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Invariant tensions from holography

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ABSTRACT: We revisit the problem of defining an invariant notion of tension in gravity. For spacetimes whose asymptotics are those of a Defect CFT we propose two independent definitions: gravitational tension given by the one-point function of the dilatation current, and inertial tension, or stiffness, given by the norm of the displacement operator. We show that both reduce to the tension of the Nambu-Goto action in the limit of classical thin-brane probes. Subtle normalisations of the relevant Witten diagrams are fixed by the Weyl and diffeomorphism Ward identities of the boundary DCFT. The gravitational tension is not defined for domain walls, whereas stiffness is not defined for point particles. When they both exist these two tensions are in general different, but the examples of line and surface BPS defects in $d = 4$ show that superconformal invariance can identify them.

KEYWORDS: AdS-CFT Correspondence, Brane Dynamics in Gauge Theories, Scale and Conformal Symmetries, Wilson, 't Hooft and Polyakov loops

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Contents

1	Introduction	1
2	Nambu-Goto probes	4
2.1	Inertial tension	4
2.2	Gravitational tension	6
3	Ward identities	9
3.1	The DCFT identities	9
3.2	The gravity calculation	10
4	Examples	12
4.1	Maldacena-Wilson lines	12
4.2	Interfaces in $d = 2$	13
4.3	Graham-Witten anomalies	14
5	Outlook and a conjecture	15
A	Constraints on DCFT correlators	17

1 Introduction

A key entry in the AdS/CFT dictionary [1–3] is the relation between the bare mass, m_0 , of a point particle in the bulk and the scaling dimension, Δ , of the dual CFT operator. For a free scalar particle in AdS_{d+1}

$$\Delta = \frac{d}{2} + \sqrt{(m_0\ell)^2 + \frac{d^2}{4}} = m_0\ell + \frac{d}{2} + \frac{d^2}{8m_0\ell} + \dots \quad (1.1)$$

where ℓ is the AdS radius which we set henceforth equal to one. In the world-line formalism one finds $\langle \mathcal{O}(x)\mathcal{O}(y) \rangle \sim e^{-m_0\mathfrak{D}}$, where \mathcal{O} is the dual CFT operator inserted at the points x and y on the Poincaré boundary, $\mathfrak{D} = 2 \log(|x - y|/\epsilon)$ is the geodesic distance between these points and ϵ is the boundary cutoff. This explains the leading term in the expansion (1.1). The next term comes from Gaussian quantum fluctuations,¹ and subleading ones from the non-linearities of the point-particle action which are conveniently resummed by the Klein-Gordon equation.

Such corrections are negligible if the Compton wavelength of the particle is much smaller than the AdS radius, $m_0^{-1} \ll 1$. In addition, for the point-particle description to stay valid the AdS radius must be much larger than the Schwarzschild radius, $r_S^{d-2} \sim G_N m_0 \ll 1$. In the language of ref. [4] one may refer to the range $1 \ll m_0 \ll 1/G_N$ as that of *heavy* but not *huge* operators. The latter correspond to black-hole micro-states for which the world-line approximation breaks down and the geometry thickens, as illustrated in figure 1. Whether or not the dual ‘banana-shaped’ geometry is a smooth horizonless fuzzball [5], a

¹Which in general depend on the particle’s spin.



Figure 1. A ‘banana-shaped’ geometry (right) replaces the geodesic world-line (left) when $G_N m_0 \gtrsim 1$. The AdS boundary is in grey. The broken line on the left is a virtual graviton that screens the bare mass parameter of the point-particle action.

bound state or a black hole, mass cannot be defined locally anymore. In the world-line formalism virtual gravitons screen the bare mass parameter m_0 . But an invariant energy does exist in global AdS [6, 7], and it coincides (up to a Casimir subtraction) with the dilatation charge Δ . The existence of this invariant charge makes it possible to count the microscopic bulk states in the dual CFT.

All this is well known. The question that we would like to address here is whether the holographic dictionary contains a similar entry for the tension of p -dimensional branes. A brane in AdS can be compact, in which case its only gravitational charge is energy, but it may also have infinite extent and intersect the boundary along a p -dimensional defect [8–11]. It should in this case be possible to give an invariant definition of tension on the AdS boundary. When a dual Defect Conformal Field Theory (DCFT) exists, the definition should only depend on its data.² As for point particles, it must also reduce to the bare Nambu-Goto tension, $T_p^{(0)}$, in the limit of a thin, heavy but not huge brane (i.e. when $1 \ll T_p^{(0)} \ll G_N T_p^{(0)}$). We will argue that two natural candidates fit the bill:

- (I) The integrated one-point function of the dilatation current,

$$T_p^{(I)} := \left(\frac{d-1}{d-p-1} \right) \oint ds^j \langle\langle \mathcal{J}_j \rangle\rangle \tag{1.2}$$

- (II) The norm, C_D , of the displacement operator

$$T_p^{(II)} := C_D \frac{\pi^{p/2} \Gamma(\frac{p}{2} + 1)}{(p+2) \Gamma(p+1)}. \tag{1.3}$$

Here $\langle\langle \dots \rangle\rangle$ denote DCFT correlation functions in \mathbb{R}^d , the defect spans a \mathbb{R}^p subspace, and the integral is over the (hyper)sphere around the defect in the transverse $\mathbb{R}^{(d-p)}$. The dilatation current is $\mathcal{J}_j = x^m T_{mj}$, and the CFT stress tensor obeys the conservation equation

$$\partial_m T^{mj} = \delta^{(d-p)}(x) D^j \tag{1.4}$$

with j labelling transverse directions. Eq. (1.4) fixes unambiguously the norm of the displacement operator D^j , which becomes a piece of DCFT data [12]. The two definitions are illustrated in figures 2 and 3.

²But the existence of such a dual is not necessary — DCFT language is just a proxy for the asymptotic AdS data.

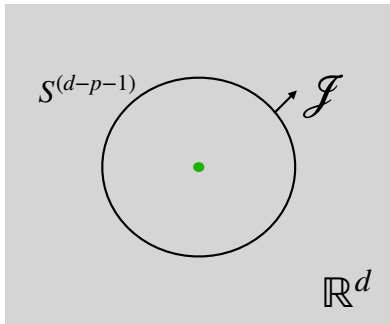


Figure 2. The gravitational tension is the integral of the dilatation current \mathcal{J} over a sphere around the defect.

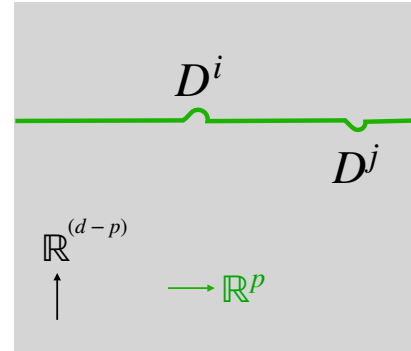
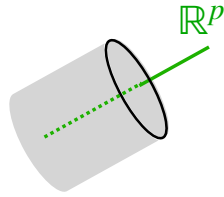


Figure 3. The inertial tension, or ‘stiffness’, is given by the 2-point, function of the displacement.

Definition (I) is the natural extension of the dilatation charge $\Delta \simeq m_0$, to which it reduces for $p = 0$. It is obtained from the graviton one-point function in AdS, so we refer to it as the *gravitational tension*. Note that for $p > 0$ the integral in (1.2) is not a topological charge, and that contrary to local operators planar extended defects do not break scale invariance. Note also that in the case of domain walls both $\langle\langle \mathcal{J}_j \rangle\rangle$ and $d - p - 1$ are zero, so the gravitational tension is not defined.

Definition (II), on the other hand, only makes sense for extended defects ($p > 0$). It is a measure of stiffness and we refer to it as kinematic or *inertial tension*. For line defects ($p = 1$) it is a multiple of the Bremsstrahlung function [13–15], and in the special case $d = 2$, where a line defect is also an interface, it is proportional to the energy reflection coefficient [16]. Our interest in the definition of tension was spurred by holographic calculations in this latter context [17–20], and part of our motivation for the present work was to generalise them to higher dimensions. We will not discuss boundaries in this work — viewed as limits of interfaces they have infinite tension [17].

We should stress that definition (I) is reminiscent of earlier proposals for an invariant tension [21–26] which also involve the asymptotic behaviour of the metric. These authors considered spacetimes whose asymptotics were ‘transverse flat’ or ‘planar empty AdS’, and they assumed the existence of an (exact or asymptotic) spacelike Killing isometry. Such an isometry is not required for our definitions. Although the AdS boundary in figures 2 and 3 is flat and the defect linear, all one really needs is that the radius of the sphere and the separation of the two displacement insertions be much smaller than all other DCFT scales.

Our main technical result will be to show, using Witten diagrams, that the right-hand sides of (1.2) and (1.3) reduce to the bare Nambu-Goto tension for classical probe branes. In this calculation normalisation factors matter and are subtle. We will fix them by verifying that the two-point function $\langle\langle T^{mn} \mathcal{D}^k \rangle\rangle$ obeys, in the above limit, the Weyl and diffeomorphism Ward identities [12] of the (putative) dual DCFT.

For defects of dimension $p = 2$ or 4 the displacement norm and one-point function of the stress tensor are related to Graham-Witten anomalies, see [27] for an updated summary. In the Nambu-Goto limit these can be obtained from the renormalised volume (alias ‘Willmore

energy') of minimal submanifolds [28, 29]. The use of Witten diagrams is a lowbrow method for computing such volumes for all p and d .³

The inertial and gravitational tensions are not, in general, equal beyond the thin-classical-probe limit. But for certain Wilson-line and surface defects in $d = 4$ theories, superconformal Ward identities lead to a linear relation between C_D and the expectation value of the stress-energy tensor [30, 31]. This in turn implies, as we will see, that the relation $T_p^{(I)} = T_p^{(II)}$ is exact. Extrapolating all known results the authors of ref. [31] proposed a linear relation between C_D and $\langle\langle T^{mn} \rangle\rangle$ for all superconformal defects with sufficient (though unspecified) supersymmetry. When expressed in terms of the tensions defined in the present paper this conjecture acquires the transparent physical meaning $T_p^{(I)} = T_p^{(II)}$.

The rest of the paper is organised as follows. In section 2 we show using Witten diagrams that (1.2) and (1.3) reduce to the Nambu-Goto tension for thin classical probes. We assume a natural but adhoc normalisation for the displacement source. In section 3 we calculate in the same limit the two-point function $\langle\langle T^{mn} \mathcal{D}^k \rangle\rangle$, and show that it obeys the DCFT Ward identities [12] which relate it to the displacement norm C_D and to $\langle\langle T^{mn} \rangle\rangle$. This confirms the normalisations of the previous section. In section 4 we consider some specific defects, check that our calculations are consistent with results obtained by other methods, and comment on the quantum and gravitational corrections. Section 5 explains how supersymmetry can protect the relation $T_p^{(I)} = T_p^{(II)}$ from such corrections and compares this relation to the conjecture of ref. [31]. This part deserves further study. The constraints on DCFT correlation functions are reviewed, for the reader's convenience, in appendix A.

2 Nambu-Goto probes

To show that the right-hand sides of (1.2) and (1.3) reduce to the Nambu-Goto tension for thin classical probes we will use Witten diagrams [3, 32]. These have been applied to DCFT [33–41] mostly in order to compare to exact results (from localisation and integrability) for superconformal Wilson lines. Our calculations are simpler but the focus is different. We are mainly concerned with the normalisation of bulk and defect sources.

2.1 Inertial tension

We begin with the definition (1.3). The norm of the displacement operator is read from the two-point function⁴

$$\langle\langle D^i(\vec{\tau}) D^j(\vec{\tau}') \rangle\rangle = \frac{C_D \delta^{ij}}{|\vec{\tau} - \vec{\tau}'|^{2p+2}}, \tag{2.1}$$

where $\vec{\tau}$ and $\vec{\tau}'$ are points on the defect. Both the norm of D^j and its scaling dimension, $\Delta = p + 1$, are part of the DCFT data. The first task is to compute them on the gravity side.

The metric of Euclidean AdS_{d+1} in Poincaré coordinates is

$$ds^2 = \frac{\delta_{\mu\nu} dy^\mu dy^\nu}{(y^0)^2} \quad \text{with} \quad \mu, \nu = 0, 1, \dots, d. \tag{2.2}$$

³It also allows in principle the systematic calculation of quantum corrections.

⁴Double brackets will always stand for normalised correlation functions in the background of the defect, both for bulk and for defect operators.

The static brane sits at $y^j = 0$ for $j = p + 1, \dots, d$. It breaks the AdS_{d+1} isometries to $\text{SO}(1, p + 1) \times \text{SO}(d - p)$, but this residual symmetry does not act simply on the coordinates y^μ . As realized in [33], the system of coordinates in which the residual isometries are manifest is⁵

$$\begin{aligned}
 y^0 &= x^0 \left(\frac{1 - \frac{1}{4}|x_\perp|^2}{1 + \frac{1}{4}|x_\perp|^2} \right), & y^a &= x^a & \text{for } a &= 1, \dots, p \\
 \text{and } y^j &= x^0 \left(\frac{x^j}{1 + \frac{1}{4}|x_\perp|^2} \right) & & & \text{for } j &= p + 1, \dots, d
 \end{aligned} \tag{2.3}$$

with $|x_\perp|^2 = \sum_j x^j x^j$. The metric in these coordinates reads

$$ds^2 = \frac{\delta_{\alpha\beta} dx^\alpha dx^\beta}{(x^0)^2} \left(\frac{1 + \frac{1}{4}|x_\perp|^2}{1 - \frac{1}{4}|x_\perp|^2} \right)^2 + \frac{\delta_{ij} dx^i dx^j}{(1 - \frac{1}{4}|x_\perp|^2)^2}. \tag{2.4}$$

The unbroken $\text{SO}(1, p + 1)$ acts now only on the x^α , not the x^j .

Let $X^\mu(\tau)$ be the embedding of the p -brane in AdS_{d+1} . The Nambu-Goto action is proportional to the bare tension $T_p^{(0)}$,

$$I_{\text{NG}} = T_p^{(0)} \int d^{p+1} \tau \sqrt{\det(\hat{g}_{\alpha\beta})} \quad \text{where} \quad \hat{g}_{\alpha\beta} = g_{\mu\nu}(X) \frac{\partial X^\mu}{\partial \tau^\alpha} \frac{\partial X^\nu}{\partial \tau^\beta} \tag{2.5}$$

is the induced world-volume metric. Working in the static gauge, $X^\alpha = \tau^\alpha$, and expanding in small transverse fluctuations gives

$$I_{\text{NG}} = T_p^{(0)} \int d^{p+1} \tau \sqrt{\bar{g}} \left[1 + \frac{1}{2} \sum_j (\bar{g}^{\alpha\beta} \partial_\alpha X^j \partial_\beta X^j + (p + 1) X^j X^j) + \dots \right]. \tag{2.6}$$

Here $\bar{g}_{\alpha\beta} = \delta_{\alpha\beta}/(\tau^0)^2$ is the metric of AdS_{p+1} and the dots are terms quartic or higher in X^j . These contribute quantum corrections to the invariant tension that are suppressed by inverse powers of $T_p^{(0)}$. The action (2.6) describes $(d - p)$ scalar fields with mass $m^2 = p + 1$ living in AdS_{p+1} . From the relation $m^2 = \Delta(\Delta - p)$ one sees that the dual operators have $\Delta = p + 1$, which is the expected dimension of D^j [33].

We extract C_D following the standard AdS/CFT recipe [32]. The on-shell p -brane coordinates are given at leading order by

$$X^j(\tau^0, \vec{\tau}) = \int d^p \tau' K_\Delta(\tau^0, \vec{\tau}; \vec{\tau}') X^j(0, \vec{\tau}'), \tag{2.7}$$

where $\vec{\tau}, \vec{\tau}'$ are p -component vectors and

$$K_\Delta(\tau^0, \vec{\tau}; \vec{\tau}') = \frac{\Gamma(\Delta)}{\pi^{\frac{p}{2}} \Gamma(\Delta - \frac{1}{2}p)} \left[\frac{\tau^0}{(\tau^0)^2 + |\vec{\tau} - \vec{\tau}'|^2} \right]^\Delta \tag{2.8}$$

⁵Here and in what follows we use letters from the beginning of the Greek and Latin alphabets for the directions along the brane, respectively the defect; early middle Latin letters (i, j, k) for the transverse directions; and late middle Greek and Latin letters for all AdS directions, respectively those of the AdS boundary. Context will hopefully help avoid confusion. A quick mnemonic is the Greek/Latin correspondence $x^\mu = (x^0, x^m)$ and $x^\alpha = (x^0, x^a)$.

is the usual bulk-to-boundary propagator normalised in order to approach $\epsilon^{p-\Delta}\delta^{(p)}(\vec{\tau} - \vec{\tau}')$ in the limit $\tau^0 = \epsilon \rightarrow 0$. The generating functional of the DCFT is found by inserting (2.7) in the Nambu-Goto action,

$$\mathcal{Z} = \frac{1}{2} T_p^{(0)} \frac{(2\Delta - p)\Gamma(\Delta)}{\pi^{p/2}\Gamma(\Delta - \frac{1}{2}p)} \int d^p \tau \int d^p \tau' X^i(0, \vec{\tau}) X^j(0, \vec{\tau}') \frac{\delta^{ij}}{|\vec{\tau} - \vec{\tau}'|^{2p+2}}, \quad (2.9)$$

with $X^j(0, \vec{\tau})$ the renormalised sources for the dual operators which we take to be the D^j . Eq. (2.9) differs from the naive on-shell action by an extra factor $(2\Delta - p)/\Delta$ which will be important for us here. It was first noticed in homogeneous CFTs by insisting that the regulated action be consistent with conformal Ward identities [32], and it was later shown to arise from the boundary limit of the bulk-to-bulk propagator [42, 43].

Setting $\Delta = p + 1$ in (2.9) leads to the following relation between C_D and the Nambu-Goto tension

$$C_D = T_p^{(0)} \frac{(p+2)\Gamma(p+1)}{\pi^{p/2}\Gamma(\frac{p}{2}+1)} + \dots \quad (2.10)$$

with dots standing for the quantum and gravitational corrections that we neglected.⁶

Comparing eq. (2.10) to eq. (1.3) we see that $T_p^{(II)} = T_p^{(0)} + \dots$, as announced in the introduction.

But there are reasons to feel uneasy about this calculation. How do we actually know that $X^j(0, \vec{\tau})$ is the properly-normalised source of the displacement operator? This should be fixed by the conservation eq. (1.4) which relates D^j to T^{mn} , and hence the brane coordinates to the graviton. The relative normalisation of these latter is fixed by their transformation under diffeomorphisms, $\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ and $\delta X^\mu = \xi^\mu(X)$. But the Fefferman-Graham expansion of the metric is singular in the coordinates x^μ , and the existence of an invariant regulator is not clear. Alternatively, one may argue following ref. [32], that the normalisation of the sources in (2.9) is determined by conformal Ward identities in AdS_{p+1} . But contrary to the homogeneous CFTs considered in this reference, the defect does not have its proper stress-energy tensor.

Despite these doubts eq. (2.10) turns out to be correct. We will show this using the DCFT Ward identities in section 3.

2.2 Gravitational tension

First we turn to the calculation of the gravitational tension. This depends on the one-point function of the bulk stress tensor which is fixed, up to an undetermined parameter a_T , by the unbroken conformal symmetry. The result [12] (see also appendix A) reads

$$\langle\langle T^{ab} \rangle\rangle = a_T \left(\frac{d-p-1}{d} \right) \frac{\delta^{ab}}{|x_\perp|^d}, \quad \langle\langle T^{aj} \rangle\rangle = 0, \quad (2.11a)$$

$$\langle\langle T^{ij} \rangle\rangle = a_T \left[\frac{x^i x^j}{|x_\perp|^{d+2}} - \left(\frac{p+1}{d} \right) \frac{\delta^{ij}}{|x_\perp|^d} \right]. \quad (2.11b)$$

⁶Note that the leading term only depends on the defect dimension p , not on d .

One can verify that $\langle\langle T^{mn} \rangle\rangle$ is traceless and conserved. Positivity of the energy density in the Lorentzian theory requires a_T to be negative. From eq. (2.11b) we find the radial component of the dilatation current in the transverse space

$$\frac{x^m x^j}{|x_\perp|} \langle\langle T_{mj} \rangle\rangle = \frac{a_T}{|x_\perp|^{d-1}} \left(\frac{d-p-1}{d} \right). \quad (2.12)$$

Integrating over the sphere in the definition (1.2) gives

$$T_p^{(I)} = -a_T \frac{2\pi^{(d-p)/2}(d-1)}{\Gamma(\frac{1}{2}(d-p))d}. \quad (2.13)$$

This expresses $T_p^{(I)}$ in terms of the DCFT parameter a_T . As already noted in the introduction, $a_T = 0$ for defects of codimension-one for which the above definition of tension fails.

We want now to calculate a_T in the thin-brane limit. To this end let $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ with $\bar{g}_{\mu\nu}$ the metric of the bulk AdS. To ensure a smooth Fefferman-Graham expansion we must use the Poincaré coordinates y^μ for the graviton. The Fefferman-Graham expansion of defect fields, on the other hand, is regular in the coordinates x^μ , whence the difficulties of regularisation mentioned in the previous subsection. Fortunately at the order at which we will work the change of coordinates (2.3) is simple,

$$y^\alpha = x^\alpha(1 + O(x_\perp^2)) \quad \text{and} \quad y^j = x^0 x^j(1 + O(x_\perp^2)). \quad (2.14)$$

Thus, up to corrections of order $O(x_\perp^2)$, the static-gauge parametrisation is unchanged and we only need to rescale the transverse brane coordinates, $Y^j = \tau^0 X^j$. Furthermore, the usual ultraviolet cutoffs of AdS_{d+1} and AdS_{p+1} are the same at this leading order, $x^0 = y^0 = \epsilon$. Expanding the Nambu-Goto action (2.5) in powers of $h_{\mu\nu}$ and Y^j , and keeping only the terms of order $O(h)$ and $O(hY^j)$, we find

$$I_{\text{NG}} = \frac{1}{2} T_p^{(0)} \int d^{p+1} \tau \sqrt{\bar{g}} \bar{g}^{\alpha\beta} \delta \hat{g}_{\alpha\beta} \\ \text{with} \quad \delta \hat{g}_{\alpha\beta} = (1 + Y^j \partial_j) h_{\alpha\beta} + h_{\alpha j} \partial_\beta Y^j + h_{\beta j} \partial_\alpha Y^j + \dots \quad (2.15)$$

Note that after replacing Y^j by $\tau^0 X^j$ manifest covariance is lost. This is because the metric perturbation transforms non-covariantly under the isometries of AdS_{p+1} .

The leading contribution to $\langle\langle T^{mn} \rangle\rangle$ comes from the graviton tadpole in (2.15). This simplest Witten diagram is shown in figure 4. It has a bulk-to-boundary graviton propagator integrated over the world-volume of the static p -brane. The graviton propagator reads [3, 32, 44]

$$K_{\mu\nu}^{mn}(x, y) = \frac{C_\Delta^{[2]}}{(y_0)^2} \left(\frac{y^0}{|x-y|^2} \right)^\Delta J_{\mu r}(x-y) J_{\nu s}(x-y) P^{rs; mn} \quad (2.16)$$

where $x = (0, \vec{x})$ is the point on the boundary,

$$C_\Delta^{[2]} = \frac{2(\Delta+1)\Gamma(\Delta-1)}{\pi^{d/2}\Gamma(\Delta-d/2)}, \quad J_{\mu\nu}(z) = \delta_{\mu\nu} - \frac{2z_\mu z_\nu}{z^2} \quad (2.17)$$

$$\text{and} \quad P^{rs; mn} = \frac{1}{2}(\delta^{rm}\delta^{sn} + \delta^{rn}\delta^{sm}) - \frac{1}{d}\delta^{rs}\delta^{mn}. \quad (2.18)$$

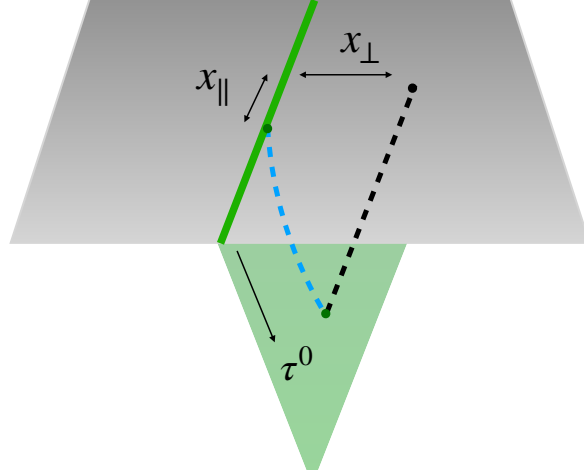


Figure 4. The Witten diagrams calculated in this and the following section. The graviton propagator (broken black line) and the displacement-field propagator (broken blue line) stretch from the thin brane in the AdS interior to a boundary point in the bulk, respectively on the defect. When the only external leg is the graviton, x_{\parallel} can be set to zero.

Recall that in our conventions $\mu, \nu \in [0, d]$ are bulk indices whereas $m, n, r, s \in [1, d]$ are boundary ones. The above propagator is valid for a spin-2 field of any mass, the massless graviton has $\Delta = d$.⁷

The tadpole in (2.15), with the AdS_{p+1} metric $\bar{g}_{\alpha\beta} = (\tau_0)^{-2}\delta_{\alpha\beta}$, gives

$$\langle\langle T^{mn}(x) \rangle\rangle = -\frac{T_p^{(0)}}{2} \int \frac{d^{p+1}\tau}{(\tau_0)^{p-1}} \delta^{\alpha\beta} K_{\alpha\beta}^{mn}(x, \tau). \quad (2.19)$$

Without loss of generality we take the boundary point to be at $x^a = 0$. The point in the interior lies on the brane at $y^\mu = (\tau^\alpha, 0)$. We perform the index contractions in two steps. First

$$\delta^{\alpha\beta} J_{\alpha r}(x-y) J_{\beta s}(x-y) = \delta_{rs}^{\parallel} + \frac{4z_r z_s |\tau|^2}{|z|^4} + \frac{2(\tau_r z_s + z_r \tau_s)}{|z|^2}, \quad (2.20)$$

where δ_{rs}^{\parallel} is the Kronecker symbol in the subspace spanned by the defect, $z^\mu = x^\mu - y^\mu = (-\tau^\alpha, x^j)$ and $|z|^2 = |\tau|^2 + |x_\perp|^2$. Contracting next with the tensor $P^{rs;mn}$ and noting that $\delta^{rs} z_r z_s = |z|^2 - (z_0)^2$ gives

$$\begin{aligned} \delta^{\alpha\beta} J_{\alpha r} J_{\beta s} P^{rs;mn} &= \delta_{\parallel}^{mn} + \frac{4z^m z^n |\tau|^2}{|z|^4} + \frac{2(\tau^m z^n + z^m \tau^n)}{|z|^2} \\ &\quad - \frac{1}{d} \delta^{mn} \left[p + \frac{8|\tau|^2}{|z|^2} - \frac{4(\tau_0)^2 |\tau|^2}{|z|^4} - \frac{4(\tau_0)^2}{|z|^2} \right]. \end{aligned} \quad (2.21)$$

For the transverse components $\langle\langle T^{ij} \rangle\rangle$ the above tensor structure reads

$$\frac{4x^i x^j |\tau|^2}{|z|^4} - \frac{1}{d} \delta^{ij} \left[p + \frac{8|\tau|^2}{|z|^2} - \frac{4(\tau_0)^2 |\tau|^2}{|z|^4} - \frac{4(\tau_0)^2}{|z|^2} \right]. \quad (2.22)$$

⁷There is a factor of 2 missing in eq. (43) of ref. [44]. The bulk-to-boundary propagator of a field dual to an operator of dimension Δ is $(2\Delta - d)$ times the bulk-to-bulk propagator instead of $(\Delta - d/2)$ as in [44].

The mixed components $\langle\langle T^{aj} \rangle\rangle$ vanish by $SO(p)$ symmetry, and in computing $\langle\langle T^{ab} \rangle\rangle$ one can replace for the same reason

$$\tau^a \tau^b \rightarrow \frac{\delta^{ab}}{p} |\vec{\tau}|^2.$$

The integrand in eq. (2.19) thus only depends on $|\vec{\tau}|^2$ and τ_0 .

It is simplest to extract the DCFT parameter a_T by looking at the term proportional to $x^i x^j$. Inserting (2.22) in (2.19) and comparing with the one-point function (2.11b) gives

$$-\frac{a_T}{|x_\perp|^{d+2}} = 2 T_p^{(0)} C_d^{[2]} \int \frac{d^{p+1} \tau}{(\tau_0)^{p-d+1}} \frac{|\tau|^2}{(|\tau|^2 + |x_\perp|^2)^{d+2}}. \quad (2.23)$$

The integral is computed using Schwinger's trick as follows:

$$\begin{aligned} -\frac{a_T}{|x_\perp|^{d+2}} &= 2 T_p^{(0)} C_d^{[2]} \int \frac{d\tau_0 d^p \tau}{(\tau_0)^{p-d+1}} \frac{(\tau_0)^2 + |\vec{\tau}|^2}{((\tau_0)^2 + |\vec{\tau}|^2 + |x_\perp|^2)^{d+2}} \\ &= \frac{2 T_p^{(0)} C_d^{[2]}}{\Gamma(d+1)} \int ds s^d \left(1 - \frac{|x_\perp|^2 s}{d+1}\right) e^{-s|x_\perp|^2} \int \frac{d\tau_0 d^p \tau}{(\tau_0)^{p-d+1}} e^{-s(\tau_0)^2 - s|\vec{\tau}|^2} \\ &= \frac{T_p^{(0)} C_d^{[2]}}{\Gamma(d+1)} \Gamma\left(\frac{d-p}{2}\right) \pi^{p/2} \int ds s^{d/2} \left(1 - \frac{|x_\perp|^2 s}{d+1}\right) e^{-s|x_\perp|^2} \\ &= \frac{1}{|x_\perp|^{d+2}} \frac{T_p^{(0)} C_d^{[2]}}{\Gamma(d+1)} \Gamma\left(\frac{d-p}{2}\right) \pi^{p/2} \frac{\Gamma(\frac{d}{2}+1)d}{2(d+1)}. \end{aligned}$$

Inserting the expression for $C_d^{[2]}$ from (2.17) leads after a little algebra to

$$-a_T = \frac{T_p^{(0)} \Gamma(\frac{d-p}{2}) d}{2(d-1) \pi^{(d-p)/2}} + \dots \quad (2.24)$$

This expresses the DCFT parameter a_T in terms of the Nambu-Goto tension in the classical thin-brane probe limit. Comparing to eq. (2.13) we see that $T_p^{(1)} = T_p^{(0)} + \dots$ as announced.

The terms in $\langle\langle T^{mn} \rangle\rangle$ proportional to δ^{ab} and δ^{ij} can be computed similarly and have the general form (2.11a), (2.11b) dictated by conformal symmetry. We leave this as an exercise for the reader, and move on to the more involved calculation of the two-point function $\langle\langle T^{mn} D^j \rangle\rangle$.

3 Ward identities

We have explained in section 2.1 why normalising the displacement operator in Witten diagrams is tricky. We will now settle the issue by checking the DCFT Ward identities that have been derived in ref. [12]. They are of the form $\langle\langle TD \rangle\rangle \sim \langle\langle T \rangle\rangle + \langle\langle DD \rangle\rangle$ and fix therefore unambiguously the normalisations of both D^j and T^{mn} .

3.1 The DCFT identities

The two-point function $\langle\langle T^{mn} D^j \rangle\rangle$ is determined by the unbroken conformal and rotation symmetry modulo three unknown parameters $\mathbf{b}_{1,2,3}$ (which are called $b_{\text{TD}}^{1,2,3}$ in ref. [12]). This is most easily derived in the lightcone formalism as reviewed in appendix A. The result is

$$\langle\langle T^{mn}(x) D^j(0) \rangle\rangle = \frac{|x_\perp|^{p-d}}{|x|^{2p+2}} F^{mn;j}(x) \quad (3.1)$$

with

$$F^{ab;j}(x) = \frac{1}{d}[(d-p-1)\mathbf{b}_2 - \mathbf{b}_1]\delta^{ab}x^j + 4\mathbf{b}_1 \frac{x^a x^b x^j |x_\perp|^2}{|x|^4}, \quad (3.2a)$$

$$F^{bi;j}(x) = -\mathbf{b}_3 \delta^{ij} x^b \frac{|x_\perp|^2}{|x|^2} + [4\mathbf{b}_3 - 2\mathbf{b}_1] \frac{x^i x^j x^b}{|x|^2} + 4\mathbf{b}_1 \frac{x^i x^j x^b |x_\perp|^2}{|x|^4}, \quad (3.2b)$$

$$F^{ik;j}(x) = -\frac{1}{d}[(p+1)\mathbf{b}_2 + \mathbf{b}_1]\delta^{ik}x^j + \frac{\mathbf{b}_3}{2}\delta^j(i x^k) \left(1 - \frac{2|x_\perp|^2}{|x|^2}\right) \\ + (\mathbf{b}_1 + \mathbf{b}_2 - \mathbf{b}_3) \frac{x^i x^k x^j}{|x_\perp|^2} + [2\mathbf{b}_3 - 4\mathbf{b}_1] \frac{x^i x^k x^j |x_\perp|^2}{|x|^2} + 4\mathbf{b}_1 \frac{x^i x^k x^j}{|x|^4}. \quad (3.2c)$$

Recall that in our notation $a, b = 1, \dots, p$ and $i, j, k = p+1, \dots, d$. Note also that because of the D^j insertion we cannot here set $x^a = 0$ as before, see figure 4. Thus $|x|^2 = |x_\parallel|^2 + |x_\perp|^2$.

Eqs. (3.1) and (3.2) follow from the unbroken $\text{SO}(1, p+1) \times \text{SO}(d-p)$. But there are also constraints coming from the fact that the displacement operator encodes the action of the broken $\text{SO}(1, d+1)$ symmetries on the defect. These relate the one-point function of any primary operator \mathcal{O} to its two-point function with the displacement operator D^j , schematically $\langle\langle \mathcal{O} D \rangle\rangle \sim \langle\langle \mathcal{O} \rangle\rangle$. When \mathcal{O} is the stress tensor these identities imply [12]

$$(p+1)\mathbf{b}_2 + \mathbf{b}_1 = \frac{d}{2}\mathbf{b}_3, \quad \mathbf{b}_3 = 2^{p+2}\pi^{-(p+1)/2}\Gamma\left(\frac{p+3}{2}\right) a_T. \quad (3.3)$$

The remaining free parameter is determined by the conservation (1.4) which relates a specific combination of $\langle\langle TD \rangle\rangle$ to $\langle\langle DD \rangle\rangle$. This gives [12]

$$2p\mathbf{b}_2 - (2d+p-2)\mathbf{b}_3 = \frac{(d-p)\Gamma(\frac{d-p}{2})}{\pi^{(d-p)/2}} C_D. \quad (3.4)$$

Together eqs. (3.3) and (3.4) can be used to express $\mathbf{b}_1, \mathbf{b}_2$ and \mathbf{b}_3 in terms of the DCFT data a_T and C_D . We want to verify that these relations are satisfied when the AdS fields sourcing T^{mn} and D^j are normalised as in section 2.

3.2 The gravity calculation

The leading-order Witten diagram for $\langle\langle TD \rangle\rangle$ has one bulk-to-boundary propagator for the graviton and one for the displacement field. They meet at a quadratic vertex ($\sim hX$) on the p -brane, as shown in figure 4. Using the vertices in (2.15) and recalling that $Y^j = \tau^0 X^j$ gives

$$\langle\langle T^{mn}(x) D_j(0) \rangle\rangle = -\frac{T_p^{(0)}}{2} \int \frac{d^{p+1}\tau}{(\tau^0)^{p-1}} \delta^{\alpha\beta} \left[\tau^0 K_{p+1}(0, \tau) \frac{\partial}{\partial w^j} K_{\alpha\beta}^{mn}(x, w) \Big|_{w=\tau} \right. \\ \left. + K_{\alpha j}^{mn}(x, \tau) \frac{\partial}{\partial \tau^\beta} (\tau^0 K_{p+1}(0, \tau)) + K_{\beta j}^{mn}(x, \tau) \frac{\partial}{\partial \tau^\alpha} (\tau^0 K_{p+1}(0, \tau)) \right]. \quad (3.5)$$

Here $K_{p+1}(0, \tau)$ is the scalar propagator for weight $\Delta = p+1$, and in the top line w^μ should be set equal to $(\tau^\alpha, 0)$ only after having taken the derivative of the propagator in the direction j .

The computation of the above diagram is straightforward but tedious. Below we present a sample calculation of the $x^i x^j x^k$ terms in (3.2c). All other terms can be handled in the same way.

The contraction $\delta^{\alpha\beta} K_{\alpha j}^{ik}$ was already performed in eq. (2.22). Only the first term in this expression will make a contribution proportional to $x^i x^j x^k$. Inserting the propagators in (3.5) and replacing $\partial/\partial w^j$ by $-\partial/\partial x^j$ we find

$$\langle\langle T^{ik}(x) D^j(0) \rangle\rangle = -4 T_p^{(0)} C_{p+1}^{[0]} C_D^{[2]} x^i x^j x^k \mathcal{I}(x), \quad (3.6)$$

where \mathcal{I} is the integral

$$\begin{aligned} \mathcal{I}(x) = \int d^{p+1}\tau (\tau^0)^{d+1} & \left[\frac{(d+2)|\tau|^2}{(|\tau|^2 + |x_{\perp}|^2)^{d+3} ((\tau^0)^2 + |\vec{\tau} + \vec{x}_{\parallel}|^2)^{p+1}} \right. \\ & \left. + \frac{(p+1)(|\tau|^2 - |\vec{x}_{\parallel}|^2) - ((\tau^0)^2 + |\vec{\tau} + \vec{x}_{\parallel}|^2)}{(|\tau|^2 + |x_{\perp}|^2)^{d+2} ((\tau^0)^2 + |\vec{\tau} + \vec{x}_{\parallel}|^2)^{p+2}} \right], \end{aligned} \quad (3.7)$$

and

$$C_{p+1}^{[0]} = \frac{\Gamma(p+1)}{\pi^{p/2} \Gamma(\frac{p}{2} + 1)}, \quad C_d^{[2]} = \frac{2(d+1)\Gamma(d-1)}{\pi^{d/2} \Gamma(\frac{d}{2})} \quad (3.8)$$

are the prefactors in the scalar and spin-2 propagators. One can decompose $\mathcal{I}(x)$ into a sum of primitive integrals

$$\begin{aligned} \mathcal{I}_{A,B,C} &= \int \frac{d^{p+1}\tau}{(\tau^0)^{p+1}} \frac{(\tau^0)^A}{((\tau^0)^2 + |\vec{\tau} - \vec{x}_{\parallel}|^2 + |x_{\perp}|^2)^B ((\tau^0)^2 + |\vec{\tau}|^2)^C} \\ &= \frac{1}{\Gamma(B)\Gamma(C)} \int \frac{d^{p+1}\tau}{(\tau^0)^{p+1-A}} \frac{ds}{s^{1-B}} \frac{dt}{t^{1-C}} e^{-(s+t)(\tau_0^2 + |\vec{\tau}|^2)} e^{-s|x_{\perp}|^2} e^{-st|\vec{x}_{\parallel}|^2/(s+t)}. \end{aligned} \quad (3.9)$$

Here we have used Schwinger's trick, and the last exponential in the lower line arises from completing the squares of $\vec{\tau}$ and \vec{x}_{\parallel} . Doing the τ integrations and inserting $1 = \int d\rho \delta(\rho - s - t)$ gives

$$\mathcal{I}_{A,B,C} = \frac{\pi^{p/2} \Gamma(\frac{A-p}{2})}{2\Gamma(B)\Gamma(C)} \int \frac{ds}{s^{1-B}} \frac{dt}{t^{1-C}} \frac{d\rho}{\rho^{A/2}} e^{-st|\vec{x}_{\parallel}|^2/\rho} e^{-s|x_{\perp}|^2} \delta(\rho - s - t).$$

Rescaling the dummy integration variables s and t by ρ , and then doing the ρ integration gives

$$\begin{aligned} \mathcal{I}_{A,B,C} &= \frac{\pi^{p/2} \Gamma(\frac{A-p}{2}) \Gamma(B+C-\frac{A}{2})}{2\Gamma(B)\Gamma(C)} \int \frac{ds dt s^{B-1} t^{C-1}}{(st|\vec{x}_{\parallel}|^2 + s|x_{\perp}|^2)^{B+C-A/2}} \delta(1-s-t) \\ &= \frac{\pi^{p/2} \Gamma(\frac{A-p}{2}) \Gamma(B+C-\frac{A}{2})}{2\Gamma(B)\Gamma(C)} \int dt \frac{(1-t)^{A/2-C-1} t^{C-1}}{(t|\vec{x}_{\parallel}|^2 + |x_{\perp}|^2)^{B+C-A/2}} \\ &= \frac{\pi^{p/2} \Gamma(\frac{A-p}{2}) \Gamma(B+C-\frac{A}{2}) \Gamma(\frac{A}{2}-C)}{2\Gamma(B)\Gamma(\frac{A}{2}) |x|^2^{(B+C)-A}} {}_2F_1 \left(\frac{A}{2} - C, B+C - \frac{A}{2}, \frac{A}{2}, \frac{|\vec{x}_{\parallel}|^2}{|x|^2} \right). \end{aligned} \quad (3.10)$$

In the last step we used Euler's representation of the hypergeometric function, valid under the assumptions $\text{Re}(A-2C) > 0$ and $\text{Re}(C) > 0$. For all $\mathcal{I}_{A,B,C}$ in the decomposition of (3.7) these hypergeometric functions reduce to simple functions.

Putting together eqs. (3.6) to (3.10) gives the $x^i x^j x^k$ term of $\langle\langle T^{ik}(x) D^j(0) \rangle\rangle$. After some straightforward algebra the result can be shown to agree with the lower line of (3.2c) for

the following values of the coefficients:

$$\mathbf{b}_1 = \frac{p+2}{2}\eta, \quad \mathbf{b}_2 = \frac{d^2 - p - 2}{2(p+1)}\eta, \quad \mathbf{b}_3 = d\eta \tag{3.11}$$

$$\text{with } \eta = -T_p^{(0)} \frac{2^{p+1}\Gamma(\frac{d-p}{2})\Gamma(\frac{p+3}{2})}{(d-1)\pi^{(d+1)/2}} \tag{3.12}$$

It takes a little more algebra to check that the above parameters $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ and the leading-order values of C_D, a_T computed in the previous section, eqs. (2.10) and (2.13), satisfy the Ward identities (3.3) and (3.4) on the nose. This confirms the source normalisations, as announced.

As a check of consistency we have computed the full Witten diagram (3.5) and verified that it reproduces the expressions (3.1), (3.2) with the above values of the parameters \mathbf{b}_i .

4 Examples

We consider now some holographic defects whose parameters C_D and a_T have been computed by other means. We will check that these reduce to (2.10) and (2.24) in the appropriate limit, and comment on the quantum ($\sim 1/T_p^{(0)}$) and gravitational ($\sim G_N$) corrections to the invariant tensions.

4.1 Maldacena-Wilson lines

The best-studied holographic defect is the half-BPS Maldacena-Wilson line of $\mathcal{N} = 4$ super Yang-Mills [8]. Its displacement norm is related to many interesting quantities, in particular to the Bremsstrahlung function $B(\lambda, N)$ which controls the energy emitted by an accelerating heavy quark. Here N is the rank of the gauge group $U(N)$, and λ is the 't Hooft coupling. As shown in refs. [14, 15] the following relations hold:

$$C_D = 12B = \frac{6}{\pi^2} \lambda \partial_\lambda \log \langle W_\odot \rangle, \tag{4.1}$$

where W_\odot is the circular Wilson loop whose expectation value was computed exactly as a matrix integral [45–47]

$$\langle W_\odot \rangle = \frac{1}{N} e^{\lambda/8N} L_{N-1}^{(1)} \left(-\frac{\lambda}{4N} \right) = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) + \frac{\lambda}{48N^2} I_2(\sqrt{\lambda}) + \dots \tag{4.2}$$

In this expression $L_n^{(a)}$ are the Laguerre polynomials and I_1, I_2 the modified Bessel functions of the first kind. Using supersymmetric localisation [47] one can derive similar expressions for the displacement norm of many other superconformal line defects in both $d = 4$ and $d = 3$.

In the planar ($N \rightarrow \infty$) limit one finds

$$C_D = \frac{3\sqrt{\lambda} I_2(\sqrt{\lambda})}{\pi^2 I_1(\sqrt{\lambda})} = \frac{3\sqrt{\lambda}}{\pi^2} - \frac{9}{2\pi^2} + O\left(\frac{1}{\sqrt{\lambda}}\right). \tag{4.3}$$

Now the relation between the 't Hooft coupling and the bare fundamental-string tension is $2\pi T_F \ell^2 = \sqrt{\lambda}$, with ℓ the AdS radius that we have set equal to one.⁸ Inserting this relation

⁸This is derived by expressing the D3-brane tension and Newton's constant in terms of the string coupling and T_F [1]. Here $T_F = T_1^{(0)}$ is the bare string tension.

in eq. (4.3) gives at the leading order $C_D = 6T_F/\pi + \dots$, in agreement with our Nambu-Goto calculation, eq. (2.10), for the case $p = 1$.

In this example all corrections to the inertial tension defined in eq. (1.3) are known exactly. Quantum fluctuations of the string, in particular, make contributions that are down by powers of $1/T_F \sim 1/\sqrt{\lambda}$, as expected. They are resummed by the Bessel functions in (4.3). Gravitational corrections, on the other hand, can be seen from (4.2) to be suppressed by powers of λ/N^2 . This is a milder suppression than the naively expected inverse Schwarzschild radius, $r_S^{-1} \sim (G_N T_F)^{-1} \sim \sqrt{\lambda}/N^2$. The emergence of this new scale is due to S duality. It can be understood by noting that

$$\lambda/N^2 \sim 1/\tilde{\lambda} \sim 1/T_D,$$

where $\tilde{\lambda}$ is the 't Hooft coupling of the S-dual gauge theory and T_D is the D-string tension. When $\lambda, N \gg 1$ the Compton wavelength of the D-string is larger than the Schwarzschild radius of the F-string, so corrections of the former to the effective supergravity dominate.

Consider next the gravitational tension defined in terms of the one-point function of the stress tensor in eq. (2.13). The latter has been also computed exactly [48, 49] with the result $a_T = -C_D/18$. This linear relation follows from superconformal Ward identities [30] which we will discuss in more detail in section 5. Inserting the above in eq. (2.13) with $d = 4, p = 1$ gives

$$T_F^{(I)} = -3\pi a_T = \frac{\pi C_D}{6} = T_F^{(II)} \tag{4.4}$$

where in the last step we used the definition (1.3). Thus the inertial and gravitational tensions of the F-string in $\text{AdS}_5 \times \text{S}^5$ are exactly equal.

4.2 Interfaces in $d = 2$

The only extended defects in $d = 2$ CFTs are line defects. Since these have codimension one, only inertial tension is defined. Integrating eq. (1.4) shows that the displacement operator is in this case the discontinuity of the stress tensor across the interface. The norm of the former can therefore be expressed in terms of the two-point function of the latter [12]. Explicitly⁹

$$C_D = \frac{1}{2\pi^2}(c_L + c_R - 2c_{LR}) \tag{4.5}$$

where c_L, c_R are the central charges of the CFTs that live on either side of the interface, and c_{LR} controls the two-point function of the stress tensor across the interface [50]. As shown in [16], c_{LR} also controls the (universal) fraction of energy transmitted across the interface: c_{LR}/c_L is the transmitted fraction of energy for excitations coming from the left, and c_{LR}/c_R for those incident from the right.

Although C_D can be readily computed in many I(nterface)CFT₂ models, much less is known about it in theories with exact holographic duals.¹⁰ One can however compute it in a

⁹The extra factor of $4\pi^2$ in this reference comes from a different definition of the energy-momentum tensor, $T_{ab} = \frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g_{ab}}$ as opposed to $\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{ab}}$. The extra 2π is convenient in two dimensions, but we use the canonical convention for arbitrary d .

¹⁰In the simplest example of Janus interfaces, c_{LR} has been computed in gravity [20] as well as at the symmetric-orbifold point of the conjectured dual CFT [51]. But how to extrapolate between these two calculations is unclear.

bottom-up model of a thin back-reacting 1-brane in AdS₃ with the result [17, 18]

$$c_{LR} = \frac{3}{G_N} \left(\frac{1}{\ell_L} + \frac{1}{\ell_R} + 8\pi G_N T_1^{(0)} \right)^{-1}, \quad (4.6)$$

where ℓ_L and ℓ_R are the AdS radii on either side of the brane. The brane is here treated as classical and thin, but its full back reaction is accounted for by solving Israel matching conditions.

Taking the zero-tension limit of these conditions can be tricky (for a recent discussion see [52]), but we avoid such subtleties by setting $\ell_L = \ell_R = \ell$, as appropriate for probe branes. Using the Brown-Henneaux formula [53] $c = 3\ell/2G_N$ and inserting (4.6) in (4.5) gives (in units $\ell = 1$)

$$C_D = \frac{6T_1^{(0)}/\pi}{1 + 4\pi G_N T_1^{(0)}}. \quad (4.7)$$

This agrees with the Nambu-Goto formula (2.10) for $G_N \rightarrow 0$. Note also that the Israel matching conditions repackage the gravitational screening of the bare tension into a simple denominator.

4.3 Graham-Witten anomalies

As one last check, consider even- p defects which are known to have new Weyl anomalies of both type-A and type-B [28, 54]. For surface defects ($p = 2$) in particular there are three irreducible anomalies

$$T_m^m|_{\text{Defect}} = \frac{1}{24\pi} (\mathbf{a}^{(2)} R + \mathbf{d}_1^{(2)} \bar{K}_{ab}^i \bar{K}_i^{ab} - \mathbf{d}_2^{(2)} W_{ab}^{ab}). \quad (4.8)$$

Here R is the Ricci scalar on the defect, K_{ab}^i is the traceless part of the extrinsic curvature, and W_{abcd} is the pullback of the bulk Weyl tensor. The coefficients $\mathbf{d}_1^{(2)}$ and $\mathbf{d}_2^{(2)}$ are proportional, respectively, to C_D and to a_T .¹¹ DCFT calculations give [55–57]

$$C_D = \frac{4\mathbf{d}_1^{(2)}}{3\pi^2} \quad (d = 4); \quad a_T = -\frac{d\Gamma(\frac{d}{2} - 1)\mathbf{d}_2^{(2)}}{12(d - 1)\pi^{d/2}} \quad (\forall d > 3), \quad (4.9)$$

while from the probe-brane holographic calculation of ref. [28] one finds

$$\mathbf{d}_1^{(2)} = \mathbf{d}_2^{(2)} = 6\pi T_2^{(0)} \quad (4.10)$$

for all $d > 3$. Eliminating the anomaly coefficients gives the $p = 2$ relations (2.10) and (2.24), which we obtained from Witten diagrams. Note that the mathematical construction of [28] fixes the normalisation of the displacement operator consistently with the Ward identities at leading order; but we don't know if the cutoff subtleties discussed in sections 2 and 3 can be incorporated so as to compute quantum and gravitational corrections. Besides giving in one stroke the leading result for all values of p and d , the expansion in terms of Witten diagrams is presumably the first step towards a systematic calculation of such corrections.

¹¹This is consistent with the fact that the Weyl tensor vanishes identically in $d = 3$ where the surface is an interface and thus a_T is zero.

We can also compare with the results of [27] for $p = 4$ defects. There are in this case 22 B-type anomalies, two of which are proportional to C_D and a_T , as in the $p = 2$ case. We quote from this reference:

$$C_D = -\frac{72}{\pi^4} \mathbf{d}_1^{(4)}; \quad a_T = \frac{d\Gamma(\frac{d}{2} - 1)}{(d - 1)\pi^{d/2}} \mathbf{d}_2^{(4)}. \quad (4.11)$$

Furthermore the calculation of the Willmore energy of 5-dimensional submanifolds gives in the holographic probe limit [27, 29]

$$\mathbf{d}_1^{(4)} = -\pi^2 T_4^{(0)}; \quad \mathbf{d}_2^{(4)} = -\frac{\pi^2 T_4^{(0)}}{d - 4}. \quad (4.12)$$

By eliminating $\mathbf{d}_1^{(4)}$ we recover our (d -independent) relation (2.10) for $p = 4$. Eliminating likewise $\mathbf{d}_2^{(4)}$ gives

$$-a_T = \frac{d\Gamma(\frac{d}{2} - 1)\pi^2 T_4^{(0)}}{(d - 1)(d - 4)\pi^{d/2}} = \frac{d\Gamma(\frac{d-4}{2})T_4^{(0)}}{2(d - 1)\pi^{(d-4)/2}} \quad (4.13)$$

which matches our eq. (2.24) for $p = 4$ at leading order.

5 Outlook and a conjecture

The take-away message of this paper is that one can define two invariant notions of holographic tension: gravitational tension, which is the analog of the ADM mass, and stiffness or inertial tension. Both reduce to the bare tension $T_p^{(0)}$ for classical probes coupled to Einstein gravity, but in general (when they are both defined) they are different.

These tensions are proportional to the DCFT parameters $-a_T$ and C_D , both of which are positive in a unitary theory.¹² They vanish for topological defects, which do not couple to the CFT stress tensor and can be deformed freely. An interesting question that we did not explore is whether these tensions obey a BPS bound when the defect couples to a conserved p -form current, i.e. when the dual brane is charged.

Another question worth investigating is how tension behaves under fusion. In the thin-brane model of section 4.2 tensions simply add up [19], but we have checked with $d = 2$ free-field interfaces [59–61] that the difference between the tension of the fusion product and the sum of the constituent tensions can have either sign. Note however that in these examples the tension is inertial since gravitational tension is not defined for $2d$ line defects. Such issues may be also relevant for the swampland conjectures that feature extended objects, see e.g. [62–64].

Here, however, we will conclude by coming back to the observation of section 4.1 that the inertial and gravitational tensions of the F-string in $\text{AdS}_5 \times \text{S}^5$ are exactly equal at all orders.

This is related to an interesting physics conundrum, as explained by Lewkowycz and Maldacena [65]. The Bremsstrahlung parameter $B = C_D/12$ controls the energy emitted by

¹²It was pointed out in ref. [27] that a_T is positive for n -fold cover boundary conditions. This is consistent with the fact that a surplus-angle defect has negative tension, and suggests that such defects might be pathological. A similar question has been raised in ref. [58].

an accelerating quark, while the parameter a_T controls the energy collected at a distance. If the two were unrelated, the emitted and collected energies would not be the same. Ref. [65] attributes this to the difficulty of separating the radiated energy from the self energy of the quark, and suggests why such a separation might be possible in the case of supersymmetric Wilson lines.

Whatever the resolution of the conundrum, the technical reason behind the relation (4.4) is understood [30, 66] and can be sketched as follows. The DCFT Ward identities relate the one-point function of *any* primary operator \mathcal{O} to its two-point function with the displacement operator, schematically $\langle\langle \mathcal{O} \rangle\rangle \sim \langle\langle \mathcal{O} D \rangle\rangle$. For a scalar operator this determines $\langle\langle \mathcal{O} D \rangle\rangle$ completely, but for a spin-2 primary an unknown parameter remains (see appendix A for details). When the spin-2 is the stress tensor, the conservation equation eq. (1.4) gives an extra relation of the form $\langle\langle DD \rangle\rangle \sim \langle\langle TD \rangle\rangle$ which determines this residual parameter in terms of C_D . So in general $\langle\langle TD \rangle\rangle$ is completely fixed, but C_D and a_T are unconstrained. Suppose however that T^{mn} has a scalar superconformal partner whose two-point function, as we saw, is fixed by its one-point function. Supersymmetry may in this case relate the missing parameter in $\langle\langle TD \rangle\rangle$ also to a_T , and thus relate this latter to the displacement norm C_D .

Bianchi and Lemos [31] realised that the above argument can apply to superconformal defects other than Wilson lines. They considered half-BPS surface defects ($p = 2$) in $d = 4, \mathcal{N} = 4$ super Yang-Mills and found that in this case $C_D = -12a_T$. Using the definitions (1.3) and (2.13) it is easy to check that the dual membranes have $T_2^{(I)} = T_2^{(II)}$. Another example are BPS surface defects in $d = 6$ for which ref. [67] found $C_D = -40\pi a_T/3$. This implies again $T_2^{(I)} = T_2^{(II)}$.

It is then natural to conjecture that whenever the superconformal Ward identities relate C_D and a_T , the inertial and gravitational tensions coincide,

$$T_p^{(I)} = T_p^{(II)} \iff C_D = -a_T \frac{2(d-1)(p+2)\Gamma(p+1)}{d \pi^{p-d/2} \Gamma(\frac{p}{2}+1)\Gamma(\frac{d-p}{2})}. \quad (5.1)$$

The authors of [31] made a similar guess by extrapolating known results on the parameters C_D and a_T of superconformal defects. They proposed that

$$C_D = h \frac{2^{p+1}(d-1)(p+2)\Gamma(\frac{p+1}{2})\pi^{(d-p)/2}}{(d-p-1)\pi^{(p+1)/2}\Gamma(\frac{d-p}{2})} \quad (5.2)$$

where $h = -a_T(d-p-1)/d$.¹³ The two conjectures look different, but they are actually the same. This follows from the identity

$$\frac{\Gamma(p+1)\sqrt{\pi}}{2^p\Gamma(\frac{p}{2}+1)\Gamma(\frac{p+1}{2})} = 1 \quad \forall \text{ integer } p, \quad (5.3)$$

which one can prove by using the following identity for integer k :

$$\Gamma\left(k + \frac{1}{2}\right) = \sqrt{\pi} \frac{1 \times 3 \times \dots \times (2k-1)}{2^k} = \sqrt{\pi} \frac{(2k)!}{k!2^{2k}} = \frac{\sqrt{\pi}\Gamma(2k+1)}{2^{2k}\Gamma(k+1)}.$$

Stating the conjecture as $T_p^{(I)} = T_p^{(II)}$ is elegant and economic, but the deeper significance, if any, of this observation is not clear.

¹³We thank Lorenzo Bianchi for a communication on this issue.

As we have seen, the two tensions are equal to the bare tension in the thin-classical-probe limit, but both receive quantum and gravitational corrections. More generally, they may depend on all the bulk and defect DCFT moduli. We expect however the superconformal Ward identities to act homogeneously in the moduli space, as in the examples of refs. [30, 31]. So if $T_p^{(I)}$ and $T_p^{(II)}$ are equal at some point they should stay equal everywhere.

A first step towards an exhaustive proof would be to classify (possibly along the lines of ref. [68]) all superconformal DCFTs in which the Ward identities lead to a linear relation between C_D and a_T .

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Note added. It was pointed out to us by Shira Chapman that the considerations of this paper could be applied to the $p = d - 2$ twist defects, τ_n , which enter in the calculation of Rényi entropies [69]. Using the results in [70, 71] one finds $T^{(I)} \neq T^{(II)}$ for general n , but $T^{(I)} = T^{(II)}$ in the limit $n \rightarrow 1$. This is consistent with the fact that, contrary to the ‘cosmic brane’ duals of general twist defects [72], the Ryu-Takayanagi branes that calculate entanglement entropy [73] do have a classical-probe limit.

A Constraints on DCFT correlators

To make the paper self-contained, we derive here the general form of the DCFT correlation functions used in the main text. The convenient tool is the embedding-space or light-cone formalism. This appendix is based on ref. [12] but we work with tensor indices (as in [74] for the homogeneous case) rather than with the auxiliary vector as in [12].

In the embedding-space formalism \mathbb{R}^d is identified with the projective $(d + 2)$ -dimensional lightcone

$$X^M X_M = -X^+ X^- + X^m X_m = 0, \quad X^M \sim \lambda X^M. \tag{A.1}$$

Here $m = 1, \dots, d$. The relation of X^M to the physical coordinates is

$$X^M = X^+(1, x^2, x^m) \iff x^m = X^m / X^+. \tag{A.2}$$

One may choose the section $X^+ = 1$, but Lorentz transformations of the embedding space need not respect this gauge.

To impose the projection $X^M \sim \lambda X^M$, one limits attention to tensors in the ambient space that obey the scaling relation

$$T^{M_1 \dots M_r}(\lambda X) = \lambda^{-\Delta} T^{M_1 \dots M_r}(X) \tag{A.3}$$

for some real Δ . We also impose the transversality conditions

$$X_{M_1} T^{M_1 \dots M_r}(X) = \dots = X_{M_r} T^{M_1 \dots M_r}(X) = 0. \quad (\text{A.4})$$

The physical-space tensor is obtained by the pullback

$$\hat{T}^{m_1 \dots m_r}(x) = (X^+)^{\Delta+r} \frac{\partial x^{m_1}}{\partial X^{M_1}} \dots \frac{\partial x^{m_r}}{\partial X^{M_r}} T^{M_1 \dots M_r}(X). \quad (\text{A.5})$$

The pullback is first defined in the ambient space before restricting to the lightcone, i.e. X^- is treated as an independent variable so that

$$X^+ \frac{\partial x^m}{\partial X^M} = (-x^m, 0, \delta_n^m). \quad (\text{A.6})$$

Since (A.5) is invariant under rescalings $X^M \rightarrow \lambda X^M$, the physical-space tensor $\hat{T}^{m_1 \dots m_r}$ only depends on the coordinate x^m as it should. Furthermore $X^M (\partial x^m / \partial X^M) = 0$, so $\hat{T}^{m_1 \dots m_r}$ is unaffected by shifts

$$T^{M_1 \dots M_r}(X) \rightarrow T^{M_1 \dots M_r}(X) + X^{M_1} \Omega^{M_2 \dots M_r}(X) \quad (\text{A.7})$$

for any tensor Ω with $r - 1$ indices. This explains why only d out of the $d + 2$ components for each index M are physical. A straightforward but tedious calculation shows that $\hat{T}^{m_1 \dots m_r}$ transforms as a conformal (quasi-)primary tensor field with scaling dimension Δ [74].

One can also show that any partial trace of $\hat{T}^{m_1 \dots m_r}$ is proportional to the corresponding trace of $T^{M_1 \dots M_r}$. Using for instance $\eta_{+-} = -\frac{1}{2}$ and the transversality condition $X_M T^{M+} = 0$ for a 2-index tensor, one finds

$$\delta_{mn} \hat{T}^{mn} = (X^+)^{\Delta+2} \eta_{MN} T^{MN}. \quad (\text{A.8})$$

Furthermore the pullback (A.5) preserves the (anti)symmetrization of the ambient-space tensor, so irreducible tensor representations of $\text{SO}\mathfrak{S}(1, d + 1)$ descend to irreducible representations of $\text{SO}(d)$.

The formalism is easily adapted to accommodate a planar p -dimensional defect. The defect breaks the symmetry to $\text{SO}(1, p + 1) \times \text{SO}(d - p)$, so we write $M = (A, i)$ where $A = +, -, 1, \dots, p$ labels the directions along the defect lifted to the ambient space and $i = p + 1, \dots, d$ labels the transverse directions. There are now two invariant tensors, η^{AB} and δ^{ij} . Following [12] we denote the corresponding inner products $X \circ Y := X^i Y^j \delta_{ij}$ and $X \bullet Y := X^A Y^B \eta_{AB}$. Clearly $X \circ X + X \bullet X = 0$ since the ambient space vector X is null.

Let us see how the unbroken symmetry constrains the one-point function of a spin-2 primary. The most general ansatz allowed by the symmetries and by the scaling (A.3) is

$$(X \circ X)^{\Delta/2} \langle\langle T^{AB} \rangle\rangle = a_1 \eta^{AB} + a_2 \frac{X^A X^B}{X \circ X}, \quad (\text{A.9a})$$

$$(X \circ X)^{\Delta/2} \langle\langle T^{Ai} \rangle\rangle = a_3 \frac{X^A X^i}{X \circ X}, \quad (\text{A.9b})$$

$$(X \circ X)^{\Delta/2} \langle\langle T^{ij} \rangle\rangle = a_4 \delta^{ij} + a_5 \frac{X^i X^j}{X \circ X}, \quad (\text{A.9c})$$

where the a_i are arbitrary coefficients. Transversality, eq. (A.4), implies

$$a_1 - a_2 + a_3 = 0 \quad \text{and} \quad a_3 - a_4 - a_5 = 0. \quad (\text{A.10})$$

Eliminating a_2 and a_5 in (A.9) gives

$$(X \circ X)^{\Delta/2} \langle\langle T^{AB} \rangle\rangle = a_1 \left(\eta^{AB} + \frac{X^A X^B}{X \circ X} \right) + a_3 \frac{X^A X^B}{X \circ X}, \quad (\text{A.11a})$$

$$(X \circ X)^{\Delta/2} \langle\langle T^{Ai} \rangle\rangle = a_3 \frac{X^A X^i}{X \circ X}, \quad (\text{A.11b})$$

$$(X \circ X)^{\Delta/2} \langle\langle T^{ij} \rangle\rangle = a_4 \left(\delta^{ij} - \frac{X^i X^j}{X \circ X} \right) + a_3 \frac{X^i X^j}{X \circ X}. \quad (\text{A.11c})$$

The terms proportional to a_3 combine to a contribution $\langle\langle T^{MN} \rangle\rangle \propto X^M X^N$ which can be dropped by using the shift symmetry (A.7). Finally the zero-trace condition gives one more relation

$$(p+1)a_1 + (d-p-1)a_4 = 0 \implies a_1 = \left(1 - \frac{d}{p+1} \right) a_4. \quad (\text{A.12})$$

Thus the one-point function depends on a single parameter, say a_4 .

To pull back to the physical stress tensor, eq. (A.5), we use the identities

$$\frac{\partial x^a}{\partial X^A} X^A = \frac{\partial x^a}{\partial X^i} X^i = 0, \quad -\frac{\partial x^i}{\partial X^A} X^A = \frac{\partial x^i}{\partial X^j} X^j = x^i \quad (\text{A.13})$$

where we separated parallel and transverse indices, $m = (a, i)$, see section 2 for conventions and notation. The result reads

$$\langle\langle \hat{T}^{ab} \rangle\rangle = \frac{a_1}{|x_\perp|^\Delta} \delta^{ab}, \quad \langle\langle \hat{T}^{ij} \rangle\rangle = \frac{a_4}{|x_\perp|^\Delta} \delta^{ij} - \frac{(a_4 - a_1)}{|x_\perp|^{\Delta+2}} x^i x^j \quad (\text{A.14})$$

with $|x_\perp|^2 = \sum_i x^i x^i$. As a check one can verify that $\partial_m \langle\langle \hat{T}^{mn} \rangle\rangle = 0$ if and only if $\Delta = d$, the canonical dimension of the stress tensor. Note that the one-point function vanishes identically in the special case $d = p + 1$, i.e. when the defect is an interface or boundary. Eqs. (A.14) are the same as (2.11) with the identification $a_T = a_4 - a_1 = a_4 d / (p + 1)$, and with hats dropped from the physical stress tensor.

We turn next to the two-point function $\langle\langle \hat{T}^{mn}(x) D^j(y) \rangle\rangle$ and its ambient-space precursor $\langle\langle T^{MN}(X) D^j(Y) \rangle\rangle$, where D^j is the displacement operator. Since Y is a point on the defect, $Y^j = 0$ and hence $X \circ Y = Y \circ Y = 0$. The only non-zero scalar products are $X \circ X = (X^+)^2 |x_\perp|^2$ and

$$X \bullet Y = -\frac{1}{2} X^+ Y^+ |x - y|^2. \quad (\text{A.15})$$

D^j has dimension $p + 1$, and we again leave the dimension of the spin-2 free for now. The most general form with the correct scaling symmetries is

$$\langle\langle T^{MN}(X) D^j(Y) \rangle\rangle = (-2X \bullet Y)^{-p-1} (X \circ X)^{\frac{1}{2}(p+1-\Delta)} P^{MNj} \quad (\text{A.16})$$

with P^{MNj} built from the scale-invariant tensors

$$\eta^{AB}, \quad \delta^{ij}, \quad \mathfrak{x}^M = \frac{X^M}{(X \circ X)^{1/2}} \quad \text{and} \quad \mathfrak{y}^A = \frac{Y^A (X \circ X)^{1/2}}{(-2X \bullet Y)}. \quad (\text{A.17})$$

The most general eleven-parameter ansatz for this tensor is

$$P^{ABj} = b_1 \eta^{AB} \mathfrak{x}^j + b_2 \mathfrak{x}^A \mathfrak{x}^B \mathfrak{x}^j + b_3 \mathfrak{y}^{(A} \mathfrak{x}^{B)} \mathfrak{x}^j + b_4 \mathfrak{y}^A \mathfrak{y}^B \mathfrak{x}^j, \quad (\text{A.18a})$$

$$P^{iBj} = b_5 \delta^{ij} \mathfrak{x}^B + b_6 \delta^{ij} \mathfrak{y}^B + b_7 \mathfrak{x}^i \mathfrak{x}^B \mathfrak{x}^j + b_8 \mathfrak{x}^i \mathfrak{y}^B \mathfrak{x}^j, \quad (\text{A.18b})$$

$$P^{ikj} = b_9 \delta^{ik} \mathfrak{x}^j + b_{10} \delta^{j(i} \mathfrak{x}^{k)} + b_{11} \mathfrak{x}^i \mathfrak{x}^k \mathfrak{x}^j, \quad (\text{A.18c})$$

where parentheses denote symmetrization.

The three shift symmetries $\delta P^{MNj} \propto \mathfrak{x}^M \mathfrak{x}^N \mathfrak{x}^j$, or $\mathfrak{x}^{(M} \mathfrak{y}^{N)} \mathfrak{x}^j$ or $\mathfrak{x}^{(M} \delta^{N)j}$ allow us to set $b_3 = b_7 = b_{10} = 0$ without affecting the physical correlator. Transversality, $X_M P^{MBj} = X_M P^{iMj} = 0$, then requires

$$\begin{aligned} b_1 - b_2 + b_5 &= 0, & -\frac{1}{2}b_4 + b_6 + b_8 &= 0, \\ -b_5 - \frac{1}{2}b_6 &= 0, & -\frac{1}{2}b_8 + b_9 + b_{11} &= 0. \end{aligned} \quad (\text{A.19})$$

Solving for b_2, b_4, b_5 and b_{11} leaves four free parameters,

$$P^{ABj} = b_1 \eta^{AB} \mathfrak{x}^j + \left(b_1 - \frac{b_6}{2}\right) \mathfrak{x}^A \mathfrak{x}^B \mathfrak{x}^j + 2(b_6 + b_8) \mathfrak{y}^A \mathfrak{y}^B \mathfrak{x}^j, \quad (\text{A.20a})$$

$$P^{iBj} = -\frac{b_6}{2} \delta^{ij} \mathfrak{x}^B + b_6 \delta^{ij} \mathfrak{y}^B + b_8 \mathfrak{x}^i \mathfrak{y}^B \mathfrak{x}^j, \quad (\text{A.20b})$$

$$P^{ikj} = b_9 \delta^{ik} \mathfrak{x}^j + \left(\frac{b_8}{2} - b_9\right) \mathfrak{x}^i \mathfrak{x}^k \mathfrak{x}^j. \quad (\text{A.20c})$$

The zero-trace condition

$$b_1(p+1) + \frac{1}{2}(b_6 + b_8) + b_9(d-p-1) = 0 \quad (\text{A.21})$$

eliminates one more leaving three.

To compute the pullback (A.5) one needs, in addition to (A.13), the following pullback identity (in the $X^+ = Y^+ = 1$ gauge)

$$\frac{\partial x^m}{\partial X^A} Y^A = y^m - x^m := -z^m. \quad (\text{A.22})$$

The physical correlation functions read

$$\langle\langle \hat{T}^{ab}(x) D^j(y) \rangle\rangle = \frac{z^j |z_\perp|^{p-\Delta}}{|z|^{2p+2}} \left[b_1 \delta^{ab} + 2(b_6 + b_8) \frac{z^a z^b |z_\perp|^2}{|z|^4} \right], \quad (\text{A.23a})$$

$$\langle\langle \hat{T}^{ib}(x) D^j(y) \rangle\rangle = \frac{z^b |z_\perp|^{p+2-\Delta}}{|z|^{2p+4}} \left[-b_6 \delta^{ij} - b_8 \frac{z^i z^j}{|z_\perp|^2} + 2(b_6 + b_8) \frac{z^i z^j}{|z|^2} \right], \quad (\text{A.23b})$$

$$\begin{aligned} \langle\langle \hat{T}^{ik}(x) D^j(y) \rangle\rangle &= \frac{|z_\perp|^{p+2-\Delta}}{|z|^{2p+2}} \left[b_9 \delta^{ik} \frac{z^j}{|z_\perp|^2} + \frac{b_6}{2} \delta^{j(i} z^{k)} \left(\frac{1}{|z_\perp|^2} - \frac{2}{|z|^2} \right) \right. \\ &\quad \left. + \left(b_1 - \frac{b_6}{2} + \frac{b_8}{2} - b_9 \right) \frac{z^i z^k z^j}{|z|^4} - 2b_8 \frac{z^i z^k z^j}{|z_\perp|^2 |z|^2} + 2(b_6 + b_8) \frac{z^i z^k z^j}{|z|^4} \right], \end{aligned} \quad (\text{A.23c})$$

where the scalar invariants are $|z_{\perp}|^2 = z^i z_i$ and $|z|^2 = z^{\mu} z_{\mu} = z^{\alpha} z_{\alpha} + |z_{\perp}|^2$. We may choose the displacement operator insertion at $y = 0$, so that $z^m = x^m$. The parameters defined in ref. [12] are

$$\mathfrak{b}_1 = \frac{1}{2}(b_6 + b_8), \quad \mathfrak{b}_2 = b_1 - b_9, \quad \mathfrak{b}_3 = b_6, \tag{A.24}$$

where b_8 can be eliminated with the help of condition (A.21). One can now check that (A.23) reduce to the expressions (3.1), (3.2) of the main text. Note that the tensor structure in (A.23) is independent of the dimension Δ of the spin-2 operator.

As explained in ref. [12], broken-conformal Ward identities constraint the parameters \mathfrak{b}_j . The first set of identities is universal, i.e. valid for all primary bulk operators. It translates the fact that the displacement operator encodes the effect of conformal transformations on correlation functions in the presence of a flat defect. For a scalar operator \mathcal{O} this determines completely $\langle\langle \mathcal{O} D^j \rangle\rangle$ in terms of the one-point function $\langle\langle \mathcal{O} \rangle\rangle$. Explicitly

$$\langle\langle \mathcal{O}_{\Delta} \rangle\rangle = \frac{a_{\mathcal{O}}}{|x_{\perp}|^{\Delta}}, \quad \langle\langle \mathcal{O}_{\Delta}(x) D^j(y) \rangle\rangle = \frac{b_{\mathcal{O}D} z^j}{|z_{\perp}|^{\Delta-p} |z|^{2p+2}}, \tag{A.25}$$

$$\text{with} \quad \Delta a_{\mathcal{O}} = \frac{\pi^{(p+1)/2}}{2^p \Gamma(\frac{1}{2}(p+1))} b_{\mathcal{O}D}. \tag{A.26}$$

For a spin-2 these identities fix only two of the three parameters \mathfrak{b}_i :

$$(p+1)\mathfrak{b}_2 = \left(\frac{\Delta}{2}\mathfrak{b}_3 - \mathfrak{b}_1\right) \quad \text{and} \quad \mathfrak{b}_3 = 2 \left(\frac{2}{\sqrt{\pi}}\right)^{p+1} \Gamma\left(\frac{p+3}{2}\right) a_T. \tag{A.27}$$

The remaining free parameter can be fixed by supersymmetry if the spin-2 has a scalar-primary superpartner and for suitable superconformal defects. When the spin-2 is the CFT stress tensor the integrated conservation eq. (1.4) gives an extra constraint [12]

$$2p \mathfrak{b}_2 - (2d - p - 2)\mathfrak{b}_3 = (d - p)\pi^{(p-d)/2} \Gamma\left(\frac{d-p}{2}\right) C_D. \tag{A.28}$$

This constraint is special to the stress tensor and hence requires $\Delta = d$. As noticed by the authors of [30, 31], it can be used to relate C_D to a_T when supersymmetry fixes all three \mathfrak{b}_j in terms of the one-point function.

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