad \forall k \in [M, +\infty).$$

- 5) We have the following characterization:

$$\mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M}) := \left\{ \sigma \in \Sigma^\omega \mid \begin{array}{l} \exists \hat{i} \in \Sigma_F^M, \forall K \in \mathbb{N}, \exists k \geq K \\ \text{s.t. } [\sigma]_{k-M, k-1} = \hat{i} \end{array} \right\}.$$

- 6) For every $\sigma \in \mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M})$, denoting $\kappa^{\sigma, \mathcal{G}_{\Sigma_F^M}}(k) =: A$, we have that

$$A \geq \left\{ r \in \mathbb{N} \mid r \geq M, r \leq k, [\sigma]_{[r-M, r-1]} \in \Sigma_F^M \right\}$$

Proof: First of all, we note that the automaton $\mathcal{G}_{\Sigma_F^M}$ is deterministic and *total*, i.e. for every node $s \in S$ and any symbol $j \in \Sigma$ there exists a (unique, by determinism) node $s' \in S$ such that $(s, s', j) \in E$. In other words, the transition map $\sigma : S \times \Sigma \rightsquigarrow S$ defined by $s' \in \sigma(s, i)$ if and only if $(s, s', j) \in E$ is a total function, (single-valued by determinism). With this property, it is easy to prove Items 1. and 2. Let us now prove Items 3. and 4. jointly. Consider an arbitrary $\sigma \in \mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M})$ and a generic node \hat{i} and let us build an accepting run $\bar{s} : \mathbb{N} \rightarrow S$, starting at \hat{i} , i.e., $\bar{s}(0) = \hat{i} = (i_0, \dots, i_M)$. By definition of $\mathcal{G}_{\Sigma_F^M}$ we have that, necessarily, $\bar{s}(1) = (i_1, i_2, \dots, \sigma(0))$; similarly, we have $\bar{s}(2) = (i_2, \dots, \sigma(0), \sigma(1))$. Iterating up to M the reasoning, we have that $\bar{s}(M) = (\sigma(0), \dots, \sigma(M-1))$. Afterwards, by determinism and definition of $\mathcal{G}_{\Sigma_F^M}$, the run \bar{s} necessarily satisfies $\bar{s}(k) = [\sigma]_{[k-M, k-1]}$, $\forall k \geq M$, independently on the initial node \hat{i} . Since the construction was made for an arbitrary $\hat{i} \in \Sigma^M$, we have proven Items

3. and 4. Items 5. and 6. then straightforwardly follow from the previous statements. ■

B. Main Stabilization Result

We use here the aforementioned automata structure to introduce feedback stabilization schemes for system (6).

Theorem 1: Let us consider an alphabet Σ and $\mathcal{A} = \{A_i\}_{i \in \Sigma} \subset \mathbb{R}^{n \times n}$, $\mathcal{B} = \{B_i\}_{i \in \Sigma} \subset \mathbb{R}^{n \times m}$. Consider an horizon $M \in \mathbb{N}$ and a subset of marked subsequences $\Sigma_F^M \subseteq \Sigma^M$. Consider the corresponding Σ_F^M -Büchi automaton $\mathcal{G}_{\Sigma_F^M}$. If there exist $\rho \in [0, 1)$, $\gamma \geq 0$, $P_{\hat{j}} \in \mathbb{S}_+^{n \times n}$ for all $\hat{j} \in \Sigma^M$ and $K_{\hat{j}, i} \in \mathbb{R}^{n \times m}$ for all $\hat{j} \in \Sigma^M$ and all $i \in \Sigma$ such that

$$(A_i + B_i K_{\hat{j}, i})^\top P_{(\hat{j}^+, i)} (A_i + B_i K_{\hat{j}, i}) \preceq \gamma^2 P_{\hat{j}}, \quad (8a)$$

$$\forall \hat{j} \in \Sigma^M \quad \forall i \in \Sigma,$$

$$(A_i + B_i K_{\hat{j}, i})^\top P_{(\hat{j}^+, i)} (A_i + B_i K_{\hat{j}, i}) \preceq \gamma^2 \rho^2 P_{\hat{j}}, \quad (8b)$$

$$\forall \hat{j} \in \Sigma^M, \quad \forall i \in \Sigma \text{ such that } (\hat{j}^+, i) \in \Sigma_F^M,$$

then, defining¹ $\Psi([\sigma]_k, x) = K_{[\sigma]_{[k-M, k]}} x$, we have that there exists $C \geq 1$ such that the solutions to (7) satisfy

$$|\Phi_\sigma(k, x)| \leq C \gamma^k |x|, \quad \forall \sigma \in \Sigma^\omega, \quad \forall x \in \mathbb{R}^n, \quad \forall k \in \mathbb{N}. \quad (9a)$$

$$|\Phi_\sigma(k, x)| \leq C \gamma^k \rho^{\kappa(k)} |x|, \quad \forall \sigma \in \mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M}), \quad (9b)$$

$$\forall x \in \mathbb{R}^n, \quad \forall k \geq M,$$

where we denote, for simplicity, $\kappa(k) := \kappa^{\sigma, \mathcal{G}_{\Sigma_F^M}}(k)$ introduced in (3).

Proof: Let us consider any $x \in \mathbb{R}^n$ and any $\sigma \in \Sigma^\omega$. Let us fix an arbitrary $j_\star \in \Sigma$. By *Item 2.* in Properties 1, we have that there is a (not necessarily accepting) run for σ , starting from the node $\hat{j}_\star = (j_\star, \dots, j_\star) \in \Sigma^M$. Let us introduce the function $V : \Sigma^M \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $V(\hat{i}, x) = \sqrt{x^\top P_{\hat{i}} x}$. We first show that the function $k \mapsto V([\sigma]_{[k-M, k-1]}, \Phi_\sigma(k, x))$ satisfies

$$V([\sigma]_{[k-M, k-1]}, \Phi_\sigma(k, x)) \leq \gamma^k V([\sigma]_{[-M, -1]}, x), \quad (10)$$

for all $k \in \mathbb{N}$. Let suppose that $\sigma(0) = i \in \Sigma$, we have that

$$\begin{aligned} V([\sigma]_{[-M+1, 0]}, \Phi_\sigma(1, x)) &= V((\hat{j}_\star^+, i), (A_i + B_i K_{\hat{j}_\star, i})x) \\ &= \sqrt{x^\top (A_i + B_i K_{\hat{j}_\star, i})^\top P_{(\hat{j}_\star^+, i)} (A_i + B_i K_{\hat{j}_\star, i}) x} \\ &\leq \gamma \sqrt{x^\top P_{\hat{j}_\star} x}, \end{aligned}$$

where, in the last inequality, we used (8a). Iterating the reasoning, we have that

$$\begin{aligned} V([\sigma]_{[k-M, k-1]}, \Phi_\sigma(k, x)) &\leq \gamma^k \sqrt{x^\top P_{\hat{j}_\star} x} \\ &= \gamma^k V([\sigma]_{[-M, -1]}, x), \quad \forall k \in \mathbb{N}. \end{aligned} \quad (11)$$

Since $P_{\hat{i}} \succ 0$ for all $\hat{i} \in \Sigma^M$, we have that there exist $a_1, a_2 > 0$ such that

$$a_1 |x| \leq V(\hat{i}, x) \leq a_2 |x|, \quad \forall \hat{i} \in \Sigma^M, \quad \forall x \in \mathbb{R}^n. \quad (12)$$

Inequalities (11) and (12) directly imply (9a) with $C = \frac{a_2}{a_1}$.

¹Once fixed an arbitrary symbol $j_\star \in \Sigma$, we suppose by convention that $\sigma(k) = j_\star$, for any $k \in [-M, -1]$ and any $\sigma \in \mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M})$.

Consider now any $\sigma \in \mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M})$. Suppose that, for some $k \in \mathbb{N}$, $k \geq M$ we have that $[\sigma]_{[k-M, k-1]} = \hat{j} \in \Sigma^M$, i.e. that σ was equal to the sequence \hat{j} in the previous M instants of time. Suppose that $\sigma(k) = i \in \Sigma$ and that $(\hat{j}^+, i) \in \Sigma_F^M$, i.e. (\hat{j}^+, i) is an accepting state in $\mathcal{G}_{\Sigma_F^M}$. Proceeding as in the previous case, we have that

$$\begin{aligned} &V([\sigma]_{[k+1-M, k]}, \Phi_\sigma(k+1, x)) \\ &= V((\hat{j}^+, i), (A_i + B_i K_{\hat{j}, i})\Phi_\sigma(k, x)) \\ &\leq \gamma \rho V(\hat{j}, \Phi_\sigma(k, x)) = \gamma \rho V([\sigma]_{[k-M, k-1]}, \Phi_\sigma(k, x)), \end{aligned}$$

where we used (8b), since \hat{i} is in Σ_F^M . Summarizing, we have proven that, for any $k \geq M$, if $[\sigma]_{[k-M-1, k]} \in \Sigma_F^M$ then the function $k \mapsto V([\sigma]_{[k-M, k-1]}, \Phi_\sigma(k, x))$ is decreasing by a factor $\gamma \rho$ at time-step k .

Moreover, we also note that, since we supposed that $k \geq M$, any accepting run for σ in $\mathcal{G}_{\Sigma_F^M}$ must pass through the node $\hat{j} \in \Sigma_F^M$ at time k and through the node (\hat{j}^+, i) at time $k+1$ (recall *Item 4.* in Properties 1). Recalling the definition of $\kappa^{\sigma, \mathcal{G}_{\Sigma_F^M}}$ in (3), and using the already proven (9a), this implies that, for all $k \in \mathbb{N}$, we have

$$V([\sigma]_{[k+1-M, k]}, \Phi_\sigma(k+1, x)) \leq \gamma^k \rho^{\kappa(k)} V([\sigma]_{[-M, -1]}, x),$$

Now recalling (12), by arbitrariness of σ and x , we obtain

$$|\Phi_\sigma(k, x)| \leq \frac{a_2}{a_1} \gamma^k \rho^{\kappa(k)} |x|,$$

for all $\sigma \in \mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M})$, for all $x \in \mathbb{R}^n$, and for all $k \in \mathbb{N}$, concluding the proof. ■

In the following statement we report how the conditions of Theorem 1 can be re-stated in the form of *linear matrix inequalities* (LMIs) depending on additional parameters $\rho \in [0, 1)$ and $\gamma > 0$.

Corollary 1: For given $\rho \in [0, 1)$ and $\gamma > 0$, there exist matrices $P_{\hat{j}} \in \mathbb{S}_+^{n \times n}$ and $K_{\hat{j}, i} \in \mathbb{R}^{n \times m}$ satisfying conditions (8a) and (8b) in Theorem 1 if and only if there exist $\bar{P}_{\hat{j}} \in \mathbb{S}_+^{n \times n}$ for all $\hat{j} \in \Sigma^M$ and $\bar{K}_{\hat{j}, i} \in \mathbb{R}^{n \times m}$ for all $\hat{j} \in \Sigma^M$ and all $i \in \Sigma$ such that the linear matrix inequalities

$$\begin{bmatrix} \bar{P}_{(\hat{j}^+, i)} & (A_i \bar{P}_{\hat{j}} + B_i \bar{K}_{\hat{j}, i}) \\ \star & \gamma^2 \bar{P}_{\hat{j}} \end{bmatrix} \succeq 0, \quad \forall \hat{j} \in \Sigma^M \quad \forall i \in \Sigma, \quad (13a)$$

$$\begin{bmatrix} \bar{P}_{(\hat{j}^+, i)} & (A_i \bar{P}_{\hat{j}} + B_i \bar{K}_{\hat{j}, i}) \\ \star & \gamma^2 \rho^2 \bar{P}_{\hat{j}} \end{bmatrix} \succeq 0 \quad (13b)$$

$$\forall \hat{j} \in \Sigma^M, \quad \forall i \in \Sigma \text{ such that } (\hat{j}^+, i) \in \Sigma_F^M$$

are feasible. Furthermore, $P_{\hat{j}}$ and $K_{\hat{j}}$ can be recovered from the solution to (13a) and (13b) by $P_{\hat{j}} = \bar{P}_{\hat{j}}^{-1}$ and $K_{\hat{j}, i} = \bar{K}_{\hat{j}, i} \bar{P}_{\hat{j}}^{-1}$.

Proof: The proof is trivial and follows from a Schur complement argument, given the fact that matrices $P_{\hat{j}}$ are positive definite and thus invertible. ■

Remark 2: We note that the feedback law $\Psi : \Sigma^* \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ introduced in Theorem 1, only depends on the past $M+1$ -values of the signal σ , and thus only uses finite and uniformly bounded information on the past of the signals, recall Remark 1.

Summarizing, if the hypotheses of Theorem 1 are satisfied (via Corollary 1), there is a prefix-dependent law which

ensures the bound (9a) for *arbitrary switching sequences*. If $\gamma \geq 1$ this is not insightful in terms of the stabilization problem, since it only provides an upper bound on the possible increase of the norm of solutions. Inequality (9b) instead provides a stricter bound, which holds only for sequences $\sigma \in \mathcal{L}(\mathcal{G}_{\Sigma_F^M})$, which relates the norm of the solutions to the return index. In order to achieve convergence to 0 we have to require a specific relation between the parameters γ , ρ and $\kappa^{\sigma, \mathcal{G}_{\Sigma_F^M}}$, and this is provided in what follows.

Corollary 2: Under the hypotheses of Theorem 1, given γ and ρ as in (9a)-(9b), consider any $\varepsilon > 0$, any $K \in \mathbb{N}$, and the set $\mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M}, \varepsilon, K) \subset \mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M})$ defined by

$$\mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M}, \varepsilon, K) := \left\{ \sigma \mid \kappa(k) \geq \left(\varepsilon - \frac{\log(\gamma)}{\log(\rho)} \right) k, \forall k \geq K \right\},$$

where we denote $\kappa(k) := \kappa^{\sigma, \mathcal{G}_{\Sigma_F^M}}(k)$, recall (3). Then, the prefix-dependent feedback map $\Psi : \Sigma^* \times \mathbb{R}^n \rightarrow \mathbb{R}^M$ defined in Theorem 1 solves Problem 1 for $\mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M}, \varepsilon, K)$, i.e., system (7) is $\mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M}, \varepsilon, K)$ -globally uniformly exponentially stable.

Proof: Let us consider any $\sigma \in \mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M}, \varepsilon, K)$, and thus

$$\kappa(k) \geq -\frac{\log(\gamma)}{\log(\rho)} k + \varepsilon k, \quad \forall k \geq K.$$

Supposing without loss of generality that $K \geq M$ using (9b) and since $\rho < 1$ we have

$$\begin{aligned} |\Phi_\sigma(k, x)| &\leq C \gamma^k \rho^{\kappa(k)} |x| \leq C \gamma^k \rho^{-\frac{\log(\gamma)}{\log(\rho)} k + \varepsilon k} |x| \\ &= C \rho^{\frac{\log(\gamma)}{\log(\rho)} k} \rho^{-\frac{\log(\gamma)}{\log(\rho)} k} \rho^{\varepsilon k} |x| \\ &= C \rho^{\varepsilon k} |x|, \quad \forall k \geq K. \end{aligned}$$

Since the growth of solutions to (7) is uniformly bounded up to time $K \in \mathbb{N}$, we can conclude that there exists a $\tilde{C} \geq 1$ such that $|\Phi_\sigma(k, x)| \leq \tilde{C} (\rho^\varepsilon)^k |x|$, $\forall \sigma \in \mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M}, \varepsilon, K)$, $\forall x \in \mathbb{R}^n$ and $\forall k \in \mathbb{N}$, concluding the proof. ■

The reasoning stemming from Corollaries 1 and 2, along with the potential trade-offs between the values of γ and ρ and their impact on the set $\kappa(k)$ (for which GUES holds), is not obvious and merits shedding light.

Upon closely inspecting inequalities (13a) and (13b), one realizes that increased values of $\gamma > 0$ tend to reduce the required value of $\rho \in (0, 1]$ ensuring satisfaction of (13b). Since ε is a free parameter in Corollary 2, choosing a larger ε minimizes the “true” decay rate given by ρ^ε . However, it should be noted that larger values of γ and ε adversely affect the size of the set $\mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M}, \varepsilon, K)$ of signals σ for which GUES holds with the designed control law. Therefore, several strategies could be developed in terms of searching values for γ and ρ satisfying (13a) and (13b), and choosing ε , either prioritizing the convergence speed or the size of the set $\mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^M}, \varepsilon, K)$. A numerical strategy that bridges these aspects with available control sequences will be discussed in Section IV.

C. Controllable sequences

The use of the class of automata introduced in previous subsections is motivated by the following discussion. Usually, the problem of stabilizing a switched system under arbitrary switching sequences can be unfeasible. For instance, a necessary condition for stabilization under arbitrary switching is the stabilizability of *all* subsystems (A_i, B_i) , since the constant signals are admissible. On the other hand, the switching among different subsystems can be beneficial in terms of reachability/stabilizability. With this motivation, inspired by [23], [24], we will briefly discuss the notions of *controllable sequences*. Given an initial state $x(0)$, an input map $u : \mathbb{N} \rightarrow \mathbb{R}^m$ and a $\sigma \in \Sigma^\omega$, the solution to (6) at time k , is denoted in what follows by $\Phi_\sigma(k, x, u)$.

Definition 4: The switched system (6) is *controllable*, if there exist a time instant $K \in \mathbb{N}$ and a switching sequence $\sigma_{[0, K-1]} \in \Sigma^K$ such that, for all $x \in \mathbb{R}^n$, there exists an input $u : \{0, \dots, K-1\} \rightarrow \mathbb{R}^m$ such that $\Phi_\sigma(K, x, u) = 0$. We refer to the finite sequence of modes $\sigma(0) \dots \sigma(K-1)$ as a *controllable sequence*.

This definition of controllability is inspired by [23]. For other notions of controllability, see [25] and references therein. We now introduce a matrix test for determining controllable sequences. Consider $\mathcal{A} = \{A_i\}_{i \in \Sigma}$ and $\mathcal{B} = \{B_i\}_{i \in \Sigma}$ and suppose that all the matrices in \mathcal{A} are non-singular. The *controllability matrix* corresponding to a sequence $\hat{i} = (i_0, i_1, \dots, i_{M-1}) \in \Sigma^M$ is defined as

$$\mathcal{C}(\hat{i}) = [A_{i_{M-1}} \dots A_{i_1} B_{i_0} \quad \dots \quad A_{i_{M-1}} B_{i_{M-2}} \quad B_{i_{M-1}}].$$

We recall the following result, which follows from [23, Theorems 3 & 4, Corollary 1].

Theorem 2: For a system (6) with A_i invertible for all $i \in \Sigma$, the following statements are equivalent:

- 1) A sequence of modes $\hat{i} \in \Sigma^M$ is a controllable sequence.
- 2) $\text{Rank}[\mathcal{C}(\hat{i})] = n$.

We thus have an algorithmic procedure to determine if a given sequence in Σ^M is controllable or not. In terms of Problem 1, controllable sequences arise as natural choices for the marked states $\hat{i} \in \Sigma_F^M$ for the numerical scheme proposed in Theorem 1 (Corollary 1 for its LMI version) and Corollary 2. In the following section we will illustrate this idea with the help of a numerical example.

IV. NUMERICAL EXAMPLE

We consider here an academic example, for which none of the subsystems is stabilizable, while all the “mixed” sequences of length 2 are controllable.

More precisely, let us consider a planar switched system (6) with 3 modes, defined by matrices $A_1 = A_2 = A_3 = I_2$, (i.e., the identity matrix of dimension 2) and $B_1 = [1 \ 0]^\top$, $B_2 = [0 \ 1]^\top$, $B_3 = [1 \ 1]^\top$. None of the pairs (A_i, B_i) , $i \in \Sigma = \{1, 2, 3\}$, is stabilizable in the classical LTI sense, and therefore the system cannot be stabilized under arbitrary switching signals $\sigma : \mathbb{N} \rightarrow \Sigma$. On the other hand, multiple *controllable sequences*, in the sense of Definition 4, exist. In particular, from the matrix

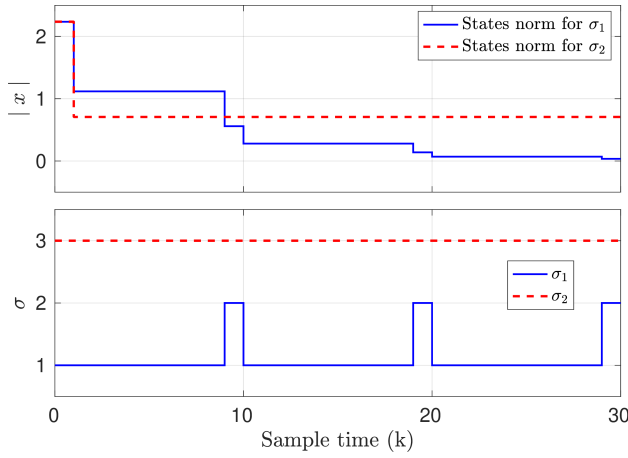


Fig. 2. Numerical example: Norms of the state and switching signals.

test defined in Theorem 2, any sequence $\hat{i} = (i_0, i_1) \in \Sigma^2$ with $i_0 \neq i_1$ is a controllable sequence. We refer to the set of such sequences as Σ_C^2 , meaning the set of all controllable sequences of length two. We study the feasibility of the LMI conditions in Corollary 1 under different scenarios concerning marked states $\hat{i} \in \Sigma_F^2 \subseteq \Sigma_C^2$. The structure of matrices (A_i, B_i) implies that the minimum feasible γ in (8a) is given by 1. Thus, we fix $\gamma = 1$ in all numerical tests and explore the influence of the set Σ_F^2 on the achievable lower-bound for ρ . Recall that, from Corollary 2, smaller ρ imply that GUES is assured for a larger set of switching signals $\sigma \in \mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^2}, \varepsilon, K)$, while also implying an improved bound on the converge rate.

At first, we consider $\Sigma_F^2 = \Sigma_C^2$. In this case, the smallest ρ for which the LMIs are feasible is 0.5. Consider another test set with $\Sigma_F^2 = \{(1, 3), (2, 3), (3, 2)\}$. In this case, $\rho = 0.013$ is the upper bound for which the LMIs are feasible. This same pattern is observed when using other sets of three sequences. When considering only one controllable sequence, for example, $\Sigma_F^2 = \{(1, 2)\}$, we can obtain the lower bound of $\rho = 0.01$.

To illustrate the effect of the designed control law, consider the simulation in Fig 2. We simulated the closed-loop system for an initial condition of $x(0) = [2 \ 1]^\top$ in two cases. In both cases, we consider the controller design with $\Sigma_F^2 = \Sigma_C^2$ and $\rho = 0.5$. In the first case, we consider a periodic switching signal (called here σ_1) that repeats the controllable sequence $\hat{i} = (1, 2)$ infinitely many times and is such that $\sigma_1 \in \mathcal{L}^\omega(\mathcal{G}_{\Sigma_F^2}, \varepsilon, K)$, for $K = 9$ and $\varepsilon = 0.001$. One can observe that every time the switching signal hits the sequence $\hat{i} = (1, 2)$, the state's norm decreases. In the same figure, we also plotted the norm of the state for a switching signal that repeats mode 3 infinitely. One can see that the corresponding trajectories are bounded, but the system does not converge to the origin since σ_2 never activates a marked sequence.

V. CONCLUSIONS

In this manuscript we provided a feedback design scheme for discrete-time switched systems. The overall control policy, valid uniformly over specific class of switching sequences, depends only on the current state and on a finite number of past values of the considered switching sequence.

As open route of future research, we aim to generalize this approach for more general classes of ω -regular language (and thus, for more general automata-structures), and to characterize the conservatism of the proposed approach.

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