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Benoît Mandelbrot in finance

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1 Introduction : the Noah and Joseph effects and the non-Gaussian and non-Brownian issues of the financial theory

Let $(S(t), t \geq 0)$ denote the price process of a given asset. The basic model in finance is the exponential Brownian motion, i.e. the stochastic process defined by

$$S(t) = S(0) \exp(\mu t + \sigma W(t)) \quad (1)$$

with $W(t)$ standard Brownian motion, that is, $W(0) = 0$, $\mathbb{E}[W_1] = 0$ et $\mathbb{E}[W_1^2] = 1$. This is the standard model of price variations in finance. It was introduced by Samuelson in 1965 and is sometimes called in the early financial literature “geometric Brownian motion”¹. We introduce the notation

$$X(t) = \ln S(t) - \ln S(0) \quad (2)$$

as the continuously compounded return of the asset. According to the standard model (1),

$$X(t) = \mu t + \sigma W(t) \quad (3)$$

is the Brownian motion introduced by Bachelier in 1900². The distribution of $X(t)$ is the normal (Gaussian) one. In other words, with the standard model, the marginal distributions of returns have to be Gaussian.

It is important to clearly distinguish between the normality assumption and the continuity assumption in (3). The two assumptions are confounded in the standard model because of the properties of Brownian motion, but they refer to different conceptual frameworks. The *normality* of asset returns has played a central role in finance for a large number of *static* models, beginning with the Markowitz’ standard portfolio theory (1952) and the Sharpe’s Capital Asset Pricing Model (1964) and more recently with the Value at Risk computations in the Basle framework. On the other hand, the *continuity* of trajectories has played a central role in the *dynamic* models built on the building blocks of the Black, Scholes and Merton’s standard option pricing theory (1973) and the default risk probabilities computed with the Vasicek’s yield curve model (1977).

It is now well documented that both the normality and continuity assumptions are contradicted by the data : the normal distribution is a poor model for the

¹Samuelson [1965].

²See Courtault and Kabanov [2002], Walter [2002b].

marginal returns, and all the trajectories exhibit a lot of jumps. Tackling these challenges has represented the aims pursued by Benoît Mandelbrot³ :

“In science, all important ideas need names and stories to fix them in the memory. It occurred to me that the market’s first wild trait, abrupt change or discontinuity, is prefigured in the Bible tale of Noah. As *Genesis* relates, in Noah’s six hundredth year God ordered the Great Flood to purify a wicked world. [...] The flood came and went – catastrophic but transient. Market crashes are like that : at times, even a great bank or brokerage house can seem like a little boat in a big storm.

The market’s second wild trait – almost cycles – is prefigured in the story of Joseph. Pharaoh dreamed that seven fat cattle were feeding in the meadows, when seven lean kine rose out of the Nile and ate them. [...] Joseph, a Hebrew slave, called the dreams prophetic : Seven years of famine would follow seven years of prosperity. [...] Of course, this is not a regular or predictable pattern. But the appearance of one is strong. Behind it is the influence of long-range dependence in an otherwise random process – or, put another way, a long-term memory through which the past continues to influence the random fluctuations of the present.

I called these two distinct forms of wild behavior the *Noah effect* and the *Joseph effect*. They are two aspects of one reality⁴”.

We present below the three main models of Benoît Mandelbrot and their legacy. The next section presents the description of the Noah effect by the α -stable motion (1962). Section 3 shows the relation between Joseph effect and fractional Brownian motion (1965). Section 4 presents the fundamental intuition of the time-changed Brownian motion (1967) for disentangling the Noah and Joseph effects.

³A short biography of Benoît Mandelbrot can be found in the Richard Hudson’s prelude to Mandelbrot [2004], p. xv-xxiv : “Introducing a Maverick in Science”. Other existing short biography is “A maverick’s apprenticeship”, by Benoît Mandelbrot himself (available on website).

⁴Mandelbrot [2004], p. 200-201.

2 The first model of Mandelbrot (1962) : α -stable motion with paretian tails

Adopting a static standpoint, the issue of non normality is exhibited by the so-called leptokurtic phenomenon⁵ which appeared very early in financial literature. To solve this puzzle, the idea of Mandelbrot can be summarized as follows: let us abandon both the continuity and normality assumptions, but keep the assumption that the process has independent increments. Hence, in 1962, Mandelbrot replaced the standard model (3) by a new model called α -stable Lévy motion⁶, where the distributions have fat or pareto-type tails, and in fact, infinite variance. This precisely allows to understand the Mandelbrot's intuition : the search for paretian tails in finance. "The power of power laws"⁷ is a tale of fat tails:

"Examine price records more closely, and you typically find a different kind of distribution than the bell curve : the tails do not become imperceptible but follow a "power law". These are *common in nature*. [...] In economics, one classic power law was discovered by italian economist Vilfredo Pareto a century ago. It describes the distribution of income in the upper reaches of society. That power laws *concentrates* much more of a society's wealth among the very few; a bell curve would be more equitable, scattering incomes more evenly around an average. Now we reach *one of my main findings*. A power law also applies to positive or negative price movements⁸ of many financial instruments. It leaves room for many more big price swings than would the bell curve. And it fits the data for many price series. *I provided the first evidence in a 1962 research report*" (our italics).

The legacy of paretian approach The sequel of the story is now well known⁹. After a period of strong arguments on the pros and cons of the infinite variance hypothesis, the 1962 model was abandoned because of the lack of second moment, considered as too strong for financial uses. However the ideas didn't disappear. On one hand, the theory of extreme values, the rebirth of which dates back to

⁵Following the terminology introduced in Walter [2002a].

⁶Mandelbrot [1962], followed by a more detailed description Mandelbrot [1963].

⁷Mandelbrot [2004], p. 13.

⁸Exactly the two sides of the Lévy measure defined in (13).

⁹See Walter [2009] for a comprehensive approach of the story of the α -stable attempts for solving the leptokurtic character of returns.

1993, kept the intuition of paretian tails but subsumed it through the generalized Pareto distribution which includes many cases of fat tails and not only the α -stable ones¹⁰. On the other hand, the general Lévy processes which were considered in the 1990s maintained the discontinuity feature but without the infinite variance characteristic of the α -stable motion¹¹. There is now a large body of research on these topics and *financial modelling under Non Gaussian distributions*¹² is becoming a new consensual approach in the financial literature.

3 The second model of Mandelbrot (1965) : fractional brownian motion with aperiodic cycles

An interesting feature of the standard model defined in (3) is that the process $X(t)$ is *self-similar* when the drift μ is zero. We say that $X(t)$ is self-similar with exponent $1/2$ (hereafter $1/2$ -ss) if

$$X(at) \stackrel{d}{=} a^{1/2} X(t) \quad (4)$$

where the notation $\stackrel{d}{=}$ defines equality in distributions. The presence of self-similarity in classical finance stems from the fact that Brownian motion is a $1/2$ -ss process. The second Mandelbrot's idea was to generalize the equation (4) and to propose representing the fluctuations of returns by a self-similar process with exponent $H \in (0, 1]$. Hence the relation

$$X(at) \stackrel{d}{=} a^H X(t) \quad (5)$$

defines the fractional brownian motion¹³ (FBM) introduced in 1965. This process can display what is called a “long-range dependence” or *long memory*, which is of great interest in modelling the complex dependency of economic variables, termed by Granger, “the typical spectral shape of an economic variable”¹⁴. In fact, the intuition behind this modification is the notion of long memory, termed “Joseph effect” by Mandelbrot, in reference to the slow and aperiodic cycles evoked by the biblical Joseph with regard to fluctuations of the harvest. This led Mandelbrot to rediscover Hurst's law¹⁵ by introducing the concept of fractional Brownian motion (FBM) which is a self-similar process with exponent H and whose increments admit

¹⁰See for instance Embrechts, Klüppelberg and Mikosch [1997].

¹¹See for instance Cont and Tankov [2004], Le Courtois et Walter [2012].

¹²According to the title of Jondeau, Poon and Rockinger [2007].

¹³Mandelbrot [1965], followed by a more detailed description : Mandelbrot and van Ness [1968].

¹⁴According to Granger [1966].

¹⁵Hurst [1951].

as autocovariance function

$$C_H(h) = \frac{1}{2} [|h + 1|^{2H} - 2|h|^{2H} + |h - 1|^{2H}] \quad (6)$$

With the FBM, the span of interdependence between the increments can be said to be infinite. There is a power law in the correlation as there was a power law in the returns (paretian tails) in the 1962 model. Thus, power laws, albeit with a different interpretation, are involved in both cases.

The legacy of long memory approach The search for long memory failed for returns but was successfully achieved for volatility. In the first case, the long memory properties of returns were hard to find, either using the Hurst R/S analysis¹⁶ or with the attempt of ARFIMA modelling¹⁷. But it appeared that long memory properties exist in volatility. This was at the origin of the FIGARCH models of volatility¹⁸ which led to an important stream of academic studies on volatility processes and volatility forecast with long memory.

4 The third model of Mandelbrot (1967) : time changed Brownian motion with stochastic clock

It was difficult, however, to reconcile both fat paretian tails (non-normal returns) and long aperiodic cycles (volatility correlations) in the same model and to isolate the two effects. In order to disentangle the Noah and Joseph contributions, Mandelbrot's third idea in 1967 was to consider the price variations in a different time grid, related to market activity (or business activity) and not to physical time¹⁹. That means to replace in the standard model (3) the physical time t by a social economically relevant time $\Theta(t)$ to design a new process $\xi(t) = X(\Theta(t))$ defined in physical time as

$$\xi(t) = \mu\Theta(t) + \sigma W(\Theta(t)) \quad (7)$$

In fact, what is called *time-changed Lévy processes* can simultaneously address these two issues. It is easy to see that the choice of the new clock $\Theta(t)$ determines the new model $\xi(t)$ in physical time. By replacing the usual clock of an independent Brownian motion by a stochastic one, Mandelbrot and Taylor paved the way for a

¹⁶About the R/S tests, see for instance Lo [1991] and Beran [1994].

¹⁷The ARFIMA process was introduced by Granger [1980] and Hosking [1981].

¹⁸The FIGARCH process was introduced by Baillie, Bollerslev and Mikkelsen [1996].

¹⁹Mandelbrot and Taylor [1967].

new interpretation of the non-normality of returns. The intuition behind it is the following :

“The key step is to introduce an auxiliary quantity called *trading time*. The term is self-explanatory and embodies two observations. While price changes over fixed clock time intervals are long-tailed, price changes between successive transactions stay near-Gaussian over sometimes long period between discontinuities. Following variations in the trading volume, the time interval between successive transactions vary greatly. This suggests that trading time is related to volume.”²⁰

In the setup of the 1967 model, $W(t)$ was Brownian motion and hence had independent increments, but this constraining hypothesis was relaxed later on. For instance, if one replaces $W(t)$ by $W_H(t)$ which is the FBM defined above, then $\xi(t)$ is called a *multifractal process*. From a general standpoint, the resulting process $\xi(t)$ is a time changed process, i.e. a process using a stochastic clock in social time.

The legacy of Brownian subordination After the first investigation of Clark in 1973²¹, the idea that return distribution should be normal when measured per unit of relevant clock was rediscovered in the end of the 1990s by several authors who generalize Clark’s results²² to be able to build a general framework for time-change modelling by using Lévy processes. The models based on Brownian subordination are now of a great interest and two recent models (variance gamma process and normal inverse Gaussian process²³) represent a good illustration of the wide applicability of this approach.

²⁰Mandelbrot [1997], p. 39.

²¹Clark [1973].

²²See for instance Ané and Geman [1997], Geman, Madan and Yor [2001], Carr and Wu [2004].

²³See Madan, Carr and Chang [1998] and Barndorff-Nielsen [1998].

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6 Appendix : a tale of fat tails

A Lévy process $(X(t), t \geq 0)$ is a process with stationary (identically distributed) and independent increments (hereafter i.i.d.). It follows that $X(t)$ is infinitely divisible. The characteristic function of the random variable X is $\Phi_X(u) = \mathbb{E}[\exp(iuX)]$. If $X(t)$ is a Lévy process, then $\Phi_{X_t} = \exp(\Psi_{X_t}(u))$ where $\Psi_{X_t}(u)$ is the characteristic exponent of $X(t)$. The characteristic exponent of the return process $X(t)$ in the case of the standard model (3) is

$$\Psi_{X_t}(u) = t \left(i\mu u - \frac{1}{2}\sigma^2 u^2 \right) \quad (8)$$

An interesting feature of the Lévy processes is that a Lévy process is completely determined by any of its one-dimensional marginal distributions : $\Psi_{X_t}(u) = t \Psi_{X_1}(u)$ and it suffices to define $\Psi_{X_t}(u)$ for $t = 1$.

In 1962, Mandelbrot replaced the standard model (8) by a new model termed the α -stable Lévy motion²⁴ and defined by

$$\Psi_{X_t}(u) = t \left(i\mu u - \gamma^\alpha |u|^\alpha \left(1 - i\beta \frac{u}{|u|} \operatorname{tg} \frac{\pi\alpha}{2} \right) \right) \quad (9)$$

The α -stable Lévy motion is another special case of the general Lévy processes, with both activity and variation infinite. For an heuristic understanding of the Mandelbrot's idea and the choice he made of this type of Lévy process, one introduces the Lévy-Khintine representation

$$\Psi_{X_t}(u) = t \left(i\mu u - \frac{1}{2}\sigma^2 u^2 \right) + t \int_{\mathbf{R}^*} \psi(u, x) \nu(dx) \quad (10)$$

with

$$\psi(u, x) = e^{iux} - 1 - iux \mathbf{1}_{|x| < 1}(x)$$

With this representation, the characteristic exponent (9) of the α -stable Lévy motion²⁵ is

$$\Psi_{X_t}(u) = t \left(i\mu u + \int_{-\infty}^0 \psi(u, x) \frac{C_-}{|x|^{1+\alpha}} dx + \int_0^{+\infty} \psi(u, x) \frac{C_+}{x^{1+\alpha}} dx \right) \quad (11)$$

From (10) and (11), it follows that the Lévy measure of the α -stable motion is

$$\nu(dx) = \frac{C_-}{|x|^{1+\alpha}} \mathbf{1}_{(-\infty, 0)}(x) dx + \frac{C_+}{x^{1+\alpha}} \mathbf{1}_{(0, +\infty)}(x) dx \quad (12)$$

²⁴Mandelbrot [1962], followed by a more detailed description Mandelbrot [1963].

²⁵See for example Samorodnitsky et Taquq [1994].

or in a more intuitive form

$$\nu(x) = \begin{cases} \frac{C_-}{|x|^{1+\alpha}} & \text{if } x < 0 \\ \frac{C_+}{x^{1+\alpha}} & \text{if } x > 0 \end{cases} \quad (13)$$

This last form (13) clearly exhibits a power paretian law in the Lévy measure. This precisely allows to understand the Mandelbrot's intuition : the search for paretian tails in finance. A tale of fat tails, he named "the power of power laws"²⁶.

²⁶Mandelbrot [2004], p. 13.