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DEVIATION AND MOMENT INEQUALITIES FOR BANACH-VALUED U -STATISTICS

DAVIDE GIRAUDDO

*Institut de Recherche Mathématique Avancée UMR 7501, Université de Strasbourg and CNRS 7 rue
René Descartes 67000 Strasbourg, France*

ABSTRACT. We show a deviation inequality for U -statistics of independent data taking values in a separable Banach space which satisfies some smoothness assumptions. We then provide applications to rates in the law of large numbers for U -statistics, a Hölderian functional central limit theorem and a moment inequality for incomplete U -statistics.

1. DEVIATION INEQUALITIES FOR U -STATISTICS

1.1. Introduction and motivations. Furnishing a bound for the probability that a random variable is bigger than some fixed numbers plays a very important role in probability theory and its applications. It can be used in order to control the convergence rates in the law of large numbers, which can translate into consistency rates for estimators. Moreover, tightness criterion can be checked via such deviation inequalities.

Since weighted series of tails can be bounded by moments of a random variable, a direct application of deviation inequalities can provide the aforementioned results, in particular, without using truncation arguments.

We plan to furnish such inequalities for U -statistics, which are defined as follows. Let m be an integer, let $(\xi_i)_{i \in \mathbb{Z}}$ be an i.i.d. sequence with values in a measurable space (S, \mathcal{S}) and for $\mathbf{i} = (i_\ell)_{\ell=1}^m$ such that $1 \leq i_1 < \dots < i_m$, let $h_{\mathbf{i}}: S^m \rightarrow \mathbb{B}$ be measurable functions, where $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ is a separable Banach space and S^m is endowed with the product σ -algebra of \mathcal{S} . The associated U -statistic of order m is defined by

$$(1.1) \quad U_{m,n}((h_{\mathbf{i}})) = \sum_{\mathbf{i} \in \text{Inc}_n^m} h_{\mathbf{i}}(\xi_{\mathbf{i}}), \quad n \geq m,$$

where

$$(1.2) \quad \text{Inc}_n^m = \left\{ \mathbf{i} = (i_\ell)_{\ell \in \llbracket 1, m \rrbracket}, 1 \leq i_1 < i_2 < \dots < i_m \leq n \right\},$$

for $k \leq m$, $\llbracket k, m \rrbracket$ denotes the set $\{k, \dots, m\}$ and $\xi_{\mathbf{i}} = (\xi_{i_1}, \dots, \xi_{i_m})$.

Such a general framework allows to consider classical U -statistics (when $h_{\mathbf{i}}$ is independent of \mathbf{i}), weighted U -statistics and incomplete U -statistics (see Subsection 2.3 for a formal definition).

E-mail address: dgiraudod@unistra.fr.

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We would like to bound the tails of maxima of the norm of U_n , that is, we would like to find a bound for

$$(1.3) \quad \mathbb{P} \left(\max_{m \leq n \leq N} \|U_{m,n}((h_i))\|_{\mathbb{B}} > t \right)$$

in terms of the tail of a suitable random variable.

Exponential bounds have been investigated in the literature, see for instance [8, 4, 2, 26, 15]. The case of not necessarily independent random variables has also been addressed in [13, 24, 42, 6]. In this paper, we shall present inequalities for U -statistics taking values in Banach spaces and such that the tail of the random variables $\|h_i(\xi_i)\|_{\mathbb{B}}$ decays like a power function.

For real-valued martingales difference sequences, it is known that we can control the tail of maxima of partial sums via the tail of maxima and the sum of conditional variances (see [37]). More precisely, given a positive q , there exists a constant C_q such that for each real-valued martingale difference sequence $(D_i)_{i \geq 1}$ with respect to a filtration $(\mathcal{F}_i)_{i \geq 0}$ and each positive t ,

$$(1.4) \quad \mathbb{P} \left(\max_{1 \leq n \leq N} \left| \sum_{i=1}^n D_i \right| > t \right) \\ \leq C_q \int_0^1 u^{q-1} \mathbb{P} \left(\max_{1 \leq i \leq N} |D_i| > tu \right) du + C_q \int_0^1 u^{q-1} \mathbb{P} \left(\left(\sum_{i=1}^N \mathbb{E} [D_i^2 | \mathcal{F}_{i-1}] \right)^{1/2} > tu \right) du.$$

A Banach-valued version with not necessarily finite variance is given in Theorem 1.3 of [19]. Such inequalities are very convenient to use, especially in the case where the random variables $|D_i|$ have the same distribution. In this case, (1.4) gives for each $q > 2$,

$$\mathbb{P} \left(\max_{1 \leq n \leq N} \left| \sum_{i=1}^n D_i \right| > t\sqrt{N} \right) \leq C_q \int_0^\infty \mathbb{P}(|D_1| > tu) \min \{u^{q-1}, u\} du \leq C'_q \mathbb{E} \left[\min \left\{ \frac{|D_1|^q}{t^q}, \frac{D_1^2}{t^2} \right\} \right],$$

where C'_q depends only on q (Theorem 1.7 of [19], where a version for random variables with infinite variance is also presented). Then one can deduce convergence of series of the form $A := \sum_{N=1}^\infty N^\alpha \mathbb{P}(\max_{1 \leq n \leq N} |\sum_{i=1}^n D_i| > N^\beta)$ by series involving only the tail of $|D_1|$ and get thanks to the elementary inequality $\sum_{N=1}^\infty 2^{Np} \mathbb{P}(Y > 2^N) \leq C_p \mathbb{E}[Y^p]$, an integrability condition on D_1 to get the convergence of the series A . This leads to the obtainment of convergence rates in the strong law of large numbers. Moreover, it is possible to check some tightness criteria in Hölder spaces for partial sum processes which are expressed in terms of tails (see equation (1.4) in [18] and Proposition 1.1 in [20]).

Motivated by these applications, we would like to formulate inequalities of the form (1.4) for U -statistics defined as in (1.1) and find bounds expressible as $\int_0^1 u^{q-1} \mathbb{P}(Y_k > tu) du$, where Y_k are non-negative random variables related to the kernels h_i and the random variables ξ_1, \dots, ξ_m . Under some conditions of degeneracy (see Definition 1.1), the considered U -statistic is a sum of martingale differences with respect to each index of summation. However, the term corresponding to condition variances can only be expressed as a random variable conditioned with respect to some σ -algebra. This make the iteration of inequality (1.4) difficult and led us to control the conditional moments of a martingale difference sequence. Also, we will use reasoning by induction on the order m of the U -statistic and it turns out that the consideration of Banach-valued random variables helps since the conditional variance term that arises can be viewed as a martingale difference sequence in an other Banach space (such an idea was used in [22] for multi-indexed martingales).

Once the wanted deviation inequality is established, one can readily derive a moment inequality which is in the spirit of the one obtained in [27], but for real-valued U -statistics of order two. For

Banach-valued random variables, a result has been obtained in [1], where the upper bound is expressed in terms of operator norm.

1.2. A deviation inequality for U -statistics. In order to formulate deviation inequalities for $\|U_n\|_{\mathbb{B}}$ and $\max_{m \leq n \leq N} \|U_n\|_{\mathbb{B}}$, one has to make assumptions on the random variables $h_i(\xi_{i_1}, \dots, \xi_{i_m})$, even if this random variable is centered. To see this, consider the case $\mathbb{B} = \mathbb{R}$ and $m = 2$. If $h_{i_1, i_2}(x, y) = x + y$ then $U_n = (n-1) \sum_{i=1}^n X_i$ while with $h_{i_1, i_2}(x, y) = xy$ we have $U_n = 1/2 (\sum_{i=1}^n X_i)^2 - \sum_{i=1}^n X_i^2$ hence the behavior of the tails will change drastically.

In order to formulate general results, we will adopt the following definition.

Definition 1.1. We say that a function $h: S^m \rightarrow \mathbb{B}$ is degenerated with respect to the i.i.d. S -valued sequence $(\xi_i)_{i \in \mathbb{Z}}$ if for each $\ell_0 \in \llbracket 1, m \rrbracket$,

$$(1.5) \quad \mathbb{E} [h(\xi_{\llbracket 1, m \rrbracket}) \mid \xi_{\llbracket 1, m \rrbracket \setminus \{\ell_0\}}] = 0,$$

where

$$(1.6) \quad \xi_J = (\xi_j)_{j \in J}.$$

We say that a U -statistic of the form (1.1) is degenerated with respect to the sequence $(\xi_i)_{i \in \mathbb{Z}}$ if all the functions h_i , $\mathbf{i} \in \text{Inc}^m := \left\{ \mathbf{i} = (i_\ell)_{\ell \in \llbracket 1, m \rrbracket}, 1 \leq i_1 < \dots < i_m \right\}$ are degenerated with respect to the sequence $(\xi_i)_{i \in \mathbb{Z}}$.

However, degeneracy of U_n does not always hold, like the previous example $h_{i_1, i_2}(x, y) = x + y$ shows. Nevertheless, it is possible to express U_n as sum of degenerated U -statistics of smaller order. Indeed, for a subset I of $\llbracket 1, m \rrbracket$, let us denote by $|I|$ the cardinal of I and define the function $h_{\mathbf{i}}^I: S^{|I|} \rightarrow \mathbb{B}$ by

$$(1.7) \quad h_{\mathbf{i}}^I((x_{i_u})_{u \in I}) = \sum_{J: J \subseteq I} (-1)^{|I| - |J|} \mathbb{E} [h_{\mathbf{i}}(V^{I, J}((x_{i_u})_{u \in I}))],$$

where the random vector $V^{I, J}((x_{i_u})_{u \in I})$ belongs to S^m , the k -th coordinates is x_{i_u} if $k = i_u$ for some (hence exactly one) $u \in J$ and ξ_k if k is not of this form. In this way, writing $\mathbf{i}_J = (i_\ell)_{\ell \in J}$ the equality

$$(1.8) \quad h_{\mathbf{i}}^I((\xi_{i_u})_{u \in I}) = \sum_{J: J \subseteq I} (-1)^{|I| - |J|} \mathbb{E} [h_{\mathbf{i}}(\xi_{\mathbf{i}}) \mid \xi_{\mathbf{i}_J}]$$

holds almost surely, where $\mathbb{E}[\cdot \mid \xi_{\mathbf{i}_0}] = \mathbb{E}[\cdot]$. Moreover, the equality

$$(1.9) \quad \sum_{I \subset \llbracket 1, m \rrbracket} h_{\mathbf{i}}^I(\xi_{\mathbf{i}_I}) = h_{\mathbf{i}}(\xi_{\mathbf{i}})$$

takes places (to see this, one can switch the sums over I and J and use the fact that $\sum_{I: J \subseteq I} (-1)^{|I| - |J|}$ equals 0 if J is not $\llbracket 1, m \rrbracket$ and 1 otherwise). As a consequence, the U -statistic U_n defined by (1.1) can be decomposed as

$$(1.10) \quad U_n = \sum_{I \subset \llbracket 1, m \rrbracket} U_n^I,$$

where $U_n^\emptyset = \sum_{\mathbf{i} \in \text{Inc}_n^m} \mathbb{E} [h_{\mathbf{i}}(\xi_{\llbracket 1, m \rrbracket})]$ and for a non-empty $I \subset \llbracket 1, m \rrbracket$,

$$(1.11) \quad U_n^I = \sum_{\mathbf{j}_I \in \text{Inc}_n^{|I|}} H_{\mathbf{j}_I}^I(\xi_{\mathbf{j}_I}),$$

$$(1.12) \quad H_{\mathbf{j}_I}^I(\xi_{\mathbf{j}_I}) = \sum_{\mathbf{i}_{\llbracket 1, m \rrbracket \setminus I}} h_{\sum_{\ell \in I} j_\ell \mathbf{e}_\ell + \sum_{\ell' \in \llbracket 1, m \rrbracket \setminus I} i_{\ell'} \mathbf{e}_{\ell'}}(\xi_{\mathbf{j}_I}),$$

\mathbf{e}_ℓ is a vector whose entry ℓ is one and the others zero, and the sum runs over the $\mathbf{i}_{[1,m] \setminus I}$ such that

$$(1.13) \quad \sum_{\ell \in I} j_\ell \mathbf{e}_\ell + \sum_{\ell' \in [1,m] \setminus I} i_{\ell'} \mathbf{e}_{\ell'} \in \text{Inc}_n^{|I|}.$$

In this way, the U -statistic U_n^I has order $|I|$ and is degenerated. When $h_{\mathbf{i}}$ does not depend on \mathbf{i} , this was done in Theorem 9.1 in [33].

For $m = 2$, the decomposition reads

$$(1.14) \quad U_n = U_n^\emptyset + U_n^{\{1\}} + U_n^{\{2\}} + U_n^{\{1,2\}},$$

where

$$(1.15) \quad U_n^\emptyset = \sum_{1 \leq i_1 < i_2 \leq n} \mathbb{E}[h_{i_1, i_2}(\xi_1, \xi_2)],$$

$$(1.16) \quad U_n^{\{1\}} = \sum_{i_1=1}^{n-1} \sum_{j_2=i_1+1}^n \mathbb{E}[h_{i_1, j_2}(\xi_{i_1}, \xi_{j_2}) \mid \xi_{i_1}],$$

$$(1.17) \quad U_n^{\{2\}} = \sum_{i_2=2}^n \sum_{j_1=1}^{i_2-1} \mathbb{E}[h_{j_1, i_2}(\xi_{j_1}, \xi_{i_2}) \mid \xi_{i_2}],$$

$$(1.18) \quad U_n^{\{1,2\}} = \sum_{1 \leq i_1 < i_2 \leq n} (h_{i_1, i_2}(\xi_{i_1}, \xi_{i_2}) - \mathbb{E}[h_{i_1, i_2}(\xi_{i_1}, \xi_{i_2}) \mid \xi_{i_1}] + \\ - \mathbb{E}[h_{i_1, i_2}(\xi_{i_1}, \xi_{i_2}) \mid \xi_{i_2}] + \mathbb{E}[h_{i_1, i_2}(\xi_{i_1}, \xi_{i_2})]).$$

Note that degenerated U -statistics enjoy a martingale property. It is known that the geometry of the involved Banach space plays a important role in the derivation of moments and deviation inequalities.

Definition 1.2. Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable Banach space. We say that \mathbb{B} is r -smooth for $1 < r \leq 2$ if there exists an equivalent norm $\|\cdot\|'_{\mathbb{B}}$ on \mathbb{B} such that

$$(1.19) \quad \sup_{t>0} \sup_{x, y \in \mathbb{B}, \|x\|'_{\mathbb{B}} = \|y\|'_{\mathbb{B}} = 1} \frac{\|x + ty\|'_{\mathbb{B}} + \|x - ty\|'_{\mathbb{B}} - 2}{t^r} < \infty.$$

By [3], we know that if \mathbb{B} is a separable r -smooth Banach space, then there exists a constant C such that for each martingale difference sequence $(D_i)_{i \geq 1}$ with values in \mathbb{B} and each n ,

$$(1.20) \quad \mathbb{E} \left[\left\| \sum_{i=1}^n D_i \right\|_{\mathbb{B}}^r \right] \leq C \sum_{i=1}^n \mathbb{E} [\|D_i\|_{\mathbb{B}}^r].$$

By definition, an r -smooth Banach space is also p -smooth for $1 < p \leq r$, hence it is possible to define

$$(1.21) \quad C_{p, \mathbb{B}} := \sup_{n \geq 1} \sup_{(D_i)_{i=1}^n \in \Delta_n} \frac{\mathbb{E} [\|\sum_{i=1}^n D_i\|_{\mathbb{B}}^p]}{\sum_{i=1}^n \mathbb{E} [\|D_i\|_{\mathbb{B}}^p]},$$

where the Δ_n denotes the set of the martingale difference sequences $(D_i)_{i=1}^n$ such that $\sum_{i=1}^n \|D_i\|_{\mathbb{B}}^p$ is not identically 0.

We will now state a general deviation inequality for Banach valued U -statistics. The obtained bound involves 2^m terms. For each subset J of $[1, m]$, we sum over the indices in J a function of the tail of sums of conditional p -th moments of $\|h_{\mathbf{i}}(\xi_{[1,m]})\|_{\mathbb{B}}$ over the coordinates in $[1, m] \setminus J$.

For a proper non-empty subset J of $\llbracket 1, m \rrbracket$, we define the subset of \mathbb{N}^m , denoted $\mathbb{N}^{J,m}$, by $\mathbb{N}^{J,m} = \left\{ \sum_{j \in J} v_j \mathbf{e}_j, v_j \in \mathbb{N} \right\}$, where \mathbf{e}_j is the element of \mathbb{N}^m having 1 at coordinate j and 0 at the others. For a subset J of $\llbracket 1, m \rrbracket$, recall the notation ξ_J given by (1.6).

The key deviation inequality on which all the results of the paper rest reads as follows.

Theorem 1.3. *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be an r -smooth Banach space for some $r \in (1, 2]$. Let $(\xi_i)_{i \geq 1}$ be an i.i.d. sequence taking values in a measurable space (S, \mathcal{S}) , let $h_i: S^m \rightarrow \mathbb{B}$ be degenerated with respect to $(\xi_i)_{i \geq 1}$, that is, such that for each $\ell_0 \in \llbracket 1, m \rrbracket$,*

$$(1.22) \quad \mathbb{E} [h_i (\xi_{\llbracket 1, m \rrbracket}) \mid \xi_{\llbracket 1, m \rrbracket \setminus \{\ell_0\}}] = 0.$$

The following inequality takes place for each $p \in (1, r]$ and each positive t and q :

$$(1.23) \quad \mathbb{P} \left(\max_{m \leq n \leq N} \left\| \sum_{\mathbf{i} \in \text{Inc}_n^m} h_{\mathbf{i}} (\xi_{\mathbf{i}}) \right\|_{\mathbb{B}} > t \right) \\ \leq K(m, p, q, \mathbb{B}) \sum_{\mathbf{i} \in \text{Inc}_N^m} \int_0^1 u^{q-1} \mathbb{P} \left(\|h_{\mathbf{i}} (\xi_{\llbracket 1, m \rrbracket})\|_{\mathbb{B}} > tu \right) du \\ + K(m, p, q, \mathbb{B}) \sum_{\emptyset \subsetneq J \subsetneq \llbracket 1, m \rrbracket} \sum_{\mathbf{i}_J \in \mathbb{N}^{J,m}} \int_0^1 u^{q-1} \mathbb{P} \left(\left(\sum_{\mathbf{i}_{J^c}: \mathbf{i}_J + \mathbf{i}_{J^c} \in \text{Inc}_N^m} \mathbb{E} \left[\|h_{\mathbf{i}_J + \mathbf{i}_{J^c}} (\xi_{\llbracket 1, m \rrbracket})\|_{\mathbb{B}}^p \mid \xi_J \right] \right)^{1/p} > tu \right) du \\ + K(m, p, q, \mathbb{B}) t^{-q} \left(\sum_{\mathbf{i} \in \text{Inc}_N^m} \mathbb{E} \left[\|h_{\mathbf{i}} (\xi_{\llbracket 1, m \rrbracket})\|_{\mathbb{B}}^p \right] \right)^{q/p},$$

where $K(m, p, q, \mathbb{B})$ depends only on m, p, q and \mathbb{B} .

For U -statistics of order two, Theorem 1.3 reads as follows: if $(\xi_i)_{i \in \mathbb{Z}}$ is an i.i.d. sequence taking values in (S, \mathcal{S}) and $h_{i_1, i_2}: S^2 \rightarrow \mathbb{B}$ is such that for each $i_1 < i_2$,

$$(1.24) \quad \mathbb{E} [h_{i_1, i_2} (\xi_1, \xi_2) \mid \xi_1] = \mathbb{E} [h_{i_1, i_2} (\xi_1, \xi_2) \mid \xi_2] = 0,$$

then

$$(1.25) \quad \mathbb{P} \left(\max_{2 \leq n \leq N} \left\| \sum_{1 \leq i_1 < i_2 \leq n} h_{i_1, i_2} (\xi_{i_1}, \xi_{i_2}) \right\|_{\mathbb{B}} > t \right) \\ \leq K(2, p, q, \mathbb{B}) \sum_{1 \leq i_1 < i_2 \leq N} \int_0^1 u^{q-1} \mathbb{P} (\|h_{i_1, i_2} (\xi_1, \xi_2)\|_{\mathbb{B}} > tu) du \\ + K(2, p, q, \mathbb{B}) \sum_{i_1=1}^{N-1} \int_0^1 u^{q-1} \mathbb{P} \left(\left(\sum_{i_2=i_1+1}^N \mathbb{E} [\|h_{i_1, i_2} (\xi_1, \xi_2)\|_{\mathbb{B}}^p \mid \xi_1] \right)^{1/p} > tu \right) du \\ + K(2, p, q, \mathbb{B}) \sum_{i_2=2}^N \int_0^1 u^{q-1} \mathbb{P} \left(\left(\sum_{i_1=1}^{i_2-1} \mathbb{E} [\|h_{i_1, i_2} (\xi_1, \xi_2)\|_{\mathbb{B}}^p \mid \xi_2] \right)^{1/p} > tu \right) du \\ + K(2, p, q, \mathbb{B}) t^{-q} \left(\sum_{1 \leq i_1 < i_2 \leq N} \mathbb{E} [\|h_{i_1, i_2} (\xi_1, \xi_2)\|_{\mathbb{B}}^p] \right)^{q/p}.$$

When the kernel functions h_i are independent of the index i , Theorem 1.3 admits the following simpler form.

Corollary 1.4. *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be an r -smooth Banach space for some $r \in (1, 2]$. Let $(\xi_i)_{i \geq 1}$ be an i.i.d. sequence taking values in a measurable space (S, \mathcal{S}) , let $h: S^m \rightarrow \mathbb{B}$ be degenerated with respect to $(\xi_i)_{i \geq 1}$, that is, for each $\ell_0 \in \llbracket 1, m \rrbracket$,*

$$(1.26) \quad \mathbb{E} [h(\xi_{\llbracket 1, m \rrbracket}) \mid \xi_{\llbracket 1, m \rrbracket \setminus \{\ell_0\}}] = 0.$$

The following inequality takes place for each $p \in (1, r]$ and each positive t and q :

$$(1.27) \quad \mathbb{P} \left(\max_{m \leq n \leq N} \left\| \sum_{i \in \text{Inc}_n^m} h(\xi_i) \right\|_{\mathbb{B}} > t \right) \\ \leq K(m, p, q, \mathbb{B}) N^m \int_0^1 u^{q-1} \mathbb{P} \left(\|h(\xi_{\llbracket 1, m \rrbracket})\|_{\mathbb{B}} > tu \right) du \\ + K(m, p, q, \mathbb{B}) \sum_{\emptyset \subsetneq J \subsetneq \llbracket 1, m \rrbracket} N^{|J|} \int_0^1 u^{q-1} \mathbb{P} \left(N^{\frac{m-|J|}{p}} \left(\mathbb{E} \left[\|h(\xi_{\llbracket 1, m \rrbracket})\|_{\mathbb{B}}^p \mid \xi_{\ell}, \ell \in J \right] \right)^{1/p} > tu \right) du \\ + K(m, p, q, \mathbb{B}) t^{-q} N^{mq/p} \left(\mathbb{E} \left[\|h(\xi_{\llbracket 1, m \rrbracket})\|_{\mathbb{B}}^p \right] \right)^{q/p},$$

where $K(m, p, q, \mathbb{B})$ depends only on m, p, q and \mathbb{B} .

In order to address the non-necessarily degenerated case, some assumptions and definitions are required.

Definition 1.5. *We say that the kernel $h: S^m \rightarrow \mathbb{B}$ is symmetric if for each bijection $\sigma: \llbracket 1, m \rrbracket \rightarrow \llbracket 1, m \rrbracket$ and each $x_1, \dots, x_m \in S$,*

$$(1.28) \quad h(x_{\sigma(1)}, \dots, x_{\sigma(m)}) = h(x_1, \dots, x_m).$$

Definition 1.6. *Let $d \in \llbracket 1, m \rrbracket$ and let $(\xi_i)_{i \in \mathbb{Z}}$ be an i.i.d. sequence taking values in S . We say that the symmetric kernel $h: S^m \rightarrow \mathbb{B}$ is degenerated of order d with respect to $(\xi_i)_{i \in \mathbb{Z}}$ if*

$$(1.29) \quad \mathbb{E} [h(\xi_{\llbracket 1, m \rrbracket}) \mid \xi_{\llbracket 1, d-1 \rrbracket}] = 0 \text{ a.s. and } \mathbb{E} \left[\left\| \mathbb{E} [h(\xi_{\llbracket 1, m \rrbracket}) \mid \xi_{\llbracket 1, d \rrbracket}] \right\|_{\mathbb{B}} \right] \neq 0.$$

If h is degenerated of order $d-1$ with respect to $(\xi_i)_{i \in \mathbb{Z}}$, it is possible to write

$$(1.30) \quad \sum_{i \in \text{Inc}_n^m} h(\xi_i) = \binom{n}{m} \sum_{c=d}^m \binom{m}{c} \binom{n}{c}^{-1} \sum_{i \in \text{Inc}_n^c} h^{(c)}(\xi_i),$$

where

$$(1.31) \quad h^{(c)}(x_1, \dots, x_c) = \sum_{k=0}^c (-1)^{c-k} \sum_{i \in \text{Inc}_c^k} \mathbb{E} [h(x_i, \xi_{\llbracket 1, m-k \rrbracket})].$$

In other words, the U -statistic of kernel h can be written as a weighted sum of $(m-d+1)$ U -statistics, each of them being degenerated.

We are now in position to provide a bound for U -statistics having a symmetric but not necessarily degenerated kernel.

Corollary 1.7. Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be an r -smooth Banach space for some $r \in (1, 2]$. Let $(\xi_i)_{i \geq 1}$ be an i.i.d. sequence taking values in a measurable space (S, \mathcal{S}) , let $h: S^m \rightarrow \mathbb{B}$ be a symmetric function with is degenerated of order d with respect to $(\xi_i)_{i \geq 1}$. Define

$$(1.32) \quad H_p := \max_{k \in \llbracket 0, m \rrbracket} \left(\mathbb{E} \left[\|h(\xi_{\llbracket 1, m \rrbracket})\|_{\mathbb{B}}^p \mid \xi_{\llbracket 1, k \rrbracket} \right] \right)^{1/p}.$$

The following inequality takes place for each $p \in (1, r]$ and each positive t and q :

$$(1.33) \quad \mathbb{P} \left(\max_{m \leq n \leq N} \left\| \sum_{\mathbf{i} \in \text{Inc}_n^m} h(\xi_{\mathbf{i}}) \right\|_{\mathbb{B}} > tN^{m-d+\frac{d}{p}} \right) \leq K(m, p, q, \mathbb{B}) \sum_{j=0}^m N^j \int_0^1 u^{q-1} \mathbb{P} \left(H_p > tN^{(\max\{d, j\}-d)\frac{p-1}{p} + \frac{j}{p}u} \right) du.$$

Notice that the term of index $j \in \llbracket 1, d \rrbracket$ in the right hand side of (1.33) the term is of the form $N^j \int_0^1 u^{q-1} \mathbb{P} \left(H_p > tN^{\frac{j}{p}u} \right) du$, which is independent of N for $j = 0$ and for $j \in \llbracket d+1, m \rrbracket$, the corresponding term is $N^j \int_0^1 u^{q-1} \mathbb{P} \left(H_p > tN^{j-d+d/p}u \right) du$. All these terms bring a different and not easily comparable contribution.

1.3. A moment inequality. One can deduce from Theorem 1.3 a moment inequality.

Corollary 1.8. Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be an r -smooth Banach space for some $r \in (1, 2]$. Let $(\xi_i)_{i \geq 1}$ be an i.i.d. sequence taking values in a measurable space (S, \mathcal{S}) , let $h_{\mathbf{i}}: S^m \rightarrow \mathbb{B}$ be degenerated with respect to $(\xi_i)_{i \geq 1}$ and such that for each $\ell_0 \in \llbracket 1, m \rrbracket$,

$$(1.34) \quad \mathbb{E} [h_{\mathbf{i}}(\xi_{\llbracket 1, m \rrbracket}) \mid \xi_{\llbracket 1, m \rrbracket} \setminus \{\ell_0\}}] = 0.$$

For each $q \geq p$, the following inequality holds:

$$(1.35) \quad \mathbb{E} \left[\max_{m \leq n \leq N} \left\| \sum_{\mathbf{i} \in \text{Inc}_n^m} h_{\mathbf{i}}(\xi_{\mathbf{i}}) \right\|_{\mathbb{B}}^q \right] \leq K(m, p, q, \mathbb{B}) \sum_{\mathbf{i} \in \text{Inc}_N^m} \mathbb{E} \left[\|h_{\mathbf{i}}(\xi_{\llbracket 1, m \rrbracket})\|_{\mathbb{B}}^q \right] + K(m, p, q, \mathbb{B}) \sum_{\emptyset \subsetneq J \subsetneq \llbracket 1, m \rrbracket} \sum_{\mathbf{i}_J \in \mathbb{N}^J} \mathbb{E} \left[\left(\sum_{\mathbf{i}_J \mathbf{c}: \mathbf{i}_J + \mathbf{i}_J \mathbf{c} \in \text{Inc}_N^m} \mathbb{E} \left[\|h_{\mathbf{i}_J + \mathbf{i}_J \mathbf{c}}(\xi_{\llbracket 1, m \rrbracket})\|_{\mathbb{B}}^p \mid \xi_{\ell}, \ell \in J \right] \right)^{q/p} \right] + K(m, p, q, \mathbb{B}) \left(\sum_{\mathbf{i} \in \text{Inc}_N^m} \mathbb{E} \left[\|h_{\mathbf{i}}(\xi_{\llbracket 1, m \rrbracket})\|_{\mathbb{B}}^p \right] \right)^{q/p}.$$

When $m = 2$, the assumption (1.34) admits the simpler form (1.24) and (1.35) reads as follows:

$$(1.36) \quad \mathbb{E} \left[\max_{2 \leq n \leq N} \left\| \sum_{1 \leq i < j \leq n} h_{i,j}(\xi_i, \xi_j) \right\|_{\mathbb{B}}^q \right] \leq K(2, p, q, \mathbb{B}) \sum_{1 \leq i < j \leq N} \mathbb{E} \left[\|h_{i,j}(\xi_1, \xi_2)\|_{\mathbb{B}}^q \right] + K(2, p, q, \mathbb{B}) \sum_{i=1}^{N-1} \mathbb{E} \left[\left(\sum_{j=i+1}^N \mathbb{E} \left[\|h_{i,j}(\xi_1, \xi_2)\|_{\mathbb{B}}^p \mid \xi_1 \right] \right)^{q/p} \right] + K(2, p, q, \mathbb{B}) \sum_{j=2}^N \mathbb{E} \left[\left(\sum_{i=1}^{j-1} \mathbb{E} \left[\|h_{i,j}(\xi_1, \xi_2)\|_{\mathbb{B}}^p \mid \xi_2 \right] \right)^{q/p} \right]$$

$$+ K(2, p, q, \mathbb{B}) \left(\sum_{1 \leq i < j \leq N} \mathbb{E} [\|h_{i,j}(\xi_1, \xi_2)\|_{\mathbb{B}}^p] \right)^{q/p}.$$

Notice also that for $q = p$, (1.35) reads as follows: for $1 < p \leq r$,

$$(1.37) \quad \mathbb{E} \left[\max_{m \leq n \leq N} \left\| \sum_{i \in \text{Inc}_n^m} h_i(\xi_i) \right\|_{\mathbb{B}}^p \right] \leq K(m, p, \mathbb{B}) \sum_{i \in \text{Inc}_N^m} \mathbb{E} [\|h_i(\xi_{[1,m]})\|_{\mathbb{B}}^p].$$

In particular, if $h_i = h$ and (1.26) holds, then

$$(1.38) \quad \mathbb{E} \left[\max_{m \leq n \leq N} \left\| \sum_{i \in \text{Inc}_n^m} h(\xi_i) \right\|_{\mathbb{B}}^p \right] \leq K(m, p, \mathbb{B}) N^m \mathbb{E} [\|h(\xi_{[1,m]})\|_{\mathbb{B}}^p].$$

Moreover, the deviation inequality (1.23) allows to derive a Rosenthal type inequality for weak \mathbb{L}^q -moments: define for a random variable Y taking values in a Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$, and $q > 1$ the quantity

$$(1.39) \quad \|Y\|_{\mathbb{B}, q, w} := \left(\sup_{t > 0} t^q \mathbb{P}(\|Y\|_{\mathbb{B}} > t) \right).$$

Then applying (1.23) with q replaced by $q + 1$ gives (under the assumptions of Theorem 1.3)

$$(1.40) \quad \left\| \max_{m \leq n \leq N} \left\| \sum_{i \in \text{Inc}_n^m} h_i(\xi_i) \right\|_{\mathbb{B}} \right\|_{\mathbb{R}, q, w}^q \leq K(m, p, q, \mathbb{B}) \sum_{i \in \text{Inc}_N^m} \|h_i(\xi_{[1,m]})\|_{\mathbb{B}, q, w}^q \\ + K(m, p, q, \mathbb{B}) \sum_{\emptyset \subsetneq J \subseteq [1, m]} \sum_{i_J \in \mathbb{N}^J} \left\| \left(\sum_{i_{J^c}: i_J + i_{J^c} \in \text{Inc}_N^m} \mathbb{E} [\|h_{i_J + i_{J^c}}(\xi_{[1,m]})\|_{\mathbb{B}}^p \mid \xi_\ell, \ell \in J] \right)^{1/p} \right\|_{\mathbb{R}, q, w}^q \\ + K(m, p, q, \mathbb{B}) \left(\sum_{i \in \text{Inc}_N^m} \mathbb{E} [\|h_i(\xi_{[1,m]})\|_{\mathbb{B}}^p] \right)^{q/p}.$$

Let us connect these results with existing ones in the literature.

- In [36], a Rosenthal type inequality for Hermitian matrix-valued U -statistics was obtained. The presented bound for the moments of such a U -statistic has explicit constant, and the terms in the spirit of second the right hand side of (1.36) involves the moment of order q of the maximum over i instead of the sum of moments. However, our result deals with vector-valued random variables and U -statistics of any order.
- In [1], a moment inequality for U -statistics taking values in a separable Banach space is given. On the one hand, the assumption on the Banach space (none in Theorem 1, type 2 in Theorem 2) is less restrictive than ours and the constants are explicit. The bounds are formulated in terms of moments of some random variable which are less explicit those involved in the right hand side of (1.35). Moreover, we deal with the case where $h_i(\xi_{[1,m]})$ does not necessarily admit a moment of order 2 hence our result can be viewed as a complement of that of [1]. A moment inequality in the same spirit has also been obtained in [15].
- In [11], a moment inequality for U -statistics having non-negative kernels is established. As mentioned by the authors in Remark 6, page 982, a Rosenthal type inequalities can be derived but it does not seem to be easily comparable with ours.

- Rosenthal type inequalities were also established in [27]. They concern the real-valued case and U -statistics of order two, while our inequality is for Banach valued U -statistics of arbitrary order. However, they do not assume that the random variables $(\xi_i)_{i \geq 1}$ have the same distribution.

2. APPLICATIONS

2.1. Rates in the law of large numbers for complete U -statistics. It is known that if $h: S^m \rightarrow \mathbb{R}$ and $(\xi_i)_{i \geq 1}$ is an i.i.d. sequence for which $\mathbb{E} [|h(\xi_{[1,m]})|] < \infty$, then $U_{m,n}(h) \rightarrow 0$ a.s. (see [25]). Convergence under the assumption of finiteness of higher moments of $h(\xi_{[1,m]})$ has been investigated in [34, 7, 23, 16, 41, 43].

The first result of this subsection is a sufficient condition for the Marcinkiewicz strong law of large numbers for Banach valued degenerated U -statistics.

Theorem 2.1. *Let $m \geq 2$ be an integer, let $(\xi_i)_{i \geq 1}$ be an i.i.d. sequence taking values in a measurable space (S, \mathcal{S}) , and let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be an r -smooth separable Banach space for $r \in (1, 2]$. Let $h: S^m \rightarrow \mathbb{B}$ be a measurable function such that for each $\ell_0 \in [1, m]$,*

$$(2.1) \quad \mathbb{E} [h(\xi_{[1,m]}) \mid \xi_{[1,m] \setminus \{\ell_0\}}] = 0.$$

For each $1 < p < r$, there exists a constant $K(p, \mathbb{B})$ depending only on p and \mathbb{B} such that

$$(2.2) \quad \sup_{t>0} t^p \mathbb{P} \left(\sup_{n \geq m} \frac{1}{n^{m/p}} \left\| \sum_{i \in \text{Inc}_n^m} h(\xi_i) \right\|_{\mathbb{B}} > t \right) \leq K(p, \mathbb{B}) \mathbb{E} \left[\|h(\xi_{[1,m]})\|_{\mathbb{B}}^p \right].$$

Moreover,

$$(2.3) \quad \lim_{n \rightarrow \infty} \frac{1}{n^{m/p}} \left\| \sum_{i \in \text{Inc}_n^m} h(\xi_i) \right\|_{\mathbb{B}} = 0 \text{ a.s.}$$

Such a result was known when $m = 2$ and $\mathbb{B} = \mathbb{R}$ (see Proposition 1.2 in [21]). Notice that we do not need symmetry of the kernel h . Note that this is not a direct application of the established deviation inequality.

We now present some results on the rates in the strong law of large numbers, which can be derived from the deviation inequality we propose.

Theorem 2.2. *Let $m \geq 2$ be an integer, let $(\xi_i)_{i \geq 1}$ be an i.i.d. sequence taking values in a measurable space (S, \mathcal{S}) , and let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be an r -smooth separable Banach space for $r \in (1, 2]$. Let $h: S^m \rightarrow \mathbb{B}$ be a measurable symmetric function which is degenerated of order d . Define for $j \in [1, m]$ the random variable*

$$(2.4) \quad H_{j,r} := \left(\mathbb{E} \left[\|h(\xi_{[1,m]})\|_{\mathbb{B}}^r \mid \xi_{[1,j]} \right] \right)^{1/r}.$$

Suppose that $0 < \alpha < (r-1)d/r$ and that for each j ,

$$(2.5) \quad H_{j,r} \in \mathbb{L}^{q(d,j,\gamma,r)}, \text{ where } q(d,j,\gamma,r) := \frac{\gamma + j + 1}{\max\{d, j\} \frac{r-1}{r} - \alpha + j/r}.$$

Then for each positive ε ,

$$(2.6) \quad \sum_{N=1}^{\infty} N^{\gamma} \mathbb{P} \left(\sup_{n \geq N} n^{\alpha} \frac{1}{\binom{n}{m}} \|U_{m,n}(h)\|_{\mathbb{B}} > \varepsilon \right) < \infty,$$

where

$$(2.7) \quad U_{m,n}(h) = \sum_{i \in \text{Inc}_n^m} h(\xi_i).$$

Kokic obtained in [30] a result in this spirit. Our result can be viewed as an extension in several directions. First, we consider the case where the random variables defined (2.5) may have different degrees of integrability (see Example 1 page 285 in [44] for examples of kernels where $H_{j,r}$ can have prescribed integration degrees), while Kokic only makes an assumption on the moments of $h(\xi_{[1,m]})$. Second, we address the case of Banach-valued U -statistics, while the result of [30] is for real valued kernels. Finally, if we compare the results on a common setting ($\mathbb{B} = \mathbb{R}$, $r = 2$ and only the assumption that $h(\xi_{[1,m]}) \in \mathbb{L}^q$ for some $q \geq 2$), we can take $\gamma = q(d/2 - \alpha) - 1$ whereas in Theorem 1 of [30], we can only take $\gamma = q(d/2 - \alpha) - 1 - \eta$ for some positive η .

2.2. Functional central limit theorems in Hölder spaces. Given a U -statistic with fixed kernel $h: S^m \rightarrow \mathbb{R}$, it is possible to associate a partial sum process by defining

$$(2.8) \quad \mathcal{U}_{n,h}(t) = \sum_{i \in \text{Inc}_{[nt]}^m} h(\xi_i), \quad t \in [0, 1],$$

where for $x \in \mathbb{R}$, $[x]$ is the unique integer satisfying $[x] \leq x < [x] + 1$. In [35], the convergence in distribution in the Skorohod space $D[0, 1]$ of the process $(n^{-r/2}U_{[n \cdot]})_{n \geq r}$ is studied. In Corollary 1, it is shown that if U_n is degenerated of order d , $d \in [2, m]$, then $(n^{-d/2}U_{[n \cdot]})_{n \geq r}$ converges in distribution to a process $I_d(h_d)$ symbolically defined as

$$(2.9) \quad I_d(h_d)(t) = \int \cdots \int h_d(x_1, \dots, x_d) \mathbf{1}_{[0,t]}(u_1) \cdots \mathbf{1}_{[0,t]}(u_d) W(dx_1, du_1) \cdots W(dx_d, du_d),$$

where W denotes the Gaussian measure (see the Appendix A.1 and A.2 of the paper [35]). For $i = 2$, the limiting process admits the expression $\sum_{j=1}^{+\infty} \lambda_j (B_j^2(t) - t)$, where $(B_j(\cdot))_{j \geq 1}$ are independent standard Brownian motions and $\sum_{j=1}^{\infty} \lambda_j^2$ is finite. Notice that for each $\alpha \in (0, 1/2)$, the trajectories of the limiting process $I_d(h_d)(\cdot)$ are almost surely α -Hölder continuous but of course, those of $t \mapsto \mathcal{U}_{n,h}(t)$ are not. For this reason, we have to consider a linearly interpolated version of $\mathcal{U}_{n,h}$, denoted by $\mathcal{U}_{n,h}^{\text{pl}}$ and defined by

$$(2.10) \quad \mathcal{U}_{m,n,h}^{\text{pl}}(t) = \begin{cases} \sum_{(i_\ell)_{\ell \in [1,m]} \in \text{Inc}_k^m} h(\xi_{i_1}, \dots, \xi_{i_m}) & \text{if } t = \frac{k}{n} \text{ for some } k \in [0, n] \\ \text{linear interpolation} & \text{on } (\frac{k}{n}, \frac{k+1}{n}), k \in [0, n-1]. \end{cases}$$

Such a process has path in Hölder spaces. Therefore, the study of the limiting behavior of (an appropriately centered and normalized version of) $(\mathcal{U}_{m,n,h}^{\text{pl}}(t))_{n \geq m}$ in Hölder spaces can be considered. We denote by \mathcal{H}_α the space of Hölder continuous functions on $[0, 1]$, that is, the set of functions $x: [0, 1] \rightarrow \mathbb{R}$ such that

$$(2.11) \quad \|x\|_\alpha := |x(0)| + \sup_{s,t \in [0,1], s < t} \frac{|x(s) - x(t)|}{(s-t)^\alpha} < \infty.$$

The space \mathcal{H}_α endowed with $\|x\|_\alpha$ is not separable and it will be more convenient to work with the separable subspace

$$(2.12) \quad \mathcal{H}_0 := \left\{ x \in \mathcal{H}_\alpha, \lim_{\delta \rightarrow 0} \sup_{s,t \in [0,1], 0 < s-t < \delta} \frac{|x(s) - x(t)|}{(s-t)^\alpha} = 0 \right\}.$$

It has been shown in [40] that if $\alpha \in (0, 1/2)$, $(\xi_i)_{i \geq 1}$ is a real-valued i.i.d. centered sequence having unit variance and W_n is the random function interpolating the points $(0, 0)$ and $(k/n, n^{-1/2} \sum_{i=1}^k \xi_i)$, $k \in \llbracket 1, n \rrbracket$, the convergence of $(W_n)_{n \geq 1}$ in \mathcal{H}_α^o is equivalent to

$$(2.13) \quad \lim_{t \rightarrow \infty} t^{p(\alpha)} \mathbb{P}(|\xi_1| > t) = 0,$$

where $p(\alpha) = (1/2 - \alpha)^{-1}$. Note that $p(\alpha)$ is strictly bigger than 2 hence (2.13) is more restrictive than having moments of order 2. Moreover, the closer α to the critical value is, the more restrictive (2.13) is.

Theorem 2.3. *Let $m \geq 2$, let (S, \mathcal{S}) be a measurable space, let $h: S^m \rightarrow \mathbb{R}$ be a symmetric measurable function and let $(\xi_i)_{i \geq 1}$ be a i.i.d. sequence taking values in S . Let $\alpha \in (0, 1/2)$ and $p(\alpha) = (1/2 - \alpha)^{-1}$. Suppose that h is degenerated of order d and that*

$$(2.14) \quad \lim_{t \rightarrow \infty} t^{p(\alpha)} \mathbb{P}(|h(\xi_{\llbracket 1, m \rrbracket})| > t) = 0.$$

Then the following convergence in distribution in \mathcal{H}_α^o takes place:

$$(2.15) \quad \left(\frac{1}{n^{m-d/2}} \mathcal{U}_{m,n,h}^{\text{pl}}(t) \right)_{t \in [0,1]} \rightarrow \left(t^{m-d} I_d(h_d)(t) \right)_{t \in [0,1]},$$

where I_d is defined as in (2.9).

This result improves that in [20], where the same conclusion was deduced under a more restrictive condition than (2.14), namely, under integrability of $|h(\xi_{\llbracket 1, m \rrbracket})|^{p(\alpha)} (\log(1 + |h(\xi_{\llbracket 1, m \rrbracket})|))^m$.

Convergence of the finite-dimensional distributions is guaranteed by Corollary 1 of [35]. The most challenging part is the proof of tightness, which rests on a criterion based on tails of differences of U -statistics. These ones can be expressed as a weighted U -statistic hence the tools developed in this paper are appropriated.

2.3. A moment inequality for incomplete U -statistics. The computation of a U -statistic of order m requires the computation of $\binom{n}{m}$ terms, which is of order n^m and can lead to practical difficulties. For this reason, Blom introduced in [5] the concept of incomplete U -statistics. The main idea is to put for each index (i) a random weight taking the value 0 or 1. There are several ways for defining such a random weights:

- sampling without replacement: we pick without replacement N m -uples of the form $\mathbf{i} = (i_\ell)_{\ell \in \llbracket 1, m \rrbracket}$ where $1 \leq i_1 < \dots < i_m \leq n$.
- sampling with replacement: we pick with replacement N m -uples of the form $\mathbf{i} = (i_\ell)_{\ell \in \llbracket 1, m \rrbracket}$ where $1 \leq i_1 < \dots < i_m \leq n$.
- Bernoulli sampling: consider for each $\mathbf{i} = (i_\ell)_{\ell \in \llbracket 1, m \rrbracket}$ satisfying $1 \leq i_1 < \dots < i_m \leq n$, a random variable $a_{n;\mathbf{i}}$ taking the value 1 with probability p_n and 0 with probability $1 - p_n$. Moreover, we assume that the family $(a_{n;\mathbf{i}})_{\mathbf{i} \in \text{Inc}_n^m}$ is independent and also independent of the sequence $(\xi_i)_{i \in \mathbb{Z}}$.

The weak convergence of incomplete U -statistics has been established in [28]. Rates in the law of large numbers were obtained in [38]. Recent papers include incomplete U -statistics in a high dimensional setting [10] and based on a triangular array [32].

It turns out that in the setting of the applications of our inequalities, the most convenient way of picking the random weights is Bernoulli sampling. Indeed, after having applied Theorem 1.3, we will have to control moments of random variables of the form $\sum a_{n;\mathbf{i}}^p$, where the sum runs over the indexes i_k

and k belongs to some subset K of $\llbracket 1, m \rrbracket$. In this case, it is possible to bound this by a sum of random variables following a binomial distribution, and this will be helpful in the sequel. We thus define

$$(2.16) \quad U_{m,n}^{\text{inc}}(h) = \sum_{i \in \text{Inc}_n^m} a_{n;i} h(\xi_i),$$

where the family $(a_{n;i})_{i \in \text{Inc}_n^m}$ is i.i.d., independent of $(\xi_i)_{i \geq 1}$, and $a_{n;i}$ takes the value 1 (respectively 0) with probability p_n (respectively $1 - p_n$).

Theorem 2.4. *Let $m \geq 2$ be an integer, $(\xi_i)_{i \in \mathbb{Z}}$ be an i.i.d. sequence taking values in a measurable space (S, \mathcal{S}) , let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable r -smooth Banach space and let $h: S^m \rightarrow \mathbb{B}$ be a symmetric function which is degenerated of order d with respect to $(\xi_i)_{i \in \mathbb{Z}}$. Let $(a_{n;i})_{i \in \text{Inc}_n^m}$ be an i.i.d. sequence of Bernoulli random variables with parameter p_n , which is independent of $(\xi_i)_{i \geq 1}$. Let $U_{m,n}^{\text{inc}}(h)$ defined as in (2.16). The following inequality takes place:*

$$(2.17) \quad \mathbb{E} \left[\left\| U_n^{\text{inc}} \right\|_{\mathbb{B}}^q \right] \leq C(\mathbb{B}, m, p, q) \left(n^{q(m-d)+dq/p} p_n^q + n^{mq/p} p_n^{q/p} + n^m p_n \right) \mathbb{E} \left[\left\| h(\xi_{\llbracket 1, m \rrbracket}) \right\|_{\mathbb{B}}^q \right],$$

where the constant $C(\mathbb{B}, m, p, q)$ depends only on \mathbb{B} , m , p and q .

Note that the degeneracy degree of h appears only in the first term of the right hand side of (2.17). When $d = m$, the second term of the right hand side of (2.17) dominates the first one. If $n^m p_n \leq 1$, then $n^{mq/p} p_n^{q/p} \leq n^m p_n$, since $q \geq p$.

In [14], a moment inequality for incomplete real-valued U -statistics has been established. No degeneracy is assumed, and the U -statistic is only centered hence this would correspond to the case $d = 1$. The authors look at the \mathbb{L}^1 norm of the U -statistic.

3. PROOFS

3.1. A "good- λ -inequality" for conditional moment of order p of some Banach-valued martingales. One of the key ingredients of the proof is a so-called "good- λ -inequality", that is, an inequality of the form $g(\beta\lambda) \leq K(\delta)g(\lambda) + h(\lambda)$, where $\beta > 1$, $\delta, \lambda > 0$, $K(\delta)$ is small as δ is close to 0 and g and h are tail functions of some random variables. Such inequalities are very helpful in order to derive moment inequalities [9, 39, 29] or deviation inequalities [12, 19]. In our context, we will need a good- λ -inequality for tails of conditional moments of partial sums of a martingale difference sequence.

Proposition 3.1. *Let $p \in (1, 2]$, $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable Banach space such that $C_{p,\mathbb{B}}$ defined as in (1.21) is finite, let $(\xi_i)_{i \in \mathbb{Z}}$ be an i.i.d. sequence and let $(\xi'_i)_{i \in \mathbb{Z}}$ be an independent copy of $(\xi_i)_{i \in \mathbb{Z}}$. For a subset J of \mathbb{Z} , denote by \mathcal{F}_J (respectively \mathcal{F}'_J) the σ -algebra generated by the random variables $\xi_i, i \in J$ (respectively $\xi'_i, i \in J$). Suppose that $(D_i)_{i \geq 1}$ is a martingale difference sequence with respect to the filtration $(\mathcal{F}_{\llbracket 1, i \rrbracket} \vee \mathcal{F}'_{\mathbb{Z}})_{i \geq 0}$ and let $S_j = \sum_{i=1}^j D_i$. Let $p \in (1, r]$ and assume that $\mathbb{E}[\|D_i\|_{\mathbb{B}}^p]$ is finite for each i . Then for each $J_0 \subset \mathbb{Z}$, each positive λ , $\beta > 1$ and $0 < \delta < 1$, the following inequality holds:*

$$(3.1) \quad \mathbb{P} \left(\max_{1 \leq j \leq n} (\mathbb{E}[\|S_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0}])^{1/p} > \beta\lambda \right) \\ \leq \left(\frac{p}{p-1} \right)^p \frac{C_{p,\mathbb{B}} \delta^p}{(\beta-1-\delta)^p} \mathbb{P} \left(\max_{1 \leq j \leq n} (\mathbb{E}[\|S_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0}])^{1/p} > \lambda \right) \\ + \mathbb{P} \left(\max_{1 \leq i \leq n} (\mathbb{E}[\|D_i\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0}])^{1/p} > \delta\lambda \right) + \mathbb{P} \left(\left(\sum_{i=1}^n \mathbb{E}[\|D_i\|_{\mathbb{B}}^p \mid \mathcal{F}_{\llbracket 1, i-1 \rrbracket} \vee \mathcal{F}'_{J_0}] \right)^{1/p} > \delta\lambda \right),$$

where $C_{p,\mathbb{B}}$ is defined as in (1.21).

Proof. We introduce the random variables

$$(3.2) \quad M^* = \left(\max_{1 \leq j \leq n} \mathbb{E} \left[\left\| \sum_{i=1}^j D_i \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \right] \right)^{1/p},$$

$$(3.3) \quad \Delta^* = \max_{1 \leq i \leq n} \mathbb{E} \left[\|D_i\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right],$$

$$(3.4) \quad \Gamma_{\ell} = \left(\sum_{i=1}^{\ell} \mathbb{E} \left[\|D_i\|_{\mathbb{B}}^p \mid \mathcal{F}_{[1, i-1]} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p},$$

$$(3.5) \quad T(t) = \min \left\{ 1 \leq j \leq n, (\mathbb{E} [\|S_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0}])^{1/p} \geq t \right\}, t > 0$$

and

$$(3.6) \quad \sigma = \min \left\{ 1 \leq \ell \leq n \mid \max \left\{ (\mathbb{E} [\|D_{\ell}\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0}])^{1/p}, \Gamma_{\ell+1} \right\} \geq \delta \lambda \right\},$$

with the convention that $\inf \emptyset = +\infty$. We now define

$$(3.7) \quad D'_i = \mathbf{1}_{T(\lambda) < i \leq \min\{T(\beta\lambda), \sigma, n\}} D_i.$$

Notice that by Lemma A.2, the set $T(\lambda) < i \leq \min\{T(\beta\lambda), \sigma, n\}$ belongs to $\mathcal{F}_{[1, i-1]} \vee \mathcal{F}'_{\mathbb{Z}}$. Consequently, the sequence $(D'_i)_{i \geq 1}$ is a martingale difference sequence with respect to the filtration $(\mathcal{F}_{[1, i]} \vee \mathcal{F}'_{\mathbb{Z}})_{i \geq 1}$. Moreover, the following equality holds by definition of $T(t)$ and σ :

$$(3.8) \quad \{M^* \geq \beta\lambda\} \cap \{\max\{\Gamma_n, \Delta^*\} \leq \delta\lambda\} = \{T(\lambda) \leq n\} \cap \{\sigma = \infty\}$$

and the following inclusion holds:

$$(3.9) \quad \{T(\lambda) \leq n\} \cap \{\sigma = \infty\} \subset \left\{ \left(\max_{1 \leq j \leq n} \mathbb{E} \left[\left\| \sum_{i=1}^j D'_i \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} > (\beta - 1 - \delta)\lambda \right\}.$$

To see this, notice that

$$(3.10) \quad \sum_{i=1}^j D'_i = \sum_{i=T(\lambda)+1}^{\min\{\sigma, T(\beta\lambda), j\}} D_i = \sum_{i=1}^{\min\{\sigma, T(\beta\lambda), j\}} D_i - \sum_{i=1}^{T(\lambda)-1} D_i - D_{T(\lambda)}$$

and applying twice the conditional Minkowski's inequality $(\mathbb{E} [|X + Y|^p \mid \mathcal{G}])^{1/p} \leq (\mathbb{E} [|X|^p \mid \mathcal{G}])^{1/p} + (\mathbb{E} [|Y|^p \mid \mathcal{G}])^{1/p}$ with $\mathcal{G} = \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0}$ gives

$$(3.11) \quad \max_{1 \leq j \leq n} \left(\mathbb{E} \left[\left\| \sum_{i=1}^j D'_i \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \right] \right)^{1/p} \geq \max_{1 \leq j \leq n} \left(\mathbb{E} \left[\left\| \sum_{i=1}^{\min\{\sigma, T(\beta\lambda), j\}} D'_i \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} \\ - \left(\mathbb{E} \left[\left\| \sum_{i=1}^{T(\lambda)-1} D'_i \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} - \left(\mathbb{E} \left[\|D_{T(\lambda)}\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p}$$

and since

$$(3.12) \quad \max_{1 \leq j \leq n} \left(\mathbb{E} \left[\left\| \sum_{i=1}^{\min\{\sigma, T(\beta\lambda), j\}} D'_i \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} \mathbf{1}_{\{T(\lambda) \leq n\} \cap \{\sigma = \infty\}} \geq \beta\lambda,$$

$$(3.13) \quad \left(\mathbb{E} \left[\left\| \sum_{i=1}^{T(\lambda)-1} D'_i \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} \mathbf{1}_{\{T(\lambda) \leq n\} \cap \{\sigma = \infty\}} \leq \lambda$$

$$(3.14) \quad \left(\mathbb{E} \left[\|D_{T(\lambda)}\|_{\mathbb{B}} \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} \mathbf{1}_{\{T(\lambda) \leq n\} \cap \{\sigma = \infty\}} \leq \delta \lambda,$$

we get (3.9) in view of (3.11). Combining (3.8) with (3.9) gives

$$\begin{aligned} \mathbb{P}(\{M^* \geq \beta \lambda\} \cap \{\max\{\Gamma_n, \Delta^*\} \leq \delta \lambda\}) &\leq \mathbb{P} \left(\max_{1 \leq j \leq n} \mathbb{E} \left[\left\| \sum_{i=1}^j D'_i \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] > (\beta - 1 - \delta)^p \lambda^p \right) \\ &\leq \mathbb{P} \left(\mathbb{E} \left[\max_{1 \leq j \leq n} \left\| \sum_{i=1}^j D'_i \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] > (\beta - 1 - \delta)^p \lambda^p \right), \end{aligned}$$

then Markov's and Doob's inequality give that

$$(3.15) \quad \mathbb{P}(\{M^* \geq \beta \lambda\} \cap \{\max\{\Gamma_n, \Delta^*\} \leq \delta \lambda\}) \leq \left(\frac{p}{p-1} \right)^p \frac{1}{(\beta - 1 - \delta)^p \lambda^p} \mathbb{E} \left[\left\| \sum_{i=1}^n D'_i \right\|_{\mathbb{B}}^p \right].$$

From (1.21), we deduce that

$$(3.16) \quad \mathbb{P}(\{M^* \geq \beta \lambda\} \cap \{\max\{\Gamma_n, \Delta^*\} \leq \delta \lambda\}) \leq \left(\frac{p}{p-1} \right)^p \frac{C_{p,\mathbb{B}}}{(\beta - 1 - \delta)^p \lambda^p} \sum_{i=1}^n \mathbb{E} \left[\|D'_i\|_{\mathbb{B}}^p \right].$$

Moreover, by definition of D'_i and $\mathcal{F}_{[1,i-1]} \vee \mathcal{F}'_{J_0}$ -measurability of $\{T(\lambda) < i \leq \min\{T(\beta\lambda), \sigma, n\}\}$, one gets that

$$(3.17) \quad \begin{aligned} \mathbb{P}(\{M^* \geq \beta \lambda\} \cap \{\max\{\Gamma_n, \Delta^*\} \leq \delta \lambda\}) \\ \leq \left(\frac{p}{p-1} \right)^p \frac{C_{p,\mathbb{B}}}{(\beta - 1 - \delta)^p \lambda^p} \sum_{i=1}^n \mathbb{E} \left[\mathbf{1}_{T(\lambda) < i \leq \min\{T(\beta\lambda), \sigma, n\}} \mathbb{E} \left[\|D_i\|_{\mathbb{B}}^p \mid \mathcal{F}_{[1,i-1]} \vee \mathcal{F}'_{J_0} \right] \right]. \end{aligned}$$

By definition of $T(\lambda)$ and σ , we infer that

$$(3.18) \quad \begin{aligned} \mathbb{P}(\{M^* \geq \beta \lambda\} \cap \{\max\{\Gamma_n, \Delta^*\} \leq \delta \lambda\}) \\ \leq \left(\frac{p}{p-1} \right)^p \frac{C_{p,\mathbb{B}}}{(\beta - 1 - \delta)^p \lambda^p} \sum_{i=1}^n \mathbb{E} \left[\mathbf{1}_{M^* > \lambda} \mathbf{1}_{\Gamma_i \leq \delta \lambda} \mathbb{E} \left[\|D_i\|_{\mathbb{B}}^p \mid \mathcal{F}_{[k,i-1]} \vee \mathcal{F}'_{J_0} \right] \right]. \end{aligned}$$

We then conclude by the elementary inequality $\sum_{i=1}^n \mathbf{1}_{\sum_{k=1}^i Y_k \leq x} Y_i \leq x$ for non-negative random variables Y_i and a positive x . This ends the proof of Proposition 3.1. \square

The previous good λ -inequality lead to the following inequality, expressing the tail of the conditional moment of a Banach-valued martingale in terms of a functional of the tails of the conditional increments and conditional moment of order p .

Corollary 3.2. *Let $p \in (1, 2]$. There exists a function $f_{1,p}: \mathbb{R}_{>0} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ such that if $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ is a separable Banach space such that $C_{p,\mathbb{B}}$ defined as in (1.21) is finite, each i.i.d. sequence $(\xi_i)_{i \in \mathbb{Z}}$ and its independent copy $(\xi'_i)_{i \in \mathbb{Z}}$, each \mathbb{B} -valued martingale difference sequence $(D_i)_{i \geq 1}$ with respect to the filtration $(\mathcal{F}_{[1,i]} \vee \mathcal{F}'_{\mathbb{Z}})_{i \geq 0}$, where $\mathcal{F}_I = \sigma(\xi_i, i \in I)$, $\mathcal{F}'_J = \sigma(\xi'_j, j \in J)$ each $J_0 \subset \mathbb{Z}$ and each $q, t > 0$,*

$$(3.19) \quad \mathbb{P} \left(\max_{1 \leq j \leq n} \left(\mathbb{E} \left[\|S_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} > t \right)$$

$$\begin{aligned} &\leq f_{1,p}(q, C_{p,\mathbb{B}}) \int_0^1 \mathbb{P} \left(\max_{1 \leq i \leq n} (\mathbb{E} [\|D_i\|_{\mathbb{B}}^p | \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0}])^{1/p} > tu \right) u^{q-1} du \\ &\quad + f_{1,p}(q, C_{p,\mathbb{B}}) \int_0^1 \mathbb{P} \left(\left(\sum_{i=1}^n \mathbb{E} [\|D_i\|_{\mathbb{B}}^p | \mathcal{F}_{[1,i-1]} \vee \mathcal{F}'_{J_0}] \right)^{1/p} > tu \right) u^{q-1} du, \end{aligned}$$

where $S_j = \sum_{i=1}^j D_i$.

Proof. Let us fix the Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ into consideration, $1 < p \leq r$, $q > 0$, $(\xi_i)_{i \in \mathbb{Z}}$ as well as $J_0 \subset \mathbb{Z}$ and $(D_i)_{i \geq 1}$. We choose $\beta_0 > 1$ and $\delta_0 < 1$ (depending only on p, q and $C_{p,\mathbb{B}}$) such that

$$(3.20) \quad \frac{\beta_0^q \delta_0^p}{(\beta_0 - 1 - \delta_0)^p} \left(\frac{p}{p-1} \right)^p C_{p,\mathbb{B}} \leq 1.$$

Defining

$$(3.21) \quad g(t) = \mathbb{P} \left(\max_{1 \leq j \leq n} (\mathbb{E} [\|S_j\|_{\mathbb{B}}^p | \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0}])^{1/p} > t \right)$$

and

$$(3.22) \quad h(t) = \mathbb{P} \left(\max_{1 \leq i \leq n} (\mathbb{E} [\|D_i\|_{\mathbb{B}}^p | \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0}])^{1/p} > t \right) \\ + \mathbb{P} \left(\left(\sum_{i=1}^n \mathbb{E} [\|D_i\|_{\mathbb{B}}^p | \mathcal{F}_{[1,i-1]} \vee \mathcal{F}'_{J_0}] \right)^{1/p} > t \right),$$

Proposition 3.1 shows that for each positive t ,

$$(3.23) \quad g(\beta_0 t) \leq \beta_0^{-q} g(t) + h(t\delta_0).$$

Observing that g is non-increasing, we infer that

$$(3.24) \quad q \int_0^1 u^{q-1} h(tu) du = \sum_{\ell \geq 0} \int_{\beta_0^{-\ell-1}}^{\beta_0^{-\ell}} q u^{q-1} h(tu) du$$

$$(3.25) \quad \geq \sum_{\ell \geq 0} h(t\beta_0^{-\ell-1}) \int_{\beta_0^{-\ell-1}}^{\beta_0^{-\ell}} q u^{q-1} du$$

$$(3.26) \quad \geq \sum_{\ell \geq 0} h(t\beta_0^{-\ell-1}) \left(\beta_0^{-\ell q} - \beta_0^{-\ell q - q} \right)$$

$$(3.27) \quad \geq \left(1 - \beta_0^{-q} \right) \sum_{\ell \geq 0} \beta_0^{-\ell q} \left(g\left(\beta_0 \beta_0^{-\ell-1} t / \delta_0\right) - \beta_0^{-q} g\left(\beta_0^{-\ell-1} t / \delta_0\right) \right)$$

and defining $c_\ell = g\left(\beta_0^{-\ell} t / \delta_0\right)$, the previous bound gives (accounting that $\beta_0^{-(\ell+1)q} c_{\ell+1} \leq \beta_0^{-(\ell+1)q} \rightarrow 0$)

$$(3.28) \quad q \int_0^1 u^{q-1} h(tu) \geq \left(1 - \beta_0^{-q} \right) \sum_{\ell \geq 0} \left(\beta_0^{-\ell q} c_\ell - \beta_0^{-(\ell+1)q} c_{\ell+1} \right) = c_0$$

hence

$$(3.29) \quad g(t/\delta_0) \leq q \int_0^1 u^{q-1} h(tu) du \frac{1}{1 - \beta_0^{-q}}.$$

To conclude, we apply this bound with t replaced by $\delta_0 t$ and do the substitution $v = \delta_0 u$. \square

3.2. Key ingredient in the proof of Theorem 1.3.

3.2.1. *Statement.* We would like to prove Theorem 1.3 by induction on the order of the considered U -statistic. An application of Corollary 3.2 would reduce to bound the tail of a U -statistic of lower order, but the term involving the conditional moment of order p cannot be treated directly with the induction assumption.

For this reason, we will consider the following assertion $A(m)$, depending on the parameter $m \geq 1$, which is defined as follows: for each $p \in (1, 2]$ and each $k \geq 0$, there exists a function $f_{m,k,p}: \mathbb{R}_{>0} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$ such that the following holds: if $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ is a separable Banach space for which $C_{p,\mathbb{B}}$ defined as in (1.21) is finite, $(\xi_i)_{i \in \mathbb{Z}}$ is an i.i.d. sequence taking values in a measurable space (S, \mathcal{S}) and $(\xi'_i)_{i \in \mathbb{Z}}$ is an independent copy of $(\xi_i)_{i \in \mathbb{Z}}$, $h_{\mathbf{i}}: S^m \times S^k \rightarrow \mathbb{B}$, $\mathbf{i} = (i_\ell)_{\ell \in \llbracket 1, m \rrbracket} \in \text{Inc}_n^m$, are measurable functions such that for each $\ell_0 \in \llbracket 1, m+k \rrbracket$,

$$(3.30) \quad \mathbb{E} [h_{\mathbf{i}}(\xi_{\llbracket 1, m+k \rrbracket}) \mid \xi_{\llbracket 1, m+k \rrbracket \setminus \{\ell_0\}}] = 0,$$

and J_0 is a subset of $\llbracket m+1, m+k+1 \rrbracket$, the following inequality takes place:

$$(3.31) \quad \mathbb{P} \left(\max_{m \leq n \leq N} \left(\mathbb{E} \left[\left\| \sum_{\mathbf{i} \in \text{Inc}_n^m} h_{\mathbf{i}}(\xi_{\mathbf{i}}, \xi'_{\llbracket m+1, m+k+1 \rrbracket}) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} > t \right) \\ \leq f_{m,k,p}(q, C_{p,\mathbb{B}}) \sum_{J \subset \llbracket 1, m \rrbracket} \sum_{\mathbf{i}_J \in \mathbb{N}^J} \int_0^1 u^{q-1} g_{\mathbf{i}_J}(ut) du,$$

where we write

$$(3.32) \quad \xi'_{\llbracket a, b \rrbracket} = (\xi'_a, \dots, \xi'_b),$$

use the small abuse of notation

$$h_{\mathbf{i}}(\xi_{\mathbf{i}}, \xi'_{\llbracket m+1, m+k+1 \rrbracket}) = h_{\mathbf{i}}(\xi_{i_1}, \dots, \xi_{i_m}, \xi'_{m+1}, \dots, \xi'_{m+k+1})$$

and define

$$(3.33) \quad g_{\mathbf{i}_J}(t) = \mathbb{P} \left(\left(\sum_{\mathbf{i}_{J^c}: \mathbf{i}_J + \mathbf{i}_{J^c} \in \text{Inc}_n^m} \mathbb{E} \left[\left\| h_{\mathbf{i}_J + \mathbf{i}_{J^c}}(\xi'_{\llbracket 1, m+k \rrbracket}) \right\|_{\mathbb{B}}^p \mid \mathcal{F}'_{J \cup J_0} \right] \right)^{1/p} > t \right),$$

$\mathcal{F}_I = \sigma(\xi_i, i \in I)$, $\mathcal{F}'_J = \sigma(\xi'_j, j \in J)$ and the convention $\mathcal{F}_\emptyset = \{\emptyset, \Omega\}$. Note that Theorem 1.3 corresponds to $A(m)$ with $k=0$ and $J_0 = \emptyset$.

In order to perform the induction step, we need to define recursively the functions $f_{m,k,p}$. We define $f_{1,k,p} = f_{1,p}$ for each $k \geq 0$, where $f_{1,p}$ is defined as in Corollary 3.2 and

$$(3.34) \quad f_{m+1,k,p}(q, K) = f_{1,k,p}(q, K) f_{m,k+1,p}(q, K), q, K \in \mathbb{R}_{>0}.$$

3.2.2. *The case $m=1$.* We will show $A(1)$. The term associated with $J = \emptyset$ in (3.31) and $J = \{1\}$ correspond to the first and second term of the right hand side of (3.19) hence it suffices to apply Corollary 3.2.

3.2.3. *The case $m=2$.* Although the case $m=2$ is not required from a purely formal point of view, it will help the understanding of the induction step. Let $p \in (1, 2]$, $k \geq 0$, let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable Banach space for which the constant $C_{p,\mathbb{B}}$ defined as in (1.21) is finite, an i.i.d. sequence $(\xi_i)_{i \in \mathbb{Z}}$, an independent copy $(\xi'_i)_{i \in \mathbb{Z}}$, functions $h_{i,j}: S^2 \times S^k \rightarrow \mathbb{B}$ such that for each $\ell_0 \in \llbracket 1, k+2 \rrbracket$,

$$(3.35) \quad \mathbb{E} [h_{i,j}(\xi_{\llbracket 1, k+2 \rrbracket}) \mid \xi_{\llbracket 1, k+2 \rrbracket \setminus \{\ell_0\}}] = 0.$$

We have to show that for each $J_0 \subset \llbracket 3, k+2 \rrbracket$ and each $q, t > 0$,

$$\begin{aligned}
 (3.36) \quad & \mathbb{P} \left(\max_{2 \leq n \leq N} \left(\mathbb{E} \left[\left\| \sum_{1 \leq i < j \leq n} h_{i,j} \left(\xi_i, \xi_j, \xi'_{\llbracket 3, k+2 \rrbracket} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} > t \right) \\
 & \leq f_{2,k,p}(q, C_{p,\mathbb{B}}) \sum_{1 \leq i < j \leq N} \int_0^1 u^{q-1} \mathbb{P} \left(\left(\mathbb{E} \left[\left\| h_{i,j} \left(\xi'_1, \xi'_2, \xi'_{\llbracket 3, k+2 \rrbracket} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}'_{\{1,2\} \cup J_0} \right] \right)^{1/p} > tu \right) du \\
 & + f_{2,k,p}(q, C_{p,\mathbb{B}}) \sum_{i=1}^{N-1} \int_0^1 u^{q-1} \mathbb{P} \left(\left(\sum_{j=i+1}^N \mathbb{E} \left[\left\| h_{i,j} \left(\xi'_1, \xi'_2, \xi'_{\llbracket 3, k+2 \rrbracket} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}'_{\{1\} \cup J_0} \right] \right)^{1/p} > tu \right) du \\
 & + f_{2,k,p}(q, C_{p,\mathbb{B}}) \sum_{j=2}^N \int_0^1 u^{q-1} \mathbb{P} \left(\left(\sum_{i=1}^{j-1} \mathbb{E} \left[\left\| h_{i,j} \left(\xi'_1, \xi'_2, \xi'_{\llbracket 3, k+2 \rrbracket} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}'_{\{2\} \cup J_0} \right] \right)^{1/p} > tu \right) du \\
 & + f_{2,k,p}(q, C_{p,\mathbb{B}}) \int_0^1 u^{q-1} \mathbb{P} \left(\left(\sum_{i=1}^{N-1} \sum_{j=i+1}^N \mathbb{E} \left[\left\| h_{i,j} \left(\xi'_1, \xi'_2, \xi'_{\llbracket 3, k+2 \rrbracket} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}'_{J_0} \right] \right)^{1/p} > tu \right) du.
 \end{aligned}$$

To do so, define

$$(3.37) \quad D_j = \sum_{i=1}^{j-1} h_{i,j} \left(\xi_i, \xi_j, \xi'_{\llbracket 3, k+2 \rrbracket} \right).$$

In this way, $\sum_{1 \leq i < j \leq n} h_{i,j} \left(\xi_i, \xi_j, \xi'_{\llbracket 3, k+2 \rrbracket} \right) = \sum_{j=2}^n D_j$ and by (3.35) the sequence $(D_j)_{j \geq 2}$ is a martingale difference sequence with respect to the filtration $(\mathcal{F}_{\llbracket 1, j \rrbracket} \vee \mathcal{F}'_{\mathbb{Z}})_{j \geq 2}$. Applying Corollary 3.2 gives

$$(3.38) \quad \mathbb{P} \left(\max_{2 \leq n \leq N} \left(\mathbb{E} \left[\left\| \sum_{1 \leq i < j \leq n} h_{i,j} \left(\xi_i, \xi_j, \xi'_{\llbracket 3, k+2 \rrbracket} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} > t \right) \leq P_1 + P_2,$$

where

$$\begin{aligned}
 P_1 & := f_{1,k,p}(q, C_{p,\mathbb{B}}) \int_0^1 u^{q-1} \mathbb{P} \left(\max_{2 \leq j \leq N} \left(\mathbb{E} \left[\left\| \sum_{i=1}^{j-1} h_{i,j} \left(\xi_i, \xi_j, \xi'_{\llbracket 3, k+2 \rrbracket} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} > tu \right) du; \\
 P_2 & := f_{1,k,p}(q, C_{p,\mathbb{B}}) \int_0^1 u^{q-1} \mathbb{P} \left(\left(\sum_{2 \leq j \leq N} \mathbb{E} \left[\left\| \sum_{i=1}^{j-1} h_{i,j} \left(\xi_i, \xi_j, \xi'_{\llbracket 3, k+2 \rrbracket} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\llbracket 1, j-1 \rrbracket} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} > tu \right) du.
 \end{aligned}$$

In order to bound P_1 , we use a union bound and find that

$$(3.39) \quad P_1 \leq \sum_{j=2}^N f_{1,k,p}(q, C_{p,\mathbb{B}}) \int_0^1 u^{q-1} \mathbb{P} \left(\left(\mathbb{E} \left[\left\| \sum_{i=1}^{j-1} h_{i,j} \left(\xi_i, \xi_j, \xi'_{\llbracket 3, k+2 \rrbracket} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} > tu \right) du.$$

Moreover, since D_j is $\mathcal{F}_{\llbracket 1, j \rrbracket} \vee \mathcal{F}'_{\mathbb{Z}}$ -measurable, Lemma A.2 gives that

$$P_1 \leq \sum_{j=2}^N f_{1,k,p}(q, C_{p,\mathbb{B}}) \int_0^1 u^{q-1} \mathbb{P} \left(\left(\mathbb{E} \left[\left\| \sum_{i=1}^{j-1} h_{i,j} \left(\xi_i, \xi_j, \xi'_{\llbracket 3, k+2 \rrbracket} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\llbracket 1, j \rrbracket} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} > tu \right) du.$$

Define for $j \in \llbracket 2, N \rrbracket$ the random variable

$$(3.40) \quad D'_j := \sum_{i=1}^{j-1} h_{i,j} \left(\xi_i, \xi'_2, \xi'_{\llbracket 3, k+2 \rrbracket} \right).$$

Notice that

$$(3.41) \quad \mathbb{E} \left[\left\| \sum_{i=1}^{j-1} h_{i,j} \left(\xi_i, \xi_j, \xi'_{\llbracket 3, k+2 \rrbracket} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\llbracket 1, j \rrbracket} \vee \mathcal{F}'_{J_0} \right] = H \left(\xi_{\llbracket 1, j \rrbracket}, (\xi'_k)_{k \in J_0} \right),$$

where

$$(3.42) \quad H \left(x_{\llbracket 1, d \rrbracket}, (x'_k)_{k \in J_0} \right) = \mathbb{E} \left[\left\| \sum_{i=1}^{j-1} h_{i,j} \left(x_i, x_j, V \left((Y_k)_{k \in \llbracket 3, k+2 \rrbracket} \right) \right) \right\|_{\mathbb{B}}^p \right],$$

where in vector $V \left((Y_k)_{k \in \llbracket 3, k+2 \rrbracket} \right)$, $Y_k = x'_k$ if $k \in J_0$ and $Y_k = \xi'_k$ otherwise. Since

$$(3.43) \quad \mathbb{E} \left[\left\| \sum_{i=1}^{j-1} h_{i,j} \left(\xi_i, \xi'_2, V \left((Y_k)_{k \in \llbracket 3, k+2 \rrbracket} \right) \right) \right\|_{\mathbb{B}}^p \right] = H \left(\xi_{\llbracket 1, j-1 \rrbracket}, \xi'_2, V \left((Y_k)_{k \in \llbracket 3, k+2 \rrbracket} \right) \right),$$

we also have that

$$(3.44) \quad \mathbb{E} \left[\|D_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{\llbracket 1, j \rrbracket} \vee \mathcal{F}'_{J_0} \right] = \mathbb{E} \left[\|D'_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{\llbracket 1, j-1 \rrbracket} \vee \mathcal{F}'_{J_0} \right],$$

which, in terms of P_1 , translates as

$$(3.45) \quad P_1 \leq \sum_{j=2}^N f_{1,k,p}(q, C_{p,\mathbb{B}}) \int_0^1 u^{q-1} \mathbb{P} \left(\left(\mathbb{E} \left[\left\| \sum_{i=1}^{j-1} h_{i,j} \left(\xi_i, \xi'_2, \xi'_{\llbracket 3, k+2 \rrbracket} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\llbracket 1, j-1 \rrbracket} \vee \mathcal{F}'_{\{2\} \cup J_0} \right] \right)^{1/p} > tu \right) du.$$

We then use Corollary 3.2 in the following setting:

- $\widetilde{D}_i = \sum_{i=1}^{j-1} h_{i,j} \left(\xi_i, \xi'_{\llbracket 2, k+2 \rrbracket} \right)$,
- $\widetilde{q} = q$,
- $\widetilde{k} = k + 1$,
- $\widetilde{J}_0 = \{2\} \cup J_0$.

After having bounded the term with the maximum by a union bound, we get

$$(3.46) \quad P_1 \leq \sum_{j=2}^N \sum_{i=1}^{j-1} f_{1,k,p}(q, C_{p,\mathbb{B}}) f_{1,k+1,p}(q, C_{p,\mathbb{B}}) \int_0^1 u^{q-1} \int_0^1 v^{q-1} \mathbb{P} \left(\left(\mathbb{E} \left[\left\| h_{i,j} \left(\xi_i, \xi'_2, \xi'_{\llbracket 3, k+2 \rrbracket} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\llbracket 1, j-1 \rrbracket} \vee \mathcal{F}'_{\{2\} \cup J_0} \right] \right)^{1/p} > tu \right) dudv \\ + \sum_{j=2}^N f_{1,k,p}(q, C_{p,\mathbb{B}}) f_{1,k+1,p}(q, C_{p,\mathbb{B}}) \int_0^1 u^{q-1} \int_0^1 v^q \mathbb{P} \left(\left(\sum_{i=1}^{j-1} \mathbb{E} \left[\left\| h_{i,j} \left(\xi_i, \xi'_2, \xi'_{\llbracket 3, k+2 \rrbracket} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\llbracket 1, i-1 \rrbracket} \vee \mathcal{F}'_{\{2\} \cup J_0} \right] \right)^{1/p} > tuv \right) dudv.$$

By Lemma A.1, (3.34), the elementary inequality

$$(3.47) \quad \int_0^1 \int_0^1 u^{q-1} v^{q-1} \mathbb{P}(Y > tuv) dudv \leq \int_0^1 u^{q-1} \mathbb{P}(Y > tu) du,$$

and (3.46), we deduce that

$$\begin{aligned}
 P_1 &\leq f_{2,k,p}(q, C_{p,\mathbb{B}}) \sum_{j=2}^N \sum_{i=1}^{j-1} \int_0^1 u^{q-1} \mathbb{P} \left(\left(\mathbb{E} \left[\left\| h_{i,j} \left(\xi_i, \xi'_2, \xi'_{[3,k+2]} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\{i\}} \vee \mathcal{F}'_{\{2\} \cup J_0} \right] \right)^{1/p} > tu \right) du \\
 &\quad + \sum_{j=2}^N f_{2,k,p}(q, C_{p,\mathbb{B}}) \int_0^1 u^{q-1} \mathbb{P} \left(\left(\sum_{i=1}^{j-1} \mathbb{E} \left[\left\| h_{i,j} \left(\xi_i, \xi'_2, \xi'_{[3,k+2]} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}'_{\{2\} \cup J_0} \right] \right)^{1/p} > tu \right) du.
 \end{aligned}$$

Since the vectors $(\xi_1, \xi'_{[2,k+2]})$ and $(\xi'_1, \xi'_{[2,k+2]})$ have the same distribution, the right hand side corresponds to the sum of the first and third terms in (3.36).

Let us now bound P_2 . By (3.44), we can write $P_2/f_{1,k,p}(q, C_{p,\mathbb{B}})$ as

$$(3.48) \quad \int_0^1 u^{q-1} \mathbb{P} \left(\left(\sum_{2 \leq j \leq N} \mathbb{E} \left[\left\| \sum_{i=1}^{j-1} h_{i,j} \left(\xi_i, \xi'_2, \xi'_{[3,k+2]} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{[1,j-1]} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} > tu \right) du.$$

In order to control this quantity, we introduce the Banach space $\widetilde{\mathbb{B}} = \{(x_j)_{j \in [2,N]}\}$ endowed with the norm $\|(x_j)_{j \in [2,N]}\|_{\widetilde{\mathbb{B}}} = \left(\sum_{j=2}^N \|x_j\|_{\mathbb{B}}^p \right)^{1/p}$. We use Corollary 3.2 in the following setting:

- $\widetilde{D}_i = \left(h_{i,j} \left(\xi_i, \xi'_{[2,k+2]} \right) \right)_{j \in [2,N]}$,
- $\widetilde{q} = q$,
- $\widetilde{k} = k + 1$,
- $\widetilde{J}_0 = J_0$.

Using (3.47) and (3.34), we get that

$$\begin{aligned}
 (3.49) \quad P_2 &\leq f_{2,k,p}(q, C_{p,\mathbb{B}}) \int_0^1 u^{q-1} \mathbb{P} \left(\max_{1 \leq i \leq N} \left(\mathbb{E} \left[\left\| \widetilde{D}_i \right\|_{\widetilde{\mathbb{B}}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} \right) du \\
 &\quad + f_{2,k,p}(q, C_{p,\mathbb{B}}) \int_0^1 u^{q-1} \mathbb{P} \left(\left(\sum_{i=1}^N \mathbb{E} \left[\left\| \widetilde{D}_i \right\|_{\widetilde{\mathbb{B}}}^p \mid \mathcal{F}_{[1,i-1]} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} \right) du \\
 &\leq f_{2,k,p}(q, C_{p,\mathbb{B}}) \sum_{i=1}^N \int_0^1 u^{q-1} \mathbb{P} \left(\left(\mathbb{E} \left[\sum_{j=i+1}^N \left\| h_{i,j} \left(\xi_i, \xi'_2, \xi'_{[3,k+2]} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} \right) du \\
 &\quad + f_{2,k,p}(q, C_{p,\mathbb{B}}) \int_0^1 u^{q-1} \mathbb{P} \left(\left(\sum_{i=1}^N \sum_{j=i+1}^N \mathbb{E} \left[\left\| h_{i,j} \left(\xi_i, \xi'_2, \xi'_{[3,k+2]} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{[1,i-1]} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} \right) du.
 \end{aligned}$$

To conclude, we notice that

$$\mathbb{E} \left[\left\| h_{i,j} \left(\xi_i, \xi'_2, \xi'_{[3,k+2]} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] = \mathbb{E} \left[\left\| h_{i,j} \left(\xi'_1, \xi'_2, \xi'_{[3,k+2]} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}'_{J_0} \right]$$

and that

$$\mathbb{E} \left[\left\| h_{i,j} \left(\xi_i, \xi'_2, \xi'_{[3,k+2]} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{[1,i-1]} \vee \mathcal{F}'_{J_0} \right] = \mathbb{E} \left[\left\| h_{i,j} \left(\xi'_1, \xi'_2, \xi'_{[3,k+2]} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}'_{\{1\} \cup J_0} \right],$$

and get the second and fourth terms of the right hand side of (3.36).

3.2.4. *Induction step: from m to $m + 1$.* Suppose that $A(m)$ is true and let us show that $A(m + 1)$ is true.

Let $p \in (1, 2]$ be fixed. Let $k \geq 0$ be fixed, as well as a separable Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ for which $C_{p, \mathbb{B}}$ defined as in (1.21) is finite, an i.i.d. sequence $(\xi_i)_{i \in \mathbb{Z}}$ and an independent copy $(\xi'_i)_{i \in \mathbb{Z}}$, functions $h_{i, i_{m+1}} : S^{m+1} \times S^k \rightarrow \mathbb{B}$ such that for each $\ell \in \llbracket 1, m + k + 1 \rrbracket$, and $\mathbf{i} \in \text{Inc}^m$ and $i_{m+1} > i_m$,

$$(3.50) \quad \mathbb{E} [h_{i, i_{m+1}} (\xi_{\llbracket 1, m+k+1 \rrbracket}) \mid \xi_{\llbracket 1, m+k+1 \rrbracket \setminus \{\ell_0\}}] = 0.$$

Finally, let $q, t > 0$, $J_0 \subset \llbracket m + 2, m + 2 + k \rrbracket$ and $N \geq m + 1$ be fixed. We have to show that

$$(3.51) \quad \mathbb{P} \left(\max_{m+1 \leq n \leq N} \left(\mathbb{E} \left[\left\| \sum_{(\mathbf{i}, i_{m+1}) \in \text{Inc}_n^{m+1}} h_{(\mathbf{i}, i_{m+1})} (\xi_{(\mathbf{i}, i_{m+1})}, \xi'_{\llbracket m+2, m+k+1 \rrbracket})} \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} > t \right) \\ \leq f_{m+1, k, p}(q, C_{p, \mathbb{B}}) \sum_{J \subset \llbracket 1, m+1 \rrbracket} \sum_{\mathbf{i}_J \in \mathbb{N}^J} \int_0^1 u^{q-1} g_{\mathbf{i}_J}(ut) du,$$

where

$$(3.52) \quad g_{\mathbf{i}_J}(t) = \mathbb{P} \left(\left(\sum_{\mathbf{i}_J \mathbf{c} : \mathbf{i}_J + \mathbf{i}_J \mathbf{c} \in \text{Inc}_N^{m+1}} \mathbb{E} \left[\left\| h_{\mathbf{i}_J + \mathbf{i}_J \mathbf{c}} (\xi'_{\llbracket 1, m+k+1 \rrbracket}) \right\|_{\mathbb{B}}^p \mid \mathcal{F}'_{J \cup J_0} \right] \right)^{1/p} > tu \right).$$

It will turn out that after an application of Corollary 3.19 to an appropriate martingale difference sequence, the contribution of the maximum of increments will correspond to the subsets of $\llbracket 1, m + 1 \rrbracket$ containing $m + 1$ while the contribution of the sum of conditional moments will correspond to the subsets of $\llbracket 1, m + 1 \rrbracket$ which do not contain $m + 1$. To do so, let us define for $j \geq m + 1$ the random variable

$$(3.53) \quad D_j := \sum_{\mathbf{i} \in \text{Inc}_{j-1}^m} h_{(\mathbf{i}, j)} (\xi_{\mathbf{i}}, \xi_j, \xi'_{\llbracket m+2, m+k+1 \rrbracket}).$$

In this way,

$$(3.54) \quad \sum_{(\mathbf{i}, i_{m+1}) \in \text{Inc}_n^{m+1}} h_{(\mathbf{i}, i_{m+1})} (\xi_{(\mathbf{i}, i_{m+1})}, \xi'_{\llbracket m+2, m+k+1 \rrbracket}) = \sum_{j=m+1}^n D_j$$

and $(D_j)_{j \geq m+1}$ is a martingale difference sequence with respect to the filtration $(\mathcal{F}_{\llbracket 1, j \rrbracket} \vee \mathcal{F}'_{J_0})_{j \geq m}$. Consequently, an application of Corollary 3.2 gives

$$(3.55) \quad \mathbb{P} \left(\max_{m+1 \leq n \leq N} \left(\mathbb{E} \left[\left\| \sum_{(\mathbf{i}, i_{m+1}) \in \text{Inc}_n^{m+1}} h_{(\mathbf{i}, i_{m+1})} (\xi_{(\mathbf{i}, i_{m+1})}, \xi'_{\llbracket m+2, m+k+1 \rrbracket})} \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0} \right] \right)^{1/p} > t \right) \\ \leq f_{1, k, p}(q, C_{p, \mathbb{B}}) \int_0^1 \mathbb{P} \left(\max_{m+1 \leq j \leq N} (\mathbb{E} [\|D_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0}])^{1/p} > tu \right) u^{q-1} du \\ + f_{1, k, p}(q, C_{p, \mathbb{B}}) \int_0^1 \mathbb{P} \left(\left(\sum_{j=m+1}^N \mathbb{E} [\|D_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{\llbracket 1, j-1 \rrbracket} \vee \mathcal{F}'_{J_0}] \right)^{1/p} > tu \right) u^{q-1} du =: P_1 + P_2.$$

Let us estimate P_1 . A union bound gives

$$(3.56) \quad P_1 \leq f_{1, k, p}(q, C_{p, \mathbb{B}}) \sum_{j=m+1}^N \int_0^1 \mathbb{P} \left((\mathbb{E} [\|D_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0}])^{1/p} > tu \right) u^{q-1} du.$$

Using the fact that D_j is $\mathcal{F}_{[1,j]} \vee \mathcal{F}'_{\mathbb{Z}}$ -measurable, one gets in view of Lemma A.2 that

$$(3.57) \quad P_1 \leq f_{1,k,p}(q, C_{p,\mathbb{B}}) \sum_{j=m+1}^N \int_0^1 \mathbb{P} \left(\left(\mathbb{E} [\|D_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{[1,j]} \vee \mathcal{F}'_{J_0}] \right)^{1/p} > tu \right) u^{q-1} du.$$

Moreover, defining

$$(3.58) \quad D'_j = \sum_{\mathbf{i} \in \text{Inc}_{j-1}^m} h_{\mathbf{i},j} \left(\xi_{\mathbf{i}}, \xi'_{[m+1, m+k+1]} \right)$$

one derive by the same arguments as those who led to (3.44) that

$$(3.59) \quad \mathbb{E} [\|D_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{[1,j]} \vee \mathcal{F}'_{J_0}] \stackrel{\text{law}}{=} \mathbb{E} [\|D'_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{[1,j-1]} \vee \mathcal{F}'_{J_0 \cup \{m+1\}}]$$

and (3.57) allows us to infer that

$$(3.60) \quad P_1 \leq f_{1,k,p}(q, C_{p,\mathbb{B}}) \sum_{j=m+1}^N \int_0^1 \mathbb{P} \left(\left(\mathbb{E} [\|D'_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{[1,j-1]} \vee \mathcal{F}'_{J_0 \cup \{m+1\}}] \right)^{1/p} > tu \right) u^{q-1} du.$$

Since D'_j is $\mathcal{F}_{[1,j-1]} \vee \mathcal{F}'_{\mathbb{Z}}$ -measurable, Lemma A.2 gives that

$$(3.61) \quad \mathbb{E} [\|D'_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{[1,j-1]} \vee \mathcal{F}'_{J_0 \cup \{m+1\}}] = \mathbb{E} [\|D'_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0 \cup \{m+1\}}]$$

hence (3.60) can be rephrased as

$$(3.62) \quad P_1 \leq f_{1,k,p}(q, C_{p,\mathbb{B}}) \sum_{j=m+1}^N \int_0^1 \mathbb{P} \left(\left(\mathbb{E} [\|D'_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0 \cup \{m+1\}}] \right)^{1/p} > tu \right) u^{q-1} du.$$

Now we are in position to use the induction assumption in the following context: for a fixed $j \in [m+1, N]$,

- $\tilde{k} = k + 1$,
- $\tilde{h}_{\mathbf{i}}: S^m \times S^{k+1} \rightarrow \mathbb{B}$ defined by

$$(3.63) \quad \tilde{h}_{\mathbf{i}}(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+k+1}) = h_{\mathbf{i},j}(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+k+1}),$$

- $\tilde{q} = q + 1$,
- $\tilde{J}_0 = J_0 \cup \{m+1\}$,
- $\tilde{N} = j - 1$,

which gives, in view of (3.34),

$$(3.64) \quad P_1 \leq f_{m+1,k,p}(q, C_{p,\mathbb{B}}) \sum_{J \subset [1,m]} \sum_{\mathbf{i}_J \in \mathbb{N}^J} \sum_{j=m+1}^N \int_0^1 \int_0^1 g_{\mathbf{i}_J}^{(j)}(tuv) u^{q-1} v^q dudv,$$

where

$$(3.65) \quad g_{\mathbf{i}_J}^{(j)}(t) = \mathbb{P} \left(\left(\sum_{\mathbf{i}_J \mathbf{c}: \mathbf{i}_J + \mathbf{i}_J \mathbf{c} \in \text{Inc}_{j-1}^m} \mathbb{E} \left[\left\| h_{\mathbf{i}_J + \mathbf{i}_J \mathbf{c} + j \mathbf{e}_{m+1}} \left(\xi'_{[1, m+k]} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}'_{J \cup J_0} \right] \right)^{1/p} > t \right).$$

Defining for $J \subset [1, m+1]$ such that $m+1 \in J$,

$$(3.66) \quad g_{\mathbf{i}_J}(t) = \mathbb{P} \left(\left(\sum_{\mathbf{i}_J \mathbf{c}: \mathbf{i}_J \cup \{m+1\} + \mathbf{i}_J \mathbf{c} \in \text{Inc}_N^{m+1}} \mathbb{E} \left[\left\| h_{\mathbf{i}_J + \mathbf{i}_J \mathbf{c} + m+1 \mathbf{e}_{m+1}} \left(\xi'_{[1, m+k]} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}'_{J \cup J_0} \right] \right)^{1/p} > t \right)$$

and using the elementary bound (3.47) gives

$$(3.67) \quad P_1 \leq f_{m+1,k,p}(q, C_{p,\mathbb{B}}) \sum_{\substack{J \subset [1,m+1] \\ m+1 \in J}} \sum_{\mathbf{i}_J \in \mathbb{N}^J} \int_0^1 g_{\mathbf{i}_J}^{(j)}(tu) u^{q-1} du.$$

Let us now bound P_2 defined by (3.55).

One can derive

$$(3.68) \quad \mathbb{E} [\|D_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{[1,j-1]} \vee \mathcal{F}'_{J_0}] = \mathbb{E} [\|D'_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{[1,j-1]} \vee \mathcal{F}'_{J_0}]$$

using that $\mathbb{E}[g(U, V_1) \mid W] = \mathbb{E}[g(U, V_2) \mid W]$ if V_1 and V_2 have the same distribution and are both independent of $\sigma(U, W)$ and D_j and D'_j only differ by the replacement of ξ_j in D_j by ξ'_m . Moreover, using Lemma A.2, the following equality holds $\mathbb{E} [\|D'_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{[1,j-1]} \vee \mathcal{F}'_{J_0}] = \mathbb{E} [\|D'_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0}]$. Consequently,

$$(3.69) \quad P_2 \leq f_{1,k,p}(q, C_{p,\mathbb{B}}) \int_0^1 \mathbb{P} \left(\left(\sum_{j=m+1}^N \mathbb{E} [\|D'_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0}] \right)^{1/p} > tu \right) u^{q-1} du.$$

In order to use the induction assumption, we need to view the term $\sum_{j=m+1}^N \mathbb{E} [\|D'_j\|_{\mathbb{B}}^p \mid \mathcal{F}_{\mathbb{Z}} \vee \mathcal{F}'_{J_0}]$ as the conditional expectation of the p -th power norm of an element of a new Banach space. This leads us to introduce the Banach space $(\tilde{\mathbb{B}}, \|\cdot\|_{\tilde{\mathbb{B}}})$ by

$$(3.70) \quad \tilde{\mathbb{B}} = \left\{ (x_j)_{j \in [m+1, N]}, x_j \in \mathbb{B} \right\}, \quad \left\| (x_j)_{j \in [m+1, N]} \right\|_{\tilde{\mathbb{B}}} = \left(\sum_{j=m+1}^N \|x_j\|_{\mathbb{B}}^p \right)^{1/p}.$$

We apply the induction assumption to the following setting

- $\tilde{k} = k + 1$,
- $\tilde{h}_{\mathbf{i}}: S^m \times S^{k+1} \rightarrow \tilde{\mathbb{B}}$ defined by

$$(3.71) \quad \tilde{h}_{\mathbf{i}}(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+k+1}) = (h_{\mathbf{i},j}(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+k+1}) \mathbf{1}_{i_m \leq j-1})_{j \in [m+1, N]},$$

- $\tilde{q} = q + 1$,
- $\tilde{J}_0 = J_0$,
- $\tilde{N} = N$.

Using (3.34) and (3.47), we find that

$$(3.72) \quad P_2 \leq f_{m+1,k,p}(q, C_{p,\mathbb{B}}) \int_0^1 \sum_{J \subset [1,m]} \sum_{\mathbf{i}_J \in \mathbb{N}^J} g_{\mathbf{i}_J}(tw) w^{q-1} dw,$$

where

$$(3.73) \quad g_{\mathbf{i}_J}(t) = \mathbb{P} \left(\left(\sum_{\mathbf{i}_{J^c}: \mathbf{i}_J + \mathbf{i}_{J^c} \in \text{Inc}_N^m} \mathbb{E} \left[\left\| \widetilde{h_{\mathbf{i}_J + \mathbf{i}_{J^c}}(\xi'_{[1,m+k]})} \right\|_{\tilde{\mathbb{B}}}^p \mid \mathcal{F}'_{J \cup J_0} \right] \right)^{1/p} > t \right).$$

Going back to the $\|\cdot\|_{\mathbb{B}}$ -norm and the expression of $\tilde{h}_{\mathbf{i}}$ finally gives

$$(3.74) \quad P_2 \leq f_{m+1,k,p}(q, C_{p,\mathbb{B}}) \int_0^1 \sum_{\substack{J \subset [1,m+1] \\ m+1 \notin J}} \sum_{\mathbf{i}_J \in \mathbb{N}^J} G_{\mathbf{i}_J}(tw) w^{q-1} dw$$

with

$$G_{i_J}(t) = \mathbb{P} \left(\left(\sum_{\substack{i_{J^c \cup \{m+1\}} \\ i_J + i_{J^c \cup \{m+1\}} \in \text{Inc}_N^{m+1}}} \mathbb{E} \left[\left\| h_{i_J + i_{J^c \cup \{m+1\}}} \left(\xi'_{[1, m+k]} \right) \right\|_{\mathbb{B}}^p \mid \mathcal{F}_{J \cup J_0} \right] \right)^{1/p} > t \right).$$

The combination of (3.67) with (3.72) concludes the proof that $A(m)$ is true for each m and that of Theorem 1.3.

3.3. Proof of Corollary 1.7. We start from (1.30), which gives

$$(3.75) \quad \max_{m \leq n \leq N} \|U_{m,n}(h)\|_{\mathbb{B}} \leq K_m \sum_{c=d}^m \max_{k \leq n \leq N} \|U_{c,n}(h^{(c)})\|_{\mathbb{B}} N^{m-c},$$

where $h^{(c)}$ is defined as in (1.31) and K_m depends only on m . As a consequence, one has

$$(3.76) \quad \mathbb{P} \left(\max_{m \leq n \leq N} \|U_{m,n}(h)\|_{\mathbb{B}} > t \right) \leq \sum_{c=d}^m \mathbb{P} \left(\max_{k \leq n \leq N} \|U_{c,n}(h^{(c)})\|_{\mathbb{B}} N^{m-c} > t / (mK_m) \right).$$

By Corollary 1.4 applied with m replaced by c and h by $h^{(c)}$, we derive that

$$(3.77) \quad \begin{aligned} \mathbb{P} \left(\max_{m \leq n \leq N} \|U_{m,n}(h)\|_{\mathbb{B}} > t \right) &\leq K(m, p, q, \mathbb{B}) \sum_{c=d}^m N^c \int_0^1 u^{q-1} \mathbb{P} \left(\|h^{(c)}(\xi_{[1,c]})\|_{\mathbb{B}} > t \frac{u}{mK_m} \right) du \\ &+ K(m, p, q, \mathbb{B}) \sum_{c=d}^m \sum_{j=1}^c N^j \int_0^1 u^{q-1} \mathbb{P} \left(N^{\frac{c-j}{p}} \left(\mathbb{E} \left[\left\| h^{(c)}(\xi_{[1,c]}) \right\|_{\mathbb{B}}^p \mid \xi_{[1,j]} \right] \right)^{1/p} > t \frac{u}{mK_m} \right) du \\ &+ K(m, p, q, \mathbb{B}) (mK_m)^q t^{-q} \sum_{c=d}^m N^{cq/p} \left(\mathbb{E} \left[\left\| h^{(c)}(\xi_{[1,c]}) \right\|_{\mathbb{B}}^p \right] \right)^{q/p}. \end{aligned}$$

We conclude using the fact that

$$\mathbb{E} \left[\left\| h^{(c)}(\xi_{[1,c]}) \right\|_{\mathbb{B}}^p \mid \xi_{[1,j]} \right] \leq \kappa_m \mathbb{E} \left[\left\| h(\xi_{[1,m]}) \right\|_{\mathbb{B}}^p \mid \xi_{[1,j]} \right].$$

3.4. Proof of the results of Subsection 2.1.

Proof of Theorem 2.1. It suffices to show that

$$(3.78) \quad \sum_{N \geq 2} \mathbb{P} \left(\max_{m \leq n \leq 2^N} \left\| \sum_{i \in \text{Inc}_n^m} h(\xi_i) \right\|_{\mathbb{B}} > 2^{1+Nm/p} \right) \leq \mathbb{E} \left[\left\| h(\xi_{[1,m]}) \right\|_{\mathbb{B}}^p \right],$$

then (2.2) can be deduced by replacing h by h/t and noticing that

$$(3.79) \quad \sup_{n \geq m} \frac{1}{n^{m/p}} \left\| \sum_{i \in \text{Inc}_n^m} h(\xi_i) \right\|_{\mathbb{B}} \leq \sup_{N \geq 1} \frac{1}{2^{m/pN}} \max_{m \leq n \leq 2^{N+1}} \left\| \sum_{i \in \text{Inc}_n^m} h(\xi_i) \right\|_{\mathbb{B}}$$

$$(3.80) \quad \leq 2^{m/p} \sup_{N \geq 2} \frac{1}{2^{m/pN}} \max_{m \leq n \leq 2^N} \left\| \sum_{i \in \text{Inc}_n^m} h(\xi_i) \right\|_{\mathbb{B}}.$$

In order to show (3.78), define for a fixed $N \geq 2$

$$(3.81) \quad p_N := \mathbb{P} \left(\max_{m \leq n \leq 2^N} \left\| \sum_{i \in \text{Inc}_n^m} h(\xi_i) \right\|_{\mathbb{B}} > 2^{1+Nm/p} \right)$$

and define the function $h_{\leq}: S^m \rightarrow \mathbb{B}$ and $h_{>}: S^m \rightarrow \mathbb{B}$ by

$$(3.82) \quad h_{\leq}(x_1, \dots, x_m) = h(x_1, \dots, x_m) \mathbf{1}_{\|h(x_1, \dots, x_m)\|_{\mathbb{B}} \leq 2^{Nm/p}} \text{ and}$$

$$(3.83) \quad h_{>}(x_1, \dots, x_m) = h(x_1, \dots, x_m) \mathbf{1}_{\|h(x_1, \dots, x_m)\|_{\mathbb{B}} > 2^{Nm/p}}.$$

Since (2.1) does not hold with h replaced by h_{\leq} , we define

$$(3.84) \quad \widetilde{h}_{\leq}(x_1, \dots, x_m) = \sum_{I \subset \llbracket 1, m \rrbracket} (-1)^{m-|I|} \mathbb{E} [h_{\leq}(x_I, \xi_{\llbracket 1, m \rrbracket \setminus I})]$$

where $(x_I, \xi_{\llbracket 1, m \rrbracket \setminus I})$ is the element of S^m whose coordinate i is ξ_i if $i \in I$ and ξ_i otherwise. We define similarly

$$(3.85) \quad \widetilde{h}_{>}(x_1, \dots, x_m) = \sum_{I \subset \llbracket 1, m \rrbracket} (-1)^{m-|I|} \mathbb{E} [h_{>}(x_I, \xi_{\llbracket 1, m \rrbracket \setminus I})].$$

By construction and assumption (2.1), $h = \widetilde{h}_{\leq} + \widetilde{h}_{>}$, which shows that

$$(3.86) \quad p_N \leq p_{N, \leq} + p_{N, >}, \text{ with } p_{N, \leq} = \mathbb{P} \left(\max_{m \leq n \leq 2^N} \left\| \sum_{i \in \text{Inc}_n^m} \widetilde{h}_{\leq}(\xi_i) \right\|_{\mathbb{B}} > 2^{Nm/p} \right),$$

$$(3.87) \quad p_{N, >} = \mathbb{P} \left(\max_{m \leq n \leq 2^N} \left\| \sum_{i \in \text{Inc}_n^m} \widetilde{h}_{>}(\xi_i) \right\|_{\mathbb{B}} > 2^{Nm/p} \right).$$

Let us bound $p_{N, \leq}$. Using Markov's inequality and noticing that

$$(3.88) \quad \forall \ell_0 \in \llbracket 1, m \rrbracket, \quad \mathbb{E} [\widetilde{h}_{\leq}(\xi_{\llbracket 1, m \rrbracket}) \mid \xi_{\llbracket 1, m \rrbracket \setminus \{\ell_0\}}] = 0,$$

we find by a use of (1.38) with p replaced by q that

$$\begin{aligned} p_{N, \leq} &\leq 2^{-Nmr/p} \mathbb{E} \left[\max_{m \leq n \leq 2^N} \left\| \sum_{(i_\ell) \in \text{Inc}_n^m} \widetilde{h}_{\leq}(\xi_{\llbracket 1, m \rrbracket}) \right\|_{\mathbb{B}}^r \right] \\ &\leq K(m, p, r, \mathbb{B}) 2^{-Nmr/p} 2^{Nm} \mathbb{E} \left[\left\| \widetilde{h}_{\leq}(\xi_{\llbracket 1, m \rrbracket}) \right\|_{\mathbb{B}}^r \right]. \end{aligned}$$

Then noticing that $\widetilde{h}_{\leq}(\xi_{\llbracket 1, m \rrbracket}) = \sum_{I \subset \llbracket 1, m \rrbracket} (-1)^{m-|I|} \mathbb{E} [h_{\leq}(\xi_{\llbracket 1, m \rrbracket}) \mid \xi_I]$ we get that

$$(3.89) \quad p_{N, \leq} \leq K(m, p, r, \mathbb{B}) 2^{Nm(1-r/p)} \mathbb{E} \left[\left\| h_{\leq}(\xi_{\llbracket 1, m \rrbracket}) \right\|_{\mathbb{B}}^r \right].$$

Denoting by H the random variable $\|h(\xi_{\llbracket 1, m \rrbracket})\|_{\mathbb{B}}$, we obtained that

$$(3.90) \quad p_{N, \leq} \leq K(m, p, r, \mathbb{B}) 2^{Nm(1-r/p)} \mathbb{E} [H^r \mathbf{1}_{H \leq 2^{mN/p}}]$$

and from the elementary bound $\sum_{N \geq 2} 2^{Nm(1-r/p)} \mathbf{1}_{H \leq 2^{mN/p}} \leq \kappa_{p,q,m} H^{p-r}$, we infer that $\sum_{N \geq 2} p_{N, \leq} < \infty$.

It remains to show the convergence of $\sum_{N \geq 2} p_{N, >}$. To do so, we use Markov's inequality combined with the observations that $\widetilde{h}_{>}(\xi_{\llbracket 1, m \rrbracket}) = \sum_{I \subset \llbracket 1, m \rrbracket} (-1)^{m-|I|} \mathbb{E} [h_{>}(\xi_{\llbracket 1, m \rrbracket}) \mid \xi_I]$ and

$$\max_{m \leq n \leq 2^N} \left\| \sum_{i \in \text{Inc}_n^m} \widetilde{h}_{>}(\xi_i) \right\|_{\mathbb{B}} \leq \sum_{i \in \text{Inc}_{2^N}^m} \left\| \widetilde{h}_{>}(\xi_i) \right\|_{\mathbb{B}},$$

we infer that

$$(3.91) \quad p_{N, >} \leq 2^{-Nm/p} 2^{mN} \mathbb{E} [H \mathbf{1}_{H > 2^{mN/p}}].$$

and the elementary bound $\sum_{N \geq 2} 2^{mN(1-1/p)} \mathbf{1}_{H > 2^{mN/p}} \leq c_p H^{p-1}$ allows to conclude. This ends the proof of Theorem 2.1. \square

Proof of Theorem 2.2. Suppose now that $\alpha > 0$. By Hoeffding's decomposition (1.30), it suffices to prove that for each positive ε ,

$$(3.92) \quad \sum_{N=0}^{\infty} 2^{N(\gamma+1)} \mathbb{P} \left(\sup_{n \geq 2^N} n^\alpha \frac{1}{\binom{n}{c}} \left\| U_{c,n} \left(h^{(c)} \right) \right\|_{\mathbb{B}} > \varepsilon \right) < \infty.$$

Writing

$$\begin{aligned} \mathbb{P} \left(\sup_{n \geq 2^N} n^\alpha \frac{1}{\binom{n}{c}} \left\| U_{c,n} \left(h^{(c)} \right) \right\|_{\mathbb{B}} > \varepsilon \right) &= \mathbb{P} \left(\bigcup_{k=1}^{\infty} \left\{ \max_{2^{N+k-1} \leq n \leq 2^{N+k}} n^\alpha \frac{1}{\binom{n}{c}} \left\| U_{c,n} \left(h^{(c)} \right) \right\|_{\mathbb{B}} > \varepsilon \right\} \right) \\ &\leq \sum_{k=1}^{\infty} \mathbb{P} \left(\max_{2^{N+k-1} \leq n \leq 2^{N+k}} n^\alpha \frac{1}{\binom{n}{c}} \left\| U_{c,n} \left(h^{(c)} \right) \right\|_{\mathbb{B}} > \varepsilon \right) \\ &\leq \sum_{k=1}^{\infty} \mathbb{P} \left(\max_{2^{N+k-1} \leq n \leq 2^{N+k}} \left\| U_{c,n} \left(h^{(c)} \right) \right\|_{\mathbb{B}} > K \varepsilon 2^{(N+k)(c-\alpha)} \right), \end{aligned}$$

where K depends only on c and α , we are reduced to prove that for each positive ε ,

$$(3.93) \quad \sum_{N=1}^{\infty} \sum_{k=1}^{\infty} 2^{N\gamma} \mathbb{P} \left(\max_{c \leq n \leq 2^{N+k}} \left\| U_{c,n} \left(h^{(c)} \right) \right\|_{\mathbb{B}} > \varepsilon 2^{(N+k)(c-\alpha)} \right) < \infty.$$

To do so, we use Corollary 1.4 in the case with m replaced by c , h by $h^{(c)}$, N by 2^{N+k} , q that will be specified later and $t = \varepsilon 2^{(N+k)(c-\alpha)}$ (note that symmetry of $h^{(c)}$ implies that the summand in the second term of the right hand side of (1.27) depends only on the cardinal of the set J). We are thus reduced to show that for each $c \in \llbracket d, m \rrbracket$ and $j \in \llbracket 0, c \rrbracket$,

$$(3.94) \quad \sum_{N=1}^{\infty} \sum_{k=1}^{\infty} a_{N,k,c} < \infty,$$

where

$$a_{N,k,c} := 2^{N(\gamma+1)} 2^{(N+k)j} \int_0^1 \mathbb{P} \left(2^{\frac{N+k}{r}(c-j)} \left(\mathbb{E} \left[\left\| h^{(c)} \left(\xi_{\llbracket 1, c \rrbracket} \right) \right\|_{\mathbb{B}}^r \mid \xi_{\llbracket 1, j \rrbracket} \right] \right)^{1/r} > \varepsilon 2^{(N+k)(c-\alpha)} u \right) u^{q-1} du.$$

Doing the change of index $\ell = N + k$ for a fixed N , switching the sums and using the fact that $\sum_{N=1}^{\ell} 2^{N(\gamma+1)} \leq c_\gamma 2^{\ell(\gamma+1)}$, we are reduced to prove that for each $c \in \llbracket d, m \rrbracket$ and $j \in \llbracket 0, c \rrbracket$,

$$(3.95) \quad \sum_{\ell=0}^{\infty} 2^{\ell(\gamma+1+j)} \int_0^1 \mathbb{P} \left(2^{\frac{\ell}{r}(c-j)} \left(\mathbb{E} \left[\left\| h^{(c)} \left(\xi_{\llbracket 1, c \rrbracket} \right) \right\|_{\mathbb{B}}^r \mid \xi_{\llbracket 1, j \rrbracket} \right] \right)^{1/r} > \varepsilon 2^{\ell(c-\alpha)} u \right) u^{q-1} du < \infty.$$

Using $\sum_{\ell=0}^{\infty} 2^{\ell\beta} \mathbb{P} \left(Y > 2^{\ell\beta'} \right) \leq C_{\beta, \beta'} \mathbb{E} \left[Y^{\beta/\beta'} \right]$, the series involved in (3.95) is convergent as long as

$$(3.96) \quad \left(\mathbb{E} \left[\left\| h^{(c)} \left(\xi_{\llbracket 1, c \rrbracket} \right) \right\|_{\mathbb{B}}^r \mid \xi_{\llbracket 1, j \rrbracket} \right] \right)^{1/r} \in \mathbb{L}^{q(\gamma, c, j)},$$

where

$$(3.97) \quad q(\gamma, c, j) = \frac{\gamma + j + 1}{c - \alpha + \frac{j-c}{r}}.$$

Since

$$(3.98) \quad \mathbb{E} \left[\left\| h^{(c)} \left(\xi_{\llbracket 1, c \rrbracket} \right) \right\|_{\mathbb{B}}^r \mid \xi_{\llbracket 1, j \rrbracket} \right] \leq K_c \mathbb{E} \left[\left\| h \left(\xi_{\llbracket 1, m \rrbracket} \right) \right\|_{\mathbb{B}}^r \mid \xi_{\llbracket 1, j \rrbracket} \right] = H_{j,r}^r,$$

(3.96) will be satisfied as only as $H_{j,r} \in \mathbb{L}^{q(\gamma,c,j)}$ for each c such that $c \geq \max\{d, j\}$. We conclude by noticing that $q(\gamma, c, j)$ is decreasing in c . \square

3.5. Proof of Theorem 2.3. As mentioned right after the statement of Theorem 2.3, the convergence of the finite-dimensional distributions is already contained in Corollary 1 of [35]. Therefore, it suffices to prove tightness of $\left(n^{d/2-m} \mathcal{U}_{m,n,h}^{\text{pl}}\right)_{n \geq m}$ in \mathcal{H}_α^o .

Using Hoeffding's decomposition (cf. (1.30)), one can decompose the process $(\mathcal{U}_{m,n}(h, t))_{t \in [0,1]}$ as a sum of similar processes associated with the kernels $h^{(c)}$, $c \in \llbracket d+1, m \rrbracket$. It suffices to prove that each of them are tight. Since required normalization for the original process, namely $n^{m-d/2}$, is bigger than $n^{m-c/2}$, it suffices to prove tightness of $\left(n^{-d/2} \mathcal{U}_{d,n,h^{(d)}}^{\text{pl}}\right)_{n \geq m}$ in \mathcal{H}_α^o . Using Proposition 1.1 in [20] with $X_k = \sum_{i \in \text{Inc}_k^{d-1}} h^{(d)}(\xi_i, \xi_k)$ and denoting $S_N = \sum_{k=1}^N X_k$, we are reduced to prove that for each positive ε ,

$$(3.99) \quad \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{j=J}^{\lfloor \log_2 n \rfloor} \sum_{k=0}^{2^j-1} \mathbb{P} \left(|S_{\lfloor n(k+1)2^{-j} \rfloor} - S_{\lfloor nk2^{-j} \rfloor}| > n^{d/2} 2^{-\alpha j} \varepsilon \right) = 0.$$

To do so, we apply Theorem 1.3 with m replaced by d , $t = n^{d/2} 2^{-\alpha j} \varepsilon$ (with fixed n, j and k), $\mathbb{B} = \mathbb{R}$, $p = 2$, $q = p(\alpha) + 1$ and for $\mathbf{i} = (i_\ell)_{\ell \in \llbracket 1, d \rrbracket}$,

$$(3.100) \quad h_{\mathbf{i}}(x_{\mathbf{i}}) = \begin{cases} h^{(d)}(x_{\mathbf{i}}) & \text{if } i_d \in \llbracket \lfloor nk2^{-j} \rfloor, \lfloor n(k+1)2^{-j} \rfloor \rrbracket, \\ 0 & \text{otherwise.} \end{cases}$$

We define $Y_0 := \left(\mathbb{E} \left[\left(h^{(d)}(\xi_{\llbracket 1, d \rrbracket}) \right)^2 \right]\right)^{1/2}$ and the random variable

$$Y := \max_{1 \leq a \leq d} \left(\mathbb{E} \left[\left(h^{(d)}(\xi_{\llbracket 1, d \rrbracket}) \right)^2 \mid \xi_{\llbracket 1, a \rrbracket} \right] \right)^{1/2},$$

which come into play in the right hand side of (1.23). Note that the assumption (2.14) implies that

$$(3.101) \quad \lim_{t \rightarrow \infty} t^{p(\alpha)} \mathbb{P}(Y > t) = 0.$$

Moreover, the sum over J in (1.23) will be split according to the case where d belongs to the set J or not. If $d \in J$, the sum over \mathbf{i}_J can be written as $\sum_{i_d \in \llbracket \lfloor nk2^{-j} \rfloor, \lfloor n(k+1)2^{-j} \rfloor \rrbracket} \sum_{\mathbf{i}_{J \setminus \{d\}}}$ and the number of summed terms does not exceed $2n2^{-j} (nk2^{1-j})^{\text{Card}(J)-1}$ while the sum over \mathbf{i}_{J^c} contains at most $(nk2^{1-j})^{d-\text{Card}(J)}$ terms.

When $d \notin J$ and J is not empty, the sum over \mathbf{i}_J contains at most $(nk2^{1-j})^{\text{Card}(J)}$ elements and that over \mathbf{i}_{J^c} at most $2n2^{-j} (nk2^{1-j})^{d-1-\text{Card}(J)}$ elements. All these considerations lead to the bound

$$(3.102) \quad \begin{aligned} P_{n,k,j} &:= \mathbb{P} \left(|S_{\lfloor n(k+1)2^{-j} \rfloor} - S_{\lfloor nk2^{-j} \rfloor}| > n^{d/2} 2^{-\alpha j} \varepsilon \right) \\ &\leq C_\alpha (n2^{-j}) (nk2^{-j})^{d-1} \int_0^1 u^{p(\alpha)} \mathbb{P} \left(Y > n^{d/2} 2^{-\alpha j} \varepsilon u \right) du \\ &\quad + C_\alpha \sum_{a=1}^d n2^{-j} (nk2^{-j})^{a-1} \int_0^1 \mathbb{P} \left((nk2^{-j})^{(d-a)/2} Y > C'_\alpha n^{d/2} 2^{-\alpha j} \varepsilon u \right) u^{p(\alpha)} du \\ &\quad + C_\alpha \sum_{a=1}^d (nk2^{-j})^a \int_0^1 \mathbb{P} \left((n2^{-j})^{1/2} (nk2^{-j})^{(d-a-1)/2} Y > C'_\alpha n^{d/2} 2^{-\alpha j} \varepsilon u \right) u^{p(\alpha)} du \end{aligned}$$

$$+ C_\alpha \left(n^{d/2} 2^{-\alpha j} \varepsilon \right)^{-p(\alpha)-1} \left((n 2^{-j}) (n k 2^{-j})^{d-1} Y_0^2 \right)^{(p(\alpha)+1)/2},$$

which can be simplified as follows

$$(3.103) \quad \begin{aligned} P_{n,k,j} &\leq C_\alpha n^d 2^{-jd} k^{d-1} \int_0^1 u^{p(\alpha)} \mathbb{P} \left(Y > n^{d/2} 2^{-\alpha j} \varepsilon u \right) du \\ &\quad + C_\alpha \sum_{a=1}^d n^a k^{a-1} 2^{-ja} \int_0^1 \mathbb{P} \left(Y > C'_\alpha n^{a/2} k^{(a-d)/2} 2^{-j(\alpha-(d-a)/2)} \varepsilon u \right) u^{p(\alpha)} du \\ &\quad + C_\alpha \sum_{a=1}^d (n k 2^{-j})^a \int_0^1 \mathbb{P} \left(Y > C'_\alpha n^{a/2} 2^{-(\alpha-(d-a)/2)j} \varepsilon k^{-(d-a-1)/2} u \right) u^{p(\alpha)} du \\ &\quad \quad \quad + C_\alpha 2^j \left(p(\alpha) + \alpha - d \frac{p(\alpha)+1}{2} \right) k^{(d-1) \frac{p(\alpha)+1}{2}} Y_0^{p(\alpha)}. \end{aligned}$$

Define

$$(3.104) \quad \tau(R) := \sup_{t \geq R} t^{p(\alpha)} \mathbb{P}(Y > t).$$

Note that assumption (2.14) combined with Lemma 1.4 in [17] guarantees that $\lim_{R \rightarrow \infty} \tau(R) = 0$. By looking at monotonicity in each variable a, j and k , we deduce that there exists a constant K_α , depending only on α , such that

$$(3.105) \quad \min_{a \in [0, d]} \min_{j \in [J, \lceil \log_2 n \rceil]} \min_{k \in [0, 2^j - 1]} C'_\alpha n^{a/2} k^{(a-d)/2} 2^{-j(\alpha-(d-a)/2)} \varepsilon \geq K_\alpha n^{1/p(\alpha)},$$

$$(3.106) \quad \min_{a \in [0, d]} \min_{j \in [J, \lceil \log_2 n \rceil]} \min_{k \in [0, 2^j - 1]} C'_\alpha n^{a/2} 2^{-(\alpha-(d-a)/2)j} \varepsilon k^{(d-a-1)/2} \geq K_\alpha n^{1/p(\alpha)}.$$

Consequently, we derive that

$$(3.107) \quad \begin{aligned} &\mathbb{P} \left(Y > C'_\alpha n^{a/2} k^{(a-d)/2} 2^{-j(\alpha-(d-a)/2)} \varepsilon u \right) \\ &\leq \left(C'_\alpha n^{a/2} k^{(a-d)/2} 2^{-j(\alpha-(d-a)/2)} \varepsilon u \right)^{-p(\alpha)} \sup_{t > C'_\alpha n^{a/2} k^{(a-d)/2} 2^{-j(\alpha-(d-a)/2)} \varepsilon u} t^{p(\alpha)} \mathbb{P}(Y > t) \\ &\leq \left(C'_\alpha n^{a/2} k^{(a-d)/2} 2^{-j(\alpha-(d-a)/2)} \varepsilon u \right)^{-p(\alpha)} \tau \left(K_\alpha n^{1/p(\alpha)} u \right) \end{aligned}$$

and with a similar reasoning, we infer that

$$(3.108) \quad \begin{aligned} &\mathbb{P} \left(Y > C'_\alpha n^{a/2} 2^{-(\alpha-(d-a)/2)j} \varepsilon k^{(d-a-1)/2} u \right) \\ &\leq \left(C'_\alpha n^{a/2} 2^{-(\alpha-(d-a)/2)j} \varepsilon k^{-(d-a-1)/2} \right)^{-p(\alpha)} \tau \left(K_\alpha n^{1/p(\alpha)} u \right) \end{aligned}$$

and plugging these bounds into (3.103) gives

$$(3.109) \quad \begin{aligned} P_{n,k,j} &\leq K_{1,\alpha} n^{d(1-p(\alpha)/2)} k^{d-1} 2^{j(\alpha p(\alpha)-d)} \int_0^1 \tau \left(K_\alpha n^{1/p(\alpha)} u \right) du \\ &\quad + K_{1,\alpha} \sum_{a=1}^d n^{a(1-p(\alpha)/2)} k^{a-1+p(\alpha)(d-a)/2} 2^{j(-a+\alpha p(\alpha)-(d-a)p(\alpha)/2)} \int_0^1 \tau \left(K_\alpha n^{1/p(\alpha)} u \right) du \\ &\quad \quad \quad + K_{1,\alpha} 2^j \left(p(\alpha) + \alpha - d \frac{p(\alpha)+1}{2} \right) k^{(d-1) \frac{p(\alpha)+1}{2}} Y_0^{p(\alpha)} \end{aligned}$$

where $K_{1,\alpha}$ depends only on α (note that since $p(\alpha) > 2$, the bound obtained via (3.108) for the third term of the right hand side of (3.103) is smaller than the one for the second term). Summing over k furnishes the bound

$$(3.110) \quad \sum_{k=0}^{2^j-1} P_{n,k,j} \leq K_{2,\alpha} n^{d(1-p(\alpha)/2)} 2^{j\alpha p(\alpha)} + K_{2,\alpha} \sum_{a=1}^d n^{a(1-p(\alpha)/2)} 2^{j\alpha p(\alpha)} \int_0^1 \tau \left(K_\alpha n^{1/p(\alpha)} u \right) du \\ + K_{2,\alpha} 2^{j(\alpha-1/2)} Y_0^{p(\alpha)},$$

then summing over j gives

$$(3.111) \quad \sum_{j=J}^{\lfloor \log_2 n \rfloor} \sum_{k=0}^{2^j-1} P_{n,k,j} \leq K_{3,\alpha} \int_0^1 \tau \left(K_\alpha n^{1/p(\alpha)} u \right) du + K_{3,\alpha} 2^{-J(1/2-\alpha)},$$

from which (3.99) follows. This ends the proof of Theorem 2.3.

3.6. Proof of Theorem 2.4. By definition of the functions $h^{(c)}$ given in (1.31) and symmetry of h , the incomplete U -statistic defined in (2.16) can be decomposed as

$$(3.112) \quad U_{m,n}^{\text{inc}}(h) = \sum_{c=d}^m \sum_{\mathbf{i} \in \text{Inc}_n^c} a_{n,c;\mathbf{i}} h^{(c)}(\xi_{\mathbf{i}}),$$

where

$$(3.113) \quad a_{n,c;\mathbf{i}} = \sum_{\substack{\mathbf{j} \in \text{Inc}_n^m \\ \{i_1, \dots, i_c\} \subset \{j_1, \dots, j_m\}}} a_{n;\mathbf{j}}.$$

The previous sum can be split according to the position of the indexes i_1, \dots, i_c . We define for $\mathbf{k} = (k_u)_{u=1}^c$ such that $1 \leq k_1 < \dots < k_c \leq m$ the random variable

$$(3.114) \quad a_{n,c,\mathbf{k};\mathbf{i}} = \sum_{\mathbf{j}} a_{n;\mathbf{j}},$$

where the sum carries over the $\mathbf{j} = (j_k)_{k \in \llbracket 1, m \rrbracket} \in \text{Inc}_n^m$ such that $j_{k_u} = i_u$ for each $u \in \llbracket 1, c \rrbracket$. As a consequence, it suffices to prove that for each $c \in \llbracket 1, m \rrbracket$, and each $\mathbf{k} = (k_u)_{u=1}^c$ such that $1 \leq k_1 < \dots < k_c \leq m$,

$$(3.115) \quad \mathbb{E} \left[\left\| \sum_{\mathbf{i} \in \text{Inc}_n^c} a_{n,c,\mathbf{k};\mathbf{i}} h^{(c)}(\xi_{\mathbf{i}}) \right\|_{\mathbb{B}}^q \right] \leq C(\mathbb{B}, m, p, q) \left(n^{q(m-d)+dq/p} p_n^q + n^{mq/p} p_n^{q/p} + n^m p_n \right).$$

To do so, we first condition on the random variables $a_{n;\mathbf{j}}$. Writing

$$(3.116) \quad a_{n,c,\mathbf{k};\mathbf{i}} = f_{c,\mathbf{k};\mathbf{i}} \left((a_{n;\mathbf{j}})_{\mathbf{j} \in \text{Inc}_n^m} \right)$$

and denoting by μ_0 the law of $a_{n;\mathbf{j}}$, we have

$$(3.117) \quad \mathbb{E} \left[\left\| \sum_{\mathbf{i} \in \text{Inc}_n^c} a_{n,c,\mathbf{k};\mathbf{i}} h^{(c)}(\xi_{\mathbf{i}}) \right\|_{\mathbb{B}}^q \right] = \int \mathbb{E} \left[\left\| \sum_{\mathbf{i} \in \text{Inc}_n^c} f_{c,\mathbf{k};\mathbf{i}} \left((x_{n;\mathbf{j}})_{\mathbf{j} \in \text{Inc}_n^m} \right) h^{(c)}(\xi_{\mathbf{i}}) \right\|_{\mathbb{B}}^q \right] d\mu_0(x_{n;\mathbf{j}}),$$

that is, the last integral is taken over the C_n^m variables $x_{n;\mathbf{j}}$, $\mathbf{j} \in \text{Inc}_n^m$. For fixed $x_{n;\mathbf{j}} \in \{0, 1\}$, $\mathbf{j} \in \text{Inc}_n^m$, we apply Corollary 1.8 in the following setting: $m = c$ and $h_{\mathbf{i}}(\xi_{\mathbf{i}}) = f_{c,\mathbf{k};\mathbf{i}} \left((x_{n;\mathbf{j}})_{\mathbf{j} \in \text{Inc}_n^m} \right) h^{(c)}(\xi_{\mathbf{i}})$. We

get, after having bounded the terms of the form $\mathbb{E} \left[\left(\mathbb{E} \left[\|h(\xi_{[1,c]})\|_{\mathbb{B}}^p \mid \xi_J \right] \right)^{q/p} \right]$ by $\mathbb{E} \left[\|h^{(c)}(\xi_i)\|_{\mathbb{B}}^q \right]$ which is in turn smaller than a constant depending only on q/p times $\mathbb{E} \left[\|h(\xi_{[1,m]})\|_{\mathbb{B}}^q \right] =: H$, that

$$(3.118) \quad \mathbb{E} \left[\left\| \sum_{i \in \text{Inc}_n^c} f_{c,\mathbf{k};i} \left((x_{n;j})_{j \in \text{Inc}_n^m} \right) h^{(c)}(\xi_i) \right\|_{\mathbb{B}}^q \right] \\ \leq K(m, p, q, \mathbb{B}) H \sum_{i \in \text{Inc}_N^m} \left| f_{c,\mathbf{k};i} \left((x_{n;j})_{j \in \text{Inc}_n^m} \right) \right|^q \\ + K(m, p, q, \mathbb{B}) H \sum_{\emptyset \subsetneq J \subsetneq [1,c]} \sum_{i_J \in \mathbb{N}^J} \mathbb{E} \left[\left(\sum_{i_{J^c}: i_J + i_{J^c} \in \text{Inc}_N^c} \left| f_{c,\mathbf{k};i_J + i_{J^c}} \left((x_{n;j})_{j \in \text{Inc}_n^m} \right) \right|^p \right)^{q/p} \right] \\ + K(m, p, q, \mathbb{B}) H \left(\sum_{i \in \text{Inc}_n^c} \left| f_{c,\mathbf{k};i} \left((x_{n;j})_{j \in \text{Inc}_n^m} \right) \right|^p \right)^{q/p}.$$

Integrating over the C_n^m variables $x_{n;j}$, $\mathbf{j} \in \text{Inc}_n^m$ with respect to μ_0 , we get that

$$(3.119) \quad \mathbb{E} \left[\left\| \sum_{i \in \text{Inc}_n^c} a_{n,c,\mathbf{k};i} h^{(c)}(\xi_i) \right\|_{\mathbb{B}}^q \right] \leq K(m, p, q, \mathbb{B}) H \sum_{i \in \text{Inc}_n^m} \mathbb{E} [|a_{n,c,\mathbf{k};i}|^q] \\ + K(m, p, q, \mathbb{B}) H \sum_{\emptyset \subsetneq J \subsetneq [1,c]} \sum_{i_J \in \mathbb{N}^J} \mathbb{E} \left[\left(\sum_{i_{J^c}: i_J + i_{J^c} \in \text{Inc}_n^c} \left| a_{n,c,\mathbf{k};i_J + i_{J^c}} \right|^p \right)^{q/p} \right] \\ + K(m, p, q, \mathbb{B}) H \mathbb{E} \left[\left(\sum_{i \in \text{Inc}_n^c} |a_{n,c,\mathbf{k};i}|^p \right)^{q/p} \right].$$

We are thus reduced to bound the moments of order q/p of random variables of the form

$$(3.120) \quad Y = \sum_{a \in A} \left(\sum_{b \in B} Y_{a,b} \right)^p,$$

where A and B are finite sets of respective cardinal $|A|$ and $|B|$, and $(Y_{a,b})_{a \in A, b \in B}$ is independent.

Lemma 3.3. *Let Y be a random variable of the form (3.120), where $Y_{a,b}$ has a Bernoulli distribution of parameter $y \in [0, 1]$, that is, $\mathbb{P}(Y_{a,b} = 1) = y$ and $\mathbb{P}(Y_{a,b} = 0) = 1 - y$. There exists a constant $C_{p,q}$ such that for $q \geq p$,*

$$(3.121) \quad \mathbb{E} \left[Y^{q/p} \right] \leq C_{p,q} \left(|A|^{q/p} |B|^q y^q + |A|^{q/p} |B|^{q/p} y^{q/p} + |A| |B| y \right)$$

Proof. We will use the following moment inequality for sums of independent non-negative random variables, given in Corollary 3 in [31]: for each $s \geq 1$, there exists a constant K_s such that for each finite set of independent non-negative random variables $(X_i)_{i \in I}$,

$$(3.122) \quad \mathbb{E} \left[\left(\sum_{i \in I} X_i \right)^s \right] \leq K_s \left(\sum_{i \in I} \mathbb{E}[X_i] \right)^s + K_s \sum_{i \in I} \mathbb{E}[X_i^s].$$

Applying (3.122) with $s = q/p$, $I = A$ and $X_i = (\sum_{b \in B} Y_{i,b})^p$ gives

$$(3.123) \quad \mathbb{E} \left[Y^{q/p} \right] \leq K_{q/p} \left(\sum_{a \in A} \mathbb{E} \left[\left(\sum_{b \in B} Y_{a,b} \right)^p \right] \right)^{q/p} + K_{q/p} \sum_{a \in A} \mathbb{E} \left[\left(\sum_{b \in B} Y_{a,b} \right)^q \right].$$

Applying (3.122) with $s = p$, $I = B$ and $X_i = Y_{a,i}$ gives

$$(3.124) \quad \mathbb{E} \left[\left(\sum_{b \in B} Y_{a,b} \right)^p \right] \leq K_p \left(\sum_{b \in B} \mathbb{E} [Y_{a,b}] \right)^p + K_p \sum_{b \in B} \mathbb{E} [Y_{a,b}^p] = K_p (|B|^p y^p + |B| y).$$

Applying (3.122) with $s = q$, $I = B$ and $X_i = Y_{a,i}$ gives

$$(3.125) \quad \mathbb{E} \left[\left(\sum_{b \in B} Y_{a,b} \right)^q \right] \leq K_q \left(\sum_{b \in B} \mathbb{E} [Y_{a,b}] \right)^q + K_q \sum_{b \in B} \mathbb{E} [Y_{a,b}^q] = K_q (|B|^q y^q + |B| y).$$

The combination of (3.123), (3.124), (3.125) and inequality $|A|^{q/p} \geq |A|$ ends the proof of Lemma 3.3. \square

We use Lemma 3.3 in order to bound each term of (3.119). Notice that the random variables $a_{n,c,\mathbf{k};i}$ have the same distribution. Moreover, each of these random variables is a sum of a number that does not exceed n^{m-c} hence by Lemma 3.3 with sets A and B of respective cardinal one and n^{m-c} gives

$$(3.126) \quad \sum_{\mathbf{i} \in \text{Inc}_n^m} \mathbb{E} [|a_{n,c,\mathbf{k};i}|^q] \leq C_{p,q} n^c \left(n^{(m-c)q} p_n^q + n^{(m-c)q/p} p_n^{q/p} + n^{m-c} p_n \right)$$

Since $q/p \geq 1$ and $q \geq 1$, the previous bound is non-increasing in c and reaches its maximum for $c = d$ hence the bound (3.126) can be converted into a bound independent of c , namely,

$$(3.127) \quad \sum_{\mathbf{i} \in \text{Inc}_n^m} \mathbb{E} [|a_{n,c,\mathbf{k};i}|^q] \leq C_{p,q} \left(n^{(m-d)q+d} p_n^q + n^{(m-d)q/p+d} p_n^{q/p} + n^m p_n \right).$$

For the second term of the right hand side of (3.119), consider $\emptyset \subsetneq J \subsetneq \llbracket 1, c \rrbracket$ of cardinal $j \in \llbracket 1, c-1 \rrbracket$. The sum over \mathbf{i}_J consists of at most n^j terms and the involved expectation can be bounded via Lemma 3.3 where A has at most n^{c-k} elements and B has at most n^{m-c} elements. We thus obtain

$$(3.128) \quad \begin{aligned} & \sum_{\emptyset \subsetneq J \subsetneq \llbracket 1, c \rrbracket} \sum_{\mathbf{i}_J \in \mathbb{N}^J} \mathbb{E} \left[\left(\sum_{\mathbf{i}_J \mathbf{c}: \mathbf{i}_J + \mathbf{i}_J \mathbf{c} \in \text{Inc}_N^c} |a_{n,c,\mathbf{k};\mathbf{i}_J + \mathbf{i}_J \mathbf{c}}|^p \right)^{q/p} \right] \\ & \leq C_{p,q} \sum_{j=1}^{c-1} n^j \left(n^{(c-j)q/p} n^{(m-c)q} p_n^q + n^{(c-j)q/p} n^{(m-c)q/p} p_n^{q/p} + n^{c-j} n^{m-c} p_n \right) \\ & = C_{p,q} \left(n^{mq} p_n^q n^{cq(1/p-1)} \sum_{j=1}^{c-1} n^{j(1-q/p)} + n^{mq/p} p_n^{q/p} \sum_{j=1}^{c-1} n^{j(1-q/p)} + n^m p_n \right). \end{aligned}$$

Since $q/p \geq 1$, monotonicity of the previous quantities in k and c leads to the bound

$$(3.129) \quad \begin{aligned} & \sum_{\emptyset \subsetneq J \subsetneq \llbracket 1, c \rrbracket} \sum_{\mathbf{i}_J \in \mathbb{N}^J} \mathbb{E} \left[\left(\sum_{\mathbf{i}_J \mathbf{c}: \mathbf{i}_J + \mathbf{i}_J \mathbf{c} \in \text{Inc}_n^c} |a_{n,c,\mathbf{k};\mathbf{i}_J + \mathbf{i}_J \mathbf{c}}|^p \right)^{q/p} \right] \\ & \leq C_{p,q} \left(n^{mq+q/p-1} p_n^q + n^{(m-1)q/p+1} p_n^{q/p} + C_{p,q} n^m p_n \right) \\ & \leq C_{p,q} \left(n^{mq+d(q/p-1)} p_n^q + n^{mq/p} p_n^{q/p} + n^m p_n \right). \end{aligned}$$

Finally, the last term of the right hand side of (3.119) is controlled via an application of Lemma 3.3 with a set A of cardinal smaller than n^c and a set B of cardinal smaller than n^{m-c} , which gives

$$(3.130) \quad \mathbb{E} \left[\left(\sum_{i \in \text{Inc}_n^c} |a_{n,c,k;i}|^p \right)^{q/p} \right] \leq C_{p,q} \left(n^{cq/p} n^{(m-c)q} p_n^q + n^{cq/p} n^{(m-c)q/p} p_n^{q/p} + n^m p_n \right),$$

and since the last quantity is maximal at $c = d$,

$$(3.131) \quad \mathbb{E} \left[\left(\sum_{i \in \text{Inc}_N^c} |a_{n,c,k;i}|^p \right)^{q/p} \right] \leq C_{p,q} \left(n^{dq/p} n^{(m-d)q} p_n^q + n^{dq/p} p_n^{q/p} + n^m p_n \right).$$

We conclude the proof of Theorem 2.4 by collecting the bounds (3.127), (3.129) and (3.131).

APPENDIX A. ELEMENTARY PROPERTIES OF CONDITIONAL EXPECTATION

We collect some lemmas which will be used in the proofs.

The following fact on conditional expectation is well-known.

Lemma A.1. *Let Y be an integrable random variable and let \mathcal{A} and \mathcal{B} be two σ -algebras such that \mathcal{B} is independent of $\sigma(Y) \vee \mathcal{A}$. Then*

$$(A.1) \quad \mathbb{E}[Y \mid \mathcal{A} \vee \mathcal{B}] = \mathbb{E}[Y \mid \mathcal{A}].$$

Lemma A.2. *Let $(\xi_i)_{i \in \mathbb{Z}}$ be an independent sequence and let $(\xi'_i)_{i \in \mathbb{Z}}$ be an independent copy of $(\xi_i)_{i \in \mathbb{Z}}$. Denote for $I, J \subset \mathbb{Z}$ by \mathcal{F}_I (respectively \mathcal{F}'_J) the σ -algebra generated by the random variables $\xi_i, i \in I$ (respectively $\xi'_j, j \in J$). For each integrable random variable Y and each subsets I_1, I_2 and J_1, J_2 of \mathbb{Z} ,*

$$(A.2) \quad \mathbb{E} \left[\mathbb{E}[Y \mid \mathcal{F}_{I_1} \vee \mathcal{F}'_{J_1}] \mid \mathcal{F}_{I_2} \vee \mathcal{F}'_{J_2} \right] = \mathbb{E} [Y \mid \mathcal{F}_{I_1 \cap I_2} \vee \mathcal{F}'_{J_1 \cap J_2}].$$

Proof. Expressing \mathcal{F}'_{J_2} as $\mathcal{F}'_{J_1 \cap J_2} \vee \mathcal{F}'_{J_2 \setminus J_1}$ and applying Lemma A.1 with $\tilde{Y} = \mathbb{E}[Y \mid \mathcal{F}_{I_1} \vee \mathcal{F}'_{J_1}]$, $\mathcal{A} = \mathcal{F}_{I_2} \vee \mathcal{F}'_{J_1 \cap J_2}$ and $\mathcal{B} = \mathcal{F}'_{J_2 \setminus J_1}$, we get that

$$(A.3) \quad \mathbb{E} \left[\mathbb{E}[Y \mid \mathcal{F}_{I_1} \vee \mathcal{F}'_{J_1}] \mid \mathcal{F}_{I_2} \vee \mathcal{F}'_{J_2} \right] = \mathbb{E} \left[\mathbb{E}[Y \mid \mathcal{F}_{I_1} \vee \mathcal{F}'_{J_1}] \mid \mathcal{F}_{I_2} \vee \mathcal{F}'_{J_1 \cap J_2} \right].$$

Then writing \mathcal{F}_{I_2} as $\mathcal{F}_{I_1 \cup I_2} \vee \mathcal{F}_{I_2 \setminus I_1}$ and applying Lemma A.1 with $\tilde{Y} = \mathbb{E}[Y \mid \mathcal{F}_{I_1} \vee \mathcal{F}'_{J_1}]$, $\mathcal{A} = \mathcal{F}_{I_1 \cap I_2} \vee \mathcal{F}'_{J_1 \cap J_2}$ and $\mathcal{B} = \mathcal{F}_{I_2 \setminus I_1}$ gives

$$(A.4) \quad \mathbb{E} \left[\mathbb{E}[Y \mid \mathcal{F}_{I_1} \vee \mathcal{F}'_{J_1}] \mid \mathcal{F}_{I_2} \vee \mathcal{F}'_{J_2} \right] = \mathbb{E} \left[\mathbb{E}[Y \mid \mathcal{F}_{I_1} \vee \mathcal{F}'_{J_1}] \mid \mathcal{F}_{I_1 \cap I_2} \vee \mathcal{F}'_{J_1 \cap J_2} \right]$$

and the inclusion $\mathcal{F}_{I_1 \cap I_2} \vee \mathcal{F}'_{J_1 \cap J_2} \subset \mathcal{F}_{I_1} \vee \mathcal{F}'_{J_1}$ allows to conclude. \square

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